The Osserman (1990) conjecture is fully resolved, and other miscellanea

Dave Goel
david44hello@gmail.com

October 28, 2026

Manuscript version 3.60, 2026-10-28.

hexagon:2611.00546v1

Abstract

The Osserman conjecture says that a Riemannian manifold whose Jacobi operators have the same eigenvalues in every unit direction is flat or locally rank-one symmetric. Chi and Nikolayevsky proved it in every dimension except one remaining case: dimension sixteen, when a Jacobi eigenvalue has multiplicity seven or eight. Nikolayevsky reduced this remaining case to a purely algebraic statement, the Nikolayevsky (2009) conjecture, his Conjecture A. We prove the Nikolayevsky (2009) conjecture. Hence the Osserman conjecture is true in every dimension, including its pointwise form in dimension sixteen. The same result completes the conformal Osserman conjecture, which is stated for dimensions greater than four.

Conjecture A says that an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ whose reduced Jacobi operator has an eigenvalue of multiplicity seven or eight has a Clifford structure or is a linear combination of the constant-curvature tensor and an orthogonal pullback of the Cayley-plane curvature tensor. For the two-eigenvalue case, we construct a Cartan–Münzner quartic and recover the original Jacobi eigenspaces from its isoparametric classification. For the remaining multiplicity patterns, we study a rational coherent factor of the Jacobi family. Its transpose pencil, polar divisors and residue images reduce the possible poles to cases excluded by degree bounds, real positivity and homogeneous factorization. A frame-independent closed-kernel identity completes the passage from local pole reductions to the algebraic classification.

2020 Mathematics Subject Classification. Primary 53C25; Secondary 53B20, 53C35, 53C40.
Key words and phrases. Osserman conjecture, Jacobi operator, algebraic curvature tensor, Clifford structure, Cayley plane, rank-one symmetric space, isoparametric hypersurface.

1 Introduction and statements

The remaining case. The Osserman conjecture, posed by Osserman in 1990 [24], says that a Riemannian manifold whose Jacobi operators have the same eigenvalues in every unit direction is flat or locally rank-one symmetric. Chi proved it in dimensions not divisible by four and in dimension four [7]. Nikolayevsky proved it, also in the pointwise form for $n \neq 2, 4$, in every other dimension except one remaining case: dimension sixteen, when a reduced Jacobi eigenvalue has multiplicity seven or eight [18, 19, 20], [20, Corollary 1.1]. He then reduced this remaining case to a purely algebraic statement, the Nikolayevsky (2009) conjecture, his Conjecture A [22, Theorem 1 and p. 679]. For surveys of Osserman manifolds see [12, 13]. We prove the Nikolayevsky (2009) conjecture (Theorem A). Hence the Osserman conjecture is true in every dimension (Corollary B and Remark 1.1). The same result completes the conformal Osserman conjecture, which is stated for dimensions greater than four [21].

Main results. The main new results of this paper are the following.

- Theorem A: every Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with a reduced Jacobi eigenvalue of multiplicity seven or eight is Clifford or of Cayley type. *This proves the Nikolayevsky (2009) conjecture (his Conjecture A), the algebraic statement to which he reduced the Osserman conjectures in dimension sixteen [22].*

- Corollary B: every pointwise Osserman 16-manifold is flat or locally rank-one symmetric, and the conformal analogue holds. *Dimension sixteen was the only open case [20, Corollary 1.1]; this completes the Osserman conjecture [24] in every dimension and, with [21], the conformal Osserman conjecture, which is stated for dimensions $n > 4$.*

- Theorem C and Corollary D: the two-eigenvalue case with multiplicities 7 and 8 is Cl(7), Cl(8) or Cayley, and the corresponding manifolds are locally the Cayley projective plane or its dual. *This proves the theorem announced by Nikolayevsky in 2006 [20, Theorem 1.2], whose proof did not appear [1, Section 4].*

- Corollary E: Jacobi-proportional Osserman tensors on $\mathbb{R}^{16}$ are Clifford or of Cayley type. *This settles the two patterns* $(8,7)$ *and* $(7,7,1)$ *to which Andrejić reduced this question [1, Theorem 5].*

- Theorem F: four three-eigenvalue patterns are Clifford. *This extends Nikolayevsky's polynomial eigenfield method [20, Lemmas 3.1–3.2] to multiplicities seven and eight.*

Notation. Let V be a Euclidean vector space. For an algebraic curvature tensor R we use the Jacobi operator

$$\mathcal{J}_x y = R(y,x)x.$$

The tensor is *Osserman* if the spectrum of $\mathcal{J}_x$ on $x^\perp$ is independent of the unit vector x. All multiplicities below refer to this reduced operator. This is also Nikolayevsky's convention [20, p. 379], [22, proof of Lemma 4], so Theorem A below is his Conjecture A. Set

$$R_1(u,v)w = \langle v,w\rangle u - \langle u,w\rangle v, \qquad \Pi(x) = \|x\|^2 I - xx^T.$$

Thus $\mathcal{J}_x(R_1) = \Pi(x)$. Here uv^T denotes the map $z \mapsto \langle v,z\rangle u$. Our Jacobi ordering is opposite to that in [19, 20, 22]; the resulting overall sign changes will be specified where their formulas are used. For an orthogonal map T we write $(T^*R)(u,v)w = T^{-1}R(Tu,Tv)Tw$.

A *Clifford structure* consists of skew-symmetric endomorphisms $J_1,\dots,J_\nu$ with

$$J_iJ_j + J_jJ_i = -2\delta_{ij}I.$$

We call R Clifford if its Jacobi family is $\lambda_0\Pi(x) + \sum_i \eta_i (J_ix)(J_ix)^T$ for constants λ_0, η_i. A *symmetric Clifford system* is a family $H_0,\dots,H_m$ of symmetric endomorphisms satisfying $H_iH_j + H_jH_i = 2\delta_{ij}I$.

Our main result is the Nikolayevsky (2009) conjecture, Conjecture A of the preprint version of [22] (arXiv:0910.1645v1, p. 2); the published version states it without a number [22, p. 679].

Theorem A (Nikolayevsky (2009) conjecture)**.** *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ one of whose reduced Jacobi eigenvalues has multiplicity 7 or 8. Then R is Clifford, or R has exactly two reduced eigenvalues and $R = aR_1 + b\,T^*R_{\mathrm{Cay}}$ for constants a,b and some $T \in O(16)$, as in Theorem C.*

Nikolayevsky showed that this algebraic statement is the only missing input for the Osserman and the conformal Osserman conjectures in dimension sixteen [22, Theorem 1 and p. 679].

Corollary B. *Let M be a connected smooth Riemannian manifold of dimension* 16*.*

1. *If M is pointwise Osserman, it is flat or locally isometric to a rank-one symmetric space.*
2. *If M is conformally Osserman, it is locally conformally equivalent to a Euclidean space or to a rank-one symmetric space.*

Proof. By [22, Theorem 1 and the paragraph following it, p. 679], both statements hold once the curvature tensor, respectively the Weyl tensor, has a Clifford structure or is a combination of R_1 and an orthogonal conjugate of R_{Cay} at every point. (In the preprint arXiv:0910.1645v1 this is Theorem 1, "Assuming Conjecture A, ..."; the dimension-sixteen argument ends in Section 3.4.) At points where some reduced eigenvalue has multiplicity 7 or 8, this is Theorem A. At the other points, the largest multiplicity of a reduced eigenvalue is at least 7, by Adams' theorem on vector fields on S^{15} [19, proof of Theorems 1 and 2, pp. 3–4 of arXiv:math/0204258v2]; hence it is at least 9, and the tensor is Clifford by [20, Proposition 2.1] if it is 9 and by [19, Proposition 1] if it is at least 10, as summarized in [22, pp. 678 and 680]. □

Remark 1.1. Together with [20, Corollary 1.1], part (1) completes the proof of the Osserman conjecture [24]: every globally Osserman manifold, and every pointwise Osserman manifold of dimension $n \neq 2, 4$, is flat or locally rank-one symmetric. Together with [21, Theorem 1], part (2) proves the conformal Osserman conjecture in every dimension $n > 4$.

Organization of the proof. We first classify the two-eigenvalue case using its associated quartic. Polynomial eigenfield reductions then leave sixteen multiplicity patterns. The coherent-factor and transpose-pencil arguments treat these by excluding all possible pole branches. Section 30 combines the reductions on a real open set and proves Theorem A. The two-eigenvalue statement below also fixes the normalization of R_{Cay} used in the main theorem.

Theorem C. *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with exactly two reduced eigenvalues, of multiplicities* 7 *and* 8*. There are constants μ and $\kappa \neq 0$ such that*

$$\mathcal{J}_x = \mu\Pi(x) + \kappa A(x),$$

where, in suitable orthogonal coordinates, A has one of the forms

$$A(x) = \sum_{i=1}^{7} (J_i x)(J_i x)^T, \qquad \textit{for a } \mathrm{Cl}(7)\textit{-structure}, \tag{1.1}$$

$$A(x) = \Pi(x) - \sum_{i=1}^{8} (J_i x)(J_i x)^T, \qquad \textit{for a } \mathrm{Cl}(8)\textit{-structure}, \tag{1.2}$$

$$A(x) = \frac{1}{2}\left(\|x\|^2 I + \sum_{i=0}^{8} \langle H_i x, x\rangle H_i \right) - xx^T, \qquad \textit{for a symmetric Clifford system on } \mathbb{R}^{16}. \tag{1.3}$$

The first two cases are Clifford. In the third,

$$R = (\mu - \kappa/3)R_1 + (\kappa/3)\, T^* R_{\mathrm{Cay}} \quad \textit{for some } T \in O(16),$$

where R_{Cay} is the curvature tensor of the Cayley projective plane, normalized to have sectional curvatures between 1 *and* 4*.*

We have not located this algebraic classification in the literature: [22, Lemma 4] takes the trichotomy from Conjecture A, and [2] classifies two-root tensors only in dimensions twice an odd number.

Corollary D. *Let M be a connected smooth pointwise Osserman Riemannian 16-manifold. Suppose that, at every point, its reduced Jacobi operator has exactly two eigenvalues of multiplicities 7 and 8. Then, up to a constant scaling, M is locally isometric to the Cayley projective plane or its noncompact dual.*

Nikolayevsky announced this statement in 2006 [20, Theorem 1.2, p. 380], with its proof deferred; it is described as an announcement without proof in [1, Section 4] and [2, Section 1, p. 2].

For an Osserman tensor with reduced eigenspace projections $\pi_i(x)$, Jacobi-proportionality means

$$\|x\|^2\langle\pi_i(x)y, y\rangle = \|y\|^2\langle\pi_i(y)x, x\rangle \quad (x, y \neq 0, \text{ every } i). \tag{1.4}$$

The radial projection is separated from the reduced zero eigenspace, if that eigenvalue occurs.

Corollary E. *Every Jacobi-proportional Riemannian Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ is Clifford or a linear combination of R_1 and an orthogonal conjugate of R_{Cay}.*

Andrejić showed that a Jacobi-proportional tensor which is not Clifford has multiplicities $(8, 7)$ or $(7, 7, 1)$ [1, Theorem 5], [3, Theorem 3]; the corollary settles both patterns. Both contain an eigenvalue of multiplicity seven, so the corollary also follows from Theorem A; the proof given below, through Theorems C and F, does not use the long argument for Theorem A.

Theorem F. *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with exactly three reduced eigenvalues. If their multiplicities are*

$$(7, 7, 1), \quad (8, 6, 1), \quad (8, 5, 2), \quad \textit{or } (7, 6, 2),$$

then R is Clifford.

Theorem F extends the polynomial eigenfield argument used by Nikolayevsky for an eigenvalue of multiplicity nine [20, Lemmas 3.1–3.2 and Proposition 2.1]; the pattern $(7, 7, 1)$ is one of Andrejić's two patterns [1, Theorem 5].

Relation to earlier work. Osserman [24] asked whether a manifold whose Jacobi operators have constant eigenvalues on the unit tangent bundle is flat or locally rank-one symmetric. Chi proved this in dimensions not divisible by four and in dimension four [7]. Nikolayevsky proved it, also in the pointwise form for $n \neq 2, 4$, except in dimension sixteen when a reduced Jacobi eigenvalue has multiplicity seven or eight [18, 19, 20]; see [20, Corollary 1.1, p. 380]. The algebraic classification behind this leaves exactly these exceptional algebraic cases [19, Proposition 1], [20, Proposition 2.1], [22, p. 680]. In [22] Nikolayevsky reduced the remaining case, and the dimension-sixteen case of the conformal Osserman conjecture, to Conjecture A. The proof of Theorem C uses Münzner's converse to the Cartan–Münzner equations, the Ferus–Karcher–Münzner construction, and Chi's classification [8] of isoparametric hypersurfaces with four principal curvatures and multiplicities $(7, 8)$. The geometric consequence uses the reduction in [22, Section 3]. Theorem F uses the polynomial eigenfield and rank-one restoration lemmas of [20]; Corollary E then follows from [1, 3].

Jacobi-proportionality is an additional condition. In contrast, Rakić duality—exchange of unit Jacobi eigenvectors with the same eigenvalue—is equivalent to the Riemannian Osserman condition [23, Section 2]. The related Jacobi-orthogonality condition is studied in [4]. Neither stronger condition is assumed in Theorem A.

2 Normalization and the quartic

Write the reduced eigenvalues as $\mu\|x\|^2$ of multiplicity 8 and $(\mu+\kappa)\|x\|^2$ of multiplicity 7, and put $A(x) = (\mathcal{J}_x - \mu\Pi(x))/\kappa$. Then A is a homogeneous quadratic symmetric matrix family satisfying

$$A(x)x = 0, \qquad A(x)^2 = \|x\|^2 A(x), \qquad \operatorname{tr} A(x) = 7\|x\|^2. \tag{2.1}$$

The Jacobi family determines its curvature tensor. In fact, for any symmetric quadratic family A with $A(x)x = 0$, let $B(u,v) = (A(u+v) - A(u) - A(v))/2$. Polarization gives $B(x,y)x = -A(x)y/2$. Since $B(x,y)$ is symmetric,

$$\langle A(x)y, y\rangle = -2\langle B(x,y)x, y\rangle = -2\langle B(y,x)y, x\rangle = \langle A(y)x, x\rangle.$$

Jacobi-family reconstruction [3, Theorem 1] therefore applies; specializing its defining formula [3, (12), p. 8468] to a quadratic family gives

$$R_A(u,v)w = \frac{2}{3}(B(v,w)u - B(u,w)v) \tag{2.2}$$

and gives a curvature tensor with Jacobi family A.

Set

$$Q(x) = A(x) + xx^T, \qquad L(x) = \mathbb{R}x + \operatorname{im} A(x).$$

For $x \neq 0$, $Q(x) = \|x\|^2 \pi_{L(x)}$ and $\dim L(x) = 8$. In particular,

$$Q(x)^2 = \|x\|^2 Q(x), \qquad Q(x)x = \|x\|^2 x, \qquad \operatorname{tr} Q(x) = 8\|x\|^2. \tag{2.3}$$

Sectional symmetry implies that the biquadratic form

$$b(x,y) = \langle Q(x)y, y\rangle = \langle A(x)y, y\rangle + \langle x, y\rangle^2 \tag{2.4}$$

is symmetric in x, y. It vanishes on either coordinate axis and satisfies $b(x,x) = \|x\|^4$.

On $\mathbb{R}^{32} = \mathbb{R}^{16} \oplus \mathbb{R}^{16}$ write $z = (x,y)$, $s = \|x\|^2 + \|y\|^2$, and define

$$F(z) = s^2 - 8b(x,y). \tag{2.5}$$

Lemma 2.1. *The polynomial F satisfies*

$$\|\nabla F\|^2 = 16s^3, \qquad \Delta F = 8s.$$

Its restriction to S^{31} takes both values 1 and -1.

Proof. Symmetry of b gives $\nabla_y b = 2Q(x)y$ and $\nabla_x b = 2Q(y)x$. By (2.3),

$$\|\nabla b\|^2 = 4sb, \qquad \langle z, \nabla b\rangle = 4b, \qquad \Delta b = 2\operatorname{tr} Q(x) + 2\operatorname{tr} Q(y) = 16s.$$

Thus

$$\|\nabla F\|^2 = 16s^3 - 64s\langle z, \nabla b\rangle + 64\|\nabla b\|^2 = 16s^3.$$

Since $\Delta s^2 = 4(32+2)s = 136s$, we obtain $\Delta F = 8s$. For unit x, $F(x,0) = 1$. If x, y are unit and $y \in L(x)$, then $F((x,y)/\sqrt{2}) = 1 - 8/4 = -1$. □

3 The isoparametric classification input

We record the exact external results needed.

Münzner. For a homogeneous polynomial of degree g satisfying $\|\nabla F\|^2 = g^2\|z\|^{2g-2}$ and $\Delta F = c\|z\|^{g-2}$ on $\mathbb{R}^{n+1}$, nonconstant on the sphere, [16, Satz 3, p. 65] shows that 0 is a regular value of $F|S^n$ and that the zero level is an isoparametric hypersurface; by its proof (p. 66) the same holds for every level in $(-1, 1)$. Its Folgerung on p. 66 identifies F itself with their Cartan polynomial when both $g(n-1) - c$ and $g(n-1) + c$ are nonzero; these conditions exclude the exceptional doubled-degree alternative. Satz 4(a), p. 67, gives connected level sets. Here $g = 4$, $n = 31$, and $c = 8$, whereas $g(n-1) = 120$. The resulting hypersurfaces have four principal curvatures, with

$$m_1 + m_2 = 15, \qquad 8(m_2 - m_1) = 8,$$

so their multiplicities are $(7, 8)$.

Chi and Ferus–Karcher–Münzner. Chi's classification [8, abstract, p. 225] gives exactly three families with unordered multiplicities $\{7, 8\}$. They are the FKM family with parameter $m = 7$ and the two FKM families with $m = 8$. The first is the Ozeki–Takeuchi family, as identified in [11, Section 5.9, p. 19 of the English translation]; the two $(8, 7)$ families are distinguished in its Section 6.6, p. 22. For a symmetric Clifford system $P_0, \dots, P_m$ on $\mathbb{R}^{32}$, [11, Theorem 4.1, p. 8; Section 4.3, p. 11] gives

$$F_P(z) = \|z\|^4 - 2\sum_{i=0}^{m}\langle P_i z, z\rangle^2, \qquad (m_1, m_2) = (m, 15 - m). \tag{3.1}$$

The page numbers for FKM throughout this paper refer to its English translation, arXiv:1112.2780. By the same Folgerung of Münzner, with $c = \pm 8 \neq \pm 120$, each F_P used here is the Cartan polynomial of its zero level.

We need congruence of the polynomials, not only of a level hypersurface. The zero level of F has principal curvatures $\cot(\pi/8 + j\pi/4)$, $0 \le j \le 3$. A congruent member of the FKM family must therefore be its zero member, with one of the two choices of normal. Let $U \in O(32)$ map the zero level of F onto the zero level of F_P. Both zero levels are connected [16, Satz 4(a), p. 67]. By [16, Folgerung (iii), p. 66], F and $F_P \circ U$ are both Cartan polynomials of this hypersurface. The Cartan polynomial of a connected oriented isoparametric hypersurface is determined by it, and reversing the orientation replaces it by its negative [16, §4, p. 65]; explicitly $F(rq) = r^4 \cos 4t(q)$ near the hypersurface [16, proof of Satz 3, p. 66]. Hence $F = \varepsilon F_P \circ U$ with $\varepsilon \in \{1, -1\}$.

Absorb U into the matrices P_i. Since $\Delta F_P = (136 - 16(m+1))s$, comparison with $\Delta F = 8s$ gives precisely

$$4b(x, y) = \sum_{i=0}^{7}\langle P_i z, z\rangle^2, \tag{3.2}$$

$$\text{or} \qquad 4b(x, y) = s^2 - \sum_{i=0}^{8}\langle P_i z, z\rangle^2. \tag{3.3}$$

These identities hold in the original coordinates (x, y); no preservation of the two coordinate summands by U has been assumed.

4 A lemma on quadratic maps between spheres

Quadratic maps between spheres have a classical literature [27, 26]. We give the dimension-threshold argument needed here. Part (b) below is the quadratic case of [27, p. 163]; we include the short proof since (a) and (c) use it.

Lemma 4.1. *Let $H_0, \dots, H_m$ be symmetric $n \times n$ matrices and $h(x) = (\langle H_i x, x\rangle)_{i=0}^m$. Suppose $\|h(x)\| = \|x\|^2$. For $a \in \mathbb{R}^{m+1}$ put $H(a) = \sum_i a_i H_i$.*

(a) *If $a \in h(S^{n-1})$, then $I - H(a) \geq 0$ and the fibre over a is the unit sphere of $U_a = \ker(I - H(a))$, where $\dim U_a \geq n - m$.*

(b) *If $n > 2m$, then h is constant on S^{n-1}.*

(c) *If $n = 2m$, $m \geq 1$, and h is not constant on S^{n-1}, then $H_i H_j + H_j H_i = 2\delta_{ij} I$.*

Proof. For unit a, x, $\langle H(a)x, x\rangle = \langle a, h(x)\rangle \leq 1$, with equality exactly when $h(x) = a$. This proves the fibre description. Fix a unit x in that fibre and rotate the index space so that $a = e_0$. Then $H_0 x = x$. Comparing the t^2 coefficients of $\|h(x + ty)\|^2 = \|x + ty\|^4$ gives

$$2\langle H_0 y, y\rangle + 4\sum_{i=0}^{m} \langle H_i x, y\rangle^2 = 2\|y\|^2 + 4\langle x, y\rangle^2.$$

The $i = 0$ terms cancel, so

$$I - H_0 = 2\sum_{i=1}^{m} (H_i x)(H_i x)^T. \tag{4.1}$$

Its rank is at most m, proving (a).

The cones U_a, U_b for distinct values meet only at zero. If $n > 2m$, two such cones would have dimensions summing to more than n. This proves (b).

Suppose $n = 2m$ and there is more than one value. For every a, comparison with another fibre gives $\dim U_a = m$. Hence the right side of (4.1) has rank m, and $H_1 x, \dots, H_m x$ are independent and perpendicular to x. On $T_x S^{n-1} = x^\perp$,

$$(dh)_x(y) = 2(\langle H_i x, y\rangle)_{i=0}^m$$

has rank m. The image of h is open in S^m and is compact; since $m \geq 1$, it is all of S^m. In particular a and $-a$ are values. Their fibres are the $+1$ and -1 eigenspaces of $H(a)$ and each has dimension m. Thus $H(a)^2 = I$ for every unit a. Polarization in a proves (c). □

5 Recovering the original eigenspaces

Proposition 5.1. *Identity* (3.2) *implies* (1.1).

Proof. Setting $y = 0$ in (3.2) shows that the upper-left diagonal block of each P_i is zero; setting $x = 0$ gives the same for the lower-right block. Hence

$$P_i = \begin{pmatrix} 0 & C_i^T \\ C_i & 0 \end{pmatrix}, \qquad Q(x) = \sum_{i=0}^{7} (C_i x)(C_i x)^T.$$

The Clifford relations give

$$C_i^T C_j + C_j^T C_i = 2\delta_{ij} I, \qquad C_i C_j^T + C_j C_i^T = 2\delta_{ij} I.$$

Moreover $Q(x)x = \|x\|^2 x$ implies $\sum_i \langle C_i x, x\rangle^2 = \|x\|^4$. Apply Lemma 4.1(b) to the symmetric parts of C_i, with $n = 16$ and $m = 7$. There is a unit $w \in \mathbb{R}^8$ such that $(\langle C_i x, x\rangle)_i = \|x\|^2 w$. Rotating the index space gives $w = e_0$. Since C_0 is orthogonal and $\langle C_0 x, x\rangle = \|x\|^2$, we have $C_0 = I$. The remaining C_i are skew, and their relations become $C_iC_j + C_jC_i = -2\delta_{ij}I$. Subtracting xx^T from $Q(x)$ proves the claim. □

Proposition 5.2. *Identity* (3.3) *implies* (1.2) *or* (1.3).

Proof. In the original splitting write $P_i = \left(\begin{smallmatrix} H_i & B_i^T \\ B_i & D_i \end{smallmatrix}\right)$. The two axes in (3.3) give

$$\sum_{i=0}^{8} \langle H_i x, x\rangle^2 = \|x\|^4, \qquad \sum_{i=0}^{8} \langle D_i y, y\rangle^2 = \|y\|^4.$$

Write $h(x) = (\langle H_i x, x\rangle)_i$ and $d(y) = (\langle D_i y, y\rangle)_i$.

If h is constant on the unit sphere, rotate the index so that $H_0 = I$, $H_i = 0$ for $i \geq 1$. The upper block of $P_0^2 = I$ gives $B_0 = 0$. Also $\operatorname{tr} P_0 = 0$, because $P_0 = -P_1P_0P_1$. Thus $D_0^2 = I$ and $\operatorname{tr} D_0 = -16$, forcing $D_0 = -I$. Anticommutation with $P_0 = \operatorname{diag}(I, -I)$ makes the other eight matrices off-diagonal. Calling their lower blocks $C_1, \ldots, C_8$ and expanding (3.3) yields

$$Q(x) = \|x\|^2 I - \sum_{i=1}^{8} (C_i x)(C_i x)^T.$$

The identity $Q(x)x = \|x\|^2 x$ gives $\sum_i \langle C_i x, x\rangle^2 = 0$. Each C_i is therefore skew, and the Clifford relations become those of a Cl(8)-structure. This is (1.2).

If h is not constant, Lemma 4.1(c) makes the H_i a symmetric Clifford system. The equation $H_i^2 + B_i^T B_i = I$ forces $B_i = 0$. Expanding (3.3) now gives

$$b(x, y) = \frac{1}{2}\left(\|x\|^2\|y\|^2 - \langle h(x), d(y)\rangle\right).$$

Since $b(x, x) = \|x\|^4$ and $\|h(x)\| = \|d(x)\| = \|x\|^2$, equality in Cauchy–Schwarz forces $d(x) = -h(x)$. Polarization gives $D_i = -H_i$. Consequently

$$Q(x) = \frac{1}{2}\left(\|x\|^2 I + \sum_{i=0}^{8} \langle H_i x, x\rangle H_i\right),$$

which proves (1.3). □

6 Identification of the Cayley tensor

We first establish the norm identity for an arbitrary symmetric Clifford system $S_0, \ldots, S_8$ on $\mathbb{R}^{16}$. For $q = \|x\|^2$ put $h_i = \langle S_i x, x\rangle$ and $S_h = \sum_i h_i S_i$. The products of zero, one, and four distinct generators are symmetric and trace-orthogonal, and their number is $1 + 9 + 126 = 136$. They therefore form a basis of the symmetric matrices. If $\mathcal{L}_k$ is their span for degree k, the operator $\mathcal{T}(Y) = \sum_i S_i Y S_i$ acts on $\mathcal{L}_k$ by $(-1)^k(9 - 2k)$, namely by $9, -7, 1$ for $k = 0, 1, 4$. These assertions also appear in [22, Lemma 3, pp. 6–7 of the preprint]. Expanding $xx^T = qI/16 + S_h/16 + Y_4$ therefore gives

$$M_x := \sum_i (S_i x)(S_i x)^T = xx^T + \frac{1}{2}(qI - S_h). \tag{6.1}$$

Contraction with x yields $\|h\|^2 = q^2 + (q^2 - \|h\|^2)/2$, so $\|h\| = q$.

For this system define $Q(x) = (qI + S_h)/2$, so that $A(x) = Q(x) - xx^T$ is the matrix in (1.3) with $H_i = S_i$. For $x \neq 0$ set $w = h/q$. Since $S_h = qS_w$, we have $Q(x) = q(I + S_w)/2$. Also $S_w x = x$, since S_w is an orthogonal involution and $\langle S_w x, x\rangle = q$. Its two eigenspaces have dimension eight. For $u \perp w$, $S_u x$ lies in its negative eigenspace. Thus

$$Q(x) = \frac{q}{2}(I + S_w), \qquad M_x = xx^T + qI - Q(x), \qquad S_h = 2Q(x) - qI.$$

In octonionic coordinates $x = (x_1, x_2)$ the standard system is

$$S_0(x_1, x_2) = (x_1, -x_2), \qquad S_i(x_1, x_2) = (e_i\overline{x_2},\, \overline{x_1}e_i), \quad 1 \le i \le 8,$$

where $(e_i)_{i=1}^8$ is an orthonormal basis of $\mathbb{O}$. With $(u \wedge v)z = \langle u, z\rangle v - \langle v, z\rangle u$, the Brown–Gray Cayley curvature formula [6, Eq. (6.12)], in the form [22, equation (7), p. 6 of the preprint], is

$$R_{\mathbb{O}}(u, v) = 3u \wedge v + \sum_{i=0}^{8} S_i u \wedge S_i v.$$

In our Jacobi convention this gives

$$\begin{aligned} \mathcal{J}_x(R_{\mathbb{O}}) &= 3(xx^T - qI) + M_x - S_h \\ &= 4xx^T - qI - 3Q(x) = -\Pi(x) - 3A(x). \end{aligned}$$

Therefore $R_{\mathrm{Cay}} = -R_{\mathbb{O}}$ has Jacobi family $\Pi(x) + 3A(x)$, with reduced eigenvalues 1 and 4 of multiplicities eight and seven.

Let $\mathrm{Cl}_+(9)$ denote the real Clifford algebra on nine generators with squares $+1$ and pairwise anticommutation. Then $\mathrm{Cl}_+(9) \cong M_{16}(\mathbb{R}) \oplus M_{16}(\mathbb{R})$. Its representations on $\mathbb{R}^{16}$ are equivalent up to orthogonal conjugacy and the sign of one generator; see [11, Sections 3.4–3.5, pp. 6–7] and [22, p. 6 of the preprint]. Changing a generator's sign leaves $\langle S_i x, x\rangle S_i$ unchanged. This identifies every family (1.3) with the standard Cayley family and finishes the proof of Theorem C.

7 The manifold consequence

Proof of Corollary D. A pointwise Osserman metric is Einstein. Its Weyl Jacobi operator is a scalar shift of its curvature Jacobi operator on $x^\perp$, so it also has exactly two roots. In particular the open set M' where the number of Weyl roots is locally constant, used in [22, Lemma 4], is all of M.

The $N = 2$ branch of that lemma (pp. 10–11 of arXiv:0910.1645) uses the algebraic conjecture only to obtain the pointwise Cl(7), Cl(8), or Cayley alternatives; in the published version, [22, Lemma 4, pp. 688–689], the alternatives are its hypothesis. Theorem C supplies that trichotomy; disjointness of the alternatives and their smooth local realizations are established inside the cited Lemma 4. The Clifford alternatives are handled by Lemmas 5 and 6 of [22] (pp. 12–18 of the preprint), including the case $\nu = 8$; its Lemma 7 and Section 3.3 handle the Cayley alternative. These yield local conformal equivalence to rank-one symmetric metrics. The final Einstein argument of Section 3.4, using [17, Theorem 4.4], gives local symmetry of the original metric, and the last paragraph of [22, Section 3.4] makes it flat or locally rank-one symmetric; flatness is excluded by the two distinct roots. (The corollary also follows from Corollary B(1); the proof given here uses only Theorem C.) The only rank-one symmetric 16-dimensional models with multiplicities $(7, 8)$ are the Cayley projective and hyperbolic planes. Connectedness fixes the type and scale, proving the corollary. □

8 Three-root tensors and proportionality

For the Cayley model let $E(x) = L(x) \cap x^{\perp}$ and $N(x) = L(x)^{\perp}$, of dimensions seven and eight.

Lemma 8.1. *There is no nonzero linear map K with $Kx \in E(x)$ for every $x \neq 0$. There is also no nonzero linear map K with $Kx \in N(x)$ for every $x \neq 0$.*

Proof. After the orthogonal coordinate change $(x_1, x_2) \mapsto (x_1, \overline{x_2})$, the Cayley lines are

$$\mathcal{C}_\omega = \{(u, \omega u) : u \in \mathbb{O}\}, \quad \omega \in \mathbb{O}, \qquad \mathcal{C}_\infty = \{(0, v) : v \in \mathbb{O}\}.$$

Here $S_{(w_0,a)} = w_0 S_0 + \sum_{i=1}^{8} a_i S_i$ for $w_0^2 + \|a\|^2 = 1$ and $a = \sum_i a_i e_i$. Its positive eigenspace becomes $\mathcal{C}_\omega$ with $\omega = \overline{a}/(1 + w_0)$ when $w_0 \neq -1$, and becomes $\mathcal{C}_\infty$ otherwise. For x in such a line, $L(x)$ is that line.

Suppose first $Kx \in E(x)$. Since $\langle Kx, x\rangle = 0$, K is skew. Preservation of $\mathcal{C}_0, \mathcal{C}_\infty, \mathcal{C}_1$ forces $K = \operatorname{diag}(D, D)$. Preservation of $\mathcal{C}_\omega$ gives $D(\omega u) = \omega D(u)$. Put $c = D(1)$; then $D(u) = uc$ and $(\omega u)c = \omega(uc)$ for all ω, u. Thus c lies in the right nucleus of $\mathbb{O}$, which is $\mathbb{R}$. Hence $K = cI$, and skewness forces $c = 0$.

Suppose next $Kx \in N(x)$. The two axes force $K = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$. The graph condition for $\mathcal{C}_\omega^{\perp}$ is $B(\omega u) = -\overline{\omega} C(u)$. Taking $\omega = 1$ gives $B = -C$; taking $u = 1$ then gives $C(\omega) = \overline{\omega}\gamma$, where $\gamma = C(1)$. Substitution yields

$$(ab)\gamma = b(a\gamma) \qquad (a, b \in \mathbb{O}).$$

Choose a quaternion subalgebra containing γ and choose noncommuting a, b in it. Associativity in that subalgebra gives $(ab - ba)\gamma = 0$. Multiplicativity of the octonion norm forces $\gamma = 0$, and therefore $K = 0$. □

Proof of Theorem F. Let $t \in \{1, 2\}$ be the smallest multiplicity, with eigenvalue ν. For the purpose of applying the polynomial eigenfield lemma, shift a different reduced eigenvalue to zero by subtracting a multiple of R_1. In the convention of [20, p. 379], the shifted tensor has $k = 3$ reduced eigenvalues, one of them zero. Nikolayevsky's [20, Lemma 3.1(1),(2), p. 383] gives an orthonormal frame for the ν-eigenspace by odd homogeneous polynomial fields of degree at most $k - 1 = 2$. They are therefore linear: $J_1 x, \ldots, J_t x$. Their norm and orthogonality identities make each J_i skew and orthogonal and make the J_i anticommute. Return to the original tensor, whose eigenspaces are unchanged by the shift.

Choose the other root μ so that the merger is

$$(7, 7, 1) \to (7, 8), \quad (8, 6, 1) \to (8, 7), \quad (8, 5, 2) \to (8, 7), \quad (7, 6, 2) \to (7, 8).$$

For a skew orthogonal J, let R_J denote the curvature tensor whose Jacobi family is $(Jx)(Jx)^T$, obtained for example from (2.2). Then

$$\widetilde{R} = R - (\nu - \mu) \sum_{i=1}^{t} R_{J_i}$$

is Osserman with two distinct roots of multiplicities seven and eight. Apply Theorem C. Its Cayley alternative is impossible: each $J_i x$ remains an eigenfield in one of the two merged distributions, contradicting Lemma 8.1 after conjugating the field to the standard model: $K = T J_i T^{-1}$. Thus $\widetilde{R}$ is Clifford.

Restore the rank-one terms successively. By [20, Lemma 3.2(1), pp. 383–384], if R is Osserman, J is orthogonal and skew, $c \neq 0$, and $R - cR_J$ is Osserman and Clifford, then R is Clifford. Each

intermediate tensor here is Osserman: only constant eigenvalues on fixed mutually orthogonal eigenlines change. The lemma consequently applies at every step. (The proof of [20, Lemma 3.2(1)] shifts the root of largest multiplicity of $R - cR_J$ to zero. For an endpoint of type (1.2), shift instead the base constant λ_0 of its Clifford structure to zero; the proof then applies verbatim, and when $\nu = 8$ its remaining coefficient c_0 vanishes because $\mathbb{R}^{16}$ carries no nine anticommuting orthogonal complex structures.) Its source convention changes the sign of the coefficient, but only its nonvanishing is required. This proves the theorem. □

Proof of Corollary E. Andrejić [1, Theorem 5], restated as [3, Theorem 3, p. 8470], proves that a Riemannian Jacobi-proportional non-Clifford algebraic curvature tensor has multiplicities $(8, 7)$ or $(7, 7, 1)$. There is no restriction on the number of roots in the hypothesis. Apply Theorems C and F. □

9 Further multiplicity reductions

The classification also completes several polynomial-eigenfield reductions from [20]. We give the additional arguments, including the exclusion needed when a linear perturbation mixes two eigenspaces.

