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OBTUSE RATIONAL LATTICE TRIANGLES

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Abstract. A rational triangle is a lattice triangle if the Veech group of its unfolding is a lattice. Answering a question of Hooper, we prove that an obtuse rational triangle is a lattice triangle if and only if its angles are $(\pi/n, \pi/n, (n-2)\pi/n)$ with $n \geq 5$, $(\pi/2k, \pi/k, (2k-3)\pi/2k)$ with $k \geq 4$, or $(\pi/12, \pi/3, 7\pi/12)$. The idea is a first-order test on relative periods. By Möller's theorem, on a Veech surface the difference of two zeros of the holomorphic form is torsion in the factor of the Jacobian with real multiplication, and stays torsion along the Teichmüller curve. So certain relative periods vanish along the Teichmüller disc, and on a triangle unfolding their first derivative is an integral over the plane that can be computed in closed form. For obtuse angle at most $2\pi/3$ we determine for every denominator where the criterion of Mirzakhani and Wright is blind, and exclude all but four of the triangles there by relative periods and by cylinders whose moduli or areas cannot occur on a Veech surface. For obtuse angle above $2\pi/3$ the proof uses a theorem of Larsen, Norton and Zykoski. Parts of the proof are computer-assisted.

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1. Introduction

A billiard ball in a polygon whose angles are rational multiples of π moves, after unfolding, along straight lines on a compact translation surface: the surface glued from the copies of the polygon under the finite group generated by the linear parts of the reflections in its sides [32]. Veech [28] singled out the translation surfaces whose affine automorphisms have derivatives forming a lattice in $\mathrm{SL}_2(\mathbb{R})$. On such a *Veech surface* the straight-line flow in

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each direction is either completely periodic or uniquely ergodic, and the number of periodic families of length at most a given bound grows like a constant times the square of the bound. A rational polygon is a *lattice polygon* when its unfolding is a Veech surface. Veech showed that the obtuse isosceles triangles with angles $(\pi/n, \pi/n, (n-2)\pi/n)$, $n \geq 5$, are lattice polygons [28, Theorem 1.1 and §8], and the question of which rational triangles are lattice polygons has been studied since.

We write $(p, q, r)/n$ for the triangle with angles $p\pi/n$, $q\pi/n$, $r\pi/n$, where $p \leq q \leq r$ are positive integers with $\gcd(p, q, r) = 1$ and $n = p + q + r$; the integer n is the *denominator*. The triangle is obtuse exactly when $r > n/2$. It is a *lattice triangle* if the Veech group of its unfolding is a lattice, the unfolding being taken as a translation surface whose only marked points are its cone points of angle above 2π; precise definitions are in Section 2.

Theorem 1.1. *An obtuse rational triangle is a lattice triangle if and only if it is* $(1, 1, n-2)/n$ *with* $n \geq 5$, $(1, 2, 2k-3)/2k$ *with* $k \geq 4$, *or* $(1, 4, 7)/12$.

The obtuse lattice triangles are those of Veech, of Vorobets and Ward, and Hooper's triangle. This answers a question of Hooper [11, Open Question 2]. McMullen lists the known lattice triangles and asks whether the list is complete [21, Theorem 3.9 and Question 3.11]. The obtuse triangles of his list are all the obtuse lattice triangles, which answers his question for obtuse triangles. Theorem 1.1 also proves the conjecture, as stated by Angdinata, Chen, Ono, Thorner, Zhang and Zhang, that there are no obtuse lattice triangles other than the known ones [1, p. 2]. By Theorem 1.1 there are at most two obtuse lattice triangles of each denominator other than 12, and three of denominator 12. The proof is assembled in Section 7.

The idea of the proof is a first-order test on relative periods. By the criterion of Mirzakhani and Wright in the form of Larsen, Norton and Zykoski (Proposition 2.4), an obtuse triangle with a witness is not a lattice triangle; a witness is a unit a modulo n with $2a \not\equiv 2 \pmod{n}$ satisfying at least two of three inequalities, defined in Section 2.3. The criterion compares two eigenforms of the cyclic symmetry of the unfolding through the variation of the period matrix. We say that the criterion is *blind* on a triangle with no witness. Where it is blind we use the relative periods instead. On a Veech surface the difference of two zeros of the holomorphic form is a torsion point of the factor of the Jacobian with real multiplication by the trace field, of order dividing a fixed integer at every surface of the Teichmüller curve, by Möller's theorem (Proposition 3.2). On the unfolding of a lattice triangle the eigenforms of the cyclic symmetry whose character is a unit modulo n lie in the cohomology of that factor, so the relative period of such an eigenform between two zeros, over a suitable rational relative class, vanishes along the whole Teichmüller disc. Its first derivative is given by integrals over the plane, which can be computed in closed form. For an integer x let $[x]_n$ be its residue in $[0, n)$.

Theorem 1.2. *Let* $\Delta = (p, q, r)/n$ *be a rational triangle.*

(a) *If* Δ *is obtuse,* n *is odd, and* $p \nmid n$ *or* $q \nmid n$, *then* Δ *is not a lattice triangle.*

(b) *Let* $x \in \{p, q, r\}$ *with* $x \nmid n$ *and* $g = \gcd(x, n) > 1$, *and let* b *be a unit modulo* n *with* $b \equiv 2 \pmod{n/g}$, $b \not\equiv 2 \pmod{n}$ *and* $[bp]_n + [bq]_n + [br]_n = n$. *Then* Δ *is not a lattice triangle.*

The condition $[bp]_n + [bq]_n + [br]_n = n$ in (b) says that the eigenspace of character b contains a holomorphic form. Part (a) places no condition on the size of the obtuse angle, and part (b) none on the angles or on the parity of n.

The proof of Theorem 1.1 has three cases, by the parity of n and the size $r\pi/n$ of the obtuse angle. The criterion rules out every triangle with a witness. When the obtuse angle is at most $2\pi/3$, the *window* of Section 4, it is blind exactly on three families and a list (Theorem 4.1): $\mathcal{A}$ and $\mathcal{B}$, in which $s = 2r - n$ divides n with odd quotient, with $p = 1$ in $\mathcal{A}$, and $2 \le p \le s/2$ and a condition on units in $\mathcal{B}$; $\mathcal{C}$, in which $s = 2$, $4 \mid n$ and p is 1, 2 or, for $n \equiv 4 \pmod 8$, 4; and the list $\mathcal{E}$ of thirty triangles with $n \le 48$. Above $2\pi/3$ it is blind exactly on the six one-parameter families (1) to (6) and the seven triples (7) of Theorem 2.5. For each case the table names the result that says where it is blind, the lattice triangles, and the result that rules out each of the other triangles; *cylinders* means two parallel cylinders whose moduli have an irrational ratio, or one whose share of the area has a Galois conjugate outside $(0, 1]$, which cannot occur on a Veech surface (Lemma 2.3).

case; lattice triangles	other triangles	ruled out by
n odd (Theorem 4.2); $(1, 1, n-2)/n$	$p \nmid n$ or $q \nmid n$	Theorem 1.2(a)
	$(1, 5, 9)/15$	Theorem 1.2(b)
	$(1, 3, 5)/9$, $(3, 7, 11)/21$	Theorem 6.1
n even, $r > 2n/3$ (Theorem 2.5); $(1, 1, n-2)/n$, $(1, 2, n-3)/n$	families (3) to (6) of Theorem 2.5	cylinders: [17, §10], Theorems 5.16, 5.9 and 5.11
	the triples of (7)	Lemma 7.1 or Theorem 6.1
n even, $r \le 2n/3$ (Theorem 4.1); $(1, 1, 4)/6$, $(1, 2, 5)/8$, $(1, 4, 7)/12$	$\mathcal{A}$, $\mathcal{B}$	Theorem 1.2 (Theorem 4.18)
	$\mathcal{C}$	cylinders: Theorems 5.15, 5.8 and 5.10
	the even members of $\mathcal{E}$	Lemma 7.1 or Theorem 6.1

Theorem 4.1, for every denominator, is the counterpart in the window of the theorem of Larsen, Norton and Zykoski for obtuse angle above $2\pi/3$ (Theorem 2.5). Theorem 4.1(a), for finitely many denominators, and Theorem 6.1 are computer-assisted; the computations are exact and are described in Appendix A.

1.1. **Earlier work.** For right and acute isosceles triangles the answer is known. Kenyon and Smillie proved that a right triangle is a lattice triangle exactly when its smallest angle is π/n with $n \ge 4$, and an acute isosceles triangle exactly when its apex angle is π/n with $n \ge 3$ [15, Theorem 2] (the 'if' parts were known before [15, p. 67]). For acute scalene triangles they reduced the question to a number-theoretic statement, which they verified for denominators up to 10,000 [15, Theorem 1]. Puchta [26] proved the statement under a coprimality assumption, and the general case is treated in a later preprint [27]. Apisa determined the orbit closure of the unfolding of every rational isosceles triangle [2, Theorems 1.4 and 7.7]; in particular the obtuse isosceles lattice triangles are Veech's. What has remained open is the case of obtuse scalene triangles (see [21, Theorem 3.9 and Corollary 3.10]).

Two kinds of obtuse scalene lattice triangles are known. The triangles with angles $(\pi/2k, \pi/k, (2k-3)\pi/2k)$, $k \ge 4$, were suggested by computations of Boshernitzan reported by Veech [28, Remark on p. 582], announced by Vorobets [29, Theorem 5], proved to be lattice triangles in [30, Proposition 4.2 and Theorem 4.4], and studied by Ward, who determined their Veech groups [31, Theorem A]. Bouw and Möller constructed, for every $m < n$, a Teichmüller curve uniformized by a group commensurable to the triangle group

(m, n, ∞), and equal to it when m or n is odd [4], with those of Veech and of Ward as special cases. The triangle with angles $(\pi/12, \pi/3, 7\pi/12)$ was found by Hooper [11, Theorem 1] (a second proof is McMullen's [19, end of §1]). Four of the obtuse lattice triangles named so far have obtuse angle at most $2\pi/3$: those with angles $(\pi/5, \pi/5, 3\pi/5)$, $(\pi/6, \pi/6, 2\pi/3)$, $(\pi/8, \pi/4, 5\pi/8)$ and $(\pi/12, \pi/3, 7\pi/12)$. In the other direction, Vorobets [30, Proposition 4.2 and Theorem 4.5] showed that the triangles with angles $(\pi/6, \pi/4, 7\pi/12)$ and $(\pi/12, \pi/4, 2\pi/3)$ are not lattice triangles, and Ward [31, Theorem B] that those with angles $(\pi/n, q\pi/n, (n-q-1)\pi/n)$ with $q \geq 2$, $4q \leq n$, and q or n odd are not, both by exhibiting two parallel cylinders whose moduli have an irrational ratio.

Larsen, Norton and Zykoski [17] answered Hooper's question when the obtuse angle is at least $3\pi/4$. Their tool is a criterion of Mirzakhani and Wright [22], which shows that the orbit closure of the unfolding has rank at least two. Building on an observation of Wright, they showed that it applies to almost all obtuse triangles [17, Proposition 7.5]. They also showed that no lower bound on the smallest numerator p makes it apply to all obtuse triangles [17, Remark 3.6 and Proposition 3.7], and determined exactly where it applies when the obtuse angle exceeds $2\pi/3$: everywhere except on six one-parameter families and seven single triangles [17, Theorem 4.1]. Angdinata, Chen, Ono, Thorner, Zhang and Zhang [1] proved with the same criterion that the obtuse lattice triangles of denominator n form at most a proportion $c(\varepsilon)n^{-1+\varepsilon}$ of all obtuse triangles of denominator n, and remark that the exponent -1 is the limit of their method [1, Remark 2]. Delecroix and Rüth showed with flatsurf that the unfoldings of the scalene triangles with $p + q + r \leq 58$ have dense orbit closures in their ambient strata (for even n, strata of quadratic differentials, as the unfolding has the rotation by π), apart from an explicit list [6, Theorem 4.1].

1.2. **Further results.** Theorem 1.3 extends the classification of Larsen, Norton and Zykoski [17, Theorem 1.1] from obtuse angle at least $3\pi/4$ to obtuse angle above $2\pi/3$, and Theorem 1.4 is the case of odd denominators of Theorem 1.1.

Theorem 1.3. *A rational triangle whose largest angle is greater than $2\pi/3$ is a lattice triangle if and only if it is $(1, 1, n-2)/n$ with $n \geq 7$ or $(1, 2, 2k-3)/2k$ with $k \geq 5$.*

Theorem 1.4. *Let n be odd. An obtuse rational triangle of denominator n is a lattice triangle if and only if it is $(1, 1, n-2)/n$ with $n \geq 5$.*

The proof of Theorem 1.4 uses neither Theorem 4.1 of Larsen, Norton and Zykoski nor any search over denominators.

Corollary 1.5. *For every prime $n \geq 5$, the only obtuse lattice triangle of denominator n is $(1, 1, n-2)/n$.*

Corollary 1.5 follows from Theorem 1.2(a) and Veech's theorem, since for prime n each of p, q divides n only if it is 1.

1.3. **Plan.** Section 2 sets up the unfolding, its cylinders and the criterion. Section 3 proves Theorem 1.2. Section 4 determines where the criterion is blind and proves Theorems 4.1 and 4.2; up to its last subsection, which applies Theorem 1.2 to two of the families, it is pure number theory and can be read on its own. Section 5 proves the cylinder theorems, and Section 6 treats the single triangles that no general theorem reaches. Section 7 proves Theorem 1.4, Corollary 1.5, Theorem 1.3 and Theorem 1.1. Appendix A describes the computations, Appendix B proves lemmas on residues for every denominator and Theorem 4.14, and Appendix C proves Theorem 4.1(a) for odd n.

2. Unfoldings, cylinders and the criterion

2.1. **Triangles and their unfoldings.** Throughout, $\Delta = (p, q, r)/n$ is a rational triangle as in Section 1, and A, B, C are its vertices with angles $p\pi/n$, $q\pi/n$, $r\pi/n$. When Δ is obtuse, $p \le q < r$ and r does not divide n. For a real number t we write $\{t\}$ for its fractional part.

Let $\mathcal{D}$ be the dihedral group of order $2n$ generated by the linear parts of the reflections in the sides of Δ. The *unfolding* X of Δ is the translation surface made of the $2n$ copies $\delta(\Delta)$, $\delta \in \mathcal{D}$, where the copy $\delta(\Delta)$ is glued along its side $\delta(e)$ to the copy $\delta\kappa_e(\Delta)$, κ_e the linear part of the reflection in the side e, by the translation that makes the two copies of the side coincide [32]. A *vertex* of X is a class of corners of copies identified by these gluings. Above the vertex of Δ with angle $x\pi/n$ there are $\gcd(x, n)$ vertices of X, each of cone angle $2\pi x/\gcd(x, n)$. A vertex of cone angle 2π, that is, a vertex above an angle $x\pi/n$ with $x \mid n$, is an ordinary point of the flat surface. We write Σ for the set of vertices of cone angle greater than 2π, and regard X as a translation surface with singular set Σ. If g_p, g_q, g_r denote $\gcd(p, n)$, $\gcd(q, n)$, $\gcd(r, n)$, the Euler characteristic gives

$$2\,\mathrm{genus}(X) - 2 = n - g_p - g_q - g_r. \tag{2.1}$$

So X has genus at least two if and only if Σ is non-empty; this is the case for every obtuse triangle, as $r \nmid n$.

The *Veech group* $\Gamma(X)$ is the group of derivatives of the orientation-preserving homeomorphisms of X that are affine in the translation charts; such a homeomorphism permutes Σ. The triangle Δ is a *lattice triangle* if $\Gamma(X)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$.

Remark 2.1. If the vertices of cone angle 2π are added to the singular set, the Veech group can only become smaller, and a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ containing a lattice is a lattice. So every statement below of the form "Δ is not a lattice triangle" holds in that convention as well. The converse statements do not: for odd $n \ge 5$ the two vertices of cone angle 2π of the unfolding of $(1, 1, n-2)/n$ are the centres of the two regular n-gons of Veech's double n-gon, and with them marked the Veech group is not a lattice. This is due to Hubert and Schmidt [12] (see [18, §2.5.1]), and for every odd $n \ge 5$ it follows from the theorem of Apisa, Saavedra and Zhang that the periodic points of the double n-gon are exactly the Weierstrass points that are not singularities [3, Theorem 1.3], since the centres of the two n-gons are not Weierstrass points for odd n.

Algebraically, X is the normalization of the curve $y^n = z^p(z-1)^q$, and the translation structure is given by

$$\omega = \frac{y\,dz}{z(z-1)} = z^{p/n-1}(z-1)^{q/n-1}\,dz, \tag{2.2}$$

since the Schwarz–Christoffel map $\int z^{p/n-1}(z-1)^{q/n-1}dz$ takes the upper half-plane onto a triangle with angles $p\pi/n$, $q\pi/n$, $r\pi/n$ at the images of 0, 1, ∞ (see [17, §2] and [22, §6]). The points of X above 0, 1, ∞ are the vertices above A, B, C, and at a point above the vertex with numerator x the form ω has a zero of order $x/\gcd(x, n) - 1$. The automorphism $T(z, y) = (z, \zeta y)$, $\zeta = e^{2\pi i/n}$, satisfies $T^*\omega = \zeta\omega$; its linear part on the flat surface is the rotation by $2\pi/n$.

2.2. **Cylinders.** Fix a direction $d \in \mathbb{C} \setminus \{0\}$. A *closed regular leaf* in direction d is a closed geodesic of X in direction d that does not meet Σ. The union of the closed regular leaves in direction d is open in X; a *cylinder* in direction d is one of its connected components. When X has genus at least two, a cylinder is an open annulus foliated by closed leaves of a common holonomy c, its two ends are unions of saddle connections, and it is a maximal

cylinder of closed leaves in Veech's sense. Its *circumference* is $|c|$, its *height* H is its width measured perpendicular to d, and its *modulus* is $H/|c|$. A cylinder ends only at points of Σ: a vertex of cone angle 2π on a boundary leaf does not stop it.

The unfolding X with its singular set Σ is a positive F-structure in the sense of Veech [28, §2], with punctures Σ, and we use the following proposition of his.

Proposition 2.2 (Veech [28, Proposition 2.10])**.** *Let X be a translation surface with finite singular set Σ, regarded as a positive F-structure with punctures Σ, whose Veech group is a lattice, and let $d \in \mathbb{C} \setminus \{0\}$ be a direction in which X has a saddle connection. Then X has an affine automorphism whose derivative is not the identity and which maps the foliation of X in direction d to itself. In particular, every leaf of that foliation is closed or a saddle connection.*

Let $\vartheta = 2\pi/n$, so that $\zeta = e^{i\vartheta}$, and $\mathbb{Q}(\zeta)^+ = \mathbb{Q}(\zeta) \cap \mathbb{R} = \mathbb{Q}(\cos\vartheta)$, a totally real field; for a unit ℓ modulo n let σ_ℓ be the automorphism of $\mathbb{Q}(\zeta)$ with $\sigma_\ell(e^{i\vartheta}) = e^{i\ell\vartheta}$. It commutes with complex conjugation and acts on $\mathbb{Q}(\zeta)^+$ by $\cos j\vartheta \mapsto \cos j\ell\vartheta$, and every embedding of $\mathbb{Q}(\zeta)^+$ in $\mathbb{R}$ is the restriction of some σ_ℓ. Place Δ with vertices 0, 1, v, $\operatorname{Im} v > 0$, and angles $p\pi/n$ at 0, $q\pi/n$ at 1. By the law of sines $v = \sin(q\pi/n)e^{ip\pi/n}/\sin(r\pi/n)$; multiplying numerator and denominator of

$$v = \frac{e^{i(p+q)\pi/n} - e^{i(p-q)\pi/n}}{e^{ir\pi/n} - e^{-ir\pi/n}}$$

by $e^{ir\pi/n}$ and using $e^{i(p+q+r)\pi/n} = -1$ gives $v = (e^{-iq\vartheta} - 1)/(e^{ir\vartheta} - 1) \in \mathbb{Q}(\zeta)$. The $2n$ copies of Δ in X are $e^{ij\vartheta}\Delta$ and $e^{ij\vartheta}\overline{\Delta}$, $j \in \mathbb{Z}/n$, so their vertices lie in $\mathbb{Q}(\zeta)$, and the holonomy of every saddle connection and of every closed geodesic of X lies in $\mathbb{Q}(\zeta)$, being the holonomy of a path along sides of the copies. This is the observation of Kenyon and Smillie [15, §5]. For a cylinder Z of X let $\rho(Z) = \operatorname{area}(Z)/\operatorname{area}(X)$ be its *share* of the area. If $c, w \in \mathbb{Q}(\zeta)$ are the holonomies of the core curve of Z and of a segment crossing Z between singular points on its two boundary components, then

$$\rho(Z) = \frac{\bar{c}w - c\bar{w}}{n(v - \bar{v})} \quad \text{(up to sign)},$$

since $\operatorname{area}(Z) = |\operatorname{Im}(\bar{c}w)|$ and $\operatorname{area}(X) = 2n \cdot \frac{1}{2}\operatorname{Im} v$. So $\rho(Z)$ is a real element of $\mathbb{Q}(\zeta)$: $\rho(Z) \in \mathbb{Q}(\zeta)^+$.

Part (ii) of the next lemma follows in two lines from the proof of Theorem 1 of Fenves with his Lemma 3 [7]; Kenyon and Smillie used a positivity condition of the same kind [15, Theorem 21 and Proposition 23].

Lemma 2.3. *Let Δ be a rational triangle whose unfolding X has genus at least two, and let Z be a maximal cylinder of X, in direction d. If Δ is a lattice triangle, then:*

(i) *any two maximal cylinders of X in direction d have moduli whose ratio is rational;*
(ii) $0 < \sigma(\rho(Z)) \leq 1$ *for every embedding σ of $\mathbb{Q}(\zeta)^+$ in $\mathbb{R}$.*

Proof. A boundary component of Z is a union of saddle connections in direction d, so by Proposition 2.2 every leaf in direction d is closed or a saddle connection. Then X is the union of the closures of finitely many maximal cylinders Z_j in direction d [28, Proposition 2.4], and their moduli μ_j satisfy $\mu_j = \lambda_j\mu_1$ with $\lambda_j \in \mathbb{Q}_{>0}$ [28, Remark on p. 582]. A closed leaf of Z meets no saddle connection, so it lies in one of the Z_j; both Z and Z_j are connected components of the union of the closed regular leaves in direction d containing that leaf, so $Z = Z_j$. The same holds for every maximal cylinder in direction d, so each of them is one

of the Z_j; this proves (i). Let $c_j \in \mathbb{Q}(\zeta)$ be the holonomy of the core curve of Z_j. Then $\operatorname{area}(Z_j) = \mu_j |c_j|^2$, and the numbers $x_j = c_j/c_1$ lie in $\mathbb{Q}(\zeta)$ and are real, since the c_j are parallel; so $x_j \in \mathbb{Q}(\zeta)^+ \setminus \{0\}$, and

$$\rho(Z_i) = \frac{\mu_i |c_i|^2}{\sum_j \mu_j |c_j|^2} = \frac{\lambda_i x_i^2}{\sum_j \lambda_j x_j^2}.$$

Both sides are elements of $\mathbb{Q}(\zeta)^+$. For an embedding σ of $\mathbb{Q}(\zeta)^+$, each $\sigma(x_j)$ is a non-zero real number, so $\sigma(\rho(Z_i))$ is a quotient of positive numbers whose numerator is one of the terms of the denominator, and $0 < \sigma(\rho(Z_i)) \le 1$. □

Conversely, a translation surface on which every direction containing a cylinder decomposes into cylinders with pairwise commensurable moduli is a Veech surface [7, Theorem 1]; so every rational triangle that is not a lattice triangle has a direction containing a cylinder in which X does not decompose into cylinders with pairwise commensurable moduli.

2.3. **The criterion of Mirzakhani and Wright.** Mirzakhani and Wright proved that the orbit closure of the unfolding has rank at least two (the rank is half the complex dimension of the image of its tangent space in absolute cohomology; a closed orbit has rank one) when the eigenspaces of T of characters a and $2-a$ both contain holomorphic forms, for a unit a with $2a \not\equiv 2 \pmod n$ [22, Theorem 7.5]. Larsen, Norton and Zykoski put this in the following form.

Proposition 2.4 (Larsen, Norton and Zykoski [17, Proposition 2.1])**.** *Let $\Delta = (p,q,r)/n$ be obtuse. If some unit a modulo n with $2a \not\equiv 2 \pmod n$ satisfies two of the three inequalities*

$$[ap]_n < [2p]_n, \quad [aq]_n < [2q]_n, \quad [ar]_n < [2r]_n,$$

then Δ is not a lattice triangle.

Following Larsen, Norton and Zykoski, for an integer x set

$$S_x = \{a \in \mathbb{Z}/n\mathbb{Z} : [ax]_n < [2x]_n\},$$

and call a unit a modulo n *usable* if $2a \not\equiv 2 \pmod n$. A *witness* for an obtuse triangle is a usable unit a for which at least two of the three inequalities $[ap]_n < [2p]_n$, $[aq]_n < [2q]_n$, $[ar]_n < [2r]_n$ hold, that is, which lies in at least two of S_p, S_q, S_r, where for $p = q$ the first two inequalities are one inequality and count as two, as in Proposition 2.4.

For an obtuse triangle write $s = 2r - n$, so that $[2p]_n = 2p$, $[2q]_n = 2q$ and $[2r]_n = s$.

Theorem 2.5 ([17, Theorem 4.1])**.** *An obtuse triangle $(p,q,r)/n$ with $r > 2n/3$ has a witness if and only if none of the following is the case:*

(1) $p = q = 1$
(2) $p = 1$, $q = 2$, *n is even*
(3) $p = 1$, $q = 4$, $r \equiv 7 \bmod 8$
(4) $p = 1$, $r = 3q + 1$
(5) $p = 2$, $r = 3q + 2$
(6) $p = 4$, $r = 3q + 4$
(7) (p,q,r) *is one of the following triples:* $(1,4,11)$, $(1,3,16)$, $(2,3,17)$, $(1,4,21)$, $(1,8,19)$, $(3,8,29)$, $(2,11,29)$.

The members of family (3) with $n > 12$ are not lattice triangles: Larsen, Norton and Zykoski find two horizontal cylinders whose moduli have the irrational ratio $4\cos^2(2\pi/n)$ [17, §10].

3. Relative periods

This section proves Theorem 1.2, which is applied in Section 7 to odd denominators and to the single triangles of Lemma 7.1, and in Section 4.4 to the families $\mathcal{A}$ and $\mathcal{B}$. On the unfolding of a lattice triangle, the relative period of an eigenform with unit character between two zeros of ω, over a suitable rational relative class, vanishes along the whole Teichmüller disc (Lemmas 3.3, 3.4 and 3.5). Its first derivative is the difference of the values at these zeros of a solution G of a $\bar\partial$-equation (Lemma 3.6), and on a triangle unfolding these values are integrals over the plane that can be computed. The obstruction is thus of first order, as is the one behind the criterion of Mirzakhani and Wright, whose proof, as they remark, can be viewed as showing that the $\mathrm{SL}_2(\mathbb{R})$-orbit of the unfolding is not even tangent to the locus of the surfaces with the endomorphisms that an orbit closure of rank one would force [22, after the proof of Theorem 7.5].

3.1. **Forms on the unfolding.** In this section we allow any placement of the vertices. Let (x_0, x_1, x_∞) be an ordering of (p, q, r), set $\theta_i = x_i/n$ and $g_i = \gcd(x_i, n)$ for $i \in \{0, 1, \infty\}$, and let X be the normalization of the curve $y^n = z^{x_0}(z-1)^{x_1}$, with $\omega = y\,dz/(z(z-1))$ and $T(z, y) = (z, \zeta y)$, $\zeta = e^{2\pi i/n}$. The substitutions $z \mapsto 1-z$ and $z \mapsto 1/z$ (with y multiplied by a suitable constant, respectively by a suitable constant times $1/z$) permute the numerators and multiply ω by a non-zero constant, so (X, ω) is the unfolding of Δ of Section 2.1 up to a constant factor, which changes neither the zeros of ω nor whether Δ is a lattice triangle. Above $z = 0$, 1, ∞ there are g_0, g_1, g_∞ points; at a point above $z = 0$ there is a local coordinate u with $z = u^{n/g_0}$ and $y/u^{x_0/g_0}$ a unit (as $(y/u^{x_0/g_0})^n = (z-1)^{x_1}$), and similarly above 1 and above ∞ (with $z = u^{-n/g_\infty}$). The form ω has a zero of order $x_i/g_i - 1$ at each point above the vertex i, so these points are zeros of ω exactly when $x_i \nmid n$; the set of zeros is the set Σ of Section 2.1.

A meromorphic 1-form η on X has *character* $b \in \mathbb{Z}/n\mathbb{Z}$ if $T^*\eta = \zeta^b\eta$; for instance ω has character 1. Let $H^1_b \subset H^1(X, \mathbb{C})$ be the ζ^b-eigenspace of T^* and $H^{1,0}_b$ the space of holomorphic forms of character b. Since T^* is real, complex conjugation maps $H^{1,0}_{-b}$ onto the ζ^b-eigenspace of T^* on antiholomorphic forms, so $H^1_b = H^{1,0}_b \oplus \overline{H^{1,0}_{-b}}$.

Lemma 3.1. *Let b be an integer.*

(a) *Every non-zero meromorphic form of character b is $y^b R(z)\,dz$ with R rational. It is holomorphic if and only if $R = z^i(z-1)^j h(z)$ with integers i, j and a polynomial h with $h(0)h(1) \neq 0$ such that, with $e_0 = b\theta_0 + i$ and $e_1 = b\theta_1 + j$,*

$$e_0 > -1, \qquad e_1 > -1, \qquad e_0 + e_1 + \deg h < -1.$$

For any $R = z^i(z-1)^j h$ its order at a point above 0 is $(n/g_0)(e_0+1) - 1$, above 1 it is $(n/g_1)(e_1+1) - 1$, and above ∞ it is $-(n/g_\infty)(e_0 + e_1 + \deg h + 1) - 1$.

(b) *If $bx_i \not\equiv 0 \pmod n$ for $i = 0, 1, \infty$, then $\dim H^{1,0}_b = 2 - (\{b\theta_0\} + \{b\theta_1\} + \{b\theta_\infty\})$. So $H^{1,0}_b$ has dimension 1 if $[bx_0]_n + [bx_1]_n + [bx_\infty]_n = n$, spanned by $y^b z^{i_b}(z-1)^{j_b}dz$ with $b\theta_0 + i_b = \{b\theta_0\} - 1$ and $b\theta_1 + j_b = \{b\theta_1\} - 1$, and it is zero otherwise. Moreover $\dim H^1_b = 1$.*

(c) *If $b \not\equiv 0$ but $bx_i \equiv 0 \pmod n$ for some i, then $H^1_b = 0$. Also $H^{1,0}_0 = 0$.*

Proof. (a) If η has character b, then $\eta/(y^b\,dz)$ is a T-invariant meromorphic function, hence a rational function of z. Above a point of $\mathbb{C} \setminus \{0, 1\}$ the map z is unramified and y is a unit, so a holomorphic η has R without poles off 0, 1, ∞, that is $R = z^i(z-1)^j h(z)$ as

stated. In the local coordinate above 0 the form is a unit times $u^{bx_0/g_0+in/g_0}\,d(u^{n/g_0})$, of order $(n/g_0)(b\theta_0+i+1)-1$, an integer; it is non-negative if and only if $e_0>-1$. The same computation applies above 1, and above ∞, where y is a unit times $u^{-(x_0+x_1)/g_\infty}$ and $z^i(z-1)^j h(z)\,dz$ is a unit times $u^{-(n/g_\infty)(i+j+\deg h+1)-1}du$.

(b), (c) For $b\not\equiv 0$ these are [22, Lemma 6.1 and Corollary 6.2], with $k=n$, $q_i=x_i$ and $t(b)=\sum_i\{b\theta_i\}$ in their notation, after the change of variable that puts the three vertices at 0, 1, ∞: a Möbius transformation sending distinct points $z_0,z_1,z_\infty\in\mathbb{C}$ to $0,1,\infty$, with y replaced by a constant times $y/(z-z_\infty)$, takes their curve $y^n=\prod_i(z-z_i)^{x_i}$ to ours, as $x_0+x_1+x_\infty=n$, and commutes with T. The form named in (b) has character b and is holomorphic by (a): for it $h=1$, $e_0=\{b\theta_0\}-1>-1$, $e_1=\{b\theta_1\}-1>-1$ and $e_0+e_1=-1-\{b\theta_\infty\}<-1$. In (c) exactly one bx_i is divisible by n (if two were, all three would be, and then $b\equiv 0$, as $\gcd(x_0,x_1,x_\infty)=1$), so $\dim H^1_b=t(b)+t(-b)-2=0$. Finally $H^{1,0}_0=0$, because a T-invariant holomorphic form descends to $X/\langle T\rangle=\mathbb{P}^1$. □

Summing (b) over b gives the genus (2.1). A holomorphic form of character b is therefore determined up to a constant when it exists, and we write η_b for it.

3.2. **The Teichmüller disc and Möller's theorem.** *The Teichmüller disc.* Let (X,ω) be a translation surface with zero set Σ, and let $\mathbb{D}$ be the unit disc. For $t\in\mathbb{D}$ let X_t be the Riemann surface whose underlying topological surface is that of X, marked by the identity, and whose holomorphic charts are $w+t\bar w$ off Σ, where w is a local primitive of ω, and $u_t=u\,(1+t\bar u^{k+1}/u^{k+1})^{1/(k+1)}$ near a zero of order k, where u is a coordinate with $u^{k+1}=(k+1)w$. Then $u_t^{k+1}=(k+1)(w+t\bar w)$, so the charts are compatible, and $\omega_t=d(w+t\bar w)$ is a holomorphic form on X_t with the same zeros. The pair (X_t,ω_t) is the image of (X,ω) under the real-linear map $v\mapsto v+t\bar v$ of $\mathbb{C}=\mathbb{R}^2$, which has positive determinant $1-|t|^2$; every element of $\mathrm{GL}_2^+(\mathbb{R})$ is a complex multiple of exactly one such map, so $t\mapsto X_t$ is the Teichmüller disc of (X,ω), and its image in the moduli space is the Teichmüller curve $C=\mathbb{H}/\Gamma$ when (X,ω) is a Veech surface with Veech group Γ (we identify $\mathbb{D}$ with the upper half-plane $\mathbb{H}$). The identity map $X\to X_t$ is a homeomorphism, not in general smooth at the zeros, and it identifies $H^1(X_t,\mathbb{Q})$ with $H^1(X,\mathbb{Q})$ for every t. We write $X_0=X$.

The complex surface $\mathbb{D}\times X$. The product $\mathbb{D}\times X$ carries the complex structure with charts $(t,w+t\bar w)$ away from the zeros of ω and (t,u_t) near a zero. Both charts are holomorphic in t at a fixed point of X, so $t\mapsto(t,P)$ is a holomorphic section for every point P of X, and the fibre over t is X_t. A holomorphic family η_t of forms, η_t holomorphic on X_t, is a holomorphic relative 1-form on this complex surface; a local primitive $\Phi(t,x)$ along the fibres is a holomorphic function, determined up to adding a holomorphic function of t.

The family over the Teichmüller curve. Let (X,ω) be a Veech surface of genus at least two whose singular set is its zero set Σ, with Veech group Γ, trace field $K=\mathbb{Q}(\mathrm{tr}\,\Gamma)$. For every subgroup $\Gamma_1\subset\Gamma$ of finite index that is small enough there is a family of curves $f:X_1\to C_1=\mathbb{H}/\Gamma_1$, the pull-back of the universal family over a finite cover of the moduli space of curves, whose fibre over the image of a point of $\mathbb{H}=\mathbb{D}$ is the corresponding surface of the Teichmüller disc [23, §1]. Let $\mathbb{V}=R^1f_*\mathbb{Z}$, with its variation of Hodge structure. By [24, Proposition 2.4] it splits as $\mathbb{V}_\mathbb{Q}=\mathbb{W}_\mathbb{Q}\oplus\mathbb{M}_\mathbb{Q}$, a direct sum of rational variations of Hodge structure of weight one, where $\mathbb{W}_\mathbb{Q}$ has rank $2[K:\mathbb{Q}]$ and the fibre of $\mathbb{W}_\mathbb{Q}\otimes\mathbb{R}$ at X contains $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$. Möller's factor A_1 of $\mathrm{Jac}(X_1/C_1)$ with real multiplication by K is the one given by this splitting: there is a homomorphism $\pi:\mathrm{Jac}(X_1/C_1)\to A_1$ of abelian schemes over C_1 whose pull-back maps the rational first cohomology of the fibres

of A_1 isomorphically onto $\mathbb{W}_{\mathbb{Q}}$ (see [23, (2), and the paragraph after Theorem 3.1] and [24, before Proposition 2.6]). By [23, Theorem 3.1] the group of sections of $A_1 \to C_1$ is finite.

