hexagon:2611.00544v1

The Osserman (1990) conjecture is fully resolved*

Dave Goel
david44hello@gmail.com

October 2026

Abstract

The Osserman conjecture says that a Riemannian manifold whose Jacobi operators have the same eigenvalues for all unit tangent vectors at all points is flat or locally rank-one symmetric. Chi and Nikolayevsky proved it in every dimension except one case in dimension sixteen: a Jacobi eigenvalue of multiplicity seven or eight. Nikolayevsky reduced this case to an algebraic statement, the Nikolayevsky (2009) conjecture: an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with a Jacobi eigenvalue of multiplicity seven or eight has a Clifford structure or is a linear combination of the constant-curvature tensor and an orthogonal conjugate of the curvature tensor of the Cayley projective plane. We prove this conjecture. Hence the Osserman conjecture is true in every dimension, and so is the conformal Osserman conjecture, which is stated for dimensions greater than four.

For two eigenvalues, a Cartan–Münzner quartic built from the Jacobi operators, together with the classification of isoparametric hypersurfaces with four principal curvatures, shows that the tensor is Clifford or comes from the Cayley projective plane. For three or more eigenvalues, polynomial eigenfields settle nineteen multiplicity patterns. In the remaining sixteen patterns, a sums-of-squares argument treats the case where the eigenvalue of multiplicity seven or eight is the smallest or the largest; otherwise we study the poles of a rational factorization of the Jacobi operators: a topological rigidity lemma for quadratic Gram matrices settles the case without poles, and degree bounds, real positivity and homogeneous factorization exclude each pole type or force a Clifford structure.

2020 Mathematics Subject Classification. Primary 53C25; Secondary 53B20, 53C35, 53C40.
Key words and phrases. Osserman conjecture, Jacobi operator, algebraic curvature tensor, Clifford structure, Cayley plane, rank-one symmetric space, isoparametric hypersurface.

1 Introduction and statements

The conjecture. Let R be the curvature tensor of a Riemannian manifold M. For a tangent vector x, the Jacobi operator $\mathcal{J}_x y = R(y, x)x$ is a symmetric operator on $x^\perp$. The manifold is *pointwise Osserman* if, at each point, the eigenvalues of $\mathcal{J}_x$ are the same for all unit vectors x, and *globally Osserman* if they are also the same at all points. Flat spaces and the rank-one symmetric spaces (spheres, the complex and quaternionic projective spaces, the Cayley projective plane, and their noncompact duals) are globally Osserman. Osserman asked in 1990 whether these are the only examples, locally [28]; this is the Osserman (1990) conjecture.

Chi proved the conjecture in dimensions not divisible by four and in dimension four [8]. Nikolayevsky proved it in all the remaining dimensions, and in its pointwise form for every $n \neq 2, 4$,

*Lean 4: `https://github.com/david44hello/papers_supporting_materials/tree/osserman-1990-conjecture-v4.00/osserman-1990-conjecture`

with a single exception: dimension sixteen, when a Jacobi eigenvalue has multiplicity seven or eight [22, 23], [24, Corollary 1.1]. He reduced this case, and the corresponding case of the conformal Osserman conjecture, to an algebraic statement, the Nikolayevsky (2009) conjecture [26, Theorem 1 and p. 679]. For surveys of Osserman manifolds see [13, 15]. We prove the Nikolayevsky (2009) conjecture (Theorem A); hence the Osserman conjecture is true in every dimension (Corollary B and Remark 1.1). Table 1 compares our results with earlier ones.

Our results.

- We prove the Nikolayevsky (2009) conjecture (Theorem A), to which Nikolayevsky reduced the last open case of the Osserman (1990) conjecture [26].
- Hence the Osserman (1990) conjecture is true in every dimension, and the conformal Osserman conjecture in every dimension greater than four (Corollary B and Remark 1.1); before this paper the only open case of each was dimension sixteen with a Jacobi eigenvalue of multiplicity seven or eight [8, 22–26].
- We classify the Osserman tensors on $\mathbb{R}^{16}$ with two Jacobi eigenvalues, of multiplicities 7 and 8 (Theorem C); Nikolayevsky announced the manifold form in 2006 [24, Theorem 1.2], and no proof appeared [1, Section 4].
- Every Jacobi-proportional Osserman tensor on $\mathbb{R}^{16}$ is Clifford or of Cayley type (Corollary D); Andrejić had reduced this question to the patterns $(8, 7)$ and $(7, 7, 1)$ [1, Theorem 5].

Table 1: Our results and what was known before. Clifford tensors are defined under Notation below, and tensors of Cayley type after Theorem A.

New result	Best result before this paper
The Nikolayevsky (2009) conjecture (Theorem A); hence the Osserman conjecture in dimension sixteen, pointwise and conformal (Corollary B).	Open. The Osserman conjecture was proved in all other dimensions, and in dimension sixteen when no eigenvalue has multiplicity seven or eight [8, 22–24], and its conformal form likewise for $n > 4$ [25, 26]; the remaining case was reduced to the Nikolayevsky (2009) conjecture [26].
Two eigenvalues of multiplicities 7 and 8: Clifford or of Cayley type (Theorem C).	Its manifold form was announced by Nikolayevsky [24, Theorem 1.2]; the proof did not appear [1, Section 4]. Osserman tensors with two eigenvalues were classified in dimensions twice an odd number [2].
Jacobi-proportional tensors (1.4) on $\mathbb{R}^{16}$ are Clifford or of Cayley type (Corollary D).	Reduced to the multiplicity patterns $(8, 7)$ and $(7, 7, 1)$ [1, Theorem 5].

Notation. Let V be a Euclidean vector space and R an algebraic curvature tensor on V. Its Jacobi operator is $\mathcal{J}_x y = R(y, x)x$. The tensor is *Osserman* if the spectrum of $\mathcal{J}_x$ on $x^\perp$ is independent of the unit vector x; all multiplicities below refer to this *reduced* operator on $x^\perp$, as in [24, p. 379], [26, proof of Lemma 4], and the *multiplicity pattern* of R is the list of these multiplicities, such as $(8, 7)$ or $(7, 7, 1)$. Set

$$R_1(u, v)w = \langle v, w\rangle u - \langle u, w\rangle v, \qquad \Pi(x) = \|x\|^2 I - xx^T,$$

so that $\mathcal{J}_x(R_1) = \Pi(x)$; here uv^T denotes the map $z \mapsto \langle v, z\rangle u$. Our sign convention for $\mathcal{J}$ is opposite to that of [23, 24, 26]; we note the sign changes where their formulas are used. For an orthogonal

map T we write $(T^*R)(u,v)w = T^{-1}R(Tu,Tv)Tw$. A *Clifford structure*, or $\mathrm{Cl}(\nu)$-structure, is a family of skew-symmetric endomorphisms $J_1,\dots,J_\nu$ with $J_iJ_j + J_jJ_i = -2\delta_{ij}I$, and R is *Clifford* if $\mathcal{J}_x = \lambda_0\Pi(x) + \sum_i \eta_i(J_ix)(J_ix)^T$ for constants λ_0, η_i. Such tensors were introduced by Gilkey [14] and by Gilkey, Swann and Vanhecke [16]; see also [23, Definition 3]. A *symmetric Clifford system* is a family $H_0,\dots,H_m$ of symmetric endomorphisms with $H_iH_j + H_jH_i = 2\delta_{ij}I$. Finally R_{Cay} is the curvature tensor of the Cayley projective plane, normalized to have sectional curvatures between 1 and 4.

Main results. The Nikolayevsky (2009) conjecture is Conjecture A of arXiv:0910.1645v1 (p. 2); the published version [26, p. 679] states it without a number.

Theorem A (The Nikolayevsky conjecture of 2009)**.** *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ one of whose reduced Jacobi eigenvalues has multiplicity 7 or 8. Then R is Clifford, or R has exactly two reduced eigenvalues and $R = aR_1 + bT^*R_{\mathrm{Cay}}$ for constants a, b and some $T \in O(16)$.*

We say that R is *of Cayley type* if $R = aR_1 + bT^*R_{\mathrm{Cay}}$ as in Theorem A.

Corollary B. *Let M be a connected smooth Riemannian manifold of dimension 16.*

1. *If M is pointwise Osserman, it is flat or locally isometric to a rank-one symmetric space.*
2. *If M is conformally Osserman (its Weyl tensor is Osserman at every point), it is locally conformally equivalent to a Euclidean space or to a rank-one symmetric space.*

Proof. By [26, Theorem 1 and the paragraph after it, p. 679], both statements hold once the curvature tensor, respectively the Weyl tensor, is Clifford or a combination of R_1 and an orthogonal conjugate of R_{Cay} at every point. (In arXiv:0910.1645v1 this is Theorem 1, "Assuming Conjecture A, ..."; the dimension-sixteen argument ends in its Section 3.4.) At points where some reduced eigenvalue has multiplicity 7 or 8, this is Theorem A. At the other points, the largest multiplicity of a reduced eigenvalue is at least 7, by Adams' theorem on vector fields on S^{15} [23, proof of Theorems 1 and 2, pp. 3–4 of arXiv:math/0204258v2]; hence it is at least 9, and the tensor is Clifford by [24, Proposition 2.1] if it is 9 and by [23, Proposition 1] if it is at least 10, as summarized in [26, pp. 678 and 680]. □

Remark 1.1. With [24, Corollary 1.1], part (1) completes the proof of the Osserman conjecture: every globally Osserman manifold, and every pointwise Osserman manifold of dimension $n \neq 2, 4$, is flat or locally rank-one symmetric. With [25, Theorem 1], part (2) proves the conformal Osserman conjecture in every dimension $n > 4$.

The proof starts with the case of two eigenvalues, which also fixes the normalization of R_{Cay}.

Theorem C. *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with exactly two reduced eigenvalues, of multiplicities 7 and 8. There are constants μ and $\kappa \neq 0$ such that $\mathcal{J}_x = \mu\Pi(x) + \kappa A(x)$, where, in suitable orthogonal coordinates, A has one of the forms*

$$A(x) = \sum_{i=1}^{7}(J_ix)(J_ix)^T, \qquad \text{for a } \mathrm{Cl}(7)\text{-structure}, \tag{1.1}$$

$$A(x) = \Pi(x) - \sum_{i=1}^{8}(J_ix)(J_ix)^T, \qquad \text{for a } \mathrm{Cl}(8)\text{-structure}, \tag{1.2}$$

$$A(x) = \frac{1}{2}\left(\|x\|^2 I + \sum_{i=0}^{8}\langle H_ix, x\rangle H_i\right) - xx^T, \quad \text{for a symmetric Clifford system } H_0,\dots,H_8. \tag{1.3}$$

The first two cases are Clifford. In the third, $R = (\mu - \kappa/3)R_1 + (\kappa/3)\,T^*R_{\mathrm{Cay}}$ *for some* $T \in O(16)$.

We have not found this classification in the literature: [26, Lemma 4] takes it from Conjecture A, and [2] treats dimensions twice an odd number only. Corollary B(1) gives the theorem announced by Nikolayevsky in 2006 [24, Theorem 1.2]: a pointwise Osserman 16-manifold with two Jacobi eigenvalues of multiplicities 7 and 8 is locally isometric, up to scaling, to the Cayley projective plane or its noncompact dual. Indeed, such a manifold is flat or locally isometric to a rank-one symmetric space, and among these only the Cayley projective plane and its noncompact dual have Jacobi eigenvalues of multiplicities 7 and 8: a flat space or a space of constant curvature has one Jacobi eigenvalue, and $\mathbb{C}\mathrm{P}^8$, $\mathbb{H}\mathrm{P}^4$ and their duals have multiplicities $1, 14$ and $3, 12$.

For an Osserman tensor, *Jacobi-proportionality* means

$$\|x\|^2\langle \pi_i(x)y, y\rangle = \|y\|^2\langle \pi_i(y)x, x\rangle \quad (x, y \neq 0, \text{ every } i), \tag{1.4}$$

where, if the reduced eigenvalues are $\varepsilon_1\|x\|^2, \dots, \varepsilon_p\|x\|^2$, $\pi_i(x)$ is the orthogonal projection onto the $\varepsilon_i\|x\|^2$-eigenspace of $\mathcal{J}_x$ on $x^\perp$, regarded as an operator on $\mathbb{R}^{16}$ that vanishes on $\mathbb{R}x$; in particular, if 0 is a reduced eigenvalue, its eigenspace does not contain x.

Corollary D. *Every Jacobi-proportional Riemannian Osserman algebraic curvature tensor on* $\mathbb{R}^{16}$ *is Clifford or a linear combination of* R_1 *and an orthogonal conjugate of* R_{Cay}.

This settles the multiplicity patterns $(8, 7)$ and $(7, 7, 1)$, to which Andrejić reduced the Jacobi-proportional case [1, Theorem 5], [3, Theorem 3]. It is proved in Section 7 from Theorem C and Theorem E below, and it is used in the final step of the proof of Theorem A.

Theorem E. *Let* R *be an Osserman algebraic curvature tensor on* $\mathbb{R}^{16}$ *with exactly three reduced eigenvalues. If their multiplicities are* $(7, 7, 1)$, $(8, 6, 1)$, $(8, 5, 2)$ *or* $(7, 6, 2)$, *then* R *is Clifford.*

Organization. Sections 2–6 prove Theorem C: the Jacobi family defines a Cartan–Münzner quartic on $\mathbb{R}^{32}$, and the classification of isoparametric hypersurfaces with four principal curvatures of multiplicities $(7, 8)$ [9, 11, 21] recovers the eigenspaces. Section 7 proves Theorem E and Corollary D with the polynomial eigenfield method of [24]. Section 8 proves Theorem F, which settles fifteen more multiplicity patterns. In the remaining sixteen patterns, Section 9 proves Theorem G, the case where the eigenvalue of multiplicity 7 or 8 is the smallest or the largest; otherwise we use a rational factorization of the Jacobi family, a coherent factor, whose poles are excluded, or shown to force a Clifford structure, case by case in Sections 10–26. Section 27 assembles the cases and proves Theorem A.

An expanded version of this paper, with many other results, is posted on Zenodo [17].

2 Normalization and the quartic

Let R be as in Theorem C. Write the reduced eigenvalues as $\mu\|x\|^2$ of multiplicity 8 and $(\mu+\kappa)\|x\|^2$ of multiplicity 7, and put $A(x) = (\mathcal{J}_x - \mu\Pi(x))/\kappa$. Then A is a homogeneous quadratic symmetric matrix family satisfying

$$A(x)x = 0, \qquad A(x)^2 = \|x\|^2 A(x), \qquad \operatorname{tr} A(x) = 7\|x\|^2. \tag{2.1}$$

The Jacobi family determines its curvature tensor. In fact, for any symmetric quadratic family A with $A(x)x = 0$, let $\mathcal{B}(u, v) = (A(u+v) - A(u) - A(v))/2$. Polarization gives $\mathcal{B}(x, y)x = -A(x)y/2$. Since $\mathcal{B}(x, y)$ is symmetric,

$$\langle A(x)y, y\rangle = -2\langle \mathcal{B}(x, y)x, y\rangle = -2\langle \mathcal{B}(y, x)y, x\rangle = \langle A(y)x, x\rangle.$$

Jacobi-family reconstruction [3, Theorem 1] therefore applies; specializing its defining formula [3, (12), p. 8468] to a quadratic family gives the curvature tensor

$$R_A(u,v)w = \frac{2}{3}(\mathcal{B}(v,w)u - \mathcal{B}(u,w)v) \tag{2.2}$$

with Jacobi family A. Conversely, if R is an algebraic curvature tensor with Jacobi family A, then $\mathcal{B}(v,w)u = \frac{1}{2}(R(u,v)w + R(u,w)v)$, and the first Bianchi identity turns (2.2) into $R_A = R$; so the Jacobi family determines the tensor.

Set

$$Q(x) = A(x) + xx^T, \qquad L(x) = \mathbb{R}x + \operatorname{Im} A(x).$$

For $x \neq 0$, $Q(x) = \|x\|^2 \pi_{L(x)}$ and $\dim L(x) = 8$. In particular,

$$Q(x)^2 = \|x\|^2 Q(x), \qquad Q(x)x = \|x\|^2 x, \qquad \operatorname{tr} Q(x) = 8\|x\|^2. \tag{2.3}$$

The symmetry $\langle A(x)y, y\rangle = \langle A(y)x, x\rangle$ shown above implies that the biquadratic form

$$b(x,y) = \langle Q(x)y, y\rangle = \langle A(x)y, y\rangle + \langle x, y\rangle^2 \tag{2.4}$$

is symmetric in x, y. It vanishes when $x = 0$ or $y = 0$ and satisfies $b(x,x) = \|x\|^4$.

On $\mathbb{R}^{32} = \mathbb{R}^{16} \oplus \mathbb{R}^{16}$ write $z = (x,y)$, $s = \|x\|^2 + \|y\|^2$, and define

$$F(z) = s^2 - 8b(x,y). \tag{2.5}$$

Lemma 2.1. *The polynomial F satisfies*

$$\|\nabla F\|^2 = 16s^3, \qquad \Delta F = 8s.$$

Its restriction to S^{31} takes both values 1 and -1.

Proof. Symmetry of b gives $\nabla_y b = 2Q(x)y$ and $\nabla_x b = 2Q(y)x$. By (2.3),

$$\|\nabla b\|^2 = 4sb, \qquad \langle z, \nabla b\rangle = 4b, \qquad \Delta b = 2\operatorname{tr} Q(x) + 2\operatorname{tr} Q(y) = 16s.$$

Thus

$$\|\nabla F\|^2 = 16s^3 - 64s\langle z, \nabla b\rangle + 64\|\nabla b\|^2 = 16s^3.$$

Since $\Delta s^2 = 4(32+2)s = 136s$, we obtain $\Delta F = 8s$. For unit x, $F(x,0) = 1$. If x, y are unit and $y \in L(x)$, then $F((x,y)/\sqrt{2}) = 1 - 8/4 = -1$. □

3 The classification of isoparametric hypersurfaces

We record the external results that are used.

Münzner. For a homogeneous polynomial of degree g satisfying $\|\nabla F\|^2 = g^2\|z\|^{2g-2}$ and $\Delta F = c\|z\|^{g-2}$ on $\mathbb{R}^{n+1}$, nonconstant on the sphere, [21, Satz 3, p. 65] shows that 0 is a regular value of $F|S^n$ and that the zero level is an isoparametric hypersurface; by its proof (p. 66) the same holds for every level in $(-1,1)$. Its Folgerung (corollary) on p. 66 identifies F with the Cartan polynomial of these hypersurfaces when both $g(n-1) - c$ and $g(n-1) + c$ are nonzero. Satz 4(a), p. 67, gives connected level sets. By Lemma 2.1 these hypotheses hold with $g = 4$, $n = 31$ and $c = 8$, whereas $g(n-1) = 120$. The resulting hypersurfaces have four principal curvatures, and Münzner's relations $g(m_1 + m_2) = 2(n-1)$ and $c = g^2(m_2 - m_1)/2$ [21, Satz 2, p. 63] give

$$m_1 + m_2 = 15, \qquad 8(m_2 - m_1) = 8,$$

so their multiplicities are $(7, 8)$.

Chi and Ferus–Karcher–Münzner. Chi's classification [9, abstract, p. 225] gives exactly three families with unordered multiplicities $\{7,8\}$. They are the FKM family with parameter $m=7$ and the two FKM families with $m=8$. The first is the Ozeki–Takeuchi family, as identified in [11, Section 5.9, p. 19] (page numbers for FKM refer to its English translation, arXiv:1112.2780); the two $(8,7)$ families are distinguished in its Section 6.6, p. 22. For a symmetric Clifford system $P_0,\dots,P_m$ on $\mathbb{R}^{32}$ with $m\geq 1$, [11, Theorem 4.1, p. 8; Section 4.3, p. 11] gives

$$F_P(z)=\|z\|^4-2\sum_{i=0}^{m}\langle P_i z,z\rangle^2,\qquad (m_1,m_2)=(m,15-m). \tag{3.1}$$

By the same Folgerung of Münzner, with $c=\pm 8\neq\pm 120$, each F_P used here is the Cartan polynomial of its zero level.

We need congruence of the polynomials, not only of a level hypersurface. The zero level of F has principal curvatures $\cot(\pi/8+j\pi/4)$, $0\leq j\leq 3$. A congruent member of the FKM family must therefore be its zero member, with one of the two choices of normal. Let $U\in O(32)$ map the zero level of F onto the zero level of F_P. Both zero levels are connected [21, Satz 4(a), p. 67]. By [21, Folgerung (iii), p. 66], F and $F_P\circ U$ are both Cartan polynomials of this hypersurface. The Cartan polynomial of a connected oriented isoparametric hypersurface is determined by it, and reversing the orientation replaces it by its negative [21, §4, p. 65]; explicitly $F(rq)=r^4\cos 4t(q)$, where $t(q)\in[0,\pi/4]$ is the spherical distance from $q\in S^{31}$ to the focal submanifold $\{F=1\}\cap S^{31}$ [21, proof of Satz 3, p. 66; Satz 4(c), p. 67]. Hence $F=\varepsilon F_P\circ U$ with $\varepsilon\in\{1,-1\}$.

Absorb U into the matrices P_i. Since $\Delta F_P=(136-16(m+1))s$, comparison with $\Delta F=8s$ gives precisely

$$4b(x,y)=\sum_{i=0}^{7}\langle P_i z,z\rangle^2, \tag{3.2}$$

$$\text{or}\qquad 4b(x,y)=s^2-\sum_{i=0}^{8}\langle P_i z,z\rangle^2. \tag{3.3}$$

These identities hold in the original coordinates (x,y); no preservation of the two coordinate summands by U has been assumed.

4 A lemma on quadratic maps between spheres

Quadratic maps between spheres have a classical literature [30, 31]. We give the short argument needed here. Part (b) below is the quadratic case of [31, p. 163]; we include its short proof.

Lemma 4.1. *Let $H_0,\dots,H_m$ be symmetric $n\times n$ matrices and $h(x)=(\langle H_i x,x\rangle)_{i=0}^m$. Suppose $\|h(x)\|=\|x\|^2$. For $a\in\mathbb{R}^{m+1}$ put $H(a)=\sum_i a_i H_i$.*

(a) *If $a\in h(S^{n-1})$, then $I-H(a)\geq 0$ and the fibre over a is the unit sphere of $U_a=\ker(I-H(a))$, where $\dim U_a\geq n-m$.*

(b) *If $n>2m$, then h is constant on S^{n-1}.*

(c) *If $n=2m$, $m\geq 1$, and h is not constant on S^{n-1}, then $H_iH_j+H_jH_i=2\delta_{ij}I$.*

Proof. For unit a, x, $\langle H(a)x, x\rangle = \langle a, h(x)\rangle \le 1$, with equality exactly when $h(x) = a$. This proves the fibre description. Fix a unit x in that fibre and rotate the index space so that $a = e_0$. Then $H_0 x = x$. Comparing the t^2 coefficients of $\|h(x+ty)\|^2 = \|x+ty\|^4$ gives

$$2\langle H_0 y, y\rangle + 4\sum_{i=0}^{m}\langle H_i x, y\rangle^2 = 2\|y\|^2 + 4\langle x, y\rangle^2.$$

The $i = 0$ terms cancel, so

$$I - H_0 = 2\sum_{i=1}^{m}(H_i x)(H_i x)^T. \tag{4.1}$$

Its rank is at most m, proving (a).

The subspaces U_a, U_b for distinct values $a \ne b$ meet only at zero. If $n > 2m$, two such subspaces would have dimensions summing to more than n. This proves (b).

Suppose $n = 2m$ and there is more than one value. For every a, comparison with another fibre gives $\dim U_a = m$. Hence the right side of (4.1) has rank m, and $H_1 x, \dots, H_m x$ are independent and perpendicular to x. On $T_x S^{n-1} = x^\perp$,

$$(dh)_x(y) = 2(\langle H_i x, y\rangle)_{i=0}^{m}$$

has rank m. The image of h is open in S^m and is compact; since $m \ge 1$, it is all of S^m. In particular a and $-a$ are values. Their fibres are the $+1$ and -1 eigenspaces of $H(a)$ and each has dimension m. Thus $H(a)^2 = I$ for every unit a. Polarization in a proves (c). □

5 Recovering the original eigenspaces

Proposition 5.1. *Identity* (3.2) *implies* (1.1).

Proof. Setting $y = 0$ in (3.2) shows that the upper-left diagonal block of each P_i is zero; setting $x = 0$ gives the same for the lower-right block. Hence

$$P_i = \begin{pmatrix} 0 & C_i^T \\ C_i & 0 \end{pmatrix}, \qquad Q(x) = \sum_{i=0}^{7}(C_i x)(C_i x)^T.$$

The Clifford relations give

$$C_i^T C_j + C_j^T C_i = 2\delta_{ij} I, \qquad C_i C_j^T + C_j C_i^T = 2\delta_{ij} I.$$

Moreover $Q(x)x = \|x\|^2 x$ implies $\sum_i \langle C_i x, x\rangle^2 = \|x\|^4$. Apply Lemma 4.1(b) to the symmetric parts of C_i, with $n = 16$ and $m = 7$. There is a unit $w \in \mathbb{R}^8$ such that $(\langle C_i x, x\rangle)_i = \|x\|^2 w$. Rotating the index space gives $w = e_0$. Since C_0 is orthogonal and $\langle C_0 x, x\rangle = \|x\|^2$, we have $C_0 = I$. The remaining C_i are skew, and their relations become $C_i C_j + C_j C_i = -2\delta_{ij} I$. Subtracting xx^T from $Q(x)$ proves the claim. □

Proposition 5.2. *Identity* (3.3) *implies* (1.2) *or* (1.3).