Theorem G. *An Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ is Clifford if its reduced multiplicity pattern is one of*

$$(7,6,1,1),\quad (8,5,1,1),\quad (7,5,2,1),\quad (8,4,2,1),$$
$$(7,5,1,1,1),\quad (8,4,1,1,1),$$

or if one root has multiplicity seven or eight and all the other roots have multiplicity at most two. The latter condition gives nine patterns:

$$\begin{array}{ll}(7,2,2,2,2), & (8,2,2,2,1),\\ (7,2,2,2,1,1), & (8,2,2,1,1,1),\\ (7,2,2,1,1,1,1), & (8,2,1,1,1,1,1),\\ (7,2,1,1,1,1,1,1), & (8,1,1,1,1,1,1,1),\\ (7,1,1,1,1,1,1,1,1). & \end{array}$$

9.1 Constant projections onto Cayley eigenspaces

Lemma 9.1. *Let $E(x), N(x)$ be the two Cayley distributions of Lemma 8.1. If a skew-symmetric linear map K satisfies*

$$\|\pi_{N(x)} Kx\|^2 = c\|x\|^2 \qquad (x \neq 0)$$

for a constant c, then $K = 0$. In particular an orthogonal skew map cannot have constant normalized projection lengths onto the two Cayley eigenspaces.

Proof. Use the symmetric Clifford system $H_0, \dots, H_8$ of Section 6. Put $q = \|x\|^2$, $h_i(x) = \langle H_i x, x\rangle$ and $a_i(x) = \langle H_i x, Kx\rangle$. The vectors $H_i x$ are orthogonal, have norm $\sqrt{q}$, and span $\mathbb{R}x + N(x)$. Since K is skew,

$$\sum_i a_i(x)^2 = cq^2, \qquad \sum_i h_i(x)^2 = q^2.$$

Differentiating the second equality in the direction Kx gives $\sum_i h_i(x)a_i(x) = 0$. The matrix of a_i is the symmetric trace-free matrix $A_i = (H_iK - KH_i)/2$.

If $c > 0$, define

$$H_i(t) = \cos t\, H_i + \frac{\sin t}{\sqrt{c}} A_i.$$

For every t, the associated quadratic map satisfies $\sum_i \langle H_i(t)x, x\rangle^2 = q^2$. This map cannot be constant on the unit sphere: otherwise $H_i(t) = b_i I$ for a unit vector b, whereas all its matrices have zero trace. Lemma 4.1(c) makes the $H_i(t)$ a symmetric Clifford system for every t. Its product $\Omega(t) = H_0(t) \cdots H_8(t)$ is a symmetric involution. Thus $\operatorname{tr} \Omega(t)$ is continuous and integer-valued, hence constant. But $\Omega(0) = \pm I$ on $\mathbb{R}^{16}$, while $\Omega(\pi) = -\Omega(0)$, a contradiction. If $c = 0$, then $Kx \in L(x)$; skewness further gives $Kx \in E(x)$. Lemma 8.1 gives $K = 0$. □

9.2 Polynomial tools and a three-column reduction

In this section $A(x)$ denotes the Jacobi matrix after shifting one reduced root to zero; it need not be the rank-seven normalization used earlier. Write $q = \|x\|^2$ and $\mathcal{K} = \mathbb{R}[x_1, \ldots, x_{16}]/(q)$. We use three published algebraic inputs. By [20, Theorem 3.1, p. 397], $\mathcal{K}$ is a UFD. By its Lemma 3.11, p. 398, if q divides a sum of at most eight real polynomial squares, then it divides every polynomial being squared. Finally its Lemma 3.5, pp. 388–390, applies to a normalized cubic simple eigenfield P satisfying

$$\langle P(x), x\rangle = 0, \quad \|P(x)\|^2 = q^3, \quad P(P(x)) = -q^4 x. \tag{9.1}$$

It says that a nonunit common divisor of the components of P in $\mathcal{K}$ forces $P(x) = qJx$ for an orthogonal skew map J. Neither this statement nor its proof requires a root of multiplicity nine. The simple eigenfields of [20, Lemma 3.1] satisfy (9.1) after multiplication by q if initially linear.

We spell out the three-column step of [20, pp. 387–389]. Suppose P_1, P_2, P_3 are mutually orthogonal cubic eigenfields of squared norm q^3, and a weighted identity modulo q has two positive and one negative coefficient. Rescale its columns by nonzero real constants and denote their images in $\mathcal{K}^{16}$ by p_i. Then

$$p_1 p_1^T + p_2 p_2^T = p_3 p_3^T. \tag{9.2}$$

Comparing diagonal and off-diagonal entries gives $(p_{1a}p_{2b} - p_{2a}p_{1b})^2 = 0$. Thus all these minors vanish. If any $p_i = 0$, the original P_i is q times a linear eigenfield. Otherwise unique factorization gives a primitive vector v and $p_1 = u_1 v$, $p_2 = u_2 v$. Equation (9.2) gives $p_3 = u_3 v$ and $u_1^2 + u_2^2 = u_3^2$. Indeed a nonzero component of v gives a square root of $u_1^2 + u_2^2$ in the fraction field; it is integral over the integrally closed ring $\mathcal{K}$, hence belongs to $\mathcal{K}$. Cross entries fix the same sign in every component.

Assume the negative column comes from a simple eigenvalue. If u_3 is a nonunit, Lemma 3.5 just quoted makes the original P_3 linear after division by q. If u_3 is a unit, homogeneity implies that p_1, p_2 are constant real multiples of p_3. Undoing the rescaling, a constant combination of the original fields therefore satisfies

$$c_1 P_1(x) + c_2 P_2(x) = qJx, \qquad c_1^2 + c_2^2 = 1, \tag{9.3}$$

where J is orthogonal and skew by norm and tangency. If P_1, P_2 have the same eigenvalue, this is a linear eigenfield. Otherwise Jx has constant coordinates in their moving orthonormal frame on the unit sphere, so every fixed rank-one update by $C(Jx)(Jx)^T$ is still Osserman.

9.3 Two simple roots

Consider $(M, N, 1, 1)$ with $(M, N) = (7, 6)$ or $(8, 5)$. Shift the M-root to zero. Let λ be the N-root and μ, ν the simple roots. These values are distinct and nonzero. Lemma 3.1 of [20] supplies cubic

normalized eigenfields P_1, P_2, with linear fields homogenized by q. Their spectral projectors give

$$\mu(\mu-\lambda)P_1P_1^T+\nu(\nu-\lambda)P_2P_2^T=qA(A-\lambda qI). \tag{9.4}$$

If the coefficients have the same sign, the sum-of-squares lemma applied to diagonal entries makes both fields q times linear ones. Move one simple root into each large root. The result has pattern $(M+1,N+1)$, either $(8,7)$ or $(9,6)$. The former is classified by Theorem C, with Cayley excluded by Lemma 8.1; the latter is Clifford by [20, Proposition 2.1, p. 382]. Restoring the two rank-one terms with [20, Lemma 3.2(1)] proves the claim.

Otherwise interchange μ,ν if necessary and put $a^2=\mu(\mu-\lambda)>0$, $b^2=-\nu(\nu-\lambda)>0$. In this subsection let p_i denote the image of the unscaled P_i in $\mathcal{K}^{16}$. Reducing (9.4) modulo q gives $(ap_1)(ap_1)^T=(bp_2)(bp_2)^T$. In a domain these vectors agree up to one global sign, by comparison of a nonzero component and then every cross entry. If both vanish the preceding linear-field argument applies. Otherwise change the sign of P_2 to obtain

$$aP_1-bP_2=\sqrt{a^2+b^2}\,qJx$$

with J orthogonal and skew. Set $C=\mu+\nu-\lambda\neq 0$; nonvanishing follows from $a^2+b^2=(\mu-\nu)C$. On the unit sphere, subtracting $C(Jx)(Jx)^T$ changes the two-dimensional block to a fixed symmetric matrix of trace λ and determinant

$$\mu\nu-C\,\frac{\nu a^2+\mu b^2}{a^2+b^2}=0,\qquad \nu a^2+\mu b^2=\mu\nu(\mu-\nu).$$

The new roots are $0,\lambda$, so the pattern is $(M+1,N+1)$. The projection lengths of Jx onto the new eigenspaces are constant, because the changed block is a constant matrix in the frame (P_1,P_2). Lemma 9.1 excludes a Cayley endpoint (for $(M,N)=(8,5)$ the endpoint $(9,6)$ is Clifford by [20, Proposition 2.1]). Thus the endpoint is Clifford, and [20, Lemma 3.2(1)] restores the original tensor.

9.4 A double and a simple root

Consider $(M,N,2,1)$ with $(M,N)=(7,5)$ or $(8,4)$. Shift M to zero, and call the other roots λ,μ,ν, where μ is double and ν is simple. There are cubic normalized fields P_1,P_2 for μ and P_3 for ν, and

$$\mu(\mu-\lambda)(P_1P_1^T+P_2P_2^T)+\nu(\nu-\lambda)P_3P_3^T=qA(A-\lambda qI).$$

For equal coefficient signs all fields are linear after division by q. For opposite signs multiply by -1 if necessary: the two double-root columns then have positive coefficients and the simple column has the negative coefficient. The three-column reduction applies. Its nonunit case gives a linear simple-root field; in its unit case (9.3) is a linear double-root eigenfield. If a column vanishes modulo q, it already gives a linear eigenfield. Thus all cases produce one.

Move that field into the zero root. The endpoints are

initial	field from simple root	field from double root
$(7,5,2,1)$	$(8,5,2)$	$(8,5,1,1)$
$(8,4,2,1)$	$(9,4,2)$	$(9,4,1,1)$.

These are Clifford by Theorem F, the preceding subsection, or [20, Proposition 2.1, p. 382]. Restore the rank-one term by [20, Lemma 3.2(1)].

9.5 Three simple roots

Consider $(M, N, 1, 1, 1)$ with $(M, N) = (7, 5)$ or $(8, 4)$. There are five reduced roots. After shifting M to zero, Lemma 3.1 of [20] gives odd degree at most four, hence at most three. Let λ be the N-root and μ_1, μ_2, μ_3 the simple roots. The cubic spectral identity has weights $w_i = \mu_i(\mu_i - \lambda)$. If all signs agree, every field is linear after division by q. Otherwise choose the global sign and ordering so that two weights are positive and the third negative. The negative column is simple, so the three-column reduction applies.

If it gives a linear eigenfield, move that field into the zero root, obtaining $(8, 5, 1, 1)$ or $(9, 4, 1, 1)$, which are Clifford by the Subsection 9.3 and by [20, Proposition 2.1]; restore by [20, Lemma 3.2(1)]. Otherwise (9.3) mixes μ_1, μ_2. Put $d = \mu_1 c_2^2 + \mu_2 c_1^2$. The characteristic equation after subtracting $C(Jx)(Jx)^T$ is

$$(\mu_1 - z)(\mu_2 - z) - C(d - z) = 0.$$

Choose $z \in \{0, \lambda\}$ with $d - z \neq 0$; at least one choice works since $\lambda \neq 0$. Then $C = (\mu_1 - z)(\mu_2 - z)/(d - z) \neq 0$. The other root is the real constant $\theta = \mu_1 + \mu_2 - C - z$. The roots $0, \lambda, \mu_3$ survive unchanged and distinctly, so there is no two-root endpoint. The complete list of endpoints, including coincidences, is as follows. The column *other large* means $\theta = \lambda$ if $z = 0$, and $\theta = 0$ if $z = \lambda$.

(M, N)	z	θ new	$\theta = \mu_3$	θ other large	$\theta = z$
$(7, 5)$	0	(8,5,1,1)	(8,5,2)	(8,6,1)	(9,5,1)
$(7, 5)$	λ	(7,6,1,1)	(7,6,2)	(8,6,1)	(7,7,1)
$(8, 4)$	0	(9,4,1,1)	(9,4,2)	(9,5,1)	(10,4,1)
$(8, 4)$	λ	(8,5,1,1)	(8,5,2)	(9,5,1)	(8,6,1)

Every entry is Clifford by the preceding subsections, Theorem F, [20, Proposition 2.1] for the entries with a root of multiplicity 9, or [19, Proposition 1] for $(10, 4, 1)$ (here $m = 10$ and $\nu = 5$ satisfy $16 \geq 3\nu$ and $16 > (\nu + 1)^2/4$). Each update has constant eigenvalues, so [20, Lemma 3.2(1)] restores the original tensor.

9.6 All other multiplicities at most two

Finally shift the root of multiplicity $M \in \{7, 8\}$ to zero. Let $r = 15 - M \leq 8$. The polynomial eigenframe lemma supplies r orthogonal normalized fields for the nonzero roots. Divide each field by every common power of q possible, then homogenize all fields to the largest remaining odd degree d. A component of a maximum-degree field is not divisible by q. Let P be the $16 \times r$ matrix of fields and let Λ be the diagonal matrix of their nonzero eigenvalues. Then

$$P^T P = q^d I, \qquad P\Lambda P^T = q^{d-1} A, \qquad P\Lambda^2 P^T = q^{d-2} A^2.$$

If $d \geq 3$, each diagonal entry on the last left side is a sum of $r \leq 8$ squares divisible by q. Lemma 3.11 of [20] forces every field component to be divisible by q, a contradiction. Thus $d = 1$. The fields are $J_i x$ with orthogonal skew, pairwise anticommuting J_i, giving a Clifford Jacobi formula. This is the proof of [20, Lemma 3.3] with its six columns replaced by at most eight. It completes Theorem G.

10 An extreme large Jacobi root

Theorem H. *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with at least three distinct reduced Jacobi eigenvalues. If an eigenvalue of multiplicity seven or eight is the smallest or the largest reduced eigenvalue, then R is Clifford.*

The two-root case has the Clifford and Cayley alternatives of Theorem C. The qualification on the number of roots is necessary: the Cayley tensor has two extreme roots. We first establish the tools used in the proof.

Let $\Pi(x) = \|x\|^2 I - xx^T$. Throughout this section a constant-curvature shift has been made, so that the reduced Jacobi operator has a zero root and nonzero rank $\nu \le 8$. Let Λ be diagonal with the nonzero Jacobi roots for unit vectors, including their multiplicities.

Proposition 10.1. *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ as above.*

1. *If $\mathcal{J}_x = M_x \Lambda M_x^T$ has a global linear factor $M_x : \mathbb{R}^\nu \to \mathbb{R}^{16}$, injective for all $x \neq 0$, then R is Clifford.*

2. *Suppose instead that $\mathcal{J}_z = M_z \Lambda M_z^T$ has a rational homogeneous factor of degree one with the following coherence property. There is a real unit vector X and an orthonormal eigenframe M_X such that, for $Y \perp X$,*
$$M_{sX+Y} = sM_X + M_Y \tag{10.1}$$
as a rational identity. If the full *Gram matrix $\Phi(z) = M_z^T M_z$ on $\mathbb{R}^{16}$ is polynomial, then R is Jacobi-proportional. Consequently it is Clifford or belongs to the Cayley affine family. If Λ is nonscalar, it is Clifford.*

The last classification uses Corollary E. Polynomiality only on $X^\perp$ is not the hypothesis in part (2): in (10.1) the full Gram is

$$\Phi(sX + Y) = s^2 I + s(M_X^T M_Y + M_Y^T M_X) + M_Y^T M_Y,$$

so its mixed coefficient must also be polynomial. Neither part asserts that the required global factor or polynomial Gram exists in the indefinite case.

10.1 Quadratic Gram rigidity

Lemma 10.2. *Let Λ be invertible symmetric of size $\nu \le 8$. Suppose $\Phi(x)$ is symmetric homogeneous quadratic on $\mathbb{R}^{16}$, positive definite for $x \neq 0$, and $\Lambda\Phi(x)$ has the spectrum of $\|x\|^2\Lambda$, with multiplicities. If $\Phi(x_0) = I$ at a unit vector x_0, then $\Phi(x) = \|x\|^2 I$ for every x.*

Proof. The argument refines [19, Section 5, Lemmas 5–8, pp. 11–15 of the preprint], whose hypothesis $n > (\nu+1)^2/4$ fails for $n = 16$, $\nu = 7, 8$; the remaining pairs are closed by the topological count below. We may diagonalize Λ orthogonally. Choose a root λ of multiplicity m whose reciprocal is extremal, and set $q = \|x\|^2$,

$$Q_\lambda(x) = \lambda q \Lambda^{-1} - \Phi(x).$$

For unit x, put $H = \Phi(x)^{1/2}$. Since $H\Lambda H$ is symmetric with the spectrum of Λ, write $H\Lambda H = O\Lambda O^T$. Then $C = O^T H$ satisfies $C\Lambda C^T = \Lambda$, $\Phi = C^T C$, and

$$Q_\lambda(x) = C^T(\lambda\Lambda^{-1} - I)C.$$

Thus $Q_\lambda(x)$ has constant kernel dimension m and is semidefinite with the same sign for all x.

For $u \in \mathbb{R}^\nu$, let $H_\lambda(u)$ be the symmetric 16×16 matrix determined by

$$x^T H_\lambda(u) x = u^T Q_\lambda(x) u.$$

Its entries are quadratic in u and it is semidefinite. Consequently

$$L_u := \{x : Q_\lambda(x)u = 0\} = \ker H_\lambda(u)$$

is a linear subspace. Write $d(u) = \dim L_u$; there is no parity assumption on this dimension.

Choose independent $u_1, \dots, u_m$ maximizing $\sum_i d(u_i)$ and put $S = \operatorname{span}(u_1, \dots, u_m)$. If $u \notin S$, then for every i the vectors u and u_j ($j \neq i$) are independent, so maximality gives $d(u) \leq d(u_i)$. Any $m+1$ independent coefficient vectors have L-intersection zero, since otherwise $\ker Q_\lambda(x)$ would have dimension at least $m+1$ at some nonzero x. The subspace intersection inequality gives

$$d(u) \leq h := \left\lfloor \frac{16m}{m+1} \right\rfloor \qquad (u \notin S). \tag{10.2}$$

The incidence space

$$\mathcal{I} = \{(x, u) : x \neq 0,\ Q_\lambda(x)u = 0\}$$

is a connected real-analytic rank-m vector bundle over $\mathbb{R}^{16} \setminus \{0\}$, of dimension $16 + m$. Its subset $\mathcal{I}_{\text{out}}$ with $u \notin S$ is open and has semialgebraic dimension at most $\nu + h$. If $\nu + h < 16 + m$, it is empty; then every $\ker Q_\lambda(x) = S$. The scalar case $m = \nu$ is immediate. For $\nu \leq 8$, the only remaining nonscalar pairs are

(ν, m)	h	$\nu + h$
$(7, 3)$	12	19
$(8, 2)$	10	18
$(8, 3)$	12	20
$(8, 4)$	12	20
$(8, 5)$	13	21.

Suppose $\mathcal{I}_{\text{out}} \neq \varnothing$ in one of these cases. Set $\ell = 16 + m - \nu$. The coefficient strata with $d(u) \leq \ell - 1$ have incidence dimension at most $\nu + \ell - 1 = 15 + m$. Hence the locus with $d(u) \geq \ell$ has full dimension in $\mathcal{I}_{\text{out}}$ and contains a relatively open subset of $\mathcal{I}$. All $(17 - \ell)$-minors of $H_\lambda(u)$ vanish there. Their pullbacks to $\mathcal{I}$ are real analytic. Connectedness and analytic continuation imply that they vanish on all of $\mathcal{I}$. Thus every *nonempty* coefficient fibre has $d(u) \geq \ell$.

Normalize both variables:

$$\mathcal{E} = \{(x, u) \in S^{15} \times S^{\nu-1} : Q_\lambda(x)u = 0\}, \qquad Z = \{u \in S^{\nu-1} : \det H_\lambda(u) = 0\}.$$

The first projection is an S^{m-1}-bundle, so $\dim \mathcal{E} = 14 + m$. The second projection is a proper surjection onto the compact real algebraic set Z, with fibres $S(L_u)$ of dimension at least $\ell - 1$. Outside S, their dimension is at most $h - 1$ by (10.2). Since $\mathcal{E}_{\text{out}}$ is nonempty and open, semialgebraic dimension inequalities give

$$15 + m - h \ \leq\ z := \dim Z \ \leq\ 15 + m - \ell = \nu - 1.$$

The relevant values are

(ν, m)	ℓ	possible z	minimum sphere-fibre dimension
$(7, 3)$	12	6	11
$(8, 2)$	10	7	9
$(8, 3)$	11	6 or 7	10
$(8, 4)$	12	7	11
$(8, 5)$	13	7	12.

In every row $z < \ell - 1$. The finite-degree Vietoris–Begle theorem [14, Theorem 1.1 (Begle), p. 383], with mod-two coefficients, gives $H^z(Z; \mathbb{F}_2) \cong H^z(\mathcal{E}; \mathbb{F}_2)$. The latter vanishes by the Gysin sequence of the S^{m-1}-bundle over S^{15}: the adjacent base groups in degrees z and $z - m + 1$ vanish in every

listed case. But a nonempty compact real algebraic set of dimension z has a nonzero top mod-two homology class [5, Theorem 1.1, p. 22], hence nonzero $H^z(Z;\mathbb{F}_2)$, a contradiction. All spaces here are compact semialgebraic and triangulable, so Čech and singular cohomology agree; no constant rank of the second projection is assumed.

We conclude that $\ker Q_\lambda(x) = S$ for all $x \neq 0$. At x_0 this is exactly the λ-eigenspace of Λ. Therefore for $u \in S$, $\Phi(x)u = \lambda q \Lambda^{-1} u = qu$. Symmetry removes the cross blocks. Restricting Φ, Λ to the Euclidean orthogonal complement preserves all hypotheses, including normalization at x_0 and the remaining spectrum. Induction on ν proves $\Phi = qI$. □

10.2 The global linear factor

Proof of Proposition 10.1(1). A constant coefficient change normalizes $M_{x_0}^T M_{x_0} = I$ at one unit vector while replacing Λ by its congruent nonzero spectral matrix. For a global linear factor, $\Phi(x) = M_x^T M_x$ is polynomial homogeneous quadratic and positive definite. The nonzero spectrum of $\mathcal{J}_x$ is that of $\Lambda\Phi(x)$, so Lemma 10.2 gives $M_x^T M_x = qI$. For a coefficient basis vector e_i, write $J_i x = M_x e_i$. The identity $\mathcal{J}_x x = 0$, injectivity of M_x, and invertibility of Λ imply $M_x^T x = 0$, so each J_i is skew. The Gram identity makes the J_i orthogonal and gives $J_iJ_j + J_jJ_i = -2\delta_{ij}I$ by polarization. Diagonalizing Λ gives the Clifford Jacobi formula. The zero-column case is simply constant curvature. □

10.3 The rational partner and disjoint pole supports

Proof of Proposition 10.1(2). On the real regular locus, $\Phi = M^T M$ is positive definite. The determinant identity $\det \Phi(z) = q(z)^\nu$ follows from the nonzero spectrum. If Φ is polynomial, continuity makes it positive semidefinite everywhere, and the determinant makes it positive definite for every nonzero real z. By (10.1) and homogeneity, $M_{X+tY} = M_X + tM_Y$ for generic $Y \perp X$ and nonzero t. Polynomial continuity at $t = 0$ gives $\Phi(X) = I$. Lemma 10.2 therefore applies and yields

$$\Phi(z) = q(z)I. \tag{10.3}$$

For a nonzero root λ_j, let I_j be the constant coefficient projector of Λ and $\pi_j(z)$ the corresponding reduced Jacobi projector. From (10.3),

$$C_j(z) := M_z I_j M_z^T = q(z)\pi_j(z) \tag{10.4}$$

on the regular locus, hence as a rational identity. In coordinates $z = sX + Y$, the left side has a denominator dividing a power of a polynomial $D(Y)$ independent of s, by (10.1). The spectral-projector formula for the right side has possible denominators only powers of $q(z) = s^2 + \|Y\|^2$. This polynomial is irreducible over $\mathbb{C}$ in sixteen variables and is coprime to every nonzero $D(Y)$. Unique factorization in the polynomial ring therefore shows that each entry of C_j has no denominator. Homogeneity makes it quadratic. The reduced zero-root family is the polynomial matrix

$$C_0(z) = q(z)I - zz^T - \sum_j C_j(z).$$

Each C_j is symmetric quadratic and satisfies $C_j(z)z = 0$. The polarization argument of Section 2 (after (2.1)) gives its sectional symmetry $\langle C_j(x)y, y\rangle = \langle C_j(y)x, x\rangle$. Thus every root satisfies the proportionality identity and R is Jacobi-proportional. Corollary E classifies it as Clifford or Cayley. If Λ is nonscalar, the shifted reduced zero root and at least two distinct nonzero roots give at least three reduced eigenvalues, excluding the Cayley affine family. □

10.4 Generic partners and the full null-cone field

Lemma 10.3. *For the shifted tensor above, with $\nu \le 8$, one can choose a real unit anchor X and a rational homogeneous partner M_Y on $X^\perp$ such that*

$$\mathcal{J}_{sX+Y} = (sM_X + M_Y)\Lambda(sM_X + M_Y)^T,$$

where M_X is an orthonormal eigenframe. No global linearity in Y is asserted.

Proof. The generic-pair and unique two-plane factorization results [19, Section 4, Lemma 2(1) and Lemma 3, pp. 7–9 of the preprint] apply when $n \ge 2\nu$, including $n = 16$, $\nu = 8$. Fix a suitable X and its orthonormal eigenframe M_X. For generic $Y \perp X$, their conclusion supplies a unique partner M_Y. Write $M_Y = M_X F(Y) + K_X P(Y)$, where K_X is an orthonormal frame of $\ker \mathcal{J}_X$, and write $2\mathcal{B}(X,Y) = \mathcal{J}_{X+Y} - \mathcal{J}_X - \mathcal{J}_Y$. Comparison of the mixed coefficient gives

$$P(Y) = K_X^T\, 2\mathcal{B}(X,Y) M_X \Lambda^{-1}.$$

This is linear. On the generic-pair locus P has full column rank, since the images of $\mathcal{J}_X$ and $\mathcal{J}_Y$ are disjoint. The lower-upper block of $\mathcal{J}_Y$ gives

$$P\Lambda F^T = K_X^T \mathcal{J}_Y M_X.$$

Solving with the rational left inverse $(P^TP)^{-1}P^T$ shows that F, and hence M_Y, is rational homogeneous of degree one. The two-plane identities hold rationally by continuation from the generic real set. Defining $M_{sX+Y} = sM_X + M_Y$ gives the required coherent factor. □

Proof of Theorem H. Shift the extreme root of multiplicity seven or eight to zero. Then $\nu = 7$ or 8, and Λ is definite. Use Lemma 10.3 and coordinates $Y = (y_1, \dots, y_{15})$ on $X^\perp$. Put

$$K = \mathbb{R}(y_1, \dots, y_{15}), \qquad d = \sum_{i=1}^{15} y_i^2, \qquad G(s) = s^2 I + sS + \Phi,$$

where $S = M_X^T M_Y + M_Y^T M_X$ and $\Phi = M_Y^T M_Y$ have entries in K. Equality of the nonzero Jacobi spectra gives

$$\det(tI - \Lambda G(s)) = \prod_{i=1}^{\nu} (t - (s^2 + d)\mu_i) \quad \text{in } K[s,t], \tag{10.5}$$

where μ_i are the nonzero roots with multiplicities.

Let $L = K(\alpha)$ with $\alpha^2 = -d$. The formally real field K does not contain a square root of $-d$, so $[L : K] = 2$. The field L is the fraction field of

$$\mathbb{R}[s, y_1, \dots, y_{15}]/(s^2 + y_1^2 + \cdots + y_{15}^2).$$

By [20, Lemma 3.11 and its proof, p. 398], a sum of at most eight squares in this field vanishes only when every term vanishes. This also follows directly by clearing denominators and applying the polynomial form of that lemma. Every constant definite real quadratic form of dimension at most eight is therefore anisotropic over L, after real diagonalization and rescaling.

Evaluate (10.5) at $s = \alpha$. The matrix $N = \Lambda G(\alpha)$ has characteristic polynomial t^ν, so it is nilpotent. It is self-adjoint for the symmetric bilinear form $H = \Lambda^{-1}$, since $HN = G(\alpha)$ is symmetric. This form is anisotropic over L by definiteness. A self-adjoint nilpotent on an anisotropic space is zero: if its nilpotency index were $k \ge 2$, choose v with $w = N^{k-1}v \ne 0$. Then

$$\langle w, w \rangle_H = \langle v, N^{2k-2} v \rangle_H = 0,$$

contradicting anisotropy. Hence $N = 0$ and, since Λ is invertible,

$$\Phi - dI + \alpha S = 0.$$

The K-linear independence of $1, \alpha$ gives separately $S = 0$ and $\Phi = dI$. Thus the full Gram is $G(s) = (s^2 + d)I$, polynomial on all of $\mathbb{R}^{16}$. Proposition 10.1(2) applies. There are at least three reduced roots, so its Cayley alternative is excluded. The original tensor is Clifford. □

Remark 10.4. Radial Gram does not make a rational frame linear; the scalar Cayley case remains consistent with the argument. The theorem settles the extreme spectral placements of the large root. When that root lies strictly between others, Λ is indefinite and the anisotropy step is unavailable.

Remark 10.5 (An intrinsic null-cone criterion). Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with at least three distinct reduced Jacobi eigenvalues and an eigenvalue of multiplicity seven or eight. Extend its Jacobi operator $R_z = R(\,\cdot\,, z)z$ complex bilinearly, and put $q(z) = z^T z$. Then R is Clifford if and only if

$$R_z^2 = 0 \qquad \text{for every } z \in \mathbb{C}^{16} \text{ with } q(z) = 0.$$

Equivalently, every entry of the polynomial matrix R_z^2 is divisible by q. This equivalence uses irreducibility of the complex quadratic polynomial q.

Here is the reduction to the preceding tools. A constant-curvature shift $J_x = R_x - \lambda\Pi(x)$ satisfies

$$R_x^2 = J_x^2 + q(x)\big(2\lambda J_x + \lambda^2 \Pi(x)\big),$$

because $J_x x = 0$ and $\Pi(x)^2 = q(x)\Pi(x)$. Thus the null-cone condition is unchanged by the shift. Shift the large root to zero and use Lemma 10.3, with the fields K, L and matrices $G(s), S, \Phi$ from the proof of Theorem H; no definiteness is needed here. Over L, set $A = M_{\alpha X + Y}$. The component $K_X P(Y)$ has full column rank, so A is injective and A^T is surjective. Consequently

$$J_{\alpha X + Y}^2 = A\Lambda G(\alpha)\Lambda A^T = 0 \quad \Longleftrightarrow \quad G(\alpha) = 0.$$

The null-cone hypothesis therefore implies $\Phi - dI + \alpha S = 0$. Independence of $1, \alpha$ over K gives $S = 0$ and $\Phi = dI$. The full Gram is polynomial, so Proposition 10.1(2) proves the Clifford conclusion; the assumption of at least three roots excludes its Cayley alternative.

Conversely, in the Clifford Jacobi formula the vectors $z, J_i z$ are mutually orthogonal and null whenever $q(z) = 0$, since $\langle J_i z, J_j z\rangle = \delta_{ij} q(z)$ and $\langle z, J_i z\rangle = 0$. All products of its rank-one terms then vanish, giving $R_z^2 = 0$. The criterion reduces the remaining indefinite cases to a polynomial divisibility statement; it does not establish that statement for general Osserman tensors.

The first differentiated spectral identity does not supply a new constraint beyond constant rank. For example, with three distinct shifted roots $0, a, c$, one has $J_z^3 = 0$ on the null cone and

$$J_z^2 B(z, w) J_z^2 = 0, \qquad 2B(z, w) = J_{z+w} - J_z - J_w.$$

At a generic null point z, the partner factorization gives $\operatorname{rank} J_z = \nu$. All $(\nu + 1)$-minors of J_x vanish identically. Differentiating these minors gives $\langle B(z, w)t, t'\rangle = 0$ for every w and $t, t' \in \ker J_z$. This tangent-space statement does not require the kernel to be nondegenerate. Since $\operatorname{Im} J_z^2 \subseteq \ker J_z$, it already implies the displayed sandwich identity. Polarized sectional symmetry also gives $B(t, t')z = 0$ on the whole kernel. These identities do not prove $J_z^2 = 0$.

11 The coherent factor and its transpose pencil

Shift the chosen reduced Jacobi root to zero. In this section its nonzero rank is $\nu \in \{7, 8\}$, and $k = 16 - \nu$. Let Λ be the diagonal matrix of the nonzero roots, with multiplicities, and put $H = \Lambda^{-1}$. Define the polarization by

$$\mathcal{B}(X, Y) = \tfrac{1}{2}(\mathcal{J}_{X+Y} - \mathcal{J}_X - \mathcal{J}_Y).$$

For a real unit anchor X, choose orthonormal frames $M_X : \mathbb{R}^\nu \to E_X = \operatorname{Im} \mathcal{J}_X$ and $K_X : \mathbb{R}^k \to \ker \mathcal{J}_X$, with $\mathcal{J}_X = M_X \Lambda M_X^T$.

11.1 Construction and normalization

The generic-pair and two-plane factorization statements in [19, Section 4, Lemmas 2(1) and 3] require $n \geq 2\nu$, which holds here, including $\nu = 8$. For a generic orthonormal pair (X, Y) with disjoint Jacobi images, fixing M_X uniquely determines a partner M_Y for which

$$\mathcal{J}_{sX+tY} = (sM_X + tM_Y)\Lambda(sM_X + tM_Y)^T. \tag{11.1}$$

We use only this two-plane result, not the three-plane construction in that source, whose stronger dimension hypothesis does not hold here. Choose X in the nonempty open set of anchors for which the generic pair construction is available, and extend in $Y \in X^\perp$ by homogeneity.

Write $M_Y = M_X F(Y) + K_X P(Y)$. Comparing the mixed coefficient in (11.1) gives the actual lower block

$$P(Y) = K_X^T \, 2\mathcal{B}(X, Y) M_X \Lambda^{-1}. \tag{11.2}$$

In particular P is linear; the right factor Λ^{-1} is part of its normalization. The other block identities are

$$F\Lambda + \Lambda F^T = A(Y) := M_X^T 2\mathcal{B}(X, Y) M_X, \tag{11.3}$$

$$P\Lambda F^T = K_X^T \mathcal{J}_Y M_X, \tag{11.4}$$

$$F\Lambda F^T = M_X^T \mathcal{J}_Y M_X, \qquad P\Lambda P^T = K_X^T \mathcal{J}_Y K_X. \tag{11.5}$$

The generic disjointness of the images makes P injective there. Solving (11.4) with a rational left inverse of P therefore makes F rational and homogeneous of degree one. All the displayed equations extend as rational identities. Consequently

$$M_{sX+Y} = sM_X + M_X F(Y) + K_X P(Y), \qquad \mathcal{J}_{sX+Y} = M_{sX+Y} \Lambda M_{sX+Y}^T. \tag{11.6}$$

Lemma 11.1. *If F has no pole along any complex hypersurface in $X^\perp$, then* (11.6) *is a global linear factor. In that event the tensor is Clifford.*

Proof. In the polynomial unique factorization domain on $X^\perp$, each entry of a rational function with no divisorial pole has constant reduced denominator. Thus F is polynomial. Its degree-one homogeneity makes it linear. Equation (11.6) is then a polynomial identity on all sixteen variables. At every nonzero real vector its factor is injective because the Jacobi operator has rank ν. Proposition 10.1(1) applies. □

11.2 Kernel directions give genuine Jacobi factors

Extend P linearly to $V = \mathbb{R}^{16}$ by $P(X) = 0$. For $g \in \mathbb{R}^k$ define $N_g : \mathbb{R}^\nu \to V$ by

$$N_g^T Y = P(Y)^T g \qquad (Y \in V). \tag{11.7}$$

This is linear in g. For a coefficient $a \in \mathbb{R}^\nu$, define the transpose pencil $B(a) : \mathbb{R}^k \to V$ by $B(a)g = N_g a$.