Proposition 3.2. *Let (X,ω) be a Veech surface of genus at least two whose singular set is its zero set Σ, with trace field K. There is a subspace $W \subset H^1(X,\mathbb{Q})$ of dimension $2[K:\mathbb{Q}]$ such that:*

(a) *$[\omega] \in W \otimes \mathbb{C}$; for every $t \in \mathbb{D}$, $W \otimes \mathbb{C} = W_t^{1,0} \oplus \overline{W_t^{1,0}}$ with $W_t^{1,0} = (W \otimes \mathbb{C}) \cap H^{1,0}(X_t)$; and the spaces $W_t^{1,0}$ form a holomorphic subbundle of the Hodge bundle over $\mathbb{D}$;*

(b) *for every divisor D of degree 0 supported on Σ there is an integer $N \geq 1$ with the following property. Let γ be a 1-chain on X with $\partial\gamma = D$. For every $t \in \mathbb{D}$ there is $\lambda_t \in H_1(X,\mathbb{Z})$ with*

$$N \int_\gamma \eta = \int_{\lambda_t} \eta \qquad \text{for every holomorphic form } \eta \text{ on } X_t \text{ with } [\eta] \in W \otimes \mathbb{C}.$$

Proof. Take $\Gamma_1 \subset \Gamma$ of finite index, acting freely on $\mathbb{H}$ (Selberg's lemma; the lattice Γ is finitely generated), and so small that f exists and that every $P \in \Sigma$ extends to a section $\tilde{P}$ of f whose value on each fibre is a zero of the form of that fibre: the orders of the zeros are constant along the $\mathrm{GL}_2^+(\mathbb{R})$-orbit, so the zeros form a curve in X_1 that is finite and unramified over C_1, and it becomes trivial over a finite unramified covering of C_1 (as in the proof of [23, Lemma 1.2]). Then $\mathbb{D} \cong \mathbb{H} \to C_1$ is a universal covering, and $C_1 \to C$ is unramified as a map of orbifolds, though not in general as a map of the underlying curves: the surface X itself is an orbifold point of C when its Veech group contains a rotation, as it does for a triangle unfolding. Every derivative below is taken in the parameter t of the disc, which is a local coordinate of C_1 everywhere. The pull-back of f along $\mathbb{D} \to C_1$ is a family of curves of genus at least two over the simply connected disc, marked at X; a marked family over $\mathbb{D}$ is the pull-back of the universal family over Teichmüller space by its classifying map, which here is the Teichmüller disc $t \mapsto X_t$. So the pull-back is the family $t \mapsto X_t$ (as a complex surface it is $\mathbb{D} \times X$, described at the start of this subsection). The sections $\tilde{P}$ pull back to the constant sections $t \mapsto P$: the pull-back of $\tilde{P}$ is continuous, its values lie in the finite set Σ of zeros of ω_t, and its value at $t = 0$ is P.

Let W be the fibre of $\mathbb{W}_{\mathbb{Q}}$ at X, a constant subspace of $H^1(X,\mathbb{Q})$ along the disc, of dimension $2[K:\mathbb{Q}]$. As $\mathbb{W}_{\mathbb{Q}}$ is a sub-variation of Hodge structure of weight one and $W \otimes \mathbb{R}$ contains $\operatorname{Re}\omega$ and $\operatorname{Im}\omega$, part (a) holds, and $W_t^{1,0}$ has dimension $[K:\mathbb{Q}]$.

Let $D = \sum_P D(P)P$ be a divisor of degree 0 on Σ, and $\sigma_D = \sum_P D(P)\,\pi(\tilde{P} - \tilde{P}_0)$ for a fixed $P_0 \in \Sigma$, a section of $A_1 \to C_1$; it has finite order N_1. Let $t \in \mathbb{D}$, with image $c \in C_1$, and write $\mathrm{Jac}(X_t) = H^{1,0}(X_t)^\vee / H_1(X,\mathbb{Z})$ and $A_{1,c} = H^{1,0}(A_{1,c})^\vee / H_1(A_{1,c},\mathbb{Z})$. The differential $d\pi_c$ is the transpose of $\pi_c^* : H^{1,0}(A_{1,c}) \to H^{1,0}(X_t)$; as π_c^* is an injective morphism of Hodge structures with image W, it maps $H^{1,0}(A_{1,c})$ onto $W_t^{1,0}$ (both have dimension $[K:\mathbb{Q}]$), so the kernel of $d\pi_c$ consists of the functionals that vanish on $W_t^{1,0}$. The cokernels of $(\pi_c)_* : H_1(X,\mathbb{Z}) \to H_1(A_{1,c},\mathbb{Z})$ are finite and form a local system on C_1; let m be the exponent of its fibres. As the sections $\tilde{P}$ are constant along the disc, $\sigma_D(c)$ is the image under π_c of the class of $\eta \mapsto \int_\gamma \eta$, for any chain γ with $\partial\gamma = D$. So $N_1\sigma_D(c) = 0$ gives $\lambda_t \in H_1(X,\mathbb{Z})$ with $d\pi_c(mN_1 \int_\gamma) = d\pi_c(\int_{\lambda_t})$, that is, $mN_1 \int_\gamma \eta = \int_{\lambda_t} \eta$ for every $\eta \in W_t^{1,0}$. This is (b) with $N = mN_1$. □

For $t \in \mathbb{D}$ let $A_t = (W_t^{1,0})^\vee / H_1(X,\mathbb{Z})$, where a class λ acts by $\eta \mapsto \int_\lambda \eta$; by (a) it is a complex torus of dimension $[K:\mathbb{Q}]$. Part (b) says that the image of D in A_t, the class of the

functional $\eta \mapsto \int_\gamma \eta$ on $W_t^{1,0}$, has order dividing N for every t. As the kernel of $d\pi_c$ consists of the functionals that vanish on $W_t^{1,0}$ and the cokernel of $(\pi_c)_*$ is finite, π_c factors through the map $\mathrm{Jac}(X_t) \to A_t$ that restricts functionals to $W_t^{1,0}$, and the induced map from A_t to the fibre of A_1 at the image of t in C_1 is an isogeny. This is the factor A of the Jacobian, of dimension $[K:\mathbb{Q}]$, in which the differences of the zeros stay torsion. When $[K:\mathbb{Q}]$ is the genus, $W = H^1(X,\mathbb{Q})$ and $A_t = \mathrm{Jac}(X_t)$, so they stay torsion in the whole Jacobian. On the unfolding of a triangle the difference of two zeros of ω is torsion in the Jacobian, as Mirzakhani and Wright remark [22, §1] (it also follows from Lemma 3.4 below); what a lattice triangle adds is that it stays torsion, in A, along the whole Teichmüller disc.

In Filip's extension of Möller's theorem to affine invariant manifolds the torsion statement holds for the combinations of zeros annihilated by the relative directions of the manifold, and for Teichmüller curves, which have none, for all of them [8, Theorem 1.3 and Remark 1.4 (ii), (iii)]. So the argument below shows that a triangle is not a lattice triangle, not that its orbit closure has rank at least two, and indeed Theorem 1.2(b) applies to $(1,3,6)/10$ and $(1,3,8)/12$ (Example 3.11). Delecroix and Rüth list these two as rank-one orbit closures that are not Teichmüller curves, with a reference to work in preparation [6, Theorem 4.1]. A criterion that proves rank at least two cannot apply to a triangle of rank one. Möller's torsion condition has been used to exclude Teichmüller curves on the stable curves over the cusps, by McMullen [20] and, for a factor of the Jacobian, by Lanneau and Möller [16, §3]; here it is read at the surface of the triangle itself.

3.3. **Relative periods that vanish along the disc.**

Lemma 3.3. *Let Δ be a lattice triangle whose unfolding (X,ω) has genus at least two, and let W be as in Proposition 3.2. Then $H_b^1 \subset W \otimes \mathbb{C}$ for every unit b modulo n. In particular $H_b^{1,0} \subset W_0^{1,0}$ for every unit b, and $b = 2$ is a unit exactly when n is odd.*

Proof. The automorphism T^* of $H^1(X,\mathbb{Q})$ is defined over $\mathbb{Q}$, so its ζ-eigenspace H_1^1 is defined over $\mathbb{Q}(\zeta)$; it contains $[\omega]$ and has dimension 1 by Lemma 3.1(b), as 1 is a unit and so is non-trivial at the three vertices. Let $v \in H^1(X,\mathbb{Q}(\zeta))$ span it. Then $v \in \mathbb{C}[\omega] \subset W \otimes \mathbb{C}$ by Proposition 3.2(a), so $v \in (W \otimes \mathbb{C}) \cap H^1(X,\mathbb{Q}(\zeta)) = W \otimes \mathbb{Q}(\zeta)$, as W is a rational subspace. For a unit b let σ_b be the automorphism of $\mathbb{Q}(\zeta)$ with $\sigma_b(\zeta) = \zeta^b$, acting on the coefficients. Then $\sigma_b(v) \in W \otimes \mathbb{Q}(\zeta)$, since W is rational, and $T^*\sigma_b(v) = \sigma_b(T^*v) = \zeta^b\sigma_b(v)$, since T^* is rational. So $\sigma_b(v)$ is a non-zero vector of H_b^1, which has dimension 1 by Lemma 3.1(b); hence $H_b^1 \subset W \otimes \mathbb{C}$. □

Lemma 3.4. *Let X be the curve of Section 3.1 for any rational triangle, and let Σ_0, Σ_1, Σ_∞ be the sums of the points of X above $z = 0$, 1, ∞.*

(a) $\mathrm{div}(z) = (n/g_0)\Sigma_0 - (n/g_\infty)\Sigma_\infty$ *and* $\mathrm{div}(z-1) = (n/g_1)\Sigma_1 - (n/g_\infty)\Sigma_\infty$.

(b) *Let P, P' be two points above the same vertex, say $z = 0$, and $g = g_0$. Then $(n/g)(P - P')$ is the divisor of a meromorphic function on X.*

Consequently every divisor D of degree 0 supported above $\{0,1,\infty\}$ is torsion in $\mathrm{Jac}(X)$, and there is a class $\hat\gamma \in H_1(X, \operatorname{supp} D; \mathbb{Q})$ with $\partial\hat\gamma = D$ such that $\int_{\hat\gamma}\eta = 0$ and $\int_{\hat\gamma}\bar\eta = 0$ for every holomorphic form η on X.

Proof. (a) follows from the local coordinates of Section 3.1. (b) The substitutions of Section 3.1 move any vertex to $z = 0$, so we may assume the points lie above 0. The quotient $X/\langle T^g\rangle$ is the normalization of $y''^g = z^{x_0}(z-1)^{x_1}$ with $y'' = y^{n/g}$, which is invariant under T^g; with $w = y''/z^{x_0/g}$ it becomes $w^g = (z-1)^{x_1}$. Here $\gcd(x_1, g) = 1$, since a prime

dividing g and x_1 divides x_0, x_1 and n, hence all three numerators. Choosing integers α, β with $\alpha x_1 + \beta g = 1$, the function $v = w^\alpha (z-1)^\beta$ satisfies $v^g = z - 1$ and $w = v^{x_1}$, so the quotient has genus 0. The stabilizer of a point above $z = 0$ in $\langle T \rangle$ has order n/g, since the g points above 0 form one orbit; it is $\langle T^g \rangle$. So the quotient map is totally ramified, of index n/g, at P and at P', and the pull-back of a function on the quotient with divisor (image of P) $-$ (image of P') has divisor $(n/g)(P - P')$.

The differences of two points above the same vertex, and the divisors $g_\infty \Sigma_0 - g_0 \Sigma_\infty$, $g_\infty \Sigma_1 - g_1 \Sigma_\infty$ (of which $n/(g_0 g_\infty)$ and $n/(g_1 g_\infty)$ times are principal; g_0, g_1, g_∞ are pairwise coprime because $\gcd(p, q, r) = 1$), generate the divisors of degree 0 supported above $\{0, 1, \infty\}$ up to finite index, so every such D has a multiple $ND = \operatorname{div}(F)$. Let γ be a 1-chain with $\partial\gamma = D$. By Abel's theorem $\eta \mapsto N \int_\gamma \eta$ is the period map of some $\lambda \in H_1(X, \mathbb{Z})$ on holomorphic forms, and $\hat{\gamma} = \gamma - \lambda/N$ has the stated property; since $\hat{\gamma}$ is a real class, $\int_{\hat{\gamma}} \bar{\eta} = \overline{\int_{\hat{\gamma}} \eta} = 0$. □

Lemma 3.5. *Let (X, ω) be a Veech surface as in Proposition 3.2, D a divisor of degree 0 supported on Σ, and $\hat{\gamma} \in H_1(X, \Sigma; \mathbb{Q})$ a class with $\partial\hat{\gamma} = D$ such that $\int_{\hat{\gamma}} \eta = 0$ for every holomorphic form η on X. Then $\int_{\hat{\gamma}} \eta = 0$ for every $t \in \mathbb{D}$ and every holomorphic form η on X_t with $[\eta] \in W \otimes \mathbb{C}$.*

The relative homology $H_1(X, \Sigma; \mathbb{Z})$ and the homology $H_1(X, \mathbb{Z})$ do not change along the disc, because the underlying surface and the points of Σ do not move; only the complex structure does.

Proof. Let $\Lambda \subset \operatorname{Hom}(W, \mathbb{R})$ be the image of $H_1(X, \mathbb{Z})$, a lattice, and for $t \in \mathbb{D}$ let $\iota_t : \operatorname{Hom}(W, \mathbb{R}) \to (W_t^{1,0})^\vee$ send a functional to the restriction of its complex-linear extension; by Proposition 3.2(a), ι_t is an $\mathbb{R}$-linear isomorphism, and it depends continuously on t. Fix a chain γ with $\partial\gamma = D$ and let N be as in Proposition 3.2(b). The functional $\eta \mapsto N \int_\gamma \eta$ on $W_t^{1,0}$ depends continuously on t (see the complex surface $\mathbb{D} \times X$ in Section 3.2), and by Proposition 3.2(b) it equals $\iota_t(\lambda(t))$ with $\lambda(t) \in \Lambda$. So $\lambda(t) = \iota_t^{-1}(N \int_\gamma |_{W_t^{1,0}})$ is a continuous function on the disc with values in the discrete set Λ, hence constant, $\lambda(t) = \lambda$. Let $\tilde{\lambda} \in H_1(X, \mathbb{Q})$ map to λ and let $\gamma' = \gamma - \tilde{\lambda}/N$, a rational relative class with $\partial\gamma' = D$ and $\int_{\gamma'} \eta = 0$ for every $\eta \in W_t^{1,0}$ and every t. The difference $\gamma' - \hat{\gamma}$ lies in $H_1(X, \mathbb{Q})$ and vanishes on $W_0^{1,0}$; being real, it vanishes on $\overline{W_0^{1,0}}$, hence on $W \otimes \mathbb{C}$. As W and $\gamma' - \hat{\gamma}$ do not depend on t, $\gamma' - \hat{\gamma}$ vanishes on $W_t^{1,0}$ for every t, and so does $\hat{\gamma}$. □

In this proof the rational relative class $\gamma' = \gamma - \tilde{\lambda}/N$ vanishes on $W_t^{1,0}$ for every t; for affine invariant manifolds the corresponding flatness is shown in the proof of [8, Theorem 5.2].

3.4. **The first-order formula.**

Lemma 3.6. *Let (X, ω) be a translation surface, X_t ($t \in \mathbb{D}$) as in Section 3.2, $S \subset X$ a finite set, and $\hat{\gamma} \in H_1(X, S; \mathbb{R})$ a class with $\int_{\hat{\gamma}} \eta = 0$ for every holomorphic form η on X_0; let $D = \partial\hat{\gamma}$. Let η_t be a holomorphic family of forms, η_t holomorphic on X_t. On X_0 write*

$$\frac{\eta_0}{\omega}\,\bar{\omega} = \bar{\partial} G + H$$

with H antiholomorphic and G a continuous function. Then

$$\frac{d}{dt}\Big|_{t=0} \int_{\hat{\gamma}} \eta_t = G(D) := \sum_P D(P)\, G(P).$$

Such G and H exist: the left side is a bounded $(0,1)$-form, smooth off the zeros of ω (at a zero of order k, η_0/ω has a pole of order at most k and $\bar\omega$ vanishes to order k), H is its antiholomorphic part in the Hodge decomposition, and G is smooth where the left side is and Hölder continuous everywhere, by elliptic regularity; G is unique up to an additive constant, which does not change $G(D)$. The derivative depends on the family only through η_0, since two families with the same η_0 differ by t times a holomorphic family plus $O(t^2)$, and $\hat\gamma$ vanishes on holomorphic forms on X_0. Kahn and Wright [14, Theorem 1.1] give a formula of the same kind for the derivative of the map from a stratum to the moduli space of curves, as an integral against a quadratic differential of the $(0,1)$-part of a closed form, divided by ω, plus terms at the zeros.

Proof. On the complex surface $\mathbb{D}\times X$ of Section 3.2 the family η_t is a holomorphic relative 1-form; let $\Phi(t,x)$ be a local primitive along the fibres, a holomorphic function determined up to adding a holomorphic function of t. Let $\dot\Phi=\partial_t\Phi|_{t=0}$ at fixed $x\in X$: it is defined locally up to a constant, $d\dot\Phi$ is a closed 1-form on X off the zeros, and for a path γ in X, $\frac{d}{dt}|_0\int_\gamma\eta_t=\int_\gamma d\dot\Phi$.

Away from the zeros, $\Phi=F(t,w+t\bar w)$ with F holomorphic in two variables, so $\dot\Phi=F_t(0,w)+F_v(0,w)\bar w$ and $\bar\partial\dot\Phi=F_v(0,w)\,d\bar w=(\eta_0/\omega)\bar\omega$, as $\eta_0=F_v(0,w)\,dw$ and $\omega=dw$. Near a zero of order k, $\Phi=\tilde F(t,u_t)$ and $\dot\Phi=\tilde F_t(0,u)+\tilde F_u(0,u)\,\bar u^{k+1}/((k+1)u^k)$, which is continuous at $u=0$. Now $\tilde\eta=d\dot\Phi-dG-H$ is a form of type $(1,0)$ off a finite set, since its $(0,1)$-part is $\bar\partial\dot\Phi-\bar\partial G-H=0$, and it is closed, since $dH=0$; so it is holomorphic there. Near each point of the finite set it is the differential of the continuous function $\dot\Phi-G-$ (a local primitive of H), which is holomorphic off the point and hence across it; so $\tilde\eta$ is a holomorphic form on X_0. Hence $d\dot\Phi=dG+H+\tilde\eta$, and

$$\frac{d}{dt}\Big|_0\int_{\hat\gamma}\eta_t=\int_{\hat\gamma}d\dot\Phi=G(D)+\int_{\hat\gamma}H+\int_{\hat\gamma}\tilde\eta=G(D),$$

the last two integrals vanishing because $\hat\gamma$ is real and vanishes on holomorphic forms on X_0. □

Corollary 3.7. *Let Δ be a lattice triangle whose unfolding (X,ω) has genus at least two, b a unit modulo n with $H_b^{1,0}\neq 0$, and write $(\eta_b/\omega)\bar\omega=\bar\partial G_b+H_b$ as in Lemma 3.6. Then $G_b(D)=0$ for every divisor D of degree 0 supported on the zeros of ω.*

Proof. By Lemma 3.4 there is $\hat\gamma\in H_1(X,\operatorname{supp}D;\mathbb{Q})$ with $\partial\hat\gamma=D$ vanishing on holomorphic forms on X_0. By Lemma 3.3, $\eta_b\in W_0^{1,0}$, and as $W_t^{1,0}$ is a holomorphic subbundle (Proposition 3.2(a)) there is a holomorphic family $\eta_t\in W_t^{1,0}$ with $\eta_0=\eta_b$. By Lemma 3.5, $\int_{\hat\gamma}\eta_t=0$ for all $t\in\mathbb{D}$. Differentiating at $t=0$ (Lemma 3.6) gives $G_b(D)=0$. □

Lemma 3.8. *Let ϖ be a bounded $(0,1)$-form on a compact Riemann surface X, smooth off a finite set, with $\varpi=\bar\partial G$ for a continuous function G, and let ξ be a meromorphic 1-form on X with at most simple poles. Then*

$$\int_X\varpi\wedge\xi=-2\pi i\sum_Q G(Q)\operatorname{Res}_Q\xi.$$

Proof. Remove a coordinate disc of radius ε about each pole of ξ and each point where ϖ is not smooth. On the rest $d(G\xi)=\bar\partial G\wedge\xi=\varpi\wedge\xi$, so by Stokes's theorem $\int\varpi\wedge\xi=-\sum\oint G\xi$ over the circles, oriented counterclockwise. As $\varepsilon\to 0$ the left side tends to $\int_X\varpi\wedge\xi$, since ϖ is bounded and ξ has at most simple poles, and $\oint G\xi\to 2\pi i\,G(Q)\operatorname{Res}_Q\xi$ by continuity of G. □

Both proofs below rest on one computation. As $T^*\omega = \zeta\omega$ and $T^*\bar\omega = \zeta^{-1}\bar\omega$, a form η_b of character b gives

$$T^*\Big(\frac{\eta_b}{\omega}\,\bar\omega\Big) = \zeta^{b-2}\,\frac{\eta_b}{\omega}\,\bar\omega.$$

So the antiholomorphic part H_b is the conjugate of a holomorphic form of the complementary character $b' = 2-b$, G_b can be taken with $G_b \circ T = \zeta^{-b'}G_b$, and for a meromorphic form ξ of character b' with at most simple poles the form $(\eta_b/\omega)\bar\omega \wedge \xi$ is T-invariant, so that its integral over X is n times an integral over the plane. When $H_b = 0$, Lemma 3.8 expresses the same integral through the values of G_b at the poles of ξ. In part (a), $b = 2$, a unit as n is odd; then $b' = 0$, $H_2 = 0$ as $H_0^{1,0} = 0$, and ξ is dz/z or $dz/(z-1)$. In part (b), $b \equiv 2 \pmod{n/g}$ and $b \not\equiv 2 \pmod{n}$ say that $b'x \equiv 0$ and $b' \not\equiv 0 \pmod{n}$. So $H_b = 0$ by Lemma 3.1(c); ξ can be given a simple pole at each of the g points above x and no other pole; and $\zeta^{-b'} \neq 1$, so that, if Δ is a lattice triangle, Corollary 3.7 applied to $D = TQ - Q$ for a point Q above x forces G_b to vanish at all the points above x. Corollary 3.7 concerns only divisors supported on the zeros of ω, and the points above a vertex x are zeros exactly when $x \nmid n$ (Section 3.1): hence the hypothesis $x \nmid n$ in part (b).

3.5. **Proof of Theorem 1.2(a).** The two integrals of the next lemma are the cases $(\varepsilon_1, \varepsilon_2) = (0,1)$ and $(1,0)$, with $q_1/k = \theta_0$ and $q_2/k = \theta_1$, of the integral of [22, Lemma 7.15], which arises there from the pairing of holomorphic forms of complementary characters in the proof of their Proposition 7.3. It is proved non-zero there, and the argument, a comparison of the integrand at symmetric points, also gives the signs.

Lemma 3.9. *Let $\theta_0, \theta_1 > 0$ with $\theta_0 + \theta_1 < 1/2$, and $\mathcal{K}(z) = |z|^{2\theta_0-2}|z-1|^{2\theta_1-2}$. The integrals of $\mathcal{K}\cdot(z-1)$ and $\mathcal{K}\cdot z$ over $\mathbb{C}$ with respect to the area form d^2z converge absolutely, and*

$$\int_{\mathbb{C}} \mathcal{K}\cdot(z-1)\,d^2z < 0, \qquad \int_{\mathbb{C}} \mathcal{K}\cdot z\,d^2z > 0.$$

Proof. Near 0 and 1 the exponents $2\theta_i - 2$ exceed -2, and at infinity the integrands are $O(|z|^{2\theta_0+2\theta_1-3})$ with $2\theta_0 + 2\theta_1 - 3 < -2$. Write I_1 and I_2 for the integrals of $\mathcal{K}\cdot(z-1)$ and $\mathcal{K}\cdot z$. As $z - (z-1) = 1$, $I_2 - I_1 = \int_{\mathbb{C}} \mathcal{K}\,d^2z > 0$. Let $f = |z|^{2\theta_0}|z-1|^{2\theta_1}/(\bar z(\bar z - 1))$, the function that appears in the proof of Theorem 1.2(a) below. Off $\{0,1\}$,

$$\frac{\partial f}{\partial z} = f\cdot\Big(\frac{\theta_0}{z} + \frac{\theta_1}{z-1}\Big) = \mathcal{K}\cdot(\theta_0(z-1) + \theta_1 z),$$

and $d(f\,d\bar z) = -2i\,(\partial f/\partial z)\,d^2z$. As $|f| = |z|^{2\theta_0-1}|z-1|^{2\theta_1-1}$, the integrals of $f\,d\bar z$ over the circles $|z| = \varepsilon$, $|z-1| = \varepsilon$ and $|z| = R$ are $O(\varepsilon^{2\theta_0})$, $O(\varepsilon^{2\theta_1})$ and $O(R^{2\theta_0+2\theta_1-1})$. So Stokes's theorem on $\{\varepsilon < |z| < R,\ |z-1| > \varepsilon\}$ and absolute convergence give, as $\varepsilon \to 0$ and $R \to \infty$, $\theta_0 I_1 + \theta_1 I_2 = 0$. Hence

$$I_1 = -\frac{\theta_1}{\theta_0+\theta_1}\,(I_2 - I_1) < 0, \qquad I_2 = \frac{\theta_0}{\theta_0+\theta_1}\,(I_2 - I_1) > 0. \qquad \square$$

Proof of Theorem 1.2(a). Suppose that Δ is a lattice triangle. Place the obtuse vertex at ∞: $(x_0, x_1, x_\infty) = (p, q, r)$, so that $\theta_0 + \theta_1 < 1/2$.

The form. Let $\eta_2 = y^2\,dz/(z(z-1))$, of character 2. In Lemma 3.1(a) it has $i = j = -1$, $e_0 = 2\theta_0 - 1 > -1$, $e_1 = 2\theta_1 - 1 > -1$ and $e_0 + e_1 = 2(\theta_0+\theta_1) - 2 < -1$, the last exactly because $r > n/2$; so η_2 is holomorphic. As n is odd, 2 is a unit and $2x \not\equiv 0 \pmod{n}$ for $x = p, q, r$, so $H_2^{1,0} = \mathbb{C}\eta_2$ by Lemma 3.1(b).

The function. On X,

$$\frac{\eta_2}{\omega}\,\bar\omega = y\bar y\,\frac{d\bar z}{\bar z(\bar z-1)} = f\,d\bar z, \qquad f=\frac{|z|^{2\theta_0}|z-1|^{2\theta_1}}{\bar z(\bar z-1)},$$

which is T-invariant. In its decomposition $f\,d\bar z=\bar\partial G+H$ the antiholomorphic form H is unique, hence T-invariant, and $\bar H\in H_0^{1,0}=0$ (Lemma 3.1(c)); averaging G over $\langle T\rangle$ we may take G to be T-invariant, that is, a continuous function of $z\in\mathbb{P}^1$. Write $G(0)$, $G(1)$, $G(\infty)$ for its values above the three vertices.

Its values. Apply Lemma 3.8 with $\varpi=f\,d\bar z$ and $\xi=dz/z$. The form ξ has simple poles above 0, with residue n/g_0 at each of the g_0 points, and above ∞, with residue $-n/g_\infty$ at each of the g_∞ points, and it is holomorphic above 1. The integrand $\varpi\wedge\xi=(f/z)\,d\bar z\wedge dz$ is T-invariant, $f/z=\mathcal{K}\cdot(z-1)$ with $\mathcal{K}$ as in Lemma 3.9, and $d\bar z\wedge dz=2i\,d^2z$; so

$$2i\,n\int_{\mathbb{C}}\mathcal{K}\cdot(z-1)\,d^2z=-2\pi i\,n\,(G(0)-G(\infty)), \qquad \text{that is,}$$
$$G(0)-G(\infty)=-\frac{1}{\pi}\int_{\mathbb{C}}\mathcal{K}\cdot(z-1)\,d^2z>0.$$

With $\xi=dz/(z-1)$ and $f/(z-1)=\mathcal{K}\cdot z$ the same computation gives $G(1)-G(\infty)=-\frac{1}{\pi}\int_{\mathbb{C}}\mathcal{K}\cdot z\,d^2z<0$ (Lemma 3.9).

The contradiction. The points above ∞ are zeros of ω, since $r\nmid n$. By hypothesis the points above 0, or those above 1, are zeros too, say those above $i\in\{0,1\}$. The divisor $D=g_\infty\Sigma_i-g_i\Sigma_\infty$ has degree 0 and is supported on zeros of ω; as ω has zeros, X has genus at least two, and Corollary 3.7 gives $0=G(D)=g_ig_\infty(G(i)-G(\infty))$, which contradicts the inequalities above. □

That the triangle is obtuse is needed for the statement and not only for the method: the acute triangles $(2,3,4)/9$ and $(1,2,2)/5$ are lattice triangles [15, Theorems 1 and 2], with n odd and a numerator not dividing n.

Example 3.10. The smallest triangle to which Theorem 1.2(a) applies and on which the criterion of Mirzakhani and Wright is blind is $(1,2,4)/7$, the member $s=1$, $n=7$ of the family $\mathcal{A}$ of Section 4. Here ω has a simple zero above 1 and a zero of order 3 above ∞, X has genus 3, and with $D=P_1-P_\infty$ the derivative of $\int_{\hat\gamma}\eta_t$ at $t=0$ is $G(1)-G(\infty)=-\frac{1}{\pi}\int_{\mathbb{C}}\mathcal{K}\cdot z\,d^2z\neq0$ (Lemma 3.9).

3.6. **Proof of Theorem 1.2(b).** We use the complex beta integral in the form given by Neretin [25, (1.4)], who attributes it to Gel'fand, Graev and Vilenkin [10, Section II.3.7] and Gel'fand, Graev and Retakh [9, Section 1.6]. With $z^{u|u'}:=z^u\bar z^{u'}$, which is single-valued when $u-u'\in\mathbb{Z}$, it reads

$$B^{\mathbb{C}}(\alpha|\alpha',\beta|\beta'):=\frac{1}{\pi}\int_{\mathbb{C}}z^{\alpha-1|\alpha'-1}(1-z)^{\beta-1|\beta'-1}\,d^2z=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(1-\alpha'-\beta')}{\Gamma(1-\alpha')\Gamma(1-\beta')}. \tag{3.1}$$

Here $z^{u|u'}=|z|^{2u'}z^{u-u'}$, and the integral converges absolutely exactly when $\operatorname{Re}(\alpha+\alpha')>0$, $\operatorname{Re}(\beta+\beta')>0$ and $\operatorname{Re}(\alpha+\alpha'+\beta+\beta')<2$; we use (3.1) only in this region.

Proof of Theorem 1.2(b). Suppose that Δ is a lattice triangle. Place the vertex x at 0: $x_0=x$, $g_0=g$, and x_1, x_∞ are the other two numerators in either order. The g_0 points $Q_j=T^jQ_0$ ($j\in\mathbb{Z}/g_0\mathbb{Z}$) above 0 are zeros of ω, as $x\nmid n$. Fix an integer representing b and let $b'=2-b$. Then $n\nmid b'$ and $(n/g_0)\mid b'$, so $b'\theta_0\in\mathbb{Z}$; and $b'\theta_1\notin\mathbb{Z}$, since otherwise $b'\theta_\infty\in\mathbb{Z}$ too, and then n divides $b'x_0$, $b'x_1$, $b'x_\infty$, hence b', as $\gcd(p,q,r)=1$.

The form and the function. As b is a unit, $bx_i \not\equiv 0 \pmod n$ for each i, and with $[bp]_n + [bq]_n + [br]_n = n$, Lemma 3.1(b) gives $H_b^{1,0} = \mathbb{C}\eta_b$, $\eta_b = y^b z^{i_b}(z-1)^{j_b}dz$, where $a_0 = \{b\theta_0\} = b\theta_0 + i_b + 1$ and $a_1 = \{b\theta_1\} = b\theta_1 + j_b + 1$ lie in $(0,1)$ and $a_0 + a_1 < 1$. The form $\varpi = (\eta_b/\omega)\bar\omega$ satisfies $T^*\varpi = \zeta^{b-2}\varpi = \zeta^{-b'}\varpi$. In its decomposition $\varpi = \bar\partial G + H$ the antiholomorphic form H is unique, so $T^*H = \zeta^{-b'}H$ and $\bar H \in H_{b'}^{1,0}$, which is 0 by Lemma 3.1(c), as $b'x_0 \equiv 0$ and $b' \not\equiv 0 \pmod n$. So $\varpi = \bar\partial G$, and $G\circ T - \zeta^{-b'}G$ is a constant; adding a constant to G (possible as $\zeta^{-b'} \neq 1$) we get $G\circ T = \zeta^{-b'}G$, so $G(Q_j) = \zeta^{-b'j}G(Q_0)$.

What the lattice property gives. The divisor $D = Q_1 - Q_0$ has degree 0 and is supported on zeros of ω; as ω has zeros, X has genus at least two, and Corollary 3.7 gives $0 = G(D) = (\zeta^{-b'} - 1)\,G(Q_0)$, that is, $G(Q_0) = 0$.

The value of $G(Q_0)$. Let $b_1' = \{b'\theta_1\} \in (0,1)$ and

$$\xi = y^{b'}\, z^{-1-b'\theta_0}(z-1)^{b_1'-1-b'\theta_1}\, dz,$$

a meromorphic form of character b' (the exponents of z and $z-1$ are integers). By the order formulas of Lemma 3.1(a), with $e_0 = -1$, $e_1 = b_1' - 1$ and $h = 1$, it has order $(n/g_1)b_1' - 1 \geq 0$ above 1 and $(n/g_\infty)(1-b_1') - 1 \geq 0$ above ∞ (both $(n/g_1)b_1'$ and $(n/g_\infty)(1-b_1')$ are positive integers, as $1 - b_1' \equiv b'\theta_\infty \pmod 1$), and a simple pole at each Q_j, where $\xi = \mathcal{U}\,dz/z$ with $\mathcal{U} = y^{b'}z^{-b'\theta_0}(z-1)^{b_1'-1-b'\theta_1}$ a function of order 0; so $\operatorname{Res}_{Q_j}\xi = (n/g_0)\mathcal{U}_j$ with $\mathcal{U}_j = \mathcal{U}(Q_j) \neq 0$. Since $T^*\xi = \zeta^{b'}\xi$ and $G\circ T = \zeta^{-b'}G$, the product $G(Q_j)\operatorname{Res}_{Q_j}\xi$ does not depend on j, and Lemma 3.8 gives

$$\int_X \varpi\wedge\xi = -2\pi i\, g_0\, G(Q_0)\,\frac{n}{g_0}\,\mathcal{U}_0 = -2\pi i\, n\,\mathcal{U}_0\, G(Q_0).$$

On the other hand $\varpi\wedge\xi$ is T-invariant, so $\int_X = n\int_{\mathbb{C}}$. As $b + b' = 2$ and $y\bar y = |z|^{2\theta_0}|z-1|^{2\theta_1}$,

$$\varpi\wedge\xi = y\bar y\,\frac{z^{k_0}(z-1)^{k_1}}{\bar z(\bar z - 1)}\,d\bar z\wedge dz,$$

$$k_0 = i_b - b'\theta_0 = a_0 - 1 - 2\theta_0, \quad k_1 = j_b + b_1' - b'\theta_1 = a_1 + b_1' - 1 - 2\theta_1,$$

with k_0, k_1 integers; in the notation of (3.1),

$$\varpi\wedge\xi = (-1)^{k_1+1}\, z^{a_0-\theta_0-1|\theta_0-1}(1-z)^{a_1+b_1'-\theta_1-1|\theta_1-1}\cdot 2i\,d^2z.$$

Here $\alpha + \alpha' = a_0 > 0$, $\beta + \beta' = a_1 + b_1' > 0$ and $\alpha + \alpha' + \beta + \beta' = a_0 + a_1 + b_1' < 2$, so the integral converges absolutely and (3.1) gives

$$\int_X \varpi\wedge\xi = 2\pi i\, n\,(-1)^{k_1+1}\varkappa, \qquad \varkappa = \frac{\Gamma(a_0-\theta_0)\,\Gamma(a_1+b_1'-\theta_1)\,\Gamma(\theta_\infty)}{\Gamma(a_0+a_1+b_1'-\theta_0-\theta_1)\,\Gamma(1-\theta_0)\,\Gamma(1-\theta_1)}.$$

Hence $G(Q_0) = (-1)^{k_1}\varkappa/\mathcal{U}_0$.

It is not zero. Modulo 1, $a_0 - \theta_0 \equiv (b-1)\theta_0 \equiv \theta_0$, because $(b-2)\theta_0 = -b'\theta_0 \in \mathbb{Z}$; $a_1 + b_1' - \theta_1 \equiv (b + b' - 1)\theta_1 = \theta_1$; and $a_0 + a_1 + b_1' - \theta_0 - \theta_1 \equiv \theta_0 + \theta_1 = 1 - \theta_\infty$. None of θ_0, θ_1, θ_∞ is an integer, so every argument of Γ in $\varkappa$ is a real number that is not an integer, and $\varkappa$ is a non-zero real number. This contradicts $G(Q_0) = 0$. □

So under the hypotheses of part (b) the derivative along the Teichmüller disc of the relative period of η_b over a rational class bounding $Q_1 - Q_0$ is $(\zeta^{-b'} - 1)(-1)^{k_1}\varkappa/\mathcal{U}_0$, a non-zero number given in closed form. As b is a unit and $b \equiv 2 \pmod{n/g}$, the number n/g must be odd: part (b) never speaks at a vertex above which the points are fixed by $T^{n/2}$.

Corollary 3.7, applied to all units b, gives nothing beyond Theorem 1.2, on any rational triangle. For a unit $b \not\equiv 2 \pmod n$ with $H_b^{1,0} \neq 0$ and $G_b\circ T = \zeta^{b-2}G_b$, the points above

a vertex x are fixed by $T^{\gcd(x,n)}$, so G_b vanishes there unless $(2-b)x \equiv 0 \pmod n$; hence $G_b = 0$ on Σ unless b satisfies the hypotheses of part (b) at some vertex, and then G_b is not constant on Σ, by the proof above. For $b \equiv 2$ (n odd, Δ obtuse) G_2 is constant above each vertex, so it is constant on Σ exactly when $p \mid n$ and $q \mid n$, the case part (a) leaves out; for even n, where 2 is not a unit, only part (b) remains.