Proof. In the original splitting write $P_i = \left(\begin{smallmatrix} H_i & B_i^T \\ B_i & D_i \end{smallmatrix}\right)$. Setting $y = 0$ and then $x = 0$ in (3.3) gives

$$\sum_{i=0}^{8}\langle H_i x, x\rangle^2 = \|x\|^4, \qquad \sum_{i=0}^{8}\langle D_i y, y\rangle^2 = \|y\|^4.$$

Write $h(x) = (\langle H_i x, x\rangle)_i$ and $d(y) = (\langle D_i y, y\rangle)_i$.

If h is constant on the unit sphere, rotate the index so that $H_0 = I$, $H_i = 0$ for $i \geq 1$. The upper block of $P_0^2 = I$ gives $B_0 = 0$. Also $\operatorname{tr} P_0 = 0$, because $P_0 = -P_1 P_0 P_1$. Thus $D_0^2 = I$ and $\operatorname{tr} D_0 = -16$, forcing $D_0 = -I$. Anticommutation with $P_0 = \operatorname{diag}(I, -I)$ makes the other eight matrices off-diagonal. Calling their lower blocks $C_1, \ldots, C_8$ and expanding (3.3) yields

$$Q(x) = \|x\|^2 I - \sum_{i=1}^{8} (C_i x)(C_i x)^T.$$

The identity $Q(x)x = \|x\|^2 x$ gives $\sum_i \langle C_i x, x\rangle^2 = 0$. Each C_i is therefore skew, and the Clifford relations become those of a Cl(8)-structure. This is (1.2).

If h is not constant, Lemma 4.1(c) makes the H_i a symmetric Clifford system. The equation $H_i^2 + B_i^T B_i = I$ forces $B_i = 0$. Expanding (3.3) now gives

$$b(x, y) = \frac{1}{2}\left(\|x\|^2\|y\|^2 - \langle h(x), d(y)\rangle\right).$$

Since $b(x, x) = \|x\|^4$ and $\|h(x)\| = \|d(x)\| = \|x\|^2$, equality in Cauchy–Schwarz forces $d(x) = -h(x)$. Polarization gives $D_i = -H_i$. Consequently

$$Q(x) = \frac{1}{2}\left(\|x\|^2 I + \sum_{i=0}^{8} \langle H_i x, x\rangle H_i\right),$$

which proves (1.3). □

6 Identification of the Cayley tensor

We first prove two identities for an arbitrary symmetric Clifford system $S_0, \ldots, S_8$ on $\mathbb{R}^{16}$: (6.1) below and the norm identity $\sum_{i=0}^{8} \langle S_i x, x\rangle^2 = \|x\|^4$. For $q = \|x\|^2$ put $h_i = \langle S_i x, x\rangle$ and $S_h = \sum_i h_i S_i$. The products of zero, one, and four distinct generators are symmetric (a product of k distinct generators is $(-1)^{k(k-1)/2}$ times its transpose) and trace-orthogonal (the product of two distinct such monomials is, up to sign, a product of j distinct generators with $1 \leq j \leq 8$, which anticommutes with a generator outside it if j is odd and with one inside it if j is even, so has trace zero), and their number is $1 + 9 + 126 = 136$. They therefore form a basis of the symmetric matrices. If $\mathcal{L}_k$ is their span for degree k, the operator $\mathcal{T}(Y) = \sum_i S_i Y S_i$ acts on $\mathcal{L}_k$ by $(-1)^k(9 - 2k)$, namely by $9, -7, 1$ for $k = 0, 1, 4$. These assertions also appear in [26, Lemma 3, pp. 6–7 of arXiv:0910.1645v1]. Expanding $xx^T = qI/16 + S_h/16 + Y_4$ therefore gives

$$Z_x := \sum_i (S_i x)(S_i x)^T = xx^T + \frac{1}{2}(qI - S_h). \tag{6.1}$$

Contraction with x yields $\|h\|^2 = q^2 + (q^2 - \|h\|^2)/2$, so $\|h\| = q$.

For this system define $Q(x) = (qI + S_h)/2$, so that $A(x) = Q(x) - xx^T$ is the matrix in (1.3) with $H_i = S_i$. For $x \neq 0$ set $w = h/q$. Since $S_h = qS_w$, we have $Q(x) = q(I + S_w)/2$. Also $S_w x = x$, since S_w is an orthogonal involution and $\langle S_w x, x\rangle = q$. Its two eigenspaces have dimension eight. For $u \perp w$, $S_u x$ lies in its negative eigenspace. With (6.1) and $S_h = qS_w$,

$$Q(x) = \frac{q}{2}(I + S_w), \qquad Z_x = xx^T + qI - Q(x), \qquad S_h = 2Q(x) - qI.$$

In octonionic coordinates $x = (x_1, x_2)$ the standard system is

$$S_0(x_1, x_2) = (x_1, -x_2), \qquad S_i(x_1, x_2) = (e_i\overline{x_2},\, \overline{x_1}e_i), \quad 1 \le i \le 8,$$

where $(e_i)_{i=1}^8$ is an orthonormal basis of $\mathbb{O}$. With $(u \wedge v)z = \langle u, z\rangle v - \langle v, z\rangle u$, the Brown–Gray Cayley curvature formula [6, Eq. (6.12)], in the form [26, equation (7), p. 6 of arXiv:0910.1645v1], is

$$R_{\mathbb{O}}(u, v) = 3u \wedge v + \sum_{i=0}^{8} S_i u \wedge S_i v.$$

In our Jacobi convention this gives

$$\begin{aligned} \mathcal{J}_x(R_{\mathbb{O}}) &= 3(xx^T - qI) + Z_x - S_h \\ &= 4xx^T - qI - 3Q(x) = -\Pi(x) - 3A(x). \end{aligned}$$

Therefore $R_{\mathrm{Cay}} = -R_{\mathbb{O}}$ has Jacobi family $\Pi(x) + 3A(x)$, with reduced eigenvalues 1 and 4 of multiplicities eight and seven.

Let $\mathrm{Cl}_+(9)$ denote the real Clifford algebra on nine generators with squares $+1$ and pairwise anticommutation. Then $\mathrm{Cl}_+(9) \cong M_{16}(\mathbb{R}) \oplus M_{16}(\mathbb{R})$. Its representations on $\mathbb{R}^{16}$ are equivalent up to orthogonal conjugacy and the sign of one generator; see [11, Sections 3.4–3.5, pp. 6–7] and [26, p. 6 of arXiv:0910.1645v1]. Changing a generator's sign leaves $\langle S_i x, x\rangle S_i$ unchanged. This identifies every family (1.3) with the standard Cayley family, so $3A(x) = \mathcal{J}_x(T^* R_{\mathrm{Cay}}) - \Pi(x)$ for some $T \in O(16)$ and $\mathcal{J}_x = (\mu - \kappa/3)\Pi(x) + (\kappa/3)\mathcal{J}_x(T^* R_{\mathrm{Cay}})$. As the Jacobi family determines the curvature tensor by (2.2), $R = (\mu - \kappa/3)R_1 + (\kappa/3)\, T^* R_{\mathrm{Cay}}$. This finishes the proof of Theorem C: by Lemma 2.1 and Section 3, b satisfies (3.2) or (3.3); Propositions 5.1 and 5.2 give the forms (1.1)–(1.3), of which the first two are Clifford and the third has just been identified.

7 Tensors with three eigenvalues, and Jacobi-proportionality

For the standard Cayley family of Section 6, that is (1.3) with $H_i = S_i$, let $E(x) = L(x) \cap x^\perp$ and $N(x) = L(x)^\perp$, of dimensions seven and eight; here $L(x) = \mathbb{R}x + \mathrm{Im}\, A(x)$ as in Section 2.

Lemma 7.1. *There is no nonzero linear map K with $Kx \in E(x)$ for every $x \neq 0$. There is also no nonzero linear map K with $Kx \in N(x)$ for every $x \neq 0$.*

Proof. After the orthogonal coordinate change $(x_1, x_2) \mapsto (x_1, \overline{x_2})$, the Cayley lines are

$$\mathcal{C}_\omega = \{(u, \omega u) : u \in \mathbb{O}\}, \quad \omega \in \mathbb{O}, \qquad \mathcal{C}_\infty = \{(0, v) : v \in \mathbb{O}\}.$$

Here $S_{(w_0,a)} = w_0 S_0 + \sum_{i=1}^8 a_i S_i$ for $w_0^2 + \|a\|^2 = 1$ and $a = \sum_i a_i e_i$. Its positive eigenspace becomes $\mathcal{C}_\omega$ with $\omega = \overline{a}/(1 + w_0)$ when $w_0 \neq -1$, and becomes $\mathcal{C}_\infty$ otherwise. Distinct lines among the $\mathcal{C}_\omega$ and $\mathcal{C}_\infty$ meet only in 0, as $\mathbb{O}$ has no zero divisors, and $x \in L(x)$; so for x in such a line, $L(x)$ is that line.

Suppose first $Kx \in E(x)$. Since $\langle Kx, x\rangle = 0$, K is skew. Preservation of $\mathcal{C}_0, \mathcal{C}_\infty, \mathcal{C}_1$ forces $K = \mathrm{diag}(D, D)$. Preservation of $\mathcal{C}_\omega$ gives $D(\omega u) = \omega D(u)$. Put $c = D(1)$; then $D(u) = uc$ and $(\omega u)c = \omega(uc)$ for all ω, u. Thus c lies in the right nucleus of $\mathbb{O}$, which is $\mathbb{R}$. Hence $K = cI$, and skewness forces $c = 0$.

Suppose next $Kx \in N(x)$. The lines $\mathcal{C}_0$ and $\mathcal{C}_\infty$ force $K = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$. The graph condition for $\mathcal{C}_\omega^\perp$ is $B(\omega u) = -\overline{\omega} C(u)$. Taking $\omega = 1$ gives $B = -C$; taking $u = 1$ then gives $C(\omega) = \overline{\omega}\gamma$, where $\gamma = C(1)$. Substitution yields

$$(ab)\gamma = b(a\gamma) \qquad (a, b \in \mathbb{O}).$$

Choose a quaternion subalgebra containing γ and choose noncommuting a, b in it. Associativity in that subalgebra gives $(ab - ba)\gamma = 0$. Multiplicativity of the octonion norm forces $\gamma = 0$, and therefore $K = 0$. □

Proof of Theorem E. Let $t \in \{1, 2\}$ be the smallest multiplicity, with eigenvalue τ. To apply [24, Lemma 3.1], shift a different reduced eigenvalue to zero by subtracting a multiple of R_1. In the convention of [24, p. 379], the shifted tensor has $k = 3$ reduced eigenvalues, one of them zero. Nikolayevsky's [24, Lemma 3.1(1),(2), p. 383] gives an orthonormal frame for the τ-eigenspace by odd homogeneous polynomial fields of degree at most $k - 1 = 2$. They are therefore linear: $J_1 x, \dots, J_t x$. Their norm and orthogonality identities make each J_i skew and orthogonal and make the J_i anticommute. Return to the original tensor, whose eigenspaces are unchanged by the shift.

Choose another reduced eigenvalue τ' so that moving τ to τ' merges the multiplicities as

$$(7, 7, 1) \to (7, 8), \quad (8, 6, 1) \to (8, 7), \quad (8, 5, 2) \to (8, 7), \quad (7, 6, 2) \to (7, 8).$$

For a skew orthogonal J, let R_J denote the curvature tensor whose Jacobi family is $(Jx)(Jx)^T$, obtained for example from (2.2). Then

$$\widetilde{R} = R - (\tau - \tau') \sum_{i=1}^{t} R_{J_i}$$

is Osserman with two distinct eigenvalues of multiplicities seven and eight. Apply Theorem C. Its Cayley alternative is impossible: each $J_i x$ remains an eigenfield in one of the two merged distributions, contradicting Lemma 7.1 after conjugating the field to the standard model: $K = T J_i T^{-1}$. Thus $\widetilde{R}$ is Clifford.

Restore the rank-one terms successively. By [24, Lemma 3.2(1), pp. 383–384], if R is Osserman, J is orthogonal and skew, $c \neq 0$, and $R - cR_J$ is Osserman and Clifford, then R is Clifford. Each intermediate tensor here is Osserman: only the constant eigenvalues on the mutually orthogonal eigenlines $\mathbb{R} J_i x$ change. The lemma consequently applies at every step, in the form of Remark 7.2 whenever $R - cR_J$ has a Clifford structure with $\nu = 8$ (an endpoint of type (1.2), or the intermediate pattern $(7, 7, 1)$ reached from $(7, 6, 2)$). The sign convention of [24] is opposite to ours, which changes the sign of c; only $c \neq 0$ is used. This proves the theorem. □

Remark 7.2. We recall the proof of [24, Lemma 3.2(1), pp. 383–384] in the form used here. The proof in [24] shifts the eigenvalue of largest multiplicity of $R - cR_J$ to zero; the version below shifts instead the base constant λ_0 of a Clifford structure of $R - cR_J$. The two choices differ only when $\nu = 8$ and another eigenvalue also has multiplicity at least seven, that is, at an endpoint of type (1.2) or at the pattern $(7, 7, 1)$. Indeed, with the generators of zero coefficient dropped, λ_0 has multiplicity $15 - \nu$; for $\nu \leq 7$ it is the unique eigenvalue of largest multiplicity, for $\nu = 8$ it has multiplicity seven, and at $(7, 7, 1)$ every Clifford structure has $\nu = 8$. Let $\mathcal{J}_x(R - cR_J) = \lambda_0 \Pi(x) + \sum_{\alpha=1}^{\nu} \mu_\alpha (J_\alpha x)(J_\alpha x)^T$ with $\mu_\alpha \neq 0$; subtracting $\lambda_0 R_1$ from R we may take $\lambda_0 = 0$. Let $\lambda_1, \dots, \lambda_p$ be the distinct values of the μ_α, and $\lambda_0 = 0$. For unit x the eigenspaces of $\mathcal{J}_x(R - cR_J)$ on $x^\perp$ are $E_i(x) = \operatorname{span}\{J_\alpha x : \mu_\alpha = \lambda_i\}$ for $i \geq 1$ and $E_0(x) = x^\perp \cap \left(\sum_\alpha \mathbb{R} J_\alpha x\right)^\perp$; let π_i be the orthogonal projections onto them. As $\mathcal{J}_x(R) = \mathcal{J}_x(R - cR_J) + c(Jx)(Jx)^T$ and $Jx \perp x$,

$$\frac{\det(tI - \mathcal{J}_x(R))}{\det(tI - \mathcal{J}_x(R - cR_J))} = 1 - c \sum_{i=0}^{p} \frac{\|\pi_i(Jx)\|^2}{t - \lambda_i}$$

on $x^\perp$, so, both tensors being Osserman, partial fractions make each $\|\pi_i(Jx)\| = c_i$ constant on the unit sphere. For $i \geq 1$, $\sum_{\mu_\alpha = \lambda_i} \langle J_\alpha x, Jx \rangle^2 = c_i^2 \|x\|^4$ is a sum of at most eight squares divisible

by $\|x\|^2$, so by [24, Lemma 3.11] each $\langle J_\alpha x, Jx\rangle$ (a quadratic form) is a constant multiple of $\|x\|^2$; after an orthogonal change of the J_α with $\mu_\alpha = \lambda_i$ we get $\pi_i(Jx) = c_i J^{(i)}x$ for one of them, $J^{(i)}$. Then $J_{\nu+1}x := \pi_0(Jx) = Jx - \sum_{i\geq 1} c_i J^{(i)}x$ is linear and skew, anticommutes with every J_α (as $J_{\nu+1}x \perp J_\alpha x$), and $\|J_{\nu+1}x\|^2 = c_0^2\|x\|^2$ with $c_0^2 = 1 - \sum_{i\geq 1} c_i^2$. If $\nu = 8$, then $c_0 = 0$: otherwise $c_0^{-1}J_9$ would be a ninth orthogonal complex structure on $\mathbb{R}^{16}$ anticommuting with $J_1, \dots, J_8$, and $\mathbb{R}^{16}$ carries at most eight (Hurwitz–Radon). Finally, write $Jx = \sum_{i\geq 1} c_i J^{(i)}x + c_0(c_0^{-1}J_{\nu+1}x)$, omitting the last term if $c_0 = 0$. An orthogonal change of $J^{(1)}, \dots, J^{(p)}$ (and of $c_0^{-1}J_{\nu+1}$ if $c_0 \neq 0$) that diagonalizes $\operatorname{diag}(\lambda_1, \dots, \lambda_p, 0) + c\,\gamma\gamma^T$, $\gamma = (c_1, \dots, c_p, c_0)^T$ (without the last row and column if $c_0 = 0$), gives a Clifford structure for R.

Proof of Corollary D. Andrejić [1, Theorem 5], restated as [3, Theorem 3, p. 8470], proves that a Riemannian Jacobi-proportional Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ that is not Clifford has multiplicity pattern $(8, 7)$ or $(7, 7, 1)$. Andrejić's hypothesis places no restriction on the number of eigenvalues. Apply Theorems C and E. □

8 Further multiplicity reductions

Theorems C and E complete several polynomial-eigenfield reductions of [24].

Theorem F. *An Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ is Clifford if its reduced multiplicity pattern is one of*

$$(7, 6, 1, 1),\ (8, 5, 1, 1),\ (7, 5, 2, 1),\ (8, 4, 2, 1),\ (7, 5, 1, 1, 1),\ (8, 4, 1, 1, 1),$$

or if one eigenvalue has multiplicity seven or eight and every other eigenvalue has multiplicity at most two (nine further patterns, fifteen in all).

8.1 Constant projections onto Cayley eigenspaces

Lemma 8.1. *Let $E(x), N(x)$ be the two Cayley distributions of Lemma 7.1. If a skew-symmetric linear map K satisfies $\|\pi_{N(x)}Kx\|^2 = c\|x\|^2$ for all $x \neq 0$ and a constant c, then $K = 0$. In particular an orthogonal skew map cannot have constant normalized projection lengths onto the two Cayley eigenspaces.*

Proof. Use the symmetric Clifford system of Section 6, written here $H_0, \dots, H_8$ (there $S_0, \dots, S_8$), and put $q = \|x\|^2$, $h_i(x) = \langle H_i x, x\rangle$, $a_i(x) = \langle H_i x, Kx\rangle$. The Clifford relations give $\langle H_i x, H_j x\rangle = \delta_{ij}q$. The identity $\sum_i (H_i x)(H_i x)^T = Z_x = xx^T + qI - Q(x)$ of Section 6 (with $H_i = S_i$) and $Q(x) = q\,\pi_{L(x)}$ (Section 2), with $x \in L(x)$, give $\sum_i (H_i x)(H_i x)^T = xx^T + q\,\pi_{N(x)} = q\,\pi_{\mathbb{R}x+N(x)}$. So the nine orthogonal vectors $H_i x$, of norm $\sqrt{q}$, span $\mathbb{R}x + N(x)$, and, K being skew, $\sum_i a_i^2 = cq^2$ and $\sum_i h_i^2 = q^2$; differentiating the latter along Kx gives $\sum_i h_i a_i = 0$. The matrix of a_i is the symmetric trace-free $A_i = (H_i K - K H_i)/2$.

If $c > 0$, put $H_i(t) = \cos t\, H_i + c^{-1/2} \sin t\, A_i$. Then $\sum_i \langle H_i(t)x, x\rangle^2 = q^2$ for every t, and this quadratic map is not constant on the unit sphere (otherwise $H_i(t) = b_i I$ with b a unit vector, but all its matrices are trace-free). By Lemma 4.1(c) the $H_i(t)$ form a symmetric Clifford system, so $\Omega(t) = H_0(t) \cdots H_8(t)$ is a symmetric involution, and $\operatorname{tr}\Omega(t)$ is continuous and integer-valued, hence constant. But $\Omega(0) = \pm I$ on $\mathbb{R}^{16}$ (as nine is odd, $\Omega(0)$ commutes with every H_i, and, by Section 6, the products of the H_i span the symmetric matrices, so a matrix commuting with every H_i is scalar) and $\Omega(\pi) = -\Omega(0)$, a contradiction. If $c = 0$, then $Kx \in L(x)$, and skewness gives $Kx \in E(x)$, so $K = 0$ by Lemma 7.1. □

8.2 Polynomial tools and a three-column reduction

From here to the end of this section, $A(x)$ denotes the Jacobi operator $\mathcal{J}_x$ after one reduced eigenvalue has been shifted to zero (not the normalized family of Section 2), $q = \|x\|^2$, $\mathcal{O} = \mathbb{R}[x_1, \dots, x_{16}]/(q)$, and $\pi\colon \mathbb{R}[x_1, \dots, x_{16}] \to \mathcal{O}$ is the projection. As q is irreducible, it is a prime element of $\mathbb{R}[x_1, \dots, x_{16}]$. We use two results of [24]: $\mathcal{O}$ is a UFD [24, Theorem 3.1, p. 397]; and if q divides a sum of at most eight real polynomial squares, it divides every polynomial being squared [24, Lemma 3.11, p. 398]. A polynomial eigenvector field of a simple eigenvalue is a *simple eigenfield*. The simple eigenfields of [24, Lemma 3.1(1), p. 383] satisfy

$$\langle P(x), x\rangle = 0, \quad \|P(x)\|^2 = q^3, \quad P(P(x)) = -q^4 x, \tag{8.1}$$

after multiplication by q if they are linear: equation (7) of that lemma states them for unit x, and homogeneity gives them for all x. We also use [24, Lemma 3.5, pp. 388–390]. It is stated there under the standing hypothesis of [24, Subsection 3.1], a Jacobi eigenvalue of multiplicity nine, and for an eigenvector field P; its proof uses neither. We state it in the form used here, with that proof.

Lemma 8.2. *Let $P\colon \mathbb{R}^{16} \to \mathbb{R}^{16}$ have homogeneous cubic components that satisfy* (8.1)*. If the images in $\mathcal{O}$ of the sixteen components of P have a nonunit common divisor, then $P(x) = qJx$ for an orthogonal skew map J.*

Proof. We follow [24, pp. 389–390]. If every component of P is divisible by q, then $P(x) = qJx$ with J linear, and (8.1) gives $\langle Jx, x\rangle = 0$ and $\|Jx\|^2 = q$, so J is skew and orthogonal. Otherwise the common divisor is homogeneous, as a divisor of a nonzero homogeneous element of the graded domain $\mathcal{O}$; its degree r is positive, since the units of $\mathcal{O}$ are the nonzero constants, and $r \le 3$, since it divides a nonzero cubic. Lifting to $\mathbb{R}[x_1, \dots, x_{16}]$, we get $P(x) = \varphi(x)\Psi(x) + qCx$ with φ homogeneous of degree r and not divisible by q, the components of Ψ homogeneous of degree $3 - r$, and C linear. It suffices to show that every component of Ψ is divisible by q.

If $r = 3$, then Ψ is constant and $\varphi\langle\Psi, x\rangle = -q\langle Cx, x\rangle$. As q is prime and does not divide φ, it divides the linear form $\langle\Psi, x\rangle$, so $\Psi = 0$.

If $r = 2$, then $\Psi(x) = Bx$ with B linear. Modulo q, $\|P(x)\|^2 \equiv \varphi(x)^2\|Bx\|^2$, so q divides $\|Bx\|^2$, and $B = cU$ with U orthogonal and $c \ge 0$. Since $\|P(x)\|^2 = q^3$ and $P(x) \equiv \varphi(x)Bx$ modulo q, we get

$$0 \equiv P(P(x)) \equiv \varphi(P(x))\, BP(x) \equiv \varphi(x)^3\varphi(Bx)\, B^2x \pmod q.$$

If $c \ne 0$, then B^2 is invertible, so q divides $\varphi(Bx)$, and the substitution $x = B^{-1}y$ shows that $c^{-2}\|y\|^2$ divides $\varphi(y)$, which is false. So $c = 0$ and $\Psi = 0$.