Lemma 11.2. *For every real g,*

$$\mathcal{J}_{K_X g} = N_g \Lambda N_g^T. \tag{11.8}$$

If $g \neq 0$, the map N_g is injective. If $a \neq 0$ is real, the map $B(a)$ is injective, equivalently $Y \mapsto P(Y)a$ is onto $\mathbb{R}^k$.

There are no nonzero fixed real coefficient space U and proper fixed real subspace $T \subset \mathbb{R}^k$ such that $P(Y)U \subset T$ for all Y.

Proof. For $z = K_X g$, the scalar identity

$$\langle \mathcal{J}_Y z, z\rangle = (P(Y)^T g)^T \Lambda (P(Y)^T g) = \langle \mathcal{J}_z Y, Y\rangle$$

follows from the coherent factor and sectional symmetry. Polarizing in Y proves (11.8). Its left side has rank ν for nonzero real g, so N_g has full column rank. If $B(a)g = N_g a = 0$ for nonzero real a, this injectivity forces $g = 0$. Transposition proves the surjectivity claim. Finally, choose nonzero $g \perp T$. The proposed containment would make N_g annihilate U, contradicting its injectivity. □

Let g_X be the unit vector with $K_X g_X = X$. The radial and skew identities used below are

$$B(a)g_X = -M_X a, \qquad K_X^T B(a) \text{ is skew-symmetric}, \qquad \operatorname{Im} B(a) \subset X^\perp. \tag{11.9}$$

Indeed injectivity of the coherent factor and $\mathcal{J}_z z = 0$ give $M_z^T z = 0$ on its domain. The coefficient of s after writing $z = sX + Y$ gives $P(Y)^T g_X = -M_X^T Y$, proving the first identity. The last follows from $P(X) = 0$. For the middle identity, (11.8) and $\mathcal{J}_{K_X g} K_X g = 0$ imply $N_g^T K_X g = 0$ at real $g \neq 0$. Hence $\langle g, K_X^T B(a) g\rangle = 0$ for all real g, a; polarization in g proves skewness.

Lemma 11.3. *For every nonzero complex coefficient a, the kernel of $B(a)$ is totally isotropic in $\mathbb{C}^k$ for the standard bilinear form. Writing $s = \operatorname{rank} B(a)$, one has*

$$s \geq \lceil k/2 \rceil, \qquad \operatorname{rank}(B(a)B(a)^T) = 2s - k.$$

Proof. If $0 \neq g \in \ker B(a)$ and $g^T g \neq 0$, the continued Osserman characteristic and minimal-polynomial identities make $\mathcal{J}_{K_X g}$ diagonalizable of rank ν. Equation (11.8), continued polynomially in g, then forces N_g to have rank ν, contrary to $N_g a = 0$. Thus every vector of $\ker B(a)$ is null. Polarization makes that kernel totally isotropic, so its dimension is at most $\lfloor k/2 \rfloor$. If $L = \ker B(a)$, then $\operatorname{Im} B(a)^T = L^\perp$ and $L \subset L^\perp$. The image of $B(a)B(a)^T$ therefore has dimension $\dim L^\perp - \dim L = 2s - k$. □

11.3 The local pole budget

Put $F_0 = \frac{1}{2} A \Lambda^{-1}$, with A from (11.3), and set $\Omega = (F - F_0)\Lambda$. Then Ω is skew-symmetric and homogeneous of degree one. Equations (11.4)–(11.5) imply that $P\Omega$ is polynomial and that the negative-order terms of $\Omega H \Omega^T$ are cancelled only by terms of at most the pole order of Ω.

Lemma 11.4. *Let f be an irreducible polynomial defining a genuine pole of Ω, of order $d > 0$. Every maximal ν-minor of P is divisible by f^{2d}. If $d = 1$ and the residue has rank r, every such minor is divisible by f^r. Consequently, if D is the reduced common denominator of Ω, then D^2 divides every maximal minor of P, and $2 \deg D \le \nu$.*

Proof. Work over the discrete valuation ring obtained by localizing the polynomial ring at (f), with uniformizer $\pi = f$. An alternating matrix over its fraction field is put, by a unimodular congruence, into two-by-two blocks $a_j \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and a zero block. One obtains this form by choosing an entry of least valuation as pivot and clearing its two rows and columns; all elimination coefficients are integral. Repeating gives the required congruence.

Write the negative block valuations as $-d_j$, with $d_j > 0$. Because $P\Omega$ is integral, the corresponding two columns of the unimodularly transformed P are divisible by π^{d_j}. Thus every maximal minor has valuation at least $2\sum_j d_j$. The largest d_j is the pole order d, proving the first bound. At a simple pole the residue rank is twice the number of negative blocks, giving the second bound. Unimodular column changes preserve the maximal-minor ideal. Distinct irreducible factors multiply in the polynomial ring, proving the assertion for D. At least one maximal minor is nonzero, since P is generically injective; that minor has degree ν. □

If W is the highest Laurent coefficient of Ω at a pole, the leading terms of the same identities give

$$PW = 0, \qquad W^T = -W, \qquad WHW^T = 0 \tag{11.10}$$

over the residue field. In particular its nonzero rank is even, and its image is H-isotropic. The upper-block identity gives the last equation because its order $2d$ term has no other possible cancellation. In the square case $\nu = 8$, the budget reads $D^2 \mid \det P$ and $\deg D \le 4$. The real factor types and the subsequent support reductions are separate steps; they do not follow from this degree bound alone.

12 Real pole types and the intrinsic denominator

We keep the notation of Section 11. All real nonzero Jacobi operators have rank $\nu \in \{7, 8\}$, and $k = 16 - \nu$. We first justify the real generic-plane condition used to control the denominator. This extends the dimension counts in [20, Lemma 3.8] to both ranks; no global three-plane factorization is assumed.

We use Rakić duality in the form recalled in [23, §§1–2]. It applies to the full kernel after a root is shifted to zero, including the radial direction. Indeed, write $a = cX + b$ with $b \perp X$ in the original μ-eigenspace at X. Duality and the curvature symmetries give

$$\mathcal{J}_a^{\text{original}} X = -c\mu|X|^2 b + \mu|b|^2 X = \mu(|a|^2 X - \langle a, X\rangle a).$$

Subtracting $\mu\Pi(a)$ gives $\mathcal{J}_a X = 0$. Exchanging the vectors proves $a \in \ker \mathcal{J}_X$ if and only if $X \in \ker \mathcal{J}_a$.

Lemma 12.1. *There is an open dense set of real orthonormal triples (X, Y, Z) such that*

$$\dim\bigl(\operatorname{Im} \mathcal{J}_X \cap \operatorname{Im} \mathcal{J}_{sY+tZ}\bigr) \le 1 \qquad ((s,t) \ne (0,0)).$$

The three pairwise Jacobi-image intersections may simultaneously be required to be zero, and the three Jacobi kernels to have zero common intersection.

Proof. Fix an orthonormal pair (Y, Z) and let w run around its unit circle. The iterated sphere bundle of triples (w, a, X) satisfying

$$|a| = |X| = 1, \qquad a \in \ker \mathcal{J}_w, \quad X \in \ker \mathcal{J}_a$$

is a compact smooth manifold of dimension $1 + 2(k-1) = 2k-1$. The kernel bundles are smooth because their rank is constant. Project it to $X \in S^{15}$. Sard's theorem and compactness show that its regular values form an open dense set. Values outside its image are regular and have empty fibre, so surjectivity is unnecessary.

Zero-eigenvalue duality identifies the fibre at X with the set of pairs (w, a) having $a \in \ker \mathcal{J}_X \cap \ker \mathcal{J}_w$. A nonempty regular fibre has dimension $2k - 16 = 16 - 2\nu$. For any fixed w, the unit sphere of an intersection of dimension $d > 0$ embeds in this fibre, so $d - 1 \le 16 - 2\nu$. Consequently

$$\dim(\operatorname{Im} \mathcal{J}_X \cap \operatorname{Im} \mathcal{J}_w) = 2\nu - 16 + d \le 1.$$

The empty-fibre case satisfies the same assertion directly. The set of triples failing this bound is closed: failure is a closed matrix-rank condition for some point of a compact circle. The regular-value argument gives density for each fixed (Y, Z), and hence global density on the independent-triple domain.

We explain why orthogonalization preserves the bound. For real vectors v_i, duality gives

$$a \in \bigcap_i \ker \mathcal{J}_{v_i} \quad \Longleftrightarrow \quad \operatorname{span}\{v_i\}_i \subset \ker \mathcal{J}_a.$$

Thus the common kernel depends only on their span. After projecting the plane of Y, Z along X to $X^\perp$, each new direction w' corresponds to an old direction w with $\operatorname{span}(X, w') = \operatorname{span}(X, w)$. The two common kernels, and therefore the image-intersection dimensions, agree. The Gram–Schmidt map to the orthonormal triple manifold is a surjective smooth submersion. It sends the open dense good set to an open dense set of orthonormal triples.

The generic-pair theorem supplies the three pairwise-disjoint conditions by intersecting three open dense sets. Finally, triples with a common unit kernel vector are the image of the compact bundle over that vector with fibre $V_3(\mathbb{R}^k)$. Its dimension is $15 + 3k - 6$, which is 36 or 33, strictly less than $\dim V_3(\mathbb{R}^{16}) = 42$. Its image is closed and has measure zero. Deleting it proves the final assertion. □

Lemma 12.2. *Let X be an admissible anchor and (X, Y, Z) a triple as in Lemma 12.1 such that* $\operatorname{span}(Y, Z)$ *is not contained in the polar hypersurface of F. Then the binary rational function $F(sY + tZ)$ has no pole at any nonzero real direction.*

Proof. Subtract a linear block F_0 with $F_0\Lambda + \Lambda F_0^T = A$ as in Section 11; the correction $\Omega = (F - F_0)\Lambda$ is skew. At a genuine real pole, its highest Laurent coefficient W is a nonzero real skew matrix, of rank at least two. The leading coefficient of the coherent factor $M_X F + K_X P$ is $M_X W \Lambda^{-1}$, so its image has that same rank and lies in E_X.

Away from the pole the columns of this factor lie in $\operatorname{Im} \mathcal{J}_{sY+tZ}$. These real image spaces have constant dimension and vary continuously through the pole. Multiplying the factor by the appropriate power of a local parameter and taking its limit therefore places the leading coefficient in the limiting Jacobi image. This gives an intersection with E_X of dimension at least two, a contradiction. □

Proposition 12.3. *At a generic regular real anchor, let D_X be the reduced common denominator of F on $X^\perp$. If $\nu = 7$, it is constant or a single positive semidefinite quadratic, irreducible over $\mathbb{R}$. If $\nu = 8$, up to nonzero real constants the possibilities are*

$$1, \qquad Q, \qquad Q^2, \qquad Q_1Q_2, \qquad f,$$

where the Q_i are distinct real irreducible positive semidefinite quadrics, and f is real irreducible of degree four. Each Q_i has rank at least two. A rank-two quadric may split into conjugate linear factors over $\mathbb{C}$; a real irreducible quartic may split into two conjugate quadrics.

Proof. The complement of the open dense set of Lemma 12.1 is a closed semialgebraic subset of $V_3(\mathbb{R}^{16})$ of dimension at most 41. Its fibre over X lies in $V_2(X^\perp)$, of dimension 27. Hence the anchors whose fibre has dimension 27 form a semialgebraic set of dimension at most 14. Outside its closure the good planes are open dense in $X^\perp$. In this proposition, *generic* means outside this closed nowhere-dense set and the proper algebraic set used for specialization. The factor D_X is homogeneous. A generic two-dimensional real subspace of $X^\perp$ preserves its degree and all its generic pole orders. To see this, write all entries over the least common denominator. On each irreducible component of that denominator, at least one numerator is not divisible by its defining factor. The locus where all these numerators vanish on that component has complex codimension at least two in $X^\perp_{\mathbb{C}}$. A generic projective line avoids the union of these loci and meets each reduced component transversally at smooth points. The same argument applies to the entries with the highest order at a repeated component.

Every real homogeneous factor of odd degree has a zero on every real projective line on which it is not identically zero. This would be a genuine real pole on a generic line, contrary to Lemma 12.2. Thus all irreducible real factors have positive even degree. A real indefinite quadratic has real zeros on an open set of two-planes; one may choose such a plane generic for the preceding cancellation condition and Lemma 12.1. It too is excluded. An irreducible real quadratic is therefore semidefinite and has rank at least two; choose its sign positive.

Lemma 11.4 gives $2 \deg D_X \le \nu$. For $\nu = 7$ this leaves degree zero or two, and for $\nu = 8$ degree zero, two or four, with exactly the listed factorizations. Finally an irreducible polynomial over $\mathbb{R}$ either remains irreducible over $\mathbb{C}$ or is a product of two conjugate irreducible factors of equal degree. This proves the last statements. □

12.1 A frame-independent polar divisor

The same pole divisor can be studied while the anchor varies, without choosing a rational orthonormal eigenframe.

Proposition 12.4. *On the open set of pairs with nonzero norms and disjoint Jacobi images there is a unique rational matrix $T(X, Y)$, regular on that set, satisfying*

$$\operatorname{Im} T \subset \operatorname{Im} \mathcal{J}_Y, \qquad \operatorname{Im} T^T \subset \operatorname{Im} \mathcal{J}_X, \qquad T + T^T = 2\mathcal{B}(X, Y).$$

Its reduced common denominator $D(X, Y)$ can be chosen real and bihomogeneous, of bidegree (d, d), where $d = 0, 2$ for $\nu = 7$ and $d = 0, 2, 4$ for $\nu = 8$. It has no nonconstant factor depending on only one slot, and it satisfies

$$D(Y, X) = D(X, Y), \qquad D(X, Y + aX) = D(X, Y).$$

For a generic anchor its fibre is exactly D_X, up to a nonzero constant. If $d = 2$, one global choice of sign makes $D(X, Y) \ge 0$ on real pairs, and $D(X, Y) = \langle A_X Y, Y\rangle$ is the sectional form of an algebraic curvature tensor.

Proof. The coherent two-plane factors give $T = M_Y \Lambda M_X^T$, proving existence. The difference of two choices is skew and has image in both disjoint Jacobi images, so it is zero. The image conditions can be written with the rational spectral projections of $\mathcal{J}_X$ and $\mathcal{J}_Y$; for example, $T = \pi_Y T \pi_X$. Solving this uniquely solvable linear system over $\mathbb{R}(X, Y)$ proves rationality. Uniqueness also gives

$$T(Y, X) = T(X, Y)^T, \qquad T(tX, sY) = tsT(X, Y), \qquad T(X, Y + aX) = T(X, Y) + a\mathcal{J}_X.$$

The last identity follows first from the coherent two-plane factor and then as a rational identity. Exchange gives the other shear.

Take the least common multiple of the reduced denominators of the entries of T in the polynomial UFD on the two slots. Separate homogeneity allows a bihomogeneous generator; exchange makes its bidegrees equal. Each shear is a polynomial automorphism and adds only a polynomial matrix to T, so it preserves the polar divisor with its orders. Hence its pullback changes D only by a nonzero real constant. Comparing highest-degree coefficients in the sheared slot makes this constant one. A pure-slot factor would acquire infinitely many distinct irreducible transforms under the other shear. Since the polar divisor has finitely many components, no such factor exists.

Fix a generic unit real X. Then $M_Y = T(X, Y) M_X \Lambda^{-1}$ and $T = M_Y \Lambda M_X^T$ recover one matrix from the other by constant coefficient maps. They have exactly the same poles; the polynomial lower block leaves exactly the poles of F. For each irreducible denominator factor g, choose a numerator N not divisible by g. The set $V(g, N)$ has codimension at least two in the pair space; outside a proper algebraic set of anchors its fibres have codimension at least two in the second slot. They therefore contain no component of the specialized denominator. Thus generic specialization creates no common factor in all numerator entries. The preceding shear removes the radial coordinate of Y. Thus the fibre degree is d, and Proposition 12.3 gives its values.

The elementary shears generate SL_2 on the two slots, so $D(Y, -X) = D(X, Y)$. The even degree d gives symmetry. At $d = 2$, also $D(X, aX) = D(X, 0) = 0$. Writing $D(X, Y) = \langle A_X Y, Y \rangle$ gives a symmetric quadratic family A; shear invariance implies $A_X X = 0$. The curvature reconstruction identity (2.2) therefore supplies its algebraic curvature tensor.

For generic real X the quadratic $D(X, \cdot)$ is semidefinite by Proposition 12.3; this persists for all X by continuity. Choose a nonzero fibre and orient its sign positively, say at X_0. For every Y outside its kernel, symmetry gives $D(Y, X_0) = D(X_0, Y) > 0$, so the semidefinite form $D(Y, \cdot)$ is positive semidefinite. Such Y are dense, and continuity proves the global sign assertion. □

The auxiliary family A is not asserted to be Osserman, to have constant rank, or to commute with the original Jacobi family. Exceptional anchors can have cancellation or a zero denominator fibre. The intrinsic divisor is used only on the generic locus where its specialization is the actual polar divisor.

13 The rank-seven indefinite case

Proposition 13.1. *Suppose the nonzero Jacobi rank is seven and Λ is indefinite. The coherent upper block at a regular anchor has no divisorial pole. Consequently the tensor is Clifford.*

Proof. Here P has nine rows and seven columns. For a nonzero complex coefficient $a \in \mathbb{C}^7$ put $s(a) = \operatorname{rank} B(a)$, using the transpose pencil of Section 11. Lemma 11.3 gives $s(a) \geq 5$ for every complex $a \neq 0$, while Lemma 11.2 gives $s(a) = 9$ for every nonzero real a.

Assume there is a pole and choose an irreducible complex component Q of its hypersurface in $X_{\mathbb{C}}^{\perp} \cong \mathbb{C}^{15}$. Its dimension is fourteen. The highest Laurent coefficient W is nonzero and skew over

its function field, so its rank r is at least two. Form the closure of the incidence

$$\mathcal{I} = \overline{\{(Y,[a]) : Y \text{ generic in } Q,\ [a] \in \mathbb{P}(\operatorname{Im} W(Y))\}} \subset \mathbb{C}^{15} \times \mathbb{P}^6.$$

It is irreducible of dimension $14 + r - 1 \geq 15$. Let A be the Zariski closure of its projection to $\mathbb{P}^6$, and write $\alpha = \dim A$. The identity $PW = 0$ in (11.10) implies $P(Y)a = 0$ throughout the incidence. For fixed $[a]$ this is a linear space in Y of dimension $15 - s(a)$: its annihilator is precisely $\operatorname{Im} B(a)$. The generic fibre dimension theorem therefore gives

$$15 - \alpha \leq \dim \mathcal{I} - \alpha \leq 15 - s(a), \qquad \text{hence} \qquad 5 \leq s(a) \leq \alpha$$

at a generic point of A.

The second residue identity $WHW^T = 0$ places A inside the smooth irreducible quadric

$$\mathcal{N}_H = \{[a] \in \mathbb{P}^6 : a^T H a = 0\}, \qquad H = \Lambda^{-1}.$$

This quadric has dimension five. Thus $\alpha = 5$ and $A = \mathcal{N}_H$. The determinantal locus $\{s(a) \leq 5\}$ is Zariski closed and contains the dense generic locus just obtained, so it contains all of A. It has no real point, since the real rank is nine. This would make $\mathcal{N}_H$ real-pointless, contradicting the indefiniteness and nondegeneracy of H.

There is therefore no pole component. Lemma 11.1 gives a global linear factor and the Clifford conclusion. □

The argument is componentwise, so a quadratic that splits over $\mathbb{C}$ causes no exception. It treats all five remaining patterns with a reduced root of multiplicity eight simultaneously, when that root is not extreme; the definite case is Theorem H. Indefiniteness is essential: for the two-root Cayley model the corresponding coefficient null quadric is real-pointless.

14 Exclusion of a simple quadratic pole of residue rank four

We work in the rank-eight coherent-partner construction of Section 11, with Λ indefinite as in Section 30. The coefficient spectral form $H = \Lambda^{-1}$ is a nondegenerate indefinite real form on the entire $\mathbb{R}^8$. The actual lower pencil $P(Y)$ is a generically invertible linear 8×8 matrix for $Y \in X^{\perp}$, and the correction $\Omega(Y)$ is skew-symmetric with $P\Omega$ polynomial. The residue identities (11.10) give $PW = 0$ and $WHW^T = 0$ for a leading polar coefficient W.

The real-plane pole classification, Proposition 12.3, used as a separate input, says that a genuine irreducible real quadratic pole factor Q is positive semidefinite of rank $\rho \geq 2$, after changing its sign if necessary. We do not infer this positivity from the denominator degree bound.

Proposition 14.1. *Let $\nu = 8$ with H indefinite, and let X be a regular real anchor at which Proposition 12.3 applies. A simple pole of Ω along a real irreducible positive semidefinite quadratic Q cannot have skew residue rank four over the fraction field of $\mathbb{R}[Y]/(Q)$.*

Proof. The argument first produces a normalized linear matrix factorization. The actual radial row then contradicts the dimensions available for its radical coefficients.

Degree saturation and a linear companion. The second bound of Lemma 11.4 makes Q^4 divide $\det P$. Both have degree eight, so $\det P = cQ^4$ for a nonzero real constant c. At the prime (Q) the four-dimensional residue image lies in $\ker P$. Thus at least four Smith exponents are positive, and their sum is four. They are exactly four ones and four zeros. Every seven-minor is divisible by Q^3. Consequently

$$P^{\sharp} = \frac{\operatorname{adj} P}{cQ^3}$$

is a real linear matrix and satisfies $PP^\sharp = P^\sharp P = QI$. Modulo Q it has rank four and

$$\operatorname{Im} P^\sharp = \ker P = \operatorname{Im} W.$$

Skewness makes the residue identity $WHW^T = 0$ equivalent to isotropy of this image for H. Therefore

$$(P^\sharp)^T H P^\sharp = QH_0 \tag{14.1}$$

with H_0 constant real symmetric. These arguments take place over the real prime (Q); they also apply when a rank-two Q splits into conjugate linear factors over $\mathbb{C}$.

Choose a real parameter e with $Q(e) = 1$ and put $P_e = P(e)$. Replace $P, P^\sharp$ by $P_e^{-1}P, P^\sharp P_e$, respectively. This is an output-row change for P; it does not change the coefficient form H. The two matrices are now the identity at e, and evaluation of (14.1) at e gives

$$PP^\sharp = P^\sharp P = QI, \qquad (P^\sharp)^T H P^\sharp = QH, \qquad P(e) = P^\sharp(e) = I. \tag{14.2}$$

Choose real parameter coordinates such that e is the first coordinate vector and

$$Q = x_0^2 + y_1^2 + \cdots + y_{\rho-1}^2,$$

with remaining coordinates z_j in its radical. Comparing coefficients in (14.2) gives

$$P = x_0 I + \sum_{i=1}^{\rho-1} y_i C_i + \sum_j z_j D_j,$$
$$P^\sharp = x_0 I - \sum_{i=1}^{\rho-1} y_i C_i - \sum_j z_j D_j.$$

Each C_i is H-skew, squares to $-I$, and anticommutes with the other C_i. Every member D of the real linear space $\mathcal{D} = \operatorname{span}_{\mathbb{R}}\{D_j\}$ is H-skew, satisfies $D^2 = 0$, and anticommutes with every C_i and every other member of $\mathcal{D}$. The radical coefficients are not assumed to vanish from their square-zero identity alone.

The actual radial row. Before the row change, let g_X be the kernel-frame vector representing the unit anchor X. The coherent radial identity (11.9) is

$$g_X^T P(Y) = -Y^T M_X. \tag{14.3}$$

The image of this row, as Y ranges over $X^\perp$, is all of $\mathbb{R}^8$, because M_X is an orthonormal image frame in $X^\perp$. After the constant row change the same assertion holds with the fixed covector $g_X^T P_e$. Hence the span of all coefficient matrices of the normalized P has dimension at least eight. The displayed expansion of P therefore implies

$$8 \le \rho + \dim_{\mathbb{R}} \mathcal{D}. \tag{14.4}$$

This condition uses the full radial row of the actual partner, rather than an arbitrary matrix factorization of Q.

Restrictions on the signature of H. Average a positive definite metric over the finite group of signed monomials in the C_i. The resulting positive metric g is invariant. The g-self-adjoint operator representing H commutes with the C_i, so its positive and negative eigenspaces are invariant modules. A single complex structure makes their dimensions even. Two anticommuting complex structures give a quaternionic action, making their dimensions multiples of four. With four generators, $C_1C_2C_3$

is an involution commuting with the first three and anticommuting with C_4. Its two eigenspaces are quaternionic under C_1, C_2 and are exchanged by C_4. Each nonzero invariant module then has dimension a multiple of eight.

Since H is indefinite in dimension eight, $\rho \geq 5$ is impossible. Up to reversing its sign, the only possibilities are

signature of H	ρ
$(2,6)$	2
$(4,4)$	$2, 3, 4.$

In particular, the signatures $(1,7)$, $(3,5)$ and their reversals are already excluded.

Anti-complex skew maps. Put $J = C_1$ and identify $\mathbb{R}^8$ with $\mathbb{C}^4$ using J. The form H is the real part of a Hermitian form whose signature is half the real signature. In a complex basis its matrix G may be chosen real diagonal or real hyperbolic. An operator anticommuting with J has the form $D(v) = B\overline{v}$, and H-skewness is

$$GB + B^T G = 0.$$

Thus GB is a complex alternating matrix. Its rank is even, and since D is complex antilinear, its real rank is twice the complex rank of B; so the real rank of D is a multiple of four. On the other hand, $D^2 = 0$ and H-skewness imply that $\operatorname{Im} D$ is totally H-isotropic. Its rank is bounded by the real Witt index of H. For signature $(2,6)$ this forces $\mathcal{D} = 0$, contrary to (14.4) with $\rho = 2$.

A single complex generator and signature $(4,4)$. Suppose $\rho = 2$ and choose $0 \neq D_0 \in \mathcal{D}$, if one exists. It has real rank four, and $L = \operatorname{Im} D_0 = \ker D_0$ is a J-invariant maximal isotropic space, of complex dimension two. In a hyperbolic complex basis, with an isotropic complement to L, we may write

$$G = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \qquad D_0(v) = B_0\overline{v}, \qquad B_0 = \begin{bmatrix} 0 & J_2 \\ 0 & 0 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Indeed, the invertible upper-right block of B_0 is a complex alternating 2×2 matrix; a change on L and the dual change on its complement normalize it to J_2.

Every $D \in \mathcal{D}$ preserves L, by $DD_0 = -D_0 D$. Writing $D(v) = B\overline{v}$, skewness gives

$$B = \begin{bmatrix} E & F \\ 0 & -E^T \end{bmatrix}, \qquad F^T = -F.$$

Anticommutation with D_0 says $EJ_2 = J_2\overline{E}^T$. Its solutions are exactly

$$E = \begin{bmatrix} a & ib \\ ic & \overline{a} \end{bmatrix}, \qquad a \in \mathbb{C}, \quad b, c \in \mathbb{R}.$$

The upper-left block of $D^2 = 0$ is $E\overline{E} = 0$, whereas

$$E\overline{E} = (|a|^2 + bc)I_2.$$

The quadratic form $|a|^2 + bc$ has real signature $(3,1)$. The image of the real linear map $D \mapsto E$ is a linear subspace of its null cone, hence has dimension at most one. Its kernel has only the alternating complex 2×2 block F, of real dimension two. Therefore $\dim \mathcal{D} \leq 3$, also trivially if $\mathcal{D} = 0$. This gives $\rho + \dim \mathcal{D} \leq 5$, again contradicting (14.4).

Two or more complex generators. It remains to treat $\rho = 3, 4$ with signature $(4,4)$. The first two generators make the real eight-space a quaternionic two-space. Put $J' = C_1 C_2$ and $T = -J'D$ for $D \in \mathcal{D}$. Then J' is H-skew, $J'^2 = -I$, and J' anticommutes with C_1 and C_2; since D also

anticommutes with them, T commutes with C_1, C_2. The operator D commutes with J', so $T^2 = 0$; it is H-self-adjoint and commutes with C_1, C_2. Thus T is quaternionic-linear. The invariant form H is the real part of a quaternionic Hermitian form of signature $(1, 1)$. In a quaternionic basis with that form diagonal, every such T has the form

$$T = \begin{bmatrix} a & b \\ -\bar{b} & d \end{bmatrix}, \qquad a, d \in \mathbb{R}, \quad b \in \mathbb{H}.$$

Its nilpotence gives zero real trace, namely $4(a + d) = 0$. Consequently $d = -a$ and

$$T^2 = (a^2 - |b|^2) I_2.$$

The real linear image of $\mathcal{D}$ lies in the null cone of this quadratic form of signature $(1, 4)$ and hence has dimension at most one. The map $D \mapsto -J'D$ is injective, so $\dim \mathcal{D} \leq 1$. For $\rho = 3$ and 4, respectively, this gives $\rho + \dim \mathcal{D} \leq 4$ and 5, both contradicting (14.4). All possibilities are excluded. □

Remark 14.2. The proof concerns a simple quadratic pole with full skew residue rank four in the rank-eight setup. It does not exclude simple rank-two residues, double poles, or quartic factors. The invariant form is the nondegenerate H on the whole coefficient space. A restriction of H to a smaller residue support can be degenerate; the anti-complex rank argument and the signature table above must not be applied to such a restriction without separate justification.

15 Exclusion of a double quadratic pole

We work in the indefinite rank-eight case. Retain the coherent factors, the real quadratic-pole condition, and the intrinsic denominator from Sections 11 and 12. At a regular real unit anchor X, choose orthonormal frames M_X of $E_X = \operatorname{Im} \mathcal{J}_X$ and K_X of $\ker \mathcal{J}_X$. Put $H = \Lambda^{-1}$ and write, for $Y \perp X$,

$$M_Y = M_X F(Y) + K_X P(Y), \qquad F = F_0 + \Omega H, \qquad \Omega^T = -\Omega.$$

Here F_0, P are linear, P is generically invertible, and $P\Omega$ is polynomial. The latter identity follows from the polynomial lower–upper Jacobi block $P\Lambda F^T$. The original lower frame K_X will remain orthonormal throughout.

Proposition 15.1. *The reduced common denominator cannot be Q_X^2 on a nonempty real open branch of regular anchors, where Q_X is an irreducible real positive semidefinite quadratic.*

15.1 A degree bound for an entire skew block

Lemma 15.2. *Let P_0 be a real polynomial $m \times b$ matrix with linear entries and generic column rank b. Let Ω_0 be a skew rational $b \times b$ matrix such that $P_0\Omega_0$ is polynomial. If Ω_0 has pole order e at a real irreducible quadratic Q, then every maximal b-minor of P_0 is divisible by Q^{2e}. In particular, $b \geq 4e$.*

Proof. Work in the discrete valuation ring at Q, and take a Smith form

$$P_0 = U \begin{bmatrix} \operatorname{diag}(Q^{a_1}, \dots, Q^{a_b}) \\ 0 \end{bmatrix} V$$

with U, V invertible over that ring. Congruence gives the skew matrix $\widehat{\Omega} = V\Omega_0 V^T$, still of pole order e. Choose an off-diagonal entry of valuation $-e$. Polynomiality of $P_0\Omega_0$, in its two corresponding rows, forces both associated Smith exponents to be at least e. Their sum is therefore at least $2e$, which is a lower bound for the valuation of every maximal minor. At least one maximal minor is nonzero; its polynomial degree is at most b, whereas $\deg Q^{2e} = 4e$. □

Under the double-pole hypothesis this lemma applied to P, Ω gives

$$\det P = c_X Q_X^4, \qquad c_X \neq 0, \tag{15.1}$$

by degree eight. Let g_X represent X in the original lower frame. The radial row is $g_X^T P(Y) = -Y^T M_X$. For $Y \in E_0(X)$, the reduced zero eigenspace, this row vanishes. Equation (15.1) implies $Q_X(Y) = 0$ there. Since Q_X is positive semidefinite, every such vector lies in its radical. Together with shear invariance, this shows that the quadratic pole form annihilates the full kernel of $\mathcal{J}_X$ and has rank at most eight.

15.2 The aligned auxiliary family

Let $D(X, Y)$ be the intrinsic reduced denominator, including pole orders. By Proposition 12.4 it has bidegree $(4, 4)$ in the present branch. We justify the square root globally, rather than applying the degree-two intrinsic-divisor statement directly to D.

Factor D in $\mathbb{R}[X, Y]$. By the same proposition there is no pure-slot factor. Thus each factor is primitive in Y, and Gauss's lemma identifies its factorization with the one over $\mathbb{R}(X)[Y]$. On a suitable nonempty Zariski-open set of anchors, the fibre of D is the actual reduced denominator (Proposition 12.4), and the nonconstant fibres of distinct such factors remain pairwise coprime and squarefree: distinct irreducible factors are coprime over $\mathbb{R}(X)$, and squarefree over its algebraic closure because the characteristic is zero; after a generic linear change of the Y-coordinates, their resultants and discriminants with respect to one coordinate are nonzero polynomials, and they stay nonzero off a proper algebraic set of anchors. This set meets the real open branch. At such an anchor, each factor fibre divides Q_X^2 and is squarefree, hence is proportional to the irreducible real quadratic Q_X. Pairwise coprimeness permits only one factor, and its exponent must be two. There is therefore exactly one global irreducible factor, with multiplicity two. Consequently

$$D = c\, \mathcal{Q}(X, Y)^2, \tag{15.2}$$

where $\mathcal{Q}$ is a real polynomial of bidegree $(2, 2)$ and c is a nonzero constant. No anchor-dependent square root has been chosen.

Both shear identities for D pass to $\mathcal{Q}$. Indeed their squares agree in the polynomial ring including the shear parameter; the integral-domain property gives a common sign, and parameter zero fixes that sign positively. The elementary shears generate the exchange $(X, Y) \mapsto (Y, -X)$, and the even bidegrees give

$$\mathcal{Q}(Y, X) = \mathcal{Q}(X, Y), \qquad \mathcal{Q}(X, Y + tX) = \mathcal{Q}(X, Y), \qquad \mathcal{Q}(X, X) = 0.$$

Write $\mathcal{Q}(X, Y) = \langle A_X Y, Y\rangle$. Then A_X is a symmetric quadratic family and $A_X X = 0$. By (2.2) it is the Jacobi family of an algebraic curvature tensor; this is not used below.

Anchors at which Proposition 12.3 applies and the fibre of D is D_X are dense. At each of them $c\,\mathcal{Q}(X, \cdot)^2 = D_X$ is one of the listed types; a square can only be Q^2, so $\mathcal{Q}(X, \cdot)$ is an irreducible positive semidefinite quadratic with a genuine double pole, and the radial-row argument above applies at X. Semidefiniteness persists everywhere by continuity. Choose a nonzero fibre positive,

at X_0. Symmetry gives $\mathcal{Q}(Y, X_0) = \mathcal{Q}(X_0, Y) > 0$ outside its kernel, so A_Y is positive semidefinite on that dense set, and then everywhere. The radial-row argument above gives

$$\ker \mathcal{J}_X \subset \ker A_X, \qquad A_X \succeq 0. \tag{15.3}$$

The inclusion extends from this dense set of anchors by continuity of the kernel projector. No assertion that A is Osserman or commutes with $\mathcal{J}$ is used.

Lemma 15.3. *For the fixed anchor X of the branch and, as a rational identity, for $Y \in X^{\perp}$, there is a constant real symmetric positive semidefinite matrix C such that*

$$A_{sX+Y} = (sM_X + M_Y)C(sM_X + M_Y)^T.$$

Moreover $\operatorname{rank} C = \operatorname{rank} Q_X$ *and*

$$\Omega HC - CH\Omega \quad \textit{is polynomial.} \tag{15.4}$$

Proof. For a generic partner Y, the matrix $B = [M_X, M_Y]$ has full column rank sixteen. Image inclusion in (15.3) gives $A_{sX+tY} = BZ(s,t)B^T$ with Z a symmetric quadratic polynomial block matrix. Its values at the axes force the form

$$Z = \begin{bmatrix} s^2C + stE & stF_1 \\ stF_1^T & t^2C_1 + stH_1 \end{bmatrix}.$$

For generic real s, t, its image lies in the image of $[sI, tI]^T$. Hence $[-tI, sI]Z = 0$ identically. Coefficient comparison gives $E = H_1 = 0$ and $F_1 = C_1 = C$. Since $C = M_X^T A_X M_X$, it is independent of Y and positive semidefinite. The fibre defining A_X is proportional to Q_X, giving the rank.