Example 3.11. For $(1,3,6)/10$, the vertex 6 has $g = 2$, $6 \nmid 10$ and $n/g = 5$, and $b = 7$ is a unit with $7 \equiv 2 \pmod 5$, $7 \not\equiv 2 \pmod{10}$ and $[7]_{10} + [21]_{10} + [42]_{10} = 7 + 1 + 2 = 10$. With 6 at 0 and 1 at 1: $\theta_0 = 3/5$, $\theta_1 = 1/10$, $\theta_\infty = 3/10$, $a_0 = 1/5$, $a_1 = 7/10$, $b' = -5$, $b'_1 = 1/2$, $k_0 = -2$, $k_1 = 0$, and

$$\varkappa = \frac{\Gamma(-2/5)\,\Gamma(11/10)\,\Gamma(3/10)}{\Gamma(7/10)\,\Gamma(2/5)\,\Gamma(9/10)}.$$

For $(1,3,8)/12$, the vertex 8 has $g = 4$ and $n/g = 3$, $b = 5$, $[5]_{12} + [15]_{12} + [40]_{12} = 12$, and with 8 at 0 and 1 at 1: $a_0 = 1/3$, $a_1 = 5/12$, $b' = -3$, $b'_1 = 3/4$, $k_0 = -2$, $k_1 = 0$ and $\varkappa = \Gamma(-1/3)\Gamma(13/12)\Gamma(1/4)/(\Gamma(3/4)\Gamma(1/3)\Gamma(11/12))$. So neither triangle is a lattice triangle.

4. Where the criterion is blind

Theorem 4.1 below says exactly where the criterion is blind when the obtuse angle is at most $2\pi/3$, for every denominator. Only its part (a) for even n is used, in the proof of Theorem 1.1.

The sets S_x, usable units and witnesses are defined in Section 2.3. We call the range $n/2 < r \le 2n/3$, where the obtuse angle lies in $(\pi/2, 2\pi/3]$, the *window*; there $1 \le s \le n/3$, with $s = 2r - n$ as in Section 2.3. When s divides n we use the *unit condition*

(U) every usable unit a modulo n with $[ap]_n < 2p$ satisfies $a \equiv 2 \pmod{n/s}$.

Three families of window triangles occur:

- $\mathcal{A}$: $p = 1$, s divides n and n/s is odd;
- $\mathcal{B}$: s divides n, $d = n/s$ is odd, $2 \le p \le s/2$, and (U) holds;
- $\mathcal{C}$: $s = 2$ and 4 divides n, and either $p \in \{1,2\}$, or $p = 4$ and $n \equiv 4 \pmod 8$.

A member of $\mathcal{A}$ is determined by n and s, namely $(1, (n-s)/2 - 1, (n+s)/2)/n$, and a member of $\mathcal{C}$ by n and p, namely $(p, n/2 - p - 1, n/2 + 1)/n$. For instance $\mathcal{A}$ with $s = 1$ is $(1, (n-3)/2, (n+1)/2)/n$, n odd. Let $\mathcal{E}$ be the list of the thirty window triangles of Table 1, all with $n \le 48$.

Table 1. The thirty triangles of $\mathcal{E}$, each a window triangle with $n \le 48$. Twenty-five of them have even denominator; the five odd ones are the first, second, third, ninth and tenth.

$(2,3,6)/11$	$(2,5,8)/15$	$(3,4,8)/15$	$(3,4,9)/16$	$(1,6,11)/18$
$(3,5,10)/18$	$(3,4,13)/20$	$(3,5,12)/20$	$(2,7,12)/21$	$(3,7,11)/21$
$(3,7,12)/22$	$(1,8,15)/24$	$(3,8,13)/24$	$(1,9,16)/26$	$(5,7,14)/26$
$(1,10,17)/28$	$(3,10,15)/28$	$(5,8,15)/28$	$(2,9,19)/30$	$(3,10,17)/30$
$(3,13,18)/34$	$(1,12,23)/36$	$(5,12,19)/36$	$(1,16,23)/40$	$(3,16,21)/40$
$(8,9,23)/40$	$(5,14,23)/42$	$(3,17,24)/44$	$(3,16,29)/48$	$(5,16,27)/48$

Theorem 4.1. *Let $\Delta = (p,q,r)/n$ be a window triangle and $s = 2r - n$.*

(a) Δ *has no witness if and only if it lies in* $\mathcal{A}$, $\mathcal{B}$ *or* $\mathcal{C}$ *or is a member of* $\mathcal{E}$ *(Table 1).*

(b) *Suppose that* s *divides* n, *that* n/s *is odd and that* $2 \le p \le s/2$. *Then* Δ *has no witness if and only if it satisfies the unit condition* (U), *that is, if and only if it lies in* $\mathcal{B}$. *If* $\gcd(p,n) = 1$, *this holds exactly when* p *is a prime,* n/s *divides* $2p-1$, *and every prime* $\ell < 2p$ *with* $\ell \ne p$ *and* $\ell \not\equiv 1 \pmod{n/s}$ *divides* n; *if* $\gcd(p,n) > 1$, *exactly when* $p = 3$ *and* $n = 3s$.

In particular, for a prime $n \ge 5$ the window triangles without a witness are $(1,(n-3)/2,(n+1)/2)/n$ and, for $n = 11$, $(2,3,6)/11$; for $n = 2\ell$ with ℓ an odd prime they are $(1,\ell-2,\ell+1)/2\ell$ and, for $n = 22,\ 26,\ 34$, also $(3,7,12)/22$; $(1,9,16)/26$, $(5,7,14)/26$; $(3,13,18)/34$. The window triangles without a witness and with $p = 1$ are the members of $\mathcal{A}$, the members of $\mathcal{C}$ with $p = 1$, and $(1,6,11)/18$, $(1,8,15)/24$, $(1,9,16)/26$, $(1,10,17)/28$, $(1,12,23)/36$, $(1,16,23)/40$.

The unit condition (U) is a condition on the units in S_p alone. The member $(3,5,12)/20$ of $\mathcal{E}$ satisfies every clause of the explicit condition in part (b) with $\gcd(p,n) = 1$ except $2 \le p \le s/2$ ($p = 3$, $s = 4$, $n/s = 5$). The family $\mathcal{A}$ contains the two lattice triangles $(1,1,3)/5$ and $(1,1,4)/6$.

Part (a) is Theorem 4.15 for even n and Theorem C.3 for odd n. For all but finitely many denominators these rest on Theorem 4.14, proved in Appendix B, and on Theorem C.2, proved in Appendix C; both use the lemmas of Appendix B.1, which hold for every denominator.

Theorem 4.2. *Let* n *be odd and let* $\Delta = (p,q,r)/n$ *be an obtuse triangle such that* p *and* q *divide* n. *Then* pq *divides* n; *put* $I = n/(pq)$ *and* $s = 2r - n$. *The triangle* Δ *has a witness if and only if* $s > pI$, *and the triangles with* $s \le pI$ *are exactly* $(1,1,n-2)/n$ *with* $n \ge 5$ *odd,* $(1,3,5)/9$, $(1,5,9)/15$ *and* $(3,7,11)/21$.

The rest of this section proves Theorems 4.1 and 4.2 and, in Section 4.4, applies Theorem 1.2 to $\mathcal{A}$ and $\mathcal{B}$. Throughout, $\Delta = (p,q,r)/n$ is a window triangle unless said otherwise; then $q \ge (p+q)/2 = (n-s)/4$.

4.1. **The region where s divides n.** In this subsection n is arbitrary, Δ is a window triangle, s divides n and $d = n/s$ is odd. Since $s \le n/3$, $d \ge 3$. As d is odd,

$$r = \frac{n+s}{2} = s\,\frac{d+1}{2}, \qquad p+q = n-r = s\,\frac{d-1}{2}.$$

Proposition 4.3. *Let* Δ *be a window triangle with* $s \mid n$ *and* $d = n/s$ *odd.*

(a) S_r *contains no unit modulo* n.

(b) *If* $2p \le s$, *a unit* a *with* $[ap]_n < 2p$ *lies in* S_q *if and only if* $a \not\equiv 2 \pmod{d}$.

(c) *If* $2 \le p \le s/2$, *then* Δ *has no witness if and only if it satisfies the unit condition* (U), *that is, if and only if it lies in* $\mathcal{B}$.

(d) *If* $p = 1$, *that is if* $\Delta \in \mathcal{A}$, *then* Δ *has no witness.*

Proof. (a) For every a, $ar = s \cdot a(d+1)/2 \equiv s\,c_1 \pmod{ds}$ with $c_1 = [a(d+1)/2]_d$, and $0 \le sc_1 \le s(d-1) < n$, so $[ar]_n = sc_1$. Thus $[ar]_n < s$ if and only if $c_1 = 0$, that is $d \mid a(d+1)/2$, that is $d \mid a$, because $2 \cdot (d+1)/2 \equiv 1 \pmod{d}$ makes $(d+1)/2$ invertible modulo d. A unit modulo n is prime to d, as $d \mid n$ and $d > 1$.

(b) Let a be a unit and $x = [ap]_n < 2p \le s$. Then $x \ge 1$, since $n \nmid ap$. Next $a(p+q) = s \cdot a(d-1)/2 \equiv s\,c \pmod{n}$ with $c = [a(d-1)/2]_d$; as $(d-1)/2$ is invertible modulo d ($2 \cdot (d-1)/2 \equiv -1$) and $d \nmid a$, $1 \le c \le d-1$. Hence $aq \equiv sc - x \pmod{n}$, and

since $1 \le x < s \le sc < n$, $[aq]_n = sc - x$. So

$$a \in S_q \iff sc - x < 2q = s(d-1) - 2p \iff x > 2p - s(d-1-c).$$

If $c = d-1$ this reads $x > 2p$, which is false. If $c \le d-2$, then $2p - s(d-1-c) \le 2p - s \le 0 < x$, and it holds. Finally $c = d-1$ if and only if $a(d-1)/2 \equiv -1 \pmod d$, that is, multiplying by -2, $a - ad \equiv 2$, that is $a \equiv 2 \pmod d$.

(c) A witness lies in two of the three sets, by (a) not in S_r, so it is a usable unit of $S_p \cap S_q$. By (b) a usable unit of S_p lies in S_q exactly when it is $\not\equiv 2 \pmod d$. So Δ has a witness if and only if some usable unit a with $[ap]_n < 2p$ is $\not\equiv 2 \pmod d$, which is the negation of (U).

(d) $S_1 = \{a : [a]_n < 2\} = \{0, 1\}$; 0 is not a unit and 1 is not usable. A witness would lie in S_1 or in S_r, which is impossible by (a). □

Parts (a) and (d) are remarks of Larsen, Norton and Zykoski [17, Remark 3.6 and p. 7124], the first with their $m = (d+1)/2$. Part (c) is Theorem 4.1(b). The unit condition (U) can be made explicit. Say that Δ satisfies *condition* (B_1) if p is a prime not dividing n, d divides $2p-1$, and every prime $\ell < 2p$ with $\ell \ne p$ and $\ell \not\equiv 1 \pmod d$ divides n.

Proposition 4.4. *Let Δ be a window triangle with $s \mid n$, $d = n/s$ odd and $2 \le p \le s/2$.*

(a) *If $\gcd(p, n) = 1$, Δ lies in $\mathcal{B}$ if and only if it satisfies (B_1).*

(b) *If $\gcd(p, n) > 1$, Δ lies in $\mathcal{B}$ if and only if $p = 3$ and $d = 3$; these are the triangles $(3, s-3, 2s)/3s$ with $s \ge 7$ and $3 \nmid s$, and for them S_3 contains the unit $(2, 1)$ of $\mathbb{Z}/n = \mathbb{Z}/3 \times \mathbb{Z}/s$.*

So on the whole region, Δ lies in $\mathcal{B}$ if and only if it satisfies (B_1) or $(p, d) = (3, 3)$.

The proof of part (b) uses that $\gcd(p, q, r) = 1$; for triples with a common factor (U) can hold with $(p, d) \ne (3, 3)$, for instance for $(p, q, r) = (2, 2, 8)$, $n = 12$: (U) holds, vacuously, since S_2 contains no usable unit, while $(p, d) = (2, 3)$. The triangles that Larsen, Norton and Zykoski construct in the proof of [17, Proposition 3.7], to show that the criterion is blind for arbitrarily large p, satisfy (B_1) with $d = 2p - 1$ and with every number below $2p$ other than p dividing n. Condition (B_1) is the exact form of the pattern they observed experimentally for $p \ne 1, 2, 4$ [17, p. 7124].

Proof. (a) Let p^{-1} be the inverse of p modulo n. As $a \mapsto ap$ permutes $\mathbb{Z}/n$, $S_p = \{kp^{-1} : 0 \le k \le 2p-1\}$, and kp^{-1} is a unit if and only if $\gcd(k, n) = 1$. Next $2kp^{-1} \equiv 2 \pmod n$ means $2k \equiv 2p \pmod n$, that is $k \equiv p$ modulo $n/\gcd(2, n)$; for $1 \le k \le 2p-1$, $|k - p| < p < n/2$, so this happens only for $k = p$. Hence the usable units of S_p are the kp^{-1} with k in

$$K = \{k : 1 \le k \le 2p-1,\ k \ne p,\ \gcd(k, n) = 1\}.$$

As $d \mid n$, p^{-1} reduces to the inverse of p modulo d, and $kp^{-1} \equiv 2 \pmod d$ if and only if $k \equiv 2p \pmod d$. So (U) says:

$$(\star) \qquad \text{every } k \in K \text{ satisfies } k \equiv 2p \pmod d.$$

Suppose $(\star)$. As $p \ge 2$, $1 \in K$, so $2p \equiv 1 \pmod d$, and $(\star)$ says that every $k \in K$ is $\equiv 1 \pmod d$. If p were composite, each prime factor ℓ of p would satisfy $\ell < p$ and $\ell \nmid n$, so $\ell \in K$ and $\ell \equiv 1 \pmod d$; multiplying, $p \equiv 1 \pmod d$, and with $2p \equiv 1$ this gives $d \mid p$, against $\gcd(p, n) = 1$. So p is a prime not dividing n. A prime $\ell < 2p$ with $\ell \ne p$ and $\ell \nmid n$ lies in K, so $\ell \equiv 1 \pmod d$: this is the last clause of (B_1). Conversely, suppose (B_1) and let $k \in K$. Each prime factor ℓ of k has $\ell < 2p$, $\ell \nmid n$ and $\ell \ne p$ (as $p \mid k$ and $k < 2p$ would force $k = p$), so $\ell \equiv 1 \pmod d$. Hence $k \equiv 1 \equiv 2p \pmod d$, and $(\star)$ holds.

(b) Let $h = \gcd(p, n) > 1$. A common divisor of p and s divides $q = (p+q) - p$ and r, so $\gcd(p, s) = 1$. Hence $h = \gcd(p, sd)$ is prime to s and divides d; in particular h is odd and $h \geq 3$. Write $p = hp'$ and $n = hn'$; then $\gcd(p', n') = 1$ and $n' = s \cdot d/h$.

Step 1. Every $a \equiv 1 \pmod{n'}$ lies in S_p: $[ap]_n = h[ap']_{n'} = hp' = p < 2p$, since $p' < n'$, as $p < n$.

Step 2. There is a usable unit $a \neq 1$ with $a \equiv 1 \pmod{n'}$. The units $\equiv 1 \pmod{n'}$ form the kernel of the surjection $(\mathbb{Z}/n)^\times \to (\mathbb{Z}/n')^\times$, of order $\varphi(n)/\varphi(n') = \prod_{\ell^e \| h} c_\ell$ with $c_\ell = \ell^e$ if $\ell \mid n'$ and $c_\ell = \ell^{e-1}(\ell - 1)$ otherwise; each factor is at least 2, as $\ell \geq 3$. Take $a \neq 1$ in it. If n is odd, a is usable. If n is even, the only $a \neq 1$ with $2a \equiv 2$ is $n/2 + 1$, and $n/2 + 1 \equiv 1 \pmod{n'}$ would need $n' \mid hn'/2$, that is h even.

Step 3. If Δ satisfies (U), then $h = d$: the a of Step 2 lies in S_p and satisfies $a \equiv 1 \pmod{d/h}$, as d/h divides n', while the condition requires $a \equiv 2 \pmod{d}$; so d/h divides 1.

Step 4. If moreover $h = d$, then $d = 3$. Now $n' = s$ and $d \mid p$, so $\gcd(d, s) = 1$ (as $\gcd(p, s) = 1$) and $\mathbb{Z}/n = \mathbb{Z}/d \times \mathbb{Z}/s$. Every $(x, 1)$ with x a unit modulo d is a unit modulo n and lies in S_p by Step 1, and for $x \neq 1$ it is usable ($n/2 + 1 \equiv 1 \pmod{s}$ would need d even). The condition forces $x \equiv 2 \pmod{d}$ for each unit $x \neq 1$ modulo d, so $\varphi(d) \leq 2$, and d odd and ≥ 3 gives $d = 3$.

Step 5. If moreover $h = d = 3$, then $p = 3$. Here $3 \nmid s$, $p = 3p'$ with $\gcd(p', s) = 1$, and $6p' \leq s$. If $p' \geq 2$, let $a = (1, {p'}^{-1}) \in \mathbb{Z}/3 \times \mathbb{Z}/s$, with ${p'}^{-1}$ the inverse of p' modulo s (${p'}^{-1} \neq 1$, as $2 \leq p' < s$). It is a unit, it lies in S_p as $[ap]_n = 3[ap']_s = 3 < 2p$, it is $\equiv 1 \pmod 3$, and $a \neq 1$. It is usable: $n/2+1$ is a unit only if $4 \mid s$, and then $n/2+1 = (1, s/2+1)$ with $(s/2+1)^2 \equiv 1 \pmod{s}$, so $a = n/2+1$ would give $p' \equiv s/2+1 \pmod{s}$, impossible as $p' \leq s/6$. So the condition fails, and $p' = 1$.

Step 6. Conversely let $(p, d) = (3, 3)$. Then $n = 3s$, $r = 2s$, $q = s - 3$, $s \geq 6$ as $2p \leq s$, and $3 \nmid s$ as $\gcd(3, s-3, 2s) = 1$; so $s \geq 7$ and $h = 3$. As $3 \mid n$, $[3a]_n$ is a multiple of 3, so $S_3 = \{a : 3a \equiv 0 \text{ or } 3 \pmod{n}\} = \{a : a \equiv 0 \text{ or } 1 \pmod{s}\}$. The $a \equiv 0 \pmod{s}$ are not units, and the units $\equiv 1 \pmod{s}$ are $(1, 1) = 1$, which is not usable, and $(2, 1) \equiv 2 \pmod 3$. So (U) holds, and the unit $(2, 1)$ lies in S_3.

For the last sentence: if $\gcd(p, n) > 1$, (B_1) fails, since it requires p to be a prime not dividing n, and a prime with $\gcd(p, n) > 1$ divides n; so part (b) is the statement. If $\gcd(p, n) = 1$, $(p, d) = (3, 3)$ is impossible, since it gives $3 \mid \gcd(p, n)$; so part (a) is the statement. □

4.2. **Odd denominators with p and q dividing n.** In this subsection n is odd and $\Delta = (p, q, r)/n$ is any obtuse triangle with $p \mid n$ and $q \mid n$; it need not lie in the window. For odd n a unit a is usable exactly when $a \neq 1$, and $[2r]_n = s$ as before.

We prove Theorem 4.2 at the end of this subsection, with the explicit witness of Proposition 4.5, which is Proposition 4.9(i) together with Proposition 4.11.

Proposition 4.5. *In Theorem 4.2, if $s > pI$, a witness is the residue a modulo n with $a \equiv 1 \pmod{pI}$ and $a \equiv 1 + I(q+t)/2 \pmod{qI}$, where t is any odd integer with $|t|\, pI < s$ and $\gcd(2 + It, q) = 1$; for it $[aq]_n = q$ and $[ar]_n = (s - pIt)/2$. Such a t exists (Proposition 4.11).*

Lemma 4.6. $\gcd(p, q) = 1$*; $n = pqI$ with I a positive odd integer; p and q are odd; and $2(p + q) < pqI$.*

Proof. A common divisor of p and q divides n, hence $r = n - p - q$, so it divides $\gcd(p, q, r) = 1$. Then $pq \mid n$, and all divisors of the odd number n are odd. Finally $2r > n$ is $2(p+q) < n$. □

Lemma 4.7. *For a unit a modulo n: $a \in S_q$ if and only if $a \equiv 1 \pmod{pI}$, if and only if $aq \equiv q \pmod{n}$; and $a \in S_p$ if and only if $a \equiv 1 \pmod{qI}$, if and only if $ap \equiv p \pmod{n}$. The only unit in $S_p \cap S_q$ is 1; so no witness lies in $S_p \cap S_q$.*

Proof. q divides n and aq, hence $[aq]_n$. If $[aq]_n < 2q$, then $[aq]_n$ is 0 or q; it is not 0, since $n \mid aq$ with a a unit would give $n \mid q$, while $q < n$. So $aq \equiv q \pmod{n}$, that is $n = pqI$ divides $(a-1)q$, that is $pI \mid a-1$. Conversely $a \equiv 1 \pmod{pI}$ gives $aq \equiv q$ and $[aq]_n = q < 2q$. The same holds for p. If both hold, $a-1$ is divisible by $\operatorname{lcm}(pI, qI) = pqI = n$. □

Lemma 4.8. *Let $a \equiv 1 \pmod{pI}$. The odd integers t with*

$$a \equiv 1 + I(q+t)/2 \pmod{qI}$$

form exactly one residue class modulo $2q$, and for each of them $aq \equiv q$ and $ar \equiv (s - pIt)/2 \pmod{n}$.

Proof. $a \equiv 1 \pmod{I}$; write $a = 1 + Im$. The congruence says $m \equiv (q+t)/2 \pmod{q}$, that is $t \equiv 2m - q \pmod{2q}$, and such t are odd as q is odd. $aq \equiv q$, since $n = pI \cdot q$ divides $(a-1)q$. Multiplying the congruence modulo qI by p gives $ap \equiv p + pI(q+t)/2 \pmod{n}$. Then, modulo n,

$$ar = an - ap - aq \equiv -p - q - \frac{pI(q+t)}{2} = -(p+q) - \frac{n + pIt}{2} \equiv n - (p+q) - \frac{n+pIt}{2} = \frac{s - pIt}{2},$$

because $s = n - 2p - 2q$. Here s and pIt are odd, so $(s - pIt)/2$ is an integer, and replacing t by $t + 2q$ changes it by n. □

Proposition 4.9. (i) *The witnesses in $S_q \cap S_r$ are in bijection with the odd integers t such that $|t|\, pI < s$ and $\gcd(2 + It, q) = 1$, the witness of t being the residue a_t modulo n with $a_t \equiv 1 \pmod{pI}$ and $a_t \equiv 1 + I(q+t)/2 \pmod{qI}$; and $[a_t q]_n = q$, $[a_t r]_n = (s - pIt)/2$.*

(ii) *The same holds with p and q exchanged: the witnesses in $S_p \cap S_r$ are in bijection with the odd t such that $|t|\, qI < s$ and $\gcd(2 + It, p) = 1$.*

(iii) *Every witness is of exactly one of these two kinds.*

Proof. (i) *From t to a witness.* The moduli pI and qI have greatest common divisor I and least common multiple n, and the two prescribed residues agree modulo I (both are $\equiv 1$), so a_t exists and is unique modulo n. It is a unit: let ℓ be a prime dividing n. If $\ell \mid pI$, then $a_t \equiv 1 \pmod{\ell}$. Otherwise $\ell \mid q$, and $2a_t \equiv 2 + Iq + It \equiv 2 + It \pmod{\ell}$, which is not 0 because $\gcd(2 + It, q) = 1$; as ℓ is odd, $\ell \nmid a_t$. By Lemma 4.8, $a_t q \equiv q$, so $[a_t q]_n = q < 2q$, and $a_t r \equiv (s - pIt)/2$; from $-s < pIt < s$ we get $0 < (s - pIt)/2 < s < n$, so $[a_t r]_n = (s - pIt)/2 < s$. Finally $a_t \neq 1$, because $[r]_n = r \geq s$, so $1 \notin S_r$. Hence a_t is a witness in $S_q \cap S_r$.

From a witness to t. Let a be a unit in $S_q \cap S_r$. By Lemma 4.7, $a \equiv 1 \pmod{pI}$. By Lemma 4.8, as t runs over the residue class modulo $2q$ attached to a, the numbers $(s - pIt)/2$ run over a full residue class modulo n, the class of ar; choose the t with $(s - pIt)/2 = [ar]_n$. Then $0 \leq (s - pIt)/2 < s$, and it is not 0, as a is a unit and $0 < r < n$; so $-s < pIt < s$. And $2a \equiv 2 + It \pmod{q}$ with a a unit gives $\gcd(2 + It, q) = 1$. By construction $a = a_t$.

Bijection. If $a_t = a_{t'}$, then $t \equiv t' \pmod{2q}$ by Lemma 4.8, and $|t|, |t'| < s/(pI) = q - 2/I - 2q/(pI) < q$, so $t = t'$.

(ii) Lemmas 4.7 and 4.8 and the argument of (i) never used $p \leq q$. (iii) A witness lies in two of S_p, S_q, S_r, and not in $S_p \cap S_q$ by Lemma 4.7; if it lay in all three it would lie in $S_p \cap S_q$. □

Lemma 4.10. *Let D be a product of $k \geq 1$ different primes, and for each prime $\ell \mid D$ let a residue class β_ℓ modulo ℓ be given. Among any N consecutive integers, more than $N\varphi(D)/D - 2^k + 1$ avoid all the classes β_ℓ.*

Proof. By inclusion and exclusion the number is $\sum_{e|D}(-1)^{k_e}A_e$, k_e the number of prime factors of e, where A_e is the number of the N integers lying in β_ℓ modulo ℓ for every $\ell \mid e$. By the Chinese remainder theorem that is one class modulo e, so $A_e = N/e + \epsilon_e$ with $|\epsilon_e| < 1$ and $\epsilon_1 = 0$. The main terms add up to $N\prod_{\ell|D}(1-1/\ell) = N\varphi(D)/D$, and the $2^k - 1$ other error terms are each less than 1 in size. □

Proposition 4.11. *If $s > pI$, there is an odd t with $|t|\,pI < s$ and $\gcd(2+It, q) = 1$. If neither $t = 1$ nor $t = -1$ is one, then $I \geq 5$ and q is divisible by two different primes that do not divide I.*

Proof. $t = \pm 1$ satisfy $|t|\,pI < s$. If $\gcd(I+2, q) = 1$ take $t = 1$; if $\gcd(I-2, q) = 1$ take $t = -1$. Otherwise there are primes $\ell_1 \mid \gcd(q, I+2)$ and $\ell_2 \mid \gcd(q, I-2)$. They are odd; they are different, since a common prime would divide $(I+2) - (I-2) = 4$; and neither divides I, since it would then divide 2. As $I - 2$ has a prime factor and I is odd and positive, $I \geq 5$ ($I - 2 = -1, 1$ for $I = 1, 3$). So $q \geq 15$, and

$$M = \frac{s}{pI} = q - \frac{2}{I} - \frac{2q}{pI} \geq q - \frac{2}{5} - \frac{2q}{5} = \frac{3q-2}{5}.$$

Let D be the product of the primes dividing q but not I, and $k \geq 2$ their number. Write the odd t as $2j+1$. The condition $|t| < M$ is $-(M+1)/2 < j < (M-1)/2$, an open interval of length M, which contains $N \geq M - 1$ consecutive integers. For a prime $\ell \mid q$: if $\ell \mid I$, then $2 + It \equiv 2 \not\equiv 0 \pmod{\ell}$; if $\ell \mid D$, then $\ell \mid 2 + I(2j+1)$ exactly when j lies in one class β_ℓ modulo ℓ, namely $-(I+2)/(2I)$. So $\gcd(2+It, q) = 1$ exactly when j avoids β_ℓ for every $\ell \mid D$, and by Lemma 4.10 the number of such j is more than $N\varphi(D)/D - 2^k + 1$. Now $N \geq M - 1 \geq (3q-7)/5 \geq (3D-7)/5 > 0$, so

$$N\varphi(D)/D \geq (3D-7)\varphi(D)/(5D) > (3\varphi(D) - 7)/5,$$

and $\varphi(D) \geq 2 \cdot 4^{k-1} \geq 2^{k+1}$, as the least prime of D is at least 3 and the others at least 5. Hence $N\varphi(D)/D > (6 \cdot 2^k - 7)/5 = 2^k - 1 + (2^k - 2)/5 \geq 2^k - 1$, and the number of good j is positive. □

The smallest triangles where $t = \pm 1$ both fail are $(1, 15, 89)/105$, with $I = 7$, where $t = 3$ gives $a = 64$, and $(1, 21, 83)/105$, with $I = 5$, where $t = 3$ gives $a = 61$.

Proposition 4.12. *$s \leq pI$ holds exactly for $(1, 1, n-2)/n$ with $n \geq 5$ odd, $(1, 3, 5)/9$, $(1, 5, 9)/15$ and $(3, 7, 11)/21$.*

Proof. $s \leq pI$ reads $pI(q-1) \leq 2(p+q)$, and the triangle is obtuse: $2(p+q) < pqI$. If $q = 1$, then $p = 1$; the first inequality is $0 \leq 4$, the second is $I > 4$, and the triangle is $(1, 1, I-2)/I$ with $I = n \geq 5$. If $q \geq 3$ and $p = 1$: $I(q-1) \leq 2q + 2 < Iq$. The right half gives $I > 2$, so $I \geq 3$; the left half gives $I \leq 2 + 4/(q-1)$, so $q \leq 5$. For $q = 3$, $I \leq 4$, so $I = 3$ and the triangle is $(1, 3, 5)/9$ ($s = 1 \leq 3$). For $q = 5$, $I \leq 3$, so $I = 3$ and the triangle is $(1, 5, 9)/15$ ($s = 3 \leq 3$). If $p \geq 3$: p and q are coprime and odd with $p \leq q$, so $q \geq p + 2$ and $2(p+q) \leq 4(q-1)$. Then $pI(q-1) \leq 4(q-1)$, so $p = 3$ and $I = 1$, and the two inequalities read $3(q-1) \leq 6 + 2q < 3q$: $6 < q \leq 9$ with q odd and prime to 3, so $q = 7$ and the triangle is $(3, 7, 11)/21$ ($s = 1 \leq 3$). □

Proof of Theorem 4.2 and Proposition 4.5. That pq divides n is Lemma 4.6. If $s > pI$, Propositions 4.11 and 4.9 give a witness. If $s \le pI$, there is no odd t with $|t|\,pI < s$, and none with $|t|\,qI < s$, as $qI \ge pI$; by Proposition 4.9 there is no witness. The triangles with $s \le pI$ are those of Proposition 4.12, and Proposition 4.5 is Proposition 4.9(i) with Proposition 4.11. □

4.3. **Even denominators.** Let $n = 2m$ be even. Then s is even; set $\varsigma = s/2 = r - m$, so that

$$p + q + \varsigma = m, \qquad 1 \le \varsigma \le m/3, \qquad L = n - 2q = 2p + 2\varsigma,$$

and $q \ge (m-\varsigma)/2 \ge m/3$, $p < m/2$. There is no window triangle for $n = 2, 4$, so $m \ge 3$. Let $J(N)$ be Jacobsthal's function: the least $J \ge 1$ such that any J consecutive integers contain one prime to N; it depends only on the primes dividing N. If $N' \mid N$, then $J(N') \le J(N)$, since an integer prime to N is prime to N'; and $J(N) \le N$, as N consecutive integers contain one $\equiv 1 \pmod N$. Let m° be the product of the odd primes dividing m ($m^\circ = 1$ if m is a power of 2), and write $J = J(m^\circ)$. Every other argument of J in this section and in Appendix B is odd and squarefree. Let $P(m) < P_2(m)$ be the two least odd primes not dividing m.

Lemma 4.13. *Let N be odd and squarefree with $k \ge 1$ prime factors. Then $J(N) \le 3$ if $k = 2$, $J(N) \le 3^k$ in general, and $J(N) \le \lfloor (2^k - 1)N/\varphi(N) \rfloor + 1$.*

Proof. For $k = 2$: among three consecutive integers each prime $\ell \ge 3$ divides at most one. In general, by Lemma 4.10 with $D = N$ and every class $\beta_\ell = 0$, among M consecutive integers more than $M\varphi(N)/N - 2^k + 1$ are prime to N; this is ≥ 0, and the number is therefore at least 1, once $M \ge (2^k - 1)N/\varphi(N)$, so $J(N) \le \lceil (2^k - 1)N/\varphi(N) \rceil \le \lfloor (2^k - 1)N/\varphi(N) \rfloor + 1$. Finally $N/\varphi(N) \le (3/2)^k$ and $(2^k - 1)(3/2)^k + 1 \le 3^k$. □

Theorem 4.14. *Let $n = 2m$, $J = J(m^\circ)$, $P_2 = P_2(m)$, and suppose $m > N_0(m) = P_2(31J^2 - 1)$. If a window triangle $(p, q, r)/n$ has no witness, then it lies in $\mathcal{A}$, in $\mathcal{B}$ or in $\mathcal{C}$.*

The idea of the proof. The proof is in Appendix B; the table at the end of this subsection lists its cases and the results that settle them. Suppose that Δ has no witness. Minkowski's theorem and Jacobsthal's function give $p \le 7J^2 - 1$, and so $7L \le 3n$ (Step 1). Step 2 shows that S_r contains no usable unit. The odd elements of S_r fall into rows of $g_\varsigma = \gcd(\varsigma, m)$ residues spaced n/g_ς apart (Proposition B.13); for $g_\varsigma \ge 2$, outside a few cases treated apart, the units of a row are spread along it and a usable one lies in S_q, a witness, and for $g_\varsigma = 1$ and r even a witness comes from doubling. For $g_\varsigma = 1$ and r odd, the products of r^{-1} with the powers of an odd prime $u \nmid m$ with $uL < n$, a *multiplier chain* (Lemma B.5), give $P_2 L \ge n$, and a rational approximation from Dirichlet's theorem gives $\varsigma \le 24J^2$; with $p \le 7J^2 - 1$ this contradicts $m > N_0(m)$, and this is where the constant $31 = 7 + 24$ comes from. Once S_r has no usable unit, either $\varsigma \mid m$ with m/ς odd, or $\varsigma = 1$ with m even (Proposition B.13(b)), and Step 3 finds a witness in $S_p \cap S_q$, from the multiplier chain and the rows of S_p, for every such triangle outside $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. The corresponding threshold of Larsen, Norton and Zykoski above $2\pi/3$ is the hypothesis $24j(n)^2 < \sqrt{n}$ of [17, Lemma 7.9]; here the multiplier chain replaces $\sqrt{n}$ by $n/P_2(m)$.

Theorem 4.15. *Let n be even. A window triangle of denominator n has no witness if and only if it lies in $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$, or is a member of $\mathcal{E}$ with even denominator (Table 1).*

Proof. There is no window triangle for $n = 2, 4$. *Only if.* By Proposition 4.3(c), a triangle without witness with $s \mid n$, n/s odd and $2 \le p \le s/2$ lies in $\mathcal{B}$; so it is enough to show that a window triangle of even denominator without witness lies in $\mathcal{A}$ or $\mathcal{C}$, or has $s \mid n$, n/s odd and $2 \le p \le s/2$, or is a member of $\mathcal{E}$ with even denominator. If $m > N_0(m)$, this is Theorem 4.14. The $m \ge 3$ with $m \le N_0(m)$ are 2,567 values, the largest 69,615 (Appendix A.2, where the list is proved complete). By Lemma B.4, which has no hypothesis on m, a window triangle of denominator $2m$ without witness has $pq \le 4mJ^2$. For the 2,567 values of m an enumeration of the window triangles of denominator $2m$ with $pq \le 4mJ^2$ finds that each of them without witness lies in $\mathcal{A}$ or $\mathcal{C}$, or has $s \mid n$, n/s odd and $2 \le p \le s/2$, or is a member of $\mathcal{E}$ (Appendix A.3). *If.* For $\mathcal{A}$ and $\mathcal{B}$ this is Proposition 4.3, and for $\mathcal{C}$ Proposition 4.16 below. The members of $\mathcal{E}$ have no witness by a direct check of their units (Appendix A.3). □

Proposition 4.16. *No member of $\mathcal{C}$ has a witness.*

Proof. Here $4 \mid n$, $s = 2$, $r = n/2 + 1$, $q = n/2 - 1 - p$, and $p \le q$ forces $n \ge 4p + 2$. S_r. r is odd and $\gcd(r, n)$ divides $2r - n = 2$, so r is a unit and $S_r = \{a : ar \equiv 0 \text{ or } 1 \pmod n\} = \{0, r^{-1}\}$. As $r^2 = n^2/4 + n + 1 \equiv 1 \pmod n$, $r^{-1} = r = n/2 + 1$, and $2r \equiv 2$: S_r has no usable unit. S_p. For $p = 1$, $S_1 = \{0, 1\}$. For $p = 2$: $[2a]_n$ is even, so $[2a]_n < 4$ means $2a \equiv 0$ or 2, that is $a \in \{0, n/2, 1, n/2 + 1\}$; 0 and $n/2$ are not units, and 1 and $n/2 + 1$ are not usable. For $p = 4$ and $n \equiv 4 \pmod 8$: $[4a]_n$ is a multiple of $4 = \gcd(4, n)$, so $[4a]_n < 8$ means $4a \equiv 0$ or 4, that is $a \equiv 0$ or $1 \pmod{n/4}$; the eight candidates are $jn/4$ and $jn/4 + 1$, $j = 0, \dots, 3$. The $jn/4$ are not units, as $n/4 > 1$ divides them and n; $n/4 + 1$ and $3n/4 + 1$ are even, as $n/4$ is odd; and 1, $n/2 + 1$ are not usable. So S_p has no usable unit. A witness lies in two of the three sets, hence in S_p or S_r: impossible. □

The statement on S_p is [17, Proposition 3.5] for $x = 1, 2, 4$.