If $r = 1$, then φ is a nonzero linear form, and $\varphi\langle\Psi, x\rangle = -q\langle Cx, x\rangle$ shows that φ divides $\langle Cx, x\rangle$, say $\langle Cx, x\rangle = \varphi(x)\langle a, x\rangle$ with $a \in \mathbb{R}^{16}$. Replacing C by $\tilde{C}x = Cx - \varphi(x)a$ and Ψ by $\tilde{\Psi} = \Psi + qa$ keeps $P = \varphi\tilde{\Psi} + q\tilde{C}x$ and makes $\tilde{C}$ skew. Modulo φ, the identity $\|P(x)\|^2 = q^3$ shows that φ divides $q - \|\tilde{C}x\|^2 = x^T(I + \tilde{C}^2)x$. A quadratic form divisible by a linear form has rank at most two and vanishes at a nonzero vector of every plane, so it is definite on no plane; as every eigenvalue of the symmetric matrix $I + \tilde{C}^2$ has even multiplicity, $I + \tilde{C}^2 = 0$, and $\tilde{C}$ is orthogonal. The third identity in (8.1) now reads $\varphi(P(x))\tilde{\Psi}(P(x)) = -q^3\varphi(x)\tilde{C}\tilde{\Psi}(x)$. On the hyperplane $\varphi = 0$ we have $P(x) = q\tilde{C}x$, so $\varphi(\tilde{C}x)\tilde{\Psi}(\tilde{C}x)$ vanishes there. Writing $\varphi(x) = \langle f, x\rangle$, the form $\varphi(\tilde{C}x) = -\langle\tilde{C}f, x\rangle$ does not vanish identically on $f^\perp$, since $\tilde{C}f \ne 0$ is orthogonal to f. Hence φ divides every component of $\tilde{\Psi}(\tilde{C}x)$, and, as $\tilde{C}^{-1} = -\tilde{C}$, we get $\tilde{\Psi}(x) = \varphi(\tilde{C}x)B'x$ with B' linear. Then $P(x) = \varphi(x)\varphi(\tilde{C}x)\, B'x + q\tilde{C}x$, where $\varphi(x)\varphi(\tilde{C}x)$ is a quadratic form not divisible by q, and the case $r = 2$ applies: $B' = 0$, so $\tilde{\Psi} = 0$, and $\Psi = -qa$ is divisible by q. □

The three-column step of [24, pp. 387–389] reads as follows. Let P_1, P_2, P_3 be mutually orthogonal cubic eigenfields of squared norm q^3 satisfying $\sum_{i=1}^3 w_i P_i P_i^T \equiv 0$ modulo q, with $w_1, w_2 > 0 > w_3$ (identities of this kind come from the spectral projectors, as in (8.4) below). After rescaling the columns by nonzero constants, their images $p_i \in \mathcal{O}^{16}$ satisfy

$$p_1 p_1^T + p_2 p_2^T = p_3 p_3^T. \tag{8.2}$$

The right side of (8.2) has rank at most one, so every 2×2 minor of the left side vanishes; by Lagrange's identity its principal minor in rows and columns a, b is $(p_{1a}p_{2b} - p_{2a}p_{1b})^2$, so, $\mathcal{O}$ being a domain, all the minors $p_{1a}p_{2b} - p_{2a}p_{1b}$ of $[p_1\ p_2]$ vanish. If some $p_i = 0$, then P_i is q times a linear eigenfield. Otherwise unique factorization gives a primitive v with $p_1 = u_1 v$, $p_2 = u_2 v$; then (8.2) gives $p_3 = u_3 v$ and $u_1^2 + u_2^2 = u_3^2$: a nonzero component of v gives a square root of $u_1^2 + u_2^2$ in the fraction field, integral over the integrally closed ring $\mathcal{O}$, hence in $\mathcal{O}$; cross entries fix one sign in all components. Suppose the negative column comes from a simple eigenvalue. If u_3 is a nonunit, Lemma 8.2 makes P_3/q linear. If u_3 is a unit, homogeneity makes p_1, p_2 constant multiples of p_3, and a constant combination of the original fields satisfies

$$c_1 P_1(x) + c_2 P_2(x) = qJx, \qquad c_1^2 + c_2^2 = 1, \tag{8.3}$$

with J orthogonal and skew (by norm and tangency). If P_1, P_2 have the same eigenvalue, this is a linear eigenfield. Otherwise Jx has constant coordinates in their moving orthonormal frame on the unit sphere, so subtracting CR_J, for any constant C, leaves an Osserman tensor.

8.3 Two simple eigenvalues

Consider $(M, N, 1, 1)$ with $(M, N) = (7, 6)$ or $(8, 5)$. Shift the eigenvalue of multiplicity M to zero; let λ be the eigenvalue of multiplicity N and μ, σ the simple eigenvalues (distinct and nonzero). By [24, Lemma 3.1] there are cubic normalized eigenfields P_1, P_2 (linear ones homogenized by q), and the spectral projectors give

$$\mu(\mu - \lambda) P_1 P_1^T + \sigma(\sigma - \lambda) P_2 P_2^T = qA(A - \lambda q I). \tag{8.4}$$

If the two coefficients have the same sign, the sum-of-squares lemma on the diagonal makes both fields q times linear ones. Moving one simple eigenvalue into each large eigenvalue gives the pattern $(M + 1, N + 1)$, that is $(8, 7)$ or $(9, 6)$: the first is classified by Theorem C, with Cayley excluded by Lemma 7.1; the second is Clifford by [24, Proposition 2.1, p. 382]. Restore the two rank-one terms with [24, Lemma 3.2(1)], first the one moved into λ and then the one moved into 0. A Clifford structure with $\nu = 8$ (the J_i of $\eta_i = 0$ dropped) gives its base eigenvalue multiplicity exactly $15 - \nu = 7$, the dimension of $x^\perp \cap (\sum_i \mathbb{R} J_i x)^\perp$. The intermediate pattern $(M + 1, N, 1)$, that is $(8, 6, 1)$ or $(9, 5, 1)$, has no eigenvalue of multiplicity seven, so its Clifford structures have $\nu \le 7$, and Remark 7.2 is needed only at an endpoint of type (1.2). This proves the claim.

Otherwise, after interchanging μ, σ, put $a^2 = \mu(\mu - \lambda) > 0$ and $b^2 = -\sigma(\sigma - \lambda) > 0$, and let p_i be the image of the unscaled P_i in $\mathcal{O}^{16}$. Modulo q, (8.4) gives $(ap_1)(ap_1)^T = (bp_2)(bp_2)^T$, so $ap_1 = \pm bp_2$ (compare a nonzero component, then every cross entry). If both vanish, the linear-field argument above applies. Otherwise, after changing the sign of P_2,

$$aP_1 - bP_2 = \sqrt{a^2 + b^2}\, qJx$$

with J orthogonal and skew. Put $C = \mu + \sigma - \lambda$; it is nonzero since $a^2 + b^2 = (\mu - \sigma)C$. On the unit sphere, subtracting $C(Jx)(Jx)^T$ turns the two-dimensional block into a fixed symmetric matrix

of trace λ and determinant $\mu\sigma - C(\sigma a^2 + \mu b^2)/(a^2 + b^2) = 0$, since $\sigma a^2 + \mu b^2 = \mu\sigma(\mu - \sigma)$. The new eigenvalues are $0, \lambda$, so the pattern is $(M + 1, N + 1)$, and the projection lengths of Jx onto the new eigenspaces are constant, because the changed block is a constant matrix in the frame (P_1, P_2). Lemma 8.1 excludes a Cayley endpoint (for $(M, N) = (8, 5)$ the endpoint $(9, 6)$ is Clifford by [24, Proposition 2.1]). So the endpoint is Clifford, and [24, Lemma 3.2(1)], with Remark 7.2 at an endpoint of type (1.2), restores the original tensor.

8.4 A double and a simple eigenvalue

Consider $(M, N, 2, 1)$ with $(M, N) = (7, 5)$ or $(8, 4)$. Shift M to zero and call the other eigenvalues λ, μ, σ, with μ double and σ simple. There are cubic normalized fields P_1, P_2 for μ and P_3 for σ with

$$\mu(\mu - \lambda)(P_1 P_1^T + P_2 P_2^T) + \sigma(\sigma - \lambda) P_3 P_3^T = qA(A - \lambda q I).$$

For equal coefficient signs all fields are linear after division by q. For opposite signs, after multiplying by -1 if necessary, the two double-eigenvalue columns have positive coefficients and the simple column the negative one, and the three-column reduction applies: its nonunit case gives a linear simple-eigenvalue field, its unit case a linear double-eigenvalue field (8.3), and a column vanishing modulo q is already a linear field. Moving that field into the zero eigenvalue gives the endpoints

initial	field from simple eigenvalue	field from double eigenvalue
$(7, 5, 2, 1)$	$(8, 5, 2)$	$(8, 5, 1, 1)$
$(8, 4, 2, 1)$	$(9, 4, 2)$	$(9, 4, 1, 1)$

which are Clifford by Theorem E, Subsection 8.3 or [24, Proposition 2.1, p. 382]. Restore the rank-one term by [24, Lemma 3.2(1)]; no endpoint in the table has an eigenvalue of multiplicity seven, so their Clifford structures have $\nu \leq 7$ and Remark 7.2 is not needed.

8.5 Three simple eigenvalues

Consider $(M, N, 1, 1, 1)$ with $(M, N) = (7, 5)$ or $(8, 4)$. After shifting M to zero there are five reduced eigenvalues, so [24, Lemma 3.1] gives fields of odd degree at most three. Let λ be the eigenvalue of multiplicity N and μ_1, μ_2, μ_3 the simple eigenvalues; the cubic spectral identity has weights $w_i = \mu_i(\mu_i - \lambda)$. If all signs agree, every field is linear after division by q, and any one of them serves as the linear eigenfield below. Otherwise choose the sign and order so that two weights are positive and the third, a simple column, is negative; the three-column reduction applies.

If we have a linear eigenfield, moving it into the zero eigenvalue gives $(8, 5, 1, 1)$ or $(9, 4, 1, 1)$, Clifford by Subsection 8.3 and [24, Proposition 2.1]; restore by [24, Lemma 3.2(1)]. Otherwise (8.3) mixes μ_1, μ_2. Put $d = \mu_1 c_2^2 + \mu_2 c_1^2$. After subtracting $C(Jx)(Jx)^T$ the characteristic equation of the block is $(\mu_1 - z)(\mu_2 - z) - C(d - z) = 0$. Choose $z \in \{0, \lambda\}$ with $d \neq z$ (possible since $\lambda \neq 0$) and $C = (\mu_1 - z)(\mu_2 - z)/(d - z)$, which is nonzero. The other eigenvalue is $\theta = \mu_1 + \mu_2 - C - z$, and $0, \lambda, \mu_3$ survive unchanged and distinct, so there is no two-eigenvalue endpoint. The complete list of endpoints, including coincidences, is as follows, where *other large* means $\theta = \lambda$ if $z = 0$ and $\theta = 0$ if $z = \lambda$:

(M, N)	z	θ new	$\theta = \mu_3$	θ other large	$\theta = z$
$(7, 5)$	0	$(8, 5, 1, 1)$	$(8, 5, 2)$	$(8, 6, 1)$	$(9, 5, 1)$
$(7, 5)$	λ	$(7, 6, 1, 1)$	$(7, 6, 2)$	$(8, 6, 1)$	$(7, 7, 1)$
$(8, 4)$	0	$(9, 4, 1, 1)$	$(9, 4, 2)$	$(9, 5, 1)$	$(10, 4, 1)$
$(8, 4)$	λ	$(8, 5, 1, 1)$	$(8, 5, 2)$	$(9, 5, 1)$	$(8, 6, 1)$

Each is Clifford by the preceding subsections, Theorem E, [24, Proposition 2.1] (an eigenvalue of multiplicity 9), or [23, Proposition 1] for $(10, 4, 1)$ (in the notation of [23, Proposition 1], $m = 10$ and $\nu = 5$ satisfy $16 \geq 3\nu$ and $16 > (\nu + 1)^2/4$). Each update has constant eigenvalues, so [24, Lemma 3.2(1)], in the form of Remark 7.2 at the endpoint $(7, 7, 1)$, restores the original tensor.

8.6 The remaining patterns with multiplicities at most two

Shift the eigenvalue of multiplicity $M \in \{7, 8\}$ to zero and let $r = 15 - M \leq 8$. By [24, Lemma 3.1, p. 383], which applies because every nonzero eigenvalue has multiplicity one or two, there are r orthogonal normalized fields for the nonzero eigenvalues. Divide each by the largest possible power of q and homogenize all to the largest remaining odd degree d; then a component of a field of maximal degree is not divisible by q. With P the $16 \times r$ matrix of fields and Λ the diagonal matrix of their eigenvalues,

$$P^T P = q^d I, \qquad P\Lambda P^T = q^{d-1} A, \qquad P\Lambda^2 P^T = q^{d-2} A^2.$$

If $d \geq 3$, each diagonal entry of $P\Lambda^2 P^T$ is a sum of $r \leq 8$ squares divisible by q, and [24, Lemma 3.11] makes every field component divisible by q, a contradiction. So $d = 1$, the fields are $J_i x$ with orthogonal skew, pairwise anticommuting J_i, and the Jacobi formula is Clifford. (This is the proof of [24, Lemma 3.3] with at most eight columns instead of six.) This proves Theorem F.

9 An extreme Jacobi eigenvalue of multiplicity seven or eight

Theorem G. *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with at least three distinct reduced Jacobi eigenvalues. If an eigenvalue of multiplicity seven or eight is the smallest or the largest reduced eigenvalue, then R is Clifford.*

The hypothesis of three eigenvalues is needed: the Cayley tensor has two extreme eigenvalues. Throughout this section a reduced eigenvalue of multiplicity at least seven has been shifted to zero by subtracting a multiple of R_1, so that the reduced Jacobi operator has nonzero rank $\nu \leq 8$, and Λ is the diagonal matrix of the nonzero Jacobi eigenvalues for unit vectors, with multiplicities.

Proposition 9.1. *Let R be an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ as above.*

1. *If $\mathcal{J}_x = M_x \Lambda M_x^T$ has a global linear factor $M_x : \mathbb{R}^\nu \to \mathbb{R}^{16}$, injective for all $x \neq 0$, then R is Clifford.*

2. *Suppose that $\mathcal{J}_z = M_z \Lambda M_z^T$ has a rational homogeneous factor of degree one with the following coherence property: there are a real unit vector X and an orthonormal eigenframe M_X such that, for all $Y \perp X$,*

$$M_{sX+Y} = sM_X + M_Y \tag{9.1}$$

as a rational identity. If the full *Gram matrix $\Phi(z) = M_z^T M_z$ on $\mathbb{R}^{16}$ is polynomial, then R is Jacobi-proportional, hence, by Corollary D, Clifford or a linear combination of R_1 and an orthogonal conjugate of R_{Cay}. If Λ is nonscalar, it is Clifford.*

In part (2) the full Gram is $\Phi(sX + Y) = s^2 I + s(M_X^T M_Y + M_Y^T M_X) + M_Y^T M_Y$, so its mixed coefficient must be polynomial as well.

9.1 Quadratic Gram rigidity

Lemma 9.2. *Let Λ be invertible symmetric of size $\nu \le 8$. Suppose $\Phi(x)$ is a symmetric $\nu \times \nu$ matrix, homogeneous quadratic in $x \in \mathbb{R}^{16}$, positive definite for $x \neq 0$, and $\Lambda\Phi(x)$ has the spectrum of $\|x\|^2\Lambda$, with multiplicities. If $\Phi(x_0) = I$ at a unit vector x_0, then $\Phi(x) = \|x\|^2 I$ for every x.*

Proof. This refines [23, Section 5, Lemmas 5–8, pp. 11–15 of arXiv:math/0204258v2], whose hypothesis $n > (\nu+1)^2/4$ fails for $n = 16$, $\nu = 7, 8$; the remaining pairs are settled by a topological count. Diagonalize Λ orthogonally, choose an eigenvalue λ of multiplicity m whose reciprocal is extremal, and set $q = \|x\|^2$, $Q_\lambda(x) = \lambda q\Lambda^{-1} - \Phi(x)$. For unit x the matrix $\Phi(x)^{1/2}\Lambda\Phi(x)^{1/2}$ is symmetric with the spectrum of Λ; write it as $O\Lambda O^T$ with O orthogonal. Then $C = O^T\Phi(x)^{1/2}$ satisfies $C\Lambda C^T = \Lambda$, $\Phi = C^T C$ and $Q_\lambda(x) = C^T(\lambda\Lambda^{-1} - I)C$, so $Q_\lambda(x)$ has kernel dimension m for all x and is semidefinite of a fixed sign.

For $u \in \mathbb{R}^\nu$ let $H_\lambda(u)$ be the symmetric 16×16 matrix with $x^T H_\lambda(u) x = u^T Q_\lambda(x) u$; it is quadratic in u and semidefinite, so $L_u := \{x : Q_\lambda(x)u = 0\} = \ker H_\lambda(u)$ is a linear subspace, of dimension $d(u)$. Choose independent $u_1, \dots, u_m$ maximizing $\sum_i d(u_i)$ and put $S = \operatorname{span}(u_1, \dots, u_m)$. For $u \notin S$ and each i, the vectors u and u_j $(j \neq i)$ are independent, so $d(u) \le d(u_i)$. For any $m+1$ independent coefficient vectors $v_0, \dots, v_m$ one has $L_{v_0} \cap \dots \cap L_{v_m} = 0$ (otherwise $\ker Q_\lambda(x)$ would have dimension at least $m+1$ at some $x \neq 0$), and the subspace intersection inequality $0 = \dim(L_u \cap L_{u_1} \cap \dots \cap L_{u_m}) \ge d(u) + \sum_i d(u_i) - 16m$, with $d(u) \le d(u_i)$ for each i, gives $(m+1)d(u) \le d(u) + \sum_i d(u_i) \le 16m$, that is,

$$d(u) \le h := \left\lfloor \frac{16m}{m+1} \right\rfloor \qquad (u \notin S). \tag{9.2}$$

The incidence space $\mathcal{I} = \{(x, u) : x \neq 0,\ Q_\lambda(x)u = 0\}$ is a connected real-analytic rank-m vector bundle over $\mathbb{R}^{16} \setminus \{0\}$, of dimension $16 + m$, and its open subset $\mathcal{I}_{\text{out}}$ where $u \notin S$ has semialgebraic dimension at most $\nu + h$. If $\nu + h < 16 + m$, then $\mathcal{I}_{\text{out}}$ is empty and $\ker Q_\lambda(x) = S$ for every x. The scalar case $m = \nu$ is immediate, and for $\nu \le 8$ the only remaining nonscalar pairs are

(ν, m)	h	$\nu + h$
$(7, 3)$	12	19
$(8, 2)$	10	18
$(8, 3)$	12	20
$(8, 4)$	12	20
$(8, 5)$	13	21

Suppose $\mathcal{I}_{\text{out}} \neq \varnothing$ in one of these cases and put $\ell = 16 + m - \nu$. The coefficient strata with $d(u) \le \ell - 1$ have incidence dimension at most $\nu + \ell - 1 = 15 + m$, so the locus with $d(u) \ge \ell$ contains a relatively open subset of $\mathcal{I}$. All $(17 - \ell)$-minors of $H_\lambda(u)$ vanish there; their pullbacks to $\mathcal{I}$ are real analytic, so by connectedness they vanish on $\mathcal{I}$, and every nonempty coefficient fibre has $d(u) \ge \ell$. Normalize both variables:

$$\mathcal{E} = \{(x, u) \in S^{15} \times S^{\nu-1} : Q_\lambda(x)u = 0\}, \qquad Z = \{u \in S^{\nu-1} : \det H_\lambda(u) = 0\}.$$

The first projection makes $\mathcal{E}$ an S^{m-1}-bundle over S^{15}, so $\dim \mathcal{E} = 14 + m$. The second projection is a proper surjection onto the compact real algebraic set Z, with fibres $S(L_u)$ of dimension at least $\ell - 1$, and at most $h - 1$ outside S by (9.2). As $\mathcal{E}_{\text{out}}$ is nonempty and open,

$$15 + m - h \ \le\ z := \dim Z \ \le\ 15 + m - \ell = \nu - 1,$$

which gives

(ν, m)	ℓ	possible z	minimum sphere-fibre dimension
$(7,3)$	12	6	11
$(8,2)$	10	7	9
$(8,3)$	11	6 or 7	10
$(8,4)$	12	7	11
$(8,5)$	13	7	12

In every row $z < \ell - 1$, so the finite-degree Vietoris–Begle theorem [19, Theorem 1.1 (Begle), p. 383], with mod-two coefficients, gives $H^z(Z;\mathbb{F}_2) \cong H^z(\mathcal{E};\mathbb{F}_2)$ (all spaces are compact semialgebraic and triangulable, so Čech and singular cohomology agree). The latter group vanishes by the Gysin sequence of the S^{m-1}-bundle over S^{15}: the base groups in degrees z and $z-m+1$ vanish in every row. But a nonempty compact real algebraic set of dimension z has a nonzero top mod-two homology class [5, Theorem 1.1, p. 22], hence $H^z(Z;\mathbb{F}_2) \neq 0$, a contradiction.

Hence $\ker Q_\lambda(x) = S$ for all $x \neq 0$; at x_0 this is the λ-eigenspace of Λ, so $\Phi(x)u = \lambda q \Lambda^{-1} u = qu$ for $u \in S$. Symmetry removes the cross blocks, and restricting Φ, Λ to the orthogonal complement preserves all hypotheses. Induction on ν proves $\Phi = qI$. □

9.2 Proof of Proposition 9.1

Proof of part (1). A constant coefficient change $M_x \mapsto M_x G_0$ normalizes $M_{x_0}^T M_{x_0} = I$ at one unit vector and replaces Λ by $G_0^{-1}\Lambda G_0^{-T}$; this replaces $\Lambda\Phi(x)$ by the similar matrix $G_0^{-1}\Lambda\Phi(x)G_0$, and at x_0 the new Λ has the spectrum of the old one, so the hypotheses of Lemma 9.2 are kept. For a global linear factor, $\Phi(x) = M_x^T M_x$ is homogeneous quadratic and positive definite, and the nonzero spectrum of $\mathcal{J}_x$ is that of $\Lambda\Phi(x)$, so Lemma 9.2 gives $M_x^T M_x = qI$. For a coefficient basis vector e_i write $J_i x = M_x e_i$. Since $\mathcal{J}_x x = 0$, M_x is injective and Λ is invertible, $M_x^T x = 0$, so each J_i is skew; the Gram identity makes the J_i orthogonal with $J_iJ_j + J_jJ_i = -2\delta_{ij}I$ by polarization. Diagonalizing Λ gives the Clifford formula (with no columns, R has constant curvature). □

Proof of part (2). On the real regular locus $\Phi = M^T M$ is positive definite, and the nonzero spectrum gives $\det\Phi(z) = q(z)^\nu$. If Φ is polynomial, it is positive semidefinite everywhere by continuity and positive definite for $z \neq 0$ by the determinant. By (9.1) and homogeneity, $M_{X+tY} = M_X + tM_Y$ for generic $Y \perp X$ and $t \neq 0$, and continuity at $t = 0$ gives $\Phi(X) = I$. Lemma 9.2 yields $\Phi(z) = q(z)I$.

For a nonzero eigenvalue λ_j let I_j be its coefficient projector and $\pi_j(z)$ the corresponding reduced Jacobi projector. Then $C_j(z) := M_z I_j M_z^T = q(z)\pi_j(z)$ on the regular locus, hence as a rational identity. In coordinates $z = sX + Y$ the left side has a denominator dividing a power of a polynomial $D(Y)$ independent of s, by (9.1), while the right side has only powers of $q(z) = s^2 + \|Y\|^2$ as possible denominators: by Lagrange interpolation over the distinct nonzero eigenvalues $\lambda_1, \dots, \lambda_p$, $\pi_j(z) = \mathcal{J}_z \prod_{k\neq j}(\mathcal{J}_z - \lambda_k q(z) I)/(\lambda_j \prod_{k\neq j}(\lambda_j - \lambda_k)\, q(z)^p)$. The polynomial $q(z)$ is irreducible over $\mathbb{C}$ in sixteen variables and coprime to every nonzero $D(Y)$, so unique factorization shows that C_j is polynomial, and homogeneity makes it quadratic; so is the reduced zero-eigenvalue family $C_0(z) = q(z)I - zz^T - \sum_j C_j(z)$. Each C_j is symmetric with $C_j(z)z = 0$, and the polarization argument of Section 2 gives $\langle C_j(x)y, y\rangle = \langle C_j(y)x, x\rangle$. So R is Jacobi-proportional, and Corollary D applies. If Λ is nonscalar, the zero eigenvalue and at least two distinct nonzero eigenvalues give three reduced eigenvalues, which excludes the Cayley alternative. □

9.3 The coherent partner

Lemma 9.3. *For the shifted tensor above, with $\nu \le 8$, there are a real unit anchor X, an orthonormal eigenframe M_X, and a rational homogeneous partner M_Y on $X^\perp$ with*

$$\mathcal{J}_{sX+Y} = (sM_X + M_Y)\Lambda(sM_X + M_Y)^T.$$

Proof. We use two results of [23, Section 4], both stated for $n \ge 2\nu$, hence valid for $n = 16$, $\nu \le 8$. By [23, Lemma 2(1) and its proof, p. 7 of arXiv:math/0204258v2], for every unit X the unit vectors $Y \perp X$ with $\operatorname{Im}\mathcal{J}_X \cap \operatorname{Im}\mathcal{J}_Y = 0$ form an open dense subset of the unit sphere of $X^\perp$. By [23, Lemma 3, p. 8 of arXiv:math/0204258v2], for every such orthonormal pair (X, Y) and every M_X with $\mathcal{J}_X = M_X\Lambda M_X^T$ there is a unique M_Y with $\mathcal{J}_{sX+tY} = (sM_X + tM_Y)\Lambda(sM_X + tM_Y)^T$ for all real s, t. Fix a unit X and an orthonormal eigenframe M_X, and put $M_{cY} = cM_Y$ for $c \ne 0$; uniqueness makes this consistent. Write $M_Y = M_X F(Y) + K_X P(Y)$, where K_X is an orthonormal frame of $\ker\mathcal{J}_X$, and $2\mathcal{B}(X, Y) = \mathcal{J}_{X+Y} - \mathcal{J}_X - \mathcal{J}_Y$. The mixed coefficient gives $P(Y) = K_X^T\, 2\mathcal{B}(X, Y)M_X\Lambda^{-1}$, which is linear, of full column rank on the generic-pair locus since the images of $\mathcal{J}_X$ and $\mathcal{J}_Y$ meet only in 0. The lower-upper block of $\mathcal{J}_Y$ gives $P\Lambda F^T = K_X^T\mathcal{J}_Y M_X$, and the rational left inverse $(P^TP)^{-1}P^T$ shows that F, hence M_Y, is rational homogeneous of degree one. The two-plane identities hold rationally by continuation, and defining $M_{sX+Y} = sM_X + M_Y$ gives the required coherent factor. □

9.4 Proof of Theorem G

Proof. Shift the extreme eigenvalue of multiplicity seven or eight to zero; then $\nu = 7$ or 8 and Λ is definite. Use Lemma 9.3 with coordinates $Y = (y_1, \dots, y_{15})$ on $X^\perp$, and put $\mathbb{F} = \mathbb{R}(y_1, \dots, y_{15})$, $d = \sum_i y_i^2$, $G(s) = s^2 I + s\Phi_1 + \Phi_0$, where $\Phi_1 = M_X^T M_Y + M_Y^T M_X$ and $\Phi_0 = M_Y^T M_Y$ have entries in $\mathbb{F}$; thus $G(s) = \Phi(sX + Y)$ is the full Gram of Proposition 9.1. Equality of the nonzero Jacobi spectra gives

$$\det(tI - \Lambda G(s)) = \prod_{i=1}^{\nu}(t - (s^2 + d)\mu_i) \quad \text{in } \mathbb{F}[s, t], \tag{9.3}$$

where the μ_i are the nonzero eigenvalues with multiplicities.