The mixed coefficient of A_{sX+Y} makes $FC + CF^T$ polynomial and linear in Y. Substitution of $F = F_0 + \Omega H$ gives (15.4). □

15.3 The whole polar correction and total isotropy

Consider the constant real linear map on skew matrices $\mathcal{L}(Z) = ZHC - CHZ$. Choose a constant complement to its kernel. On that complement $\mathcal{L}$ is injective, so (15.4) shows that the complementary part of Ω is polynomial. Subtract it. The remaining skew correction, still denoted by Ω, has the same genuine double pole and satisfies

$$\Omega HC = CH\Omega, \qquad P\Omega \text{ polynomial.} \tag{15.5}$$

This is an identity for the *entire rational correction*, not only its highest residue.

Completion of Proposition 15.1. Let $r = \operatorname{rank} C = \operatorname{rank} Q_X$ and $s = 8 - r$. Under a constant coefficient change G, the simultaneous transformations are

$$C' = G^{-1}CG^{-T}, \quad H' = G^T HG, \quad \Omega' = G^{-1}\Omega G^{-T}, \quad P' = PG.$$

They preserve skewness, the exact identities (15.5), and generic injectivity of column subpencils. They do not change the lower output frame.

First normalize $C = \mathrm{diag}(I_r, 0_s)$. Let k be the nullity of the restriction of H to the first r-space and set $d = r - k$, $\ell = s - k$. Nondegeneracy of H gives $k \le s$. Further C-preserving changes put the two forms, on spaces of dimensions d, k, k, ℓ, in the form

$$C = \mathrm{diag}(I_d, I_k, 0_k, 0_\ell), \qquad H = \begin{bmatrix} H_0 & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & H_1 \end{bmatrix}, \tag{15.6}$$

where H_0, H_1 are nondegenerate and empty blocks are allowed. Here is an explicit justification. Orthogonally split the first r-space into its nondegenerate part and radical. The radical's pairing with the last s-space has row rank k, or H would be singular. Changes of shape $\left[\begin{smallmatrix} O & T \\ 0 & N \end{smallmatrix}\right]$, with O orthogonal and N invertible, preserve C. The first d rows of T remove the cross terms from H_0; N normalizes the radical pairing to $[I_k, 0]$. The remaining rows of T remove the diagonal and off-diagonal terms on the paired k-space. This gives (15.6); nondegeneracy gives that of H_1.

Direct comparison in $\Omega H C = C H \Omega$ now gives the exact rational block form

$$\Omega = \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & B_0 & M & T_0 \\ 0 & M & 0 & 0 \\ 0 & -T_0^T & 0 & V_0 \end{bmatrix}, \qquad Z H_0 = H_0 Z, \tag{15.7}$$

where Z, B_0, M, V_0 are skew. In particular, both displayed off-diagonal copies of M have the same sign because $M^T = -M$.

Diagonalize H_0 orthogonally. Its commuting block Z is a direct sum of skew blocks on the nonzero eigenspaces of H_0. Each such space is definite for the full indefinite form H, and therefore has dimension at most seven. The remaining skew block, on the last three spaces of (15.7), has dimension $s + k = 8 - d$. If $d > 0$, it also has dimension at most seven.

Because these are exact constant block decompositions of the entire correction, the corresponding columns of $P\Omega$ are separately $P_{\mathrm{block}}\Omega_{\mathrm{block}}$ and are polynomial. Each P_{block} is linear and generically injective. Lemma 15.2 excludes an order-two quadratic pole in every block of dimension at most seven. If $d > 0$, no block could carry the genuine double pole. Thus $d = 0$, or equivalently

$$CHC = 0. \tag{15.8}$$

The range of C is a real totally H-isotropic space, so $r \le \mathrm{Witt}(H) \le 4$. This excludes all ranks five through eight without an anisotropy argument over a residue field.

Ranks two and three. The quadratic Q_X is real irreducible and semidefinite, so $r \ge 2$. With $d = 0$, we have $k = r$, $\ell = 8 - 2r$, and

$$\Omega = \begin{bmatrix} B_0 & M & T_0 \\ M & 0 & 0 \\ -T_0^T & 0 & V_0 \end{bmatrix}.$$

Write $P = [P_R, P_{R'}, P_L]$ for the column blocks on the three spaces. The columns indexed by the middle r-space in $P\Omega$ are exactly $P_R M$. For $r = 2, 3$, even a simple pole of the skew block M would require $r \ge 4$ by Lemma 15.2. All its possible denominators divide Q_X^2, so M is polynomial. Subtract its polynomial skew off-diagonal block. The remaining entire polar correction is supported on the first and last spaces, of combined dimension $8 - r < 8$. It still has a genuine double pole, contradicting the same degree lemma. Hence ranks two and three are impossible.

The last rank-four case. Now $r = 4$, H has signature $(4,4)$, and the constant coefficient normalization gives

$$H' = \Lambda' = \begin{bmatrix} 0 & I_4 \\ I_4 & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Omega' = \begin{bmatrix} B_0 & M \\ M & 0 \end{bmatrix}, \quad P' = [P_R, P_N].$$

The four-column identity $P_R M$ polynomial excludes a double pole of M. If M were polynomial, subtracting it would leave a double-polar skew block supported on the first four-space, also impossible. Thus M has a genuine simple Q_X-pole.

Every maximal four-minor of P_R is consequently divisible by Q_X^2. Its degree is four, so all those minors are constant multiples of Q_X^2, with at least one nonzero. Their Plücker coordinates describe one fixed real four-dimensional output space T. Thus $\operatorname{Im} P_R(Y) \subset T$ on the generic locus, and then for every Y by polynomial continuation.

Choose a nonzero vector $\eta \in T^\perp$ in the *original* Euclidean lower output coordinates, and put $z = K_X \eta \neq 0$. Since only coefficient coordinates were changed,

$$\eta^T P'(Y) = (0, b(Y)^T), \qquad P' \Lambda' P'^T = P \Lambda P^T.$$

The hyperbolic matrix Λ' therefore gives

$$\langle \mathcal{J}_{tX+Y} z, z \rangle = \eta^T P'(Y) \Lambda' P'(Y)^T \eta = 0$$

for all real t, Y. These are polynomial identities, so rational frame singularities do not remove any directions. Sectional symmetry gives $\langle \mathcal{J}_z V, V \rangle = 0$ for every real V. Symmetry and polarization imply $\mathcal{J}_z = 0$, contradicting rank eight at the nonzero vector z. This is the sectional-symmetry fixed-output obstruction of the actual lower pencil.

All possible ranks $2 \leq r \leq 8$ have been excluded. At no point was a nonorthogonal lower output change identified with a physical orthonormal frame, or a highest-residue block substituted for an entire rational polar block. The double-quadratic branch is impossible. □

16 Closed kernels and a fixed ten-dimensional image

Throughout this section, $V = \mathbb{R}^{16}$ is Euclidean and $\mathcal{J}_x$ is the Jacobi family of a shifted Osserman algebraic curvature tensor of rank eight. Write $q(x) = \langle x, x \rangle$, and extend all polynomial identities complex linearly. The distinct nonzero reduced eigenvalues are the constants $\lambda_1, \ldots, \lambda_t$ times $q(x)$. For a real nonzero anchor x, put $E_x = \operatorname{Im} \mathcal{J}_x$ and $K_x = \ker \mathcal{J}_x$.

We use the coherent-partner construction established earlier. At a regular real unit anchor e, choose orthonormal image and kernel frames, with image frame M_e and kernel frame K_e. In these frames, the rational factor at $z = te + y$, $y \perp e$, has the form

$$M_z = \begin{bmatrix} tI + F(y) \\ P(y) \end{bmatrix}, \qquad \mathcal{J}_z = M_z \Lambda M_z^T, \tag{16.1}$$

where Λ is an invertible constant diagonal matrix, F is rational, and P is a generically invertible linear 8×8 matrix. This construction is an input to the lemmas below.

Lemma 16.1 (Closed-kernel criterion)**.** *Suppose that, on a nonempty Euclidean-open set of real anchors x,*

$$\mathcal{J}_y z = 0 \qquad \text{for every } y, z \in K_x. \tag{16.2}$$

Then $\mathcal{J}_x^2 = 0$ for every complex x with $q(x) = 0$. Moreover, at every regular real anchor e, the full coherent-partner Gram matrix is radial: $M_{te+y}^T M_{te+y} = (t^2 + q(y))I$ as a rational identity.

Proof. We first check that the complex null quadric contains a dense set of rank-eight Jacobi operators. Over $\mathbb{C}(y)$, adjoin α with $\alpha^2 = -q(y)$. The polynomial $q(y)$ has rank fifteen, so it is not a square up to sign in this rational function field. The lower block $P(y)$ in (16.1) remains invertible after this extension. Consequently $M_{\alpha e+y}$ is injective and its transpose is surjective. Since Λ is invertible, $\mathcal{J}_{\alpha e+y}$ has rank eight. This assertion requires no nondegeneracy of its Gram matrix.

Every nine-minor of $\mathcal{J}_x$ vanishes identically. On its rank-eight locus, choose a rational kernel basis using a nonzero eight-minor. Substitution of arbitrary linear combinations of this basis for y, z in (16.2) gives finitely many rational identities in x. They vanish on a nonempty real open subset, hence identically. Such minor charts cover the generic null locus just found. Thus (16.2) holds at its generic complex points. In particular, we have not evaluated a spectral projector with denominator $q(x)$ on the null quadric.

Fix such a null point x, and set $K = \ker \mathcal{J}_x$. Then $\dim K = 8$, and the Osserman characteristic polynomial makes $\mathcal{J}_x$ nilpotent. If $q|_K$ is not identically zero, choose $y \in K$ with $q(y) \neq 0$. The continued closed-kernel identity gives $\mathcal{J}_y K = 0$. The polynomial identity

$$\mathcal{J}_y \prod_{j=1}^{t} (\mathcal{J}_y - \lambda_j q(y) I) = 0$$

holds over $\mathbb{C}$, by continuation from the real family. Its roots are distinct when $q(y) \neq 0$, so $\mathcal{J}_y$ is diagonalizable. Its characteristic polynomial gives rank eight, whence $K = \ker \mathcal{J}_y$. The kernel of a self-adjoint diagonalizable operator on a nondegenerate bilinear space is nondegenerate. Hence K is nondegenerate. But self-adjointness of $\mathcal{J}_x$ then gives $\operatorname{Im} \mathcal{J}_x = K^\perp$, and $\mathcal{J}_x$ restricts to an invertible operator on $K^\perp$, contradicting nilpotency.

Therefore $q|_K = 0$. Polarization makes K totally isotropic, and its dimension gives $K = K^\perp$. It follows that $\operatorname{Im} \mathcal{J}_x = \ker \mathcal{J}_x$, and hence $\mathcal{J}_x^2 = 0$. The null quadric is irreducible, so this polynomial identity extends to all its points, including those of smaller Jacobi rank.

Finally put $G_z = M_z^T M_z$. At $z = \alpha e + y$,

$$0 = \mathcal{J}_z^2 = M_z \Lambda G_z \Lambda M_z^T$$

implies $G_z = 0$, by left and right inverses. Expanding the Gram matrix in the radial variable gives

$$G_{te+y} = t^2 I + tS(y) + \Phi(y).$$

Thus $\alpha S(y) + \Phi(y) - q(y) I = 0$. Independence of $1, \alpha$ over $\mathbb{C}(y)$ yields $S = 0$ and $\Phi = q(y) I$, as required. □

Remark 16.2. The last lemma supplies the full radial Gram, not just its restriction to $e^\perp$. Applying the previously established radial-Gram classification is a separate step. In the remaining nonscalar patterns it gives a Clifford structure; the scalar two-root Cayley alternative is handled separately.

For completeness, the connection with the transpose pencil is direct. At a real anchor, define N_g by $N_g^T y = P(y)^T g$, extending P to vanish in the anchor direction. Sectional symmetry gives $\mathcal{J}_{K_e g} = N_g \Lambda N_g^T$. If the full transpose pencil $B(a)g = N_g a$ has image contained in E_e for every coefficient a, then $\operatorname{Im} N_g \subset E_e$. It follows that $\mathcal{J}_{K_e g}$ kills K_e, proving the closed-kernel condition at this anchor. Applying this on a real open set permits Lemma 16.1. One may also propagate this condition away from the null cone using the rational projector

$$\Pi_0(x) = \prod_{j=1}^{t} \frac{\mathcal{J}_x - \lambda_j q(x) I}{-\lambda_j q(x)}$$

and the identity $\mathcal{J}_{\Pi_0(x)y}\Pi_0(x)z = 0$. Clearing denominators makes propagation frame independent; the minor charts in the proof above supply the subsequent passage to null points.

Lemma 16.3 (Fixed ten-dimensional image)**.** *Fix a regular real anchor e. Let $B(a)$ be its actual linear transpose pencil, with eight real coefficient variables, and set $C(a) = B(a)B(a)^T$. Thus $C(a)$ is positive semidefinite of rank eight for every real $a \neq 0$. Suppose that all images of $B(a)$ lie in a fixed real subspace T of dimension at most ten.*

Suppose also that the coefficient residue locus is a reduced complete intersection $A = V(h, q_0) \subset \mathbb{P}^7_{\mathbb{C}}$, where h, q_0 are independent real quadrics and q_0 is positive definite. Assume that C has generic rank two on each component of A, and that no nonzero real linear form vanishes identically on any component. Then $\operatorname{Im} B(a) = E_e$ *for every real $a \neq 0$.*

Proof. The radial column $B(a)g_e = -M_e a$ puts $E_e \subset T$. The lower projection $\Gamma(a) = K_e^T B(a)$ is skew-symmetric and has image in a fixed real space R of dimension at most two. If $\dim R$ is at most one, skewness gives $\Gamma = 0$, and the assertion follows from rank eight. Otherwise, every such skew map is a multiple of a fixed rank-two skew map. Write $\Gamma(a) = \ell(a)J$, with ℓ real linear. We may assume $\ell \neq 0$.

On $E_e \oplus R$ the Gram has block form

$$C(a) = \begin{bmatrix} A_0(a) & \ell(a)L(a) \\ \ell(a)L(a)^T & \ell(a)^2 D \end{bmatrix},$$

where D is constant positive definite of size two and L is linear. These degrees follow from the linear factor B and its lower block ℓJ. The component hypothesis ensures that ℓ is nonzero generically on each component of A. There the lower block has rank two, so the rank-two condition on C makes its Schur complement zero. Consequently the quadratic matrix

$$H = A_0 - LD^{-1}L^T$$

vanishes on the reduced complete intersection. Its degree-two ideal is spanned by h, q_0, and therefore $H = hH_0 + q_0H_1$ with constant real symmetric matrices H_0, H_1.

At real points with $\ell \neq 0$, the Schur complement is positive semidefinite of rank six. Continuity gives positivity and rank at most six everywhere. At real nonzero points of $\ker \ell$, the image of B lies in E_e, so A_0 is positive definite of rank eight. Since $LD^{-1}L^T$ has rank at most two, H has rank at least six there as well. Thus H has constant rank six for all real nonzero coefficients.

Normalize $q_0 = 1$. The range of h is a nontrivial closed interval, and the pencil $tH_0 + H_1$ is positive semidefinite of rank six on that interval. The kernel of a strict positive combination of two positive semidefinite matrices is the intersection of their kernels. Constancy of rank therefore makes these kernels equal, including at the endpoints. Hence all $H(a)$ have one fixed six-dimensional image $F \subset E_e$.

On the seven-dimensional real space $\ker \ell$, compress $A_0 = H + LD^{-1}L^T$ to $F^\perp \cap E_e$, a two-dimensional space. The result is $L_2(a)D^{-1}L_2(a)^T$, positive definite for every nonzero $a \in \ker \ell$. Thus the real 2×2 matrix $L_2(a)$ would be invertible at every such point. But L_2 is a linear map from a seven-dimensional space into a four-dimensional matrix space, so it has a nonzero kernel. This contradiction gives $\Gamma = 0$. Rank eight then proves the assertion. □

Remark 16.4. The component hypothesis in Lemma 16.3 holds for an integral full-span residue locus. It also holds for a conjugate pair whose components span distinct nonreal hyperplanes: a real linear form vanishing on one would be proportional both to that hyperplane equation and to its conjugate. Thus the lemma applies to that reduced union without assuming its coordinate ring is a domain.

17 Quadratic matrices with a small scalar span

Lemma 17.1. *Let $n \geq 3$, and let $C(a)$ be a real symmetric homogeneous quadratic matrix on $\mathbb{R}^n$. Suppose that $C(a)$ is positive semidefinite of one fixed positive rank for every $a \neq 0$. If its scalar quadratic entries span a real vector space of dimension at most three, then its image is independent of $a \neq 0$.*

Proof. The quadratic form $q(a) = \operatorname{tr} C(a)$ is positive definite and belongs to the scalar-entry span. Complete it to a basis q, p, r, allowing unused forms to be zero. Thus $C = qD_0 + pD_1 + rD_2$ for constant symmetric matrices D_i. Change real coefficient coordinates so that $q(a) = \|a\|^2$. Brickman's theorem states that the joint range of two real quadratic forms on a unit sphere in dimension at least three is convex; see [15, Theorem 1.1]. Consequently $\{C(a) : q(a) = 1\}$ is convex.

For any two matrices in this normalized family, their midpoint is another member of the same rank. For positive semidefinite matrices A, B one has $\ker(A + B) = \ker A \cap \ker B$. Equal ranks therefore force the endpoint kernels to agree. This holds for every pair, and symmetry gives a common image. Homogeneity extends the conclusion from $q = 1$ to all nonzero coefficients. □

Remark 17.2. For a full eight-coefficient transpose pencil $B(a)$ with $C(a) = B(a)B(a)^T$ of real rank eight and radial column $B(a)e = -La$, where L maps the coefficient space isomorphically onto the Jacobi image E, the fixed image in Lemma 17.1 is exactly E. Indeed, over the reals the images of $C(a)$ and $B(a)$ agree, and the radial vectors span E. The conclusion is an image statement; it does not by itself identify the entire Gram matrix with a radial scalar matrix. The closed-kernel criterion supplies that subsequent step in the actual curvature setting.

18 Small supports and the residue-image incidence

Use the normalization of Section 11. The shifted Jacobi rank is $\nu \in \{7, 8\}$, the lower dimension is $k = 16 - \nu$, and the actual lower pencil $P(Y) : \mathbb{R}^\nu \to \mathbb{R}^k$ is linear and generically injective for $Y \in X^\perp$. Its extension to $\mathbb{R}^{16}$ has $P(X) = 0$. The maps N_g and $B(a)g = N_g a$ are as in (11.7). In particular, Lemma 11.2 gives

$$P(Y)U \subset T \text{ for all } Y, \quad U \neq 0 \quad \Longrightarrow \quad T = \mathbb{R}^k \tag{18.1}$$

for fixed real coefficient and output subspaces U, T. This identity is valid through exceptional partner directions by polynomial continuation; no limiting rational frame is required.

18.1 Four- and five-dimensional quadratic supports

For a specified simple irreducible real quadratic pole Q, let W be its nonzero rank-two skew residue over the fraction field of $\mathbb{R}[Y]/(Q)$. Its *real support* is the smallest constant real coefficient subspace whose complexified or residue-field extension contains $\operatorname{im} W$. This includes both geometric components when Q splits over $\mathbb{C}$.

Lemma 18.1. *A simple quadratic component with rank-two residue cannot have real support of dimension at most five. This is componentwise: other polar factors may be present.*

Proof. Suppose the support is contained in a fixed real d-space U. At the prime (Q), the kernel of $P|_U$ contains the rank-two residue image. Rectangular Smith form therefore gives at least two positive valuations. Every maximal d-minor of $P|_U$ is divisible by Q^2. Some such minor is nonzero by generic injectivity, and has degree d; thus $d \geq 4$.

If $d = 4$, all maximal minors are constant multiples of Q^2. Their nonzero Plücker vector has constant projective direction, so the generic images $P(Y)U$ are one fixed real four-space. Projection to its complement is polynomial and vanishes generically, hence for every Y. This contradicts (18.1), since $k \geq 8$.

If $d = 5$, the maximal minors give

$$\bigwedge^5 (P(Y)|_U) = Q(Y)^2 \ell(Y) \tag{18.2}$$

with ℓ a nonzero real homogeneous linear map into $\bigwedge^5 \mathbb{R}^k$. Its generic values are decomposable. The Plücker relations are polynomial, so every value is decomposable. The resulting five-spaces have one of three arrangements: constant; all contain a fixed four-space T_0; or all lie in a fixed six-space T_1.

Here is the elementary linear-Grassmannian argument. Two independent decomposable five-vectors whose sum is decomposable determine five-spaces meeting in dimension four. Fix two distinct members, with intersection T_0 and sum T_1. A third member adjacent to both either contains T_0 or lies in T_1. If one member contains T_0 but is not contained in T_1, a member in T_1 not containing T_0 would meet it in dimension at most three, a contradiction. Thus the family is a star through T_0 or a top inside T_1; the one-dimensional Plücker image is the constant case.

Constant output five and a top inside output six both contradict (18.1). In the star case project $P|_U$ modulo T_0. The resulting linear matrix family has rank at most one, first generically and then everywhere by its two-minors. A linear space of rank-one matrices has either a common image line or a common coefficient kernel of codimension one. Indeed, a sum of two rank-one tensors with independent left and right factors has rank two; applying this to all pairs gives the dichotomy. If $u_1 \otimes v_1, u_2 \otimes v_2$ have parallel left factors but independent right factors, any third member $u_3 \otimes v_3$ with u_3 independent of u_1 would need v_3 parallel to both v_1 and v_2, which is impossible. A common image line confines $P(Y)U$ to a fixed five-space. A common kernel is a fixed four-space $U_0 \subset U$ with $P(Y)U_0 \subset T_0$ for all Y. Both violate (18.1).

Only divisibility by the specified factor Q^2 was used. No sole-denominator or minimal-support refinement enters the star case. □

Proposition 18.2 (Isolated skew corrections)**.** *Let $U \subset \mathbb{R}^\nu$ be a fixed real coefficient space of dimension at most five, and write $P_U = P|_U$ in a fixed orthonormal basis of U. If $\Gamma(Y)$ is a rational skew-symmetric endomorphism of U and $P_U(Y)\Gamma(Y)$ is polynomial, then Γ has no pole along an irreducible real quadratic. No assertion that Γ is the full actual residue or correction is required.*

Proof. If Γ had a pole of any order $e > 0$ at a quadratic Q, its leading coefficient $W = (Q^e\Gamma) \bmod Q$ would be nonzero and skew, hence of rank at least two. Polynomiality of $P_U\Gamma$ gives $P_U W = 0$ modulo Q. Thus P_U has corank at least two there, and every maximal minor is divisible by Q^2. The degree count, constant-output case, and linear Plücker star/top argument in the proof of Lemma 18.1 now apply without change. Each gives a nonzero fixed real coefficient subspace mapped by the *actual* pencil P into a proper fixed output, contradicting (18.1). Neither H-isotropy nor a simple-pole assumption is used in this proposition; the essential inputs are skewness, generic injectivity of P_U, and the actual output property (18.1) of P. □

18.2 A real-irreducible quartic has full real support

Lemma 18.3. *In the case $\nu = 8$, a genuine simple real-irreducible quartic pole has real residue support of dimension eight, including when it splits into two conjugate complex quadratics.*

Proof. Let the quartic be f. The local budget of Lemma 11.4 and $\deg \det P = 8$ give

$$\det P = cf^2, \qquad c \in \mathbb{R} \setminus \{0\}.$$

The nonzero simple skew residue has positive even rank. The same budget forces rank two and Smith exponents $(1, 1, 0^6)$ at (f). Every seven-minor is divisible by f, so

$$A = \frac{\operatorname{adj} P}{cf}$$

is a homogeneous cubic polynomial matrix with $AP = PA = fI$. Modulo f, its image is the two-dimensional kernel of P and therefore equals the image of the rank-two residue W.

If a constant real row b^T annihilates the real support of W, it annihilates A modulo f. Each entry of $b^T A$ has degree three, less than $\deg f = 4$, so $b^T A = 0$ identically. Multiplication by P gives $fb^T = 0$, hence $b = 0$. The argument takes place over the irreducible real prime (f). It makes no claim that either individual complex component has full complex support. □

18.3 Dimension and rank of the closed image

From now on let $\nu = k = 8$, and let $H = \Lambda^{-1}$ be indefinite. Put $h(a) = a^T Ha$ and $G(a) = B(a)B(a)^T$. Fix one irreducible *complex* pole hypersurface $Z \subset \mathbb{P}(X_{\mathbb{C}}^{\perp}) = \mathbb{P}^{14}$ with generic residue rank two. Let I be the closure of its projectivized residue-image bundle in $Z \times \mathbb{P}^7$, and let A be the closed image of I in $\mathbb{P}^7$. These are irreducible projective varieties, and $\dim I = 13 + 1 = 14$. Write $\alpha = \dim A$ and let s be the generic rank of $B(a)$ on A.

Lemma 18.4. *One has*

$$4 \le s \le \alpha \le 5.$$

Moreover A has no real point, $A \subset V(h)$, and $\operatorname{rank} G(a) = 2 \operatorname{rank} B(a) - 8$ *for every nonzero complex a.*

Proof. The rank identity and the lower bound four are Lemma 11.3. The residue identities (11.10) imply $A \subset V(h)$, so initially $\alpha \le 6$. They also put every fibre of $I \to A$ inside

$$L_a = \{[Y] \in \mathbb{P}^{14} : P(Y)a = 0\}.$$

The linear map $Y \mapsto P(Y)a$ has rank $\operatorname{rank} B(a)$, since its transpose is $B(a)$ and $P(X) = 0$. Generically, $\dim L_a = 14 - s$. The generic incidence fibre has dimension $14 - \alpha$, whence $s \le \alpha$.

The minors giving $\operatorname{rank} B \le \alpha$ vanish generically on A and therefore everywhere on A. A nonzero real coefficient has rank eight by Lemma 11.2, so A has no real point. If $\alpha = 6$, it would equal the irreducible six-dimensional quadric $V(h)$, which has real points because h is indefinite. Hence $\alpha \le 5$. □

18.4 Eliminating the unequal full-support type

Proposition 18.5. *Let a genuine simple irreducible real quadratic or quartic pole have total real residue support eight. If a complex residue-image component has $\alpha = 5$ and $s = 4$, then the full transpose image is E_X at this anchor:* $\operatorname{Im} B(a) = E_X$ *for every real $a \neq 0$.*

Proof. The rank identity makes $G = 0$ on that closed image A. If the real pole polynomial is geometrically irreducible, conjugation preserves its unique complex component, so A is real-defined. Its span is then a real-defined subspace containing the residue image, hence equals $\mathbb{P}^7$ by total support eight. If A is real-defined and spans $\mathbb{P}^7$, the bound $\deg A \ge 1 + \operatorname{codim} A$ for nondegenerate varieties [10, p. 1, (∗)] gives $\deg A \ge 3$. Since A has no real points, intersection with a generic transverse real complementary plane makes its degree even, hence at least four. If every entry of G is

a multiple of h, evaluation at a nonzero real h-null vector contradicts rank eight. Otherwise choose a real entry f independent of h. The degree-four complete intersection $V(h, f)$ contains A. The complete intersection (h, f) is Cohen–Macaulay, hence unmixed. Since $\deg V(h, f) = 4 \le \deg A$, the variety A is its only component, of multiplicity one. Thus (h, f) is generically reduced and unmixed, hence radical, and its degree-two part is $\operatorname{span}(h, f)$. The scalar-span criterion, Lemma 17.1 and Remark 17.2, gives the claimed image.

An irreducible real pole polynomial is geometrically irreducible or has exactly two conjugate irreducible factors. In the latter case write $A, \overline{A}$ for the two images and $U, \overline{U}$ for their vector spans. Total real support eight means $U + \overline{U} = \mathbb{C}^8$. Both images have dimension five and G vanishes on both. Put $d = \dim_{\mathbb{C}} U$.

If $d \le 6$, dimension forces $d = 6$ and $A = \mathbb{P}(U)$. The conjugation-invariant intersection $U \cap \overline{U}$ has dimension at least four and contains a nonzero real vector, giving a forbidden real point of A. If $d = 7$, the restriction $h|_U$ has rank at least six and its irreducible quadric fivefold equals A. The two spans are distinct hyperplanes, say $\ell = 0$ and $\overline{\ell} = 0$. Their two degree-two quadrics exhaust the degree-four complete intersection $V(h, \ell\overline{\ell})$, which is their reduced union by the same unmixedness argument. Its quadratic ideal has scalar span two, so the criterion again applies.

If $d = 8$ and $A = \overline{A}$, use the first paragraph. If $d = 8$ and they are distinct, each has degree at least three [10, p. 1, $(*)$]. Any quadratic entry f independent of h would put their union of degree at least six inside $V(h, f)$ of degree four, impossible. Thus all entries would be multiples of h, already contradicted. These alternatives also cover a real quartic splitting into conjugate quadratics; no individual full-span assumption was made in the split cases. □

The proposition gives a pointwise full-image identity. On a nonempty real open branch of anchors, the argument following Remark 16.2 (see the proof in Section 30) and Lemma 16.1 turn it into the classification conclusion. It is enough here to use the earlier scalar-span criterion; the stronger two-form radial-Gram theorem is not needed for this incidence reduction.

18.5 Saturation is conditional on equal dimension and rank

Lemma 18.6. *For the incidence above, suppose $\alpha = s$. For generic $a \in A$ the fibre I_a equals the entire projective linear space L_a. Consequently two distinct complex pole hypersurfaces with this equality cannot have the same residue image.*

Proof. The closed fibre I_a lies in L_a and has dimension $14 - \alpha = 14 - s = \dim L_a$ at a generic point. A proper closed subset of an irreducible projective linear space has smaller dimension, so $I_a = L_a$.

If two pole incidences had the same image A and satisfied this equality, their fibres would coincide over one common dense open subset of A. Their irreducible closures would therefore coincide. Projecting those closures to $\mathbb{P}^{14}$ would give the same pole hypersurface, contrary to distinctness. No reality or full-span assumption on an individual image enters this lemma. □

Remark 18.7. The order of use is essential. First remove proper real supports, using the small-support lemma above and Propositions 20.2, 20.9 and 20.5, with Lemma 18.3 for quartics. Then every remaining real component has support eight. Lemma 18.4 gives the possibilities $(4, 4)$, $(5, 4)$, $(5, 5)$; Proposition 18.5 reduces the unequal one to full transpose alignment, which ends in the Clifford conclusion on an open branch by Lemma 16.1. Only at this point is $\alpha = s$ available, and Lemma 18.6 gives saturation and distinct images. The saturation lemma does not itself prove the equality.

19 Aligned quadratic families and two quadratic poles

We use the coherent construction at Jacobi rank eight, in the remaining nonscalar patterns. At a regular unit anchor X, let $E = \operatorname{Im} \mathcal{J}_X$, $K = \ker \mathcal{J}_X$, and use orthonormal frames M_X, K_X. Write $Y = M_X v + u \in X^\perp$, with $u \in K \cap X^\perp$, and

$$M_Y = M_X F(Y) + K_X P(Y), \qquad H = \Lambda^{-1}.$$

The lower pencil P is linear and generically invertible; its radial row is $-v^T$. The transpose pencil will be denoted $B(a)$, so $B(a)^T Y = P(Y)a$ and $B(a)g = N_g a$. It has rank eight for nonzero real a, satisfies $B(a)g_X = -M_X a$, and $K_X^T B(a)$ is skew-symmetric.

The earlier intrinsic-denominator construction is used here in its frame-free form. Its cross matrix $\mathcal{T}(X,Y)$ satisfies $\mathcal{T} + \mathcal{T}^T = 2\mathcal{B}(X,Y)$, has its two images in E_Y, E_X, and obeys the two shear identities. Its reduced common denominator $D(X,Y)$ is bihomogeneous, has no pure-slot factor, and is invariant under both shears. Generic specialization has no common cancellation. A whole quadratic denominator is a nonnegative symmetric biquadratic sectional form, hence determines an auxiliary algebraic-curvature Jacobi family. These facts (Proposition 12.4, where the cross matrix is written T), the generic-plane absence of real poles (Lemma 12.2), and the componentwise exclusion of quadratic residue supports of dimension at most five (Lemma 18.1) are incoming results. All global branch arguments below take place on a nonempty real open anchor region, as in Lemma 30.2.

19.1 Common coefficients and a whole aligned quadratic

Lemma 19.1. *Let A_z be a quadratic symmetric family with $\operatorname{Im} A_z \subset \operatorname{Im} \mathcal{J}_z$ for all real nonzero z. In the canonical coherent frame there is a constant symmetric matrix $C = M_X^T A_X M_X$ such that $A_z = M_z C M_z^T$. If $A_z \succeq 0$, then $C \succeq 0$.*

Proof. At a generic pair X, Y, the concatenation $\widehat{M} = [M_X, M_Y]$ is injective. A constant left inverse expresses the quadratic family A_{sX+tY} uniquely as $\widehat{M} Z(s,t) \widehat{M}^T$, with Z symmetric quadratic. Its values at the coordinate axes give

$$Z(s,t) = \begin{bmatrix} s^2 C + st Z_{11} & st Z_{12} \\ st Z_{12}^T & t^2 C_1 + st Z_{22} \end{bmatrix}.$$

Image inclusion implies $[-tI, sI]Z = 0$. Comparing its coefficients gives $Z_{11} = Z_{22} = 0$ and $Z_{12} = C_1 = C$. This proves the identity on the generic pair locus and then rationally everywhere. The coefficient is fixed at X, and positivity there proves the last assertion. □

We record the anchor-change step used for a whole quadratic. The coherent-pair construction supplies a nonempty open set of disjoint partners at each fixed nonzero real anchor: the proof of [19, Lemma 2(1), arXiv:math/0204258v2, p. 7] fixes an arbitrary unit X before proving openness and density in Y. The intrinsic matrix $\mathcal{T}$ is regular near such a pair.

Lemma 19.2. *Suppose the tensor is not Clifford and its whole intrinsic denominator is quadratic in each slot. Fix a regular unit X and put $Q(Y) = D(X,Y)$. Suppose Q is unchanged by translations in a real subspace $U \subset X^\perp$, and the full coherent factor is affine in those directions. If A is the auxiliary family, then*

$$A_V = c(V) A_X \quad (V \in \mathbb{R}X + U),$$

where c is a positive definite quadratic form and $c(X) = 1$.

Proof. First no nonzero real anchor V has $D(V,Y) \equiv 0$. Indeed, nonnegativity in the first slot would give $D(V+tZ,Y) = t^2 D(Z,Y)$. Put $N = D\mathcal{T}$, a polynomial. At a disjoint pair (V,Y), the coefficient $N_2(Z,Y)$ of t^2 in $N(V+tZ,Y)$ is $D(Z,Y)\mathcal{T}(V,Y)$. Let $p(Y)$ be an irreducible factor of a reduced denominator of an entry of $\mathcal{T}(V,Y)$. Then p divides $D(Z,Y)$ for Z in a Zariski-dense set. Reduce D modulo p, as a polynomial in Z over the domain $\mathbb{R}[Y]/(p)$: this shows $p \mid D$ in $\mathbb{R}[Z,Y]$. That is a pure-slot factor, which Proposition 12.4 excludes. Consequently $\mathcal{T}(V,Y)$ is polynomial and homogeneous linear in Y, giving a global linear Jacobi factor and the Clifford conclusion of Proposition 10.1(1), a contradiction.