Remark 4.17. Let $4 \mid n$ and $u = 1 + n/2$. Then u is a unit, $2u \equiv 2$ and $u^2 \equiv 1 \pmod n$, so $S_{ux} = uS_x$ and multiplication by u preserves usable units. If ς is odd, exactly one of p, q is even, say e, the other o is odd, and $[up]_n$, $[uq]_n$, $[ur]_n$ are e, $o + n/2$, ς in some order, with sum n: a triangle $u\Delta$ which has a witness if and only if Δ has one, and whose unfolding is the same Riemann surface as that of Δ with the eigenform of character u. The members $(2, m - 3, m + 1)/2m$ $(m \ge 6)$ and $(4, m - 5, m + 1)/2m$ $(m \equiv 2 \bmod 4,\ m \ge 10)$ of $\mathcal{C}$ are sent to the triangles $(1, 2, n - 3)/n$ and $(1, 4, n - 5)/n$ of families (2) and (3) of Theorem 2.5; $(3, 4, 9)/16$, $(5, 8, 15)/28$, $(8, 9, 23)/40$ to $(1, 4, 11)/16$, $(1, 8, 19)/28$, $(3, 8, 29)/40$ of item (7).

Proof of Theorem 4.1. Part (a) is Theorem C.3 for odd n ($\mathcal{C}$ is empty for odd n, and the members of $\mathcal{E}$ with odd denominator are the five of Theorem C.3) and Theorem 4.15 for even n. Part (b) is Proposition 4.3(c), and its explicit form is Proposition 4.4. In the sentence after Theorem 4.1, the prime case is proved at the end of Appendix C, and the cases $n = 2\ell$ and $p = 1$ follow from part (a). For $n = 2\ell$, ℓ an odd prime: $\mathcal{C}$ needs $4 \mid n$; a member of $\mathcal{B}$ has $s \mid 2\ell$ with n/s odd and $s \ge 4$, so $s = 2\ell > n/3$, which is impossible; $\mathcal{A}$ needs $s = 2$, giving $(1, \ell - 2, \ell + 1)/2\ell$; and the members of $\mathcal{E}$ with denominator twice an odd prime are $(3, 7, 12)/22$, $(1, 9, 16)/26$, $(5, 7, 14)/26$ and $(3, 13, 18)/34$. For $p = 1$: a member of $\mathcal{B}$ has $p \ge 2$, and the members of $\mathcal{E}$ with $p = 1$ are the six listed there. □

case	condition	settled by
Step 1	every Δ without witness	Corollary B.11: $p \le 7J^2-1$; hence $7L \le 3n$
Step 2: S_r contains a usable unit, $g_\varsigma = \gcd(\varsigma, m)$; a witness is found, or $m \le N_0(m)$		
(2a)	$\varsigma \mid m$	$\varsigma \le 8$: Lemma B.12; $\varsigma \ge 9$: Lemma B.8
(2b)	$\varsigma \nmid m$, $g_\varsigma \ge 2$	$g_\varsigma \ge 4$: Lemma B.8; $g_\varsigma = 2$, or $g_\varsigma = 3$ with $9 \mid m$ or $3L \le n$: Lemma B.7(iv); $g_\varsigma = 3$, $9 \nmid m$, $3L > n$: Lemma B.17
(2c)	$g_\varsigma = 1$, m and ς odd	$\varsigma \ge 9$: Lemma B.16; $\varsigma \le 7$: Lemma B.12
(2d)	$g_\varsigma = 1$, r odd	Corollary B.6(a) and Lemma B.15: $m \le N_0(m)$
Step 3: S_r has no usable unit: (i) $\varsigma \mid m$ with m/ς odd, or (ii) $\varsigma = 1$ with m even		
	(i), $p \le \varsigma$	Δ lies in $\mathcal{A}$ if $p = 1$, in $\mathcal{B}$ if $p \ge 2$ (Proposition 4.3(c))
	(ii), $p \in \{1,2\}$, or $p = 4$ and $m \equiv 2 \pmod 4$	Δ lies in $\mathcal{C}$
the other triangles, $p > \varsigma$, $p \ge 3$, $g = \gcd(p, m)$: a witness in $S_p \cap S_q$, except in (3d)		
(3a)	$g = 1$	Corollary B.6(b)
(3b)	$g \ge 2$, $p/g \ge 3$	$g \ge 4$: Lemma B.8; $g \in \{2,3\}$: Lemma B.7(iv)
(3c)	$g \ge 3$, $p/g \le 2$	Lemma B.10, with Lemma 4.13
(3d)	$g = 2$, $p/g \le 2$	$p = 4$, $m \equiv 2 \pmod 4$: in (ii) Δ lies in $\mathcal{C}$, and (i) cannot occur

4.4. **The families $\mathcal{A}$ and $\mathcal{B}$.** For a member of $\mathcal{A}$ or $\mathcal{B}$ write $n = sd$ with d odd, $d \ge 3$; then $r = s(d+1)/2$ and $p + q = s(d-1)/2$. Since $\gcd((d+1)/2, d) = 1$, $\gcd(r, n) = s$: the s points above the obtuse vertex are zeros of ω ($r \nmid n$), and $n/\gcd(r,n) = d$. Also $\gcd(p, q, r) = \gcd(p, s)$, since $(d-1)/2$ and $(d+1)/2$ are coprime.

Theorem 4.18. (a) *Part (a) or part (b) of Theorem 1.2 applies to every member of $\mathcal{A}$ other than $(1,1,3)/5$, $(1,1,4)/6$ and $(1,3,5)/9$.*

(b) *Part (b) of Theorem 1.2 applies to every member of $\mathcal{B}$, at the obtuse vertex.*

(c) *Neither part applies to $(1,1,3)/5$, $(1,1,4)/6$ or $(1,3,5)/9$.*

Consequently no member of $\mathcal{A} \cup \mathcal{B}$ other than $(1,1,3)/5$, $(1,1,4)/6$ and $(1,3,5)/9$ is a lattice triangle; the first two are, by Veech's theorem; the third is not, by Theorem 6.1.

Proof. A unit $a \equiv 2 \pmod d$ that is small on p gives a holomorphic form. Let a be a unit with $a \equiv 2 \pmod d$ and $x = [ap]_n < n - s$. Then $ar = s \cdot a(d+1)/2$ and $a(d+1)/2 \equiv d+1 \equiv 1 \pmod d$, so $[ar]_n = s$; and $aq \equiv -ap - ar \equiv -(x+s)$ with $0 < x + s < n$, so $[aq]_n = n - x - s$. The three residues add up to n, so $H_a^{1,0} \ne 0$ by Lemma 3.1(b). We refer to this as $(*)$.

(b) Let $\Delta \in \mathcal{B}$. By Proposition 4.4, either $\gcd(p, n) = 1$, and then the inverse of p modulo n is a usable unit of S_p (it is the element kp^{-1} with $k = 1 \ne p$ in the proof of Proposition 4.4(a)); or $(p, d) = (3, 3)$, and then the unit $(2, 1)$ of $\mathbb{Z}/n = \mathbb{Z}/3 \times \mathbb{Z}/s$ lies in S_3 (Proposition 4.4(b)) and is usable, since $2a \equiv 2 \pmod n$ would give $a \equiv 1 \pmod 3$, as 3 divides $n/\gcd(2, n)$. So S_p contains a usable unit a, and by (U), $a \equiv 2 \pmod d$. Also $a \not\equiv 2 \pmod n$, because $[2p]_n = 2p$ is not less than $2p$, and $[ap]_n < 2p \le s \le n - s$. By $(*)$, $H_a^{1,0} \ne 0$. As $\gcd(r, n) = s \ge 2p > 1$, $r \nmid n$ and $n/s = d$, part (b) applies at the vertex r with $b = a$.

(a), n odd. Here s is odd. Part (a) applies unless $q \mid n$, as $p = 1$. Since $q = s(d-1)/2 - 1$ is prime to s, $q \mid n$ means $q \mid d$; as d is odd, either $q = d$ or $q \le d/3$. If $q = d$, then

$s(d-1) = 2(d+1)$, $s = 2+4/(d-1)$, and s odd forces $d = 5$, $s = 3$: the triangle $(1,5,9)/15$. If $q \le d/3$, then $s(d-1) \le 2d/3+2$, which forces $s = 1$ and $d \le 9$, with $q = (d-3)/2$: $d = 5$ gives $(1,1,3)/5$, $d = 9$ gives $(1,3,5)/9$, $d = 7$ gives $q = 2 \nmid 7$, and $d = 3$ gives no triangle. On $(1,5,9)/15$ part (b) applies at the vertex 9 ($\gcd(9,15) = 3$) with $b = 7$: $[7]_{15} + [35]_{15} + [63]_{15} = 7+5+3 = 15$.

(a), n even. Here s is even. Take $b = 2-kd$ with $1 \le k \le s-1$; then $b \equiv 2 \pmod d$, $b \not\equiv 2 \pmod n$, and $[b]_n = n+2-kd$, which is less than $n-s$ exactly when $kd > s+2$; then $(*)$ gives $H_b^{1,0} \neq 0$, once b is a unit. As $\gcd(kd-2,d) = \gcd(2,d) = 1$, b is a unit exactly when $\gcd(kd-2,s) = 1$, that is, when k is odd and $k \not\equiv 2d^{-1} \pmod \ell$ for every odd prime $\ell \mid s$ with $\ell \nmid d$ (for $\ell \mid d$, $kd-2 \equiv -2$). Call such k *valid*. Validity depends only on k modulo s^*, the product of the primes dividing s and not d (so $2 \mid s^*$); by the Chinese remainder theorem valid classes exist modulo every divisor of s^*, and $k = s$ is not valid, as s is even. We need a valid k with $(s+2)/d < k < s$.

- $s = 2$: $k = 1$ is valid, and $d > 4$ is needed; every $d \ge 5$ works, and $(s,d) = (2,3)$ is $(1,1,4)/6$.
- $s \ge 4$ and $s^* < s$: then $s \ge 2s^*$, and the s^* consecutive integers in $(s-s^*, s]$ contain a valid k, which is less than s; it satisfies $k > s/2 \ge (s+2)/3 \ge (s+2)/d$.
- $s = s^* > 2$, so s is squarefree and prime to d: let ℓ be the least odd prime dividing s, and $k_1 \in [1, s/\ell]$ a class valid for the primes of s/ℓ. The ℓ numbers $k_1 + js/\ell$, $0 \le j < \ell$, are valid for the primes of s/ℓ and run over all classes modulo ℓ, of which exactly one is not valid; so each class modulo s/ℓ valid for its primes contains $\ell - 1$ valid classes modulo s, and $j = \ell-1$ or $j = \ell-2$ gives a valid $k > (\ell-2)s/\ell$, with $k < s$. If $\ell = 3$, then $3 \nmid d$, so $d \ge 5$ and $k > s/3 \ge (s+2)/5$; if $\ell \ge 5$, then $k > 3s/5 \ge (s+2)/3$.

So part (b) applies at the vertex r ($\gcd(r,n) = s > 1$, $n/s = d$) to every member of $\mathcal{A}$ with n even except $(1,1,4)/6$.

(c) On $(1,1,3)/5$ both p and q divide n and every $\gcd(x,n)$ is 1. On $(1,1,4)/6$, n is even, and at the vertex 4 ($\gcd(4,6) = 2$) the only unit $b \equiv 2 \pmod 3$ with $b \not\equiv 2 \pmod 6$ is $b = 5$, for which $[5]_6 + [5]_6 + [20]_6 = 12 \neq 6$. On $(1,3,5)/9$ the numerators 1 and 3 divide 9, and $\gcd(5,9) = 1$. □

5. Cylinder families

In this section we show by Lemma 2.3 that several one-parameter families of obtuse triangles are not lattice triangles, exhibiting two parallel maximal cylinders whose moduli have an irrational ratio, or one maximal cylinder whose share of the area has a Galois conjugate outside $(0,1]$. The model of Lemma 5.1 with $(p,q) = (4,1)$ is the unfolding drawn as a regular n-gon with four regular $(n/4)$-gons attached, in which Larsen, Norton and Zykoski find the two cylinders of family (3) [17, §10], after [11, Figure 1]; the star of Section 5.3 is the model in which Ward finds the two cylinders of [31, Theorem B]. Theorems 5.8 and 5.9 use an irrational ratio of moduli in the first model for $p = 2$, Theorems 5.15 and 5.16 one in the star, and Theorems 5.10 and 5.11 a share of the area in the first model for $p = 4$.

5.1. **Two families with an angle $2\pi/n$.** In this subsection $\Delta = PQR$ has angles $p\psi$, $q\psi$, $r\psi$ at P, Q, R, where $\psi = \pi/n$ (the order of p,q,r is not assumed). We use complex coordinates $P = 0$, $Q = 1$, $R = |PR|\, e^{ip\psi}$, where by the law of sines $|PR| = \sin q\psi / \sin r\psi$ and $|QR| = \sin p\psi / \sin r\psi$. Let Θ_k be the rotation by $2k\psi$ ($k \in \mathbb{Z}/n$) and κ complex conjugation. The dihedral group $\mathcal{D} = \{\Theta_k, \Theta_k\kappa\}$ is generated by the linear parts of the

reflections in the sides, $\kappa_{PQ} = \kappa$, $\kappa_{PR} = \Theta_p\kappa$, $\kappa_{QR} = \Theta_{-q}\kappa$ (the reflection in a line of direction ϕ is $z \mapsto e^{2i\phi}\bar{z}$).

Lemma 5.1. *Assume $p \mid n$, $n/p \geq 3$, $p < n/2$ and $q < n/2$. Then X is the union of the following pieces, glued by translations as stated.*

- *For $i \in \mathbb{Z}/p$, the regular (n/p)-gon G_i with vertices $\Theta_k(R)$, $k \equiv i \pmod p$, centre 0 and circumradius $|PR|$; its side from $\Theta_k(\bar{R}) = \Theta_{k-p}(R)$ to $\Theta_k(R)$ is called* side k*, and its outward normal has direction $2k\psi$.*
- *For $k \in \mathbb{Z}/n$, the isosceles triangle (*ear*) E_k with vertices $\Theta_k(\bar{R})$, $\Theta_k(Q)$, $\Theta_k(R)$: its apex is $O_k = \Theta_k(Q) = e^{2ik\psi}$, its apex angle is $2q\psi$ and its legs have length $|QR|$.*
- *The base $\Theta_k(\bar{R}R)$ of E_k is glued to side k of $G_{k \bmod p}$, and the leg $\Theta_k(QR)$ of E_k to the leg $\Theta_{k-q}(Q\bar{R})$ of E_{k-q}.*

The centres of the G_i are the points over P, the apexes are the points over Q, and the vertices of the G_i are the points over R. At its apex, E_k occupies the directions from $(2k+n-q)\psi$ to $(2k+n+q)\psi$; across its leg of direction $(2k+n-q)\psi$ lies E_{k-q}, across the other E_{k+q}.

Proof. For $k \in \mathbb{Z}/n$ let $F_k = \Theta_k\Delta \cup \Theta_k\kappa\Delta$. Since $\kappa_{PQ} = \kappa$, these two copies are glued along $\Theta_k(PQ)$, and F_k is the quadrilateral $\Theta_k(P\bar{R}QR)$. Its angles are $2p\psi$ at P, $2q\psi$ at Q and $r\psi$ at R and $\bar{R}$, all less than π, so F_k is convex and its diagonal $\Theta_k(\bar{R}R)$ cuts it into the triangle $\Theta_k(P\bar{R}R)$ and the ear E_k. Each element of $\mathcal{D}$ is Θ_k or $\Theta_k\kappa$ for exactly one k, so X is the union of the n quadrilaterals F_k, and the gluings along the copies of PQ are used inside them. The side $\Theta_k(PR)$ of $\Theta_k\Delta$ is glued to the copy $\Theta_k\kappa_{PR}\Delta = \Theta_{k+p}\kappa\Delta \subset F_{k+p}$ along $\Theta_{k+p}\kappa(PR) = \Theta_{k+p}(P\bar{R})$, which is the same segment from 0 to $\Theta_k(R) = \Theta_{k+p}(\bar{R})$. Hence $F_i, F_{i+p}, \dots, F_{i+(n/p-1)p}$ are glued cyclically around one point over P, each with angle $2p\psi$, in total 2π; they form a star centred at 0, and the triangles $\Theta_k(P\bar{R}R)$, $k \equiv i \pmod p$, tile G_i, whose sides are the bases of the ears. The side $\Theta_k(QR)$ of $\Theta_k\Delta$ is glued to $\Theta_k\kappa_{QR}\Delta = \Theta_{k-q}\kappa\Delta \subset F_{k-q}$ along $\Theta_{k-q}\kappa(QR) = \Theta_{k-q}(Q\bar{R})$, a leg of E_{k-q}; the side $\Theta_k\kappa(QR)$ of $\Theta_k\kappa\Delta$ is glued to $\Theta_{k+q}\Delta$ along $\Theta_{k+q}(QR)$, which is the same rule for $k+q$. Every copy has three sides, so these are all the gluings. Finally $\Theta_k(R-Q)$ has direction $2k\psi + \pi - q\psi$ and $\Theta_k(\bar{R}-Q)$ has direction $2k\psi + \pi + q\psi$. □

From now on in this subsection assume, besides the hypotheses of Lemma 5.1, that n is even and q is odd. *Horizontal* means the direction of PQ, and leaves are oriented by increasing x. For an apex O and a leaf not through O, the *height above* O of the leaf is its y-coordinate in the chart of an ear at O with O at the origin; let $\chi = (\text{height})/|QR|$. The legs at an apex have directions $(2k+n\pm q)\psi$, odd multiples of ψ since $n+q$ is odd. For an odd integer i write $v(i)$ for the endpoint (a point over R) of the leg of direction $i\psi$ at the apex under consideration, and let

$$\mathcal{I}_\chi = \{i \text{ odd} : 0 < i < n,\ \sin(i\psi) > \chi\}, \qquad \iota(i) = n - i.$$

The map ι preserves $\mathcal{I}_\chi$. For an odd i with $-n < i < 2n$, $\sin(i\psi) > \chi > 0$ already forces $0 < i < n$. A *chain* of $\mathcal{I}_\chi$ is a set $\mathcal{W} = \{i_{\text{bot}}, i_{\text{bot}} + 2q, \dots, i_{\text{top}}\} \subset \mathcal{I}_\chi$ with $i_{\text{top}} + 2q \notin \mathcal{I}_\chi$ and $i_{\text{bot}} - 2q \notin \mathcal{I}_\chi$ (Figure 1); then $\iota(\mathcal{W})$ is a chain, with top $n - i_{\text{bot}}$ and bottom $n - i_{\text{top}}$. Since $2q < n$, we have $i_{\text{top}} + 2q < 2n$ and $i_{\text{bot}} - 2q > -n$. A value of χ is *critical* if it equals $\sin(i\psi)$ for some integer i.

A horizontal leaf that meets no vertex and enters the ears around an apex O through a base, at height $|QR|\chi$ above O, turns clockwise about O, crossing one leg after another while the far end $v(j)$ of the next leg lies above it, that is, while $\sin(j\psi) > \chi$, and then leaves

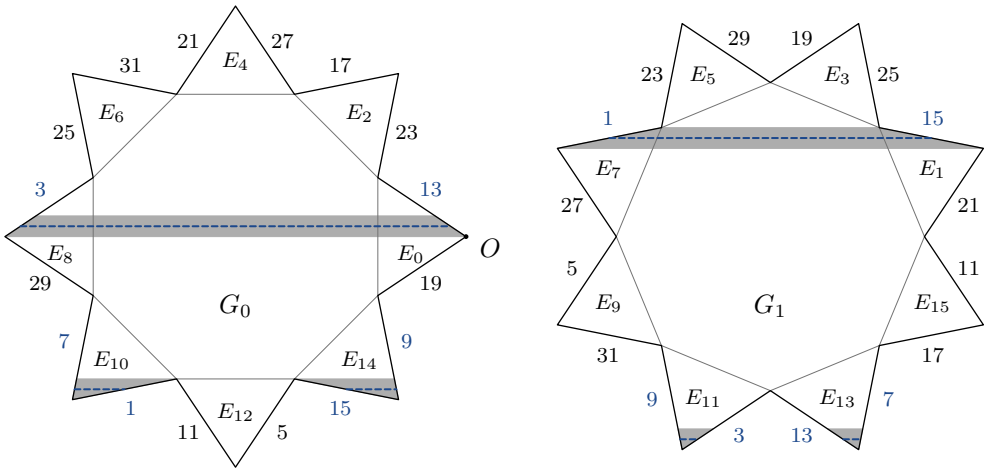


FIGURE 1. The model of Lemma 5.1 for $(2,3,11)/16$: the octagons G_0, G_1 with their ears E_k. A leg is labelled i when its direction from the apex is $i\psi$; legs with the same label are glued. Shaded: the cylinder $Z(\mathcal{W})$ of the chain $\mathcal{W} = \{1,7,13\}$ (Theorem 5.9, $q = 3$); its lower boundary passes through the apex O.

through a base. A chain of $\mathcal{I}_\chi$ is the set of indices of the legs crossed in one such passage (Lemma 5.2, Figure 1). If the polygon the leaf crosses next is symmetric, it re-enters the ears at the next apex at the same height and crosses the legs of $\iota(\mathcal{W})$, and if the polygon after that is symmetric too, it is back at its starting point (Lemmas 5.3 and 5.4; after one passage if $\iota(\mathcal{W}) = \mathcal{W}$). Since the chains stay the same while χ moves in an open interval, these closed leaves, all of one length, fill an annulus whose upper boundary leaf runs into a point over R at which the chain changes, and whose lower boundary leaf runs into such a point or, when $a(\mathcal{W}) = 0$, into the apex O. It is a maximal cylinder when those points are singular (Proposition 5.5).

Lemma 5.2. *Let $y > 0$ be a height with $\chi = y/|QR|$ not critical, and suppose a horizontal leaf crosses the base of E_k from $G_{k \bmod p}$ into E_k at height y above $O = O_k$. Let i_1 be the representative of $2k+n-q$ in $(0,2n)$, so that the clockwise leg of E_k has direction $i_1\psi$. Then $0 < i_1 < n$, $i_1 \in \mathcal{I}_\chi$ and $i_1 + 2q \notin \mathcal{I}_\chi$; the leaf crosses, in this order, the legs of directions $i_1\psi, (i_1-2q)\psi, \dots, i_N\psi$, where $\{i_N, \dots, i_1\}$ is the chain of $\mathcal{I}_\chi$ with top i_1; and it then leaves the ears around O through the base of the ear $E_{k'}$ with $2k' \equiv i_N - n - q \pmod{2n}$, at height y above O, into $G_{k' \bmod p}$.*

Proof. Develop the ears around O into the plane, O at the origin, one ear after the other across their common legs. The leaf develops into a segment of the line $\{y = \text{const} > 0\}$, along which the polar angle decreases strictly as x increases and stays in $(0,\pi)$; since this range of angles is shorter than π, the development meets each ear at most once and is injective. Inside an ear (the triangle $O, v(j), v(j+2q)$, with the sector $[j\psi, (j+2q)\psi]$) the line enters through one side and leaves through another. In E_k the leaf enters through the base; the other side it crosses has smaller polar angle, so it is the clockwise leg, of direction $i_1\psi$. The line separates $v(i_1)$ (above it) from O (below), so $|QR|\sin(i_1\psi) > y$ and $i_1\psi \in (0,\pi)$; it also separates $v(i_1+2q)$ from $v(i_1)$, so $\sin((i_1+2q)\psi) < \chi$. After crossing a leg of direction $j\psi$ the leaf is in the next ear clockwise, with sector $[(j-2q)\psi, j\psi]$, entered through its counterclockwise leg; it leaves through the clockwise leg, of direction $(j-2q)\psi$,

if $v(j-2q)$ lies strictly above the line, that is, if $j-2q \in \mathcal{I}_\chi$, and through the base otherwise (χ is not critical, so the leaf meets no vertex). The ear left through its base has sector $[(i_N-2q)\psi, i_N\psi]$, so $2k'+n+q \equiv i_N \pmod{2n}$. □

Lemma 5.3. *The polygon G_j is symmetric under $(x,y)\mapsto(-x,y)$ about its centre if and only if $2j \equiv n/2 \pmod p$; for $p=2$ this holds for both polygons if and only if $4 \mid n$. If G_j is symmetric, a horizontal leaf entering G_j through side k' leaves it through side $k''=n/2-k'$, its heights above $O_{k'}$ and above $O_{k''}$ are equal, and in the chart of G_j we have $O_{k''}-O_{k'}=(-2\cos 2k'\psi, 0)$.*

Proof. The reflection maps the direction ϕ to $\pi-\phi$, so the outward normal $2k\psi$ to $2(n/2-k)\psi$, and the set $\{k \equiv j \pmod p\}$ is preserved if and only if $n/2-j \equiv j \pmod p$. For $p=2$ and $j=0,1$ this says that $n/2$ is even. A horizontal line meets a symmetric convex polygon in a segment symmetric about the vertical axis, whose ends lie on mirror sides k' and $n/2-k'$. Finally $e^{2ik''\psi} = e^{i(\pi-2k'\psi)} = -\cos 2k'\psi + i\sin 2k'\psi$. □

Lemma 5.4. *In Lemma 5.2, if $G_{k' \bmod p}$ is symmetric, the leaf next enters $E_{k''}$, $k''=n/2-k'$, through its base at the same height y above $O_{k''}$, and the first leg it crosses there has direction $(n-i_N)\psi$; so its next passage traverses $\iota(\mathcal{W})$, where $\mathcal{W}$ is the chain of the first passage.*

Proof. We have $2k''+n-q = 2n-2k'-q \equiv 2n-(i_N-n-q)-q = 3n-i_N \equiv n-i_N \pmod{2n}$, and $0<n-i_N<n$. Apply Lemma 5.2 to the new passage: it traverses the chain of $\mathcal{I}_\chi$ with top $n-i_N$, which is $\iota(\mathcal{W})$. □

For a set $\mathcal{W}=\{i_{\mathrm{bot}},\dots,i_{\mathrm{top}}\}$ of odd integers in $(0,n)$ in arithmetic progression with difference $2q$ let

$$b(\mathcal{W}) = \min(\sin i_{\mathrm{bot}}\psi,\ \sin i_{\mathrm{top}}\psi), \qquad a(\mathcal{W}) = \max\bigl(0,\ \sin(i_{\mathrm{top}}+2q)\psi,\ \sin(i_{\mathrm{bot}}-2q)\psi\bigr).$$

Since sin is concave on $[0,\pi]$, $b(\mathcal{W})$ is the minimum of $\sin(i\psi)$ over $\mathcal{W}$, and for non-critical χ the set $\mathcal{W}$ is a chain of $\mathcal{I}_\chi$ if and only if $a(\mathcal{W})<\chi<b(\mathcal{W})$.

Proposition 5.5. *Let $\mathcal{W}=\{i_{\mathrm{bot}},\dots,i_{\mathrm{top}}\}$ be as above, with $a(\mathcal{W})<b(\mathcal{W})$; let E_k be the ear with $2k+n-q \equiv i_{\mathrm{top}} \pmod{2n}$, and O its apex. Define $k_1,k_2\in\mathbb{Z}/n$ by $2k_1 \equiv i_{\mathrm{bot}}-n-q$ and $2k_2 \equiv -i_{\mathrm{top}}-q \pmod{2n}$, and suppose that*

$$2k_1 \equiv 2k_2 \equiv n/2 \pmod p, \tag{5.1}$$

that is, $i_{\mathrm{bot}} \equiv q+n/2$ and $i_{\mathrm{top}} \equiv -q-n/2 \pmod p$. Then the horizontal leaves through the base of E_k at heights $y \in (|QR|\,a(\mathcal{W}), |QR|\,b(\mathcal{W}))$ above O are closed and fill an open annulus of closed leaves $Z(\mathcal{W})=Z(\iota(\mathcal{W}))$, of height $|QR|\,(b(\mathcal{W})-a(\mathcal{W}))$ and circumference

$$\mathcal{L}(\mathcal{W}) = \begin{cases} 2\bigl[\cos((i_{\mathrm{bot}}-q)\psi) - \cos((i_{\mathrm{top}}+q)\psi)\bigr], & \iota(\mathcal{W})\neq\mathcal{W}, \\ 2\cos((i_{\mathrm{bot}}-q)\psi), & \iota(\mathcal{W})=\mathcal{W}. \end{cases}$$

Its upper boundary contains a point over R; its lower boundary contains a point over R if $a(\mathcal{W})>0$, and the apex O if $a(\mathcal{W})=0$. If moreover $r\nmid n$, then $Z(\mathcal{W})$ is a maximal cylinder if $a(\mathcal{W})>0$, and also if $a(\mathcal{W})=0$ and $q\nmid n$.

Proof. For $a(\mathcal{W})<\chi<b(\mathcal{W})$ the line at height $y=|QR|\chi$ meets the base of E_k, which runs from $v(i_{\mathrm{top}}+2q)$ (below the line) to $v(i_{\mathrm{top}})$ (above it), in one interior point, and the leaf crosses there into E_k. Let $\mathcal{T}$ be the open subsegment of the base of these points. For non-critical χ, by Lemma 5.2 the leaf traverses $\mathcal{W}$ and leaves the ears around O through the base of E_{k_1} into $G_{k_1 \bmod p}$, which is symmetric by (5.1) and Lemma 5.3; by Lemma 5.4 it next traverses $\iota(\mathcal{W})$, whose bottom is $n-i_{\mathrm{top}}$, and leaves through the base of E_{k_2} (as

$2k_2 \equiv (n - i_{\text{top}}) - n - q \pmod{2n}$) into $G_{k_2 \bmod p}$, also symmetric; and by Lemma 5.4 again it re-enters under $n - (n - i_{\text{top}}) = i_{\text{top}}$, that is, through the base of E_k at height y above its apex: the point it started from. If $\iota(\mathcal{W}) = \mathcal{W}$ it is back after one passage.

Length. In a passage the leaf runs, in the developed coordinates of its apex, from (x_{in}, y) to (x_{out}, y); then, by Lemma 5.3, across the polygon a length $-2\cos 2k'\psi + x'_{\text{in}} - x_{\text{out}}$. Over one period the x's telescope, and the length is the sum of $-2\cos 2k'\psi$ over the exit ears k' of the period. By Lemma 5.2, $2k'\psi \equiv (i_{\text{bot}} - q)\psi - \pi$ for the exit after $\mathcal{W}$, and $2k'\psi \equiv (n - i_{\text{top}} - q)\psi - \pi$ after $\iota(\mathcal{W})$; this gives $\mathcal{L}(\mathcal{W})$.

Annulus. Along these leaves the only points over Q or R that can be met are the points $v(j)$, at heights $|QR|\sin(j\psi)$ above the apex, for j in $\mathcal{W}$, in $\iota(\mathcal{W})$ and their neighbours in the progressions; the apexes, at height 0; and in the polygons the ends of the sides crossed, which are ends of bases of the ears involved and so again points $v(j)$ of this list. For j in $\mathcal{W}$ or in $\iota(\mathcal{W})$ the height is at least $|QR|\,b(\mathcal{W})$, for a neighbour it is at most $|QR|\,a(\mathcal{W})$ (since $\sin(n-i)\psi = \sin i\psi$, ι preserves the sines, and $a(\iota(\mathcal{W})) = a(\mathcal{W})$, $b(\iota(\mathcal{W})) = b(\mathcal{W})$), and $0 \le |QR|\,a(\mathcal{W})$. So for every $\chi \in (a(\mathcal{W}), b(\mathcal{W}))$, critical or not, $\mathcal{W}$ and $\iota(\mathcal{W})$ are chains of $\mathcal{I}_\chi$, the leaf meets no point over Q or R, and the proofs of Lemmas 5.2 and 5.4, which use that χ is not critical only to know that the leaf meets no such point, apply unchanged. A leaf may pass through the centre of a polygon, a point over P; this point is regular, since $p \mid n$, so the leaf is still a closed regular leaf. So the map $\mathcal{T} \times \mathbb{R}/\mathcal{L}(\mathcal{W})\mathbb{Z} \to X$ that sends a point of $\mathcal{T}$ and a time to the point reached at that time on the leaf from that point is a local homeomorphism, affine in the charts. It is injective: a rightward leaf crosses the base of E_k only into E_k, which happens once per period, at the same height above O each time (Lemma 5.3). Its image is an open annulus of closed leaves, of circumference $\mathcal{L}(\mathcal{W})$ and height $|QR|(b(\mathcal{W}) - a(\mathcal{W}))$, and the leaves of $\iota(\mathcal{W})$ are the same leaves.

Boundary. As $y \uparrow |QR|\,b(\mathcal{W})$ the leaves converge to the leaf from the upper end of $\mathcal{T}$, which runs into $v(i)$ with $\sin(i\psi) = b(\mathcal{W})$; as $y \downarrow |QR|\,a(\mathcal{W})$, to the leaf through O (if $a = 0$) or through the neighbouring vertex (if $a > 0$). An open annulus of closed leaves is a maximal cylinder if each of its boundary components contains a singular point. If $r \nmid n$, the $\gcd(r, n)$ points over R have cone angle $2\pi r/\gcd(r, n) > 2\pi$, and if $q \nmid n$ so have the points over Q; this gives the last sentence of the statement. □

Lemma 5.6. *If $p = 2$ and $4 \mid n$, then (5.1) holds for every $\mathcal{W}$, and $r \nmid n$.*

Proof. Both sides of (5.1) are even, as $4 \mid n$. Next, $r = n - 2 - q$ is odd, as n is even and q is odd; $r > n/2 - 2$, as $q < n/2$; and $n \ge 8$, as $4 \mid n$ and $n/p \ge 3$. An odd divisor of n divides $n/4$, so it is at most $n/4 < n/2 - 1 \le r$. Hence $r \nmid n$. □

The map $\delta\Delta \to \kappa\delta\Delta$, $z \mapsto \bar{z}$ in each copy, respects the gluings, so it is an isometry $\hat{\kappa}$ of X with derivative $\operatorname{diag}(1, -1)$. It sends E_k to E_{-k}, apexes to apexes, polygons to polygons, rightward horizontal leaves to rightward horizontal leaves, and height y above O_k to height $-y$ above O_{-k}; if the clockwise leg of E_k has index i_1, that of E_{-k} has index $2n - i_1 - 2q$.

Proposition 5.7. *In Proposition 5.5, suppose that $r \nmid n$, that $q \mid n$, that $a(\mathcal{W}) = 0$ with $\sin(i_{\text{top}} + 2q)\psi < 0$, and that for small $\chi > 0$ the chain of $\mathcal{I}_\chi$ with top $2n - i_{\text{top}} - 2q$ is $\mathcal{W}$ or $\iota(\mathcal{W})$. Then $Z(\mathcal{W}) \cup \gamma_0 \cup \hat{\kappa}(Z(\mathcal{W}))$, where γ_0 is the leaf at height 0 through O, is a maximal cylinder of circumference $\mathcal{L}(\mathcal{W})$ and height $2|QR|\,b(\mathcal{W})$.*

Proof. The base of E_k has its ends at heights $|QR|\sin(i_{\text{top}}+2q)\psi < 0$ and $|QR|\sin(i_{\text{top}}\psi) > 0$, so it contains a point at height 0 in its interior. The leaf γ_0 from that point is the limit of the leaves of $Z(\mathcal{W})$; it meets no point over R (their heights above an apex are $|QR|\sin(j\psi) \ne 0$

with j odd, and polygon chords meet vertices only at ends of bases of ears), only regular points over Q, and possibly centres of polygons, points over P, which are regular since $p \mid n$; so it is a regular closed leaf, of length $\mathcal{L}(\mathcal{W})$. The leaves just below γ_0 cross the base of E_k at small negative heights; they are the $\hat{\kappa}$-images of leaves entering E_{-k} at small positive heights, whose first leg has index $2n - i_{\text{top}} - 2q$, so by hypothesis they are leaves of $Z(\mathcal{W})$, and the leaves just below γ_0 lie in $\hat{\kappa}(Z(\mathcal{W}))$. The annuli $Z(\mathcal{W})$ and $\hat{\kappa}(Z(\mathcal{W}))$ are disjoint: a leaf of $Z(\mathcal{W})$ enters each ear of its period at the same positive height above the apex, while its image enters the image ears at the negative height. The union is an annulus foliated by closed leaves of length $\mathcal{L}(\mathcal{W})$, of height $2|QR|\, b(\mathcal{W})$, whose two boundary components contain points over R, which are singular since $r \nmid n$. □

In the irrationality steps below we use the standard fact that, for $N \geq 3$ and $\gcd(j, N) = 1$, $\cos(2\pi j/N)$ has degree $\varphi(N)/2$ over $\mathbb{Q}$.

Theorem 5.8. *For every even $m \geq 6$ the triangle $(2, m-3, m+1)/2m$ is not a lattice triangle. Let $\psi = \pi/2m$, so that $|QR| = \sin 2\psi / \sin(m+1)\psi$.*

(a) *If $m \geq 8$, in the direction of the side joining the vertices of angles 2ψ and $(m-3)\psi$, with that side of length 1, the unfolding has two maximal cylinders, of circumference $4 \sin 5\psi \cos 3\psi$ and height $|QR|(\sin 5\psi - \sin \psi)$, and of circumference $4 \cos 3\psi \cos \psi$ and height $|QR| \cos \psi$; the ratio of their moduli is $1 - \sin \psi / \sin 5\psi$, which is irrational.*

(b) *If $m = 6$, the triangle is $(2, 3, 7)/12$; in the same direction it has a maximal cylinder of circumference 2 and height $2|QR| \sin 3\psi$, and the first cylinder of (a); the ratio of their moduli is $(\sqrt{3} - 1)/2$.*

These triangles form the part of $\mathcal{C}$ with $p = 2$. For $m = 4$ the triangle is $(1, 2, 5)/8$, a lattice triangle. For $m = 6$ the statement is Vorobets's [30, Theorem 4.5], proved there with two horizontal cylinders. For odd m the integers 2, $m-3$, $m+1$ have greatest common divisor 2, and the triangle is $(1, (m-3)/2, (m+1)/2)/m$, which is treated in Sections 4 and 6.