Let $\mathbb{E} = \mathbb{F}(\alpha)$ with $\alpha^2 = -d$; as $\mathbb{F}$ is formally real, $[\mathbb{E} : \mathbb{F}] = 2$, and $\mathbb{E}$ is the fraction field of $\mathbb{R}[s, y_1, \dots, y_{15}]/(s^2 + y_1^2 + \cdots + y_{15}^2)$. By [24, Lemma 3.11 and its proof, p. 398], a sum of at most eight squares in $\mathbb{E}$ vanishes only when every term vanishes, so every constant definite real quadratic form of dimension at most eight is anisotropic over $\mathbb{E}$. At $s = \alpha$, (9.3) shows that $N = \Lambda G(\alpha)$ is nilpotent, and it is self-adjoint for the anisotropic form $\langle v, w\rangle_{\Lambda^{-1}} = v^T\Lambda^{-1}w$, since $\Lambda^{-1}N = G(\alpha)$ is symmetric. Such an N is zero: if its nilpotency index were $e \ge 2$, then $w = N^{e-1}v \ne 0$ for some v and $\langle w, w\rangle_{\Lambda^{-1}} = \langle v, N^{2e-2}v\rangle_{\Lambda^{-1}} = 0$, contradicting anisotropy. As Λ is invertible, $\Phi_0 - dI + \alpha\Phi_1 = 0$, and the independence of $1, \alpha$ over $\mathbb{F}$ gives $\Phi_1 = 0$ and $\Phi_0 = dI$. So the full Gram is $G(s) = (s^2 + d)I$, polynomial on $\mathbb{R}^{16}$, and Proposition 9.1(2) makes R Clifford (there are at least three reduced eigenvalues). □

10 The coherent factor and its transpose pencil

From here to the end of Section 26, R is an Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ with a reduced eigenvalue μ of multiplicity seven or eight that is neither the smallest nor the largest reduced eigenvalue (the extreme case is Theorem G). Shift μ to zero; the shifted Jacobi operators then have rank $\nu \in \{7, 8\}$; put $k = 16 - \nu$. Let Λ be the diagonal matrix of the nonzero eigenvalues, with

multiplicities; it is indefinite because μ is neither the smallest nor the largest reduced eigenvalue. Put $H = \Lambda^{-1}$ and $\mathcal{B}(X,Y) = \frac{1}{2}(\mathcal{J}_{X+Y} - \mathcal{J}_X - \mathcal{J}_Y)$. For a real unit anchor X choose orthonormal frames $M_X : \mathbb{R}^\nu \to E_X = \operatorname{Im}\mathcal{J}_X$ and $K_X : \mathbb{R}^k \to \ker\mathcal{J}_X$, with $\mathcal{J}_X = M_X \Lambda M_X^T$. A *regular anchor* is a point of the nonempty Euclidean-open set U_0 constructed in Section 27: the generic-pair construction below applies there, and U_0 avoids the proper algebraic exceptional loci named in the following sections. Section 27 partitions U_0 into finitely many semialgebraic sets $\mathcal{U}_j$ on which all pole invariants are constant (Lemma 27.1); we call them *branches*. It also lists the cases, (L1)–(L8), and names the section that settles each; a first reading may start there.

10.1 Construction and normalization

Take X in the nonempty open set of anchors where the generic-pair construction of Lemma 9.3 applies (only the two-plane result of [23, Section 4] is used, not its three-plane construction, whose dimension hypothesis fails here). As in that proof, $M_Y = M_X F(Y) + K_X P(Y)$ with

$$P(Y) = K_X^T\, 2\mathcal{B}(X,Y) M_X \Lambda^{-1} \tag{10.1}$$

linear, F rational and homogeneous of degree one, and the block identities

$$F\Lambda + \Lambda F^T = M_X^T\, 2\mathcal{B}(X,Y) M_X, \tag{10.2}$$

$$P\Lambda F^T = K_X^T \mathcal{J}_Y M_X, \tag{10.3}$$

$$F\Lambda F^T = M_X^T \mathcal{J}_Y M_X, \qquad P\Lambda P^T = K_X^T \mathcal{J}_Y K_X \tag{10.4}$$

hold as rational identities; P, which we call the *lower pencil*, is injective for generic Y. Thus

$$M_{sX+Y} = sM_X + M_X F(Y) + K_X P(Y), \qquad \mathcal{J}_{sX+Y} = M_{sX+Y}\Lambda M_{sX+Y}^T. \tag{10.5}$$

Lemma 10.1. *If F has no pole along any complex hypersurface in $X^\perp$, then* (10.5) *is a global linear factor, and the tensor is Clifford.*

Proof. In the polynomial ring on $X^\perp$, a rational function with no divisorial pole is polynomial; by homogeneity F is linear, and (10.5) is a polynomial identity in all sixteen variables. Its factor is injective at every nonzero real vector, as the Jacobi operator has rank ν, so Proposition 9.1(1) applies. □

10.2 Kernel directions give Jacobi factors

Extend P linearly to $V = \mathbb{R}^{16}$ by $P(X) = 0$. For $g \in \mathbb{R}^k$ define $N_g : \mathbb{R}^\nu \to V$ by

$$N_g^T Y = P(Y)^T g \qquad (Y \in V), \tag{10.6}$$

linear in g, and for $a \in \mathbb{R}^\nu$ define the *transpose pencil* $B(a) : \mathbb{R}^k \to V$ by $B(a)g = N_g a$.

Lemma 10.2. *For every real g,*

$$\mathcal{J}_{K_X g} = N_g \Lambda N_g^T. \tag{10.7}$$

If $g \neq 0$, then N_g is injective. If $a \neq 0$ is real, then $B(a)$ is injective; equivalently $Y \mapsto P(Y)a$ is onto $\mathbb{R}^k$. There are no nonzero fixed real coefficient space U and proper fixed real subspace $T \subset \mathbb{R}^k$ with $P(Y)U \subset T$ for all Y.

Proof. For $z = K_X g$ and $v = sX + Y$ with $Y \perp X$, the relations $M_X^T z = 0$ and $P(X) = 0$ give $M_v^T z = P(Y)^T g = N_g^T v$ for every real s. So the coherent factor and the symmetry $\langle \mathcal{J}_v z, z\rangle = \langle \mathcal{J}_z v, v\rangle$ give

$$\langle \mathcal{J}_z v, v\rangle = (N_g^T v)^T \Lambda (N_g^T v) \qquad (v \in \mathbb{R}^{16});$$

polarizing in v proves (10.7). Its left side has rank ν for real $g \neq 0$, so N_g is injective, and $N_g a = 0$ with $a \neq 0$ forces $g = 0$; transposition gives surjectivity. Finally, for nonzero $g \perp T$ the containment would make N_g vanish on U. □

Let g_X be the unit vector with $K_X g_X = X$. Then

$$B(a) g_X = -M_X a, \qquad K_X^T B(a) \text{ is skew-symmetric}, \qquad \operatorname{Im} B(a) \subset X^\perp. \tag{10.8}$$

Indeed, injectivity of the coherent factor and $\mathcal{J}_z z = 0$ give $M_z^T z = 0$; the coefficient of s for $z = sX + Y$ gives $P(Y)^T g_X = -M_X^T Y$, the first identity. The last follows from $P(X) = 0$. For the middle one, (10.7) and $\mathcal{J}_{K_X g} K_X g = 0$ give $N_g^T K_X g = 0$, so $\langle g, K_X^T B(a) g\rangle = 0$ for all real g, a; polarize in g. We call the first identity in (10.8), in either form, the *radial identity*; $B(a) g_X = -M_X a$ is the *radial column* and $g_X^T P(Y) = -Y^T M_X$ the *radial row*. For $\nu = 8$ (so $k = \nu$) we say that the transpose pencil is *aligned* at X if $\operatorname{Im} B(a) = E_X$ for every real $a \neq 0$.

Lemma 10.3. *For every nonzero complex coefficient a, the kernel of $B(a)$ is totally isotropic in $\mathbb{C}^k$ for the standard bilinear form, and $s(a) = \operatorname{rank} B(a)$ satisfies $s(a) \geq \lceil k/2 \rceil$ and $\operatorname{rank}(B(a)B(a)^T) = 2s(a) - k$.*

Proof. If $0 \neq g \in \ker B(a)$ with $g^T g \neq 0$, the Osserman characteristic and minimal-polynomial identities, continued to complex g, make $\mathcal{J}_{K_X g}$ diagonalizable of rank ν, so by (10.7), continued polynomially in g, N_g has rank ν, contrary to $N_g a = 0$. So $L = \ker B(a)$ consists of null vectors, is totally isotropic by polarization, and has dimension at most $\lfloor k/2 \rfloor$. As $\operatorname{Im} B(a)^T = L^\perp \supset L$, the image of $B(a)B(a)^T$ has dimension $\dim L^\perp - \dim L = 2s(a) - k$. □

10.3 The local pole budget

Put $F_0 = M_X^T \mathcal{B}(X, Y) M_X \Lambda^{-1}$, linear in Y, and $\Omega = (F - F_0)\Lambda$, a skew-symmetric matrix, homogeneous of degree one. Since Ω is skew and Λ symmetric, $P\Omega = -P\Omega^T = -P\Lambda F^T + P\Lambda F_0^T$, which is polynomial by (10.3), as P and F_0 are linear; and, as $F = F_0 + \Omega H$ and $H\Lambda = I$, the first identity in (10.4) makes $\Omega H \Omega^T + \Omega F_0^T + F_0 \Omega^T = F\Lambda F^T - F_0 \Lambda F_0^T$ polynomial; so the negative-order terms of $\Omega H \Omega^T$ are cancelled only by terms whose pole order is at most that of Ω.

Lemma 10.4. *Let f be an irreducible polynomial, real or complex, defining a pole of Ω of order $d > 0$. Every maximal ν-minor of P is divisible by f^{2d}; if $d = 1$ and the residue has rank r, every such minor is divisible by f^r. Let the* reduced common denominator *D of Ω be the least common multiple of the denominators of its entries, each entry written in lowest terms; a pole of order d along f contributes f^d, so D need not be squarefree. Then D^2 divides every maximal minor of P, and $2 \deg D \leq \nu$.*

Proof. Localize the polynomial ring at (f), a discrete valuation ring with uniformizer $\pi = f$. An alternating matrix over its fraction field is unimodularly congruent to two-by-two blocks $a_j \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and a zero block (pivot on an entry of least valuation and clear its two rows and columns; all coefficients are integral). Let the negative block valuations be $-d_j$. As $P\Omega$ is integral, the corresponding two columns of the transformed P are divisible by π^{d_j}, so every maximal minor has valuation at least $2\sum_j d_j \geq 2d$, since the largest d_j is the pole order d. At a simple pole the residue rank is twice the

number of negative blocks. Unimodular column changes preserve the maximal-minor ideal, and distinct irreducible factors multiply, which gives the claim for D. Some maximal minor is nonzero, of degree ν, since P is generically injective. □

If Ω has a pole of order $d > 0$ along an irreducible f, call $W = (f^d\Omega) \bmod f$, a matrix over the residue field of (f), the *leading Laurent coefficient* of Ω at f. The leading terms of the same identities give, over the residue field,

$$PW = 0, \qquad W^T = -W, \qquad WHW^T = 0; \tag{10.9}$$

the last because the order-$2d$ term of the upper-block identity has no other possible cancellation. So the rank of W is even and its image is H-isotropic. For $\nu = 8$ the budget reads $D^2 \mid \det P$ and $\deg D \le 4$.

11 Real pole types and the intrinsic denominator

We keep the notation of Section 10: real nonzero Jacobi operators have rank $\nu \in \{7, 8\}$ and $k = 16 - \nu$. The first lemma extends the dimension counts of [24, Lemma 3.8] to both ranks. We use Rakić duality [27, §§1–2]; it applies to the full kernel after an eigenvalue is shifted to zero, including the radial direction: for $a = cX + b$ with $b \perp X$ in the $\mu\|X\|^2$-eigenspace of the unshifted $\mathcal{J}_X$, duality and the curvature symmetries give for the unshifted tensor $\mathcal{J}_a X = -c\mu\|X\|^2 b + \mu\|b\|^2 X = \mu(\|a\|^2 X - \langle a, X\rangle\, a)$, so $\mathcal{J}_a X = 0$ after subtracting $\mu\Pi(a)$. Exchanging the vectors gives the converse, so $a \in \ker \mathcal{J}_X$ if and only if $X \in \ker \mathcal{J}_a$.

11.1 Generic triples and real pole types

Lemma 11.1. *There is an open dense set of real orthonormal triples (X, Y, Z) with*

$$\dim(\operatorname{Im} \mathcal{J}_X \cap \operatorname{Im} \mathcal{J}_{sY+tZ}) \le 1 \qquad ((s,t) \neq (0,0)),$$

on which, in addition, the three pairwise Jacobi-image intersections are zero and the three Jacobi kernels have zero common intersection.

Proof. Fix an orthonormal pair (Y, Z) and let w run over its unit circle. The triples (w, a, X) with $\|a\| = \|X\| = 1$, $a \in \ker \mathcal{J}_w$ and $X \in \ker \mathcal{J}_a$ form a compact smooth iterated sphere bundle (the kernel bundles have constant rank) of dimension $1 + 2(k-1) = 2k - 1$. By Sard's theorem and compactness the regular values of its projection to $X \in S^{15}$ form an open dense set (values outside the image are regular, with empty fibre). By duality the fibre at X is the set of pairs (w, a) with $a \in \ker \mathcal{J}_X \cap \ker \mathcal{J}_w$, of dimension $2k - 16 = 16 - 2\nu$ when nonempty and regular. For fixed w the unit sphere of an intersection of dimension $d > 0$ embeds in it, so $d - 1 \le 16 - 2\nu$ and $\dim(\operatorname{Im} \mathcal{J}_X \cap \operatorname{Im} \mathcal{J}_w) = 2\nu - 16 + d \le 1$ (if $d = 0$, including an empty fibre, then $\nu = 8$, since for $\nu = 7$ the two nine-dimensional kernels meet nontrivially, and this dimension is 0). Failure of this bound is a closed condition (a matrix-rank condition at some point of a compact circle), so the good set is open, and, by the regular-value argument, dense for each fixed (Y, Z), hence on the independent-triple domain.

Orthogonalization preserves the bound: by duality, $a \in \bigcap_i \ker \mathcal{J}_{v_i}$ if and only if $\operatorname{span}\{v_i\} \subset \ker \mathcal{J}_a$, so common kernels depend only on spans; projecting the plane of Y, Z along X to $X^\perp$ keeps each $\operatorname{span}(X, w)$, so the common kernels, and therefore the image-intersection dimensions, agree; the Gram–Schmidt map is a surjective submersion onto orthonormal triples and sends the open dense

good set to an open dense set of orthonormal triples. The generic-pair theorem [23, Section 4, Lemma 2(1)] gives the three pairwise conditions (an intersection of three open dense sets). Finally, triples with a common unit kernel vector form the image of a compact bundle with fibre $V_3(\mathbb{R}^k)$ over that vector, of dimension $15 + 3k - 6 \in \{33, 36\}$, less than $\dim V_3(\mathbb{R}^{16}) = 42$; this closed image has measure zero. □

Lemma 11.2. *Let X be a regular anchor and (X, Y, Z) a triple as in Lemma 11.1 such that* $\operatorname{span}(Y, Z)$ *is not contained in the polar hypersurface of F. Then $F(sY + tZ)$ has no pole at any nonzero real direction.*

Proof. With $\Omega = (F - F_0)\Lambda$ as in Section 10, the leading Laurent coefficient W at a real pole is a nonzero real skew matrix, of rank at least two, and the leading coefficient of the coherent factor $M_X F + K_X P$ is $M_X W \Lambda^{-1}$, with image of that rank in E_X. Away from the pole the columns of the factor lie in $\operatorname{Im} \mathcal{J}_{sY+tZ}$, real spaces of constant dimension that vary continuously through the pole; rescaling by a power of a local parameter and passing to the limit puts the leading coefficient in the limiting Jacobi image. Its intersection with E_X then has dimension at least two, a contradiction. □

Proposition 11.3. *At a generic regular real anchor, let D_X be the reduced common denominator of F on $X^\perp$. If $\nu = 7$, it is constant or a single positive semidefinite quadratic, irreducible over $\mathbb{R}$. If $\nu = 8$, up to nonzero real constants it is one of*

$$1, \qquad Q, \qquad Q^2, \qquad Q_1 Q_2, \qquad f,$$

where the Q_i are distinct real irreducible positive semidefinite quadrics of rank at least two and f is real irreducible of degree four. A rank-two quadric may split into conjugate linear factors over $\mathbb{C}$, and a real irreducible quartic into two conjugate quadrics.

Proof. The complement of the good set of Lemma 11.1 is a closed semialgebraic subset of $V_3(\mathbb{R}^{16})$ of dimension at most 41, with fibre over X in $V_2(X^\perp)$, of dimension 27; the anchors with a 27-dimensional fibre form a set of dimension at most 14. Outside its closure the good planes are open dense in $X^\perp$; we call an anchor *generic* if it lies outside this set and outside the proper algebraic set of anchors excluded in the proof of Proposition 11.4, where specialization of the intrinsic denominator could create a common factor. That set is defined from the intrinsic denominator alone; the present proof does not use the second exclusion, which is made here so that both propositions use one notion of genericity. A generic real two-plane of $X^\perp$ preserves the degree of the homogeneous D_X and its pole orders: write all entries over the least common denominator; on each irreducible component some numerator is not divisible by its defining factor, the common zero locus of these numerators on the component has complex codimension at least two, and a generic projective line avoids it and meets each reduced component transversally at smooth points (likewise for the entries of highest order at a repeated component).

A real homogeneous factor of odd degree vanishes somewhere on every real projective line on which it is not identically zero, which would give a real pole on a generic line, contrary to Lemma 11.2; so every irreducible real factor has even degree. A real indefinite quadratic has real zeros on an open set of two-planes, among which a generic one can be chosen, so it is excluded too: an irreducible real quadratic factor is semidefinite, of rank at least two, and we take it positive. Lemma 10.4 gives $2 \deg D_X \le \nu$, leaving degree 0 or 2 for $\nu = 7$ and 0, 2 or 4 for $\nu = 8$, with the listed factorizations. A polynomial irreducible over $\mathbb{R}$ stays irreducible over $\mathbb{C}$ or splits into two conjugate factors of equal degree. □

11.2 A frame-independent polar divisor

Proposition 11.4. *On the open set of pairs with nonzero norms and* $\operatorname{Im}\mathcal{J}_X \cap \operatorname{Im}\mathcal{J}_Y = 0$ *there is a unique rational matrix* $T(X,Y)$*, regular there, with*

$$\operatorname{Im} T \subset \operatorname{Im}\mathcal{J}_Y, \qquad \operatorname{Im} T^T \subset \operatorname{Im}\mathcal{J}_X, \qquad T + T^T = 2\mathcal{B}(X,Y).$$

Its reduced common denominator $D(X,Y)$ *can be chosen real and bihomogeneous of bidegree* (d,d)*, with* $d \in \{0,2\}$ *for* $\nu = 7$ *and* $d \in \{0,2,4\}$ *for* $\nu = 8$*; it has no nonconstant factor depending on one slot only, and* $D(Y,X) = D(X,Y) = D(X,Y+aX)$*. For a generic anchor its fibre is* D_X *up to a nonzero constant. If* $d = 2$*, one global choice of sign makes* $D(X,Y) \geq 0$ *on real pairs, and* $D(X,Y) = \langle A_X Y, Y\rangle$ *is the sectional form of an algebraic curvature tensor.*

Proof. The coherent two-plane factors give $T = M_Y \Lambda M_X^T$. The difference of two solutions is skew with image in both Jacobi images, which meet only in 0, so it is zero. Writing the image conditions with the rational spectral projections ($T = \pi_Y T \pi_X$) gives a uniquely solvable linear system over $\mathbb{R}(X,Y)$, so T is rational, and uniqueness gives $T(Y,X) = T(X,Y)^T$, $T(tX,sY) = tsT(X,Y)$ and $T(X,Y+aX) = T(X,Y) + a\mathcal{J}_X$ (first from the coherent factor, then as a rational identity), and by exchange the other shear.

Take the least common multiple of the reduced denominators of the entries of T; separate homogeneity gives a bihomogeneous generator, and exchange makes the bidegrees equal. Each shear is a polynomial automorphism adding only a polynomial matrix to T, so it preserves the polar divisor with its orders and changes D by a constant factor; the part of $D(X,Y+aX)$ of top degree in Y is $D(X,Y)$, so this factor is 1. A one-slot factor would have infinitely many distinct irreducible transforms under the other shear, which is impossible for the finitely many polar components.

For generic unit real X, $M_Y = T(X,Y)M_X\Lambda^{-1}$ and $T = M_Y\Lambda M_X^T$ have the same poles, those of F (the lower block is polynomial). For each irreducible denominator factor g choose a numerator N not divisible by g; $V(g,N)$ has codimension at least two in the pair space, so outside a proper algebraic set of anchors its fibres have codimension at least two and contain no component of the specialized denominator. So generic specialization creates no common factor, the shear removes the radial coordinate of Y, the fibre degree is d, and Proposition 11.3 gives its values.

The elementary shears generate SL_2 on the two slots, so $D(Y,-X) = D(X,Y)$, and the even degree gives symmetry. For $d = 2$, $D(X,aX) = D(X,0) = 0$; writing $D(X,Y) = \langle A_X Y, Y\rangle$, shear invariance gives $A_X X = 0$, and (2.2) gives the curvature tensor. For generic real X the form $D(X,\cdot)$ is semidefinite by Proposition 11.3, hence for all X by continuity. Orient a nonzero fibre positively at X_0; for Y outside its kernel, $D(Y,X_0) = D(X_0,Y) > 0$, so $D(Y,\cdot)$ is positive semidefinite, and density and continuity give the global sign. □

The intrinsic divisor is used only at generic anchors, where it specializes to the polar divisor of F.

12 Auxiliary lemmas on the transpose pencil

This section collects arguments used in several of the later case exclusions. Throughout, $\nu = 8$, the anchor is regular, and the transpose pencil of Section 10 is written $B(a)$, with Gram matrix $G(a) = B(a)B(a)^T$. By Lemma 10.2 and (10.8) it satisfies the following conditions.

Setting (T). $V = \mathbb{R}^{16} = E \oplus E^\perp$ with $\dim E = 8$; the map $B : \mathbb{R}^8 \to \operatorname{Hom}(\mathbb{R}^8, V)$ is real linear and $\operatorname{rank} B(a) = 8$ for every real $a \neq 0$; there are a linear isomorphism $L : \mathbb{R}^8 \to E$, an isometry $K : \mathbb{R}^8 \to E^\perp$ and a vector $g_0 \neq 0$ such that

$$B(a)g_0 = -La, \qquad K^T B(a) \text{ is skew-symmetric} \qquad (a \in \mathbb{R}^8). \tag{12.1}$$

At a regular anchor X one takes $E = E_X$, $L = M_X$, $K = K_X$ and $g_0 = g_X$. A real linear change of the coefficient variable a preserves (T), with L replaced by its composite with the change; L need not stay an isometry. On coefficient subspaces we write $|x|^2$ for a Euclidean quadratic form and also for its complex-bilinear extension to complex vectors.