For $V \in \mathbb{R}X + U$, affineness gives $M_{Y+tV} = M_Y + tC_V(Y)$. Here $C_V(Y)$ does not depend on t, is unchanged by translations in $\mathbb{R}X + U$, and is rational with possible poles only on Q. Comparison of the quadratic coefficient gives $C_V \Lambda C_V^T = \mathcal{J}_V$. Choose a fixed real factor M_V^0 of $\mathcal{J}_V$. Then $C_V(Y) = M_V^0 R_V(Y)$ with $R_V(Y)\Lambda R_V(Y)^T = \Lambda$. Both R_V and $R_V^{-1} = \Lambda R_V^T \Lambda^{-1}$ have possible poles only on Q. Since $R_V(Y+tV) = R_V(Y)$, the factor $M_Y' = M_Y R_V(Y)^{-1}$ satisfies $M'_{Y+tV} = M'_Y + tM_V^0$ and $M'_Y \Lambda M_Y'^T = \mathcal{J}_Y$. Uniqueness in Proposition 12.4 gives $\mathcal{T}(V,Y) = M'_Y \Lambda M_V^{0T}$. The factor M' is coherent at V and has possible poles only on Q. It must have a genuine pole, since otherwise the tensor would be Clifford by Lemma 11.1. Its denominator divides the nonzero quadratic $D(V,Y)$, so $D(V,Y) = c(V)Q(Y)$. Nonnegativity gives $c(V) > 0$ for $V \neq 0$. Evaluating at one Y_0 with $Q(Y_0) > 0$ shows that c is quadratic. □

Proposition 19.3. *A genuine whole simple quadratic denominator cannot have an auxiliary family $A \succeq 0$ whose image equals E_X and whose rank is eight on a nonempty real open anchor region.*

Proof. Suppose, for contradiction, that R is not Clifford. Fix a generic regular unit X in the region, so that $D(X,\cdot)$ is the actual reduced denominator. Image inclusion extends by polynomial continuation of the spectral projector identities. Lemma 19.1 gives $A_z = M_z C M_z^T$ with $C > 0$, and $Q(Y) = v^T C v$. The inequality $\lambda_{\min}(C)\|M_z\|_F^2 \le \operatorname{tr} A_z$ bounds the partner on the regular real unit sphere. Write $F(v,u) = N(v,u)/Q(v)$ with N homogeneous cubic. Under $v \mapsto tv$, its terms of bidegrees $(0,3)$ and $(1,2)$ would give poles of orders two and one. Boundedness forces both terms to vanish. Thus F is affine in all seven u directions.

Lemma 19.2 gives $A_V = c(V)A_X$ on K. On the regular locus, Cauchy–Binet and the original scaled spectrum give

$$e_8(A_z) = \det C \det(M_z^T M_z) = \det C\, |z|^{16},$$

where e_8 is the sum of principal eight-minors. This polynomial identity holds also at exceptional directions. On K it gives $c(V)^8 = |V|^{16}$; positivity gives $c(V) = |V|^2$. Sectional symmetry of A now yields

$$P(Y) C P(Y)^T = (v^T C v) I.$$

At $v = 0$ positivity of C gives $P = 0$, so $P = P(v)$. For $v \neq 0$, $P(v)^T P(v) = (v^T C v) C^{-1}$.

At $u = 0$, the Gram determinant identity and positivity give

$$|v|^{16} = \det\big(F(v)^T F(v) + (v^T C v) C^{-1}\big) \ge \frac{(v^T C v)^8}{\det C}.$$

The largest eigenvalue of C is at most their geometric mean, forcing $C = cI$. Equality of determinants then gives $F(v) = 0$. Affineness writes $F(v,u) = L_u(v)$, with L linear in u and $L_u(tv) = L_u(v)$. At $xX + u + tM_X v$, the upper frame block is $xI + L_u(v)$ and the lower block is $tP(v)$. Taking the polynomial limit of $A = cMM^T$ as $t \to 0$ gives

$$(xI + L_u(v))(xI + L_u(v))^T = (x^2 + |u|^2) I.$$

The transpose product is the same. Adding the lower Gram produces the full radial Gram. Proposition 10.1(2) gives Clifford, a contradiction; coherent-pair uniqueness removes the presumed genuine pole. The limit used only A, not a value of the rational frame at an exceptional direction. □

19.2 A semidefinite aligned family excludes two quadrics

Proposition 19.4. *Suppose the genuine denominator at rank eight is Q_1Q_2, with distinct real irreducible semidefinite quadrics. There is no nonzero quadratic curvature family $A \succeq 0$ with $\operatorname{Im} A_z \subset \operatorname{Im} \mathcal{J}_z$ for every real $z \neq 0$.*

Proof. By Lemma 19.1, write $A_z = M_z C M_z^T$, $C \succeq 0$, $r = \operatorname{rank} C \geq 1$. For the skew rational correction Ω, the mixed coefficient of A implies that $\Omega HC - CH\Omega$ is polynomial. Choose a constant linear complement to the kernel of this linear map on skew matrices. The component of Ω in that complement is polynomial; subtracting it leaves the whole polar part satisfying

$$\Omega HC = CH\Omega, \qquad P\Omega \text{ polynomial.}$$

Put $s = 8 - r$ and $k = \dim \ker(H|_{\operatorname{Im} C})$. Constant real changes of coefficient coordinates give, on spaces of dimensions $r-k, k, k, s-k$,

$$C = \operatorname{diag}(I_{r-k}, I_k, 0_k, 0_{s-k}), \qquad H = \begin{bmatrix} H_0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & H_1 \end{bmatrix},$$

where H_0, H_1 are nondegenerate. To obtain this, first normalize C on its image, diagonalize the restriction of H, pair its radical with k complementary directions, and remove the remaining cross terms by shears preserving C. Nondegeneracy gives $k \leq \min(r, s) \leq 4$. Call the coefficient blocks U_1, R, R', Z_0, of dimensions $r-k, k, k, s-k$, and write P_R for the columns of the normalized P on R. Comparing the blocks of the commutator yields exactly

$$\Omega = \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & B & M & T \\ 0 & M & 0 & 0 \\ 0 & -T^T & 0 & V \end{bmatrix}, \qquad ZH_0 = H_0 Z,$$

with Z, B, M, V skew. The repeated M blocks have the same sign because $M^T = -M$.

The columns corresponding to the third block of $P\Omega$ are $P_R M$. By Proposition 18.2, this isolated skew block, whose support has dimension $k \leq 4$, has no quadratic pole. This application is unchanged by the constant, possibly nonorthogonal, coefficient normalization: if its matrix is G_0, embed a transformed isolated block Γ back as $G_0 \Gamma G_0^T$. This matrix is skew in the original coefficient space and its product with the original P is polynomial. Restricting to an orthonormal basis of its fixed real support gives precisely the hypotheses of Proposition 18.2. Subtract the polynomial block. The remaining polar blocks are Z and the skew block on $R + Z_0$, of dimension $s = 8 - r$. Diagonalizing H_0 splits Z into blocks on its nonzero eigenspaces. Each is definite for H, hence has dimension at most seven because H is indefinite; the last block also has dimension at most seven.

These blocks have disjoint coefficient supports with total dimension $(r-k) + s \leq 8$. Each isolated product with its column subpencil of P is polynomial. The same isolated-skew proposition therefore permits at most one pole-bearing block, of size six or seven. Both Q_1, Q_2 must occur in it. At each prime, skewness requires at least two positive Smith exponents of that generically injective linear subpencil (Lemma 15.2 with $e = 1$). Thus $Q_1^2 Q_2^2$ divides every maximal minor of degree at most seven, a contradiction to its degree eight. No positivity over a pole function field is used. □

19.3 Descent and alignment of the two components

We make explicit a factor-descent principle. Work over the generic anchor field $\mathbb{R}(X)$. A Galois-invariant collection of geometric factors of total Y-degree two descends, after coefficient normalization and primitive Gauss descent, to a real polynomial factor $F(X, Y)$ of D. Every irreducible factor of a bihomogeneous D is bihomogeneous. Under $Y \mapsto Y + tX$, a factor maps to a factor of D, while its highest Y-degree part is unchanged. Comparison with that bihomogeneous factor proves invariance; the same holds in the other slot. The two shears generate $(X, Y) \mapsto (tX, t^{-1}Y)$, forcing equal degrees, and generate $(X, Y) \mapsto (Y, -X)$, giving symmetry in bidegree $(2, 2)$. This also applies to a descended product, by applying it to its irreducible factors.

At a generic real anchor, $F(X, \cdot)$ is a real quadratic factor of D_X. By Proposition 12.3, D_X is then Q^2 or Q_1Q_2, so $F(X, \cdot)$ is proportional to a semidefinite quadric. Continuity extends semidefiniteness to all fibres. Symmetry fixes one global sign: if $F(X_0, \cdot) \geq 0$ and $F(X_0, Y) > 0$, then $F(Y, X_0) = F(X_0, Y) > 0$, so the fibre $F(Y, \cdot)$ is nonnegative. Such Y are dense. Thus F is the sectional form of a nonnegative curvature family. If its factors vanish on the rational kernel subspace at generic anchors, this family is aligned with E_X. Generic factor covers preserve algebraic invariants; no claim that real irreducibility persists at every specialization is made.

Proposition 19.5. *In a surviving generic two-quadratic branch, both quadrics are aligned with K_X, their intrinsic global denominator is irreducible over $\mathbb{R}[X, Y]$, and their geometric factors have uniform intrinsic algebraic invariants. The case in which both quadrics have rank eight is excluded.*

Proof. The determinant budget is tight: $\det P = cQ_1^2Q_2^2$. At $v = 0$ the radial row is zero, so $Q_1(0, u)Q_2(0, u) = 0$ identically. One of the two restrictions is identically zero, and semidefiniteness puts that real subspace in its radical. At least one quadric is therefore aligned.

If exactly one is aligned, distinguish the geometric factors that vanish on K_X. This is a rational, Galois-invariant condition. A rank-two real semidefinite quadric contributes either both of its conjugate linear factors or neither: a complex linear form vanishes on a real subspace exactly when its conjugate does. Thus the aligned collection has total degree two even in this split case. The preceding descent gives a nonzero nonnegative aligned family, contradicting Proposition 19.4. Both components are aligned.

If D had a proper real factor, its specialization on the branch would divide Q_1Q_2. With pure-slot factors excluded, its degree must be two. The same shear and sign argument again produces a forbidden aligned family. Hence D is irreducible over $\mathbb{R}[X, Y]$, and its primitive fibre is irreducible over $\mathbb{R}(X)[Y]$. Its geometric factors form one Galois orbit. After passage to the factor cover their degrees, ranks, support dimensions and incidence invariants are equal; these equalities persist on the generic strata. In particular a mixed geometric degree list $2, 1, 1$ is impossible. A geometrically irreducible quartic is also impossible: absolute irreducibility persists on a nonempty Zariski-open set of specializations, which meets the real open branch, and on that branch $D_X = Q_1Q_2$. The possibilities are two quadratics or four lines.

Finally suppose both aligned quadrics have rank eight. Then $\det P(v, u) = cQ_1(v)^2Q_2(v)^2 \neq 0$ whenever real $v \neq 0$. For a fixed nonzero real coefficient a, the map $v \mapsto P(v, 0)a$ is an isomorphism of eight-spaces. If $P(0, u)a \neq 0$, choose its unique cancelling $v \neq 0$; then $P(v, u)a = 0$, contradicting the determinant identity. Thus $P(0, u) = 0$. The full transpose image is E_X, which supplies the closed-kernel condition. On the real open branch Lemma 16.1 and Proposition 10.1(2) give Clifford, excluding the genuine poles. □

20 Quadratic poles with proper coefficient support

Retain the notation of Section 19. For a simple real irreducible quadratic pole Q, let $U \subset \mathbb{R}^8$ be the total real span of its rank-two skew residue, of dimension d. For each geometric pole component, let $\mathcal{A}$ be the closed projective residue image, and write $\alpha = \dim \mathcal{A}$ and $s(a) = \operatorname{rank} B(a)$. Lemmas 18.4 and 11.3 give

$$4 \le s(a) \le \alpha \le 5 \quad (a \in \mathcal{A}), \qquad \mathcal{A} \subset V(a^T H a), \qquad \operatorname{rank}(B(a)B(a)^T) = 2s(a) - 8.$$

The image has no real points. Small supports $d \le 5$ have already been excluded componentwise (Lemma 18.1). This section handles $d = 6, 7$, including rank-two real quadrics which split over $\mathbb{C}$. Write $G(a) = B(a)B(a)^T$.

20.1 Six-dimensional support

Proposition 20.1. *If $d = 6$, then $H|_U$ is definite. For its positive choice of sign h, one has $G(a) = h(a)D$ on U, where $D \succeq 0$ has rank eight and image E. The physical pole quadric is proportional to $Y^T DY$, has rank eight, and has kernel K_X. Two distinct quadratic poles cannot both have support dimension six.*

Proof. The restriction $H|_U$ is not zero, since a nondegenerate eight-form has no totally isotropic six-space. Its projective null locus in $\mathbb{P}(U_{\mathbb{C}})$ has dimension four, so each residue image fills a component. If the restricted rank is at least three, this quadric is irreducible. Any indefinite or degenerate real restriction would supply a real point. At ranks one or two every hyperplane component contains the nonzero real radical. Both alternatives are impossible. Thus $H|_U$ is definite of rank six.

The image is its whole null quadric, with $\alpha = s = 4$. Hence $G = 0$ there, and quadratic divisibility gives $G = hD$. Evaluation at a real nonzero coefficient makes D positive semidefinite of rank eight. All $B(a)$ for real $a \in U \setminus 0$ have the same image $F = \operatorname{Im} D$. Radial columns put $M_X U \subset F \cap E$, so the lower projection of F has dimension at most two. The lower skew pencil has exactly that fixed image at every nonzero a. If it had rank two, a six-parameter linear map into the one-dimensional space of skew maps on a fixed two-plane would have to have no nonzero kernel. This is impossible. Thus $F = E$.

Contraction and polarization give

$$(P(Y)|_U)^T (P(Y)|_U) = (Y^T DY) H_0,$$

where $H_0 > 0$ represents h. Modulo Q, the restricted pencil has corank at least two. Its Gram determinant therefore vanishes, so $Q \mid (Y^T DY)^6$. Equality of degrees gives proportionality of Q and $Y^T DY$. Its rank and kernel follow from $\operatorname{Im} D = E$. For two six-spaces, choose a real nonzero vector in their intersection. Evaluating the two identities $G = h_i D_i$ there makes D_1, D_2 proportional, and therefore their quadrics proportional, contrary to distinctness. □

Proposition 20.2. *No generic sole-quadratic or two-quadratic branch has a component of support dimension six.*

Proof. For a whole simple quadratic, Proposition 20.1 gives full-rank alignment of its intrinsic auxiliary family on the branch. Proposition 19.3 excludes it.

For two quadrics there is at most one six-support component. It has rank eight, hence is geometrically irreducible. Its support dimension is intrinsic: multiply the frame-free cross matrix

by the pole factor, reduce at its generic point, and take the constant span of its image. This equals the coefficient support dimension in any eigenframe. On the generic factor cover, Galois automorphisms preserve this dimension and the factor degree. A nontrivial conjugate of the unique six-support factor would be the other quadratic and would also have support six, impossible on the branch. Thus this factor descends over $\mathbb{R}(X)$. Primitive descent and the shear argument of Section 19 give a nonzero nonnegative aligned biquadratic family. Proposition 19.4 excludes it. This uses generic algebraic invariants on the factor cover, not preservation of real factorization at every specialization. □

20.2 Seven-dimensional support with transpose rank four

Lemma 20.3. *Suppose $d = 7$ and the generic transpose rank on the residue image components is four. Then the quadratic entries of $G|_U$ span at most two scalar forms. For every real nonzero $a \in U$, $\operatorname{Im} B(a) = E$. Moreover Q has rank eight and kernel K_X.*

Proof. Put $h = H|_U$, of rank at least six. On the residue image, $G = 0$. First suppose Q is geometrically irreducible. Its image is real, irreducible and spans $\mathbb{P}(U_{\mathbb{C}}) = \mathbb{P}^6$. If $\alpha = 5$ it is the whole h-quadric, and every entry of G is proportional to h. If $\alpha = 4$, its degree is at least three by the degree bound [10, p. 1, inequality $(*)$], and is even because a generic real complementary plane intersects it in conjugate pairs. An independent quadratic equation f among the entries of G puts it in the degree-four complete intersection (h, f). Thus its degree is four and it exhausts that intersection, including the scheme structure: the complete intersection is unmixed and generically reduced, with no embedded components. All quadratic entries are in $\operatorname{span}\{h, f\}$.

If Q has rank two and both components have the same image, that image is real-defined and spans $\mathbb{P}(U_{\mathbb{C}})$, so the preceding paragraph applies. Otherwise take the conjugate image pair and let d_1 be the complex span dimension of one component. Dimension and the absence of real points exclude $d_1 \le 5$: the only possible image there is a projective linear space, whose intersection with its conjugate contains real vectors. If $d_1 = 6$, the image is the whole irreducible quadric in that hyperplane, of degree two; the conjugate hyperplanes are distinct. Their union is the reduced complete intersection $(h, \ell\bar{\ell})$, giving the same two-form bound. If $d_1 = 7$, dimension five again gives the whole h-quadric. In dimension four, a real image uses the preceding parity argument; two distinct nondegenerate conjugate images have total degree at least six and cannot lie in a degree-four complete intersection. Consequently no independent quadratic f exists and all entries are multiples of h.

A positive semidefinite quadratic matrix of constant rank whose entries span at most two forms has fixed image. Indeed, on a real ball where one form has positive sign, the ratio of the other has an interval of values; the positive-semidefinite kernel-intersection identity and constant rank give equal kernels at two distinct parameters. Both coefficient matrices then have that fixed image. Apply this to G, of real rank eight. Its fixed image contains the radial seven-space $M_X U$, so its lower projection has dimension at most one. Skewness makes that projection zero, proving alignment.

Thus $P(Y)|_U$ depends only on v. For every nonzero real v it is injective: $B(a)$ maps onto E for every nonzero real $a \in U$, so $B(a)^T M_X v = P(v)a \neq 0$. Every maximal seven-minor is divisible by Q and some such minor is nonzero. Since that minor depends only on v, so does Q. A nonzero real vector in a smaller-rank semidefinite $Q(v)$ would contradict the injectivity. Hence $\operatorname{rank} Q = 8$ and its full physical kernel is K_X. In particular Q cannot have rank two. □

Lemma 20.4. *In the preceding situation $G|_U = h_0 D_0$ for a positive definite quadratic h_0 and a fixed rank-eight positive semidefinite D_0. Moreover $(P|_U)^T (P|_U) = Q H_1$ after a constant rescaling, with H_1 nondegenerate. Such a component cannot coexist with a distinct quadratic pole.*

Proof. Identify E with the original orthonormal coefficient space. The square matrix $B_E(a)$ is invertible at real nonzero $a \in U$ and $B_E(a)g_X = -a$, with $|g_X| = 1$. Hence $a^T G(a)^{-1} a = 1$. If the scalar span is genuinely two-dimensional, then, since the pencil contains the positive definite $G_E(a_0)$ ($a_0 \in U$ real nonzero), simultaneous congruence diagonalizes it as

$$G(a) = S \operatorname{diag}(f_j(a) I_{b_j}) S^T, \qquad \sum b_j = 8,$$

with pairwise nonproportional quadratic forms f_j in the span. Every f_j is positive definite on the seven-space, hence irreducible, and the f_j are pairwise nonassociate. The inverse identity is

$$\sum_j \frac{n_j(a)}{f_j(a)} = 1, \qquad n_j = |\operatorname{pr}_j S^{-1} a|^2.$$

Reduction modulo f_j after clearing denominators gives $f_j \mid n_j$. If $n_j \neq 0$, its rank is seven, whereas its sum-of-squares presentation has rank at most b_j. At least one is nonzero, so the only two-block possibility is sizes seven and one. But then

$$(\det B_E)^2 = (\det S)^2 f^7 g$$

has an odd valuation, a contradiction. Thus the scalar span is one. Contraction gives $(P|_U)^T (P|_U) = (Y^T D_0 Y) H_1$; its determinant modulo Q makes Q proportional to $Y^T D_0 Y$.

Modulo Q, the image of the eight-by-seven $P|_U$ is totally isotropic in the standard eight-space, hence has dimension at most four. Adding the last coefficient column gives $\operatorname{rank} P \leq 5$, so $Q^3 \mid \det P$. A distinct quadratic pole Q' requires $(Q')^2 \mid \det P$ (Lemma 11.4). The divisor $Q^3 (Q')^2$ has degree ten, contradicting $\deg \det P = 8$. □

Proposition 20.5. *A two-quadratic branch has no component of support dimension seven.*

Proof. Write $\det P = c Q_1^2 Q_2^2$. The local Smith exponents at either prime are $(1, 1, 0^6)$, and every seven-minor is divisible by both. Thus $A = \operatorname{adj} P / (c Q_1 Q_2)$ is cubic and $AP = PA = Q_1 Q_2 I$. If the first residue lies in $U = \ker f$, then $fA = Q_1 \ell(Y)^T$, with ℓ linear, because the image of A modulo Q_1 is exactly the residue kernel plane. Multiplying by P gives $P(Y)^T \ell(Y) = Q_2(Y) f^T$.

Write $L : Y \mapsto \ell(Y)$. For real nonzero $a \in U$, put $P_a Y = P(Y) a$. The last identity says $P_a^T L$ is skew; surjectivity of P_a gives its kernel $\ker L$ and rank $m = \operatorname{rank} L$. Its image is the same m-space for every such a, and m is even. For any nonzero real $g \in \operatorname{Im} L$, the vectors $N_g a$ with $a \in U$ lie in that space. Since N_g is injective, it has dimension at least seven. Thus $m = 8$ and L is surjective. Over $\mathbb{C}$, $\operatorname{rank}(P_a^T L) = \operatorname{rank} P_a = s(a)$ is therefore even for every $a \in U_{\mathbb{C}}$. On each residue image the incidence bounds force $s = 4$. Lemma 20.4 now supplies the degree-ten contradiction. This argument also treats conjugate geometric factors of a real rank-two quadric. □

20.3 Seven-dimensional support with transpose rank five

Here $\alpha = s = 5$. Since U is a hyperplane, $H|_U$ has rank at least six and its null quadric in $\mathbb{P}^6$ is irreducible. The residue image fills it. Absence of real points makes $H|_U$ definite of rank seven. Reverse the curvature sign if necessary and put $h = H|_U > 0$; then H has signature $(7, 1)$.

We first record an elementary seven-coefficient version of the fixed-image observation. If all images of $B(a)$, $a \in U$, lie in a fixed space T of dimension at most nine, then they equal E. Indeed $M_X U \subset T \cap E$, so the lower projection R of T has dimension at most two. The lower skew pencil takes values in $\bigwedge^2 R$, of dimension at most one, and hence vanishes at some nonzero real coefficient a_0. Its rank-eight image there is E, putting $E \subset T$. The lower projection of T now has dimension at most one, so the entire lower skew pencil is zero.

Lemma 20.6. *In this rank-five case, either $B(U)$ has image E, or*

$$G = hD + uu^T - vv^T \tag{20.1}$$

with D constant symmetric and u, v real linear columns. The extraction includes divisorial rank-zero loci.

Proof. The complex cone ring $S = \mathbb{C}[U]/(h)$ is factorial. For example write $h = x_1x_2 + x_3^2 + \cdots + x_7^2$; x_1 is prime and inverting it gives a Laurent polynomial ring, so Nagata's criterion [25, Corollary to Theorem 6.3′, p. 18] applies. Alternatively, [25, Theorem 8.2, p. 26] applies directly, since h is nondegenerate in seven variables. At the generic point the transpose kernel is an isotropic three-space in the standard eight-form. Its binary quotient has determinant class -1 and is hyperbolic over the real function field.

The two saturated isotropic line modules are rank-one reflexive and therefore principal over S. Choose homogeneous generators p, r. Then $G = t(pr^T + rp^T)$ over the fraction field. At every height-one prime, both generators are primitive and their symmetric cross product has a nonzero reduction. Thus t has no negative valuation, and normality gives $t \in S$. Homogeneity gives $\deg t + \deg p + \deg r = 2$. Each line is individually real-defined; conjugation changes its generator by a constant unit, removable by rescaling. Thus all generators and t may be taken real.

If both vector degrees are positive, they are one and t is constant. Lifting and taking real sums and differences gives (20.1). Otherwise the lift is $G = hD + cb^T + bc^T$ with c constant and b quadratic. Compression to $c^\perp$ has the fixed form hD_0. Positivity of the full matrix puts its cross block in $\operatorname{Im} D_0$, so all images lie in a fixed space of dimension at most nine. The preceding observation gives alignment. The scalar t retains all common divisorial factors, so no rank-zero divisor has been discarded. □

Lemma 20.7. *For a nonaligned form* (20.1)*, $D \succeq 0$ has rank exactly eight. A constant hyperbolic change of the two columns gives*

$$G = hD - ww^T + zz^T, \qquad w \in F := \operatorname{Im} D, \qquad w^T D^+ w = h, \tag{20.2}$$

where the projection of z to $F^\perp$ has rank at least two.

Proof. For any fixed real physical vector b, choose nonzero $a \in U$ with $b^T u(a) = b^T v(a) = 0$. Positivity gives $b^T Db \geq 0$. Put $r = \operatorname{rank} D$ and $G_+ = hD + uu^T$. Positivity of $G = G_+ - vv^T$ gives $v \in \operatorname{Im} G_+$, and rank eight gives $\operatorname{rank} G_+ \in \{8, 9\}$. Therefore $7 \leq r \leq 9$. If $r = 9$, both u, v lie in F and the fixed-nine-space observation aligns the image.

Suppose $r = 7$. The quotient map $\bar{u} : U \to V/F$ is injective, and $\operatorname{Im} B(a) = F + \mathbb{R}u(a)$ for all real $a \neq 0$. The radial vector $M_X a$ belongs to this space. Two linear maps which are pointwise proportional, one of rank at least two, have constant proportionality; compare two independent images and their sums. Thus $\overline{M_X a} = c\bar{u}(a)$ for a constant c. If $c = 0$, $F = M_X U \subset E$; each lower image has dimension at most one and skewness kills it. This would put the injective $\bar{u}(U)$ in the one-dimensional quotient E/F, impossible. If $c \neq 0$, the lower skew pencil has one fixed image $R = K_X^T F$ at all nonzero real coefficients. Its even rank is $0, 2, 4$, or 6. Ranks two and four are impossible because their skew-operator spaces have dimensions one and six, less than seven. Rank six would give a cubic Pfaffian nowhere zero on the real unit sphere, impossible for an odd function on a connected sphere. Rank zero puts $F \subset E$, again contradicting the rank of the quotient radial map. Thus $r \neq 7$.

We have $r = 8$. If $\operatorname{rank} \bar{u} \le 1$, all images lie in a fixed nine-space and align. Otherwise positivity gives $\bar{v} = c\bar{u}$ with constant c, so $v = cu + w_0$ with $w_0 \in F$. At points where $\bar{u} \neq 0$, the rank drop from nine to eight is

$$c^2 + h^{-1} w_0^T D^+ w_0 = 1.$$

This polynomial identity extends everywhere. Since $h > 0$ on $U \setminus 0$ and $D^+ \succeq 0$ on F, we have $c^2 \le 1$. If $|c| = 1$, then $w_0 = 0$ and the image is fixed and aligned. Otherwise set $d = 1 - c^2 > 0$, $w = w_0/\sqrt{d}$, $z = \sqrt{d}\, u - cw$. Expansion gives (20.2); its quotient column retains rank at least two. □

Proposition 20.8. *A nonaligned rank-five seven-support branch forces a positive nonzero Jacobi root of multiplicity at least five.*

Proof. Project B to $F^\perp$ in (20.2), writing the projection as T and that of z as b. Then $TT^T = bb^T$. Every row of T with squared norm b_i^2 vanishes on the real hyperplane $b_i = 0$, hence is $b_i e_i^T$ for a constant unit vector e_i. Cross-row identities give $e_i^T e_j = 1$, so all are the same unit e. Thus $T = be^T$. The mixed block then gives $B(a)e = z(a)$. In orthonormal input coordinates beginning with e, write $B = [z, B_0]$, with B_0 valued in F and

$$B_0 B_0^T = hD - ww^T.$$

Choose $S : \mathbb{R}^8 \to F$ with $SS^T = D$, put $A_0 = S^{-1}B_0$, $\eta_0 = S^{-1}w$, and use $|\eta_0|^2 = h$. The seven nonzero eigenvalues of $A_0 A_0^T = hI - \eta_0 \eta_0^T$ are h, so $A_0^T A_0 = hI_7$.

For $g \in e^\perp$, use $B(a)g = N_g a$ and an h-orthonormal basis of U. The resulting eight-by-seven matrix $A(g)$, linear in the seven variables g, satisfies

$$A(g)^T A(g) = q(g) I_7, \qquad q(g) = |g|^2.$$

At the prime (q) its image is isotropic in the standard eight-space, of dimension at most four. Each maximal seven-minor is therefore divisible by q^3. The signed cofactor vector has the form $q^3 \eta(g)$, with η linear. Cauchy–Binet gives $|\eta|^2 = q$, and the normal-vector identity gives

$$A(g) A(g)^T = q(g) I_8 - \eta(g) \eta(g)^T.$$

Complete the h-orthonormal basis of U by its H-orthogonal negative unit vector. In these coefficient coordinates $H = \Lambda = \operatorname{diag}(I_7, -1)$, with the reciprocal congruence on the factors leaving the physical Jacobi family unchanged. Let $n(g)$ be the last column of N_g in this basis and $p(g) = S\eta(g)$. Then

$$\mathcal{J}_{K_X g} = q(g) D - p(g) p(g)^T - n(g) n(g)^T, \qquad p(g)^T D^+ p(g) = q(g). \tag{20.3}$$

In $\operatorname{Frac}(\mathbb{R}[g]/(q))$, the left side is nilpotent. The determinant identity for $V = [p, n]$ makes the symmetric two-by-two matrix $V^T V$ nilpotent. It has the form $\left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right]$ with $a^2 + b^2 = 0$. The rank-seven quadric is geometrically integral over $\mathbb{R}$, so its function field does not contain a square root of -1. Hence $V^T V = 0$ in that field, and quadratic divisibility gives, in particular, $p^T p = \alpha q$ with $\alpha > 0$.

The image $W_0 = \operatorname{Im} p$ is a seven-space in F. The two norm identities in (20.3) give $D^+|_{W_0} = \alpha^{-1} I$ as a quadratic compression. A symmetric matrix with scalar compression on a hyperplane has that eigenvalue on the intersection of the hyperplane with the orthogonal complement of its one cross-block vector. Thus D has eigenvalue α on a space L_α of dimension at least six. If this dimension is at least seven, subtracting the two rank-one terms leaves multiplicity at least five. If

it is six, choose nonzero g in the kernel of the linear map $g \mapsto \mathrm{pr}_{L_\alpha} p(g)$ from seven variables to six. The vectors of L_α orthogonal to $n(g)$ then give an $\alpha q(g)$-eigenspace of dimension at least five. Constancy of the real Osserman spectrum proves the assertion. □

Proposition 20.9. *No remaining generic sole-quadratic branch has support dimension seven.*

Proof. For generic transpose rank four, Lemma 20.3 gives rank-eight alignment of the whole pole, and Proposition 19.3 excludes it. For rank five, Lemma 20.6 either gives the same alignment or a balanced form. Any aligned alternative again has pole rank eight by the maximal-minor argument in Lemma 20.3, which uses alignment only. The nonaligned balanced alternative has signature $(7, 1)$ and Proposition 20.8 forces a positive root of multiplicity at least five. Among the eleven remaining patterns with a reduced root of multiplicity seven (Section 30), only $(7, 5, 3)$ has a nonzero block of size five; it has no simple nonzero root and therefore cannot have negative index one. Every other applicable placement has positive multiplicities at most four. This contradiction finishes the case. □

Together the propositions exclude every proper support remaining after the small-support reduction. The whole-pole criterion was used only for a sole quadratic. The two-quadratic conclusions instead used componentwise composition, algebraic descent, and the tight degree-eight determinant. Alignment on a proper coefficient space was never substituted directly for the full-coefficient closed-kernel hypothesis.

21 Real four-dimensional residue images

We exclude the real, geometrically integral, full-span residue image of dimension four. The proof uses the quadratic equations of that image and the radial and skew identities of the actual transpose pencil. The conjugate-image alternative is treated separately.

Fix a real regular anchor in the rank-eight case. Let $V = \mathbb{R}^{16}$, let $E \subset V$ be the eight-dimensional Jacobi image, and let $\iota : \mathbb{R}^8 \to E$ and $K : \mathbb{R}^8 \to E^\perp$ be orthogonal isomorphisms. Let B be the real linear transpose pencil, and write

$$B : \mathbb{R}^8 \longrightarrow \mathrm{Hom}(\mathbb{R}^8, V), \qquad G(a) = B(a)B(a)^T, \qquad L(a) = K^T B(a).$$

The transpose-pencil construction supplies the following identities:

$$\operatorname{rank} B(a) = 8 \quad (a \in \mathbb{R}^8 \setminus \{0\}), \qquad B(a)e = -\iota(a), \qquad L(a)^T = -L(a), \tag{21.1}$$

where $e \in \mathbb{R}^8$ is fixed. We also have a nondegenerate indefinite quadratic form $h(a) = a^T H a$ on the coefficient space. In particular, $G(a)$ is positive semidefinite of rank eight at every nonzero real a.

We use Lemma 17.1 and Remark 17.2: if the real span of the scalar quadratic entries of G has dimension at most three, then

$$\operatorname{im} B(a) = E \qquad (a \in \mathbb{R}^8 \setminus \{0\}). \tag{21.2}$$

This criterion is an input from the actual-pencil analysis, not an assertion about arbitrary symmetric quadratic matrices. All complex varieties below carry their reduced structure unless a scheme is specified explicitly.

Proposition 21.1. *Suppose that a geometrically integral real-defined fourfold $A \subset \mathbb{P}^7_{\mathbb{C}}$ spans $\mathbb{P}^7_{\mathbb{C}}$ and is contained in the common zero locus of h and every entry of G. Then* (21.2) *holds.*

We first record a consequence of skewness which will be used twice.

Lemma 21.2. *Let $U \subset \mathbb{R}^8$ have dimension at least four. If $\operatorname{im} B(a)$ is one fixed eight-space T for all $a \in U \setminus \{0\}$, then $T = E$.*

Proof. The radial column in (21.1) gives $\iota(U) \subset T \cap E$. Therefore $R = K^T T$ has dimension $r \leq 8 - \dim U \leq 4$. For every nonzero $a \in U$, the skew map $L(a)$ has image exactly R. A real skew map with image R vanishes on $R^\perp$ and restricts to an invertible skew map on R. Thus $r \in \{0, 2, 4\}$.

If $r = 2$, the linear family $L|_U$ takes values in the one-dimensional space of skew maps supported on R. It has a nonzero kernel, contradicting its constant rank two. If $r = 4$, then $\dim U = 4$ and $a \mapsto L(a)|_R$ is injective. The Pfaffian on skew four-by-four matrices is

$$A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23},$$

a quadratic form of signature $(3, 3)$. Its restriction to the four-dimensional image of U would have no nonzero real zero, hence would be definite. A form of signature $(3, 3)$ has no definite four-dimensional subspace. Therefore $r = 0$, and $T = E$. □

21.1 The quadratic base and its degree

Set

$$\mathcal{V} = \operatorname{span}_{\mathbb{R}}\{h,\ G_{ij} : 1 \leq i, j \leq 16\}, \qquad Z(\mathcal{V}) = \{h = G_{ij} = 0\} \subset \mathbb{P}^7_{\mathbb{C}}.$$

This base has no real point, since $G(a)$ has rank eight at every nonzero real coefficient vector.

Lemma 21.3. *Under the hypotheses of Proposition 21.1, either* (21.2) *holds or $Z(\mathcal{V})$ has dimension four. In the latter case $\mathcal{V}$ contains a real regular sequence of three quadrics, and* $\deg A \in \{4, 6, 8\}$.

Proof. The inclusion of A gives dimension at least four. A six-dimensional component would be the entire irreducible quadric $V(h)$, which has real points. Suppose that a component Y has dimension five. Its span cannot have dimension at most five: this would make Y a linear $\mathbb{P}^5$ contained in $V(h)$, whereas a nondegenerate complex eight-form has maximal isotropic vector dimension four.

If Y spans a hyperplane $\ell = 0$, the restriction of h to that hyperplane has rank at least six. Thus $Y = V(h, \ell)$, an irreducible quadric. For every quadratic entry g of G,

$$g = c_g h + \ell m_g$$

with m_g linear. Restriction to A gives $\ell m_g = 0$. Its integral coordinate ring and full span imply first $m_g|_A = 0$, then $m_g = 0$. All entries of G are multiples of h, impossible at a nonzero real h-null vector.