Proof. Let $m \geq 6$ be even, $n = 2m$, $p = 2$, $q = m - 3$, $r = m + 1$; then $4 \mid n$, the hypotheses of Lemma 5.1 hold, and by Lemma 5.6 so does (5.1), for every $\mathcal{W}$, and $r \nmid n$. The vertex P gives two regular points, R one point of cone angle $2\pi(m+1)$ (as $\gcd(m+1, 2m) = 1$), and Q $\gcd(m-3, 6) \in \{1, 3\}$ points of cone angle $2\pi(m-3)/\gcd(m-3, 6)$, which are regular exactly for $m \in \{4, 6\}$. By (2.1), $2\,\text{genus}(X) - 2 = 2m - 3 - \gcd(m-3, 6) \geq 6$. Since $m\psi = \pi/2$, $\sin(m+j)\psi = \cos j\psi$.

The chain $\mathcal{W}_c = \{2m-5\}$: its neighbours in the progression are $2m - 5 + 2q = 2n - 11$, with $\sin(2n-11)\psi = -\sin 11\psi < 0$ as $n \geq 12$, and $2m - 5 - 2q = 1$. So $a(\mathcal{W}_c) = \sin \psi$ and $b(\mathcal{W}_c) = \sin(2m-5)\psi = \sin 5\psi > \sin \psi$; $\iota(\mathcal{W}_c) = \{5\} \neq \mathcal{W}_c$, and $\mathcal{L}(\mathcal{W}_c) = 2[\cos(m-2)\psi - \cos(3m-8)\psi] = 2[\sin 2\psi + \sin 8\psi] = 4 \sin 5\psi \cos 3\psi$. By Proposition 5.5, $Z_c = Z(\mathcal{W}_c)$ is a maximal cylinder of height $|QR|(\sin 5\psi - \sin \psi)$ and modulus $\mu_c = |QR|(\sin 5\psi - \sin \psi)/(4 \sin 5\psi \cos 3\psi)$.

Part (a): let $m \geq 8$ and $\mathcal{W}_d = \{m+1\}$. Its neighbours are $m + 1 + 2q = 3m - 5$, with $\sin(3m-5)\psi = -\cos 5\psi < 0$, and $m+1-2q = 7-m \in (-n, 0)$. So $a(\mathcal{W}_d) = 0$, $b(\mathcal{W}_d) = \cos \psi$, $\iota(\mathcal{W}_d) = \{m-1\} \neq \mathcal{W}_d$, and $\mathcal{L}(\mathcal{W}_d) = 2[\cos 4\psi - \cos(2m-2)\psi] = 4 \cos 3\psi \cos \psi$. The points over Q are singular, so $Z_d = Z(\mathcal{W}_d)$ is maximal, of modulus $\mu_d = |QR|/(4 \cos 3\psi)$. Then $\mu_c/\mu_d = 1 - \sin \psi / \sin 5\psi$. With $x = 2 \cos 2\psi = 2 \cos(\pi/m)$ we have $4 \cos^2 \psi = 2 + x$ and $\sin 5\psi / \sin \psi = 16 \cos^4 \psi - 12 \cos^2 \psi + 1 = x^2 + x - 1$. If the ratio were rational, x would have degree at most 2 over $\mathbb{Q}$; but $[\mathbb{Q}(2 \cos(2\pi/2m)) : \mathbb{Q}] = \varphi(2m)/2$, and $\varphi(2m) \leq 4$ with $m \geq 3$

only for $2m \in \{6,8,10,12\}$. So for even $m \geq 8$ the ratio is irrational, and Lemma 2.3(i) applies.

Part (b): $m=6$, $n=12$, $q=3$, and the points over Q are regular. Take $\mathcal{W}_a=\{3,9\}$: its neighbours 15 and -3 have negative sines, so $a(\mathcal{W}_a)=0$, $b(\mathcal{W}_a)=\sin 3\psi$, $\iota(\mathcal{W}_a)=\mathcal{W}_a$ and $\mathcal{L}(\mathcal{W}_a)=2$. The index $2n-i_{\text{top}}-2q=24-9-6$ is $9=i_{\text{top}}$, so Proposition 5.7 gives a maximal cylinder $\hat{Z}_a$ of circumference 2, height $2|QR|\sin 3\psi$ and modulus $|QR|\sin 3\psi$. Since $\sin 6\psi=1$ and $\sin 5\psi=\cos\psi$,

$$\frac{\mu_c}{\mu(\hat{Z}_a)}=\frac{1-\sin\psi/\sin 5\psi}{4\sin 3\psi\cos 3\psi}=\frac{1-\tan(\pi/12)}{2}=\frac{\sqrt{3}-1}{2},$$

which is irrational, and Lemma 2.3(i) applies. □

At $(1,2,5)/8$ ($m=4$) the chains $\{3\}$ and $\{5\}$ satisfy $a=b$ and give no cylinder.

Theorem 5.9. *For every odd $q\geq 3$ the triangle $(2,q,3q+2)/(4q+4)$ is not a lattice triangle. With $\psi=\pi/(4q+4)$ and $|QR|=\sin 2\psi/\sin(3q+2)\psi$, in the direction of the side joining the vertices of angles 2ψ and $q\psi$, with that side of length 1, the unfolding has two maximal cylinders, of circumference $4\cos\psi\sin(\pi/4+3\psi)$ and height $|QR|\sin\psi$, and of circumference $2\cos 2\psi$ and height $|QR|\sin(\pi/4+\psi)$; the ratio of their moduli is $\tan\psi/(1+2\sin 2\psi)$, which is irrational.*

Proof. Here $n=4q+4$, $m=n/2=2q+2$, so $4\mid n$ and $2q=m-2$; $q\psi=\pi/4-\psi$ and $(q+2)\psi=\pi/4+\psi$. The hypotheses of Lemma 5.1 hold, and by Lemma 5.6 so do (5.1) and $r\nmid n$. The points over Q are singular, since $\gcd(q,4q+4)=1$ and $q\geq 3$, and the point over R is singular, since $\gcd(3q+2,4q+4)=\gcd(q+2,4)=1$. By (2.1), $2\,\text{genus}(X)-2=4q$.

The chain $\mathcal{W}_T=\{1,2q+1,4q+1\}$: the sines are $\sin\psi$, $\cos\psi$ and $\sin 3\psi$, so $b(\mathcal{W}_T)=\sin\psi$; the neighbours are $6q+1=3m-5$, with sine $-\cos 5\psi<0$ as $n>10$, and $1-2q\in(-n,0)$; so $a(\mathcal{W}_T)=0$. Also $\iota(\mathcal{W}_T)=\{3,2q+3,4q+3\}\neq\mathcal{W}_T$, and $\mathcal{L}(\mathcal{W}_T)=2[\cos(q-1)\psi+\cos(q-3)\psi]=4\cos\psi\cos(q-2)\psi=4\cos\psi\sin(\pi/4+3\psi)$. So $Z_T=Z(\mathcal{W}_T)$ is maximal, with height $|QR|\sin\psi$.

The chain $\mathcal{W}_S=\{q+2,3q+2\}$ satisfies $\iota(\mathcal{W}_S)=\mathcal{W}_S$; its neighbours $5q+2\in(n,2n)$ and $2-q<0$ have negative sines, so $a(\mathcal{W}_S)=0$, $b(\mathcal{W}_S)=\sin(q+2)\psi=\sin(\pi/4+\psi)$ and $\mathcal{L}(\mathcal{W}_S)=2\cos 2\psi$. So $Z_S=Z(\mathcal{W}_S)$ is maximal, with height $|QR|\sin(\pi/4+\psi)$.

Since $2\sin(\pi/4+3\psi)\sin(\pi/4+\psi)=\cos 2\psi-\cos(\pi/2+4\psi)=\cos 2\psi(1+2\sin 2\psi)$, the ratio of the moduli is

$$\varrho=\frac{\tan\psi}{1+2\sin 2\psi}=\frac{\sin 2\psi}{(1+\cos 2\psi)(1+2\sin 2\psi)}.$$

Let $\zeta=e^{2\pi i/n}$, so that $\cos 2\psi,\sin 2\psi\in\mathbb{Q}(\zeta)$ (as $i=\zeta^{n/4}$). Since q is odd, $8\mid n$, and $n/2+1$ is odd and prime to n; so $\sigma_{n/2+1}:\zeta\mapsto\zeta^{n/2+1}=-\zeta$ is an automorphism of $\mathbb{Q}(\zeta)$. It fixes i (as $n/4$ is even) and changes the signs of $\cos 2\psi$ and $\sin 2\psi$, so

$$\sigma_{n/2+1}(\varrho)=-\frac{\sin 2\psi}{(1-\cos 2\psi)(1-2\sin 2\psi)}<0,$$

because $\sin 2\psi<1/2$ for $n>12$, while $\varrho>0$. So ϱ is irrational, and Lemma 2.3(i) applies. □

For $q=1$ the triangle is $(1,2,5)/8$ again. For odd $q\geq 3$, $(2,q,3q+2)/(4q+4)$ is family (5) of Theorem 2.5, and its obtuse angle exceeds $2\pi/3$.

5.2. **Two families with an angle $4\pi/n$: one cylinder and its share.** We keep the notation of Section 5.1, now with $p = 4$; Propositions 5.5 and 5.7 are stated for every p.

Theorem 5.10. *Let $m \equiv 2 \pmod 4$ with $m \geq 10$, $n = 2m$ and $\vartheta = 2\pi/n = \pi/m$. The triangle $(4, m-5, m+1)/2m$ is not a lattice triangle. In the direction of the side joining its vertices of angles $4\pi/n$ and $(m-5)\pi/n$, its unfolding has a maximal cylinder Z with*

$$n\,\rho(Z) = 4\cos\vartheta\,(\cos\vartheta - \cos 4\vartheta) \quad (m \geq 14), \qquad n\,\rho(Z) = 8\cos\vartheta\,(\cos\vartheta - \cos 4\vartheta) \quad (m = 10),$$

and $\sigma_\ell(\rho(Z)) < 0$ for $\ell = m/2 - 2$ if $m \geq 22$, and for $\ell = 7, 11, 13$ if $m = 10, 14, 18$.

These triangles form the part of $\mathcal{C}$ with $p = 4$. For $m = 6$ the triangle is $(1, 4, 7)/12$, Hooper's lattice triangle; the proof below needs $n \geq 20$.

Proof. Here $p = 4$, $q = m - 5$, $r = m + 1$, $\psi = \pi/n$, so that $m\psi = \pi/2$ and $\vartheta = 2\psi$. The hypotheses of Lemma 5.1 hold, n is even and q is odd; since $n/2 = m \equiv 2 \pmod 4$, the polygon G_j is symmetric if and only if j is odd (Lemma 5.3). The vertex R gives one point of cone angle $2\pi(m+1)$, since $\gcd(m+1, 2m) = 1$; it is singular, and $r \nmid n$, as Propositions 5.5 and 5.7 require. The points over Q are singular if and only if $q \nmid n$, that is, $(m-5) \nmid 10$, that is, $m \neq 6, 10$.

Let $\mathcal{W} = \{3, n-7\}$ (Figure 2, left, for $m = 14$), a progression with difference $n - 10 = 2q$. Since $10\psi < \pi$ we have $\sin(n-7)\psi = \sin 7\psi > \sin 3\psi > 0$. The neighbours are $(n-7)+2q = 2n-17$, with $\sin(2n-17)\psi = -\sin 17\psi < 0$ because $17\psi < \pi$, and $3 - 2q = 13 - n < 0$. So $\mathcal{W}$ is a chain of $\mathcal{I}_\chi$ for $0 < \chi < \sin 3\psi$, with $a(\mathcal{W}) = 0$ and $b(\mathcal{W}) = \sin 3\psi$, and $\iota(\mathcal{W}) = \{7, n-3\} \neq \mathcal{W}$. The exit ears are given by $2k_1 \equiv 3 - n - q = 8 - 3m$ and $2k_2 \equiv 7 - n - q = 12 - 3m \pmod{4m}$, so $k_1 \equiv m/2 + 4$ and $k_2 \equiv m/2 + 6 \pmod n$; both are odd, as $m/2$ is odd, so $2k_1 \equiv 2k_2 \equiv 2 \equiv n/2 \pmod 4$: this is (5.1), and Proposition 5.5 applies, with

$$\mathcal{L}(\mathcal{W}) = 2\big[\cos(3-q)\psi - \cos(n-7+q)\psi\big] = 2[\sin 8\psi + \sin 12\psi] = 4\sin 10\psi\cos 2\psi,$$

since $(3-q)\psi = 8\psi - \pi/2$ and $(n-7+q)\psi = 3\pi/2 - 12\psi$.

The share. The area of X is $2n$ times the area $\frac{1}{2}|PQ|\,|QR|\sin q\psi$ of Δ, that is, $n|QR|\sin q\psi = n|QR|\cos 5\psi$. If $m \geq 14$, the points over Q are singular and $Z(\mathcal{W})$ is a maximal cylinder of height $|QR|\sin 3\psi$, so

$$n\,\rho = \frac{4\sin 3\psi\,\sin 10\psi\,\cos 2\psi}{\cos 5\psi} = 8\sin 3\psi\sin 5\psi\cos 2\psi$$
$$= 4\cos 2\psi\,(\cos 2\psi - \cos 8\psi) = 4\cos\vartheta\,(\cos\vartheta - \cos 4\vartheta).$$

If $m = 10$, the points over Q are regular, and we apply Proposition 5.7 ($r \nmid n$ and $q \mid n$): $\sin(i_{\text{top}} + 2q)\psi = -\sin 17\psi < 0$, and $2n - i_{\text{top}} - 2q = 17 = n - 3$ is the top of $\iota(\mathcal{W})$, which is the chain of $\mathcal{I}_\chi$ with top 17 for $0 < \chi < \sin 3\psi$ (its neighbours 27 and -3 have negative sines). So the maximal cylinder containing $Z(\mathcal{W})$ has the same circumference and height $2|QR|\sin 3\psi$, and $n\rho = 8\cos\vartheta(\cos\vartheta - \cos 4\vartheta)$.

The conjugates. $n\rho$ is a polynomial in $\cos\vartheta$ with rational coefficients, so $\rho \in \mathbb{Q}(\zeta)^+$, and for a unit ℓ modulo n, $\sigma_\ell(n\rho)$ is the same expression with ϑ replaced by $\ell\vartheta$; it has the sign of $\sigma_\ell(\rho)$. (i) Let $m \geq 22$ and $\ell = m/2 - 2$. It is odd, and $\gcd(\ell, m/2) = \gcd(2, m/2) = 1$, so ℓ is a unit modulo $2m$. As $\ell\vartheta = \pi/2 - 2\vartheta$, $\cos\ell\vartheta = \sin 2\vartheta$ and $\cos 4\ell\vartheta = \cos(2\pi - 8\vartheta) = \cos 8\vartheta$, so $\sigma_\ell(n\rho) = 4\sin 2\vartheta\,(\sin 2\vartheta - \cos 8\vartheta)$. Since $10\vartheta < \pi/2$, we have $0 < 2\vartheta < \pi/2 - 8\vartheta < \pi/2$, so $\sin 2\vartheta < \cos 8\vartheta$ and $\sigma_\ell(n\rho) < 0$. (ii) $m = 14$, $\ell = 11$: $\cos(11\pi/14) = -\cos(3\pi/14)$ and $\cos(44\pi/14) = \cos(8\pi/7) = -\cos(\pi/7)$, so $\sigma_{11}(n\rho) = -4\cos(3\pi/14)\,(\cos(\pi/7) - \cos(3\pi/14)) < 0$. (iii) $m = 18$, $\ell = 13$: $\cos(13\pi/18) = -\cos(5\pi/18)$ and $\cos(52\pi/18) =$

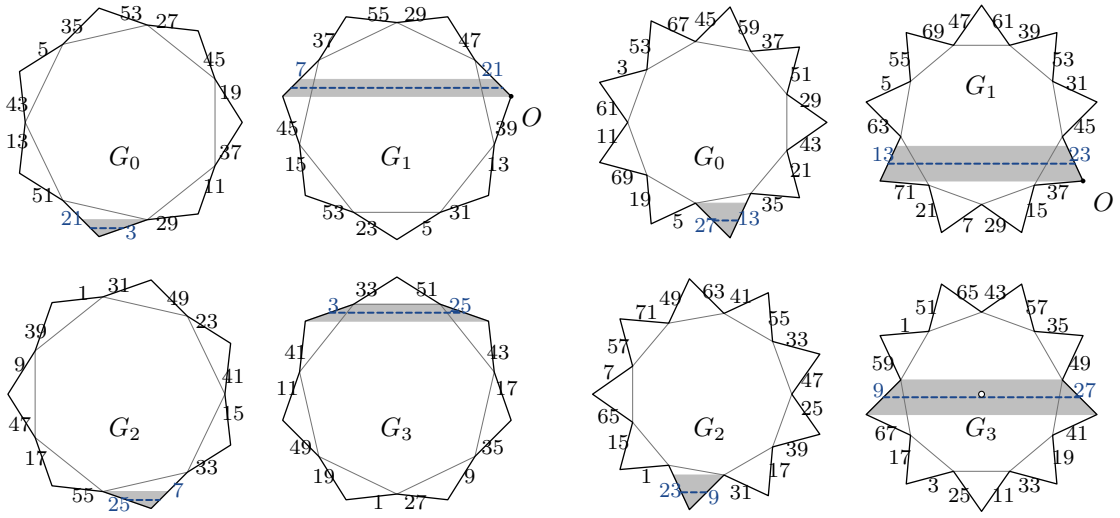


FIGURE 2. The model of Lemma 5.1 with $p = 4$: the regular $(n/4)$-gons $G_0, \dots, G_3$ with their ears; a leg is labelled i when its direction from its apex is $i\psi$, and legs with the same label are glued. Left, $(4, 9, 15)/28$ (Theorem 5.10, $m = 14$); right, $(4, 7, 25)/36$ (Theorem 5.11, $q = 7$): shaded, the maximal cylinder $Z(\mathcal{W})$ of the chain $\mathcal{W} = \{3, 21\}$, resp. $\{9, 23\}$, whose share of the area has a negative conjugate; its lower boundary passes through the apex O, and on the right one of its leaves crosses the centre of G_3. For $(1, 4, 7)/12$ ($n = 12$, $q = 1$) the ears close up into a regular 12-gon and the model is Hooper's Figure 1 [11], turned through $13\pi/12$.

$\cos(8\pi/9) = -\cos(\pi/9)$, so $\sigma_{13}(n\rho) = -4\cos(5\pi/18)\,(\cos(\pi/9) - \cos(5\pi/18)) < 0$. (iv) $m = 10$, $\ell = 7$: $\cos(7\pi/10) = -\cos(3\pi/10)$ and $\cos(28\pi/10) = -\cos(\pi/5)$, so $\sigma_7(n\rho) = -8\cos(3\pi/10)\,(\cos(\pi/5) - \cos(3\pi/10)) < 0$. In each case Lemma 2.3(ii) applies. □

In general $4\cos a\,(\cos a - \cos 4a) = 8\cos a\,\sin(5a/2)\sin(3a/2)$ is negative for $a \in (0, \pi)$ exactly when $a \in (2\pi/5, \pi/2) \cup (2\pi/3, 4\pi/5)$; so $\sigma_\ell(\rho(Z)) < 0$ exactly when $\pm\ell/n$ is congruent modulo 1 to a number in $(1/5, 1/4) \cup (1/3, 2/5)$.

Theorem 5.11. *For every odd $q \geq 3$ the triangle $(4, q, 3q + 4)/(4q + 8)$ is not a lattice triangle. Let $n = 4q + 8$, $\psi = \pi/n$ and $\vartheta = 2\psi$.*

(a) *If $q \geq 7$, in the direction of the side joining the vertices of angles 4ψ and $q\psi$ the unfolding has a maximal cylinder Z with $n\,\rho(Z) = 2(1 + \cos 2\vartheta + \sin 2\vartheta)$, and $\sigma_\ell(\rho(Z)) < 0$ for $\ell = q$ if $q \equiv 3 \pmod 4$ and for $\ell = q - 2$ if $q \equiv 1 \pmod 4$.*

(b) *If $q = 5$, in the same direction the unfolding has a maximal cylinder Z with $n\,\rho(Z) = 8\cos^2 4\psi\,\cos 2\psi\,\sin 3\psi/\sin 5\psi$, and $\sigma_{11}(\rho(Z)) < 0$.*

(c) *If $q = 3$, the triangle is $(3, 4, 13)/20$, which is not a lattice triangle by Theorem 6.1.*

For $q \geq 5$ these triangles form family (6) of Theorem 2.5: their obtuse angle exceeds $2\pi/3$ exactly when $q > 4$. For $q = 3$ the obtuse angle is $13\pi/20 < 2\pi/3$, and $(3, 4, 13)/20$ is a member of $\mathcal{E}$. For $q = 1$ the triangle is $(1, 4, 7)/12$ once more.

Proof. Here $p = 4$, $r = 3q + 4$, $(q + 2)\psi = \pi/4$ and $q\psi = \pi/4 - 2\psi$. The hypotheses of Lemma 5.1 hold, n is even and q is odd; $n/2 = 2q + 4 \equiv 2 \pmod 4$, so G_j is symmetric if

and only if j is odd. Since $\gcd(3q+4, 4q+8) = \gcd(q+4, 8) = 1$, the vertex R gives one singular point, and $r \nmid n$, as Proposition 5.5 requires; since $\gcd(q, 4q+8) = \gcd(q, 8) = 1$, the vertex Q gives one point of cone angle $2\pi q$, singular for $q \geq 3$.

Let $q \geq 5$ and $\mathcal{W} = \{q+2, 3q+2\}$ (Figure 2, right, for $q = 7$). Its sines are $\sin(\pi/4)$ and $\sin(3q+2)\psi = \sin(q+6)\psi > \sin(\pi/4)$, because $\pi/4 < (q+6)\psi < 3\pi/4$. The neighbours are $(3q+2)+2q = n+(q-6)$, with sine $-\sin(q-6)\psi$, and $2-q < 0$. So for $q \geq 7$, $a(\mathcal{W}) = 0$; for $q = 5$, $a(\mathcal{W}) = \sin\psi$; in both cases $b(\mathcal{W}) = \sin(\pi/4) > a(\mathcal{W})$, and $\iota(\mathcal{W}) = \{q+6, 3q+6\} \neq \mathcal{W}$. The exit ears are given by $2k_1 \equiv (q+2) - n - q = 2 - n$ and $2k_2 \equiv -(3q+2) - q = 6 - n \pmod{2n}$, so $k_1 \equiv -2q - 3$ and $k_2 \equiv -2q - 1 \pmod{n}$, both odd, so $2k_1 \equiv 2k_2 \equiv 2 \equiv n/2 \pmod 4$, which is (5.1). So Proposition 5.5 applies:

$$\mathcal{L}(\mathcal{W}) = 2[\cos 2\psi - \cos(4q+2)\psi] = 2[\cos 2\psi + \cos 6\psi] = 4\cos 2\psi \cos 4\psi,$$

as $(4q+2)\psi = \pi - 6\psi$, and $Z(\mathcal{W})$ is a maximal cylinder: for $q \geq 7$ its lower boundary contains the apex O, a point over Q, and for $q = 5$ a point over R.

Part (a). The height is $|QR|\sin(\pi/4)$ and the area of X is $n|QR|\sin q\psi$, so

$$n\rho = \frac{4\sin(\pi/4)\cos 2\psi \cos 4\psi}{\sin(\pi/4 - 2\psi)} = \frac{4\cos 2\psi \cos 4\psi}{\cos 2\psi - \sin 2\psi}$$
$$= 4\cos 2\psi(\cos 2\psi + \sin 2\psi) = 2(1 + \cos 2\vartheta + \sin 2\vartheta).$$

Since $(q+2)\vartheta = \pi/2$, $\sin 2\vartheta = \cos q\vartheta$, so $n\rho = 2 + 2\cos 2\vartheta + 2\cos q\vartheta \in \mathbb{Q}(\zeta)^+$ and $\sigma_\ell(n\rho) = 2 + 2\cos 2\ell\vartheta + 2\cos q\ell\vartheta$. For odd ℓ, $\cos q\ell\vartheta = \cos(\ell\pi/2 - 2\ell\vartheta) = \varepsilon_\ell \sin 2\ell\vartheta$ with $\varepsilon_\ell = 1$ if $\ell \equiv 1$ and $\varepsilon_\ell = -1$ if $\ell \equiv 3 \pmod 4$. If $q \equiv 3 \pmod 4$, $\ell = q$ is a unit ($\gcd(q, 8) = 1$), $2q\vartheta = \pi - 4\vartheta$, and $\sigma_q(n\rho) = 2(1 - \cos 4\vartheta - \sin 4\vartheta) = 2 - 2\sqrt{2}\sin(4\vartheta + \pi/4) < 0$, because $0 < 4\vartheta < \pi/2$. If $q \equiv 1 \pmod 4$, then $q \geq 9$ and $\ell = q - 2 \equiv 3 \pmod 4$ is a unit (it is odd and $\gcd(q-2, q+2)$ divides 4); $2(q-2)\vartheta = \pi - 8\vartheta$, and $\sigma_{q-2}(n\rho) = 2(1 - \cos 8\vartheta - \sin 8\vartheta) < 0$, because $0 < 8\vartheta < \pi/2$ for $n \geq 44$.

Part (b). Now $\psi = \pi/28$ and $\pi/4 = 7\psi$; the height is $|QR|(\sin 7\psi - \sin\psi) = 2|QR|\cos 4\psi \sin 3\psi$ and $\sin q\psi = \sin 5\psi$, so $n\rho = 8\cos^2 4\psi \cos 2\psi \sin 3\psi / \sin 5\psi$. With $\zeta = e^{i\vartheta}$ we have $\sin 3\psi / \sin 5\psi = \zeta(\zeta^3 - 1)/(\zeta^5 - 1) \in \mathbb{Q}(\zeta)$, so $\rho \in \mathbb{Q}(\zeta)^+$, and σ_ℓ maps $\sin 3\psi / \sin 5\psi$ to $\sin 3\ell\psi / \sin 5\ell\psi$. So $\sigma_{11}(n\rho) = 8\cos^2(44\psi)\cos(22\psi)\sin(33\psi)/\sin(55\psi)$, where $\cos 22\psi = \cos(11\pi/14) < 0$, $\sin 33\psi = \sin(33\pi/28) < 0$ and $\sin 55\psi = -\sin(\pi/28) < 0$; so $\sigma_{11}(\rho) < 0$.

Part (c) is Theorem 6.1.

In parts (a) and (b), Lemma 2.3(ii) applies. □

At $q = 3$ the cylinder $Z(\mathcal{W})$ of the proof still exists, with $a(\mathcal{W}) = \sin 3\psi$, but its share $n\rho = 4\cos^2(\pi/10) - 2\cos(\pi/10)$ is totally positive, and we find that all eight maximal cylinders in the direction of PQ have the same modulus, so that direction is consistent with Lemma 2.3 and Theorem 6.1 uses another one.

5.3. **Two families with an angle π/n: the star.** In this subsection $\Delta = (1, q, r)/n$ has vertices A, B, C with angles ψ, $q\psi$, $r\psi$, where $\psi = \pi/n$ as before, is obtuse at C, that is, $r > n/2$, and has $|BC| = 1$. Place A at the origin O and, for a real number ϕ,

$$B_k = |AB|\, e^{i(2k\psi + \phi)}, \quad C_k = |AC|\, e^{i((2k+1)\psi + \phi)} \quad (k \in \mathbb{Z}/n),$$
$$|AB| = \frac{\sin r\psi}{\sin\psi}, \quad |AC| = \frac{\sin q\psi}{\sin\psi}.$$

The *star* $\mathcal{P}$ is the union of the $2n$ triangles $\Delta_{2k} = OB_kC_k$ and $\Delta_{2k+1} = OC_kB_{k+1}$, each a copy of Δ with the angle ψ at O. Its sides are $b_k = B_kC_k$ and $c_k = C_kB_{k+1}$, all of length 1; its interior angles are $2q\psi$ at the B_k and $2r\psi > \pi$ at the C_k, which we call the *inner vertices*. For $\phi = 0$ the star is $|AB|$ times the model of Lemma 5.1 for $p = 1$ and $(P, Q, R) = (A, B, C)$ (its hypotheses hold, as $q < n/2$): $B_k = |AB|\,\Theta_k(Q)$ and $C_k = |AB|\,\Theta_k(R)$, the regular n-gon G_0 is the union of the triangles $OC_{k-1}C_k$, the ear E_k is $C_{k-1}B_kC_k$, and the gluing of the leg $\Theta_k(QR)$ to $\Theta_{k-q}(Q\bar{R})$ is that of b_k to c_{k-q-1} in Lemma 5.12.

Lemma 5.12. *The unfolding X is $\mathcal{P}$ with b_k glued by translation to c_{k-q-1}, the gluing matching B_k with B_{k-q} and C_k with C_{k-q-1}; the point O is a regular point. The B_k form* $\gcd(q, n)$ *points of cone angle* $2\pi q/\gcd(q, n)$ *and the C_k form* $\gcd(r, n)$ *points of cone angle* $2\pi r/\gcd(r, n)$.

Proof. It suffices to take $\phi = 0$. The linear part of Δ_{2k} is the rotation Θ_k and that of Δ_{2k+1} is $\Theta_{k+1}\kappa$, κ complex conjugation; these $2n$ elements are distinct, so they are all of the dihedral group, and every copy of Δ in X is one Δ_j. The gluings across the sides through A put Δ_j next to $\Delta_{j\pm1}$ around A, which gives $\mathcal{P}$, and A is a regular point. Crossing the side BC of the copy with linear part δ leads to the copy with linear part $\delta\kappa_{BC}$, where $\kappa_{BC} = \Theta_{-q}\kappa$ is the linear part of the reflection in BC (which has direction $\pi - q\psi$). Since $\Theta_k\kappa_{BC} = \Theta_{(k-q-1)+1}\kappa$, the side b_k of Δ_{2k} is glued to the side $c_{k-q-1} = C_{k-q-1}B_{k-q}$ of $\Delta_{2(k-q-1)+1}$, matching the copies of B and the copies of C. The cone angles are as in Section 2.1. □

Two properties of $\mathcal{P}$ are used. (a) *$\mathcal{P}$ is star-shaped about O*: Δ_j fills the closed sector between the arguments of its two outer vertices, an angle $\psi < \pi$; these sectors tile the full turn, and inside its sector a ray from O meets the outer side of Δ_j once. So each ray from O meets $\partial\mathcal{P}$ in exactly one point $\beta(\alpha)$, α the argument of the ray, and the open segment from O to $\beta(\alpha)$ lies in the interior of $\mathcal{P}$. (b) *Mirrors*: the reflection in the line through O and any vertex maps $\mathcal{P}$ to itself (it sends the vertex at argument $\beta + j\psi$ to the one at $\beta - j\psi$, which is of the same type, B or C).

Lemma 5.13 (chords)**.** *Let $h > 0$ and let $p_+, p_- \in \partial\mathcal{P}$ be points of height h, neither a vertex, lying on two different sides of $\mathcal{P}$. Suppose that every vertex of $\mathcal{P}$ met when going counterclockwise along $\partial\mathcal{P}$ from p_+ to p_- has height greater than h. Then the open horizontal segment from p_- to p_+ lies in the interior of $\mathcal{P}$, and p_- is to the left of p_+.*

Proof. Height is affine along each side, equal to h at p_+ and p_- and greater than h at the vertices between, so the open boundary arc from p_+ to p_- lies in $\{y > h\}$; by (a) it is the set of $\beta(\alpha)$ with α in an interval $(\alpha_+, \alpha_-) \subset (0, \pi)$, where α_+, α_- are the arguments of p_+, p_-. On the line $y = h$ the argument decreases strictly from left to right, so the open segment between p_- and p_+ is $\{w : \operatorname{Im} w = h,\ \alpha_+ < \arg w < \alpha_-\}$, and for such w, $|w| = h/\sin(\arg w) < |\beta(\arg w)|$, because $\beta(\arg w)$ has height greater than h. So w lies on the open segment from O to $\beta(\arg w)$, which is in the interior of $\mathcal{P}$ by (a). Finally $\alpha_- > \alpha_+$ gives $x(p_-) < x(p_+)$. □

Strips. Let e_+ be a side of $\mathcal{P}$ that rises when $\partial\mathcal{P}$ is traversed counterclockwise, e_- a side that falls, and $0 \le a < b$ such that both sides cross every height in $[a, b]$. If every vertex on the counterclockwise arc from the upper end of e_+ to the upper end of e_- (both ends included) has height at least b, then by Lemma 5.13 (the two sides are different, one rising and one falling), for every $h \in (a, b)$, the open horizontal segment at height h from e_- to e_+

lies in the interior of $\mathcal{P}$. The union of these segments is the *strip* of heights (a, b) with left side e_- and right side e_+. By (b), the same holds after reflection in a horizontal mirror of $\mathcal{P}$.

Lemma 5.14 (annuli)**.** *Let $S_1, \dots, S_k$ be disjoint strips, S_i of heights $(a_i, a_i + h)$ with left side e_i^- and right side e_i^+, and suppose that for each i the gluing of Lemma 5.12 maps e_i^+ to e_{i+1}^- (indices modulo k), sending the point of e_i^+ at height $a_i + t$ to the point of e_{i+1}^- at height $a_{i+1} + t$ for $0 < t < h$. Then the S_i, with the glued open pieces of the e_i^+, form an open annulus $Y \subset X$ foliated by closed horizontal leaves, of height h and circumference $\sum_i \ell_i(a_i + t)$, where $\ell_i(y)$ is the length of the chord of S_i at height y; this sum does not depend on t. If each boundary component of Y contains a point of Σ, then Y is a maximal cylinder.*

Proof. The horizontal line through S_1 at height $a_1 + t$ crosses $S_1, \dots, S_k$ in turn and re-enters S_1 at height $a_1 + t$: it is closed. Distinct values of t give disjoint leaves, and each strip is crossed once per turn, so Y is an embedded annulus. The leaves are homotopic in Y and the holonomy of a closed curve is a homotopy invariant, so the circumference does not depend on t. □

Theorem 5.15. *For every even $m \geq 8$ the triangle $(1, m-2, m+1)/2m$ is not a lattice triangle. With $\psi = \pi/2m$, in the direction of the side joining the vertices of angles ψ and $(m+1)\psi$ the unfolding has two maximal cylinders Z and Z' of moduli*

$$\mu(Z) = \frac{\sin\psi}{4\cos\psi\cos 2\psi}, \qquad \mu(Z') = \frac{\sin\psi}{16\cos^3\psi\cos 2\psi},$$

whose ratio $4\cos^2\psi = 2 + 2\cos(\pi/m)$ is irrational.

These triangles form the part of $\mathcal{C}$ with $p = 1$; for $m = 4$ and $m = 6$ the triangle is the lattice triangle $(1, 2, 5)/8$, respectively $(1, 4, 7)/12$. For odd m the triangle lies in $\mathcal{A}$ (with $s = 2$): for $m = 3$ it is the lattice triangle $(1, 1, 4)/6$, and for odd $m \geq 5$ it is treated in Section 4 (Theorem 4.18).

Proof. Here $n = 2m$, $q = m-2$, $r = m+1$, $|AB| = \cot\psi$ and $|AC| = \cos 2\psi/\sin\psi$. Position the star so that an inner vertex, the *notch*, lies on the positive y-axis (Figure 3), and let V_j ($j \in \mathbb{Z}/4m$) be the vertex at argument $\pi/2 - j\psi$: $V_j = |V_j|(\sin j\psi, \cos j\psi)$, with $|V_j| = |AC|$ for j even (a copy of C) and $|V_j| = |AB|$ for j odd (a copy of B). Then V_0 is the notch, and V_m lies on the positive x-axis. Write y_j for the height of V_j and Ξ_j for the side with ends V_j, V_{j-1}.

Mirrors. The reflection $x \mapsto -x$ sends V_j to V_{-j}, and $y \mapsto -y$ sends V_j to V_{2m-j}.

Gluing. For j odd, Ξ_j is glued to Ξ_{j+2m-3}, with $V_j \leftrightarrow V_{j+2m-4}$ and $V_{j-1} \leftrightarrow V_{j+2m-3}$; equivalently, a side Ξ_i with i even is glued to Ξ_{i+2m+3}. Indeed, in the labels of Lemma 5.12, $B_k = V_{j_0-2k}$ and $C_k = V_{j_0-2k-1}$ for a fixed odd j_0, so $b_k = \Xi_{j_0-2k}$ and $c_{k-q-1} = \Xi_{j_0-2k+2q+1}$, with $B_k \mapsto B_{k-q} = V_{j_0-2k+2q}$ and $C_k \mapsto C_{k-q-1} = V_{j_0-2k+2q+1}$, and $2q + 1 = 2m - 3$.

Heights. For $1 \leq j \leq m$, $w_j := y_{j-1} - y_j$ equals $\sin(j+1)\psi$ if j is even and $\sin(j-2)\psi$ if j is odd: for j even, $y_{j-1} - y_j = [\cos\psi\cos(j-1)\psi - \cos 2\psi\cos j\psi]/\sin\psi = [\cos j\psi - \cos(j+2)\psi]/(2\sin\psi)$, and for j odd, $[\cos 2\psi\cos(j-1)\psi - \cos\psi\cos j\psi]/\sin\psi = [\cos(j-3)\psi - \cos(j-1)\psi]/(2\sin\psi)$. Hence

$$0 = y_m < y_{m-1} < \cdots < y_3 < y_2 < y_0 < y_1,$$

with $y_1 - y_0 = \sin\psi$ and $y_0 - y_2 = 2\sin\psi\cos 2\psi$.

Strips. For $3 \leq j \leq m$ let S_j be the strip of heights (y_j, y_{j-1}) with right side Ξ_j (rising) and left side its mirror image Ξ_{1-j}; the vertices on the arc between their upper ends are

the V_i with $|i| \le j-1$, of height at least y_{j-1} by the order above. The *right tip* is the strip of heights (y_0, y_1) with right side Ξ_2 and left side Ξ_1, and the *left tip* its mirror image, with right side Ξ_0 and left side Ξ_{-1}. Reflecting in the x-axis, S'_j is the strip of heights $(-y_{j-1}, -y_j)$ with right side Ξ_{2m+1-j} and left side Ξ_{2m+j}. S_j and S'_j have height w_j. Strips of disjoint height ranges, or on the two sides of the y-axis, are disjoint.