12.1 Positive semidefinite pencils

Lemma 12.1. *Let D_0, D_1 be real symmetric matrices and let $I \subset \mathbb{R}$ be an interval with nonempty interior such that $D(t) = D_1 + tD_0 \succeq 0$ for all $t \in I$. Then $\ker D(t) = \ker D_0 \cap \ker D_1$ for every interior point t of I. Hence $D(t)$ has one image $\mathcal{F} = \operatorname{Im} D_0 + \operatorname{Im} D_1$ for all interior t. If $\operatorname{rank} D(t) = \operatorname{rank} D(t')$ for an endpoint t and an interior point t', then $\ker D(t) = \ker D_0 \cap \ker D_1$ as well.*

Proof. The kernel of a sum of two positive semidefinite matrices is the intersection of their kernels. An interior point t is a strict convex combination of two points $t_1 < t < t_2$ of I, and D is affine in t, so $\ker D(t) = \ker D(t_1) \cap \ker D(t_2)$. A vector killed by $D(t_1)$ and $D(t_2)$ with $t_1 \neq t_2$ is killed by D_0 and D_1; the converse inclusion is clear. For symmetric matrices $(\ker D_0 \cap \ker D_1)^\perp = \operatorname{Im} D_0 + \operatorname{Im} D_1$. At an endpoint $\ker D(t) \supseteq \ker D_0 \cap \ker D_1$, and equal ranks give equality. □

Lemma 12.2. *Let $C(a)$, $a \in \mathbb{R}^n$, be a real symmetric matrix of any size whose entries are quadratic forms in a, positive semidefinite of one fixed positive rank for every $a \neq 0$. Suppose that its scalar quadratic entries span a real vector space of dimension at most three and $n \geq 3$, or of dimension at most two and $n \geq 2$. Then $\operatorname{Im} C(a)$ is independent of $a \neq 0$.*

Proof. The form $q(a) = \operatorname{tr} C(a)$ is positive definite and lies in the scalar-entry span; choose quadratic forms p, r, possibly zero, such that q, p, r span a space containing that span, and write $C = qD_0 + pD_1 + rD_2$ with constant symmetric D_i. After a real change of coordinates $q(a) = \|a\|^2$. If the span has dimension at most three and $n \geq 3$, the joint range of (p, r) on the unit sphere is convex by Brickman's theorem [20, Theorem 1.1], so $\{C(a) : q(a) = 1\}$ is convex. If the span has dimension at most two (so $r = 0$) and $n \geq 2$, the range of p on the connected unit sphere is an interval, with the same consequence. The midpoint of two members of this convex family is a member of the same rank, so the two members have the same kernel, as in the proof of Lemma 12.1. Symmetry gives a common image, and homogeneity extends it to all $a \neq 0$. □

Remark 12.3. If $C = G$ for a pencil in Setting (T), the fixed image in Lemma 12.2 is E, by Lemma 12.5(1) with $U = \mathbb{R}^8$; over the reals $\operatorname{Im} G(a) = \operatorname{Im} B(a)$.

12.2 Fixed images and the Jacobi image

Lemma 12.4. *The Pfaffian $A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$ of real skew 4×4 matrices is a quadratic form of signature $(3, 3)$. Hence no four-dimensional real space of skew 4×4 matrices consists, apart from 0, of invertible matrices.*

Proof. Each of the three products is a hyperbolic plane in its two variables, so the form has signature $(3, 3)$. A four-dimensional space whose nonzero members are invertible carries a nowhere-vanishing restriction of the Pfaffian, which would then be definite on a four-dimensional subspace; a form of signature $(3, 3)$ has none. □

Lemma 12.5. *Let B be as in Setting (T) and let $U \subset \mathbb{R}^8$ be a real subspace.*

1. *If $\operatorname{Im} B(a) = V_1$ for all $a \in U \setminus \{0\}$, with V_1 fixed, and $\dim U \geq 4$, then $V_1 = E$.*

2. *If* $\operatorname{Im} B(a) \subset V_1$ *for all* $a \in U \setminus \{0\}$*, with* V_1 *fixed,* $\dim V_1 \le 9$ *and* $\dim U \ge 7$*, then* $\operatorname{Im} B(a) = E$ *for every* $a \in U \setminus \{0\}$*.*

Proof. In both parts the radial identity in (12.1) gives $L(U) \subset V_1 \cap E$. Put $\mathcal{W} = K^T V_1$; then $\dim \mathcal{W} = \dim V_1 - \dim(V_1 \cap E) \le \dim V_1 - \dim U$, and the skew map $K^T B(a)$ has image in $\mathcal{W}$ for $a \in U$.

(1) Here $\dim \mathcal{W} \le 4$, and $K^T B(a)$ has image exactly $\mathcal{W}$ for every nonzero $a \in U$. A skew map with image $\mathcal{W}$ vanishes on $\mathcal{W}^\perp$ and is invertible on $\mathcal{W}$, so $r = \dim \mathcal{W} \in \{0, 2, 4\}$. If $r = 2$, the linear map $a \mapsto K^T B(a)$ sends U into the one-dimensional space of skew maps supported on $\mathcal{W}$, so it has a nonzero kernel, contradicting rank two. If $r = 4$, then $\dim U = 4$ and $a \mapsto K^T B(a)|_{\mathcal{W}}$ embeds U as a four-dimensional space of invertible skew maps of $\mathcal{W}$, which Lemma 12.4 excludes. Hence $r = 0$, so $V_1 \subset E$, and $\dim V_1 = 8$ gives $V_1 = E$.

(2) Here $\dim \mathcal{W} \le 2$, so every $K^T B(a)$ with $a \in U$ lies in a space of skew maps of dimension at most one. Since $\dim U \ge 2$, some $a_0 \ne 0$ in U has $K^T B(a_0) = 0$; then $\operatorname{Im} B(a_0) = E$ by rank eight, so $E \subset V_1$ and $\dim \mathcal{W} \le 1$. A skew map with image of dimension at most one is zero, so $K^T B(a) = 0$ on U, and rank eight gives the assertion. □

Lemma 12.6. *A linear space of matrices over a field, all of rank at most one, either has all images in one fixed line or has all rows in one fixed line.*

Proof. Write two nonzero members as $w_1 e_1^*$ and $w_2 e_2^*$. If w_1, w_2 are independent, rank at most one of the sum forces e_1, e_2 to be proportional; a third member $w_3 e_3^*$ has w_3 independent of w_1 or of w_2, so the same argument makes e_3 proportional to e_1 or to e_2, hence to both. If no such pair exists, the image line is common. □

12.3 Isotropic lines of a rank-two Gram matrix

Lemma 12.7. *Let* $A \subset \mathbb{P}^7_{\mathbb{C}}$ *be an irreducible variety in the coefficient space on which* G *has generic rank two, and let* $\Bbbk$ *be its function field (or that of its affine cone). Then* $K_0 = \ker B$ *over* $\Bbbk$ *is a totally isotropic three-space for the standard bilinear form on* $\Bbbk^8$*, and* B *induces an isometry from* $K_0^\perp / K_0$ *onto* $\operatorname{Im} G$ *with the pairing* $\langle Gx, Gx' \rangle = x^T G x'$*. This binary pairing has determinant class* -1 *and is hyperbolic over* $\Bbbk$*. If* A *is real-defined, the same holds over its real function field; then its two isotropic lines are individually defined over the reals.*

Proof. Lemma 10.3 gives $\operatorname{rank} G = 2 \operatorname{rank} B - 8$, so $\operatorname{rank} B = 5$ and K_0 has dimension three; it is totally isotropic by the same lemma. Since $\operatorname{Im} B^T = K_0^\perp$, we have $\operatorname{Im} G = B(K_0^\perp)$ and $x^T G x' = (B^T x)^T (B^T x')$, which is the stated isometry. The standard form on $\Bbbk^8$ is the orthogonal sum of three hyperbolic planes containing K_0 and of $K_0^\perp / K_0$; its determinant class is 1, so that of $K_0^\perp / K_0$ is -1. A nondegenerate binary form of determinant class -1 is hyperbolic. For a real-defined A the same computation runs over the real function field. □

Lemma 12.8. *Let* S *be a normal standard graded domain over* $\mathbb{C}$ *with degree-zero part* $\mathbb{C}$*, and let* C *be a symmetric* $N \times N$ *matrix of homogeneous quadratic elements of* S *with generic rank two, whose induced pairing* $\langle Cx, Cx' \rangle = x^T C x'$ *on its image is hyperbolic over* $\operatorname{Frac}(S)$*. Suppose that the saturated rank-one submodules of* S^N *along its two isotropic lines are principal. Then*

$$C = t\,(p r^T + r p^T) \tag{12.2}$$

with homogeneous $t \in S$ *and homogeneous vectors* p, r *of degrees* $d_p, d_r \ge 0$*, and* $\deg t + d_p + d_r = 2$*. If moreover* $S = S_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ *for a geometrically integral normal standard graded real domain* $S_{\mathbb{R}}$*, the*

entries of C lie in $S_{\mathbb{R}}$, and the pairing is hyperbolic over $\mathrm{Frac}(S_{\mathbb{R}})$*, then p, r, t may be taken real, and over $S_{\mathbb{R}}$*

$$C = uu^T - vv^T \qquad or \qquad C = cb^T + bc^T, \tag{12.3}$$

with u, v homogeneous linear, or c a nonzero constant vector and b homogeneous quadratic.

Proof. The scaling $\sigma_\tau : a \mapsto \tau a$, $\tau \in \mathbb{C}^*$, multiplies C by τ^2 and so permutes the two isotropic lines. The stabilizer of a line has index at most two, and the divisible group $\mathbb{C}^*$ has no subgroup of index two, so each saturated module is σ_τ-stable and therefore graded. A principal graded rank-one module has a homogeneous generator (graded Nakayama at $S_{>0}$); call the two generators p, r. Over the fraction field, $\operatorname{Im} C$ is spanned by p, r, so $C = [p\ r]\, M\, [p\ r]^T$ with M symmetric and invertible; in the basis (p, r) the induced pairing has Gram matrix M^{-1}, and isotropy of p and r makes M^{-1}, hence M, antidiagonal: $C = t(pr^T + rp^T)$ with $0 \neq t \in \mathrm{Frac}(S)$. At each height-one prime both generators are primitive, so their reductions are nonzero, and $pr^T + rp^T$ has a nonzero reduction in characteristic zero (if $u \neq 0$ and $uv^T + vu^T = 0$, a nonzero coordinate of u, its diagonal entry and its row force $v = 0$). Hence some entry of $pr^T + rp^T$ is a local unit, t has no negative height-one valuation, and normality gives $t \in S$. Homogeneity gives the degree identity.

In the real case the two lines are individually real-defined, so conjugation preserves each saturated module and changes its generator by a unit of S, a nonzero constant; conjugating twice shows it has modulus one, and a constant phase makes the generator real. Then t is real. If $d_p, d_r > 0$, both equal one and t is a real constant; with $u = (tp + r)/\sqrt{2}$ and $v = (tp - r)/\sqrt{2}$ one has $t(pr^T + rp^T) = uu^T - vv^T$. Otherwise one generator is a constant c and the other, multiplied by t, is a quadratic b. □

12.4 Complex arcs through a quadric stratum

Lemma 12.9. *Let B be as in Setting (T). Split the coefficient space as $\mathbb{R}^8 = W \oplus Z$, $a = (x, z)$, with $w = \dim W \geq 4$, put $q(x, z) = |x|^2 + |z|^2$ for Euclidean norms on W and Z, and let ρ be a real quadratic form on Z. Extend $|\cdot|^2$ complex bilinearly, and suppose:*

(i) G vanishes on $\Sigma = \{(x, 0) : |x|^2 = 0\} \subset W_{\mathbb{C}}$;

(ii) $\operatorname{rank} G \leq 2$ at every point of $V(q, \rho) \subset (W \oplus Z)_{\mathbb{C}}$.

Then the following hold.

1. *$\operatorname{Im} B(x, 0) = E$ for every real $x \in W \setminus \{0\}$, and $K^T B(x, z) = T(z)$ for a real linear map T from Z to skew matrices. For a real unit $x_* \in W$ the matrix $S = B(x_*, 0)$ maps $\mathbb{R}^8$ onto E, and $\Xi(x) = S^{-1} B(x, 0)$ satisfies $\Xi(x)^T \Xi(x) = \Xi(x)\Xi(x)^T = |x|^2 I$ and $\Xi(x_*) = I$; the operators $\Gamma_y = \Xi(y)$, $y \in W$, $y \perp x_*$, satisfy $\Gamma_y^T = -\Gamma_y$ and $\Gamma_y \Gamma_{y'} + \Gamma_{y'} \Gamma_y = -2\langle y, y' \rangle I$.*

2. *For every nonzero complex $z_0 \in Z_{\mathbb{C}}$ with $\rho(z_0) = 0$, either $T(z_0) = 0$, or $U = \operatorname{Im} T(z_0)^T \subset \mathbb{C}^8$ has dimension two, is invariant under every Γ_y, and $\operatorname{tr}(\Gamma_y|_U) = 0$.*

3. *If $w \geq 5$, $\dim Z \geq 2$ and ρ is nondegenerate, then $T = 0$ and $\operatorname{Im} B(a) = E$ for every real $a \neq 0$.*

Proof. (1) The quadric $|x|^2 = 0$ in $W_{\mathbb{C}}$ is irreducible, so by (i) every entry of $G(x, 0)$ is a multiple of $|x|^2$: $G(x, 0) = |x|^2 D$ with D real symmetric, positive semidefinite of rank eight. All real $x \neq 0$ in W give the same image $\operatorname{Im} D$, which is E by Lemma 12.5(1). Thus $K^T B(x, 0) = 0$ and $K^T B(x, z) = K^T B(0, z) = T(z)$, skew by (12.1). Since $SS^T = G(x_*, 0) = D$, we get $\Xi(x)\Xi(x)^T = |x|^2 I$; for square matrices also $\Xi(x)^T \Xi(x) = |x|^2 I$. Polarizing at $x_* + y$ and $y + y'$ gives the relations for Γ_y.

(2) Fix also a nonzero complex $x_0 \in W_{\mathbb{C}}$ with $|x_0|^2 = 0$. As the gradient of $|x|^2$ at x_0 is nonzero, there is an analytic arc $x(t) = x_0 + O(t^2)$ with $|x(t)|^2 = -t^2|z_0|^2$, so $a(t) = (x(t), tz_0) \in V(q,\rho)$ and $G(a(0)) = 0$ by (i). The coefficient of t in $G(a(t))$ is $G_1 = B(x_0,0)B(0,z_0)^T + B(0,z_0)B(x_0,0)^T$. The three-minors of $G(a(t))$ vanish by (ii), so their t^3 coefficients give $\operatorname{rank} G_1 \le 2$. In the splitting $V = E \oplus E^\perp$ the lower diagonal block of G_1 is zero and its off-diagonal block is $B(x_0,0)T(z_0)^T$. A symmetric block matrix whose lower diagonal block is zero has rank at least twice the rank of its off-diagonal block, hence

$$\operatorname{rank}\bigl(\Xi(x_0)T(z_0)^T\bigr) \le 1. \tag{12.4}$$

Suppose $T(z_0) \neq 0$. Then $U = \operatorname{Im} T(z_0)^T$ has positive even dimension r, and by (12.4) $\Xi(x_0)|_U$ has rank at most one for every null x_0. The entries of $\bigwedge^2 \Xi(x)|_{\wedge^2 U}$ are quadratic and vanish on the irreducible quadric $|x|^2 = 0$, so they are multiples of $|x|^2$; evaluation at x_* gives $\bigwedge^2 \Xi(x)|_{\wedge^2 U} = |x|^2 \operatorname{incl}_{\wedge^2 U}$. Differentiating at x_* in a direction $y \perp x_*$ gives $\Gamma_y u \wedge v + u \wedge \Gamma_y v = 0$ for $u, v \in U$, and the components outside U show $\Gamma_y U \subset U$. If $r \ge 3$, the induced action of $\operatorname{End}(U)$ on $\wedge^2 U$ is faithful (a third basis vector detects each off-diagonal entry, and diagonal entries with $d_i + d_j = 0$ for all pairs vanish), so $\Gamma_y|_U = 0$, contrary to $\Gamma_y^2 = -|y|^2 I$. Hence $r = 2$, and the identity says $\operatorname{tr}(\Gamma_y|_U) = 0$.

(3) If $T(z_0) \neq 0$, choose orthonormal $y_1, \dots, y_4 \in W$ orthogonal to x_* and put $\Gamma_i = \Gamma_{y_i}$. By (1) and (2), $\operatorname{tr}\bigl((\Gamma_i|_U)(\Gamma_j|_U)\bigr) = -2\delta_{ij}$, so the four restrictions are linearly independent; but they lie in the three-dimensional space of traceless endomorphisms of U. Hence $T(z_0) = 0$ for every complex null vector of ρ. These span $Z_{\mathbb{C}}$ when ρ is nondegenerate and $\dim Z \ge 2$, so $T = 0$, $K^T B = 0$, and rank eight gives the image. □

13 The rank-seven indefinite case

This section treats the five patterns left open by Theorems C, E and F in which a reduced Jacobi eigenvalue has multiplicity eight, namely $(8,4,3)$, $(8,3,3,1)$, $(8,3,2,2)$, $(8,3,2,1,1)$ and $(8,3,1^4)$, in the case where that eigenvalue is neither the smallest nor the largest.

Proposition 13.1. *Suppose the nonzero Jacobi rank is seven and Λ is indefinite. At a regular anchor the rational matrix F of* (10.5) *has no divisorial pole. Consequently the tensor is Clifford.*

Proof. Here $\nu = 7$ and $k = 9$, so P is 9×7. For $0 \neq a \in \mathbb{C}^7$ let $s(a) = \operatorname{rank} B(a)$ (Section 10); Lemma 10.3 gives $s(a) \ge 5$, and Lemma 10.2 gives $s(a) = 9$ for real $a \neq 0$. Suppose there is a pole, and let Q be an irreducible complex component of its hypersurface in $X_{\mathbb{C}}^{\perp} \cong \mathbb{C}^{15}$, of dimension fourteen (the argument is componentwise, so a quadratic that splits over $\mathbb{C}$ is no exception). The leading Laurent coefficient W is nonzero and skew over its function field, of rank $r \ge 2$. The closure of the incidence

$$\mathcal{I} = \overline{\{(Y,[a]) : Y \text{ generic in } Q,\ [a] \in \mathbb{P}(\operatorname{Im} W(Y))\}} \subset \mathbb{C}^{15} \times \mathbb{P}^6$$

is irreducible of dimension $14 + r - 1 \ge 15$; let $\mathcal{Z}$ be the Zariski closure of its projection to $\mathbb{P}^6$ and $\alpha = \dim \mathcal{Z}$. By $PW = 0$ in (10.9), $P(Y)a = 0$ on $\mathcal{I}$; for fixed $[a]$ this is a linear space in Y of dimension $15 - s(a)$ (its annihilator is $\operatorname{Im} B(a)$). The fibre dimension theorem gives $15 - \alpha \le \dim \mathcal{I} - \alpha \le 15 - s(a)$, so $5 \le s(a) \le \alpha$ at a generic point of $\mathcal{Z}$.

By $WHW^T = 0$, $\mathcal{Z}$ lies in the smooth irreducible five-dimensional quadric $\mathcal{N}_H = \{[a] \in \mathbb{P}^6 : a^T H a = 0\}$, $H = \Lambda^{-1}$. So $\alpha = 5$ and $\mathcal{Z} = \mathcal{N}_H$. The Zariski-closed locus $\{s(a) \le 5\}$ contains a dense subset of $\mathcal{Z}$, hence all of $\mathcal{Z}$; it has no real point, since the real rank is nine. So $\mathcal{N}_H$ has no real point, which contradicts the indefiniteness and nondegeneracy of H. There is therefore no pole, and Lemma 10.1 makes the tensor Clifford. □

14 Exclusion of a simple quadratic pole of residue rank four

We use the rank-eight construction of Section 10 with Λ indefinite, so $H = \Lambda^{-1}$ is a nondegenerate indefinite form on $\mathbb{R}^8$. The lower pencil $P(Y)$ is a generically invertible linear 8×8 matrix, Ω is skew with $P\Omega$ polynomial, and the leading Laurent coefficient W at a pole satisfies $PW = 0$ and $WHW^T = 0$ by (10.9). By Proposition 11.3, a real irreducible quadratic pole factor Q is, up to sign, positive semidefinite of rank $r_Q \geq 2$.

Proposition 14.1. *Let $\nu = 8$ with H indefinite, and let X be a regular real anchor at which Proposition 11.3 applies. A simple pole of Ω along a real irreducible positive semidefinite quadratic Q cannot have skew residue rank four over the fraction field of $\mathbb{R}[Y]/(Q)$.*

Proof. A linear companion. By Lemma 10.4, Q^4 divides $\det P$; both have degree eight, so $\det P = cQ^4$ with $c \neq 0$. At the prime (Q) the four-dimensional residue image lies in $\ker P$, so at least four Smith exponents are positive; their sum is four ($\det P = cQ^4$), so they are four ones and four zeros, and every seven-minor is divisible by Q^3. Hence $P^\sharp = \operatorname{adj} P/(cQ^3)$ is a real linear matrix with $PP^\sharp = P^\sharp P = QI$, and modulo Q it has rank four and $\operatorname{Im} P^\sharp = \ker P = \operatorname{Im} W$. As W is skew, $WHW^T = 0$ says that this image is H-isotropic, so $(P^\sharp)^T H P^\sharp = QH_0$ with H_0 constant. (These arguments take place over the real prime (Q); they also apply when a rank-two Q splits over $\mathbb{C}$.) Choose e with $Q(e) = 1$ and replace $P, P^\sharp$ by $P(e)^{-1}P$, $P^\sharp P(e)$; this does not change H, both matrices are now I at e, and evaluating $(P^\sharp)^T H P^\sharp = QH_0$ at e gives $H_0 = H$:

$$PP^\sharp = P^\sharp P = QI, \qquad (P^\sharp)^T H P^\sharp = QH, \qquad P(e) = P^\sharp(e) = I. \tag{14.1}$$

In coordinates with $Q = x_0^2 + y_1^2 + \cdots + y_{r_Q-1}^2$, e the first basis vector and z_j spanning the radical of Q, (14.1) gives

$$P = x_0 I + \sum_{i=1}^{r_Q-1} y_i C_i + \sum_j z_j D_j, \qquad P^\sharp = x_0 I - \sum_{i=1}^{r_Q-1} y_i C_i - \sum_j z_j D_j,$$

where the C_i are H-skew, square to $-I$ and anticommute, and every $D \in \mathcal{D} = \operatorname{span}_{\mathbb{R}}\{D_j\}$ is H-skew, satisfies $D^2 = 0$, and anticommutes with every C_i and every member of $\mathcal{D}$.

A dimension bound. Let g_X be the unit vector with $K_X g_X = X$ (Section 10). The radial identity (10.8) reads $g_X^T P(Y) = -Y^T M_X$, whose values fill $\mathbb{R}^8$ as Y ranges over $X^\perp$, since M_X is an orthonormal frame. This survives the row change, so the coefficient matrices of P span at least eight dimensions:

$$8 \leq r_Q + \dim \mathcal{D}. \tag{14.2}$$

Signatures. Averaging a positive metric over the signed monomials in the C_i gives an invariant metric, for which the operator representing H commutes with the C_i; its eigenspaces are therefore modules. A single generator makes their dimensions even, two generators make them multiples of four, and four make them multiples of eight ($C_1C_2C_3$ is an involution commuting with C_1, C_2, C_3 and anticommuting with C_4; its two eigenspaces are quaternionic under C_1, C_2 and exchanged by C_4). As H is indefinite on $\mathbb{R}^8$, $r_Q \leq 4$, and up to sign either H has signature $(2, 6)$ and $r_Q = 2$, or H has signature $(4, 4)$ and $r_Q \in \{2, 3, 4\}$.

Antilinear skew maps. Identify $\mathbb{R}^8 = \mathbb{C}^4$ by $J = C_1$. Then H is the real part of a Hermitian form whose signature is half the real signature; in a complex basis its matrix S_H may be chosen real diagonal or real hyperbolic. A map anticommuting with J is $D(v) = \Theta\overline{v}$, and H-skewness reads $S_H\Theta + \Theta^T S_H = 0$, so $S_H\Theta$ is a complex alternating matrix; its rank is even, and the real rank of D

is twice the complex rank of Θ, hence a multiple of four. Since $D^2 = 0$ and D is H-skew, the image of D is totally H-isotropic, so its rank is at most the real Witt index of H.

Signature $(2,6)$. The Witt index is 2, so every $D \in \mathcal{D}$ is zero; $\mathcal{D} = 0$ contradicts (14.2) with $r_Q = 2$.

Signature $(4,4)$, $r_Q = 2$. Let $0 \neq D_0 \in \mathcal{D}$ (if $\mathcal{D} = 0$, (14.2) already fails). By the paragraph on antilinear skew maps, D_0 has real rank four, so $L = \operatorname{Im} D_0 = \ker D_0$ is a J-invariant maximal isotropic subspace. In a hyperbolic basis adapted to L and an isotropic complement, the invertible upper-right block of Θ_0 is alternating, and a change of basis of L (with the dual change on the complement) gives

$$S_H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \quad \Theta_0 = \begin{bmatrix} 0 & J_2 \\ 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Every $D \in \mathcal{D}$ preserves L, by $DD_0 = -D_0 D$; skewness gives $\Theta = \left[\begin{smallmatrix} \Theta_{11} & \Theta_{12} \\ 0 & -\Theta_{11}^T \end{smallmatrix}\right]$ with $\Theta_{12}^T = -\Theta_{12}$, and $DD_0 = -D_0 D$ gives $\Theta_{11} J_2 = J_2 \overline{\Theta}_{11}^T$, whose solutions are $\Theta_{11} = \left[\begin{smallmatrix} a & ib \\ ic & \overline{a} \end{smallmatrix}\right]$ with $a \in \mathbb{C}$, $b, c \in \mathbb{R}$. The block $\Theta_{11}\overline{\Theta}_{11} = (|a|^2 + bc) I_2$ of D^2 vanishes; the form $|a|^2 + bc$ has signature $(3,1)$, so the image of the real linear map $D \mapsto \Theta_{11}$ is a linear subspace of its null cone, hence has dimension at most one, and its kernel, where $\Theta_{11} = 0$ and $\Theta_{12} = f J_2$ with $f \in \mathbb{C}$, has dimension at most two. Thus $r_Q + \dim \mathcal{D} \leq 5$, a contradiction.