It remains that Y spans all $\mathbb{P}^7$. Its degree is at least three by the nondegenerate degree bound [10, p. 1]. There is a real entry f of G independent of h, by the preceding real-null-vector argument. The complete intersection $V(h, f)$ has pure dimension five and degree four. If $Y \neq \overline{Y}$, its two conjugate components would have total degree at least six. Hence Y is real-defined. A generic transverse real complementary plane meets it only in conjugate nonreal pairs, so its degree is even. Therefore $\deg Y = 4$.

The degree of the complete intersection is exhausted by Y with generic multiplicity one. A complete intersection has no embedded associated primes, so generic reducedness makes it reduced. It is exactly Y as a scheme. Its homogeneous ideal is saturated in this positive dimension; consequently every quadratic entry of G lies in $\operatorname{span}_{\mathbb{R}}\{h, f\}$. The small scalar-span criterion gives (21.2).

Otherwise the ideal generated by $\mathcal{V}$ has height three. Start with h, choose a real quadric independent of it, and choose a third real element outside the finitely many height-two associated

primes of the first two. None contains all of $\mathcal{V}$; over the infinite real field, a finite union of proper linear subspaces cannot cover $\mathcal{V}$. This gives the required regular sequence. Its degree-eight complete intersection contains A. Since A is nondegenerate of codimension three, $\deg A \geq 4$. Its degree is even by the same conjugation argument. Thus its only possibilities are four, six, and eight. □

If $\deg A = 8$, it exhausts the three-quadric complete intersection. The same unmixedness and generic-reducedness argument identifies the intersection with A as a scheme. Hence all quadratic entries of G lie in the span of its three defining quadrics, and (21.2) follows. We now treat the other two degrees.

21.2 The degree-four image

Lemma 21.4. *If* $\deg A = 4$*, then* (21.2) *holds.*

Proof. The variety has minimal degree. By the Del Pezzo–Bertini classification, as stated in [10, Theorem 0.1], it is a rational normal scroll or a cone over the Veronese surface. A singular variety in this list has a nonempty intrinsic linear vertex. Since A is real-defined, such a vertex would be real-defined and would contain a real point. This is impossible. Thus A is smooth. The four positive scroll integers sum to four, so all are one, and A is the Segre variety

$$\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^4).$$

Conjugation preserves its two rulings, which have different dimensions. It therefore lifts, up to a scalar, to $J \otimes K_0$ with conjugate-linear maps on the two factors. Absorb that scalar into one factor. Then $(J \otimes K_0)^2 = I$ forces $J^2 = \alpha I$ and $K_0^2 = \beta I$ with $\alpha\beta = 1$. Commuting each map with its square makes its scalar real. Rescaling normalizes both squares to signs. The ambient real structure has square one, so those signs agree. If both signs were positive, fixed vectors in the two factors would give a real Segre point. Therefore both signs are negative.

The standard $\mathrm{SU}(2)$ action on the first factor commutes with its quaternionic real structure J. It induces a real action on the coefficient eight-space; its three quaternion units give that space a left $\mathbb{H}$-module structure. The quadratic ideal of the Segre embedding is

$$\bigwedge^2(\mathbb{C}^2)^* \otimes \bigwedge^2(\mathbb{C}^4)^*.$$

Indeed it is the kernel of restriction of quadratic polynomials to rank-one tensors. The first $\mathrm{SL}(2,\mathbb{C})$ acts trivially on this space. Every entry of G is consequently $\mathrm{SU}(2)$-invariant. By homogeneity,

$$G(ba) = |b|^2 G(a) \qquad (b \in \mathbb{H}).$$

For every nonzero real a, the space $U = \mathbb{H}a$ has real dimension four, since nonzero quaternions act invertibly. The displayed identity makes $\operatorname{im} B$ one fixed eight-space on $U \setminus \{0\}$. Lemma 21.2 identifies it with E. Since a was arbitrary, the conclusion holds on the entire coefficient space. This argument does not require the quaternion action to preserve the original coefficient metric. □

21.3 The degree-six image

Lemma 21.5. *If* $\deg A = 6$ *and alignment has not already been obtained, there is a reduced real complete intersection* $C = A \cup Q$ *of three quadrics, where* Q *is a real-defined quadric fourfold spanning a real vector six-space* U *on which* h *is definite.*

Proof. The quadratic $t = \operatorname{tr} G$ is positive definite and belongs to $\mathcal{V}$. It is independent of the indefinite form h. The height-three conclusion of Lemma 21.3 allows a real $f \in \mathcal{V}$ such that (h, t, f) is a regular sequence. Its complete intersection C has degree eight and has no real points.

Every four-dimensional irreducible component of C has degree at least two: degree one would give a linear $\mathbb{P}^4$ in $V(h)$, contradicting the maximal isotropic dimension four. The component A consumes degree six with multiplicity one. The remaining degree is exactly one component Q of degree two and multiplicity one. Unmixedness and generic reducedness make $C = A \cup Q$ reduced. The unique residual component is preserved by conjugation.

The degree bound puts the span of Q in a projective five-space; it cannot be a projective four-space, since then Q would be that linear space. Thus Q is a quadric hypersurface in $\mathbb{P}(U_{\mathbb{C}})$ for a real six-space U. Its equation is proportional to $h|_U$, which is nonzero by the maximal isotropic dimension bound. Since Q has no real point, this real quadratic form is definite. □

Lemma 21.6. *In Lemma 21.5, choose real independent linear forms u, v with $U = \ker u \cap \ker v$. There are real quadrics s, r, Δ such that*

$$I_C = (h, s, r), \qquad (I_A)_2 \subset \operatorname{span}_{\mathbb{R}}\{h, s, r, \Delta\}, \qquad s|_U = r|_U = 0.$$

Here $(I_A)_2$ denotes the real degree-two part of the homogeneous ideal.

Proof. Since $I_Q = (h, u, v)$, subtracting constant multiples of h from t, f gives

$$s = ua_0 + vb_0, \qquad r = uc_0 + vd_0, \qquad \Delta = a_0d_0 - b_0c_0,$$

where a_0, b_0, c_0, d_0 are real linear forms. Work in the graded domain $S = \mathbb{R}[x_1, \dots, x_8]/(h)$ in coefficient coordinates x_i. Both (s, r) and (u, v) are regular sequences; the latter follows from $h|_U \neq 0$ and the Cohen–Macaulay property of the polynomial ring.

For $g \in (I_A)_2$, reducedness gives $ug, vg \in I_A \cap I_Q = I_C$. Modulo h, write

$$ug = a_1s + b_1r, \qquad vg = c_1s + d_1r$$

with real linear coefficients. The Koszul syzygies of (s, r), with degrees taken into account, give a real constant k such that

$$va_1 - uc_1 = kr, \qquad vb_1 - ud_1 = -ks.$$

Substitute the expressions for s, r and use the Koszul syzygies of (u, v). There are real constants e_1, e_2 with

$$a_1 - kd_0 = e_1u, \quad c_1 + kc_0 = e_1v, \qquad b_1 + kb_0 = e_2u, \quad d_1 - ka_0 = e_2v.$$

It follows that

$$ug = k(d_0s - b_0r) + u(e_1s + e_2r) = u(k\Delta + e_1s + e_2r).$$

Cancel u in the domain S. Lifting back adds a constant multiple of h, proving the assertion. No general linkage theorem is needed. □

Completion of Proposition 21.1. Only degree six remains. Lemma 21.6 gives constant real symmetric matrices D_1, D_2 such that

$$G(b) = h(b)D_1 + \Delta(b)D_2 \qquad (b \in U).$$

Choose the sign of $h|_U$ positive, absorbing it into D_1. On the connected ellipsoid $h(b) = 1$, the continuous scalar function Δ has an interval as its image. Thus the midpoint of any two normalized

Gram values is another such Gram value. All have rank eight and are positive semidefinite. The kernel of a sum of positive semidefinite matrices is the intersection of their kernels; the common rank therefore forces all these kernels to agree. Homogeneity makes $\operatorname{im} B(b)$ one fixed eight-space for every nonzero $b \in U$. Lemma 21.2 shows that this image is E, so $L|_U = 0$.

Linearity now gives $L(a) = u(a)L_1 + v(a)L_2$ for constant real matrices. The polynomial

$$Q(a) = \operatorname{tr}(L(a)L(a)^T)$$

is a homogeneous binary quadratic in u, v. On A, the identity $G = 0$ gives $LL^T = K^T GK = 0$, so Q vanishes there. If Q were nonzero, it would factor over $\mathbb{C}$ into two nonzero linear forms, possibly repeated. The integral coordinate ring of A would force one factor to vanish on A, contrary to full complex span. Hence Q is the zero polynomial. On real coefficients it is a sum of squares of real entries, so $L(a) = 0$ for every a. The rank condition in (21.1) now gives (21.2). □

Remark 21.7. For a residue image of type $(\alpha, s) = (4, 4)$, the previously established rank identity $\operatorname{rank} G = 2s - 8$ gives exactly the vanishing hypothesis used above. Full real support becomes full complex span when this image is geometrically integral and real-defined. At anchors in this branch, the proposition supplies $K^T B = 0$. The frame-free closed-kernel propagation from a nonempty real open branch, followed by Lemma 16.1, gives the Clifford conclusion. The residual quadric Q in the degree-six proof is not assumed to be a residue image, and no transpose-rank condition on Q is used. The two nonreal conjugate-image case requires its separate argument.

22 Unions and conjugate pairs of four-dimensional images

Retain the actual real linear transpose pencil $B(a)$, its Gram $G(a) = B(a)B(a)^T$, the image space E, and the nondegenerate indefinite coefficient form h from Section 21. Thus $G(a)$ is positive semidefinite of rank eight for every nonzero real a, and its radial columns span E. Individual complex image components need not have full span in this section.

Lemma 22.1. *Let $A_1, \dots, A_k \subset \mathbb{P}^7_{\mathbb{C}}$ be irreducible fourfolds on which h and every entry of G vanish. Suppose that their union spans $\mathbb{P}^7_{\mathbb{C}}$. Then either $\operatorname{im} B(a) = E$ for every nonzero real a, or the real span of h and the entries of G contains a regular sequence of three quadrics whose degree-eight complete intersection contains all the A_i.*

Proof. Let Z be the common complex zero locus of those quadrics. It has no real point. Its dimension is at least four, while a six-dimensional component would equal $V(h)$ and contain real points. Suppose Y is a five-dimensional component. It cannot span a projective space of dimension at most five: then it would be a linear $\mathbb{P}^5$ inside a nondegenerate eight-form, whose maximal isotropic vector dimension is only four.

If Y spans all of $\mathbb{P}^7$, its degree is at least three. Choose a real entry f of G independent of h; otherwise all entries are multiples of h, contradicting real rank eight on a nonzero h-null vector. The degree-four complete intersection $V(h, f)$ contains Y. Two distinct conjugate components of degree at least three cannot fit in it, so Y is real-defined. Its degree is even, since a generic transverse real complementary plane meets it only in nonreal conjugate pairs. Hence its degree is four. Degree exhaustion and complete-intersection unmixedness make $V(h, f)$ exactly the reduced scheme Y. Its quadratic ideal is $\operatorname{span}\{h, f\}$, so Lemma 17.1 and Remark 17.2 give the asserted image.

Otherwise Y spans a hyperplane $\ell = 0$. The restriction of h has rank at least six, and $Y = V(h, \ell)$ is its irreducible quadric. Each quadratic entry g of G has the form

$$g = c_g h + \ell m_g$$

with m_g linear. Since the union of the A_i spans all of $\mathbb{P}^7$, some fixed A_j is not contained in $\ell = 0$. Integrality gives $m_g|_{A_j} = 0$ for every g. The vector span of A_j has dimension at least six: dimension five would make A_j a linear $\mathbb{P}^4$ inside $V(h)$, again impossible. Its linear annihilator therefore has dimension at most two. All entries g lie in one complex space of quadrics of dimension at most three. Their real span has the same dimension bound, since real linearly independent coefficient vectors remain complex linearly independent. The scalar-span criterion again gives the asserted image.

In the remaining case $\dim Z = 4$. The generated homogeneous ideal has height three. Begin with h and a real independent quadric; then avoid the finitely many height-two associated primes of their complete intersection by a real element of the given quadratic span. None contains that entire span. Real prime avoidance produces the claimed regular sequence and the degree-eight complete intersection. □

Proposition 22.2. *Under the hypotheses of Lemma 22.1, assume the A_i are distinct and $\sum_i \deg A_i \geq 8$. Then $\operatorname{im} B(a) = E$ for every nonzero real a.*

Proof. Unless the conclusion already holds, the complete intersection from the lemma contains all the distinct fourfolds. Its total degree is eight, so the assumed degree sum must equal eight. These are exactly its components, each with multiplicity one. Unmixedness and generic reducedness make it their reduced union; its positive-dimensional homogeneous complete-intersection ideal is saturated. Every entry of G belongs to its degree-two ideal, spanned by its three defining quadrics. The scalar-span criterion proves the result. □

In particular, this proposition applies to two distinct full-span fourfolds, each of degree at least four, or to four distinct fourfolds in $V(h)$, each of degree at least two. In the two-quadratic pole case these are the two geometric quadratics and four geometric lines, respectively. The earlier generic-field uniformity and saturated-fibre lemmas supply these alternatives and the distinctness of the images; neither conclusion is inferred from the degree argument itself.

Proposition 22.3. *Suppose A and $\overline{A}$ are distinct irreducible fourfolds in $V(h, G)$ whose combined complex span is $\mathbb{P}^7$. Then $\operatorname{im} B(a) = E$ for every nonzero real a.*

Proof. Let U be the vector span of A and put $d = \dim_{\mathbb{C}} U$. The isotropic-dimension argument in Lemma 22.1 gives $6 \leq d \leq 8$.

If $d = 8$, the two images each have degree at least four, and Proposition 22.2 applies. If $d = 7$, the restriction $h|_U$ has rank at least six. The affine coordinate ring of its quadric is factorial by the Klein–Nagata theorem [25, Theorem 8.2, p. 26], with a polynomial extension for a possible one-dimensional radical. The cone over A is a height-one homogeneous prime, hence is generated modulo $h|_U$ by one homogeneous element f. Full span in U excludes $\deg f = 1$. Thus $\deg A = 2 \deg f \geq 4$, and the same degree-budget argument applies.

Suppose $d = 6$. The restriction $h|_U$ has rank at least four, so its null locus is an irreducible quadric fourfold and equals A. The quadratic entries of G restricted to U are consequently constant multiples of $h|_U$:

$$G(a) = h(a)D \qquad (a \in U)$$

for one constant complex symmetric matrix D. Since $U + \overline{U} = \mathbb{C}^8$, the intersection $U \cap \overline{U}$ has complex dimension four. It is invariant under conjugation and contains a nonzero real vector w. Real rank eight of $G(w)$ forces $h(w) \neq 0$ and $D = G(w)/h(w)$ to be real of rank eight. Put $T = \operatorname{im}_{\mathbb{R}} D$.

For complex $a \in U$ with $h(a) \neq 0$, the Gram has image $T_{\mathbb{C}}$ and rank eight. Since $B(a)$ has eight columns, its image is also $T_{\mathbb{C}}$. The projection of the linear pencil $B|_U$ to $V_{\mathbb{C}}/T_{\mathbb{C}}$ vanishes on a dense open set, hence everywhere. Conjugation gives the same conclusion on $\overline{U}$. Their sum is all of $\mathbb{C}^8$, so every $B(a)$ maps into this fixed eight-space. The radial columns identify its real form with E. □

Remark 22.4. For a simple real-irreducible quadratic or quartic pole splitting over $\mathbb{C}$, the two geometric components give distinct images by saturated-fibre uniqueness when $\alpha = s = 4$; full real residue support gives the combined span required above. No full individual complex span is assumed. These propositions give image identities at the anchor under consideration. Their use in classification still invokes the separate real-open branch propagation and closed-kernel criterion of Lemma 16.1.

23 The divisor budget for five-dimensional residue images

Throughout this section $\nu = 8$, the anchor is regular, and $B(a)$ is the actual real linear transpose pencil, with 8 coefficient variables $a = (a_1, \dots, a_8)$. By Lemma 11.2, $B(a)$ has rank 8 for every real $a \neq 0$. Let $h(a) = a^T H a$ be the real coefficient form; it is nondegenerate and indefinite. Every residue image A lies in the projective quadric $V(h) \subset \mathbb{P}^7_{\mathbb{C}}$, by the isotropy in (11.10). We consider poles of type $(\alpha, s) = (5, 5)$: each geometric pole component has an irreducible closed residue image A of dimension five, and B has generic rank five on A. Distinct geometric pole components have distinct images (Lemma 18.6).

Since rank is lower semicontinuous, $\operatorname{rank} B(a) \leq 5$ at *every* point of A. Hence A has no real point.

The real residue support is the smallest real subspace $U \subset \mathbb{R}^8$ whose complexification contains every residue image. If a real-defined image A lies in a complex hyperplane $\{c \cdot a = 0\}$, it also lies in $\{\overline{c} \cdot a = 0\}$, hence in a real hyperplane. So a real-defined image of a component with real support 8 spans $\mathbb{P}^7$.

23.1 The budget

Let $S = \mathbb{C}[a_1, \dots, a_8]/(h)$ with its standard grading. By the Klein–Nagata theorem [25, Theorem 8.2, p. 26], a nondegenerate quadratic form in at least five variables over a field of characteristic not two gives a factorial quotient; thus S is a UFD. The affine cone over a five-dimensional irreducible $A \subset V(h)$ is a height-one prime of S, so its ideal is generated by a homogeneous irreducible f_A of some degree $e_A \geq 1$. If A spans $\mathbb{P}^7$, then $e_A \geq 2$.

Lemma 23.1 (Divisor budget)**.** *Let $A_1, \dots, A_k$ be distinct residue images of type $(5, 5)$. Then*

$$e_{A_1} + \cdots + e_{A_k} \leq 2.$$

Proof. Over the fraction field of S the matrix B has rank 8: the real cone of h contains nonzero points, where B has rank 8, so some maximal minor of B is nonzero in S. Fix i and localize S at the height-one prime (f_{A_i}); the result is a discrete valuation ring. The reduction of B has rank five at the generic point of A_i, so in the local Smith form of the 15×8 matrix B (its image lies in $X^{\perp}$) at least three of the eight invariant factors lie in the maximal ideal. Hence every maximal minor of B is divisible in S by $f_{A_i}^3$. The f_{A_i} are pairwise nonassociate primes, so every maximal minor is divisible by $\prod_i f_{A_i}^3$, of degree $3 \sum_i e_{A_i}$. A maximal minor has degree 8 and some maximal minor is nonzero in S. Therefore $3 \sum_i e_{A_i} \leq 8$. □

23.2 Consequences

Proposition 23.2 (Two quadratic poles)**.** *Suppose that the reduced denominator is $Q_1 Q_2$ with two distinct real quadratic factors, that every real residue support is 8, and that all geometric pole components have type $(5, 5)$. This configuration does not occur.*

Proof. Each real quadratic factor contributes at least two to the sum of the image-generator degrees. If it is geometrically irreducible, its single image is real-defined and has full span, so $e \geq 2$. If it splits, it has two distinct conjugate geometric components with distinct images, each with $e \geq 1$; no individual full span is needed. Images belonging to different pole components are also distinct by Lemma 18.6. Thus the two real quadratic factors contribute a total of at least four, contrary to Lemma 23.1. This count also covers the mixed pattern of one geometrically irreducible and one split factor. □

Proposition 23.3 (A single real pole)**.** *Suppose that the pole has one real component, with full real support and type* $(5, 5)$*.*

1. *If its image A is real-defined and geometrically integral, then $A = V(h, f)$ for a real quadric f; it is a complete intersection of degree four with no real point.*
2. *Otherwise there are exactly two images, A and $\overline{A}$, and $A = V(h, \ell)$, $\overline{A} = V(h, \overline{\ell})$ for a complex linear form ℓ that is not proportional to a real form.*

Proof. A real irreducible polynomial is geometrically irreducible or splits into one pair of complex conjugate factors.

(1) Full real support gives full complex span (see the start of this section), so $e_A \geq 2$, and Lemma 23.1 gives $e_A = 2$. Thus $A = V(h, f_A)$ with f_A of degree two. Since A is real-defined, $\overline{f_A}$ generates the same prime, so $\overline{f_A} = cf_A$ for a nonzero constant c, the only units of S in degree zero. Applying conjugation twice gives $|c| = 1$; write $c = \omega/\overline{\omega}$ and replace f_A by ωf_A, which is fixed by conjugation. Any real lift f of this class defines the same complete intersection. Its degree is $2 \cdot 2 = 4$, and it has no real point by the remark before the lemma.

(2) The two images are distinct by Lemma 18.6 and conjugate, so $e_A = e_{\overline{A}}$, and Lemma 23.1 gives $e_A = e_{\overline{A}} = 1$. Thus $A = V(h, \ell)$ with ℓ linear. If ℓ were proportional to a real form, A would be real-defined and equal to $\overline{A}$. □

23.3 The conjugate pair: definiteness and the positive model

We now normalize case (2) of Proposition 23.3. Write $\ell = \ell_1 + i\ell_2$ with real linear forms ℓ_1, ℓ_2; they are linearly independent. Put

$$W = \{a \in \mathbb{R}^8 : \ell_1(a) = \ell_2(a) = 0\}, \qquad r = \ell_1^2 + \ell_2^2 = \ell\,\overline{\ell}.$$

Then $\dim W = 6$, and r is a real positive semidefinite form of rank two.

Lemma 23.4. *The restriction $h|_W$ is definite. Consequently the signature of h is* $(6, 2)$ *or* $(7, 1)$*, up to an overall sign.*

Proof. A real point of $V(h, \ell)$ is a real $a \neq 0$ with $h(a) = 0$ and $\ell(a) = 0$, that is, $\ell_1(a) = \ell_2(a) = 0$. So the real points of $A = V(h, \ell)$ are exactly the real zeros of h on W. Since A has no real point, h has no nonzero real zero on W, and a real quadratic form without nontrivial real zeros is definite. A nondegenerate form that is definite on a six-dimensional subspace has at least six positive or at least six negative squares; since h is indefinite, its signature is $(6, 2)$ or $(7, 1)$ up to sign. □

Proposition 23.5 (Positive model of the conjugate pair)**.** *There are a positive definite real quadratic form q and a real constant $\lambda \neq 0$ such that $\rho = h - \lambda q$ is a semidefinite form of rank two with radical W, whose restriction to $W^{\perp_q}$ is definite. In q-orthonormal coordinates $a = (x, z)$ with $x \in W$ and $z \in W^{\perp_q}$,*

$$q = |x|^2 + |z|^2, \qquad h = \lambda q + \rho(z),$$

and

$$A \cup \overline{A} = V(h, r) = V(h, q) = V(q, \rho).$$

In particular the pencil spanned by h and q contains the rank-two member ρ, and λ is a relative eigenvalue of h with respect to q of multiplicity six.

Proof. The forms h and r have no common nonzero real zero: such a zero would lie in W and be a real zero of h there, which is excluded by Lemma 23.4. The joint range of (h, r) on the real unit sphere of $\mathbb{R}^8$ is compact, and it is convex by Brickman's theorem [15, Theorem 1.1]. It avoids the origin. If v is its point closest to the origin, then $\langle v, w\rangle \geq \langle v, v\rangle > 0$ for every point w of the range. This gives a positive definite combination $q = \alpha h + \beta r$. Here $\alpha \neq 0$, because r has rank two, and $\beta \neq 0$, because h is indefinite. Put $\lambda = 1/\alpha$ and $\rho = -(\beta/\alpha)r = h - \lambda q$. Then ρ is a nonzero multiple of r, so it is semidefinite of rank two with radical W, and definite on $W^{\perp_q}$.

Choose q-orthonormal coordinates x on W and z on its q-orthogonal complement. The forms ℓ_1, ℓ_2 vanish on W, so they, and hence ρ, depend on z alone. This gives the displayed model. Finally $V(h, q) = V(h, \alpha h + \beta r) = V(h, r)$ since $\beta \neq 0$, and $V(h, r) = V(h, \ell\overline{\ell}) = V(h, \ell) \cup V(h, \overline{\ell})$. Also $V(h, q) = V(q, h - \lambda q) = V(q, \rho)$. On W the relative eigenvalue of h with respect to q is λ, and on $W^{\perp_q}$ the form $h - \lambda q = \rho$ is nondegenerate, so the multiplicity of λ is exactly six. □

Remark 23.6. Proposition 23.5 is the input (28.1) of Section 28. The union $A \cup \overline{A}$ is a *reducible* complete intersection, so the factorial and normality arguments for a geometrically integral image (Section 26) do not apply to it; the conjugate pair is treated in Section 28 and Section 29. Each component $V(h, \ell)$ is a quadric of rank six or seven in its hyperplane, since restricting the rank-eight form h to a hyperplane lowers its rank by at most two; in particular each component is irreducible and spans its hyperplane.

24 Alignment of a balanced quadratic normal form

We isolate the linear algebra needed after a real balanced normal form has been constructed. All coefficient directions are retained in this argument.

Notation. In this section and in Section 25, the pencil is written $M(a)$, its Gram matrix is $C(a) = M(a)M(a)^*$, ι is an isometry onto the kernel space K, and L is the radial isomorphism onto E. In Sections 21 and 22 the same objects are written $B(a)$, $G(a) = B(a)B(a)^T$, K and ι, and $L(a)$ there denotes the lower skew block $K^T B(a)$. Let A and G be eight-dimensional Euclidean spaces, and let $V = E \oplus K$ be an orthogonal decomposition with $\dim E = \dim K = 8$. Fix an isometry $\iota : G \longrightarrow K$, a nonzero vector $g_0 \in G$, and an isomorphism $L : A \longrightarrow E$. Put $x = \iota g_0$. Consider a linear map

$$M : A \longrightarrow \mathrm{Hom}(G, V).$$

We assume that, for all $a \in A$,

$$M(a)g_0 = -La, \qquad \iota^* M(a) \text{ is skew-adjoint}, \tag{24.1}$$

and that

$$N_g : A \longrightarrow V, \qquad N_g(a) = M(a)g, \quad \text{is injective for every } g \neq 0. \tag{24.2}$$

In particular, $M(a)$ has rank eight for every $a \neq 0$: a nonzero vector in its kernel would contradict (24.2). Thus $C(a) = M(a)M(a)^*$ is positive semidefinite of rank eight. The two identities in (24.1) also give

$$\mathrm{Im}\, M(a) \subset x^{\perp}, \qquad \langle \iota g, M(a)g\rangle = 0. \tag{24.3}$$

Indeed, $\iota^* M(a)g_0 = 0$, so skew-adjointness makes g_0 orthogonal to the image of $\iota^* M(a)$.

Lemma 24.1. *If all images* $\operatorname{Im} M(a)$ *lie in one fixed subspace* $T \subset V$ *of dimension at most nine, then* $\operatorname{Im} M(a) = E$ *for every* $a \neq 0$.

Proof. The radial identity gives $E = L(A) \subset T$. Consequently the projection of T onto K has dimension at most one. The skew-adjoint map $\iota^* M(a)$ therefore has rank at most one, and hence is zero. This puts every image in E; its rank is eight. □

Proposition 24.2. *Suppose that* q *is a positive definite quadratic form on* A *and* h *is a real quadratic form. Let* $\lambda_1 \leq \cdots \leq \lambda_8$ *be the eigenvalues of* h *relative to* q. *Assume there are constant self-adjoint endomorphisms* D_0, D_1 *of* V *and real linear maps* $u, v : A \longrightarrow V$ *such that*

$$C(a) = h(a)D_0 + q(a)D_1 + u(a)u(a)^* - v(a)v(a)^*. \tag{24.4}$$

If $\lambda_2 < \lambda_7$, *then* $\operatorname{Im} M(a) = E$ *for every nonzero real* a.

Proof. Write $D(t) = D_1 + tD_0$. Fix $b \in V$ and impose the single linear condition $\langle b, u(a)\rangle = 0$. Its solution space $A_b \subset A$ has codimension at most one. On $A_b \setminus \{0\}$, positivity of (24.4) gives

$$q(a)\langle D(h(a)/q(a))b, b\rangle \geq \langle b, v(a)\rangle^2 \geq 0.$$

The minimum of h/q on A_b is at most λ_2, and its maximum is at least λ_7, by the min–max principle. Its Rayleigh range is an interval, so the displayed inequality proves

$$D(t) \succeq 0 \quad \text{for every } t \in [\lambda_2, \lambda_7]. \tag{24.5}$$

Here b was arbitrary; the vector a used to test a given t may depend on b.

On the interior of this interval, $D(t)$ has one common kernel and therefore one common image F. To see this, express one interior parameter as a strict convex combination of another and a third parameter in the interval. The kernel of a positive convex combination of two positive semidefinite matrices is the intersection of their kernels. Reversing the two interior parameters proves equality. Let $r = \dim F$. The set

$$\Omega = \{a \neq 0 : \lambda_2 < h(a)/q(a) < \lambda_7\}$$

is nonempty and open. For $a \in \Omega$, set $B = q(a)D(h(a)/q(a)) + u(a)u(a)^*$. Both B and $C = B - v(a)v(a)^*$ are positive semidefinite. Testing a vector in $\ker B$ shows that $v(a) \in \operatorname{Im} B$. As C has rank eight, $\operatorname{rank} B$ is eight or nine. Also $r \leq \operatorname{rank} B \leq r + 1$, whence $7 \leq r \leq 9$.

If $r = 9$, then $u(a) \in F$ for $a \in \Omega$: otherwise B would have rank ten and subtracting a rank-one matrix could not give rank eight. It follows that $v(a) \in F$ and $\operatorname{Im} C(a) \subset F$. Over the reals $\operatorname{Im} C(a) = \operatorname{Im} M(a)$. Since M is linear, its image lies in F for every a. Lemma 24.1 applies.

Suppose now $r \in \{7, 8\}$, and let $\pi : V \longrightarrow F^\perp$ be the orthogonal projection. For $a \in \Omega$,

$$(\pi M(a))(\pi M(a))^* = (\pi u(a))(\pi u(a))^* - (\pi v(a))(\pi v(a))^* \succeq 0.$$

A positive semidefinite difference of two matrices of rank at most one has rank at most one. Indeed, a vector orthogonal to $\pi u(a)$ must also be orthogonal to $\pi v(a)$. Hence every two-minor of the linear matrix $\pi M(a)$ vanishes on Ω, and therefore identically.

A real linear space of matrices of rank at most one has a common image line or a common row line. For completeness, write two nonzero members as $w_1 e_1^*$ and $w_2 e_2^*$. If their image lines are independent, rank at most one of their sum forces e_1, e_2 to be proportional. Comparison of any third member with these two forces its row into the same line. If no such pair exists, the image line is common. The zero space is included in the common-image alternative.

In the common-image case the images of M lie in F plus one fixed line, a space of dimension at most nine, and Lemma 24.1 applies again. In the common-row case there is a fixed nonzero covector e^* on G such that $\pi M(a) = w(a)e^*$. For every nonzero $g \in \ker e^*$,

$$N_g(A) \subset F.$$

Injectivity of N_g excludes $r = 7$ and gives $N_g(A) = F$ when $r = 8$. Equation (24.3) now implies $F \perp \iota(\ker e^*)$ and $F \perp x$. If $g_0 \notin \ker e^*$, these spaces span K, so $F = E$. If $g_0 \in \ker e^*$, the identity $N_{g_0} = -L$ directly gives $F = E$. Thus in either case the projection of $M(a)$ onto K has rank at most one. Its skew-adjointness makes it zero, and rank eight gives the result. □

Proposition 24.3. *Under the hypotheses of Proposition 24.2, its conclusion holds in either of the following situations:*

1. *Every eigenvalue of h relative to q has multiplicity at most five.*
2. *There is a q-orthogonal decomposition $A = W \oplus Z$, with $\dim W = 6$, $\dim Z = 2$, such that $h = \lambda q + \rho$, where ρ vanishes on W and its restriction to Z is definite and nondegenerate.*

Proof. Equality $\lambda_2 = \lambda_7$ would require an eigenvalue of multiplicity at least six, excluding the first situation. In the second situation, λ is an extreme eigenvalue of multiplicity six. If $\rho|_Z$ is positive definite, $\lambda_2 = \lambda < \lambda_7$; if it is negative definite, $\lambda_2 < \lambda = \lambda_7$. □

The second situation is the positive model for the conjugate pair of residue images. The proposition applies only after the component forms have been glued to the global *real* identity (24.4). It does not supply that gluing. Likewise, the first situation requires a real normal form from the preceding extraction arguments. In the transpose construction, (24.1) and (24.2) are the radial, tangency, and injectivity identities for the full eight-dimensional coefficient space. At anchors in a real open set, the resulting alignment feeds into the separate frame-free closed-kernel propagation and Clifford criterion.

25 Unbalanced forms and vanishing on a singular stratum

We retain all eight coefficient directions, and we use the notation of Section 24 (see the notation paragraph there; the letters ι, L and G differ from their use in Sections 21 and 22). Let A, G be real Euclidean spaces of dimension eight and $V = E \oplus K$ an orthogonal sum of two eight-dimensional spaces. Fix an isometry $\iota : G \to K$, a nonzero $g_0 \in G$, and an isomorphism $L : A \to E$. Let $M : A \to \mathrm{Hom}(G, V)$ be linear, and suppose

$$\operatorname{rank} M(a) = 8 \quad (a \neq 0), \qquad M(a)g_0 = -La, \qquad \iota^* M(a) \text{ is skew-adjoint.} \tag{25.1}$$

Set $C(a) = M(a)M(a)^*$. Over the complex numbers, the adjoint denotes the transpose for the complex-bilinear extension, not the Hermitian adjoint. The earlier transpose construction supplies (25.1); the statements below are conditional linear-algebra tools and do not construct a normal form.

We will use the following immediate consequence of the radial and skew identities. If all images of M lie in one fixed space T of dimension at most nine, then they equal E for every nonzero real coefficient. Indeed, $L(A) = E \subset T$, so the projection of T to K has dimension at most one. The skew map $\iota^* M(a)$ has rank at most one and is zero. Its upper image then has dimension eight and equals E.

Proposition 25.1 (Unbalanced normal form)**.** *Let h, q be independent real quadratic forms on A, with q positive definite. Suppose that D_0, D_1 are constant self-adjoint endomorphisms of V, $u \in V$ is a fixed nonzero vector, and $b : A \to V$ is quadratic, such that*

$$C(a) = h(a)D_0 + q(a)D_1 + ub(a)^* + b(a)u^*.$$

Then $\operatorname{Im} M(a) = E$ *for every nonzero real* a*.*

Proof. Compress to $u^\perp$. The resulting matrix is $C_\perp(a) = h(a)B_0 + q(a)B_1$, positive semidefinite of rank at most eight. On $q = 1$, the Rayleigh quotient h/q ranges over a nondegenerate closed interval I. Thus $B_1 + tB_0 \succeq 0$ for $t \in I$. On the interior of I its kernel is constant: expressing either of two interior parameters as a strict convex combination of the other and a third parameter proves this by the positive-semidefinite kernel-intersection identity. Let F be the common image and $r = \dim F \leq 8$.

Two distinct interior values show that both B_0 and B_1 annihilate $F^\perp$. Therefore $\operatorname{Im} C_\perp(a) \subset F$ for every a, including endpoint and exceptional values. Positivity of $C(a)$ puts its cross block in $\operatorname{Im} C_\perp(a)$: a vector in the kernel of a positive semidefinite principal block must be annihilated by the corresponding cross block. It follows that $\operatorname{Im} C(a) \subset \mathbb{R}u + F$. Over the reals $\operatorname{Im} C(a) = \operatorname{Im} M(a)$. The fixed ambient space has dimension at most nine, so the preceding observation finishes. □

For the singular-stratum statement, choose coefficient coordinates $A = W \oplus Z$ with $\dim W = m \in \{4, 5\}$ and

$$q(x, z) = |x|^2 + |z|^2, \qquad h(x, z) = \lambda q(x, z) + \rho(z),$$

where ρ is a nondegenerate real quadratic form on Z. Its signature is unrestricted. Let

$$\mathcal{A} = \{(x, z) \in A_{\mathbb{C}} : q(x, z) = \rho(z) = 0\}, \qquad \mathcal{S} = \{(x, 0) : |x|^2 = 0\}.$$

We assume $\operatorname{rank} C \leq 2$ on $\mathcal{A}$. In the residue construction this follows from the generic rank bound on every component, since the three-minors are polynomials. In particular this assumption is not restricted to the smooth points of $\mathcal{A}$.