The cylinder Z' (for $m \ge 5$) is made of the right tip, the top part of S'_5 of height $\sin\psi$, the left tip, and S'_3. The side Ξ_2 of the right tip is glued to Ξ_{2m+5}, the left side of S'_5, with $V_1 \leftrightarrow V_{2m+5}$, so the apex V_1 goes to the upper end of Ξ_{2m+5}, at the top height $-y_5$ of S'_5; the tip has height $\sin\psi < w_5 = \sin 3\psi$, so it is glued to the top part of S'_5. The right side Ξ_{2m-4} of S'_5 is glued to Ξ_{-1}, the left side of the left tip, with $V_{2m-5} \leftrightarrow V_{-1}$. The side Ξ_0 of the left tip is glued to Ξ_{2m+3}, the left side of S'_3, apex to top, and $w_3 = \sin\psi$. Finally Ξ_{2m-2}, the right side of S'_3, is glued to Ξ_1, the left side of the right tip; both have height $\sin\psi$. By Lemma 5.14, Z' is an annulus of height $\sin\psi$. On its upper boundary the tips have width zero and the chords of S'_5 and S'_3 join the mirror images of $V_{\pm5}$ and $V_{\pm3}$, so its circumference is $2x(V_5) + 2x(V_3) = 2\cot\psi(\sin 5\psi + \sin 3\psi) = 16\cos^3\psi\cos 2\psi$. Its upper boundary contains V_1 and its lower boundary the notch V_0.

The cylinder Z. $Z = S_m \cup S_{m-2}$: the side Ξ_m is glued to $\Xi_{3m+3} = \Xi_{1-(m-2)}$, the left side of S_{m-2}, with $V_m \leftrightarrow V_{2-m}$, bottom to bottom; and Ξ_{m-2} is glued to $\Xi_{3m+1} = \Xi_{1-m}$, the left side of S_m. Both strips have height $\cos\psi$ ($w_m = \sin(m+1)\psi$, $w_{m-2} = \sin(m-1)\psi$), and on the lower boundary the circumference is $2x(V_m) + 2x(V_{m-2}) = 2|AC|(1+\cos 2\psi) = 4\cos^2\psi\cos 2\psi/\sin\psi$. So $\mu(Z) = \sin\psi/(4\cos\psi\cos 2\psi)$; this needs $m-2 \ge 3$. The two boundary components of Z contain V_m and V_{m-1}.

Maximality. The C_k form one point of cone angle $2\pi(m+1)$, since $\gcd(m+1, 2m) = 1$, and the B_k form $\gcd(m-2, 4)$ points of cone angle $2\pi(m-2)/\gcd(m-2,4) \ge 4\pi$, since $m \ge 8$. So every vertex of the star is in Σ, and by Lemma 5.14 Z and Z' are maximal cylinders, both horizontal.

Irrationality. $\mu(Z)/\mu(Z') = 4\cos^2\psi = 2 + 2\cos(2\pi/2m)$, and $\cos(2\pi/2m)$ has degree $\varphi(2m)/2 \ge 2$ for $m \ge 4$. By Lemma 2.3(i), Δ is not a lattice triangle.

The direction. V_m is a copy of C, and the horizontal direction is that of the side AC. □

At $m = 6$, that is at $(1, 4, 7)/12$, the annuli Z and Z' exist with the moduli of the theorem, but the copies of B are regular points and Z and Z' are not maximal; the maximal cylinders in that direction are four, with moduli $1/(5+3\sqrt{3})$ and $1/(10+6\sqrt{3})$ in pairs [11, (5)].

Theorem 5.16. *For every $q \ge 3$ the triangle $(1, q, 3q+1)/(4q+2)$ is not a lattice triangle. With $\varepsilon = \sin(\pi/(4q+2))$, in the direction of the side joining the vertices of angles $\pi/(4q+2)$ and $q\pi/(4q+2)$ the unfolding has two maximal cylinders U and V of moduli*

$$\mu(U) = \frac{\varepsilon^2}{\cos\psi\,(1+2\varepsilon)}, \qquad \mu(V) = \frac{\varepsilon}{4\cos\psi\,(1+\varepsilon)}, \qquad \psi = \frac{\pi}{4q+2},$$

whose ratio $\mu(V)/\mu(U) = (1+2\varepsilon)/(4\varepsilon(1+\varepsilon))$ is irrational.

This is family (4) of Theorem 2.5; its obtuse angle exceeds $2\pi/3$ for $q \ge 2$. For q odd it is also a theorem of Ward [31, Theorem B]. For $q = 1$ and $q = 2$ the triangle is $(1, 1, 4)/6$, respectively $(1, 2, 7)/10$, both lattice triangles.

Proof. Here $n = 4q+2$, $\varepsilon = \sin\psi$, $\Psi := q\psi = \pi/4 - \psi/2$ and $r\psi = 3\pi/4 - \psi/2$; so $|AB| = \cos\Psi/\varepsilon$, $|AC| = \sin\Psi/\varepsilon$ and $\tan\Psi = \cos\psi/(1+\varepsilon)$. Take $\phi = 0$: $B_k = |AB|e^{2ik\psi}$ and $C_k = |AC|e^{i(2k+1)\psi}$. Since $(2q+1)\psi = \pi/2$, the inner vertex C_q lies on the positive y-axis, B_0 on the positive x-axis and B_{2q+1} on the negative x-axis. The side b_k rises by

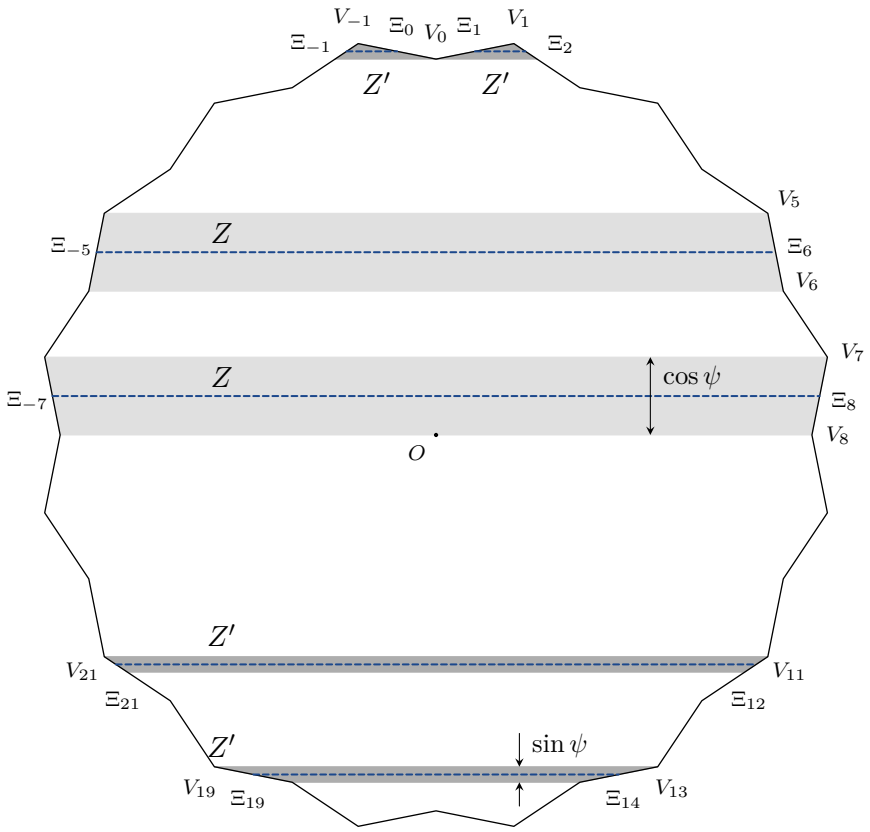


FIGURE 3. Ward's star for $m = 8$, that is for $(1, 6, 9)/16$, placed as in the proof of Theorem 5.15, the notch V_0 on the positive y-axis; for odd j the side Ξ_j is glued to Ξ_{j+2m-3}. Shaded: the cylinders Z' (dark), of height $\sin\psi$, and Z (light), of height $\cos\psi$; dashed: their core curves.

$\sin(q-2k)\psi$ and c_k by $\sin(2k+q+2)\psi$ when traversed counterclockwise. By Lemma 5.12, c_k is glued to b_{k+q+1}, with $C_k \leftrightarrow C_{k+q+1}$ and $B_{k+1} \leftrightarrow B_{k+q+1}$. The mirror $x \mapsto -x$ maps $B_k \mapsto B_{2q+1-k}$, $C_k \mapsto C_{2q-k}$, and $y \mapsto -y$ maps $B_k \mapsto B_{-k}$, $C_k \mapsto C_{-k-1}$. The heights are $y(B_k) = |AB| \sin 2k\psi$ and $y(C_k) = |AC| \sin(2k+1)\psi$.

Heights used. (i) $y(C_0) = \sin\Psi$ and $y(B_1) - y(C_0) = \sin(q+2)\psi = \sin\Psi + 2\varepsilon\sin\Psi$, because $\sin(q+2)\psi - \sin q\psi = 2\sin\psi\cos(q+1)\psi$ and $(q+1)\psi = \pi/2 - \Psi$. (ii) $y(C_1) - y(B_1) = [\sin\Psi\sin 3\psi - \cos\Psi\sin 2\psi]/\varepsilon = \cos\Psi\cos\psi\,(1 - 2\varepsilon - 4\varepsilon^2)/(1+\varepsilon)$, using $\sin\Psi(1+\varepsilon) = \cos\Psi\cos\psi$. The root of $1 - 2\varepsilon - 4\varepsilon^2$ in $(0,1)$ is $\sin(\pi/10)$, so $y(C_1) > y(B_1)$ exactly when $\psi < \pi/10$, that is, $q \geq 3$. (iii) For $1 \leq k \leq 2q$, $y(B_k) \geq y(B_1)$; for $1 \leq k \leq 2q-1$, $y(C_k) \geq y(C_1) > y(B_1)$ by (ii); for $0 \leq k \leq 2q$, $y(C_k) \geq y(C_0) = \sin\Psi$. (iv) $y(B_q) - y(C_q) = \sin\Psi$ and $y(C_q) - y(C_{q-1}) = |AC|(1 - \cos 2\psi) = 2\varepsilon\sin\Psi > 0$.

Strips (Figure 4). M: heights $(\sin\Psi, y(B_1))$, right side c_0, left side b_{2q}, height $\sin\Psi + 2\varepsilon\sin\Psi$; the arc from B_1 to B_{2q} passes through $B_1, C_1, \ldots, C_{2q-1}, B_{2q}$, all of height at least $y(B_1)$ by (iii). M_0: heights $(0, \sin\Psi)$, right side b_0, left side c_{2q}; the vertices on the arc from C_0 to C_{2q} have height at least $\sin\Psi$ by (iii). Its mirror image M' in the x-axis has heights $(-\sin\Psi, 0)$, left side b_{2q+1} and right side c_{-1}. The *slab*: heights $(y(C_{q-1}), y(C_q))$, right side c_{q-1}, left side b_{q+1}; the arc from B_q to B_{q+1} is B_q, C_q, B_{q+1}, of heights at least $y(C_q)$ by (iv). The *tips*: heights $(y(C_q), y(B_q))$; the right tip has right side c_{q-1} and left side b_q, the

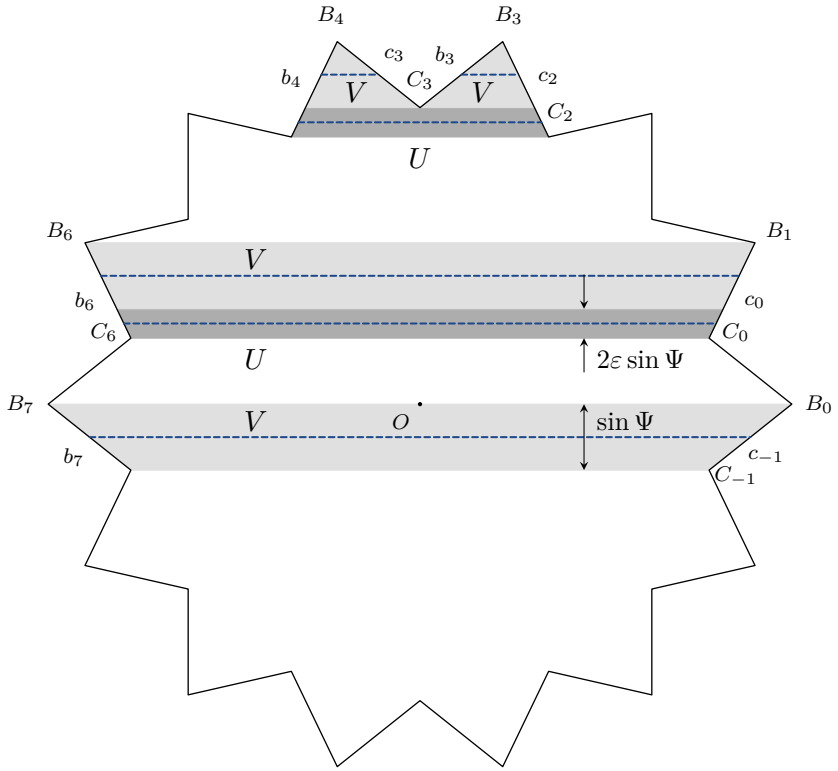


FIGURE 4. The star of Lemma 5.12 for $(1, 3, 10)/14$ ($q = 3$ in Theorem 5.16), B_0 on the positive x-axis; b_k is glued to c_{k-4}. Shaded: the cylinders U (dark), of height $2\varepsilon \sin \Psi$, and V (light), of height $\sin \Psi$; dashed: their core curves.

left tip right side c_q and left side b_{q+1}. These strips are pairwise disjoint: the slab and the two tips lie at heights above $y(C_{q-1}) \geq y(C_1) > y(B_1)$, the top of M; the tips lie in $x > 0$ and $x < 0$; M' lies below the x-axis; and U and V below use complementary parts of M.

The cylinder U is the lower part of M, of heights $(\sin \Psi, \sin \Psi + 2\varepsilon \sin \Psi)$, together with the slab. The gluing $c_0 \leftrightarrow b_{q+1}$ takes C_0 to C_{q+1}, and $y(C_{q+1}) = y(C_{q-1})$ by the mirror, so the bottom of M goes to the bottom of the slab and the lower part of M onto the slab (iv). The gluing $c_{q-1} \leftrightarrow b_{2q}$ takes C_{q-1} to C_{2q}, of height $y(C_0)$: bottom to bottom, back into M through its left side. By Lemma 5.14, U is an annulus of height $2\varepsilon \sin \Psi$; on its lower boundary its circumference is $2x(C_0) + 2x(C_{q-1}) = 2|AC|(\cos \psi + \sin 2\psi) = 2 \sin \Psi \cos \psi(1 + 2\varepsilon)/\varepsilon$, as $(2q - 1)\psi = \pi/2 - 2\psi$. So $\mu(U) = \varepsilon^2/(\cos \psi(1 + 2\varepsilon))$. The lower boundary passes through C_0, the upper boundary through C_q.

The cylinder V is the right tip, the upper part of M (heights $(y(B_1) - \sin \Psi, y(B_1))$), the left tip and M'. The gluing $c_{q-1} \leftrightarrow b_{2q}$ takes B_q to B_{2q}, of height $y(B_1)$: the apex of the right tip goes to the top of M, and the tip (of height $\sin \Psi$) onto the upper part of M. The gluing $c_0 \leftrightarrow b_{q+1}$ takes B_1 to B_{q+1}, of height $y(B_q)$: the upper part of M goes onto the left tip. The gluing $c_q \leftrightarrow b_{2q+1}$ takes B_{q+1} to B_{2q+1} (height 0, the top of M') and C_q to C_{2q+1} (height $-\sin \Psi$): the left tip goes onto M'. The gluing $c_{-1} \leftrightarrow b_q$ takes B_0 to B_q: M' goes onto the right tip. By Lemma 5.14, V has height $\sin \Psi$; on its upper boundary the tips have width zero and the chords are $B_{2q}B_1$ and $B_{2q+1}B_0$, so its circumference is

$2x(B_1)+2x(B_0) = 2|AB|(1+\cos 2\psi) = 4\cos\Psi\cos^2\psi/\varepsilon$, and $\mu(V) = \varepsilon/(4\cos\psi(1+\varepsilon))$. The upper boundary passes through B_q, the lower boundary through C_q.

Maximality. For q even the C_k form one point of cone angle $2\pi(3q+1)$ and the B_k two points of cone angle πq; for q odd the B_k form one point of cone angle $2\pi q$ and the C_k two points of cone angle $\pi(3q+1)$. For $q \geq 3$ all exceed 2π, so U and V are maximal cylinders, both horizontal; the horizontal direction is that of the side AB.

Irrationality. Let $\varrho = \mu(V)/\mu(U) = (1+2\varepsilon)/(4\varepsilon(1+\varepsilon))$. If $\varrho \in \mathbb{Q}$, then ε is a root of $4\varrho x^2 + (4\varrho - 2)x - 1$, a non-zero rational polynomial of degree 2. But $\varepsilon = \cos(q\pi/(2q+1))$ is, up to sign, $\cos(2\pi j/(2q+1))$ with $\gcd(j, 2q+1) = 1$ ($j = q/2$ for q even, and $\varepsilon = -\cos(2\pi\frac{q+1}{2}/(2q+1))$ for q odd), so its degree is $\varphi(2q+1)/2 \geq 3$ for $q \geq 3$. So $\varrho \notin \mathbb{Q}$, and Lemma 2.3(i) applies. □

6. Single triangles

The list $\mathcal{F}$ of Table 2 consists of the triangles that no general theorem of this paper reaches: $(1,3,5)/9$ of $\mathcal{A}$, to which Theorem 1.2 does not apply (Theorem 4.18(c)); the members of $\mathcal{E}$ to which Theorem 1.2 does not apply, among them $(3,4,13)/20$, the case $q = 3$ of Theorem 5.11; and the triangles $(1,4,11)/16$, $(2,3,17)/22$, $(3,8,29)/40$ and $(2,11,29)/42$ of item (7) of Theorem 2.5, to which it does not apply either. All of them have denominator at most 48.

Theorem 6.1. *No triangle of Table 2 is a lattice triangle. For each of them, in the direction d given in the table, the unfolding has two maximal cylinders Z_1, Z_2 such that $\mu(Z_1)/\mu(Z_2)$ has the minimal polynomial given in the table, of degree between 2 and 8. The direction d, as an element of $\mathbb{Z}[z]$ with $z = e^{i\pi/n}$, is in the deposit* [13].

Proof. The cylinders Z_1 and Z_2, their maximality, and the minimal polynomial of the ratio of their moduli are verified for each triangle by an exact computation, described in Appendix A.1. A polynomial of degree at least 2 that is irreducible over $\mathbb{Q}$ has no rational root, so the ratio is irrational, and Lemma 2.3(i) applies. □

The certificates for the members of $\mathcal{E}$ in Table 2 with even denominator are used in the window case of the proof of Theorem 1.1, those for the triangles of item (7) in Theorem 1.3, and those for $(1,3,5)/9$ and $(3,7,11)/21$ in Theorem 1.4. Figure 5 shows the certificate of $(1,3,5)/9$.

The unfolding of $(1,3,5)/9$ has genus 3, and Σ is the one point over C. In its certificate, with $z = e^{i\pi/9}$, $d = -1 + z + 2z^3 - 2z^4$ is a positive multiple of $1+z$, at the angle $\pi/18$ to AB. The cylinder Z_1 consists of three strips, each with an itinerary of sixteen steps, which meet along a leaf through a point over B and along a leaf through the point over A. The cylinder Z_2 consists of two strips, each with an itinerary of eight steps, which meet along a leaf through a point over B. The outer boundary leaves of both pass through the point over C. By Lemma A.1,

$$\mu(Z_1) = \sin\tfrac{\pi}{9} + \sin\tfrac{2\pi}{9} - \sin\tfrac{\pi}{3}, \qquad \mu(Z_2) = \sin\tfrac{2\pi}{9} - \sin\tfrac{\pi}{9},$$

and $\mu(Z_1)/\mu(Z_2) \approx 0.3949$ has the minimal polynomial $x^3 - x + \frac{1}{3}$, of degree 3, as in row 1 of Table 2.

That the triangles of Table 2 are not lattice triangles also follows from [6, Theorem 4.1] for the nineteen of them other than $(1,4,11)/16$ and $(3,4,13)/20$. These two Delecroix and Rüth list as orbit closures of rank two, and for $(3,4,13)/20$ they prove that the orbit closure has complex dimension at least 4 [6, Theorem 1.2], so that this triangle is not a lattice

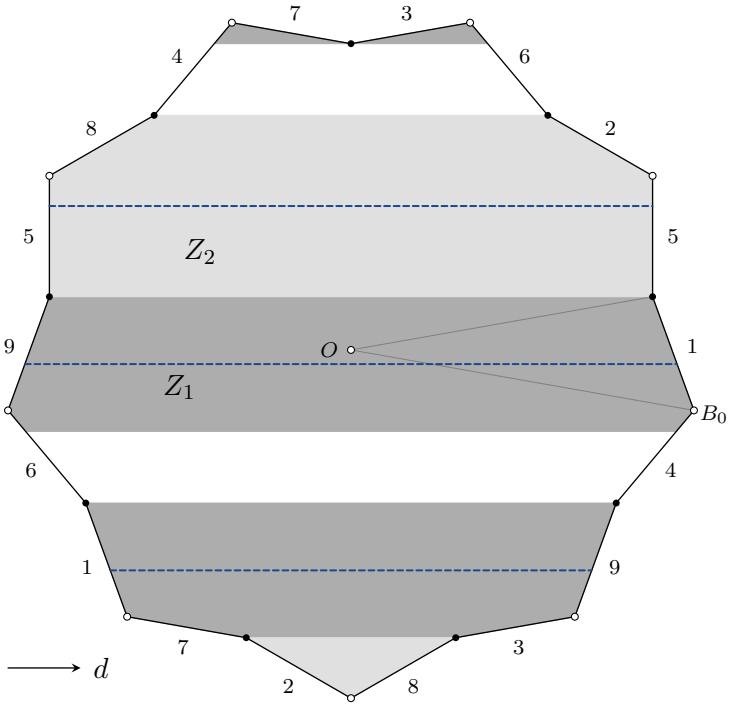


FIGURE 5. The unfolding of $(1,3,5)/9$, the star of Lemma 5.12 turned so that the direction d of Table 2, row 1, is horizontal. Sides with the same numeral are glued. Filled dots: the copies of the vertex C, which form the one point of Σ, of cone angle 10π. Hollow dots: the copies of B and the point O over A, which form four points of cone angle 2π. Shaded: the cylinders Z_1 (dark) and Z_2 (light) of the certificate.

triangle. That the triangles of item (7) of Theorem 2.5 are not lattice triangles was shown with the program of Rüth, Delecroix and Eskin [17, pp. 7120 and 7125].

7. The classification

In this section we prove Theorem 1.4, Corollary 1.5, Theorem 1.3 and Theorem 1.1, in this order.

The members of $\mathcal{E}$ and the triples of item (7) of Theorem 2.5 that are not in Table 2 are reached by Theorem 1.2: $(2,3,6)/11$, $(2,5,8)/15$, $(3,4,8)/15$ and $(2,7,12)/21$ by part (a), and the triangles of the following lemma by part (b).

TABLE 2. The triangles of Theorem 6.1, those that no general theorem of the paper reaches. Column 3: the list or family of the triangle ((7): item (7) of Theorem 2.5). Column 4: the direction d, as the angle $\theta \in [0, \pi)$ from AB to the line $\mathbb{R}d$, counterclockwise, in the coordinates of Appendix A.1, with $A = 0$ and B on the positive real axis; in row 5, where θ/π is irrational, d itself, as a polynomial in $z = e^{i\pi/n}$. Column 5: the minimal polynomial of $\mu(Z_1)/\mu(Z_2)$ over $\mathbb{Q}$, Z_1 and Z_2 being the first two cylinders of the certificate.

	triangle	in	direction	minimal polynomial
1	$(1,3,5)/9$	$\mathcal{A}$	$\pi/18$	$x^3 - x + \frac{1}{3}$
2	$(1,4,11)/16$	(7)	$3\pi/4$	$x^2 - 4x + 2$
3	$(3,4,9)/16$	$\mathcal{E}$	$\pi/4$	$x^4 - 4x^3 - 2x^2 + 4x - 1$
4	$(1,6,11)/18$	$\mathcal{E}$	0	$x^3 - 3x^2 + 2x - \frac{1}{3}$
5	$(3,4,13)/20$	$\mathcal{E}$	$2z^2 + z^4 - 3z^6 - 2z^8 + 2z^{12}$	$x^2 - x + \frac{1}{5}$
6	$(3,7,11)/21$	$\mathcal{E}$	$5\pi/14$	$x^6 - 14x^5 + 63x^4 - 126x^3 + 119x^2 - 49x + 7$
7	$(2,3,17)/22$	(7)	$19\pi/22$	$x^5 - 6x^4 - x^3 + 10x^2 - 6x + 1$
8	$(1,8,15)/24$	$\mathcal{E}$	0	$x^2 - x + \frac{1}{6}$
9	$(3,8,13)/24$	$\mathcal{E}$	$7\pi/8$	$x^4 - 4x^2 + 1$
10	$(1,10,17)/28$	$\mathcal{E}$	$5\pi/14$	$x^3 - 7x^2 + 14x - 7$
11	$(3,10,15)/28$	$\mathcal{E}$	0	$x^6 - 10x^5 - 12x^4 + 50x^3 + 11x^2 - 12x + 1$
12	$(2,9,19)/30$	$\mathcal{E}$	$\pi/10$	$x^4 + 3x^3 - 21x^2 + 27x - 9$
13	$(3,10,17)/30$	$\mathcal{E}$	$5\pi/6$	$x^2 - 4x - 1$
14	$(1,12,23)/36$	$\mathcal{E}$	$5\pi/6$	$x^3 - 9x + 9$
15	$(5,12,19)/36$	$\mathcal{E}$	$\pi/2$	$x^6 - 6x^5 - 18x^4 + 58x^3 - 21x^2 - 18x + 1$
16	$(3,8,29)/40$	(7)	$13\pi/40$	$x^8 + 4x^7 - 36x^6 + 48x^5 + 44x^4 - 104x^3 + 16x^2 + 32x - 4$
17	$(8,9,23)/40$	$\mathcal{E}$	$11\pi/20$	$x^8 - 28x^7 + 116x^6 + 716x^5 - 4711x^4 + 2048x^3 + 21334x^2 - 12036x - 31679$
18	$(2,11,29)/42$	(7)	$31\pi/42$	$x^6 - 13x^5 + 50x^4 - 69x^3 + 15x^2 + 22x + 1$
19	$(5,14,23)/42$	$\mathcal{E}$	$25\pi/42$	$x^6 + \frac{92}{41}x^5 - \frac{43}{41}x^4 - \frac{125}{41}x^3 - \frac{22}{41}x^2 + \frac{15}{41}x - \frac{1}{41}$
20	$(3,16,29)/48$	$\mathcal{E}$	0	$x^8 + \frac{424}{191}x^7 - \frac{2708}{191}x^6 - \frac{1808}{191}x^5 + \frac{8094}{191}x^4 - \frac{3672}{191}x^3 - \frac{132}{191}x^2 + \frac{40}{191}x + \frac{2}{191}$
21	$(5,16,27)/48$	$\mathcal{E}$	$\pi/8$	$x^8 + 100x^7 + 2362x^6 + 12380x^5 + 10851x^4 - 5156x^3 - 3758x^2 - 52x + \frac{191}{2}$

Lemma 7.1. *Theorem 1.2(b) applies to each of the thirteen triangles below at the vertex x with the unit b, so none of them is a lattice triangle. Three of them, $(1,3,16)/20$, $(1,4,21)/26$ and $(1,8,19)/28$, are triangles of item (7) of Theorem 2.5, and the other ten are members of $\mathcal{E}$ with even denominator. Here $g = \gcd(x,n)$, and the last column gives $[bp]_n$, $[bq]_n$, $[br]_n$.*

Δ	x	g	n/g	b	residues	Δ	x	g	n/g	b	residues
$(3,5,10)/18$	10	2	9	11	15, 1, 2	$(1,8,19)/28$	8	4	7	9	9, 16, 3
$(1,3,16)/20$	16	4	5	7	7, 1, 12	$(5,8,15)/28$	8	4	7	23	3, 16, 9
$(3,5,12)/20$	12	4	5	7	1, 15, 4	$(3,13,18)/34$	18	2	17	19	23, 9, 2
$(3,7,12)/22$	12	2	11	13	17, 3, 2	$(1,16,23)/40$	16	8	5	7	7, 32, 1
$(1,4,21)/26$	4	2	13	15	15, 8, 3	$(3,16,21)/40$	16	8	5	27	1, 32, 7
$(1,9,16)/26$	16	2	13	15	15, 5, 6	$(3,17,24)/44$	24	4	11	13	39, 1, 4
$(5,7,14)/26$	14	2	13	15	23, 1, 2						

Proof. In each row $1 < g < x$, so $g > 1$ and $x \nmid n$. Also $\gcd(b,n) = 1$, and $b - 2$ is n/g, except for $(5,8,15)/28$ and $(3,16,21)/40$, where it is $3n/g$ and $5n/g$; so $b \equiv 2 \pmod{n/g}$,

and $0 < b - 2 < n$ gives $b \not\equiv 2 \pmod{n}$. The three residues add up to n. These are the hypotheses of Theorem 1.2(b). □

Proof of Theorem 1.4. Let n be odd and $\Delta = (p, q, r)/n$ obtuse. If $p \nmid n$ or $q \nmid n$, then Δ is not a lattice triangle by Theorem 1.2(a). Otherwise p and q divide n. By Theorem 4.2, Δ has a witness, and so is not a lattice triangle (Proposition 2.4), unless it is $(1, 1, n-2)/n$, $(1, 3, 5)/9$, $(1, 5, 9)/15$ or $(3, 7, 11)/21$. The first is a lattice triangle for every $n \geq 5$ by Veech's theorem. $(1, 5, 9)/15$ is the member $(s, d) = (3, 5)$ of $\mathcal{A}$, and Theorem 1.2(b) applies to it at the vertex 9 with $b = 7$ (proof of Theorem 4.18); $(1, 3, 5)/9$ and $(3, 7, 11)/21$ are not lattice triangles by Theorem 6.1. So the obtuse lattice triangles of odd denominator n are exactly the triangles $(1, 1, n-2)/n$ with $n \geq 5$. □

Proof of Corollary 1.5. There is no obtuse triangle of denominator 2 or 3. Let $n \geq 5$ be prime and $\Delta = (p, q, r)/n$ obtuse. Then $p, q < n$, so $p \mid n$ and $q \mid n$ only if $p = q = 1$. Otherwise Theorem 1.2(a) applies and Δ is not a lattice triangle; and $(1, 1, n-2)/n$ is a lattice triangle by Veech's theorem. □

Proof of Theorem 1.3. Let $\Delta = (p, q, r)/n$ with $r > 2n/3$. For n odd this is Theorem 1.4 ($n - 2 > 2n/3$ exactly when $n \geq 7$); let n be even. If Δ has a witness, it is not a lattice triangle (Proposition 2.4). Otherwise, by Theorem 2.5 in the direction "no witness implies listed", Δ lies in one of the families (1) to (6) or is one of the seven triangles of (7).

- Family (1): $(1, 1, n-2)/n$, with $n - 2 > 2n/3$, that is $n \geq 7$. It is a lattice triangle by Veech's theorem.
- Family (2): $(1, 2, 2k-3)/2k$, with $2k - 3 > 4k/3$, that is $k \geq 5$. It is a lattice triangle by the theorem of Vorobets and Ward.
- Family (3): $(1, 4, r)/(r+5)$ with $r \equiv 7 \pmod{8}$. For $n = 12$, $r = 7 < 2n/3$; so $n \geq 20$, and the triangle is not a lattice triangle by [17, §10].
- Family (4): $(1, q, 3q+1)/(4q+2)$, with $3q + 1 > 2(4q+2)/3$, that is $q \geq 2$. For $q = 2$ it is $(1, 2, 7)/10$, of family (2). For $q \geq 3$ it is not a lattice triangle by Theorem 5.16.
- Family (5): $(2, q, 3q+2)/(4q+4)$. In lowest terms q is odd (for even q the triple has the common factor 2 and the triangle lies in family (4)), and $p \leq q$ gives $q \geq 3$. It is not a lattice triangle by Theorem 5.9.
- Family (6): $(4, q, 3q+4)/(4q+8)$, with q odd in lowest terms (for $q \equiv 2 \pmod{4}$ the triangle lies in family (5), for $4 \mid q$ in family (4)), and $3q + 4 > 2(4q+8)/3$, that is $q \geq 5$. It is not a lattice triangle by Theorem 5.11.
- The seven triangles of (7) are not lattice triangles: $(1, 3, 16)/20$, $(1, 4, 21)/26$ and $(1, 8, 19)/28$ by Lemma 7.1, and $(1, 4, 11)/16$, $(2, 3, 17)/22$, $(3, 8, 29)/40$ and $(2, 11, 29)/42$ by Theorem 6.1.

So Δ is a lattice triangle exactly when it is $(1, 1, n-2)/n$ with $n \geq 7$ or $(1, 2, 2k-3)/2k$ with $k \geq 5$. □

Proof of Theorem 1.1. Let $\Delta = (p, q, r)/n$ be obtuse. If n is odd, the statement is Theorem 1.4, since $(1, 2, 2k-3)/2k$ and $(1, 4, 7)/12$ have even denominator. If n is even and $r > 2n/3$, it is Theorem 1.3, since the triangles of the statement with n even and $r \leq 2n/3$ are $(1, 1, 4)/6$, $(1, 2, 5)/8$ and $(1, 4, 7)/12$. Let n be even and $n/2 < r \leq 2n/3$. These three triangles are lattice triangles, by the theorems of Veech, of Vorobets and Ward, and of Hooper. Let Δ be any other triangle. If Δ has a witness, it is not a lattice triangle (Proposition 2.4). If it has none, then by Theorem 4.1(a) it lies in $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ or is a member of $\mathcal{E}$ with even

denominator. The members of $\mathcal{A}$ with n even other than $(1,1,4)/6$, and the members of $\mathcal{B}$, are not lattice triangles by Theorem 4.18. The members of $\mathcal{C}$ are $(1, m-2, m+1)/2m$ with $m \geq 4$ even, $(2, m-3, m+1)/2m$ with $m \geq 6$ even, and $(4, m-5, m+1)/2m$ with $m \equiv 2 \pmod 4$, $m \geq 10$; the first with $m = 4$ and $m = 6$ are $(1,2,5)/8$ and $(1,4,7)/12$, and the others are not lattice triangles by Theorems 5.15, 5.8 and 5.10. Of the members of $\mathcal{E}$ with even denominator, those of Lemma 7.1 are not lattice triangles by that lemma, and the others, which are in Table 2, by Theorem 6.1. □

Appendix A. The computations

This appendix describes the computations the proofs use; the programs and their outputs are in the deposit [13]. The arithmetic on which a proof rests is exact, in integers, rationals or elements of $\mathbb{Q}(\zeta)$ for a root of unity ζ, and signs of real elements of $\mathbb{Q}(\zeta)$ are decided by interval arithmetic with an exact zero test.

A.1. **Cylinder certificates.** This subsection proves that the cylinders of Theorem 6.1 exist and are maximal, and that the ratio of their moduli has the minimal polynomial given in Table 2. Let $\Delta = (p,q,r)/n$ and let X be its unfolding, with singular set Σ as in Section 2.1; every surface of Table 2 has genus between 3 and 20.

Coordinates. Let $z = e^{i\pi/n}$. The copy of Δ with the identity as linear part has vertices $A = 0$, $B = -(z^r - z^{-r})^2 = 4\sin^2(r\pi/n)$ and $C = -(z^r - z^{-r})(z^q - z^{-q})z^p$, so that $C/B = (\sin(q\pi/n)/\sin(r\pi/n))e^{ip\pi/n}$ and the angles at A, B, C are $p\pi/n$, $q\pi/n$, $r\pi/n$. The other copies are its images under the maps $w \mapsto z^{2k}w$ and $w \mapsto z^{2k}\bar{w}$, and all their vertices lie in $\mathbb{Z}[z]$. For a direction $d \in \mathbb{Q}(z) \setminus \{0\}$ the *height* of a point w of a copy is $h(w) = w\bar{d} - \bar{w}d$; it is purely imaginary, and $\operatorname{Im} h(w)/(2|d|)$ is the signed distance from w to the line $\mathbb{R}d$, positive on the left. Crossing a side out of a copy changes heights by a fixed element of $\mathbb{Q}(z)$ (the height of the translation part of the gluing, with a minus sign) and adds that translation part to the holonomy.

A certificate for a cylinder in direction d consists of a holonomy c, a number $\mathcal{V}$, and a list of *strips*. A strip is given by a start (a copy, a side of it through which the leaf enters, and a height $a_0 + b\epsilon$ with $a_0 \in \mathbb{Q}(z)$, $b \in \{-1, 0, 1\}$ and ϵ a positive infinitesimal; heights of this form are compared lexicographically), an itinerary (the list of triples: copy, entry side, exit side), and two heights lo $<$ hi. The conditions on a certificate are:

(v1) the leaf entering the start copy through the start side at height $a_0 + b\epsilon$, followed copy by copy (it leaves each copy through the other side whose two end heights lie on opposite sides of its own height), returns for the first time to the start side of the start copy at the same height after exactly the stored itinerary;

(v2) lo (respectively hi) is the largest (respectively smallest) height, carried back to the start by the accumulated height changes, of a corner of a copy of the itinerary lying below (respectively above) $a_0 + b\epsilon$;

(v3) the accumulated holonomy is c, $h(c) = 0$ and $\operatorname{Re}(c\bar{d}) > 0$;

(v4) exactly one strip has a corner in Σ among the corners attaining its lo, and exactly one has a corner in Σ among those attaining its hi; starting from the first, each next strip has no corner in Σ attaining its lo and one of the corners attaining its lo is the same point of X as one of the corners attaining the current hi; the chain uses every strip once and ends at the second;

(v5) $\mathcal{V}$ is the sum of hi − lo over the strips.