Signature $(4,4)$, $r_Q \in \{3,4\}$. Put $J' = C_1 C_2$ and $T = -J'D$ for $D \in \mathcal{D}$. As J' and D are H-skew, D commutes with J', and both anticommute with C_1, C_2, the map T is H-self-adjoint, satisfies $T^2 = 0$ and commutes with C_1, C_2, so it is quaternionic-linear. The form H is the real part of a quaternionic Hermitian form of signature $(1,1)$, and in a quaternionic basis with that form diagonal, $T = \left[\begin{smallmatrix} a & b \\ -\overline{b} & d \end{smallmatrix}\right]$ with $a, d \in \mathbb{R}$, $b \in \mathbb{H}$. Nilpotence gives zero real trace, $4(a+d) = 0$, so $d = -a$ and $T^2 = (a^2 - |b|^2) I_2$. The real linear image of $\mathcal{D}$ lies in the null cone of this form of signature $(1,4)$ and hence has dimension at most one; the map $D \mapsto -J'D$ is injective, so $\dim \mathcal{D} \leq 1$ and $r_Q + \dim \mathcal{D} \leq 5$, a contradiction. □

15 Exclusion of a double quadratic pole

We work in the indefinite case $\nu = 8$ with the notation of Sections 10 and 11: at a regular anchor X, for $Y \perp X$,

$$M_Y = M_X F(Y) + K_X P(Y), \qquad F = F_0 + \Omega H, \qquad \Omega^T = -\Omega, \qquad H = \Lambda^{-1},$$

where F_0, P are linear, P is generically invertible, and $P\Omega$ is polynomial, because the lower–upper Jacobi block $P\Lambda F^T$ is.

Proposition 15.1. *There is no nonempty open set of regular anchors X at which the reduced common denominator of F on $X^\perp$ is Q_X^2 for an irreducible real positive semidefinite quadratic Q_X (in particular, no branch $\mathcal{U}_j$ of Lemma 27.1 with nonempty interior is of this type).*

15.1 A degree bound for an entire skew block

Lemma 15.2. *Let P_0 be a real polynomial $m \times b$ matrix with linear entries and generic column rank b. Let Ω_0 be a skew rational $b \times b$ matrix such that $P_0\Omega_0$ is polynomial. If Ω_0 has pole order e at a real irreducible quadratic Q, then every maximal b-minor of P_0 is divisible by Q^{2e}. In particular, $b \geq 4e$.*

Proof. Over the discrete valuation ring at Q take a Smith form $P_0 = U \left[\begin{smallmatrix} \operatorname{diag}(Q^{a_1},\dots,Q^{a_b}) \\ 0 \end{smallmatrix}\right] V$ with U, V invertible over that ring. The congruent skew matrix $\widehat{\Omega} = V\Omega_0 V^T$ still has pole order e; choose an off-diagonal entry $\widehat{\Omega}_{ij}$ of valuation $-e$. Polynomiality of $P_0\Omega_0$ makes $\operatorname{diag}(Q^{a_1},\dots,Q^{a_b})\,\widehat{\Omega}$ integral; its rows i and j, which contain the entries $\pm\widehat{\Omega}_{ij}$, force $a_i, a_j \geq e$, so every maximal minor has valuation at least $2e$. Some maximal minor is nonzero, of degree at most b, whereas $\deg Q^{2e} = 4e$. $\square$

Under the double-pole hypothesis this lemma, applied to P, Ω, gives by degree eight

$$\det P = c_X Q_X^4, \qquad c_X \neq 0. \tag{15.1}$$

Let g_X be the unit vector with $K_X g_X = X$ (Section 10). The radial row $g_X^T P(Y) = -Y^T M_X$ vanishes for Y in the reduced zero eigenspace $E_0(X) = \ker \mathcal{J}_X \cap X^\perp$, so (15.1) gives $Q_X(Y) = 0$ there, and semidefiniteness puts these vectors in the radical of Q_X. With the shear invariance of the intrinsic denominator (Proposition 11.4), the quadratic pole form annihilates the full kernel of $\mathcal{J}_X$ and has rank at most eight.

15.2 The auxiliary quadratic family

Let $D(X,Y)$ be the intrinsic reduced denominator of Proposition 11.4. Its fibre at a generic anchor of the open set in Proposition 15.1 is Q_X^2, so by Proposition 11.4 it has bidegree $(4,4)$ and no factor in one slot only, so each irreducible factor of D in $\mathbb{R}[X,Y]$ is primitive in Y, and Gauss's lemma identifies the factorization with the one over $\mathbb{R}(X)[Y]$. Distinct irreducible factors are coprime over $\mathbb{R}(X)$ and squarefree over its algebraic closure, the characteristic being zero; after a generic linear change of the Y-coordinates their resultants and discriminants in one coordinate are nonzero polynomials, which stay nonzero off a proper algebraic set of anchors. On the nonempty Zariski-open set of anchors where these resultants and discriminants are nonzero and the fibre of D is the reduced denominator (Proposition 11.4), a set that meets that open set, each factor fibre divides Q_X^2 and is squarefree, hence is proportional to the irreducible quadratic Q_X, and coprimeness leaves one factor, with exponent two. Hence

$$D = c\,\mathcal{Q}(X,Y)^2 \tag{15.2}$$

with $\mathcal{Q}$ real of bidegree $(2,2)$ and $c \neq 0$. The two shear identities of D pass to $\mathcal{Q}$: their squares agree in the polynomial ring with the shear parameter, the domain property gives a common sign, and parameter zero makes it positive. The shears generate $(X,Y) \mapsto (Y,-X)$, and the even bidegrees give $\mathcal{Q}(Y,X) = \mathcal{Q}(X,Y)$, $\mathcal{Q}(X,Y+tX) = \mathcal{Q}(X,Y)$ and $\mathcal{Q}(X,X) = 0$. Write $\mathcal{Q}(X,Y) = \langle A_X Y, Y\rangle$; then A_X is a symmetric quadratic family with $A_X X = 0$.

Anchors at which Proposition 11.3 applies and the fibre of D is D_X are dense. At each of them $c\,\mathcal{Q}(X,\cdot)^2 = D_X$, and among the factor types of Proposition 11.3 the only square is the square of an irreducible quadratic, so $\mathcal{Q}(X,\cdot)$ is an irreducible positive semidefinite quadratic with a double pole and the argument after (15.1) applies at X. Semidefiniteness persists by continuity. Fix one such anchor X_0 and the sign of $\mathcal{Q}$ so that $\mathcal{Q}(X_0,\cdot) \succeq 0$; symmetry gives $\mathcal{Q}(Y,X_0) = \mathcal{Q}(X_0,Y) > 0$ for Y outside its kernel, so $A_Y \succeq 0$ on a dense set and then everywhere. The argument after (15.1) and continuity of the kernel projector give

$$\ker \mathcal{J}_X \subset \ker A_X, \qquad A_X \succeq 0. \tag{15.3}$$

Lemma 15.3. *Let A_z be a symmetric quadratic family with $\operatorname{Im} A_z \subset \operatorname{Im} \mathcal{J}_z$ for all real $z \neq 0$. For the anchor X and, as a rational identity, for $Y \in X^\perp$,*

$$A_{sX+Y} = (sM_X + M_Y)C(sM_X + M_Y)^T, \qquad C = M_X^T A_X M_X,$$

with C constant, $C \succeq 0$ if $A \succeq 0$, and

$$\Omega HC - CH\Omega \quad \textit{polynomial.} \tag{15.4}$$

Proof. For a generic partner Y the matrix $\widehat{M} = [M_X, M_Y]$ is injective, and $A_{sX+tY} = \widehat{M}\mathcal{M}(s,t)\widehat{M}^T$ for a unique symmetric quadratic block matrix $\mathcal{M}$. Its values at the axes give

$$\mathcal{M} = \begin{bmatrix} s^2C + st\Theta_{11} & st\Theta_{12} \\ st\Theta_{12}^T & t^2C_1 + st\Theta_{22} \end{bmatrix}.$$

Since $\operatorname{Im} A_{sX+tY} \subset \operatorname{Im} \widehat{M}[sI, tI]^T$, we get $[-tI, sI]\mathcal{M} = 0$ for generic real s, t, hence identically, and comparing coefficients gives $\Theta_{11} = \Theta_{22} = 0$ and $\Theta_{12} = C_1 = C$. This holds on the generic pair locus and then rationally. The mixed coefficient of A_{sX+Y} makes $FC + CF^T$ polynomial, and $F = F_0 + \Omega H$ gives (15.4). □

By (15.3) the lemma applies here, with $\operatorname{rank} C = \operatorname{rank} Q_X$, since the fibre defining A_X is proportional to Q_X.

15.3 The polar part and total isotropy

Lemma 15.4. *Let $C \succeq 0$ have rank r, put $r' = 8 - r$, and let Ω be skew rational with $P\Omega$ and $\Omega HC - CH\Omega$ polynomial. After subtracting a polynomial skew matrix from Ω,*

$$\Omega HC = CH\Omega, \qquad P\Omega \textit{ polynomial.} \tag{15.5}$$

After a constant change S of coefficient coordinates, $C' = S^{-1}CS^{-T}$, $H' = S^THS$, $\Omega' = S^{-1}\Omega S^{-T}$, $P' = PS$ (this keeps skewness, (15.5), generic injectivity of column subpencils, and the lower output frame), on spaces U_1, U_2, U_3, U_4 of dimensions d, n_0, n_0, ℓ with $d = r - n_0$, $\ell = r' - n_0$, $n_0 \le \min(r, r')$,

$$C = \operatorname{diag}(I_d, I_{n_0}, 0_{n_0}, 0_\ell), \qquad H = \begin{bmatrix} H_0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_0} & 0 \\ 0 & I_{n_0} & 0 & 0 \\ 0 & 0 & 0 & H_1 \end{bmatrix}, \tag{15.6}$$

$$\Omega = \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & B_0 & \Psi & T_0 \\ 0 & \Psi & 0 & 0 \\ 0 & -T_0^T & 0 & V_0 \end{bmatrix}, \qquad ZH_0 = H_0Z, \tag{15.7}$$

with H_0, H_1 nondegenerate, Z, B_0, Ψ, V_0 skew, and empty blocks allowed.

Proof. The constant linear map $\mathcal{L}(\Omega_1) = \Omega_1 HC - CH\Omega_1$ on skew matrices is injective on a constant complement of its kernel, so the component of Ω in that complement is polynomial; subtracting it gives (15.5). Normalize $C = \operatorname{diag}(I_r, 0_{r'})$ and let n_0 be the nullity of H on the first r-space; split that space orthogonally into its nondegenerate part and radical. The radical pairs with the last r'-space with row rank n_0, since H is nondegenerate, so $n_0 \le r'$. Changes $\left[\begin{smallmatrix} O & T \\ 0 & N \end{smallmatrix}\right]$ with O orthogonal and N invertible preserve C: the first d rows of T remove the cross terms of H_0, N normalizes the radical pairing to $[I_{n_0}, 0]$, and the remaining rows of T remove the terms on the paired n_0-space. This gives (15.6), and nondegeneracy of H gives that of H_1; comparing blocks in $\Omega HC = CH\Omega$ gives (15.7); the two copies of Ψ have the same sign because $\Psi^T = -\Psi$. □

Completion of Proposition 15.1. Apply Lemma 15.4 with $r = \operatorname{rank} Q_X$; the subtracted matrix is polynomial, so Ω keeps its double pole. Diagonalize H_0 orthogonally. Since $ZH_0 = H_0Z$ by (15.7), Z splits into skew blocks on the nonzero eigenspaces of H_0, each definite for the indefinite form H and so of dimension at most seven; the skew block on the last three spaces of (15.7) has dimension $r' + n_0 = 8 - d$, also at most seven if $d > 0$. These are exact constant block decompositions, so the corresponding columns of $P\Omega$ are polynomial products of linear, generically injective column subpencils with the blocks, and Lemma 15.2 excludes a double pole in every block of dimension at most seven. If $d > 0$, every block has dimension at most seven, so none carries the double pole of Ω; hence $d = 0$, that is

$$CHC = 0, \tag{15.8}$$

so the range of C is totally H-isotropic and r is at most the Witt index of H; hence $r \le 4$.

Ranks two and three. Since Q_X is real irreducible and semidefinite, $r \ge 2$. With $d = 0$ we have $n_0 = r$, $\ell = 8 - 2r$ and

$$\Omega = \begin{bmatrix} B_0 & \Psi & T_0 \\ \Psi & 0 & 0 \\ -T_0^T & 0 & V_0 \end{bmatrix}.$$

Write $P = [P_2, P_3, P_4]$ by the column blocks on U_2, U_3, U_4; the middle columns of $P\Omega$ are $P_2\Psi$. For $r = 2, 3$ a pole of the skew block Ψ, even a simple one, would need $r \ge 4$ by Lemma 15.2, and all its denominators divide Q_X^2, so Ψ is polynomial. Subtracting it leaves the double pole on the first and last spaces, of total dimension $8 - r < 8$, which the same lemma excludes.

Rank four. Now H has signature $(4, 4)$ and the normalization gives

$$H' = \Lambda' = \begin{bmatrix} 0 & I_4 \\ I_4 & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Omega' = \begin{bmatrix} B_0 & \Psi \\ \Psi & 0 \end{bmatrix}, \quad P' = [P_2, P_3].$$

Polynomiality of the four columns $P_2\Psi$ excludes a double pole of Ψ; if Ψ were polynomial, subtracting it would leave a double pole in a four-dimensional block, again excluded. Thus Ψ has a simple Q_X-pole, so every maximal four-minor of P_2 is divisible by Q_X^2. These minors have degree four, so they are constant multiples of Q_X^2, not all zero; their Plücker vector gives one fixed real four-space T with $\operatorname{Im} P_2(Y) \subset T$, first generically and then for all Y by polynomial continuation.

Choose $0 \ne \eta \in T^\perp$ in the original Euclidean lower output coordinates and put $z = K_X\eta$. Only coefficient coordinates were changed, so $\eta^T P'(Y) = (0, b(Y)^T)$ and $P'\Lambda' P'^T = P\Lambda P^T$. The hyperbolic Λ' gives $\langle \mathcal{J}_{tX+Y} z, z\rangle = \eta^T P'(Y)\Lambda' P'(Y)^T\eta = 0$ for all real t, Y; these are polynomial identities, so no direction is lost at the frame singularities. The symmetry $\langle \mathcal{J}_z V, V\rangle = \langle \mathcal{J}_V z, z\rangle$ of the curvature tensor gives $\langle \mathcal{J}_z V, V\rangle = 0$ for every real V, and polarization gives $\mathcal{J}_z = 0$, contradicting rank eight at $z \ne 0$. So each of the ranks $r = 2, 3, 4$ is excluded, which proves Proposition 15.1. □

16 Closed kernels and a fixed ten-dimensional image

Throughout this section $\mathcal{J}_x$ is the Jacobi family of a shifted Osserman algebraic curvature tensor of rank eight on $V = \mathbb{R}^{16}$, $q(x) = \langle x, x\rangle$, polynomial identities are extended complex linearly, and the distinct nonzero reduced eigenvalues are $\lambda_1 q(x), \dots, \lambda_m q(x)$. For a real anchor $x \ne 0$ put $E_x = \operatorname{Im} \mathcal{J}_x$ and $\mathcal{K}_x = \ker \mathcal{J}_x$. At a regular real unit anchor e (written X in the other sections), in orthonormal image and kernel frames M_e, K_e, the coherent factor at $z = te + Y$, $Y \perp e$, is

$$M_z = \begin{bmatrix} tI + F(Y) \\ P(Y) \end{bmatrix}, \qquad \mathcal{J}_z = M_z \Lambda M_z^T, \tag{16.1}$$

with Λ invertible constant diagonal, F rational, and P a generically invertible linear 8×8 matrix.

Lemma 16.1 (Closed-kernel criterion)**.** *Suppose that, on a nonempty Euclidean-open set of real anchors x,*

$$\mathcal{J}_y z = 0 \qquad \text{for every } y, z \in \mathcal{K}_x. \tag{16.2}$$

Then $\mathcal{J}_x^2 = 0$ for every complex x with $q(x) = 0$, and at every regular real anchor e the full coherent-partner Gram matrix is radial: $M_{te+Y}^T M_{te+Y} = (t^2 + q(Y))I$ as a rational identity.

Proof. Over $\mathbb{C}(Y)$ adjoin α with $\alpha^2 = -q(Y)$; as $q(Y)$ has rank fifteen it is not a square up to sign there. The lower block $P(Y)$ stays invertible, so $M_{\alpha e+Y}$ is injective and its transpose surjective, and, Λ being invertible, $\mathcal{J}_{\alpha e+Y} = M_{\alpha e+Y}\Lambda M_{\alpha e+Y}^T$ has rank eight. Hence the complex null quadric contains a dense set of rank-eight Jacobi operators.

Every nine-minor of $\mathcal{J}_x$ vanishes identically. On the rank-eight locus choose a rational kernel basis using a nonzero eight-minor; substituting linear combinations of it for y, z in (16.2) gives finitely many rational identities in x, which vanish on a nonempty real open set and hence identically. These minor charts cover the generic null locus, so (16.2) holds at its generic complex points.

Fix such a null point x and set $\mathcal{K} = \ker \mathcal{J}_x$, so $\dim \mathcal{K} = 8$ and $\mathcal{J}_x$ is nilpotent by the Osserman characteristic polynomial. If $q|_{\mathcal{K}} \neq 0$, choose $y \in \mathcal{K}$ with $q(y) \neq 0$. Then (16.2), continued above, gives $\mathcal{J}_y\mathcal{K} = 0$, and $\mathcal{J}_y \prod_{j=1}^{m}(\mathcal{J}_y - \lambda_j q(y) I) = 0$ holds over $\mathbb{C}$ by continuation; its roots are distinct because $q(y) \neq 0$, so $\mathcal{J}_y$ is diagonalizable; its characteristic polynomial gives rank eight, whence $\mathcal{K} = \ker \mathcal{J}_y$. The kernel of a self-adjoint diagonalizable operator on a nondegenerate bilinear space is nondegenerate, so $\mathcal{K}$ is nondegenerate, and self-adjointness of $\mathcal{J}_x$ gives $\operatorname{Im} \mathcal{J}_x = \mathcal{K}^\perp$ with $\mathcal{J}_x$ invertible on $\mathcal{K}^\perp$, contradicting nilpotency. Therefore $q|_{\mathcal{K}} = 0$; by polarization $\mathcal{K}$ is totally isotropic, and $\dim \mathcal{K} = 8$ gives $\mathcal{K} = \mathcal{K}^\perp$, so $\operatorname{Im} \mathcal{J}_x = \mathcal{K}^\perp = \ker \mathcal{J}_x$ and $\mathcal{J}_x^2 = 0$. The null quadric is irreducible, so this extends to all its points, including those of smaller Jacobi rank.

Put $\Phi(z) = M_z^T M_z$. At $z = \alpha e + Y$, $0 = \mathcal{J}_z^2 = M_z \Lambda \Phi(z) \Lambda M_z^T$ gives $\Phi(z) = 0$ by left and right inverses. With $\Phi(te + Y) = t^2 I + tS(Y) + \Phi(Y)$ this reads $\alpha S(Y) + \Phi(Y) - q(Y)I = 0$, and independence of $1, \alpha$ over $\mathbb{C}(Y)$ gives $S = 0$ and $\Phi(Y) = q(Y)I$. □

Lemma 16.2. *Let e be a real anchor with transpose pencil $B(a)g = N_g a$ as in* (10.6)*. If $\operatorname{Im} B(a) \subset E_e$ for every real a, then* (16.2) *holds at e.*

Proof. By (10.7), $\mathcal{J}_{K_e g} = N_g \Lambda N_g^T$, and $\operatorname{Im} N_g \subset E_e$ by hypothesis, so $\mathcal{J}_{K_e g}$ annihilates $E_e^\perp = \mathcal{K}_e$ for every g. □

Lemma 16.3 (Fixed ten-dimensional image)**.** *At a regular real anchor e, let $B(a)$ be the transpose pencil and $G(a) = B(a)B(a)^T$, positive semidefinite of rank eight for real $a \neq 0$. Suppose that all images of $B(a)$ lie in a fixed real subspace T with $\dim T \leq 10$, that a possibly reducible projective variety $A \subset \mathbb{P}^7_{\mathbb{C}}$ is a reduced complete intersection $A = V(h, q_0)$ of independent real quadrics with q_0 positive definite, that G has generic rank two on each component of A, and that no nonzero real linear form vanishes on any component. Then $\operatorname{Im} B(a) = E_e$ for every real $a \neq 0$.*

Proof. The radial column $B(a)g_e = -M_e a$ gives $E_e \subset T$, so the lower projection $\Gamma(a) = K_e^T B(a)$ is skew with image in a fixed real space $\mathcal{W}$, $\dim \mathcal{W} \leq 2$. If $\dim \mathcal{W} \leq 1$, then skewness gives $\Gamma = 0$, and rank eight gives the claim. Otherwise $\Gamma(a) = \ell(a)J$ for a fixed rank-two skew J and a real linear form ℓ; if $\ell = 0$, rank eight gives the claim as before, so let $\ell \neq 0$. On $E_e \oplus \mathcal{W}$,

$$G(a) = \begin{bmatrix} A_0(a) & \ell(a)\Theta(a) \\ \ell(a)\Theta(a)^T & \ell(a)^2 D \end{bmatrix}$$

with D constant positive definite of size two and Θ linear (the degrees follow from the linearity of B and $\Gamma = \ell J$). By the component hypothesis $\ell \neq 0$ generically on each component of A, where

the lower block has rank two; rank two of G makes the Schur complement $\Psi = A_0 - \Theta D^{-1}\Theta^T$ vanish there. The degree-two ideal of the reduced complete intersection is spanned by h, q_0, so $\Psi = h\Psi_0 + q_0\Psi_1$ with constant real symmetric Ψ_0, Ψ_1.

At real points with $\ell \neq 0$ the Schur complement is positive semidefinite of rank six; by continuity $\Psi \succeq 0$ of rank at most six everywhere. At real nonzero points of $\ker \ell$ the image of B lies in E_e, so A_0 is positive definite and, as $\Theta D^{-1}\Theta^T$ has rank at most two, Ψ has rank at least six. Thus Ψ has rank six at every real $a \neq 0$. Normalize $q_0 = 1$: the range of h is a nontrivial closed interval, on which $t\Psi_0 + \Psi_1 \succeq 0$ has rank six, so by Lemma 12.1 all $\Psi(a)$ have one image $\mathcal{F} \subset E_e$ of dimension six.

On the seven-dimensional real space $\ker \ell$, compressing $A_0 = \Psi + \Theta D^{-1}\Theta^T$ to the two-dimensional space $\mathcal{F}^\perp \cap E_e$ gives $\Theta_2(a)D^{-1}\Theta_2(a)^T$, positive definite for every nonzero $a \in \ker \ell$. So the real 2×2 matrix $\Theta_2(a)$ is invertible there; but Θ_2 is linear from a seven-dimensional space to a four-dimensional matrix space and has a nonzero kernel. So $\dim \mathcal{W} = 2$ is impossible; hence $\Gamma = 0$, and rank eight proves the claim. □

The lemma is applied in Proposition 25.2, with A the union of the two conjugate residue images of Section 25.

17 Small supports and the residue-image incidence

We use the normalization of Section 10: $\nu \in \{7, 8\}$, $k = 16 - \nu$, and the lower pencil $P(Y) : \mathbb{R}^\nu \to \mathbb{R}^k$ is linear and generically injective for $Y \in X^\perp$, extended by $P(X) = 0$; N_g and $B(a)g = N_g a$ are as in (10.6). By Lemma 10.2, for fixed real coefficient and output subspaces U, T,

$$P(Y)U \subset T \text{ for all } Y, \quad U \neq 0 \quad \Longrightarrow \quad T = \mathbb{R}^k. \tag{17.1}$$

17.1 Four- and five-dimensional quadratic supports

For a simple pole along a real irreducible polynomial f (a quadric or a quartic), let W be its nonzero skew residue over the fraction field of $\mathbb{R}[Y]/(f)$. Its *real support* is the smallest constant real coefficient subspace whose extension to $\mathbb{C}$ or to the residue field contains $\operatorname{Im} W$; when f splits over $\mathbb{C}$, it contains the residue images on both geometric components.

Lemma 17.1. *A simple quadratic pole component with rank-two residue cannot have real support of dimension at most five, whatever other polar factors are present.*

Proof. Let the support lie in a fixed real d-space U. At the prime (Q) the kernel of $P|_U$ contains the rank-two residue image, so the rectangular Smith form has at least two positive valuations and every maximal d-minor of $P|_U$ is divisible by Q^2. Some such minor is nonzero by generic injectivity, and has degree d; thus $d \geq 4$.

If $d = 4$, all maximal minors are constant multiples of Q^2, so their Plücker vector has a constant direction and $P(Y)U$ is one fixed real four-space, first generically, and then, the projection to a complement being polynomial, for every Y. This contradicts (17.1), as $k \geq 8$.