Proposition 25.2 (Vanishing on a multiplicity-four or -five stratum)**.** *In this setting, if C vanishes on $\mathcal{S}$, then* $\operatorname{Im} M(a) = E$ *for every nonzero real* a*.*

Proof. We first align the restriction to W, then use complex arcs to constrain the remaining lower block.

The restriction to W. The quadric $|x|^2 = 0$ is irreducible over $\mathbb{C}$ for $m \geq 4$. Every quadratic entry of $C(x, 0)$ is therefore a scalar multiple of $|x|^2$. Write $C(x, 0) = |x|^2 D$, where D is real positive semidefinite of rank eight, and put $F = \operatorname{Im} D$. For every nonzero real x, $\operatorname{Im} M(x, 0) = F$. The radial identity puts $L(W) \subset F \cap E$, so the projection R of F onto K has dimension at most $8 - m$.

The skew pencil $\iota^* M(x, 0)$ has the same image $\iota^{-1}R$ for every nonzero real x. Its rank is even, and a skew map with image in $\iota^{-1}R$ is supported on that space. Rank two is impossible: skew maps on a fixed two-plane form a one-dimensional space, and a linear map from W into that space cannot be nonzero at every nonzero x. The only other nonzero possibility is rank four when $m = 4$. It would give an injective four-dimensional linear space of skew 4×4 matrices whose Pfaffian never vanishes away from zero. The restriction of the Pfaffian would be a definite quadratic form. But on the six-dimensional space of skew matrices the Pfaffian is

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23},$$

which has signature $(3, 3)$ and has no four-dimensional definite subspace. Thus $R = 0$ and $F = E$.

It follows that the lower pencil is $\iota^* M(x,z) = T(z)$, with T real linear and skew-adjoint. The map $M(x,0) : G \to E$ can be normalized on its *output* by a fixed isomorphism $S : G \to E$ so that

$$M(x,0) = SO(x), \qquad O(x)^* O(x) = O(x)O(x)^* = |x|^2 I, \qquad O(x_*) = I \tag{25.2}$$

for a chosen real unit vector $x_* \in W$. For example, take $S = M(x_*, 0)$, since $SS^* = D$. This normalization changes no coordinates on G, so T remains skew-adjoint in the same domain as the endomorphisms $O(x)$. For $y \perp x_*$ set $J_y = O(y)$. Polarization gives

$$J_y^* = -J_y, \qquad J_y J_{y'} + J_{y'} J_y = -2\langle y, y' \rangle I. \tag{25.3}$$

Complex arcs and exterior squares. Fix any nonzero complex z_0 with $\rho(z_0) = 0$, and any nonzero complex x_0 with $|x_0|^2 = 0$. The differential of $|x|^2$ at x_0 is nonzero, so there is a local complex analytic arc $x(t) = x_0 + O(t^2)$ satisfying $|x(t)|^2 = -t^2 |z_0|^2$. Then $a(t) = (x(t), tz_0)$ lies in $\mathcal{A}$. Since $C(a(0)) = 0$, the coefficient of t in $C(a(t))$ is

$$C_1 = M(x_0, 0) M(0, z_0)^* + M(0, z_0) M(x_0, 0)^*.$$

Every three-minor of $C(a(t))$ is zero, so its coefficient of t^3 gives $\operatorname{rank} C_1 \leq 2$. Relative to $E \oplus K$ the lower diagonal block of C_1 is zero, while the cross block is $M(x_0, 0) T(z_0)^*$, after identifying K with G. The rank of a symmetric block matrix $\left[\begin{smallmatrix} A & B \\ B^* & 0 \end{smallmatrix}\right]$ is at least $2 \operatorname{rank} B$. Hence

$$\operatorname{rank}\bigl(O(x_0) T(z_0)^*\bigr) \leq 1. \tag{25.4}$$

Suppose $T(z_0) \neq 0$, and put $U = \operatorname{Im} T(z_0)^* \subset G_{\mathbb{C}}$. Its dimension r is positive and even. A fixed complex right inverse onto U transfers (25.4) to $O(x_0)|_U$. The entries of $\bigwedge^2 O(x)\big|_{\bigwedge^2 U}$ are quadratic and vanish on $|x|^2 = 0$, so they are constant multiples of $|x|^2$. Evaluation at x_* identifies those constants:

$$\bigwedge{}^2 O(x)\Big|_{\bigwedge^2 U} = |x|^2 \operatorname{inc}_{\bigwedge^2 U}.$$

Differentiating at x_* in a real direction $y \perp x_*$ yields

$$J_y u \wedge v + u \wedge J_y v = 0 \qquad (u, v \in U). \tag{25.5}$$

Projection to the terms with one factor outside U first shows that $J_y U \subset U$: for independent u, v, their distinct U factors force both outside components to vanish.

If $r \geq 3$, the induced action of an endomorphism of U on $\bigwedge^2 U$ is faithful. In a basis, a third basis vector detects every off-diagonal entry; the diagonal entries satisfy $d_i + d_j = 0$ for every pair and therefore all vanish. Equation (25.5) would give $J_y|_U = 0$, contrary to $J_y^2 = -|y|^2 I$ for any nonzero real y. Thus $r = 2$, and (25.5) says

$$\operatorname{tr}(J_y|_U) = 0. \tag{25.6}$$

This argument is over the full complex endomorphism algebra; it does not require the bilinear form on U to be nondegenerate.

Multiplicity five. If $m = 5$, choose four orthonormal real vectors in $x_*^\perp$ and denote their operators by $J_1, \ldots, J_4$. On the invariant complex two-space U, equations (25.3) and (25.6) give

$$\operatorname{tr}(J_i J_j|_U) = 0 \ (i \neq j), \qquad \operatorname{tr}(J_i^2|_U) = -2.$$

The four restrictions are therefore linearly independent, although they lie in the three-dimensional space of traceless 2×2 complex matrices. This contradiction proves $T(z_0) = 0$ for every complex ρ-null z_0. The null cone of the nondegenerate rank-three form spans $Z_{\mathbb{C}}$, so linearity gives $T = 0$.

Multiplicity four. If $m = 4$, there are three such operators J_1, J_2, J_3. For a complex null z_0 with nonzero $T(z_0)$, its rank-two image U is invariant and each $J_i|_U$ has trace zero. A skew map with image U corresponds to a bivector in $\bigwedge^2 U$. The induced action of J_i on that line is its trace and is zero. In matrix terms this gives

$$J_i T(z_0) + T(z_0) J_i^* = J_i T(z_0) - T(z_0) J_i = 0.$$

The same equation holds if $T(z_0) = 0$. Each commutator is linear in z, and the nondegenerate rank-four null cone spans $Z_{\mathbb{C}}$. Consequently $T(z)$ commutes with all three J_i for every z.

The actual radial identity in (25.1) gives $T(z)g_0 = 0$. Commutation with J_1, J_2 makes $T(z)$ zero on the real four-space

$$H = \operatorname{span}_{\mathbb{R}}\{g_0, J_1 g_0, J_2 g_0, J_1 J_2 g_0\}.$$

These vectors divided by $|g_0|$ are orthonormal by (25.3). The space H and its orthogonal complement are invariant under J_1, J_2. Skewness makes T preserve $H^\perp$. On this four-dimensional complement, J_1, J_2 give the left action of the quaternions on a quaternionic line. A commuting real endomorphism is right multiplication by a quaternion; skew-adjointness restricts it to an imaginary quaternion. Explicitly, choose a unit $v \in H^\perp$. A commuting map is determined by its value at v in the basis $v, J_1 v, J_2 v, J_1 J_2 v$, and skewness removes its v coefficient. Hence there are three real linear forms t_1, t_2, t_3 on Z such that

$$T(z)^2\Big|_{H^\perp} = -\tau(z) I, \qquad \tau = t_1^2 + t_2^2 + t_3^2. \tag{25.7}$$

We have already proved $\operatorname{rank} T(z) \le 2$ whenever $\rho(z) = 0$ over $\mathbb{C}$. If $\tau(z) \neq 0$ at such a point, (25.7) would make the restriction to $H^\perp$ invertible, of rank four. Therefore τ vanishes on the irreducible rank-four quadric $\rho = 0$. Both polynomials have degree two, so $\tau = c\rho$ for a real constant c. The rank of τ is at most three, whereas ρ has rank four, forcing $c = 0$. Over the reals a sum of three squares can vanish identically only if each t_i is zero. Thus $T = 0$ also when $m = 4$.

In both cases the lower projection of M is zero. Its real rank eight identifies its image with E, completing the proof. □

Remark 25.3. The proof uses complex transverse null directions even when ρ is definite over the reals. It never assumes that their rank-two image spaces are real or nondegenerate. The multiplicity-four finish uses the actual radial kernel vector, in addition to the Clifford relations. No integrality or divisor-class argument is needed once the stated rank bound on every component and vanishing on $\mathcal{S}$ are known. If C has generic rank two on that stratum instead, extracting global real normal forms is a separate task. Finally, alignment at one anchor is the conclusion here; the established real-open propagation and closed-kernel criterion supply the later Clifford conclusion when these hypotheses hold along a branch of anchors.

26 Positive quadratic models and the factorial case

Throughout, $\nu = 8$ and the anchor is regular, as in Section 23. Proposition 23.3 supplies the following input: a real-defined, geometrically integral, full-span residue fivefold is a complete intersection $A = V(h, f) \subset \mathbb{P}^7_{\mathbb{C}}$ of two real quadrics. Here h is nondegenerate and indefinite, and A has no real point. We first normalize this intersection, then treat the case in which every relative eigenspace has dimension at most three.

Proposition 26.1. *There is a positive definite real quadric q with $\operatorname{span}_{\mathbb{R}}\{h, f\} = \operatorname{span}_{\mathbb{R}}\{h, q\}$. Every eigenvalue of h relative to q has multiplicity at most five. Both A and its affine cone are normal.*

Proof. The joint range of (h, f) on the real unit sphere is compact, and it is convex by Brickman's theorem [15, Theorem 1.1]. It avoids the origin, since A has no real point. If v is its point closest to zero, the supporting inequality $\langle v, w\rangle \geq \langle v, v\rangle > 0$ for every point w of the range gives a positive definite combination $q = \alpha h + \beta f$. Indefiniteness of h forces $\beta \neq 0$.

Choose real coefficient coordinates with

$$q = \sum_{i=1}^{8} a_i^2, \qquad h = \sum_{i=1}^{8} \lambda_i a_i^2.$$

The λ_i are real and nonzero. If an eigenvalue λ had multiplicity at least six, the nonzero quadratic $h - \lambda q$ would have rank at most two. Over $\mathbb{C}$ it factors into two linear forms, possibly equal. The ideal $(h, f) = (h, f_A)$ is the prime ideal of the affine cone of A (Section 23). Primality would put one linear factor in that ideal, contradicting full span.

The Jacobian criterion identifies the projective singular locus as the union, over relative eigenspaces W_λ, of

$$\mathbb{P}(W_\lambda) \cap V(q).$$

An eigenspace of dimension $m \geq 2$ contributes dimension $m - 2$; one of dimension one contributes nothing. Thus the projective singular locus has dimension at most three. The affine cone has dimension six, and its singular locus, including the vertex, has dimension at most four. Both spaces are regular in codimension one. As integral complete intersections they are Cohen–Macaulay, hence satisfy Serre's second condition. Serre's normality criterion applies. □

These multiplicities belong to the coefficient pencil (h, q), not to the original Jacobi spectrum. Normalizing q does not identify it with the original coefficient metric. In particular, a radial coefficient map remains an isomorphism and need not remain an isometry.

Proposition 26.2. *If every relative eigenspace has dimension at most three, the standard graded cone ring*

$$S = \mathbb{C}[a_1, \ldots, a_8]/(h, q)$$

is a unique factorization domain.

Proof. The preceding singular-locus calculation now bounds its affine dimension by two. Since $\dim S = 6$, every localization at a prime of height at most three is regular, and therefore factorial [25, Ch. 2, Theorem 1.6, p. 40]. Let $\mathfrak{m} = S_{>0}$ be the vertex maximal ideal. The local complete-intersection factoriality theorem applies to $S_{\mathfrak{m}}$: its localizations at primes of height at most three are factorial. We use the form of this theorem recorded in [25, §8, Remark 3 following Theorem 8.2 (Grothendieck's theorem), p. 27]. Finally, the localization map

$$\operatorname{Cl}(S) \longrightarrow \operatorname{Cl}(S_{\mathfrak{m}})$$

is an isomorphism for this positively graded normal domain [25, Proposition 7.4, p. 21]. The target is zero, so the normal Noetherian domain S is factorial. □

26.1 Extraction from principal isotropic lines

We give the homogeneous extraction in a form also usable after the local class-group calculations for larger eigenspaces. All pairings on an image below are the ones induced by the symmetric matrix as an isomorphism from the dual image; they are not assumed to be restrictions of the ambient Euclidean pairing.

Lemma 26.3. *Let $S_{\mathbb{R}}$ be a geometrically integral, normal, standard graded real domain with degree-zero part $\mathbb{R}$, and put $S = S_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Let C be a symmetric matrix of homogeneous degree two over $S_{\mathbb{R}}$ with generic rank two and hyperbolic induced pairing over $\mathrm{Frac}(S_{\mathbb{R}})$. Suppose the saturated rank-one modules of its two generic complex isotropic lines in S^N are principal. Then, over $S_{\mathbb{R}}$, C has one of the forms*

$$C = uu^T - vv^T, \qquad or \qquad C = cb^T + bc^T, \tag{26.1}$$

where u, v are homogeneous linear columns, or c is a constant column and b is homogeneous quadratic, respectively.

Proof. The scaling $\sigma_t : a \mapsto ta$, $t \in \mathbb{C}^*$, multiplies C by t^2 and so permutes the two isotropic lines. The stabilizer of a line has index at most two, and the divisible group $\mathbb{C}^*$ has no subgroup of index two. Each saturated module is therefore σ_t-stable for all t, and comparison of homogeneous components shows that it is graded. A principal graded rank-one module has a homogeneous generator; for example, choose a homogeneous generator of its one-dimensional fiber at $S_{>0}$ and use graded Nakayama. Denote the two polynomial vector generators by p, r, of nonnegative degrees d_p, d_r.

Over the fraction field, symmetry and isotropy give

$$C = t(pr^T + rp^T), \qquad 0 \neq t \in \mathrm{Frac}(S).$$

At each height-one prime, saturation makes both vectors primitive. Their reductions are nonzero, and their symmetric cross product is nonzero over the residue field: this holds both for independent vectors and for proportional vectors in characteristic zero. Some entry of $pr^T + rp^T$ is therefore a unit locally. Since C is polynomial, t has no negative height-one valuation. Normality gives $t \in S$. Homogeneity then yields

$$\deg t + d_p + d_r = 2. \tag{26.2}$$

This step includes any common divisorial scalar factor of C.

Hyperbolicity over the real fraction field makes the two lines individually real-defined. Conjugation preserves each saturated module, so it changes its homogeneous generator by a unit of S. Every such unit is a nonzero complex constant. Rescaling the generator makes it real: conjugating twice gives modulus one for that constant, which is removed by a scalar phase. Thus p, r and then t can be taken real.

If $d_p, d_r > 0$, both are one and t is constant. Absorb t into one column, then use their real sum and difference to obtain the first form in (26.1). Otherwise one generator is constant, and its partner multiplied by t is quadratic by (26.2), giving the second form. □

Proposition 26.4. *In the factorial case of Proposition 26.2, suppose $C = B(a)B(a)^T$ is the actual quadratic transpose Gram (the matrix G of Lemma 18.4), of generic rank two on A, whose generic transpose kernel is a totally isotropic three-space for the standard eight-form. Then, as an ambient real quadratic matrix, it has one of the forms*

$$C = hD_0 + qD_1 + uu^T - vv^T, \tag{26.3}$$

$$C = hD_0 + qD_1 + cb^T + bc^T, \tag{26.4}$$

where D_0, D_1 are constant real symmetric matrices and the columns have the degrees specified in Lemma 26.3, with $c \neq 0$ in (26.4)*.*

Proof. Let $K_0 = \ker B$ be the generic transpose kernel over the real function field of A. Since $\operatorname{Im} B^T = K_0^{\perp}$, we have $\operatorname{Im} C = B(K_0^{\perp})$, and for $y = Cx$, $y' = Cx'$ the induced pairing is $x^T C x' = (B^T x)^T (B^T x')$. Thus B induces an isometry from $K_0^{\perp}/K_0$, with the standard form, onto $\operatorname{Im} C$ with its induced pairing. Splitting off the three hyperbolic planes associated with the isotropic three-space leaves determinant square class -1 from the standard eight-form. A nonsingular binary form of this determinant class is hyperbolic. The same holds on the affine cone, whose function field is a purely transcendental extension by the scaling parameter.

The saturated modules of the two isotropic lines are rank-one reflexive modules over the normal domain S: equivalently, they are the intersections of their height-one localizations inside their fraction-field lines. Factoriality makes their divisor classes zero, so they are principal. Since S is normal, so is its conjugation-fixed subring $S_{\mathbb{R}}$, and Lemma 26.3 applies. Lift its homogeneous columns to ambient real polynomials. The difference between C and the lifted expression has degree two and belongs to the ideal (h, q). It is therefore $hD_0 + qD_1$ with constant real symmetric coefficient matrices, as asserted. □

For the actual full eight-coefficient pencil, Proposition 25.1 applies to (26.4). Proposition 24.2 applies to (26.3): relative multiplicities at most three imply $\lambda_2 < \lambda_7$. For the unbalanced form, the coefficient space A of Proposition 25.1 is the eight-dimensional coefficient space, as in Sections 24 and 25. Thus the full image equals the Jacobi image at every anchor of this case. On a real-open branch of such anchors, the frame-free closed-kernel criterion (Lemma 16.1) and Proposition 10.1(2) make R Clifford; see Section 30. No smoothness or factoriality assertion for relative multiplicities four or five is used here; their local class-group detectors are required in the subsequent section.

27 Extraction at the singular integral residue images

In the situation of Proposition 23.3(1), we consider the real geometrically integral residue image

$$A = V(q, \rho) \subset \mathbb{P}^7, \qquad q(w, z) = q_W(w) + q_Z(z) > 0,$$

arising from a relative eigenvalue of multiplicity $m \in \{4, 5\}$. Here $\dim W = m$, $\dim Z = 8 - m$, and ρ is nondegenerate on Z. Proposition 26.1 supplies these coordinates and normality of the standard graded complex cone ring

$$S = \mathbb{C}[w, z]/(q_W + q_Z, \rho).$$

Thus normality is an input; we will prove the required divisor-class statements rather than infer factoriality from normality.

Let C be the real symmetric quadratic transpose Gram matrix. For type $(\alpha, s) = (5, 5)$, Lemmas 18.4 and 11.3 give rank zero or two on A and generic rank two. The generic transpose kernel is a totally isotropic three-space in the standard eight-form. Its binary quotient has determinant square class -1, after splitting off three hyperbolic planes, and is therefore hyperbolic over the real function field. If C vanishes on a relevant singular stratum, Proposition 25.2 applies. Our task is to extract a real normal form in the remaining case, where C has rank two at the generic points of those strata.

27.1 A local divisor detector

We use divisor classes of affine normal rings, not projective Picard groups. Localization of a normal domain induces a surjection on divisor class groups whose kernel is generated by the height-one primes removed by localization [25, Theorem 6.3 and formula (6.4), p. 17]. We also use its factoriality corollary, often called Nagata's criterion [25, Corollary to Theorem 6.3′, p. 18].

Lemma 27.1. *Let F be a field of characteristic zero, and set*

$$B = F[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4), \qquad \mathfrak{m} = (z_1, z_2, z_3, z_4), \quad P = (z_1, z_3).$$

The ring $B \cong F[su, sv, tu, tv]$ is a normal domain (a quadric of rank four), so its divisor classes are defined. The class of $PB_{\mathfrak{m}}$ has infinite order in $\mathrm{Cl}(B_{\mathfrak{m}})$.

Proof. Use the Segre presentation

$$z_1 = su, \quad z_3 = sv, \quad z_4 = tu, \quad z_2 = tv, \qquad B = F[su, sv, tu, tv].$$

The valuation at P is the exponent of s. A homogeneous element of standard degree d has bidegree (d, d) in (s, t) and (u, v); its s-order is at most d. Consequently the symbolic power $P^{(n)}$ has no nonzero homogeneous element of degree less than n. Its degree-n part contains the $n+1$ independent monomials $z_1^{n-i}z_3^i$, $0 \le i \le n$.

Localization at $\mathfrak{m}$ only introduces denominators with nonzero constant term, so it preserves initial degree. Every element of $\mathfrak{m}P^{(n)}B_{\mathfrak{m}}$ has initial degree at least $n+1$. The displayed monomials therefore remain independent modulo that submodule, and $P^{(n)}B_{\mathfrak{m}}$ needs at least $n+1$ generators. It is not principal for any $n \ge 1$. In a normal domain, vanishing of $n[P]$ would make this divisorial symbolic power principal. Hence $[P]$ has infinite order. □

27.2 Multiplicity five

Lemma 27.2. *For $m = 5$, let $\eta = (z_1, z_2, z_3)$ be the generic singular-stratum prime of S. Then* $\mathrm{Cl}(S) \simeq \mathbb{Z}/2$, *and*

$$\mathrm{Cl}(S) \longrightarrow \mathrm{Cl}(S_\eta)$$

is injective.

Proof. Choose complex coordinates so that

$$\rho = z_1z_2 - z_3^2, \qquad q_W = w_1w_2 + w_3^2 + w_4^2 + w_5^2.$$

After inverting z_1 and eliminating z_2, we obtain

$$S[z_1^{-1}] = R[w_1, \dots, w_5]/(w_1w_2 + g), \quad R = \mathbb{C}[z_1, z_1^{-1}, z_3],$$

where

$$g = w_3^2 + w_4^2 + w_5^2 + q_Z(z_1, z_3^2/z_1, z_3).$$

Over $\mathrm{Frac}(R)$, the quadratic leading part of g in the three w variables has rank three. A factorization into affine linear factors would have leading rank at most two, so g is irreducible. It is primitive over the factorial ring R. Thus w_1 is prime in $S[z_1^{-1}]$. Inverting w_1 also eliminates w_2 and gives a Laurent polynomial ring. Nagata's criterion shows that $S[z_1^{-1}]$ is factorial.

The only height-one prime above z_1 is $P = (z_1, z_3)$: the relation modulo z_1 is $z_3^2 = 0$, and the reduced quotient has a rank-five w-quadratic part and is a domain. At P, z_2 is a unit and $z_1 = z_3^2/z_2$. Hence $\mathrm{div}(z_1) = 2P$, and localization shows that $\mathrm{Cl}(S)$ is generated by $[P]$ with $2[P] = 0$.

At η, w_2 is invertible. Eliminating w_1 identifies S_η exactly with the vertex local ring of $z_1z_2 = z_3^2$ over $F = \mathbb{C}(w_2, w_3, w_4, w_5)$. The three z variables are independent in its cotangent space. The ideal (z_1, z_3) has a two-dimensional image in that space, so it cannot be principal. It is a height-one prime in a normal local domain and is divisorial. Thus its class is nonzero locally. A cyclic group of order dividing two whose generator has nonzero image maps injectively. This proves both assertions. □

27.3 Multiplicity four and the two-block exception

Lemma 27.3. *Suppose $m = 4$ and q_Z is not proportional to ρ. For $\eta = (z_1, z_2, z_3, z_4)$, the map*

$$\mathrm{Cl}(S) \longrightarrow \mathrm{Cl}(S_\eta)$$

is injective, and $\mathrm{Cl}(S) \simeq \mathbb{Z}$.

Proof. Choose

$$\rho = z_1z_2 - z_3z_4, \qquad q_W = w_1w_2 + w_3w_4.$$

Inverting z_1 gives

$$S[z_1^{-1}] = R[w_1, w_2, w_3, w_4]/(w_1w_2 + w_3w_4 + \psi), \qquad R = \mathbb{C}[z_1, z_1^{-1}, z_3, z_4],$$

where $\psi = q_Z(z_1, z_3z_4/z_1, z_3, z_4)$. Our hypothesis gives $\psi \neq 0$: otherwise q_Z would vanish on the irreducible ρ-quadric and, being quadratic, would be proportional to ρ. The polynomial $w_3w_4 + \psi$ is irreducible over $\mathrm{Frac}(R)$ and primitive over R. Thus w_1 is prime; inverting it produces a Laurent polynomial ring. As before, $S[z_1^{-1}]$ is factorial.

The two height-one primes above z_1 are

$$P = (z_1, z_3), \qquad P' = (z_1, z_4).$$

Their quotients are domains because the w-quadratic has rank four. The equation $z_1z_2 = z_3z_4$ gives valuation one for z_1 at each prime, so $\mathrm{div}(z_1) = P + P'$. Consequently $\mathrm{Cl}(S)$ is cyclic, generated by $[P]$.

At η, invert w_2 and eliminate $w_1 = -(w_3w_4 + q_Z(z))/w_2$. The resulting local ring is exactly

$$\left(F[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4)\right)_{(z_1,z_2,z_3,z_4)}, \qquad F = \mathbb{C}(w_2, w_3, w_4).$$

The generator P maps to the plane prime of Lemma 27.1, of infinite order. This makes the map from the cyclic global class group injective and proves the claim. □

When q_Z is proportional to ρ, the relative spectrum has two four-dimensional eigenspaces. The preceding one-detector argument does not apply. The two defining equations instead give a product of two rank-four quadric cones.

Lemma 27.4. *Let*

$$S = \mathbb{C}[w_1, \dots, w_4, z_1, \dots, z_4]/(w_1w_2 - w_3w_4,\ z_1z_2 - z_3z_4).$$

Put $\eta_W = (w_1, \dots, w_4)$ and $\eta_Z = (z_1, \dots, z_4)$. Then $\mathrm{Cl}(S) \simeq \mathbb{Z}^2$, and the combined map

$$\mathrm{Cl}(S) \longrightarrow \mathrm{Cl}(S_{\eta_W}) \oplus \mathrm{Cl}(S_{\eta_Z})$$

is injective.

Proof. Inverting both w_1 and z_1 eliminates w_2, z_2, giving a Laurent polynomial ring. The removed height-one primes are

$$P_W = (w_1, w_3), \quad P'_W = (w_1, w_4), \qquad P_Z = (z_1, z_3), \quad P'_Z = (z_1, z_4).$$

The principal divisors of w_1, z_1 give the relations $[P'_W] = -[P_W]$ and $[P'_Z] = -[P_Z]$. Thus $[P_W], [P_Z]$ generate the class group.

At η_W the local ring is the split w-node over the function field of the z-quadric. The class of P_W has infinite order by Lemma 27.1, while P_Z becomes the unit ideal since $z_1 \notin \eta_W$. At η_Z the roles reverse. Therefore a class $n[P_W] + k[P_Z]$ vanishes in both local groups only if $n = k = 0$. This proves injectivity and also excludes every relation between the two global generators. □

27.4 A common extraction argument

Lemma 27.5 (Local splitting)**.** *Let T be a normal local domain in which 2 is invertible. A unimodular symmetric bilinear form on a free T-module of rank two which is hyperbolic over* Frac(T) *splits over T into two free isotropic lines.*

Proof. There is a vector of unit norm: otherwise all diagonal entries and, by evaluation on sums of basis vectors, all matrix entries would lie in the maximal ideal. Splitting off this vector diagonalizes the form as $\mathrm{diag}(a, b)$ with a, b units. Generic hyperbolicity gives $r^2 = -b/a$ in the fraction field. The root r is integral over T, hence lies in T by normality; it is a unit. The vectors $(r, 1), (-r, 1)$ are isotropic and form a basis, since $2r$ is a unit. □

Proposition 27.6. *In the setup of this section, suppose C has rank two at the generic singular-stratum prime in Lemma 27.2 or Lemma 27.3, or at* both *primes in Lemma 27.4, as applicable. Then, in ambient real coefficient coordinates, C has one of the forms*

$$C = qD_0 + \rho D_1 + uu^T - vv^T, \tag{27.1}$$

$$C = qD_0 + \rho D_1 + cb^T + bc^T, \tag{27.2}$$

where D_0, D_1 are constant real symmetric matrices, u, v are real linear columns, c is a nonzero constant real column, and b is a real quadratic column. All common divisorial factors of C are allowed; no primitivity assumption on C is imposed.

Proof. Over Frac(S) the two isotropic image lines of C are defined by the hyperbolic binary pairing. For each line L, take its saturated closure

$$\mathcal{L} = L \cap S^N \subset \mathrm{Frac}(S)^N,$$

where N is the physical matrix size. This is a rank-one reflexive module, graded by the scaling argument in the proof of Lemma 26.3: its height-one localizations are saturated line lattices, and normality identifies their intersection with the displayed module.

At each designated detector, a unit two-minor and vanishing three-minors of C make its image a free rank-two direct summand. The induced symmetric pairing is unimodular. Explicitly, view C as a symmetric map from the dual free module to the free module; after quotienting its kernel it is an isomorphism onto its image, and its inverse defines the pairing there. By the isometry $K_0^{\perp}/K_0 \to \mathrm{Im}\, C$ induced by B (proof of Proposition 26.4), this is the pairing whose generic isotropic lines were specified above, rather than an assumed nondegenerate restriction of the ambient Euclidean form. Lemma 27.5 makes each line closure free at that detector. The injectivity proved in the

preceding class-group lemmas therefore makes each global class zero. In the two-block case this uses *both* detectors.

Choose homogeneous polynomial vector generators u_0, v_0 of the two closures. Such homogeneous generators exist because a class-zero rank-one reflexive module is free; graded Nakayama at the homogeneous vertex gives a single homogeneous generator. Their degrees are nonnegative, since they are polynomial vectors. Over the fraction field, symmetry and isotropy of the two lines give

$$C = t(u_0 v_0^T + v_0 u_0^T)$$

for a nonzero scalar t. At every height-one prime, saturation makes both generators primitive. Their reductions are nonzero, and their symmetric cross product cannot vanish over a field of characteristic different from two. Indeed $uv^T + vu^T = 0$ with $u \neq 0$ forces $v = 0$ by a nonzero coordinate, its diagonal entry, and its row. At least one matrix entry of the cross product is therefore a unit locally. Regularity of C forces every height-one valuation of t to be nonnegative. Normality gives $t \in S$.

All objects are homogeneous, and hence

$$\deg t + \deg u_0 + \deg v_0 = 2. \tag{27.3}$$

The two generic lines are individually defined over the real function field, so conjugation preserves each saturated closure. Its homogeneous generator is changed by a unit of S, necessarily a nonzero complex constant. Conjugating twice and rescaling by a constant allow both generators to be chosen real. The scalar t is then real as well.

If both vector degrees are positive, (27.3) forces both to be one and t constant. With $u = (tu_0 + v_0)/\sqrt{2}$ and $v = (tu_0 - v_0)/\sqrt{2}$ one has $t(u_0 v_0^T + v_0 u_0^T) = uu^T - vv^T$, giving (27.1) modulo the defining ideal. If a vector has degree zero, it is a nonzero constant c; absorbing t into the other vector gives a quadratic vector b and (27.2) modulo that ideal. Real homogeneous elements lift to real ambient polynomials. Since the degree-two part of the ideal of A is precisely $\operatorname{span}_{\mathbb{R}}\{q, \rho\}$, the remainders are $qD_0 + \rho D_1$, with the required real symmetric coefficient matrices. □

Remark 27.7. For a single multiplicity-four or multiplicity-five block the relevant stratum is $z = 0$, $q_W(w) = 0$. For relative spectrum $(4, 4)$ there are two strata, one for each cone vertex. For the stratum $w = 0$, $q_Z(z) = 0$, apply Proposition 25.2 with W and Z exchanged; there $h = \lambda' q + \rho'$ with $\rho' = (\lambda - \lambda')q_W$ nondegenerate on W. If the Gram vanishes identically on any required stratum, Proposition 25.2 applies. Otherwise its pointwise ranks zero or two give rank two at every required generic point, and Proposition 27.6 applies. The separate balanced and unbalanced alignment results, Propositions 24.3 and 25.1, then give full alignment at this anchor; Section 30 supplies the real-open closed-kernel conclusion. This section proves extraction only: it neither replaces those alignment results nor applies its domain arguments to a reducible conjugate pair.

28 Preliminary reductions for the conjugate five-dimensional images

Let $B(a)$ be the actual real linear transpose pencil at a regular rank-eight anchor, and write $G = BB^T$. For every nonzero real coefficient a, it has rank eight. Its radial column is $B(a)e = -\iota(a)$, where ι is an isomorphism onto the physical Jacobi image E, and $K^T B(a)$ is skew for the orthogonal lower frame K. A real change of coefficient coordinates is allowed: no isometry property of the resulting radial map ι is used.

Proposition 23.5 supplies real coordinates $a = (x, z) \in W \oplus Z$, with $\dim W = 6$, $\dim Z = 2$, and

$$q(x, z) = |x|^2 + |z|^2, \qquad h(x, z) = \lambda q(x, z) + \rho(z). \tag{28.1}$$

Here q is positive definite, h is nondegenerate indefinite, and ρ is a definite nondegenerate binary form. Write ρ as a nonzero real multiple of $\ell\overline{\ell}$, with ℓ the complex linear form of Proposition 23.3(2). Over $\mathbb{C}$ the two null lines of ρ give distinct conjugate components

$$A_+ = V(h, \ell), \qquad A_- = V(h, \overline{\ell}), \qquad A_+ \cup A_- = V(h, q) = V(q, \rho).$$

This union is a reduced complete intersection: (q, ℓ) and $(q, \overline{\ell})$ are prime, and in the factorial ring $\mathbb{C}[a]/(q)$ [25, Theorem 8.2, p. 26] the nonassociate primes $\ell, \overline{\ell}$ satisfy $(\ell) \cap (\overline{\ell}) = (\ell\overline{\ell})$. Hence the ideal of $A_+ \cup A_-$ is $(q, \rho) = (h, q)$. Each component spans its nonreal hyperplane and is a quadric of rank six or seven. The Gram has rank at most two everywhere on the union, generically two on each component. These geometric and rank statements are consequences of that proposition, Remark 23.6 and Lemmas 18.4, 18.6 and 11.3 in type $(\alpha, s) = (5, 5)$; by the last lemma the rank of the Gram is even at every point. Set

$$\Sigma = A_+ \cap A_- = \{z = 0,\ |x|^2 = 0\}.$$

28.1 Vanishing on the intersection

Proposition 28.1. *If G vanishes identically on Σ, then $\operatorname{im} B(a) = E$ for every nonzero real a.*

Proof. The complex quadric $|x|^2 = 0$ in six variables is irreducible. Thus

$$G(x, 0) = |x|^2 D$$

for a constant real positive semidefinite matrix D of rank eight. All nonzero real $x \in W$ give the same image $F = \operatorname{im} D$. The radial column puts a six-space in $F \cap E$, so the lower projection of F has dimension at most two. For every such x the lower skew map has precisely that image. Rank two would give a linear map from six parameters to the one-dimensional skew space on a fixed two-plane with no nonzero kernel, which is impossible (a skew map has even rank, so dimension one is impossible). Hence $F = E$ and $K^T B(x, z) = T(z)$ is a linear skew pencil in z alone.