Lemma A.1. *Let X have genus at least two. If a certificate in direction d passes* (v1)–(v5)*, then the union of the leaves of its strips and of the leaves along which consecutive strips meet is a cylinder Z of X in direction d in the sense of Section 2.2, with core holonomy c, height $H = \mathcal{V}/(2i|d|)$, modulus $\mu = \mathcal{V}/(2i\,c\bar{d})$, and* $\operatorname{area}(Z)/\operatorname{area}(X) = \mathcal{V}\,c\bar{d}/(n\,d\bar{d}(\bar{B}C - B\bar{C}))$.

Proof. (i) *Each strip is an annulus of closed leaves.* Let η lie strictly between lo and hi. By (v2), at every step of the itinerary each corner of the copy lies strictly on the same side of η as of $a_0 + b\epsilon$, so the leaf at height η leaves each copy through the same side as the leaf at $a_0 + b\epsilon$ and follows the same itinerary. The height changes depend only on the steps, so after the whole itinerary the leaf at η is back on the start side at height η, the total change being 0 by (v1); its holonomy is c by (v3), and it meets no corner, so it misses Σ.

(ii) *Embedded.* Let ℓ be the length of the itinerary, and suppose the leaf at η enters the start copy through the start side at a step k with $0 < k < \ell$, at a height η' strictly between lo and hi. By (i) it follows the itinerary from there, so the sequence of steps, which has period ℓ, is invariant under the shift by k and so has period $e = \gcd(k, \ell) < \ell$. The height changes are functions of the steps, so their sum over e steps is e/ℓ times their sum over ℓ steps, which is 0. Then the leaf at $a_0 + b\epsilon$ returns to the start side at the same height after $e < \ell$ steps, against the first return in (v1). So distinct heights give distinct leaves (two leaves in direction d through one regular point coincide), and $\eta \mapsto$ leaf is an embedding of an annulus of height $(\text{hi} - \text{lo})/(2i|d|)$ foliated by closed leaves of holonomy c.

(iii) *Joints.* Let $\partial^+ S$ be the limit leaf at the upper end hi of a strip S whose corners attaining hi are all regular. The vertices on $\partial^+ S$ are the corners attaining hi; through these regular points $\partial^+ S$ is a straight line, so $\partial^+ S$ is a closed geodesic in direction d missing Σ, of holonomy c (a closed regular leaf has a neighbourhood of parallel closed leaves with its own holonomy, and the leaves of S near $\partial^+ S$ have holonomy c). The next strip S' of the chain has a corner attaining its lo at the same point of X as a corner attaining hi (v4); that corner is a regular point and lies on exactly one leaf in direction d, so the lower limit leaf of S' is $\partial^+ S$. Heights increase to the left of d, so S lies to the right of $\partial^+ S$ and S' to its left. By induction the union Z_0 of the strips and the joint leaves is a connected open set of closed regular leaves of holonomy c, so it lies in one cylinder Z.

(iv) *Maximal.* The frontier of Z_0 lies in the two outer limit leaves, the lower one of the first strip and the upper one of the last. By (v4) each of them passes through a point of Σ, and from any of its points the leaf in direction d follows it (it is straight through regular points) until it meets Σ; so it is not a closed regular leaf. Hence Z_0 is closed in the union of the closed regular leaves, and being open and connected, $Z_0 = Z$.

(v) *Height and the formulas.* The leaves of Z are parametrized, strip after strip, by heights; inside a strip the parametrization is $\eta \mapsto \eta$, and at each joint the next strip starts at the height where the previous one ended, so in the transverse distance the parametrization is a translation $t \mapsto t + \text{const}$ on each piece, with the pieces abutting. It is therefore an isometry from an interval of length $\sum(\text{hi} - \text{lo})/(2i|d|)$ onto the space of leaves of Z, and $H = \mathcal{V}/(2i|d|)$. As $c = \lambda d$ with $\lambda > 0$ by (v3), $|c| = c\bar{d}/|d|$, so $\mu = H/|c| = \mathcal{V}/(2i\,c\bar{d})$ and $\operatorname{area}(Z) = H|c| = \mathcal{V}\,c\bar{d}/(2i\,d\bar{d})$. The copy of Δ has area $\operatorname{Im}(\bar{B}C)/2 = (\bar{B}C - B\bar{C})/(4i)$, and X is made of $2n$ copies. □

A vertex of cone angle 2π on a boundary leaf does not stop the cylinder (Section 2.2): the chain of (v4) continues through it.

By Lemma A.1, if two cylinders in direction d have certificates with holonomies c_k, c_l and numbers $\mathcal{V}_k$, $\mathcal{V}_l$, the ratio of their moduli is $\mu_k/\mu_l = \mathcal{V}_k\,c_l\bar{d}/(\mathcal{V}_l\,c_k\bar{d}) \in \mathbb{Q}(z)$, and its minimal polynomial over $\mathbb{Q}$ is computed exactly. The certificates of the triangles of Table 2, each

with its direction and two or three cylinders, and the programs that verify (v1)–(v5) and compute the minimal polynomials are in the deposit [13]. The directions were found with the flatsurf software [5].

A.2. **The denominators not covered by the general theorems.** Theorems 4.14 and C.2 leave finitely many denominators, which are determined here, with a proof that the lists are complete; the lists are in the deposit [13]. Let π_i be the i-th odd prime, $\Omega_k = \pi_1 \cdots \pi_k$ and $\beta_k = \prod_{i \le k} \pi_i/(\pi_i - 1)$. If N is odd and squarefree with k prime factors, then $N \ge \Omega_k$ and $N/\varphi(N) \le \beta_k$, so by Lemma 4.13

$$J(N) \le \mathcal{J}_k, \qquad \mathcal{J}_1 = 2, \quad \mathcal{J}_2 = 3, \quad \mathcal{J}_k = \lfloor (2^k - 1)\beta_k \rfloor + 1 \ (k \ge 3).$$

For $k = 0$ we have $N = 1$ and $J(1) = 1$, and we put $\Omega_0 = 1$, $\mathcal{J}_0 = 1$.

Odd n. Theorem C.2 applies unless $5n \le 324J(n)^2$. Let $n \ge 5$ satisfy this inequality, with radical N having $k \ge 1$ primes. Then $5\Omega_k \le 5N \le 5n \le 324\mathcal{J}_k^2$. For $k = 8, 9, 10$ one has $\mathcal{J}_k = 780$, 1,618, 3,347 and $5\Omega_k > 324\mathcal{J}_k^2$; for $k = 11$, $5\Omega_{11} > 324 \cdot 9^{11}$, and since $\pi_{k+1} > 9$ for $k \ge 11$, $5\Omega_k > 324 \cdot 9^k \ge 324J(N)^2$ for every $k \ge 11$ (Lemma 4.13, $J(N) \le 3^k$). So $k \le 7$ and $5N \le 324\mathcal{J}_k^2$. Each odd squarefree N with $k \le 7$ primes and $5N \le 324\mathcal{J}_k^2$ is discarded if $5N > 324E(N)^2$, where $E(N)$ is the least of the bounds of Lemma 4.13 at N itself; otherwise $J(N)$ is computed exactly, as the largest gap between consecutive integers prime to N, and every odd n with radical N and $5n \le 324J(N)^2$ is listed. The result is the list of 320 values $n \ge 5$ of Theorem C.3, the largest 3,135, with $J = 7$.

Even $n = 2m$. Theorem 4.14 applies unless $m \le N_0(m) = P_2(m)(31J(m^\circ)^2 - 1)$. Let m satisfy this inequality, with $k \ge 0$ odd primes and odd radical $N = m^\circ$. Then $P_2(m) \le \pi_{k+2}$ and $m \ge N \ge \Omega_k$, so $\Omega_k \le \pi_{k+2}(31\mathcal{J}_k^2 - 1)$. This fails for $9 \le k \le 12$, by exact computation; and for $k \ge 13$, $J(N) \le 3^k$ gives $m < 31 \cdot 9^k \pi_{k+2}$, which fails at $k = 13$, while from k to $k + 1$ the left side Ω_k is multiplied by $\pi_{k+1} \ge 47$ and the right side by $9\pi_{k+3}/\pi_{k+2} < 18$ (Bertrand's postulate). So $k \le 8$ and $N \le \pi_{k+2}(31\mathcal{J}_k^2 - 1)$. One enumerates every set of $k \le 8$ odd primes whose product satisfies this bound (for $k = 2$, the larger range given by the third bound of Lemma 4.13), discards a set when its product exceeds $P_2(31E^2 - 1)$, with P_2 the second least odd prime outside the set and E the bound of Lemma 4.13 at the set itself (for the empty set, $E = J(1) = 1$), computes J exactly for the rest by an exhaustive search over the ways of covering an initial run of integers by one residue class per prime, and lists every $m \ge 3$ with exactly those odd primes and $m \le N_0(m)$. The result is the list of 2,567 values of Theorem 4.15, the largest $69{,}615 = 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17$.

A.3. **Enumerations of the criterion.** *Odd n.* For the 320 values of Appendix A.2 an enumeration of the window triangles finds that those without witness lie in $\mathcal{A}$ or $\mathcal{B}$ or are the five odd members of $\mathcal{E}$.

Even n. For the 2,567 values of m of Appendix A.2 an enumeration of the window triangles of denominator $2m$ with $pq \le 4mJ^2$ finds that those without witness lie in $\mathcal{A}$ or $\mathcal{C}$, have $s \mid n$, n/s odd and $2 \le p \le s/2$ (and so lie in $\mathcal{B}$, by Proposition 4.3(c)), or are the twenty-five even members of $\mathcal{E}$.

The thirty members of $\mathcal{E}$. Each has no witness, by a direct test of all usable units, and none lies in $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$.

Appendix B. Residues for every denominator, and the proof of Theorem 4.14

This appendix proves Theorem 4.14: for $m > N_0(m)$, a window triangle of denominator $n = 2m$ without witness lies in $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$. Appendix B.1 proves the lemmas on residues that hold for every denominator; Appendix C uses them as well. Appendix B.2 proves the lemmas special to even n and the theorem. The plan of the proof follows the statement in Section 4.3, and a table of its cases follows Proposition 4.16.

B.1. **Lemmas for every denominator.** In this subsection n is any integer and $\Delta = (p,q,r)/n$ is a window triangle (so $n \geq 5$), $s = 2r - n$ and $L = n - 2q = 2p + s$; the thresholds of S_p, S_q, S_r are $2p$, $2q$, s, and $2p + 2q + s = n$. Let n° be the product of the odd primes dividing n ($n^\circ = 1$ if n is a power of 2), and $J = J(n^\circ)$. For $n = 2m$ this is the $J = J(m^\circ)$ of Section 4.3, and for odd n it is $J(n)$, as J depends only on the primes dividing its argument.

Lemma B.1. *The units b modulo n with $2b \equiv 2 \pmod n$ are $b = 1$ and, if $4 \mid n$, $b = 1 + n/2$. Such a unit lying in S_p has $[bp]_n = p$. If one lies in S_r, then $4 \mid n$, r is odd, $b = 1 + n/2$ and $[br]_n = s/2$.*

Proof. $2b \equiv 2 \pmod n$ means $b \equiv 1$ modulo $n/\gcd(2,n)$, so $b = 1$, or n is even and $b = 1 + n/2$, which is prime to $n/2$, and so a unit exactly when it is odd, that is when $n/2$ is even. $[1 \cdot x]_n = x$ for $x \in \{p, r\}$, and $r \geq s$, so $1 \notin S_r$. For $4 \mid n$, $(1 + n/2)x \equiv x$ if x is even and $\equiv x + n/2$ if x is odd. So $[(1 + n/2)p]_n$ is p or $p + n/2 \geq 2p$ (as $p < n/2$), and $[(1 + n/2)r]_n$ is $r \geq s$ or, for r odd, $r + n/2 - n = s/2$. □

Good progressions. For integers c, b and an odd squarefree N, the progression $c + ib$ ($i \in \mathbb{Z}$) is *good* modulo N if $\gcd(c, b, N) = 1$ (the term is from [17, Definition 7.4]). Any $J(N)$ consecutive terms of a good progression contain one prime to N. Indeed, for a prime $\ell \mid N$: if $\ell \mid b$, then $\ell \nmid c + ib$ for every i; if $\ell \nmid b$, the i with $\ell \mid c + ib$ form one class modulo ℓ. Choosing i_0 in all these classes, $c + ib$ is prime to N if and only if $i - i_0$ is prime to the product N' of the primes $\ell \mid N$ with $\ell \nmid b$, and $J(N') \leq J(N)$.

Lemma B.2. *Let (u,v,w) be (p,q,r) in any order, and $\bar u = [2u]_n$, $\bar v = [2v]_n$, $\bar w = [2w]_n$, so that $\bar u + \bar v + \bar w = n$. Let $a \in \mathbb{Z}/n$ with $x = [au]_n < \bar u$, and $y = [av]_n$. Then $a \notin S_v \cup S_w$ if and only if $\bar v \leq y \leq \bar v + \bar u - x$ or $n - x < y < n$.*

Proof. Let $z = [aw]_n$. As $u + v + w = n$, $x + y + z \in \{0, n, 2n\}$. If the sum is 0, then $y = 0 < \bar v$ and $a \in S_v$. If it is n, then $z = n - x - y$, and $z \geq \bar w$ if and only if $y \leq \bar u + \bar v - x$; together with $y \geq \bar v$ this is the first interval. If it is $2n$, then $z = 2n - x - y < n$ forces $y > n - x > n - \bar u \geq \bar v$, and $z \geq n - x + 1 > \bar v + \bar w \geq \bar w$, so $a \notin S_v \cup S_w$; conversely $n - x < y < n$ forces $x + y > n$, so the sum is $2n$. Finally y in the first interval gives $x + y \leq \bar u + \bar v < n$ and $y \geq 1$, so the sum is n. □

For $x \in \{p, r\}$ let $\tau = [2x]_n$ and let τ' be the other of $2p$, s, so that $S_x = \{a : [ax]_n < \tau\}$ and $\tau + \tau' = L$. For a residue a write $\mathcal{X} = [ax]_n$ and $\mathcal{Y} = [-aq]_n$.

Lemma B.3. (a) *$a \notin S_q$ if and only if $\mathcal{Y} \in [1, L]$.*

(b) *$L\mathcal{X} \equiv \tau\mathcal{Y} \pmod n$.*

(c) *If $\mathcal{X} < \tau$, then $a \notin S_q \cup S_{x'}$, x' the other of p, r, if and only if $1 \leq \mathcal{Y} \leq \mathcal{X} - 1$ or $\mathcal{X} + \tau' \leq \mathcal{Y} \leq L$.*

(d) *If* $1 \le \mathcal{X} < \tau$ *and* $\mathcal{X} + \tau' \le \mathcal{Y} \le L$ *("above"), then* $\tau\mathcal{Y} - L\mathcal{X}$ *is a multiple of* n *with* $n \le \tau\mathcal{Y} - L\mathcal{X} \le L(\tau - \mathcal{X})$. *If* $1 \le \mathcal{Y} \le \mathcal{X} - 1$ *("below"), then* $L\mathcal{X} - \tau\mathcal{Y}$ *is a multiple of* n *with* $n \le L\mathcal{X} - \tau\mathcal{Y} \le L\mathcal{X} - \tau$.

Proof. (a) $a \notin S_q$ means $[aq]_n \ge 2q$, that is $aq \not\equiv 0$ and $n - [aq]_n \le n - 2q = L$. (b) $L\mathcal{X} - \tau\mathcal{Y} \equiv a(Lx + \tau q)$; for $x = p$ this is $ap(L + 2q) = apn$, and for $x = r$, with $L = n - 2q$ and $\tau = 2r - n$, it is $a((n-2q)r + (2r-n)q) = an(r - q)$. (c) Apply Lemma B.2 with $(u, v, w) = (x, x', q)$, so $(\bar{u}, \bar{v}) = (\tau, \tau')$ and the x of that lemma is $\mathcal{X}$: with $y = [ax']_n$, $a \notin S_{x'} \cup S_q$ if and only if $\tau' \le y \le L - \mathcal{X}$ or $n - \mathcal{X} < y < n$. As $-aq \equiv ax + ax'$, $\mathcal{Y} \equiv \mathcal{X} + y$. In the first case $\mathcal{X} + y \in [\mathcal{X} + \tau', L]$ and $L < n$, so $\mathcal{Y} = \mathcal{X} + y$; in the second $\mathcal{X} + y \in (n, n + \mathcal{X})$, so $\mathcal{Y} = \mathcal{X} + y - n \in [1, \mathcal{X} - 1]$. Conversely $\mathcal{Y} \in [\mathcal{X} + \tau', L]$ gives $y = \mathcal{Y} - \mathcal{X} \in [\tau', L - \mathcal{X}]$, and $\mathcal{Y} \in [1, \mathcal{X} - 1]$ gives $y = n + \mathcal{Y} - \mathcal{X} \in (n - \mathcal{X}, n)$. (d) Above: $\tau\mathcal{Y} - L\mathcal{X} \ge \tau(\mathcal{X} + \tau') - (\tau + \tau')\mathcal{X} = \tau'(\tau - \mathcal{X}) > 0$ and $\tau\mathcal{Y} - L\mathcal{X} \le \tau L - L\mathcal{X}$. Below: $L\mathcal{X} - \tau\mathcal{Y} \ge L\mathcal{X} - \tau(\mathcal{X} - 1) = \tau'\mathcal{X} + \tau > 0$ and $L\mathcal{X} - \tau\mathcal{Y} \le L\mathcal{X} - \tau$. Both are multiples of n by (b). □

Lemma B.4. *Let* $\epsilon = \gcd(2, n)$. *If* $pq > \epsilon n J^2$, *there is a witness in* $S_p \cap S_q$.

Proof. Let $\Lambda = \{(x, z) \in \mathbb{Z}^2 : (x, z) \equiv (bp, bq) \pmod n \text{ for some } b \equiv 0 \pmod \epsilon\}$. As $\gcd(p, q, n)$ divides $r = n - p - q$, it is 1, so $b \mapsto (bp, bq) \bmod n$ is injective on $\mathbb{Z}/n$; the $b \equiv 0 \pmod \epsilon$ form a subgroup of order n/ϵ, so Λ is a lattice of index $n^2/(n/\epsilon) = \epsilon n$ in $\mathbb{Z}^2$. The open box $|x| < p/J$, $|z| < q/J$ has area $4pq/J^2 > 4\epsilon n$, so by Minkowski's theorem it contains a point $(x, z) \ne 0$ of Λ; let $b \equiv 0 \pmod \epsilon$ be such that $(x, z) \equiv (bp, bq)$. For $1 \le i \le J$ let $a_i = 1 + ib$. Then $a_i p \equiv p + ix$ and $a_i q \equiv q + iz$, with $0 < p + ix < 2p$ and $0 < q + iz < 2q$ as $|ix| < p$ and $|iz| < q$; so $a_i \in S_p \cap S_q$. If $2a_i \equiv 2$, then $2ib \equiv 0$, so $2ix \equiv 2iz \equiv 0 \pmod n$, and $|2ix| < 2p < n$, $|2iz| < 2q < n$ give $x = z = 0$; so $2a_i \not\equiv 2$. The a_i are J consecutive terms of the progression $1 + ib$, so one of them is prime to n° (as $1 + ib$ is a good progression modulo n°); it is odd when n is even, as b is then even, so it is a unit, and a witness. □

Lemma B.4 is a variant of [17, Proposition 7.5], which finds a usable unit in $S_p \cap S_q$ when $4\lfloor p/2j(n)\rfloor\lfloor q/2j(n)\rfloor > n$, j being Jacobsthal's function, by the same progression through $a = 1$.

Lemma B.5. *Let* $x \in \{p, r\}$ *and let* a_0 *be a unit of* S_x *with* $\mathcal{X}_0 = [a_0 x]_n \ge 1$ *and* $\mathcal{X}_0 + \tau' \le \mathcal{Y}_0 = [-a_0 q]_n \le L$. *Let* $u \ge 2$ *be an integer prime to* n *with* $uL < n$, *let* $a_j = u^j a_0$, *and let* $\kappa \ge 0$ *be the largest integer with* $u^\kappa \mathcal{X}_0 < \tau$. *If* $0 \le j_1 \le \kappa$ *and* $a_1, \dots, a_{j_1}$ *are usable and not witnesses, then* $[a_j x]_n = u^j \mathcal{X}_0$ *and* $[-a_j q]_n = u^j \mathcal{Y}_0 \le L$ *for* $0 \le j \le j_1$, *and*

$$n \le (u^{\kappa+1-j_1} - 1) L \mathcal{X}_0.$$

In particular, if $(u - 1)L\mathcal{X}_0 < n$, *then* $\kappa \ge 1$ *and some* a_j *with* $1 \le j \le \kappa$ *is a witness or is not usable; for the least such* j, $[a_j x]_n = u^j \mathcal{X}_0$.

Proof. For $j = 0$ the two equalities are the hypothesis. Suppose they hold for some $j < j_1$. Then $j + 1 \le \kappa$, so $u^{j+1}\mathcal{X}_0 < \tau$; hence $[a_{j+1}x]_n = u^{j+1}\mathcal{X}_0$ and $a_{j+1} \in S_x$. Also $1 \le u \cdot u^j \mathcal{Y}_0 \le uL < n$, so $[-a_{j+1}q]_n = u^{j+1}\mathcal{Y}_0$. As a_{j+1} is a usable unit of S_x and not a witness, it is not in S_q, so $u^{j+1}\mathcal{Y}_0 \le L$ by Lemma B.3(a). This proves the equalities. Next, $u^{\kappa+1}\mathcal{X}_0 \ge \tau$ and $u^{j_1}\mathcal{Y}_0 \le L$ give $\tau\mathcal{Y}_0 \le u^{\kappa+1}\mathcal{X}_0\mathcal{Y}_0 = u^{\kappa+1-j_1}\mathcal{X}_0 \cdot u^{j_1}\mathcal{Y}_0 \le u^{\kappa+1-j_1}\mathcal{X}_0 L$, and by Lemma B.3(d), $n \le \tau\mathcal{Y}_0 - L\mathcal{X}_0 \le (u^{\kappa+1-j_1} - 1)L\mathcal{X}_0$. If $(u-1)L\mathcal{X}_0 < n$, this rules out $j_1 = \kappa$; as $j_1 = 0$ is always allowed, $\kappa \ge 1$. Let j_1 be the largest integer in $[0, \kappa]$ such that $a_1, \dots, a_{j_1}$ are usable and not witnesses; then $j_1 < \kappa$, and $j = j_1 + 1 \le \kappa$ is the least index with a_j a witness or not usable, and $a_j x \equiv u \cdot u^{j_1}\mathcal{X}_0 = u^j \mathcal{X}_0 < \tau$, so $[a_j x]_n = u^j \mathcal{X}_0$. □

Corollary B.6. *Suppose that Δ has no witness, and let u be a prime not dividing n with $uL < n$.*

(a) *If $\gcd(r, n) = 1$ and $s \geq 3$, then $4 \mid n$ and $s/2$ is a power of u.*

(b) *If $\gcd(p, n) = 1$ and $p \geq 2$, then p is a power of u, and $(u^2 - 1)L \geq n$.*

In both the exponent is at least 1. Hence, if (a) or (b) applies, at most one prime $u \nmid n$ has $uL < n$; for $n = 2m$, where the two least primes not dividing n are $P(m) < P_2(m)$, this gives $P_2(m)L \geq n$.

Proof. Let $x = r$ in (a) and $x = p$ in (b), and $a_0 = x^{-1} \bmod n$: a unit of S_x with $\mathcal{X}_0 = 1$, as $\tau = s \geq 3$ or $\tau = 2p \geq 4$. It is usable by Lemma B.1, since $\mathcal{X}_0 = 1$ differs from $s/2$ in (a) and from p in (b). It is not a witness, so it is not in $S_q \cup S_{x'}$, and by Lemma B.3(c), $\mathcal{Y}_0 = [-a_0 q]_n \in [1 + \tau', L]$: above. Lemma B.5 applies with u, and $(u-1)L\mathcal{X}_0 < uL < n$: some $a_j = u^j a_0$ with $1 \leq j \leq \kappa$ is a witness or not usable, and for the least such j, $[a_j x]_n = u^j$. There is no witness, so a_j is a unit of S_x that is not usable, and by Lemma B.1: in (a), $4 \mid n$ and $u^j = s/2$; in (b), $u^j = p$. In (b) moreover $\kappa = j$, as $u^{j+1} = up \geq 2p = \tau$, and $a_1, \dots, a_{j-1}$ are usable and not witnesses, so Lemma B.5 with $j_1 = j - 1 = \kappa - 1$ gives $n \leq (u^2 - 1)L\mathcal{X}_0 = (u^2 - 1)L$. Finally, a number at least 2 is not a power of two different primes; for $n = 2m$, $P_2(m)L < n$ would give $P(m)L < n$ as well. □

Lemma B.7. *Let g be a divisor of n, with n/g even if n is even, such that $\gcd(q, g) = 1$. Let a_0 be a residue modulo n with $\gcd(a_0, n/g) = 1$, $\mathcal{R} = \{a_0 + (n/g)i : i \in \mathbb{Z}/g\}$ (the* row *of a_0), $\mathcal{Y}_0 = [-a_0 q]_n$, and $\Upsilon = \lfloor (L-1)g/n \rfloor + 1$. Let g^* be the product of the primes $\ell \mid g$ with $\ell \nmid n/g$; they are odd.*

(i) *$a \mapsto [-aq]_n$ maps $\mathcal{R}$ bijectively onto $\{[\mathcal{Y}_0 + (n/g)t]_n : t \in \mathbb{Z}/g\}$, the element $a_0 + (n/g)i$ going to $t \equiv -qi \pmod g$, which we call its* index.

(ii) *The elements of $\mathcal{R}$ outside S_q have cyclically consecutive values of t, at most Υ of them.*

(iii) *For each prime $\ell \mid g^*$ there is a class t_ℓ modulo ℓ such that the element of index t is a unit if and only if $t \not\equiv t_\ell \pmod \ell$ for every $\ell \mid g^*$. In particular any $J(g^*)$ consecutive values of t contain one whose element is a unit.*

(iv) *Let ν be the number of units of $\mathcal{R}$ that are not usable, and suppose $g - \Upsilon \geq (1+\nu)J(g^*)$. Then $\mathcal{R}$ contains a usable unit lying in S_q.*

If g divides n and one of p, r, then $\gcd(q, g) = 1$.

Proof. A prime $\ell \mid g$ with $\ell \nmid n/g$ is odd: if $\ell = 2$, then n is even and n/g odd. (i) $-(a_0 + (n/g)i)q \equiv \mathcal{Y}_0 + (n/g)(-qi) \pmod n$, and $i \mapsto -qi$ is a bijection of $\mathbb{Z}/g$. (ii) The g points $[\mathcal{Y}_0 + (n/g)t]_n$ are equally spaced on $\mathbb{Z}/n$ with gap n/g; outside S_q means in $[1, L]$ (Lemma B.3(a)), an arc that holds at most $\lfloor (L-1)/(n/g) \rfloor + 1$ of them, consecutive ones. (iii) For a prime $\ell \mid n/g$, $a_0 + (n/g)i \equiv a_0 \not\equiv 0 \pmod \ell$. For a prime $\ell \mid g^*$, n/g is invertible modulo ℓ, so $\ell \mid a_0 + (n/g)i$ for exactly one class of i modulo ℓ, that is, q being invertible modulo ℓ, for one class t_ℓ of t. Every prime of n divides n/g or g^*. The last sentence follows by the Chinese remainder theorem: with $t_0 \equiv t_\ell \pmod \ell$ for every $\ell \mid g^*$, the element of index t is a unit if and only if $t - t_0$ is prime to g^*, and $J(g^*) \leq g^* \leq g$. (iv) By (ii) the t whose element lies in S_q include $g - \Upsilon$ cyclically consecutive values; cut $1 + \nu$ disjoint blocks of $J(g^*)$ consecutive values out of them. By (iii) each block holds a unit; these $1 + \nu$ units are distinct and lie in S_q, and at most ν of them are not usable. For the last sentence: a common prime of q and g divides n, q and one of p, r, hence the third numerator, against $\gcd(p, q, r) = 1$. □

For $g \geq 4$ the units can be named, and Jacobsthal's function is not needed. On $\mathbb{Z}/g$ the *gap* between two cyclically consecutive points of a set is their circular distance.

Lemma B.8. *In Lemma B.7 let $g \geq 4$, and suppose $7L \leq 3n$ and either $\nu = 0$, or $\nu = 1$ and $g \geq 7$. Then $\mathcal{R}$ contains a usable unit lying in S_q.*

Proof. Choose $t_0 \in \mathbb{Z}/g$ with $t_0 \equiv t_\ell \pmod{\ell}$ for each $\ell \mid g^*$ (Chinese remainder theorem); then the element of index t is a unit if and only if $\gcd(t - t_0, g^*) = 1$. Let $g' = \lfloor (g-1)/2 \rfloor$. Each of ± 1, ± 2, $\pm g'$ is prime to every odd prime dividing g, as $2g'$ is $g-1$ or $g-2$; so the elements of indices $t_0 \pm 1$, $t_0 \pm 2$, $t_0 \pm g'$ are units. Let $\Pi = \{t_0 \pm 1, t_0 \pm g'\}$ if $\nu = 0$, and $\Pi = \{t_0 \pm 1, t_0 \pm 2, t_0 \pm g'\}$ without the index of the unit that is not usable if $\nu = 1$; every element with index in Π is a usable unit. The gaps of Π are at most $\max(2, g'-1)$ if $\nu = 0$ (they are $g'-1$, $g-2g'$, $g'-1$, 2, and $g-2g' \leq 2$), and at most $\max(3, g')$ if $\nu = 1$ (for $g \geq 7$ the six points are distinct, with gaps 1, $g'-2$, $g-2g'$, $g'-2$, 1, 2, and removing one point adds two neighbouring gaps). Both bounds are less than $\lceil 4g/7 \rceil$, since $g' < g/2 < 4g/7$, $\lceil 4g/7 \rceil \geq 3$ for $g \geq 4$ and $\lceil 4g/7 \rceil \geq 4$ for $g \geq 6$. As $7L \leq 3n$, $\Upsilon \leq \lfloor 3g/7 \rfloor + 1$. If no element with index in Π were in S_q, then by Lemma B.7(ii) Π would lie in a set of at most Υ cyclically consecutive indices, and the at least $g - \Upsilon$ indices outside it would lie between two cyclically consecutive points of Π, which would have a gap at least $g - \Upsilon + 1 \geq g - \lfloor 3g/7 \rfloor = \lceil 4g/7 \rceil$. This is a contradiction. □

Lemma B.9. *Suppose that 3 divides r and n, that $9 \nmid n$, $3s < n$ and $3(L-1) < 2n$. Let a_0 be a residue with $\gcd(a_0, n/3) = 1$ and $\mathcal{X} = [a_0 r]_n$, $1 \leq \mathcal{X} < s$, $2\mathcal{X} \neq s$, and let $\mathcal{R}$ be its row for $g = 3$. If no element of $\mathcal{R}$ is a witness, then $\mathcal{R}$ contains exactly two units, and the smaller value $\mathcal{Y}$ of $[-aq]_n$ on them satisfies $1 \leq \mathcal{Y} \leq \mathcal{X} - 1$ and $\mathcal{Y} \leq L - n/3$, and $L\mathcal{X} - s\mathcal{Y}$ is a multiple of n with $n \leq L\mathcal{X} - s\mathcal{Y} \leq L\mathcal{X} - s$.*

Proof. As $3 \mid r$, every $a \in \mathcal{R}$ has $ar \equiv a_0 r$, so $[ar]_n = \mathcal{X} < s$ and $\mathcal{R} \subset S_r$. The divisor $g = 3$ satisfies the hypotheses of Lemma B.7 ($n/3$ is even if n is, and $\gcd(q, 3) = 1$ as 3 divides r and n), and $g^* = 3$ as $9 \nmid n$; by Lemma B.7(iii) exactly one element of $\mathcal{R}$ is divisible by 3, and the other two are units. They are usable by Lemma B.1, as $\mathcal{X} \neq s/2$. They are not witnesses, so they are not in S_q, and their values $\mathcal{Y} \leq \mathcal{Y}'$ lie in $[1, L]$ (Lemma B.3(a)) and differ by $\pm n/3$ modulo n (Lemma B.7(i)). As $\mathcal{Y}' - \mathcal{Y} \leq L - 1 < 2n/3$, $\mathcal{Y}' - \mathcal{Y} = n/3$, so $\mathcal{Y} \leq L - n/3 = 2p + s - n/3 < 2p$. By Lemma B.3(c) with $x = r$, $\tau = s$, $\tau' = 2p$, $\mathcal{Y} \in [1, \mathcal{X} - 1] \cup [\mathcal{X} + 2p, L]$, so $\mathcal{Y} \leq \mathcal{X} - 1$, and Lemma B.3(d) gives the rest. □

Lemma B.10. *Let $g \geq 3$ divide p and n, with n/g even if n is even; let $a_i = 1 + (n/g)i$ for $i \in \mathbb{Z}/g$, $t = [iq]_g$, and g^* as in Lemma B.7. Then $[a_i p]_n = p$, so $a_i \in S_p$; $a_i \in S_q$ if and only if $(n/g)t < q$ or $(n/g)t \geq n - q$; and a unit a_i is usable if and only if $g \nmid 2i$.*

(a) *If $n/g < q$ and g^* is 1 or a prime, some a_i is a usable unit in $S_p \cap S_q$.*

(b) *If $t_{\max} = \max\{t \geq 0 : (n/g)t < q\} \geq J(g^*)$, some a_i is a usable unit in $S_p \cap S_q$.*

Proof. $a_i p \equiv p \pmod{n}$ as $g \mid p$. By Lemma B.7, $\gcd(q, g) = 1$, so $i \mapsto t$ is a bijection of $\mathbb{Z}/g$, and $a_i q \equiv q + (n/g)t = v \in [q, n+q)$. If $v < n$, $[a_i q]_n = v < 2q$ if and only if $(n/g)t < q$; if $v \geq n$, $[a_i q]_n = v - n < q$. Next, $2a_i \equiv 2$ if and only if $n \mid (2n/g)i$, that is $g \mid 2i$, that is $g \mid 2t$. The row of 1 satisfies the hypotheses of Lemma B.7, so by its part (iii) a_i is a unit if and only if i, equivalently t, avoids one class modulo ℓ for each prime $\ell \mid g^*$, and that class is not 0, as $a_0 = 1$. (a) $t = 1$ and $t = g - 1$ both give elements of S_q, as $n/g < q$ and $(n/g)(g-1) = n - n/g > n - q$; they are $i = \pm q^{-1} \bmod g$, and $g \nmid 2i$ as $g \geq 3$ and $\gcd(i, g) = 1$. If $g^* = 1$ both are units; if $g^* = \ell$ is a prime, i and $-i$ lie in different classes modulo ℓ, as ℓ is odd and $\ell \nmid i$, so one of them avoids the excluded class. (b) $t = 1, \ldots, t_{\max}$

give elements of S_q; they are at least $J(g^*)$ consecutive values, so, as in Lemma B.7(iii), one of them gives a unit; and $0 < 2t < g$, because $(n/g)t < q < n/2$, so $g \nmid 2t$. □

B.2. **Even denominators.** In this subsection $n = 2m$. We keep the notation and the inequalities stated at the start of Section 4.3: $\Delta = (p, q, r)/n$ is a window triangle unless said otherwise, $\varsigma = s/2 = r - m$, $L = n - 2q = 2p + 2\varsigma$, $J = J(m^\circ)$, and $P(m) < P_2(m)$ are the two least odd primes not dividing m.

A unit modulo n is an odd a with $\gcd(a, m) = 1$; by Lemma B.1 it is usable unless $a \equiv 1 \pmod m$, that is unless $a = 1$ or (m even and $a = 1 + m$). For odd a, $am \equiv m \pmod n$, so $ar \equiv m + a\varsigma$ and

$$a \in S_r \iff [a\varsigma]_n \in [m, m + 2\varsigma). \tag{B.1}$$

Corollary B.11. *If $m \geq 74J^2$ and Δ has no witness, then $p \leq 7J^2 - 1$.*

Proof. By Lemma B.4, with $\epsilon = 2$, $pq \leq 4mJ^2$. Now $q = m - \varsigma - p \geq 2m/3 - p$, and $p \mapsto p(m - \varsigma - p)$ is increasing for $p \leq (m - \varsigma)/2$, which holds as $p \leq q$. So for $p \geq 7J^2$, $pq \geq 7J^2(2m/3 - 7J^2)$, which exceeds $4mJ^2$ when $m > 73.5J^2$. □

Lemma B.12. *If $2\varsigma(p + \varsigma) < m$, every odd $a \in S_r$ lies in $S_p \cup S_q$. If $2p(p + \varsigma) < m$, every odd $a \in S_p$ lies in $S_q \cup S_r$.*

Proof. Let a be odd in S_r, $c = [a\varsigma]_n - m \in [0, 2\varsigma)$ by (B.1), so that $[ar]_n = c$, and $x = [ap]_n$. Suppose $a \notin S_p \cup S_q$. By Lemma B.2 with $(u, v, w) = (r, p, q)$, either $2p \leq x \leq 2p + 2\varsigma - c$ or $n - c < x < n$. Multiplying $a\varsigma \equiv m + c$ by p and $ap \equiv x$ by ς gives $\varsigma x \equiv pm + pc \pmod{2m}$, so $\varsigma x - pc \equiv 0$ or $m \pmod{2m}$. In the first case $0 < p(2\varsigma - c) \leq \varsigma x - pc \leq \varsigma(2p + 2\varsigma) < m$, impossible. In the second $x = n - j$ with $1 \leq j \leq c - 1$, and $\varsigma x - pc \equiv -(j\varsigma + pc)$ with $0 < j\varsigma + pc < 2\varsigma(\varsigma + p) < m$, impossible.