If $d = 5$, the maximal minors give

$$\bigwedge^5 (P(Y)|_U) = Q(Y)^2 \ell(Y) \tag{17.2}$$

with ℓ a nonzero real linear map into $\bigwedge^5 \mathbb{R}^k$. Its generic values are decomposable, and, the Plücker relations being polynomial, so are all its values. Two independent decomposable five-vectors with

decomposable sum span five-spaces meeting in dimension four. Fix two members, with intersection T_0 and sum T_1: a third member meeting both in dimension four contains T_0 or lies in T_1, and a member containing T_0 but not in T_1 meets a member in T_1 not containing T_0 in dimension at most three, whereas any two members meet in dimension four. So the five-spaces are constant, or all contain the four-space T_0, or all lie in the six-space T_1. The first and last contradict (17.1). In the second, $P|_U$ modulo T_0 is a linear family of matrices of rank at most one, generically and then everywhere by its two-minors. By Lemma 12.6 either all $P(Y)U$ lie in a fixed five-space, or a fixed four-space $U_0 \subset U$ has $P(Y)U_0 \subset T_0$ for all Y; both violate (17.1). □

Proposition 17.2 (Isolated skew corrections)**.** *Let $U \subset \mathbb{R}^\nu$ be a fixed real coefficient space of dimension at most five, and write $P_U = P|_U$ in a fixed orthonormal basis of U. If $\Gamma(Y)$ is a rational skew-symmetric endomorphism of U and $P_U(Y)\Gamma(Y)$ is polynomial, then Γ has no pole along an irreducible real quadratic.*

Proof. If Γ had a pole of order $e > 0$ at a quadratic Q, its leading coefficient $W = (Q^e\Gamma) \bmod Q$ would be nonzero and skew, hence of rank at least two, and polynomiality of $P_U\Gamma$ gives $P_U W = 0$ modulo Q. So P_U has corank at least two there and every maximal minor is divisible by Q^2. The degree count, the constant-output case and the Plücker argument for $d = 5$ in the proof of Lemma 17.1 then apply unchanged, each giving a nonzero fixed coefficient subspace mapped by P into a proper fixed output space, contrary to (17.1). □

17.2 A real-irreducible quartic has full real support

Lemma 17.3. *For $\nu = 8$, a simple real-irreducible quartic pole has real residue support of dimension eight, also when it splits into two conjugate complex quadratics.*

Proof. Let the quartic be f. Lemma 10.4 and $\deg\det P = 8$ give $\det P = cf^2$ with $c \neq 0$; the nonzero simple skew residue has positive even rank, and the same budget forces rank two and Smith exponents $(1, 1, 0^6)$ at (f). Every seven-minor is divisible by f, so $P^\sharp = \operatorname{adj} P/(cf)$ is a homogeneous cubic polynomial matrix with $P^\sharp P = PP^\sharp = fI$, and modulo f its image is the two-dimensional kernel of P, which contains the image of the rank-two residue W and therefore equals it. If a constant real row b^T annihilates the real support of W, it annihilates $P^\sharp$ modulo f; the entries of $b^T P^\sharp$ have degree three $< \deg f$, so $b^T P^\sharp = 0$, and multiplying by P gives $fb^T = 0$, so $b = 0$. □

17.3 Dimension and rank of the closed image

From now on $\nu = k = 8$ and $H = \Lambda^{-1}$ is indefinite. Put $h(a) = a^T Ha$ and $G(a) = B(a)B(a)^T$. Fix one irreducible complex pole hypersurface $Z \subset \mathbb{P}(X_{\mathbb{C}}^{\perp}) = \mathbb{P}^{14}$ with generic residue rank two, let $\mathcal{I}$ be the closure of its projectivized residue-image bundle in $Z \times \mathbb{P}^7$, and let A be the image of $\mathcal{I}$ in $\mathbb{P}^7$. Both are irreducible, $\dim \mathcal{I} = 13 + 1 = 14$ (the fibres over Z are projective lines); write $\alpha = \dim A$ and let s be the generic rank of $B(a)$ on A; we say that the pole component has *type* (α, s).

Lemma 17.4. *One has $4 \le s \le \alpha \le 5$. Moreover A has no real point, $A \subset V(h)$, and $\operatorname{rank} G(a) = 2\operatorname{rank} B(a) - 8$ for every nonzero complex a.*

Proof. The rank identity and $s \ge 4$ are Lemma 10.3. The residue identities (10.9) give $A \subset V(h)$, so $\alpha \le 6$, and put every fibre of $\mathcal{I} \to A$ inside $L_a = \{[Y] \in \mathbb{P}^{14} : P(Y)a = 0\}$. The map $Y \mapsto P(Y)a$ has transpose $B(a)$ (as $P(X) = 0$), so generically $\dim L_a = 14 - s$; the generic fibre has dimension $14 - \alpha$, whence $s \le \alpha$. The minors giving $\operatorname{rank} B \le \alpha$ vanish generically on A, hence on all of A; a nonzero real coefficient gives rank eight (Lemma 10.2), so A has no real point. If $\alpha = 6$, A would be the irreducible sixfold $V(h)$, which has real points since h is indefinite. Hence $\alpha \le 5$. □

17.4 Full real support with $\alpha = 5$ and $s = 4$

Proposition 17.5. *Let a simple irreducible real quadratic or quartic pole have total real residue support eight. If a complex residue-image component has $\alpha = 5$ and $s = 4$, then $\operatorname{Im} B(a) = E_X$ for every real $a \neq 0$.*

Proof. The rank identity makes $G = 0$ on the image A. If the real pole polynomial is geometrically irreducible, conjugation preserves its unique component, so A is real-defined, and its span, a real-defined space containing the residue image, is $\mathbb{P}^7$ by total support eight. For such A the bound $\deg A \geq 1 + \operatorname{codim} A$ for nondegenerate varieties [10, p. 1, $(*)$] gives $\deg A \geq 3$, and a generic transverse real complementary plane meets the real-pointless A in conjugate pairs, so $\deg A \geq 4$. If every entry of G is a multiple of h, evaluation at a nonzero real h-null vector contradicts rank eight. Otherwise choose a real entry f independent of h. The complete intersection $V(h, f)$ has degree four and contains A; being Cohen–Macaulay it is unmixed, and since $\deg V(h, f) = 4 \leq \deg A$, A is its only component, with multiplicity one, so (h, f) is generically reduced and unmixed, hence radical, and its degree-two part is $\operatorname{span}(h, f)$. Lemma 12.2 and Remark 12.3 give the claim.

Otherwise the real pole polynomial, being real irreducible, has exactly two conjugate irreducible factors, with images $A, \overline{A}$ spanning $U, \overline{U}$; total real support eight means $U + \overline{U} = \mathbb{C}^8$. Both images are fivefolds on which G vanishes. Put $d = \dim_{\mathbb{C}} U$. If $d \leq 6$, then, A being a fivefold, $d = 6$ and $A = \mathbb{P}(U)$, and $U \cap \overline{U}$ is conjugation-invariant of dimension at least four, so it contains a real point of A, contrary to Lemma 17.4. If $d = 7$, $h|_U$ has rank at least six and its irreducible quadric fivefold is A; the spans are distinct hyperplanes $\ell = 0$, $\overline{\ell} = 0$, and the two quadrics exhaust the degree-four complete intersection $V(h, \ell\overline{\ell})$, their reduced union by unmixedness, whose quadratic ideal has scalar span two; Lemma 12.2 and Remark 12.3 apply again. If $d = 8$ and $A = \overline{A}$, the first paragraph applies; if $d = 8$ and they are distinct, each has degree at least three [10, p. 1, $(*)$], so an entry f independent of h would put a union of degree at least six inside $V(h, f)$ of degree four, and all entries are multiples of h, already excluded. □

On a nonempty real open set of anchors this gives (16.2) by Lemma 16.2, so Lemma 16.1 applies, and Proposition 9.1(2) makes R Clifford (Section 27).

17.5 Saturated fibres

Lemma 17.6. *For the incidence above, suppose $\alpha = s$. For generic $a \in A$ the fibre $\mathcal{I}_a$ is the whole projective linear space L_a. Consequently two distinct complex pole hypersurfaces with this equality cannot have the same residue image.*

Proof. The closed fibre $\mathcal{I}_a \subset L_a$ has dimension $14 - \alpha = 14 - s = \dim L_a$ at a generic point, and a proper closed subset of an irreducible projective linear space has smaller dimension. If two pole incidences had the same image and satisfied the equality, their fibres would agree over a common dense open subset of A, so their irreducible closures, and their projections to $\mathbb{P}^{14}$, would agree, and the two pole hypersurfaces would coincide. □

18 Aligned quadratic families and two quadratic poles

We work at Jacobi rank eight and, as in the proof of Theorem A in Section 27, assume throughout this section and Section 19 that R is not Clifford. We consider the eleven patterns with a reduced eigenvalue of multiplicity seven that Theorems C, E and F leave open, namely $(7, 5, 3)$, $(7, 4, 4)$, $(7, 4, 3, 1)$, $(7, 4, 2, 2)$, $(7, 4, 2, 1, 1)$, $(7, 4, 1^4)$, $(7, 3, 3, 2)$, $(7, 3, 3, 1, 1)$, $(7, 3, 2, 2, 1)$, $(7, 3, 2, 1^3)$ and

$(7,3,1^5)$, where Λ is not a multiple of the identity. At a regular unit anchor X, let $E = \operatorname{Im} \mathcal{J}_X$, $\mathcal{K} = \ker \mathcal{J}_X$, with orthonormal frames M_X, K_X; write $Y = M_X v + u \in X^\perp$ with $u \in \mathcal{K} \cap X^\perp$, and $M_Y = M_X F(Y) + K_X P(Y)$, $H = \Lambda^{-1}$. The lower pencil P is linear and generically invertible with radial row $-v^T$; the transpose pencil $B(a)$, $B(a)^T Y = P(Y)a$, is as in Section 12. We use the cross matrix $T(X,Y)$ and its reduced denominator $D(X,Y)$ of Proposition 11.4: D is bihomogeneous, shear invariant and has no factor in one slot only, and if D is quadratic in each slot, it is a nonnegative symmetric biquadratic sectional form $D(X,Y) = \langle A_X Y, Y\rangle$ of an auxiliary curvature family A. Branch arguments take place on a nonempty real open anchor region as in Lemma 27.1; a branch is *sole-quadratic* if its reduced denominator is one real quadric Q and *two-quadratic* if it is $Q_1 Q_2$. Besides an aligned transpose pencil (Section 10), we call a semidefinite quadric $Q(X,\cdot)$ *aligned* if it vanishes on $\ker \mathcal{J}_X$, and a quadratic family A *aligned* if $\operatorname{Im} A_X \subset E_X$.

18.1 A single quadratic denominator

The partners Y of a fixed real anchor X with $\operatorname{Im} \mathcal{J}_X \cap \operatorname{Im} \mathcal{J}_Y = 0$ (disjoint pairs) form a nonempty open set: the proof of [23, Lemma 2(1), p. 7 of arXiv:math/0204258v2] fixes an arbitrary unit X before proving openness and density in Y; and T is regular near such a pair.

Lemma 18.1. *Suppose the intrinsic denominator is quadratic in each slot. Fix a regular unit X and put $Q(Y) = D(X,Y)$. Suppose Q is unchanged by translations in a real subspace $U \subset X^\perp$ and the full coherent factor is affine in those directions. Then $A_V = c(V) A_X$ for $V \in \mathbb{R}X + U$, where c is a positive definite quadratic form with $c(X) = 1$.*

Proof. First, no nonzero real V has $D(V,Y) \equiv 0$. Otherwise nonnegativity in the first slot gives $D(V + tZ, Y) = t^2 D(Z,Y)$; with the polynomial matrix $\widetilde{T} = DT$, at a disjoint pair (V,Y) the coefficient of t^2 in $\widetilde{T}(V + tZ, Y)$ is $D(Z,Y)T(V,Y)$. An irreducible factor $p(Y)$ of a reduced denominator of an entry of $T(V,Y)$ then divides $D(Z,Y)$ for Z in a Zariski-dense set; reducing D modulo p over the domain $\mathbb{R}[Y]/(p)$ gives $p \mid D$ in $\mathbb{R}[Z,Y]$, a factor in one slot only, excluded by Proposition 11.4. So $T(V,Y)$ is polynomial and homogeneous linear in Y. Write $T(V,Y) = M'_Y \Lambda M_V^T$ with a fixed real factor $\mathcal{J}_V = M_V \Lambda M_V^T$, as in the proof of Proposition 11.4. Then $M'_Y = T(V,Y) M_V (M_V^T M_V)^{-1} \Lambda^{-1}$ is linear in Y, and $\mathcal{J}_Y = M'_Y \Lambda M_Y'^T$ on a dense set of Y, hence for all Y; M'_Y is injective for real $Y \neq 0$ because $\mathcal{J}_Y$ has rank eight. So Proposition 9.1(1) makes the tensor Clifford, a contradiction.

For $V \in \mathbb{R}X + U$, affineness gives $M_{Y+tV} = M_Y + t C_V(Y)$ with C_V independent of t, invariant under translations in $\mathbb{R}X + U$, and rational with poles only on Q. The quadratic coefficient gives $C_V \Lambda C_V^T = \mathcal{J}_V$. For a fixed real factor M_V^0 of $\mathcal{J}_V$ write $C_V(Y) = M_V^0 R_V(Y)$ with $R_V \Lambda R_V^T = \Lambda$; both R_V and $R_V^{-1} = \Lambda R_V^T \Lambda^{-1}$ have poles only on Q, and $R_V(Y + tV) = R_V(Y)$. So $M'_Y = M_Y R_V(Y)^{-1}$ satisfies $M'_{Y+tV} = M'_Y + t M_V^0$ and $M'_Y \Lambda M_Y'^T = \mathcal{J}_Y$, and uniqueness in Proposition 11.4 gives $T(V,Y) = M'_Y \Lambda M_V^{0T}$. The factor M' is coherent at V with poles only on Q, and it has a pole: otherwise M'_Y, homogeneous of degree one without poles, would be linear in Y, a global linear factor of $\mathcal{J}$ injective at every real $Y \neq 0$, and Proposition 9.1(1) would make the tensor Clifford. Its denominator divides the nonzero quadratic $D(V,Y)$, so $D(V,Y) = c(V)Q(Y)$; nonnegativity gives $c(V) > 0$ for $V \neq 0$, and evaluation at a Y_0 with $Q(Y_0) > 0$ shows that c is quadratic. □

Proposition 18.2. *Suppose the intrinsic denominator is one quadric with a simple pole, $D(X,Y) = \langle A_X Y, Y\rangle$. Then $A \succeq 0$ with $\operatorname{Im} A_X = E_X$ cannot hold at every anchor X of a nonempty real open region.*

Proof. Fix a generic regular unit X in the region, so that $D(X,\cdot)$ is the reduced denominator. Image inclusion holds for all z by continuation of the spectral projector identities, so Lemma 15.3 gives

$A_z = M_z C M_z^T$ with $C > 0$ and $Q(Y) = v^T C v$. The inequality $\lambda_{\min}(C)\|M_z\|_F^2 \le \operatorname{tr} A_z$ bounds the partner on the regular real unit sphere. Write $F(v,u) = n(v,u)/Q(v)$ with n homogeneous cubic; under $v \mapsto tv$ its terms of bidegrees $(0,3)$ and $(1,2)$ would give poles of orders two and one, so boundedness removes them, and F is affine in all seven u directions.

Lemma 18.1 gives $A_V = c(V)A_X$ on $\mathcal{K}$. On the regular locus, Cauchy–Binet and the scaled spectrum give $e_8(A_z) = \det C \det(M_z^T M_z) = \det C\, \|z\|^{16}$, where e_8 is the sum of principal eight-minors; this polynomial identity holds everywhere, and on $\mathcal{K}$ it gives $c(V)^8 = \|V\|^{16}$, so, c being positive, $c(V) = \|V\|^2$. The symmetry $\langle A_V Y, Y\rangle = \langle A_Y V, V\rangle$ of the sectional form yields $P(Y)CP(Y)^T = (v^T C v)I$. At $v = 0$ positivity of C gives $P = 0$, so $P = P(v)$, and $P(v)^T P(v) = (v^T C v)C^{-1}$ for $v \neq 0$. At $u = 0$ the Gram determinant identity and positivity give

$$\|v\|^{16} = \det\big(F(v)^T F(v) + (v^T C v)C^{-1}\big) \ge \frac{(v^T C v)^8}{\det C},$$

so the largest eigenvalue of C is at most their geometric mean, $C = \gamma I$, and equality gives $F(v) = 0$. Affineness writes $F(v,u) = L_u(v)$ with L linear in u and $L_u(tv) = L_u(v)$. At $xX + u + tM_X v$ the upper frame block is $xI + L_u(v)$ and the lower block is $tP(v)$; the polynomial limit of $A = \gamma M M^T$ as $t \to 0$ gives $(xI + L_u(v))(xI + L_u(v))^T = (x^2 + \|u\|^2)I$, and the transpose product is the same. Adding the lower Gram gives the full radial Gram, and Proposition 9.1(2) makes R Clifford, a contradiction. □

18.2 Two quadratic poles and semidefinite families

Proposition 18.3. *Suppose the denominator at rank eight is Q_1Q_2, with distinct real irreducible semidefinite quadrics. There is no nonzero quadratic curvature family $A \succeq 0$ with $\operatorname{Im} A_z \subset \operatorname{Im} \mathcal{J}_z$ for every real $z \neq 0$.*

Proof. Lemma 15.3 gives $A_z = M_z C M_z^T$ with $C \succeq 0$ of rank $r \ge 1$ and $\Omega HC - CH\Omega$ polynomial, so Lemma 15.4 applies: with $r' = 8 - r$ and blocks U_1, U_2, U_3, U_4 of dimensions $r - n_0, n_0, n_0, r' - n_0$, $n_0 \le \min(r, r') \le 4$, the polar part has the form (15.7). Write P_2 for the columns of the normalized P on U_2; the third block column of $P\Omega$ is $P_2\Psi$. This skew block has no pole along a real irreducible quadratic. Indeed, if S_0 is the constant, possibly nonorthogonal, normalizing change of coefficients, then $P_2(Y) = P(Y)S_0|_{U_2}$, so P_2 is P restricted to the fixed real coefficient space $U' = S_0U_2$, of dimension $n_0 \le 4$. At such a pole the leading Laurent coefficient of Ψ is nonzero and skew, hence of rank at least two, and P_2 annihilates it modulo the quadratic; so $P|_{U'}$ has corank at least two there, which the proof of Proposition 17.2 excludes: for $n_0 \le 3$ by the degree count, and for $n_0 = 4$ because $P(Y)U'$ would be one fixed four-space, contrary to (17.1). Subtract the polynomial block. The remaining polar blocks are Z and the skew block on $U_2 + U_4$, of dimension r'. Diagonalizing H_0 splits Z into blocks on its nonzero eigenspaces, each definite for the indefinite H and so of dimension at most seven; the last block also has dimension at most seven.

These blocks have disjoint supports of total dimension $(r - n_0) + r' \le 8$, and each product with its column subpencil of P is polynomial. Proposition 17.2 therefore permits at most one pole-bearing block, of size six or seven, and both Q_1, Q_2 occur in it. At each prime, skewness requires at least two positive Smith exponents of that generically injective linear subpencil (Lemma 15.2 with $e = 1$), so $Q_1^2Q_2^2$ divides every maximal minor, of degree at most seven, while $Q_1^2Q_2^2$ has degree eight and some maximal minor is nonzero: a contradiction. □

18.3 Descent and alignment of the two components

Work over the field $\mathbb{R}(X)$. Let $\mathcal{C}$ be a set of geometric factors of D, that is, irreducible factors over an algebraic closure of $\mathbb{R}(X)$, of total Y-degree two, that is invariant under the Galois group.

Their product, normalized so that a chosen nonzero coefficient equals 1, is Galois-invariant and so has coefficients in $\mathbb{R}(X)$; clearing denominators and dividing by the content in $\mathbb{R}[X]$ gives, by Gauss's lemma, a real polynomial factor $\mathcal{Q}(X,Y)$ of D of Y-degree two. Every irreducible factor of the bihomogeneous D is bihomogeneous. Under $Y \mapsto Y + tX$ a factor maps to a factor of D with the same highest Y-degree part; being bihomogeneous, the image factor equals that part, so the factor is invariant, and likewise in the other slot. The shears generate $(X,Y) \mapsto (tX, t^{-1}Y)$, forcing equal degrees, and $(X,Y) \mapsto (Y,-X)$, giving symmetry in bidegree $(2,2)$; this extends to descended products, factor by factor. At a generic real anchor $\mathcal{Q}(X,\cdot)$ is a real quadratic factor of D_X, which is Q^2 or Q_1Q_2 by Proposition 11.3, so $\mathcal{Q}(X,\cdot)$ is proportional to a semidefinite quadric; semidefiniteness extends to all fibres by continuity. If $\mathcal{Q}(X_0,\cdot) \geq 0$ and $\mathcal{Q}(X_0,Y) > 0$, then $\mathcal{Q}(Y,X_0) = \mathcal{Q}(X_0,Y) > 0$ by symmetry, so the semidefinite fibre $\mathcal{Q}(Y,\cdot)$ is nonnegative; such Y are dense. So $\mathcal{Q}$ is the sectional form of a nonnegative curvature family, which is aligned if its factors are aligned at generic anchors. As in Remark 27.2, algebraic invariants of geometric factors (ranks, degrees, spans) persist under generic specialization.

Proposition 18.4. *On a two-quadratic branch, both quadrics are aligned, the intrinsic denominator D is irreducible over $\mathbb{R}[X,Y]$, and its geometric factors have equal algebraic invariants. The case in which both quadrics have rank eight is excluded.*

Proof. By Lemma 10.4, $Q_1^2Q_2^2$ divides $\det P$, and both have degree eight, so $\det P = cQ_1^2Q_2^2$. At $v = 0$ the radial row vanishes, so $Q_1(0,u)Q_2(0,u) = 0$ identically. One of the two restrictions is identically zero, and semidefiniteness puts that real subspace in its radical, so at least one quadric is aligned. If exactly one is aligned, the geometric factors vanishing on $\ker \mathcal{J}_X$ form a rational, Galois-invariant collection of total degree two (a rank-two real semidefinite quadric contributes both conjugate linear factors or neither, since a complex linear form vanishes on a real subspace exactly when its conjugate does), and the descent gives a nonzero nonnegative aligned family, contrary to Proposition 18.3. So both are aligned.

A proper real factor of D would specialize to a divisor of Q_1Q_2 of degree two (factors in one slot only are excluded), and the same shear and sign argument would give an aligned family excluded by Proposition 18.3. So D is irreducible over $\mathbb{R}[X,Y]$, its primitive fibre is irreducible over $\mathbb{R}(X)[Y]$, and its geometric factors form one Galois orbit, whose degrees, ranks, support dimensions and incidence invariants are equal; by Remark 27.2 these equalities persist at generic anchors of the branch. In particular a geometric degree list $2,1,1$ is impossible, and so is a geometrically irreducible quartic, since absolute irreducibility persists on a nonempty Zariski-open set of specializations meeting the branch, where $D_X = Q_1Q_2$. This leaves two quadratics or four lines.

Finally, if both aligned quadrics have rank eight, then $\det P(v,u) = cQ_1(v)^2Q_2(v)^2 \neq 0$ for real $v \neq 0$. For a fixed nonzero real a the map $v \mapsto P(v,0)a$ is an isomorphism of eight-spaces; if $P(0,u)a \neq 0$, the vector $v \neq 0$ with $P(v,0)a = -P(0,u)a$ gives $P(v,u)a = 0$, contradicting the determinant identity. So $P(0,u) = 0$, the full transpose image is E_X, and on the real open branch Lemmas 16.2 and 16.1 and Proposition 9.1(2) make R Clifford, contrary to our assumption. □

19 Quadratic poles with proper coefficient support

Retain the notation of Section 18, including sole-quadratic and two-quadratic branches. For a simple real irreducible quadratic pole Q, let $U \subset \mathbb{R}^8$ be the real support (Section 17) of its skew residue, $d = \dim U$. The residue has rank two: it is skew, so its rank is even; Q raised to that rank divides $\det P$, of degree eight (Lemma 10.4), so the rank is at most four; and rank four is excluded by Proposition 14.1. For each geometric pole component let A now denote the closed projective residue

image, $\alpha = \dim A$, $s(a) = \operatorname{rank} B(a)$, and $G(a) = B(a)B(a)^T$. Lemmas 17.4 and 10.3 give

$$4 \le s(a) \le \alpha \le 5 \quad (a \in A), \qquad A \subset V(a^T H a), \qquad \operatorname{rank} G(a) = 2s(a) - 8,$$

and A has no real points. Supports $d \le 5$ are excluded by Lemma 17.1; this section treats $d = 6, 7$, including rank-two real quadrics that split over $\mathbb{C}$.