Choose an isomorphism $S : \mathbb{R}^8 \to E$ with $SS^T = D$. The real square pencil $O(x) = S^{-1} B(x, 0)$ satisfies

$$O(x)O(x)^T = O(x)^T O(x) = |x|^2 I_8. \tag{28.2}$$

Fix any nonzero complex z_0 with $\rho(z_0) = 0$ and any nonzero complex x_0 with $|x_0|^2 = 0$. The nonzero gradient in the x variables gives a formal or analytic arc $x(t) = x_0 + O(t^2)$ satisfying

$$|x(t)|^2 + t^2 |z_0|^2 = 0.$$

Throughout, $|\cdot|^2$ on complex vectors denotes the complex-bilinear extension of the real quadratic form. The arc $(x(t), t z_0)$ lies in $V(q, \rho)$, and the Gram vanishes at $t = 0$. Its coefficient of t is

$$G_1 = B(x_0, 0) B(0, z_0)^T + B(0, z_0) B(x_0, 0)^T.$$

The three-minors vanish along the arc, so their leading coefficients give $\operatorname{rank} G_1 \le 2$. Its lower diagonal block is zero and its upper–lower block is $B(x_0, 0) T(z_0)^T$. Since a symmetric block matrix $\left[\begin{smallmatrix} A & C \\ C^T & 0 \end{smallmatrix}\right]$ has rank at least $2 \operatorname{rank} C$, we obtain

$$\operatorname{rank}\big(O(x_0) T(z_0)^T\big) \le 1. \tag{28.3}$$

This argument is valid on either complex transverse null line.

Suppose $T(z_0) \neq 0$, and put $U = \operatorname{im} T(z_0)^T$. Its complex dimension r is positive and even. A constant right inverse onto U transfers (28.3) to $O(x_0)|_U$. Its quadratic two-minors vanish on the irreducible quadric $|x|^2 = 0$, so they are constant multiples of $|x|^2$. Normalize at a real unit x_* by replacing $O(x)$ with $O(x_*)^T O(x)$. Evaluation at x_* then yields the identity

$$\bigwedge^2 O(x)\big|_{\wedge^2 U} = |x|^2 \operatorname{incl}_{\wedge^2 U}. \tag{28.4}$$

For real $y \perp x_*$, let $J_y = O(y)$. Polarization of (28.2) gives $J_y^T = -J_y$ and $J_y^2 = -|y|^2 I$. Differentiating (28.4) gives

$$J_y u \wedge v + u \wedge J_y v = 0 \qquad (u, v \in U).$$

The components outside U show that $J_y U \subset U$. If $r \geq 3$, the infinitesimal exterior-square action of $\operatorname{End}(U)$ is faithful: a third basis vector detects each off-diagonal entry, while diagonal entries satisfy $d_i + d_j = 0$ for all pairs, forcing all to vanish. Thus $J_y|_U = 0$, contrary to its square for nonzero y.

If $r = 2$, the same identity says $\operatorname{tr}(J_y|_U) = 0$. Choose five real orthonormal vectors $y_i \in x_*^\perp \cap W$ and put $J_i = J_{y_i}$. Polarization gives

$$J_i J_j + J_j J_i = -2\delta_{ij} I, \qquad \operatorname{tr}\big((J_i|_U)(J_j|_U)\big) = -2\delta_{ij}.$$

The five restrictions are therefore complex linearly independent, but all lie in the three-dimensional space of traceless two-by-two complex matrices. This contradiction proves $T(z_0) = 0$. No nondegeneracy or reality of the subspace U was assumed.

The two distinct complex null lines of the nondegenerate binary form ρ span $Z_{\mathbb{C}}$. Since T is linear and vanishes on both, it is zero. Real rank eight now gives the claimed image. □

28.2 A constant component column

Proposition 28.2. *Suppose that on A_+ one has*

$$G = ub^T + bu^T,$$

where u is a fixed nonzero complex physical vector and b is a homogeneous quadratic vector in the component coordinate ring. Then $\operatorname{im} B(a) = E$ for every nonzero real a.

Proof. Let $F = \operatorname{span}_{\mathbb{R}}\{\operatorname{Re} u, \operatorname{Im} u\}$ and let Π be real orthogonal projection to $F^\perp$. Then $\dim F \leq 2$. The compressed Gram $\Pi G \Pi$ vanishes on A_+, and reality makes it vanish on A_- as well. The reduced union has degree-two ideal $\operatorname{span}\{h, q\}$. Therefore

$$C(a) := \Pi G(a) \Pi = h(a) D_0 + q(a) D_1$$

for constant real symmetric matrices. It is positive semidefinite of rank at most eight at every real coefficient; constant rank is not needed.

On $q = 1$, the Rayleigh range of h is a nontrivial interval $[\alpha, \beta]$. Every $tD_0 + D_1$ on that interval is positive semidefinite. At interior parameters it is a strict positive combination of the endpoint matrices, so all interior kernels are their common intersection. The corresponding fixed image V_0 has dimension at most eight. On the nonempty real open set of interior Rayleigh values, $C = (\Pi B)(\Pi B)^T$ has image V_0. Projection of the linear pencil ΠB to $V_0^\perp$ vanishes there, and therefore identically. All images of B lie in the fixed real space $F + V_0$ of dimension at most ten.

Lemma 16.3 applies to this reduced union. To check its component hypothesis, a nonzero real linear form vanishing on A_+ would vanish on its span $\ker \ell$, hence would be a complex multiple of ℓ. Reality would make it a multiple of $\overline{\ell}$ as well, impossible because the two forms are independent. The same holds for A_-. This proves the conclusion. In particular, no constancy of the compressed rank at the Rayleigh endpoints has been inserted into the argument. □

28.3 The remaining component columns are linear

We recall the actual-pencil pairing input for the following extraction. At a generic point of A_+, the kernel of B is a totally isotropic three-space in its standard eight-dimensional domain. The induced rank-two symmetric pairing is the binary quotient of that form by three hyperbolic planes; over the complex component field it is hyperbolic. This is the isotropic-kernel identity of Lemma 11.3: over the component field, $K_0^{\perp}/K_0$ has determinant class -1, and -1 is a square there. It is not a consequence of a rank-two matrix alone.

Proposition 28.3. *If the alignment conclusion of Proposition 28.2 does not already hold, then on* A_+

$$G = pr^T + rp^T$$

with complex homogeneous linear columns p, r*. On* A_- *the factorization is their conjugate.*

Proof. The coordinate ring $S_+ = \mathbb{C}[a_1, \dots, a_8]/(h, \ell)$ is a UFD: the restricted quadric has rank six or seven, so Klein–Nagata [25, Theorem 8.2, p. 26] applies, with a polynomial extension if there is a radical variable. The two lines are stable under coefficient scaling, so their saturated modules are graded and have homogeneous generators, as in the proof of Lemma 26.3. Thus the two homogeneous isotropic lines of the generic rank-two pairing have primitive homogeneous polynomial generators p, r in this ring. Equivalently, their saturated rank-one reflexive modules are principal over the UFD. Thus over its fraction field

$$G = t(pr^T + rp^T)$$

for a homogeneous scalar t.

At a height-one prime, each primitive vector has a nonzero reduction. Over a field of characteristic zero, the symmetric cross product of two nonzero vectors cannot be the zero matrix, even if the vectors are proportional: choose a nonzero coordinate in the first vector and use its diagonal and row entries. Hence some entry of the cross product is a unit at that prime. Regularity of G forces the valuation of t to be nonnegative there. Factoriality therefore gives $t \in S_+$.

The nonnegative degrees satisfy

$$\deg t + \deg p + \deg r = 2.$$

If either column has degree zero, absorb t and the other column into the quadratic vector of Proposition 28.2. Otherwise both column degrees are one and t is constant. Absorbing this nonzero complex constant into one column gives the stated form. □

Remark 28.4. The conclusions in this section are pointwise image statements or componentwise factorizations. Generic-anchor closed-kernel propagation remains a separate input. Likewise, the two complex linear columns must still be glued to a real global identity, and alignment of a real balanced form is a separate criterion. No extension theorem on the reducible, nonnormal union is used here. A component column may acquire a common factor when restricted to Σ; the later real-gluing argument must handle that possibility.

29 Gluing the conjugate residue components

We use the actual transpose pencil from the preceding reduction. To make the hypotheses of this last gluing step explicit, let $V = E \oplus K$ be an orthogonal decomposition with $\dim E = \dim K = 8$,

and let $\iota : \mathbb{R}^8 \to K$ be an isometry. The coefficient space is another copy A of $\mathbb{R}^8$. We have a linear pencil

$$M(a) : \mathbb{R}^8 \longrightarrow V, \qquad C(a) = M(a)M(a)^T, \qquad a \in A,$$

of rank eight for every nonzero real a. There are a fixed nonzero vector $e \in \mathbb{R}^8$ and an isomorphism $L : A \to E$ such that

$$M(a)e = -La, \qquad \iota^T M(a) \text{ is skew-symmetric.} \tag{29.1}$$

The map L need not be an isometry: a change of coefficient coordinates may have been used to normalize a positive quadratic form.

29.1 A determinant obstruction on six coefficient directions

Lemma 29.1. *Let $W \subset A$ have dimension six, and let q_W be positive definite on W. There do not exist constant real symmetric operators D, H on V and a nonzero real quadratic form s on W such that*

$$C(x) = q_W(x)D + s(x)H \quad (x \in W), \tag{29.2}$$

where $\operatorname{rank} s \le 2$ *and H has rank two and signature* $(1,1)$.

Proof. The range of s on the ellipsoid $q_W = 1$ is a nondegenerate closed interval, since s and q_W are independent. Every matrix $D + tH$ on this interval is positive semidefinite of rank eight. For positive semidefinite matrices, the kernel of a strict convex combination is the intersection of their kernels. Equality of the ranks therefore shows that these matrices have a common kernel, including at the endpoints. Thus $M(x)$ has a fixed image F of dimension eight for every real $x \in W \setminus \{0\}$.

The radial identity in (29.1) gives $L(W) \subset F \cap E$. Consequently $R = \iota^T F$ has dimension at most two. For every such x, the skew operator $\iota^T M(x)$ has image exactly R. If R were nonzero, its dimension would be two. Every skew operator with image in this fixed two-plane is a scalar multiple of a fixed operator. A linear map from the six-dimensional space W to this one-dimensional space has a nonzero kernel, contradicting its constant positive rank. Hence $R = 0$ and $F = E$.

Independence of q_W, s now implies that D, H are supported on E. The radical of s has dimension at least four. At a nonzero vector in that radical, (29.2) shows that $D|_E$ is positive definite. The two nonzero eigenvalues of $D^{-1/2}HD^{-1/2}|_E$ are therefore $\alpha > 0 > \beta$. Taking determinants on E gives

$$\det(C(x)|_E) = \det(D|_E)q_W(x)^6\big(q_W(x) + \alpha s(x)\big)\big(q_W(x) + \beta s(x)\big). \tag{29.3}$$

Both last quadratic factors are positive definite: each is positive on the radical of s, and neither vanishes on the connected set $W \setminus 0$, because their product is positive there. They are irreducible real quadratics of rank six. Independence of q_W, s and $\alpha \neq \beta$ makes the three displayed quadratic factors pairwise nonassociate. The last two thus occur to odd multiplicity. On the other hand, in orthonormal coordinates on E, the matrix $M(x)$ is square, so $\det(C(x)|_E) = (\det M(x))^2$. This contradicts unique factorization in the real polynomial ring on W. □

29.2 The real balanced form

Split $A = W \oplus Z$, where $\dim W = 6$, $\dim Z = 2$, and write

$$q(x,z) = q_W(x) + q_Z(z) > 0 \quad \text{for } (x,z) \neq 0.$$

Let ρ be a definite quadratic form of rank two on Z. Over $\mathbb{C}$ it is a nonzero scalar multiple of $\ell\bar{\ell}$ for independent conjugate linear forms. Set

$$U_+ = \ker \ell, \quad U_- = \ker \bar{\ell}, \qquad A_\pm = V(q) \cap \mathbb{P}(U_\pm), \qquad \Sigma = V(q_W) \subset \mathbb{P}(W_{\mathbb{C}}).$$

Each component quadric has rank at least six. Its homogeneous coordinate ring, and that of Σ, are factorial. For the latter, for example, write $q_W = x_1x_2 + x_3x_4 + x_5x_6$. The element x_1 is prime, and inverting it gives a Laurent polynomial ring. Nagata's criterion applies [25, Theorem 6.3 and its Corollary, pp. 17–18]; alternatively, [25, Theorem 8.2, p. 26] applies directly. The same argument allows the additional transverse variable on either component.

The constant-column exclusion and homogeneous factorization in Propositions 28.2 and 28.3 leave two complex linear columns on a component. The next proposition is the precise gluing statement used here.

Proposition 29.2. *Assume that on A_+ there are complex linear columns p, r with*

$$C = pr^T + rp^T. \tag{29.4}$$

Assume that C has generic rank two on Σ and that its induced binary pairing there is hyperbolic over $\mathbb{R}(\Sigma)$. Then there exist real linear columns P, R on A and constant real symmetric operators D_0, D_1 such that

$$C = qD_0 + \rho D_1 + PR^T + RP^T. \tag{29.5}$$

Proof. Write $p_0 = p|_W$, $r_0 = r|_W$. They are nonzero and independent over $\mathbb{C}(\Sigma)$, since C has rank two there. Their lines are the two isotropic lines of the induced binary pairing. Hyperbolicity over $\mathbb{R}(\Sigma)$ makes each line individually invariant under conjugation.

Let $S_0 = \mathbb{C}[W]/(q_W)$. Suppose first that p_0 is primitive over the factorial ring S_0. Its conjugate is primitive and spans the same fraction-field line. Their proportionality factor and its inverse have nonnegative valuation at every height-one prime, so that factor is a unit of S_0, hence a nonzero constant. Conjugating twice shows that its modulus is one. Rescale p by a constant and r inversely so that p_0 is real. Reality of C on Σ gives

$$p_0(r_0 - \bar{r}_0)^T + (r_0 - \bar{r}_0)p_0^T = 0.$$

For a nonzero vector u over a field of characteristic different from two, the map $v \mapsto uv^T + vu^T$ is injective, as follows by using a nonzero coordinate of u, its diagonal entry, and its row. Thus $r_0 = \bar{r}_0$ in S_0. Since the ideal (q_W) has no degree-one part, both restrictions are real linear columns on W. The argument is symmetric if r_0 is primitive.

It remains to exclude the case where neither restriction is primitive. Homogeneity then gives

$$p_0 = fc, \qquad r_0 = gd,$$

with nonzero complex linear forms f, g and independent constant vectors c, d. Conjugation invariance of each constant line permits us to take c, d real by rescaling f, g. Put $H = cd^T + dc^T$; this real symmetric matrix has rank two and signature $(1, 1)$. Reality of C on Σ implies

$$fg - \overline{fg} \in \mathbb{C}q_W.$$

The left side has rank at most four, while q_W has rank six. It follows that $s = fg$ is a real polynomial, not merely a real class modulo q_W. It is nonzero and has rank at most two. Lifting the quadratic identity on Σ gives

$$C(x, 0) = q_W(x)D + s(x)H$$

with D constant real symmetric. This contradicts Lemma 29.1. At least one restriction is therefore primitive, and the preceding paragraph makes both restrictions real.

Choose a nonzero transverse vector $\zeta \in U_+ \cap Z_{\mathbb{C}}$. There is a unique real linear column P with

$$P|_W = p_0, \qquad P(\zeta) = p(\zeta), \qquad P(\bar{\zeta}) = \overline{p(\zeta)}.$$

Indeed $\zeta, \bar{\zeta}$ are independent and the prescriptions commute with conjugation. Define R similarly. These columns restrict to p, r on U_+ and to their conjugates on U_-. Hence the quadratic matrix $C - PR^T - RP^T$ vanishes on both component quadrics. In the factorial ring $\mathbb{C}[A]/(q)$, the distinct prime hyperplane elements $\ell, \bar{\ell}$ have intersection of principal ideals equal to their product. The reduced union consequently has quadratic ideal $\operatorname{span}_{\mathbb{C}}\{q, \rho\}$. Reality and symmetry give (29.5) with real symmetric coefficient matrices. □

In the actual transpose-pencil application both hypotheses hold. By Lemma 11.3, $\operatorname{rank} C = 2 \operatorname{rank} M - 8$ is even at every point of $\Sigma \subset A_+$ and is at most two. Since Σ is irreducible, either C vanishes identically on Σ, the case of Proposition 28.1, or C has generic rank two on Σ. In the latter case the hyperbolicity hypothesis has already been supplied by the kernel identity. At the generic point of Σ, the space $K_0 = \ker M$ is a totally isotropic three-space for the standard eight-dimensional form. The induced binary pairing is the nondegenerate quotient $K_0^{\perp}/K_0$. Removing three hyperbolic planes leaves determinant square class -1, which makes the binary quotient hyperbolic over $\mathbb{R}(\Sigma)$. This argument fixes the two isotropic lines individually; it does not assume that a chosen pair of polynomial generators is already real.

Finally, writing

$$u = (P + R)/\sqrt{2}, \qquad v = (P - R)/\sqrt{2}$$

turns the last term of (29.5) into $uu^T - vv^T$. Since $\operatorname{span}\{q, \rho\} = \operatorname{span}\{q, h\}$, this is the real balanced form required by Proposition 24.2; its hypothesis (24.2) is the injectivity of N_g in Lemma 11.2, which a real change of coefficient coordinates preserves. For this conjugate configuration the sixfold relative eigenvalue is extreme, so Proposition 24.3 supplies the strict interval $\lambda_2 < \lambda_7$.

30 Assembly of the proof of Conjecture A

We now combine the preceding results to prove Theorem A.

30.1 The anchor set and the pole alternative

Fix R as in Theorem A. If R has two reduced eigenvalues, Theorem C applies. Theorem C, together with Theorems F and G, classifies twenty of the thirty-six patterns, leaving exactly sixteen:

$$(8,4,3),\ (8,3,3,1),\ (8,3,2,2),\ (8,3,2,1,1),\ (8,3,1^4);$$
$$(7,5,3),\ (7,4,4),\ (7,4,3,1),\ (7,4,2,2),\ (7,4,2,1,1),\ (7,4,1^4),$$
$$(7,3,3,2),\ (7,3,3,1,1),\ (7,3,2,2,1),\ (7,3,2,1^3),\ (7,3,1^5).$$

Shift the root of multiplicity 8 or 7 to zero. The shifted family has constant rank $\nu = 7$ or $\nu = 8$ on $\mathbb{R}^{16} \setminus 0$, and its nonzero spectral matrix Λ is nonscalar. If Λ is definite, Theorem H shows that R is Clifford. We assume from now on that Λ is indefinite.

The coherent-partner construction of Lemma 10.3 applies at an anchor x whenever the generic-pair conditions of [19, Section 4, Lemmas 2(1) and 3, pp. 7–9 of arXiv:math/0204258v2] hold for (x, y) with y in a dense subset of $x^{\perp}$; the condition $\operatorname{Im} \mathcal{J}_x \cap \operatorname{Im} \mathcal{J}_y = 0$ is the nonvanishing of the sum of squares of the 2ν-minors of the 16×32 matrix $[\mathcal{J}_x\ \mathcal{J}_y]$, that is, $\operatorname{rank}[\mathcal{J}_x\ \mathcal{J}_y] = 2\nu$. These are polynomial nonvanishing conditions $g(x, y) \neq 0$. Substitute $y = (\|x\|^2 I - xx^T)z$ and expand g in z;

the anchors for which $g(x,\cdot)$ vanishes identically form a real algebraic subset, which is proper because a good generic pair exists. Let $U_0 \subset \mathbb{R}^{16} \setminus 0$ be its complement; it is a nonempty Euclidean-open cone. The construction at a nonzero anchor x is understood after normalizing to $x/\|x\|$, and its rank and image conditions extend conically to U_0. In the coherent block notation the image block has size ν and the lower block has size $(16-\nu)\times\nu$; (16.1) is the instance $\nu = 8$. For $x \in U_0$ we write F_x for the rational upper block and P_x for the linear lower block. We also shrink U_0 away from the proper algebraic exceptional loci used below. These include the non-generic anchors of Lemma 12.1 and Propositions 12.3 and 12.4, which form a semialgebraic set of dimension less than sixteen and therefore lie in a proper algebraic subset. They also include the loci used for spreading out geometric factors (Remark 30.3). The result is a nonempty open set. In the preceding sections a *regular anchor* means a point of this U_0, and a *generic branch* means a branch of Lemma 30.2 for this U_0.

Lemma 30.1 (Pole alternative)**.** *If F_x is polynomial for one $x \in U_0$, then R is Clifford.*

Proof. A polynomial homogeneous rational map of degree one is linear. Then $M_{sx+y} = sM_x + M_y$ is linear on $\mathbb{R}^{16}$, and $\mathcal{J}_z = M_z \Lambda M_z^T$ holds identically by continuation of the rational identity. For real $z \neq 0$ the operator $\mathcal{J}_z$ has rank ν, so M_z has rank ν and is injective. This is a global linear factor, and Proposition 10.1(1) applies. □

Thus we may assume that F_x has a pole for every $x \in U_0$. The rest of the proof shows that this alternative again leads to a Clifford tensor.

30.2 A finite semialgebraic case partition

At each $x \in U_0$ the pole data of F_x consist of finitely many discrete invariants: the real factorization type of the reduced denominator and its pole orders; the ranks of the skew residues; the real residue support d; the residue-variety dimension α and the generic transpose rank s; the number, degrees and dimensions of the complex spans of the residue images; the relative multiplicities of the coefficient pencil and the signatures involved; and the vanishing or not of the transpose Gram on the singular stratum. They do not depend on the choice of orthonormal eigenframes at x: two such choices differ by constant orthogonal block transformations commuting with Λ, which preserve every rank, dimension, support, degree and image inclusion.

Lemma 30.2 (Real-open branch)**.** *There is a finite partition $U_0 = B_1 \sqcup \cdots \sqcup B_N$ into semialgebraic sets such that every invariant above is constant on each B_j. At least one B_j has nonempty Euclidean interior.*

Proof. All objects have bounded degree in (x,y), and bounded-degree algebraic incidence families admit finite algebraic stratifications on which their ranks, dimensions and degrees are constant; this identifies the geometric invariants with algebraic predicates. Include the image and kernel frames at x as auxiliary real variables, constrained by the polynomial orthonormality and spectral equations; by frame independence, an existential and a universal quantifier over the frames give the same predicate. Ranks and signatures are sign conditions on minors. A factorization of bounded degree is a system of coefficient equations with auxiliary real and imaginary coefficients. The absence of real points is a quantified polynomial condition; the positive pencil member and its relative multiplicities have the same description. Each value of each invariant therefore defines a first-order formula over $\mathbb{R}$ in x. By Tarski–Seidenberg elimination (see, for example, the statement recalled in [9, pp. 1–2]), each such set is semialgebraic, and the common refinement $B_1, \dots, B_N$ is finite.

Choose quantifier-free descriptions of the B_j and collect the finitely many nonzero polynomials occurring in them. Outside the union of their zero sets, every sign is locally constant, so membership in each B_j is locally constant. A finite union of proper real zero sets does not contain a nonempty open set. Hence some point of U_0 has a neighbourhood contained in a single B_j. □

The branches need not be Zariski open, and signature boundaries may form lower-dimensional strata; the proof below never uses them.

Remark 30.3 (Galois arguments). Several branch results use the action of a Galois group on the geometric factors of the pole data over the function field $\mathbb{R}(x)$. Irreducibility over $\mathbb{R}(x)$, or a particular orbit over this field, need not survive as real irreducibility, or as a real orbit, after specialization at a real anchor: $y^2 - x_1$ is irreducible over $\mathbb{R}(x)$, but it splits over $\mathbb{R}$ when $x_1 > 0$ and not when $x_1 < 0$. What does persist, after a finite algebraic cover and the removal of a proper algebraic subset, are the geometric components and their algebraic invariants (ranks, degrees, spans, and (α, s)). If Galois transitivity forces such an invariant to be equal on all conjugate geometric factors, the equality persists at the specialized anchors, even where the real conjugation orbits split. Real splittings and signatures are recorded by Lemma 30.2. This is the only form in which the branch results use Galois transitivity.

30.3 The leaves

Every branch is covered by one of the results below. Each applicable result either classifies R directly, contradicts the branch under consideration, or supplies full transpose alignment throughout the real-open region of that branch: the full transpose pencil has image $E_x = \operatorname{Im} \mathcal{J}_x$ for every real coefficient $a \neq 0$, at every anchor x of the region. We list the branches in the order of the case analysis.

(L1) $\nu = 7$: there is no pole at any anchor (Proposition 13.1).

(L2) $\nu = 8$, pole orders. The reduced denominator D satisfies $D^2 \mid \det P_x$ and has no real linear or odd-degree factor, so it is a quadratic Q, its square, a product $Q_1 Q_2$ of distinct quadratics, or a real-irreducible quartic f (Proposition 12.3). Double quadratic poles are excluded (Proposition 15.1), as is a simple quadratic pole with residue rank four (Proposition 14.1). A quartic residue has rank two by the determinant budget. For $Q_1 Q_2$ the budget $Q_1^{r_1} Q_2^{r_2} \mid \det P_x$ gives $r_1 = r_2 = 2$. Every remaining pole is simple, with residue rank two.

(L3) Proper real support $d \leq 7$. For a quadratic component, Q^2 divides the maximal d-minors, so $d \geq 4$; $d = 4, 5$ are excluded componentwise (Lemma 18.1). For a sole quadratic, $d = 6$ is excluded by Proposition 20.2, and both $d = 7$ alternatives, $s = 4$ and $s = 5$, by Proposition 20.9. For two quadratics, a component with $d = 6$ is excluded by Proposition 20.2, and $d = 7$ by Proposition 20.5. A real-irreducible quartic pole always has $d = 8$ (Lemma 18.3).

(L4) Full support. Incidence gives $4 \leq s \leq \alpha \leq 5$; the pair $(\alpha, s) = (5, 4)$ is excluded by Proposition 18.5 and the closed-kernel criterion. Hence $\alpha = s$, and then Lemma 18.6 makes distinct pole components have distinct residue images.

(L5) Two quadratic poles. The alignment and descent results of Proposition 19.5 leave two geometric quadratics or four geometric lines in one Galois orbit, so all geometric components have one type (α, s) (Remark 30.3). Type $(5, 5)$ is excluded by the divisor budget (Proposition 23.2), and type $(4, 4)$ aligns by the union-height argument (Proposition 22.2).

(L6) A single real pole, type $(4,4)$. A real geometrically integral image of full span has degree 4, 6 or 8, and each is treated in Proposition 21.1. A conjugate pair of images aligns (Proposition 22.3).

(L7) A single real pole, type $(5,5)$, one real integral image $V(h,q)$ with q positive and maximal relative multiplicity of the pencil (h,q) at most five (Propositions 23.3 and 26.1). The maximal multiplicity is at most three (Propositions 26.2 and 26.4), five (Propositions 25.2 and 27.6, using Lemma 27.2), or four (Propositions 25.2 and 27.6, using Lemmas 27.3 and 27.4). The normal forms produced there align: unbalanced forms by Proposition 25.1, balanced forms by Proposition 24.3, which covers every relative multiplicity at most five; the extraction uses Lemma 26.3, Proposition 25.2 and Lemma 27.1.

(L8) A single real pole, type $(5,5)$, a conjugate pair of images $V(h,l)$, $V(h,\bar{l})$ (Proposition 23.3, Lemma 23.4, and Proposition 23.5). A transpose Gram vanishing on $\Sigma = V(h,l) \cap V(h,\bar{l})$ gives alignment (Proposition 28.1), and so does a constant column (Proposition 28.2, using Lemma 16.3). Otherwise Proposition 28.3 supplies complex linear columns, and Proposition 29.2 gives a global real balanced form, which aligns by Proposition 24.3.

The list is exhaustive. Items (L2)–(L4) reduce every pole at $\nu = 8$ to simple poles with rank-two residues, full real support, and type $(\alpha, s) = (4,4)$ or $(5,5)$. The denominator has one real pole component (Q or f) or two (Q_1Q_2); the latter is (L5). A single real pole component is geometrically irreducible or splits over $\mathbb{C}$ into one conjugate pair, so it has one geometrically integral real image or a conjugate pair of images; the two conjugate components of a split real pole have the same (α, s), since B is real. With the two types this gives the four alternatives (L6), (L7) and (L8); in (L7) the maximal relative multiplicity takes one of the three listed values.

30.4 Proof of Theorem A

Proof. By the reductions above we may assume that Λ is indefinite. Suppose, for contradiction, that R is not Clifford. By Lemma 30.1, F_x has a pole at every $x \in U_0$. The case $\nu = 7$ is excluded by (L1), so $\nu = 8$. Direct Clifford conclusions among the leaves are now contradictions to our assumption. By Lemma 30.2, some branch B_j contains a nonempty open set W of anchors. Since W is nonempty, the result covering B_j cannot end in a contradiction. Hence, at every $x \in W$, the full transpose pencil has image E_x for every real $a \neq 0$. As shown after Remark 16.2, this gives $\mathcal{J}_y z = 0$ for all $y, z \in \ker \mathcal{J}_x$ at every $x \in W$. Lemma 16.1 applies on the open set W; it makes the full Gram matrix of the coherent partner radial, hence polynomial. Since Λ is nonscalar, Proposition 10.1(2) shows that R is Clifford, contrary to our assumption. □

No frame and no pole factor is propagated away from W: the closed-kernel identity is frame independent, and its continuation to generic real and complex anchors is part of Lemma 16.1. A pole-free regular anchor is handled directly by Lemma 30.1.

31 Remarks

Remark 31.1 (The quartic in other dimensions). For a two-root Osserman tensor on $\mathbb{R}^n$ with multiplicities $(\nu, n-1-\nu)$, the same construction has $\operatorname{rank} Q = \nu + 1$ and

$$\|\nabla F\|^2 = 16 s^3, \qquad \Delta F = 8(n-1-2\nu)s.$$

Thus it yields a Cartan–Münzner quartic on $\mathbb{R}^{2n}$ with those multiplicities. A classification of the quartic alone does not recover the original splitting: a further argument such as Propositions 5.1–5.2

is required. In particular the homogeneous non-FKM multiplicities $(2, 2)$ and $(4, 5)$ would require their own such argument.

Remark 31.2 (The definite FKM family is excluded). For nine generators on $\mathbb{R}^{32}$ the Clifford product $P_0 \cdots P_8$ is central. The definite $(8, 7)$ family has this product equal to I or $-I$; the indefinite family has trace zero [11, Sections 3.5 and 6.6]. In the constant branch of Proposition 5.2 it is $\operatorname{diag}(C_1 \cdots C_8, -C_1 \cdots C_8)$. In the nonconstant branch it is $\operatorname{diag}(H_0 \cdots H_8, -H_0 \cdots H_8)$. Both have trace zero. Hence the definite FKM$(8, 7)$ family cannot arise from the quartic of a two-root tensor. An orthogonal change of the generator index changes the product only by a sign.

Acknowledgments

This work was done overwhelmingly by AI models and supervised by the author. The models were Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

Statements and Declarations

Funding. No funding was received for conducting this study.

Competing interests. The author has no competing interests to declare that are relevant to the content of this article.

Data availability. Data sharing is not applicable to this article, as no datasets were generated or analysed during the current study.

References

[1] V. Andrejić, *The proportionality principle for Osserman manifolds*, J. Geom. Phys. **176** (2022), 104516. doi:10.1016/j.geomphys.2022.104516.

[2] V. Andrejić, *Two-root Riemannian manifolds*, Mediterr. J. Math. **20** (2023), article 100. doi:10.1007/s00009-023-02340-2; arXiv:2009.12834v4.

[3] V. Andrejić and K. Lukić, *On the existence of a curvature tensor for given Jacobi operators*, Filomat **37**:25 (2023), 8465–8471. doi:10.2298/FIL2325465A. `https://www.pmf.ni.ac.rs/filomat-content/2023/37-25/37-25-6-19416.pdf`.

[4] V. Andrejić and K. Lukić, *The orthogonality principle for Osserman manifolds*, Acta Math. Hungar. **173** (2024), 246–252. doi:10.1007/s10474-024-01434-x; arXiv:2308.14851.

[5] J. Bochnak and W. Kucharz, *On homology classes represented by real algebraic varieties*, Banach Center Publ. **44** (1998), 21–35. `https://eudml.org/doc/208886`.

[6] R. B. Brown and A. Gray, *Riemannian manifolds with holonomy group* Spin(9), in *Differential Geometry (in Honor of Kentaro Yano)*, Kinokuniya, Tokyo, 1972, 41–59.

[7] Q.-S. Chi, *A curvature characterization of certain locally rank-one symmetric spaces*, J. Differential Geom. **28** (1988), 187–202. doi:10.4310/jdg/1214442277.

[8] Q.-S. Chi, *Isoparametric hypersurfaces with four principal curvatures, IV*, J. Differential Geom. **115** (2020), 225–301. doi:10.4310/JDG/1589853626; arXiv:1605.00976.

[9] T. Drescher, T. Netzer and A. Thom, *On projections of free semialgebraic sets*, preprint (2017), introduction, pp. 1–2. arXiv:1709.08424.

[10] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Small schemes and varieties of minimal degree*, arXiv:math/0404517. `https://arxiv.org/abs/math/0404517`.

[11] D. Ferus, H. Karcher and H.-F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502. English translation by T. E. Cecil, *Clifford algebras and new isoparametric hypersurfaces*, arXiv:1112.2780.

[12] E. García-Río, D. N. Kupeli and R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry*, Lecture Notes in Math. **1777**, Springer, Berlin, 2002. doi:10.1007/b83213.

[13] P. B. Gilkey, *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*, World Scientific, River Edge, NJ, 2001.

[14] W. Kryszewski, *Remarks to the Vietoris theorem*, Topol. Methods Nonlinear Anal. **8** (1996), 383–405. doi:10.12775/TMNA.1996.041; `https://projecteuclid.org/journals/topological-methods-in-nonlinear-analysis/volume-8/issue-2/Remarks-to-the-Vietoris-theorem/tmna/1479265248.full`.

[15] J. E. Martínez-Legaz, *On Brickman's theorem*, J. Convex Anal. **12** (2005), 139–143. `https://www.heldermann-verlag.de/jca/jca12/jca0474_b.pdf`.

[16] H.-F. Münzner, *Isoparametrische Hyperflächen in Sphären. I*, Math. Ann. **251** (1980), 57–71.

[17] H. K. Nickerson, *On conformally symmetric spaces*, Geom. Dedicata **18** (1985), 87–99.

[18] Y. Nikolayevsky, *Osserman manifolds of dimension 8*, Manuscripta Math. **115** (2004), 31–53. doi:10.1007/s00229-004-0480-y; arXiv:math/0310387.

[19] Y. Nikolayevsky, *Osserman conjecture in dimension $n \neq 8, 16$*, Math. Ann. **331** (2005), 505–522. doi:10.1007/s00208-004-0580-8; arXiv:math/0204258.

[20] Y. Nikolayevsky, *On Osserman manifolds of dimension 16*, in *Contemporary Geometry and Related Topics* (N. Bokan et al., eds.), Proceedings of the Conference, Belgrade, June 26–July 2, 2005, Faculty of Mathematics, University of Belgrade, 2006, 379–398. `https://www.maths.tcd.ie/EMIS/proceedings/CGRT2005/Articles/cgrt24.pdf`.

[21] Y. Nikolayevsky, *Conformally Osserman manifolds*, Pacific J. Math. **245** (2010), 315–358. doi:10.2140/pjm.2010.245.315; arXiv:0810.5621.

[22] Y. Nikolayevsky, *Conformally Osserman manifolds of dimension 16 and a Weyl–Schouten theorem for rank-one symmetric spaces*, Ann. Mat. Pura Appl. **191** (2012), 677–709 (published online 2011). doi:10.1007/s10231-011-0201-6. Preprint: *Osserman manifolds and the Weyl–Schouten theorem for rank-one symmetric spaces*, arXiv:0910.1645v1.

[23] Y. Nikolayevsky and Z. Rakić, *A note on Rakić duality principle for Osserman manifolds*, Publ. Inst. Math. (Beograd) (N.S.) **94**(108) (2013), 43–45. doi:10.2298/PIM1308043N; arXiv:1204.1600.

[24] R. Osserman, *Curvature in the eighties*, Amer. Math. Monthly **97** (1990), 731–756. doi:10.1080/00029890.1990.11995659.

[25] P. Samuel, *Lectures on Unique Factorization Domains*, Tata Institute of Fundamental Research, Bombay, 1964. https://mathweb.tifr.res.in/Documents/Publications/Lectures/tifr30.pdf.

[26] D. B. Shapiro, *Compositions of Quadratic Forms*, de Gruyter Expositions in Mathematics **33**, Walter de Gruyter, Berlin–New York, 2000.

[27] R. Wood, *Polynomial maps from spheres to spheres*, Invent. Math. **5** (1968), 163–168. doi:10.1007/BF01425547.