For the second statement let a be odd, $x = [ap]_n < 2p$, $y = [ar]_n$, and suppose $a \notin S_q \cup S_r$. By Lemma B.2 with $(u, v, w) = (p, r, q)$, $2\varsigma \leq y \leq 2\varsigma + 2p - x$ or $n - x < y < n$. From $y \equiv m + a\varsigma$, $py - \varsigma x \equiv pm \pmod{2m}$, which is 0 or m modulo $2m$. In the first case $0 < \varsigma(2p - x) \leq py - \varsigma x \leq p(2\varsigma + 2p) < m$; in the second $y = n - j$ with $1 \leq j < x$ and $py - \varsigma x \equiv -(pj + \varsigma x)$ with $0 < pj + \varsigma x < 2p(p + \varsigma) < m$. Both are impossible. □

Proposition B.13. *Let $g_\varsigma = \gcd(\varsigma, m)$, $\varsigma = g_\varsigma\varsigma'$, $m = g_\varsigma m'$, and $C_0' = \{c' \in \mathbb{Z} : 0 \leq c' < 2\varsigma',\ c' \equiv m' + \varsigma' \pmod 2\}$.*

(a) *For each $c' \in C_0'$ there is exactly one odd residue $\beta(c')$ modulo $2m'$ with $\beta(c')\varsigma' \equiv m' + c' \pmod{2m'}$. The odd elements of S_r are exactly the a with $a \equiv \beta(c') \pmod{2m'}$ for some $c' \in C_0'$, and then $[a\varsigma]_n = m + g_\varsigma c'$. For such an a, $\gcd(a, m/g_\varsigma) = 1$ if and only if $\gcd(c', m') = 1$; so a is a unit if and only if $\gcd(c', m') = 1$ and $\gcd(a, g_\varsigma) = 1$; and $a \equiv 1 \pmod m$ is possible only for $c' = \varsigma'$.*

(b) *S_r contains no usable unit if and only if $\varsigma \mid m$ and m/ς is odd, or $\varsigma = 1$ and m is even.*

Proof. (a) By (B.1) an odd a is in S_r if and only if $a\varsigma \equiv m + c \pmod{2m}$ with $0 \leq c < 2\varsigma$. Reducing modulo g_ς gives $g_\varsigma \mid c$, and with $c = g_\varsigma c'$ the congruence is $a\varsigma' \equiv m' + c' \pmod{2m'}$, $0 \leq c' < 2\varsigma'$, where $\gcd(\varsigma', m') = 1$. If an odd a solves it, then $c' \equiv m' + \varsigma' \pmod 2$. Conversely let $c' \equiv m' + \varsigma' \pmod 2$. If m' is odd, $a \equiv c'\varsigma'^{-1} \pmod{m'}$ has exactly one odd lift modulo $2m'$, and the congruence holds modulo m' and modulo 2. If m' is even, ς' is odd and a unit modulo $2m'$, the only solution is $(m' + c')\varsigma'^{-1}$, and it is odd as $m' + c'$ is. For a prime $\ell \mid m'$, $\ell \mid a$ if and only if $\ell \mid m' + c'$, that is $\ell \mid c'$ (for $\ell = 2$ both are false, c' being odd); the primes of g_ς not dividing m' are accounted for by $\gcd(a, g_\varsigma) = 1$.

Finally $a \equiv 1 \pmod m$ gives $g_\varsigma c' \equiv a\varsigma \equiv \varsigma = g_\varsigma \varsigma' \pmod{g_\varsigma m'}$, so $c' \equiv \varsigma' \pmod{m'}$, and $|c' - \varsigma'| \le \varsigma' < m'$.

(b) $\varsigma' = 1$, m' *odd.* $C_0' = \{0\}$ and $\gcd(0, m') = m' \ge 3$: no unit at all. $\varsigma' = 1$, m' *even.* $C_0' = \{1\}$ and $\beta(1) = m' + 1$, so the odd elements of S_r are the $1 + m'j$ with j odd modulo $2g_\varsigma$. If $g_\varsigma = 1$ the only one is $1 + m$, which is not usable. If $g_\varsigma \ge 2$, choose an odd j with $m'j \equiv 1 \pmod \ell$ for every odd prime $\ell \mid g_\varsigma$ with $\ell \nmid m'$ ($j = 1$ if there is none). Then $\gcd(1 + m'j, g_\varsigma) = 1$ ($1 + m'j$ is $\equiv 1 \pmod \ell$ if $\ell \mid m'$ and $\equiv 2$ otherwise, and $2 \mid g_\varsigma$ implies $2 \mid m'$ here), so $a = 1 + m'j$ is a unit; and $a \equiv 1 \pmod m$ would need $g_\varsigma \mid j$, which fails (if some $\ell \mid g_\varsigma$ has $\ell \nmid m'$, then $\ell \nmid j$; otherwise $j = 1 < g_\varsigma$). So a is a usable unit. $\varsigma' \ge 2$. Set $c' = 1$ if $m' + \varsigma'$ is odd and $c' = 2$ otherwise (then m', ς' are both odd, being coprime). In both cases $c' \in C_0'$, $\gcd(c', m') = 1$ and $c' \ne \varsigma'$. Among the g_ς lifts $a = \beta(c') + 2m'i$ choose i with $\ell \nmid a$ for every prime $\ell \mid g_\varsigma$ with $\ell \nmid m'$ (one class modulo the odd prime ℓ to avoid); for $\ell \mid g_\varsigma$ with $\ell \mid m'$, $\ell \nmid a$ already. By (a), a is a unit and $a \not\equiv 1 \pmod m$. □

The case $\varsigma' = 1$, m' odd of (b) is Proposition 4.3(a). The corresponding statement for S_p is [17, Proposition 3.5].

Lemma B.14. *Let* $g = \gcd(p, m)$, $p = gp''$, $m = gm''$, *and* $C_0'' = \{x \in \mathbb{Z} : 0 \le x < 2p'',\ x \equiv p'' \pmod 2\}$. *For each* $x \in C_0''$ *there is exactly one odd residue* $\gamma(x)$ *modulo* $2m''$ *with* $\gamma(x)p'' \equiv x \pmod{2m''}$. *The odd elements of* S_p *are exactly the* a *with* $a \equiv \gamma(x) \pmod{2m''}$ *for some* $x \in C_0''$, *and then* $[ap]_n = gx$. *Such an* a *is a unit if and only if* $\gcd(x, m'') = 1$ *and* $\gcd(a, g) = 1$; *and* $a \equiv 1 \pmod m$ *is possible only for* $x = p''$.

Proof. An odd a is in S_p if and only if $ap \equiv \mathcal{X} \pmod{2m}$ with $0 \le \mathcal{X} < 2p$. Reducing modulo g gives $g \mid \mathcal{X}$, and with $\mathcal{X} = gx$ the congruence is $ap'' \equiv x \pmod{2m''}$, $0 \le x < 2p''$, where $\gcd(p'', m'') = 1$. If an odd a solves it, then $x \equiv p'' \pmod 2$. Conversely let $x \equiv p'' \pmod 2$. If m'' is odd, $a \equiv xp''^{-1} \pmod{m''}$ has exactly one odd lift modulo $2m''$, and the congruence holds modulo m'' and modulo 2. If m'' is even, p'' is odd and a unit modulo $2m''$, the only solution is xp''^{-1}, and it is odd as x is. For a prime $\ell \mid m''$, $\ell \mid a$ if and only if $\ell \mid x$ (for $\ell = 2$ both are false); the primes of g not dividing m'' are accounted for by $\gcd(a, g) = 1$. Finally $a \equiv 1 \pmod m$ gives $gx \equiv ap \equiv p = gp'' \pmod{gm''}$, so $x \equiv p'' \pmod{m''}$, and $|x - p''| \le p'' < m''$ as $p < m/2$. □

A *row* of S_p is the set of the g residues $a \equiv \gamma(x) \pmod{2m''}$ for one $x \in C_0''$, and a row of S_r the set of the g_ς residues $a \equiv \beta(c') \pmod{2m'}$ for one $c' \in C_0'$. In both cases a row is the row $\{a_0 + (n/g)i : i \in \mathbb{Z}/g\}$ of Lemma B.7, with $g = \gcd(p, m)$, respectively $g = g_\varsigma$, and g satisfies its hypotheses: $g \mid m$, so n/g is even, and g divides n and one of p, r ($r = m + \varsigma$), so $\gcd(q, g) = 1$. As a_0 is odd, $\gcd(a_0, n/g) = 1$ means $\gcd(a_0, m/g) = 1$: for a row of S_r this holds if and only if $\gcd(c', m') = 1$, and then $\nu = 0$ unless $c' = \varsigma'$; for a row of S_p, if and only if $\gcd(x, m'') = 1$, and then $\nu = 0$ unless $x = p''$ (Proposition B.13(a), Lemma B.14).

Lemma B.15. *Let* $\gcd(\varsigma, m) = 1$, r *odd,* $7L \le 3n$ *and* $\varsigma \ge 24J^2 + 1$. *Then* Δ *has a witness in* S_r.

Proof. The argument is that of [17, Lemma 7.9], with q and r exchanged. Let $M = 2J$. Any M consecutive terms of a progression $c + ik$ with $\gcd(c, k, n) = 1$ contain a unit of n: if k is even, all terms are odd and J consecutive ones suffice; if k is odd, the odd terms among M consecutive ones are J consecutive terms of a progression with difference $2k$. As $\gcd(r, n) = 1$, let $w = [-qr^{-1}]_n$ and, for an integer $\mathcal{X}$, $a_\mathcal{X} = \mathcal{X}r^{-1} \bmod n$; then $[a_\mathcal{X} r]_n = [\mathcal{X}]_n$ and $[-a_\mathcal{X} q]_n = [\mathcal{X}w]_n$. For $1 \le \mathcal{X} < \varsigma$ with $\gcd(\mathcal{X}, n) = 1$, $a_\mathcal{X}$ is a unit of S_r, usable by Lemma B.1 (as $\mathcal{X} < \varsigma = s/2$), and it lies in S_q if and only if $[\mathcal{X}w]_n \notin [1, L]$.

Suppose a_1 is not a witness. Then $w \in [1, L]$, and by Lemma B.3(c), $w \geq 1 + 2p \geq 3$. Let $\alpha = w/n \leq L/n \leq 3/7$ and $T = 6M$. By Dirichlet's theorem there are coprime integers h, k with $1 \leq k \leq T$ and $|k\alpha - h| < 1/T$.

$k = 1$. $h = 1$ would give $\alpha > 1 - 1/T > 3/7$; so $h = 0$ and $w < n/T$. For $L/w < \mathcal{X} < n/w$ we have $L < \mathcal{X}w < n$, so $a_{\mathcal{X}} \in S_q$. The interval $(L/w, \min(n/w, \varsigma))$ has length more than M: $(n-L)/w > (4/7)T > M$, and $\varsigma - L/w \geq \varsigma - (2p+2\varsigma)/(2p+1) \geq \varsigma - (2\varsigma+2)/3 = (\varsigma-2)/3 > M$. So it contains M consecutive integers, one of them a unit $\mathcal{X}$ of n with $1 \leq \mathcal{X} < \varsigma$, and $a_{\mathcal{X}}$ is a witness.

$k = 2$. Then $h = 1$ and $\alpha > 1/2 - 1/(2T) > 3/7$, which is impossible.

$k \geq 3$. Choose an integer e prime to k: if $k\alpha < h$, $e = k - 1$; if $k\alpha \geq h$, $e = 2, 3, 3, 5$ for $k = 3, 4, 5, 6$, and for $k \geq 7$, $e = (k+1)/2$ (k odd), $k/2+1$ ($4 \mid k$), $k/2+2$ ($k \equiv 2 \bmod 4$). The choices of e are prime to k: $\gcd((k+1)/2, k) = 1$; $\gcd(k/2+1, k)$ divides $\gcd(k/2+1, 2) = 1$ for $4 \mid k$; $\gcd(k/2+2, k)$ divides $\gcd(k/2+2, 4) = 1$ for $k \equiv 2 \pmod 4$; and $e/k \leq 1/2+2/7 < 5/6$ for $k \geq 7$. In the first case $e/k \geq 2/3 \geq 3/7+1/6$; in the second $3/7 < 1/2 < e/k \leq 5/6$. Let $c \in [1, k-1]$ with $ch \equiv e \pmod k$, so $\gcd(c, k) = 1$, and $\mathcal{X}_i = c + ik$ for $0 \leq i \leq M-1$. Then $1 \leq \mathcal{X}_i < Mk \leq 6M^2 \leq \varsigma - 1$, and $\mathcal{X}_i\alpha = \mathcal{X}_i h/k + \mathcal{X}_i(k\alpha - h)/k \equiv e/k + \mathcal{X}_i(k\alpha - h)/k \pmod 1$ with $|\mathcal{X}_i(k\alpha - h)/k| < M/T = 1/6$. So the fractional part of $\mathcal{X}_i\alpha$ is $e/k + \mathcal{X}_i(k\alpha - h)/k \in (3/7, 1)$ in both cases, that is $L < [\mathcal{X}_i w]_n < n$, and $a_{\mathcal{X}_i} \in S_q$. One of the $\mathcal{X}_i$ is a unit of n, and $a_{\mathcal{X}_i}$ is a witness. □

Lemma B.16. *Let m and ς be odd, $\gcd(\varsigma, m) = 1$, $\varsigma \geq 9$ and $8L \leq 7m$. Then $S_q \cap S_r$ contains a usable unit.*

Proof. $r = m + \varsigma$ and $p + q = m - \varsigma$ are even, so p and q are odd. For an odd residue a let $a^* = 2a + m \bmod n$; it is odd, $\gcd(a^*, m) = \gcd(a, m)$, $[a^*r]_n = [2ar]_n$ as r is even, and $[-a^*q]_n = [2(-aq) + m]_n$ as q is odd. As m is odd, every unit other than 1 is usable, and $1 \notin S_r$. Let a_0 be the odd residue with $a_0\varsigma \equiv m + 2 \pmod{2m}$ (Proposition B.13(a) with $g_\varsigma = 1$, $c' = 2$): a unit with $[a_0 r]_n = 2$. Set $a_{j+1} = a_j^*$; then $[a_j r]_n = 2^{j+1} < s$ for $j \leq 3$, so $a_0, \dots, a_3$ are units of S_r. Suppose none is in S_q: $\mathcal{Y}_j = [-a_j q]_n \in [1, L]$, with $L < m$. If $\mathcal{Y}_j < m/2$, then $\mathcal{Y}_{j+1} = 2\mathcal{Y}_j + m > L$; so $\mathcal{Y}_j > m/2$ and $\mathcal{Y}_{j+1} = 2\mathcal{Y}_j - m$ for $j = 0, 1, 2$. Then $\mathcal{Y}_1 > m/2$ gives $\mathcal{Y}_0 > 3m/4$, and $\mathcal{Y}_2 = 4\mathcal{Y}_0 - 3m > m/2$ gives $\mathcal{Y}_0 > 7m/8 \geq L$, a contradiction. □

Lemma B.17. *Let $\gcd(\varsigma, m) = 3$, $9 \nmid m$, $3L > n$, $\varsigma \geq 9$, $\varsigma > 6p$ and $18p < m$. Then Δ has a witness.*

Proof. Write $m = 3m'$, $\varsigma = 3\varsigma'$ with $3 \nmid m'$, $\gcd(\varsigma', m') = 1$, $\varsigma' \geq 3$, and $f = m - 3\varsigma$. Here $3 \mid f$, and $f > 0$, since $\varsigma = m/3$ would give $\gcd(\varsigma, m) = m/3 = 3$ and $\varsigma = 3$; so $3s < n$. Let $c' = 1$ if $m' + \varsigma'$ is odd and $c' = 2$ otherwise (then m' is odd); $c' \in C_0'$ and $\gcd(c', m') = 1$. Its row is the row of an odd a_0 with $\gcd(a_0, n/3) = 1$ and $\mathcal{X} = [a_0 r]_n = 3c' \in \{3, 6\}$ (Proposition B.13(a) and (B.1)); $\mathcal{X} < s$ and $2\mathcal{X} \neq s$, as $\varsigma \geq 9$. Also $3 \mid r = m + \varsigma$, $9 \nmid n$, and $L < 2p + 2m/3 < 7m/9$, so $3(L-1) < 2n$. Suppose no element of the row is a witness. By Lemma B.9, $L\mathcal{X} - s\mathcal{Y}$ is a positive multiple of n, at most $L\mathcal{X} - s$, for an integer $\mathcal{Y} \geq 1$. For $\mathcal{X} = 3$, $L\mathcal{X} - s = 6p + 4\varsigma < 2m$, because $2\varsigma \leq 2m/3$ and $3p < m/3$; a contradiction. For $\mathcal{X} = 6$, $L\mathcal{X} - s = 12p + 10\varsigma < 4m$, so $L\mathcal{X} - s\mathcal{Y} = n$ and $\mathcal{Y} = (6L - n)/s = (6p + 6\varsigma - m)/\varsigma = 3 + (6p - f)/\varsigma$. But $3L > n$ means $f < 3p$, so $0 < 6p - f < 6p < \varsigma$, and $\mathcal{Y}$ is not an integer; a contradiction. □

Proof of Theorem 4.14. As $P_2 \geq 5$, $m > 155J^2 - 5 \geq 150$; and $J = 1$ only for m a power of 2, when $m \geq 256$. Suppose Δ has no witness.

Step 1: p is small. $m \geq 74J^2$, so $p \leq 7J^2 - 1$ by Corollary B.11. Hence $21p < m$, and $7L = 14p + 14\varsigma \leq 14p + 14m/3 < 6m$: so $7L \leq 3n$.

Step 2: S_r has no usable unit. Suppose it has one, and let $g_\varsigma = \gcd(\varsigma, m)$, $\varsigma = g_\varsigma \varsigma'$, $m = g_\varsigma m'$. By Proposition B.13(b) we are not in the cases ($\varsigma' = 1$, m' odd) and ($\varsigma = 1$, m even).

(2a) $\varsigma' = 1$. Then m' is even, $g_\varsigma = \varsigma \geq 2$, and the only row of S_r is $c' = 1$, with $\nu \leq 1$, as $1 \notin S_r$. If $\varsigma \leq 8$, then $2\varsigma(p + \varsigma) \leq 16(7J^2 + 7) < m$ (for $J \geq 2$, $112J^2 + 112 < 155J^2 - 5$; for $J = 1$, $p \leq 6$ and $224 < 256$), and by Lemma B.12 the usable unit of S_r is a witness. If $\varsigma \geq 9$, Lemma B.8 ($g = g_\varsigma = \varsigma$, the row $c' = 1$, $\nu \leq 1$) gives a witness.

(2b) $\varsigma' \geq 2$ and $g_\varsigma \geq 2$. Take the row $c' = 1$ if $m' + \varsigma'$ is odd and $c' = 2$ otherwise: $\gcd(c', m') = 1$ and $c' \neq \varsigma'$, so $\nu = 0$. If $g_\varsigma \geq 4$, Lemma B.8 gives a witness. If $g_\varsigma = 2$: $\Upsilon = 1$ as $(L - 1)g_\varsigma/n < 6/7$, and $g_\varsigma - \Upsilon = 1 = J(1)$. If $g_\varsigma = 3$ and $9 \mid m$: $g_\varsigma^* = 1$, $\Upsilon \leq 2$ and $g_\varsigma - \Upsilon \geq 1$. In these two cases Lemma B.7(iv) gives a witness. If $g_\varsigma = 3$ and $9 \nmid m$: if $3L \leq n$, then $\Upsilon = 1$ and $g_\varsigma - \Upsilon = 2 = J(3)$, and Lemma B.7(iv) gives a witness; if $3L > n$, then $\varsigma > m/3 - p > 6p$ (as $21p < m$), so $\varsigma \geq 9$, and $18p < m$, and Lemma B.17 gives a witness.

(2c) $g_\varsigma = 1$ and m, ς both odd; then $\varsigma \geq 3$. If $\varsigma \geq 9$, $8L \leq 7m$ follows from $7L \leq 3n$, and Lemma B.16 gives a witness. If $\varsigma \leq 7$, then $2\varsigma(p + \varsigma) \leq 14(7J^2 + 6) < m$ ($J \geq 2$ as m is odd), and Lemma B.12 gives one.

(2d) $g_\varsigma = 1$ and r odd, the remaining case; here $\varsigma \geq 2$, and $\gcd(r, n) = 1$. By the last sentence of Corollary B.6, which applies by its part (a), $P_2 L \geq n$; and by Lemma B.15, $\varsigma \leq 24J^2$. So $n \leq P_2(2p + 2\varsigma) \leq P_2(62J^2 - 2) = 2N_0(m)$, that is $m \leq N_0(m)$, a contradiction.

Step 3. So S_r has no usable unit, and by Proposition B.13(b): (i) $\varsigma \mid m$ with $d = m/\varsigma$ odd, or (ii) $\varsigma = 1$ and m is even. In case (i) with $p = 1$, $\Delta \in \mathcal{A}$; in case (i) with $2 \leq p \leq \varsigma$, it lies in $\mathcal{B}$ by Proposition 4.3(c); in case (ii) with $p \in \{1, 2\}$, or $p = 4$ and $m \equiv 2 \pmod 4$, $\Delta \in \mathcal{C}$. In all other cases $p > \varsigma$, and we show that there is a witness in $S_p \cap S_q$. Then $L = 2p + 2\varsigma \leq 4p - 2 \leq 28J^2 - 6$, so $10L < n$ and $q = m - p - \varsigma > 2m/3$. The case $p = 2$ does not occur: in (i), $\varsigma = 1 < p$ makes m odd and p, $q = m - 3$, $r = m + 1$ all even; in (ii), $p = 2$ is in $\mathcal{C}$. So $p \geq 3$. Let $g = \gcd(p, m)$, $p = gp''$, $m = gm''$.

(3a) $g = 1$. Then p is odd: if p were even, m would be odd, which excludes (ii), and in (i) ς, a divisor of the odd m, would be odd, so $r = m + \varsigma$ and $q = m - \varsigma - p$ would be even together with p. So $\gcd(p, n) = 1$. Now $P_2 L \leq P_2(28J^2 - 6) < P_2(62J^2 - 2) = 2N_0(m) < n$, against the last sentence of Corollary B.6, which applies by its part (b).

(3b) $g \geq 2$ and $p'' \geq 3$. Take the row $x = 1$ if p'' is odd and $x = 2$ if p'' is even (then m'' is odd): $\gcd(x, m'') = 1$ and $x \neq p''$, so $\nu = 0$. If $g \geq 4$, Lemma B.8 gives a usable unit of S_p in S_q. If $g \in \{2, 3\}$, $\Upsilon = 1$ as $10L < n$, and $g - \Upsilon = g - 1 \geq J(g^*)$; Lemma B.7(iv) gives one.

(3c) $g \geq 3$ and $p'' \leq 2$. Then $n/g \leq 2m/3 < q$, and g divides p and n with n/g even. If g^* is 1 or a prime, Lemma B.10(a) gives a witness. If g^* has $k \geq 2$ primes, then $g \geq 15 \cdot 7^{k-2}$ and $t_{\max} \geq \lfloor g/3 \rfloor \geq 5 \cdot 7^{k-2} \geq J(g^*)$ ($J(g^*) \leq 3$ for $k = 2$ and $\leq 3^k$ for $k \geq 3$, by Lemma 4.13), and Lemma B.10(b) gives one.

(3d) $g = 2$ and $p'' \leq 2$. Then $p = 4$, and $m'' = m/2$ is odd, so $m \equiv 2 \pmod 4$. In case (ii) this is $\mathcal{C}$. In case (i), m is even and m/ς odd, so ς is even, and p, $q = m - p - \varsigma$, $r = m + \varsigma$ are all even: impossible. □

Appendix C. Odd denominators

This appendix proves Theorem 4.1(a) for odd n (Theorem C.3), with the lemmas of Appendix B.1. As in Section 4, $\Delta = (p, q, r)/n$ is a window triangle, $n/2 < r \leq 2n/3$, unless said otherwise, and $s = 2r - n$, so that $1 \leq s \leq n/3$.

Let n be odd, $n \geq 5$. Then s is odd, $L = n-2q = 2p+s$, $\gcd(s,n) = \gcd(2r,n) = \gcd(r,n)$, and the J of Appendix B.1 is $J = J(n)$, with J Jacobsthal's function as defined in Section 4.3. A unit is a unit modulo n; by Lemma B.1 every unit other than 1 is usable, and $1 \notin S_r$. For $x \in \{p,r\}$ and $g = \gcd(x,n)$, the residues a with $ax \equiv g \pmod{n}$ form the row (Lemma B.7) of a_0, the inverse of x/g modulo n/g, so that $\gcd(a_0, n/g) = 1$; they all have $[ax]_n = g$. The divisor g satisfies the hypotheses of Lemma B.7.

Corollary C.1. *If Δ has no witness and $5n > 324J^2$, then $p < n/18$.*

Proof. By Lemma B.4, with $\epsilon = 1$, $pq \leq nJ^2$. Suppose $p \geq n/18$. As $p \leq q$ and $p+q = (n-s)/2 \geq n/3$ (because $s \leq n/3$), $q \geq \max(p, n/3-p)$. For $n/18 \leq p \leq n/6$ the concave function $p(n/3-p)$ is at least $\min(5n^2/324, n^2/36) = 5n^2/324$, and for $p > n/6$, $pq \geq p^2 > n^2/36$. So $nJ^2 \geq 5n^2/324$, which contradicts $5n > 324J^2$. □

Theorem C.2. *Let n be odd with $5n > 324J(n)^2$. If a window triangle $(p,q,r)/n$ has no witness, then $s \mid n$; if moreover $p \geq 2$, then $2p \leq s$ and Δ lies in $\mathcal{B}$.*

Proof. Suppose Δ has no witness. By Corollary C.1, $p < n/18$, so $L = 2p+s < n/9+n/3 < n/2$.

s divides n. Suppose not. Then $s \geq 3$ (as s is odd), $s \neq n/3$, and $g = \gcd(s,n) = \gcd(r,n) < s$ is odd. If $g = 1$, Corollary B.6(a) with $u = 2$, a prime not dividing n with $2L < n$, would give $4 \mid n$. If $g = 3$ and $9 \nmid n$, apply Lemma B.9 to the row of the a with $ar \equiv 3$: 3 divides r and n, $3s < n$, $3(L-1) < 2n$, and $\mathcal{X} = 3 < s$ with $2\mathcal{X} \neq s$; it gives $n \leq 3L - s = 6p + 2s < n/3 + 2n/3 = n$, which is false. Otherwise take the row of the a with $ar \equiv g$: its elements have $[ar]_n = g < s$, so they lie in S_r, and 1 is not among them, as $[1 \cdot r]_n = r \neq g$; so $\nu = 0$. As $(L-1)g/n < g/2$, $\Upsilon \leq (g+1)/2$ and $g - \Upsilon \geq (g-1)/2$; and $(g-1)/2 \geq J(g^*)$: if $g^* = 1$, as $g \geq 3$; if g^* is a prime, then $g \geq 5$ (since $g = 3$ with $g^* = 3$ means $9 \nmid n$) and $J(g^*) = 2$; if g^* has two primes, $g \geq g^* \geq 15$, so $(g-1)/2 \geq 7$, and $J(g^*) \leq 3$; if g^* has $k \geq 3$ primes, $g \geq g^* \geq 3 \cdot 5 \cdot 7 \cdot 11^{k-3}$ and $(g-1)/2 \geq 52 \cdot 11^{k-3} \geq 3^k \geq J(g^*)$ (Lemma 4.13). So Lemma B.7(iv) gives a usable unit in $S_r \cap S_q$, a witness. In each case this is a contradiction.

If $p \geq 2$, then $2p \leq s$. Suppose $s \mid n$, $p \geq 2$ and $2p > s$. Then $L < 4p < 2n/9$, so $3L < n$. Let $g = \gcd(p,n)$. If $g = 1$, Corollary B.6(b) with $u = 2$ would give $3L \geq n$. If $3 \leq g < p$, take the row of the a with $ap \equiv g$: its elements lie in S_p, as $g < 2p$, and 1 is not among them, as $[1 \cdot p]_n = p \neq g$; so $\nu = 0$. If $g \geq 5$, Lemma B.8 applies, as $7L < 3n$; if $g = 3$, then $\Upsilon = 1$ as $3L < n$, and $g - \Upsilon = 2 \geq J(g^*)$, so Lemma B.7(iv) applies. Either gives a usable unit in $S_p \cap S_q$, a witness. If $g = p$, so $p \mid n$ and $p \geq 3$: $q = (n-s)/2 - p > n/2 - 2p > 7n/18 > n/3 \geq n/p$ (as $s < 2p < n/9$), and $t_{\max} = \lceil qp/n \rceil - 1 > 7p/18 - 1$ in Lemma B.10 with $g = p$. If g^* has at most one prime, part (a) of that lemma applies; if two, $p \geq 15$ and $t_{\max} \geq 5 \geq J(g^*)$; if $k \geq 3$, $p \geq 105 \cdot 11^{k-3}$ and $t_{\max} > 39 \cdot 11^{k-3} \geq 3^k \geq J(g^*)$; in these two cases part (b) applies. It gives a witness. In each case this is a contradiction.

The unit condition. As n is odd, n/s is odd; with $s \mid n$ and $2 \leq p \leq s/2$, Proposition 4.3(c) gives (U), and Δ lies in $\mathcal{B}$. □

For $p = 1$ the theorem says that a window triangle $(1,q,r)/n$ without a witness has $s \mid n$, that is, lies in $\mathcal{A}$.

Theorem C.3. *Let n be odd. A window triangle of denominator n has no witness if and only if it lies in $\mathcal{A}$ or in $\mathcal{B}$, or is one of* $(2,3,6)/11$, $(2,5,8)/15$, $(3,4,8)/15$, $(2,7,12)/21$, $(3,7,11)/21$.

Proof. Only if. If $5n > 324J(n)^2$, this is Theorem C.2. The odd $n \geq 5$ with $5n \leq 324J(n)^2$ are 320 values, the largest $3{,}135 = 3 \cdot 5 \cdot 11 \cdot 19$, with $J = 7$ ($n = 3$ also satisfies the inequality and has no window triangle); for them an enumeration of all window triangles finds no witness-free triangle other than those stated (Appendix A.3; the list of 320 values is proved complete in Appendix A.2). *If.* Members of $\mathcal{A}$ and of $\mathcal{B}$ have no witness by Proposition 4.3. The five single triangles have no witness by a direct check of the units (Appendix A). □

For prime n, Theorem C.3 gives the prime case stated after Theorem 4.1: $s \mid n$ forces $s = 1$, which excludes $\mathcal{B}$, and the only single triangle of prime denominator is $(2,3,6)/11$.

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Fable 5.2 in Claude Code and OpenAI's GPT-6 Astra in Codex. The author chooses the problems and prioritizes the work. Separate AI agents checked each result, and a second program confirmed the computations the proofs use, with any exceptions listed in the public deposit. The system also searched the literature and checked the citations against reliable sources. The author read the paper, reran the computations from the deposit and takes full responsibility for all of the paper's contents.

Data availability. The programs and data used in this paper, with instructions for rerunning them, are deposited at Zenodo [13].

References

1. David Kurniadi Angdinata, Evan Chen, Ken Ono, Jesse Thorner, Jiaxin Zhang, and Jujian Zhang, *On the paucity of lattice triangles*, 2026, Preprint, arXiv:2603.23928v2.
2. Paul Apisa, *Billiards in right triangles and orbit closures in genus zero strata*, 2021, Preprint, arXiv:2110.07540v1.
3. Paul Apisa, Rafael M. Saavedra, and Christopher Zhang, *Periodic points on the regular and double n-gon surfaces*, Geom. Dedicata **216** (2022), Paper No. 69, doi:10.1007/s10711-022-00730-6; arXiv:2011.02668v1.
4. Irene I. Bouw and Martin Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Ann. of Math. (2) **172** (2010), no. 1, 139–185, doi:10.4007/annals.2010.172.139.
5. Vincent Delecroix, W. Patrick Hooper, and Julian Rüth, *sage-flatsurf, version 0.8.0*, 2025, Zenodo, doi:10.5281/zenodo.17650831.
6. Vincent Delecroix and Julian Rüth, *A new orbit closure in genus 8?*, Exp. Math. **34** (2025), no. 2, 254–259, doi:10.1080/10586458.2024.2361608; arXiv:2110.05407v2.
7. Inbo Gottlieb Fenves, *Completely parabolic surfaces are Veech*, 2026, Preprint, arXiv:2610.20566v1.
8. Simion Filip, *Splitting mixed Hodge structures over affine invariant manifolds*, Ann. of Math. (2) **183** (2016), no. 2, 681–713, doi:10.4007/annals.2016.183.2.5; arXiv:1311.2350v3.
9. Israel M. Gel'fand, Mark I. Graev, and Vladimir S. Retakh, *Hypergeometric functions over an arbitrary field*, Russian Math. Surveys **59** (2004), no. 5, 831–905, doi:10.1070/RM2004v059n05ABEH000771.
10. Israel M. Gel'fand, Mark I. Graev, and Naum Ya. Vilenkin, *Generalized functions. Vol. 5: Integral geometry and representation theory*, Academic Press, New York, 1966, doi:10.1016/C2013-0-12222-2.
11. W. Patrick Hooper, *Another Veech triangle*, Proc. Amer. Math. Soc. **141** (2013), no. 3, 857–865, doi:10.1090/S0002-9939-2012-11379-6; arXiv:1107.5054v1.
12. Pascal Hubert and Thomas A. Schmidt, *Veech groups and polygonal coverings*, J. Geom. Phys. **35** (2000), no. 1, 75–91, doi:10.1016/S0393-0440(99)00080-7.
13. Manvir Jaswal, *Programs and data for "Obtuse rational lattice triangles"*, Zenodo deposit, 2026, doi:10.5281/zenodo.24223587.

14. Jeremy Kahn and Alex Wright, *Hodge and Teichmüller*, J. Mod. Dyn. **18** (2022), 149–160, doi:10.3934/jmd.2022007; arXiv:2106.04665v1.
15. Richard Kenyon and John Smillie, *Billiards on rational-angled triangles*, Comment. Math. Helv. **75** (2000), no. 1, 65–108, doi:10.1007/s000140050113.
16. Erwan Lanneau and Martin Möller, *Non-existence and finiteness results for Teichmüller curves in Prym loci*, Exp. Math. **31** (2022), 621–636, doi:10.1080/10586458.2019.1671920; arXiv:1704.03210v2.
17. Anne Larsen, Chaya Norton, and Bradley Zykoski, *Strongly obtuse rational lattice triangles*, Trans. Amer. Math. Soc. **374** (2021), no. 10, 7119–7142, doi:10.1090/tran/8415.
18. Samuel Lelièvre, *Veech surfaces associated with rational billiards*, 2002, Preprint, arXiv:math/0205249v1.
19. Curtis T. McMullen, *Prym varieties and Teichmüller curves*, Duke Math. J. **133** (2006), no. 3, 569–590, doi:10.1215/S0012-7094-06-13335-5.
20. ———, *Teichmüller curves in genus two: torsion divisors and ratios of sines*, Invent. Math. **165** (2006), no. 3, 651–672, doi:10.1007/s00222-006-0511-2.
21. ———, *Billiards and Teichmüller curves*, Bull. Amer. Math. Soc. (N.S.) **60** (2023), no. 2, 195–250, doi:10.1090/bull/1782.
22. Maryam Mirzakhani and Alex Wright, *Full-rank affine invariant submanifolds*, Duke Math. J. **167** (2018), no. 1, 1–40, doi:10.1215/00127094-2017-0036; arXiv:1608.02147v2.
23. Martin Möller, *Periodic points on Veech surfaces and the Mordell–Weil group over a Teichmüller curve*, Invent. Math. **165** (2006), no. 3, 633–649, doi:10.1007/s00222-006-0510-3.
24. ———, *Variations of Hodge structures of a Teichmüller curve*, J. Amer. Math. Soc. **19** (2006), no. 2, 327–344, doi:10.1090/S0894-0347-05-00512-6; arXiv:math/0401290v2.
25. Yury A. Neretin, *Barnes–Ismagilov integrals and hypergeometric functions of the complex field*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 072, doi:10.3842/SIGMA.2020.072.
26. Jan-Christoph Puchta, *On triangular billiards*, Comment. Math. Helv. **76** (2001), no. 3, 501–505, doi:10.1007/PL00013215.
27. Jan-Christoph Schlage-Puchta, *Addendum to "On triangular Billard"*, 2021, Preprint, arXiv:2107.14025v1.
28. William A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), no. 3, 553–583, doi:10.1007/BF01388890; Erratum: Invent. Math. 103 (1991), 447, doi:10.1007/BF01239521.
29. Yaroslav B. Vorobets, *Planar structures and billiards in rational polygons*, Russian Math. Surveys **51** (1996), no. 1, 177–178, doi:10.1070/RM1996v051n01ABEH002769.
30. ———, *Planar structures and billiards in rational polygons: the Veech alternative*, Russian Math. Surveys **51** (1996), no. 5, 779–817, doi:10.1070/RM1996v051n05ABEH002993.
31. Clayton C. Ward, *Calculation of Fuchsian groups associated to billiards in a rational triangle*, Ergodic Theory Dynam. Systems **18** (1998), no. 4, 1019–1042, doi:10.1017/S0143385798117479.
32. Alexander N. Zemljakov and Anatole B. Katok, *Topological transitivity of billiards in polygons*, Math. Notes **18** (1975), no. 2, 760–764, doi:10.1007/BF01818045.

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