19.1 Six-dimensional support

Proposition 19.1. *If $d = 6$, then $H|_U$ is definite, and with $h = \pm H|_U$ chosen positive definite one has $G(a) = h(a)D$ on U with $D \succeq 0$ of rank eight and image E. The pole quadric Q is proportional to $Y^T DY$, has rank eight and has kernel $\mathcal{K} \cap X^\perp$. Two distinct quadratic poles cannot both have support dimension six.*

Proof. $H|_U \neq 0$, since a nondegenerate eight-form has no totally isotropic six-space, so its projective null locus in $\mathbb{P}(U_\mathbb{C})$ has dimension four and each residue image fills a component. If the restricted rank is at least three this quadric is irreducible, and an indefinite or degenerate real restriction would give a real point; at ranks one or two every hyperplane component contains the nonzero real radical, contrary to the absence of real points in either case. So $H|_U$ is definite of rank six. The image is its whole null quadric, $\alpha = s = 4$, so $G = 0$ there, and, each entry vanishing on the irreducible quadric $h = 0$, quadratic divisibility gives $G = hD$; evaluation at a real $a \neq 0$ makes D positive semidefinite of rank eight. All $B(a)$, $a \in U \setminus 0$, have the image $\operatorname{Im} D$, which is E by Lemma 12.5(1).

Contraction and polarization give $(P(Y)|_U)^T (P(Y)|_U) = (Y^T DY)H_0$ with $H_0 > 0$ representing h. Modulo Q the restricted pencil has corank at least two, so its Gram determinant vanishes and $Q \mid (Y^T DY)^6$; equal degrees make Q proportional to $Y^T DY$, and $\operatorname{Im} D = E$ gives its rank and kernel. For two six-spaces, evaluating $G = h_i D_i$ at a real nonzero vector of their intersection (two six-spaces in $\mathbb{R}^8$ meet in dimension at least four) makes D_1, D_2, and so the two quadrics, proportional; proportional quadrics define the same pole. □

Proposition 19.2. *No sole-quadratic or two-quadratic branch has a component of support dimension six.*

Proof. For a sole quadratic, Proposition 19.1 gives an aligned auxiliary family of rank eight on the branch, excluded by Proposition 18.2. For two quadrics there is at most one six-support component; it has rank eight, so it is geometrically irreducible. Its support dimension is intrinsic (multiply the frame-free cross matrix by the pole factor, reduce at its generic point and take the constant span of the image, which equals the coefficient support dimension in any eigenframe), so the Galois automorphisms permuting the geometric factors over $\mathbb{R}(X)$ (Remark 27.2) preserve it and the degree. A nontrivial conjugate of this factor would be the other quadric, also of support six, which is impossible; so the factor descends over $\mathbb{R}(X)$, and primitive descent and the shear argument of Section 18 give a nonzero nonnegative aligned family, excluded by Proposition 18.3. □

19.2 Seven-dimensional support with transpose rank four

Lemma 19.3. *Suppose $d = 7$ and the generic transpose rank on the residue image components is four. Then the quadratic entries of $G|_U$ span at most two scalar forms, $\operatorname{Im} B(a) = E$ for every real nonzero $a \in U$, and Q has rank eight and kernel $\mathcal{K} \cap X^\perp$.*

Proof. Put $h = H|_U$, of rank at least six; $G = 0$ on the residue image. If Q is geometrically irreducible, its image is real, irreducible and spans $\mathbb{P}(U_{\mathbb{C}}) = \mathbb{P}^6$. If $\alpha = 5$ it is the whole h-quadric and every entry of G is a multiple of h. If $\alpha = 4$, its degree is at least three [10, p. 1, $(*)$] and even, since a generic real complementary plane meets it in conjugate pairs; an entry f of G independent of h puts it in the degree-four complete intersection (h, f), so its degree is four and it exhausts that intersection with its scheme structure (unmixed, generically reduced), so all entries lie in $\operatorname{span}\{h, f\}$. If Q has rank two and both components have the same image, that image is real-defined and spans $\mathbb{P}(U_{\mathbb{C}})$, as before. Otherwise let d_1 be the complex span dimension of one of the two conjugate images. $d_1 \le 5$ is impossible: by dimension the image would be linear, and its intersection with its conjugate would contain real vectors, that is, real points. If $d_1 = 6$, the image is the whole irreducible quadric in its hyperplane, the conjugate hyperplanes are distinct, and their union is the reduced complete intersection $(h, \ell\bar{\ell})$, so all entries lie in $\operatorname{span}\{h, \ell\bar{\ell}\}$. If $d_1 = 7$, dimension five gives the whole h-quadric; in dimension four a real image gives $\operatorname{span}\{h, f\}$ by the parity argument above, while two distinct nondegenerate conjugate images have total degree at least six, too much for $V(h, f)$, so then no entry is independent of h. In all cases the entries span at most two forms.

By Lemma 12.2 (two forms, $n = 7$) all $G(a)$, $a \in U \setminus 0$, have one image, which is E by Lemma 12.5(1). So $P(Y)|_U$ depends only on v and is injective for real $v \neq 0$: $B(a)^T M_X v = P(v)a \neq 0$. Every maximal seven-minor is divisible by Q and some is nonzero, and it depends only on v, so Q does too; a nonzero real vector in a smaller-rank semidefinite $Q(v)$ would contradict injectivity. Hence $\operatorname{rank} Q = 8$ with kernel $\mathcal{K} \cap X^{\perp}$; in particular Q does not have rank two. □

Lemma 19.4. *In the preceding situation $G|_U = h_0 D_0$ for a positive definite quadratic h_0 and a fixed rank-eight $D_0 \succeq 0$, and $(P|_U)^T(P|_U) = QH_1$ after a constant rescaling, with H_1 nondegenerate. Such a component cannot coexist with a distinct quadratic pole.*

Proof. Identify E with the coefficient space $\mathbb{R}^8$ through the orthonormal frame M_X, and put $B_E(a) = M_X^T B(a)$ and $G_E(a) = B_E(a)B_E(a)^T$; for $a \in U$ the image of $B(a)$ is E (proof of Lemma 19.3), so $G(a) = M_X G_E(a) M_X^T$. The square matrix $B_E(a)$ is invertible at real nonzero $a \in U$ and $B_E(a)g_X = -a$ with $\|g_X\| = 1$, so $a^T G_E(a)^{-1} a = 1$. If the scalar span is two-dimensional, the pencil G_E contains the positive definite $G_E(a_0)$, and simultaneous congruence gives $G_E(a) = S \operatorname{diag}(f_j(a) I_{b_j}) S^T$, $\sum b_j = 8$, with pairwise nonproportional forms f_j of the span, each positive definite on the seven-space, hence irreducible and pairwise nonassociate. The inverse identity reads $\sum_j n_j(a)/f_j(a) = 1$ with $n_j = \|\mathrm{pr}_j\, S^{-1}a\|^2$; clearing denominators and reducing modulo f_j gives $f_j \mid n_j$. A nonzero n_j has rank seven but a sum-of-squares presentation of rank at most b_j, and some $n_j \neq 0$, so the blocks have sizes seven and one, and $(\det B_E)^2 = (\det S)^2 f_1^7 f_2$, for the forms f_1, f_2 of the blocks of sizes seven and one, has odd valuation, a contradiction. So the span is one form, contraction gives $(P|_U)^T(P|_U) = (Y^T D_0 Y)H_1$, and its determinant modulo Q makes Q proportional to $Y^T D_0 Y$. Modulo Q the image of the eight-by-seven $P|_U$ is totally isotropic in the standard eight-space, of dimension at most four; with the last column, $\operatorname{rank} P \le 5$ there, so $Q^3 \mid \det P$. A distinct quadratic pole Q' needs $(Q')^2 \mid \det P$ (Lemma 10.4), and $Q^3(Q')^2$ has degree ten $> \deg \det P = 8$. □

Proposition 19.5. *A two-quadratic branch has no component of support dimension seven.*

Proof. Write $\det P = cQ_1^2 Q_2^2$. The Smith exponents at either prime are $(1, 1, 0^6)$, so every seven-minor is divisible by both and $P^\sharp = \operatorname{adj} P/(cQ_1Q_2)$ is cubic with $P^\sharp P = PP^\sharp = Q_1Q_2 I$. If the first residue lies in $U = \ker f$, then $fP^\sharp = Q_1 \ell(Y)^T$ with ℓ linear, since the image of $P^\sharp$ modulo Q_1 is the residue kernel plane; multiplying by P gives $P(Y)^T \ell(Y) = Q_2(Y) f^T$. Let $\Theta : Y \mapsto \ell(Y)$ and, for real nonzero $a \in U$, $P_a Y = P(Y)a$. By the last identity, as $fa = 0$, $P_a^T \Theta$ is skew, and

surjectivity of P_a gives it kernel $\ker\Theta$ and even rank $m = \operatorname{rank}\Theta$, with the same image for every such a. For nonzero real $g \in \operatorname{Im}\Theta$ the vectors $N_g a$, $a \in U$, lie in that image, and N_g is injective, so $m \geq 7$, hence $m = 8$ and Θ is surjective. Over $\mathbb{C}$, $\operatorname{rank}(P_a^T\Theta) = s(a)$ is then even for every $a \in U_{\mathbb{C}}$, so, as $4 \leq s \leq 5$ (Lemma 17.4), $s = 4$ on each residue image, and Lemma 19.4 gives the degree contradiction. This covers conjugate geometric factors of a rank-two quadric as well. □

19.3 Seven-dimensional support with transpose rank five

Here $\alpha = s = 5$. As U is a hyperplane, $H|_U$ has rank at least six and its null quadric in $\mathbb{P}^6$ is irreducible and filled by the residue image; absence of real points makes $H|_U$ definite of rank seven. Reversing the curvature sign if necessary, $h = H|_U > 0$ and H has signature $(7, 1)$.

Lemma 19.6. *In this rank-five case, either* $\operatorname{Im} B(a) = E$ *for all real* $a \in U \setminus 0$*, or*

$$G = hD + uu^T - vv^T \tag{19.1}$$

on U, with D constant symmetric and u, v real linear columns.

Proof. The cone ring $S = \mathbb{C}[U]/(h)$ is factorial by the Klein–Nagata theorem [29, Theorem 8.2, p. 26], h being nondegenerate in seven variables. By Lemma 12.7 the induced pairing of G is hyperbolic over the real function field, and the saturated isotropic line modules are rank-one reflexive, hence principal. So Lemma 12.8 applies over $\mathbb{R}[U]/(h)$; lifting its columns to real polynomials gives (19.1) or $G = hD + cb^T + bc^T$ with $c \neq 0$ constant and b quadratic. In the second case, compression to $c^\perp$ is hD_0, and positivity puts the cross block in $\operatorname{Im} D_0$, so all images lie in a fixed space of dimension at most nine and Lemma 12.5(2) gives the first alternative. □

Lemma 19.7. *If the first alternative of Lemma 19.6 fails, then $D \succeq 0$ in* (19.1) *has rank exactly eight. A constant hyperbolic change of the two columns gives*

$$G = hD - ww^T + zz^T, \qquad w \in \mathcal{F} := \operatorname{Im} D, \qquad w^T D^+ w = h, \tag{19.2}$$

where the projection of z to $\mathcal{F}^\perp$ has rank at least two.

Proof. For a fixed real vector $b \in V$ choose nonzero $a \in U$ with $b^T u(a) = b^T v(a) = 0$; positivity gives $b^T D b \geq 0$. Put $r = \operatorname{rank} D$ and $G_+ = hD + uu^T$. Positivity of $G = G_+ - vv^T$ gives $v \in \operatorname{Im} G_+$, and rank eight gives $\operatorname{rank} G_+ \in \{8, 9\}$, so $7 \leq r \leq 9$. If $r = 9$, u, v lie in $\mathcal{F}$ and Lemma 12.5(2) aligns the image.

Suppose $r = 7$. The quotient map $\bar{u} : U \to V/\mathcal{F}$ is injective and $\operatorname{Im} B(a) = \mathcal{F} + \mathbb{R}u(a)$ for real $a \neq 0$, a space containing the radial vector $M_X a$. Two pointwise proportional linear maps, one of rank at least two, have constant ratio (compare two independent images and their sum), so $\overline{M_X a} = c\,\bar{u}(a)$ for a constant c. If $c = 0$, then $\mathcal{F} = M_X U \subset E$, each lower image has dimension at most one and, being skew, it is zero; then the injective $\bar{u}(U)$ would lie in the one-dimensional $E/\mathcal{F}$, which is impossible. If $c \neq 0$, the lower skew pencil has one fixed image $K_X^T\mathcal{F}$ at all nonzero real coefficients, of even rank $0, 2, 4$ or 6. Ranks two and four are impossible, as their skew-operator spaces have dimensions one and six, less than seven; rank six would give a cubic Pfaffian, an odd function, nowhere zero on the connected real unit sphere, which is impossible; rank zero puts $\mathcal{F} \subset E$, contradicting the rank of the quotient radial map. So $r \neq 7$.

Thus $r = 8$. If $\operatorname{rank}\bar{u} \leq 1$, all images lie in a fixed nine-space and Lemma 12.5(2) aligns them. Otherwise positivity gives $\bar{v} = c\bar{u}$ with a constant c, so $v = cu + w_0$ with $w_0 \in \mathcal{F}$, and where $\bar{u} \neq 0$ the rank drop from nine to eight reads $c^2 + h^{-1} w_0^T D^+ w_0 = 1$: write $G_+ = M_+ M_+^T$ with

$M_+ = [\sqrt{h}\,S,\ u]$, where $SS^T = D$ and S maps $\mathbb{R}^8$ isomorphically onto $\mathcal{F}$; M_+ is injective since $u \notin \mathcal{F}$, $v = M_+\xi$ with $\xi = (S^{-1}w_0/\sqrt{h},\ c)$, and $G_+ - vv^T = M_+(I - \xi\xi^T)M_+^T$ has rank eight exactly when $\|\xi\|^2 = 1$. Multiplied by h, this is a polynomial identity valid everywhere. As $h > 0$ on $U \setminus 0$ and $D^+ \succeq 0$ on $\mathcal{F}$, $c^2 \le 1$. If $|c| = 1$, then $w_0 = 0$ and the image is fixed and aligned. Otherwise put $\delta = 1 - c^2 > 0$, $w = w_0/\sqrt{\delta}$, $z = \sqrt{\delta}\,u - cw$; expanding gives (19.2), and the quotient column keeps rank at least two. □

Proposition 19.8. *If the first alternative of Lemma 19.6 fails on a branch with $d = 7$ and generic transpose rank five, then there is a positive Jacobi eigenvalue of multiplicity at least five.*

Proof. Project B to $\mathcal{F}^\perp$ in (19.2), writing the projection as $B_\perp$ and that of z as b; then $B_\perp B_\perp^T = bb^T$. Every row of $B_\perp$ has squared norm b_i^2, so it vanishes on the real hyperplane $b_i = 0$ and is $b_i e_i^T$ for a constant unit vector e_i; the cross-row identities give $e_i^T e_j = 1$, so all e_i equal one unit vector e and $B_\perp = be^T$. The mixed block gives $B(a)e = z(a)$. In orthonormal input coordinates beginning with e, $B = [z, B_0]$ with B_0 valued in $\mathcal{F}$ and $B_0 B_0^T = hD - ww^T$. Choose $S : \mathbb{R}^8 \to \mathcal{F}$ with $SS^T = D$, put $\Xi_0 = S^{-1}B_0$ and $\eta_0 = S^{-1}w$; then $\|\eta_0\|^2 = w^T D^+ w = h$ by (19.2), and the seven nonzero eigenvalues of $\Xi_0\Xi_0^T = hI - \eta_0\eta_0^T$ are h, so $\Xi_0^T\Xi_0 = hI_7$.

For $g \in e^\perp$, using $B(a)g = N_g a$ and an h-orthonormal basis of U, the eight-by-seven matrix $\Xi(g)$, linear in the seven variables g, satisfies $\Xi(g)^T\Xi(g) = q(g)I_7$ with $q(g) = \|g\|^2$. At the prime (q) its image is isotropic in the standard eight-space, of dimension at most four, so each maximal seven-minor is divisible by q^3; the signed cofactor vector is $q^3\eta(g)$ with η linear, Cauchy–Binet gives $\|\eta\|^2 = q$, and the normal-vector identity gives $\Xi(g)\Xi(g)^T = q(g)I_8 - \eta(g)\eta(g)^T$.

Complete the h-orthonormal basis of U by its H-orthogonal negative unit vector; in these coefficient coordinates $H = \Lambda = \operatorname{diag}(I_7, -1)$, the reciprocal congruence on the factors leaving the Jacobi family unchanged. With $n(g)$ the last column of N_g in this basis and $p(g) = S\eta(g)$,

$$\mathcal{J}_{K_X g} = q(g)D - p(g)p(g)^T - n(g)n(g)^T, \qquad p(g)^T D^+ p(g) = q(g). \tag{19.3}$$

In $\operatorname{Frac}(\mathbb{R}[g]/(q))$ the left side is nilpotent, and the determinant identity makes the symmetric 2×2 matrix $[p, n]^T[p, n]$ nilpotent, of the form $\left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right]$ with $a^2 + b^2 = 0$. The rank-seven quadric is geometrically integral over $\mathbb{R}$, so its function field has no square root of -1; hence $[p, n]^T[p, n] = 0$ there, and quadratic divisibility gives $p^T p = \beta q$ with $\beta > 0$.

The image $W_0 = \operatorname{Im} p$ is a seven-space in $\mathcal{F}$, and the two norm identities in (19.3) give $D^+|_{W_0} = \beta^{-1}I$ as a quadratic compression. A symmetric matrix with scalar compression on a hyperplane has that eigenvalue on the intersection of the hyperplane with the orthogonal complement of its cross-block vector, so D has eigenvalue β on a space L_β of dimension at least six. If this dimension is at least seven, subtracting the two rank-one terms leaves multiplicity at least five. If it is six, choose nonzero g in the kernel of $g \mapsto \operatorname{pr}_{L_\beta} p(g)$, from seven variables to six; the vectors of L_β orthogonal to $n(g)$ give a $\beta q(g)$-eigenspace of dimension at least five. Constancy of the real Osserman spectrum proves the claim. □

Proposition 19.9. *No sole-quadratic branch whose simple pole has a rank-two residue has support dimension seven.*

Proof. For generic transpose rank four, Lemma 19.3 makes the pencil aligned on U and the pole of rank eight, which Proposition 18.2 excludes. For rank five, Lemma 19.6 gives either the same alignment, whose pole has rank eight by the maximal-minor argument of Lemma 19.3 (which uses alignment only), or the form (19.2), with signature $(7, 1)$, where Proposition 19.8 forces a positive eigenvalue of multiplicity at least five. Among the eleven patterns listed at the start of Section 18,

only $(7,5,3)$ gives Λ an eigenvalue of multiplicity five, and then Λ has no simple eigenvalue, so $H = \Lambda^{-1}$ cannot have signature $(7,1)$ up to sign; in every other pattern each eigenvalue of Λ has multiplicity at most four: a contradiction in every case. □

20 Four-dimensional residue images

We work in Setting (T) of Section 12, with $\nu = 8$ and Λ indefinite, and let $h(a) = a^T H a$, $H = \Lambda^{-1}$, be the nondegenerate indefinite coefficient form. So $G(a)$ is positive semidefinite of rank eight at every nonzero real a. Let $\mathcal{H}$ be the real span of h and the entries G_{ij}, and let $V(h, G) \subset \mathbb{P}^7_{\mathbb{C}}$ be their common zero locus; it has no real point, since $G(a)$ has rank eight at every nonzero real a. A maximal isotropic subspace of the complex form h has dimension four, so $V(h)$ contains no linear $\mathbb{P}^4$. Complex varieties carry their reduced structure unless a scheme is specified. By Lemma 12.2 and Remark 12.3 (the scalar-span criterion), if the real span of the quadratic entries of G has dimension at most three, then

$$\operatorname{Im} B(a) = E \qquad (a \in \mathbb{R}^8 \setminus \{0\}). \tag{20.1}$$

20.1 The quadratic base

Lemma 20.1. *Let $A_1, \dots, A_k \subset V(h, G)$ be irreducible fourfolds whose union spans $\mathbb{P}^7_{\mathbb{C}}$. Then either* (20.1) *holds, or the ideal generated by $\mathcal{H}$ has height three and $\mathcal{H}$ contains a real regular sequence of three quadrics whose degree-eight complete intersection contains every A_i.*

Proof. The locus $Z = V(h, G)$ contains the A_i, so it has dimension at least four. A six-dimensional component would be the irreducible quadric $V(h)$, which has real points. Some real entry f of G is independent of h: otherwise every entry is a multiple of h, against rank eight at a nonzero real h-null vector. Let Y be a five-dimensional component. It is not a linear $\mathbb{P}^5$, since $V(h)$ contains no linear $\mathbb{P}^4$, so it spans either $\mathbb{P}^7$ or a hyperplane.

If Y spans $\mathbb{P}^7$, then $\deg Y \geq 3$ by the nondegenerate degree bound [10, p. 1, $(*)$], and Y lies in the complete intersection $V(h, f)$, of pure dimension five and degree four. If $Y \neq \overline{Y}$, the two conjugate components would have total degree at least six, more than $\deg V(h, f) = 4$; so Y is real-defined; a generic transverse real complementary plane meets it only in conjugate nonreal pairs, so $\deg Y$ is even, hence four. Thus Y exhausts $V(h, f)$ with multiplicity one. A complete intersection has no embedded associated primes, so generic reducedness makes $V(h, f)$ the reduced scheme Y; its homogeneous ideal, a complete-intersection ideal of positive dimension, is saturated, so every quadratic entry of G lies in $\operatorname{span}_{\mathbb{R}}\{h, f\}$, and the scalar-span criterion gives (20.1).

If Y spans a hyperplane $\ell = 0$, the restriction of h to it has rank at least six, so $Y = V(h, \ell)$ is an irreducible quadric and every quadratic entry of G is $g = c_g h + \ell m_g$ with m_g linear. Since the union of the A_i spans $\mathbb{P}^7$, some A_j is not contained in $\ell = 0$; as g and h vanish on A_j, so does ℓm_g, and integrality gives $m_g|_{A_j} = 0$ for every g. The vector span of A_j has dimension at least six, since $V(h)$ contains no linear $\mathbb{P}^4$, so the linear forms vanishing on A_j span at most two dimensions. All entries $g = c_g h + \ell m_g$ therefore lie in one complex space of quadrics of dimension at most three; real linearly independent vectors stay complex linearly independent, so their real span has the same bound, and the scalar-span criterion gives (20.1).

In the remaining case $\dim Z = 4$, so the ideal generated by $\mathcal{H}$ has height three. Begin with h and a real element of $\mathcal{H}$ independent of it, and choose a third real element of $\mathcal{H}$ outside the finitely many height-two associated primes of their complete intersection. None of these primes contains $\mathcal{H}$, and over the infinite field $\mathbb{R}$ a finite union of proper linear subspaces cannot cover $\mathcal{H}$. This gives the regular sequence; its degree-eight complete intersection contains every A_i. □

Proposition 20.2. *Under the hypotheses of Lemma 20.1, assume the A_i are distinct and $\sum_i \deg A_i \geq 8$. Then* (20.1) *holds.*

Proof. Otherwise the degree-eight complete intersection of Lemma 20.1 contains the distinct A_i, so the degree sum is eight and the A_i are exactly its components, each of multiplicity one. Unmixedness and generic reducedness make it their reduced union, and its homogeneous ideal, a complete-intersection ideal of positive dimension, is saturated. Every entry of G therefore lies in its degree-two part, spanned by the three defining quadrics, and the scalar-span criterion applies. □

In particular, the proposition applies to two distinct fourfolds in $V(h, G)$ that each span $\mathbb{P}^7$ (each then has degree at least four), and to four distinct fourfolds in $V(h, G)$ whose union spans $\mathbb{P}^7$ (each has degree at least two, since $V(h)$ contains no linear $\mathbb{P}^4$). In the two-quadratic pole case these are the images of the two geometric quadratics and of the four geometric lines; Proposition 18.4 supplies these alternatives, Lemma 17.6 the distinctness of the images, and Remark 20.9 the vanishing of G on them and their combined span.

20.2 A real integral image

Proposition 20.3. *Suppose that a geometrically integral real-defined fourfold $A \subset \mathbb{P}^7_{\mathbb{C}}$ spans $\mathbb{P}^7_{\mathbb{C}}$ and is contained in $V(h, G)$. Then* (20.1) *holds.*

Lemma 20.4. *Under the hypotheses of Proposition 20.3, either* (20.1) *holds, or the ideal generated by $\mathcal{H}$ has height three and $\deg A \in \{4, 6\}$.*

Proof. Apply Lemma 20.1 with $k = 1$. In its second alternative, A is nondegenerate of codimension three, so $\deg A \geq 4$. As A is real-defined and has no real point, a generic transverse real complementary plane meets it only in conjugate nonreal pairs, so $\deg A$ is even. The degree is at most eight, since A lies in the degree-eight complete intersection of that lemma, and $\deg A = 8$ gives (20.1) by Proposition 20.2 with $k = 1$. □

Lemma 20.5. *If $\deg A = 4$, then* (20.1) *holds.*

Proof. The variety A has minimal degree, so by the Del Pezzo–Bertini classification [10, Theorem 0.1] it is a rational normal scroll or a cone over the Veronese surface. A singular variety in this list has a nonempty intrinsic linear vertex, which, since A is real-defined, would be real-defined and contain a real point; but A has no real point. So A is smooth, its four positive scroll integers sum to $\deg A = 4$, so all are one, and A is the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^4)$.

Conjugation preserves the two rulings, which have different dimensions, so it lifts, up to a scalar absorbed into one factor, to $\sigma_1 \otimes \sigma_2$ with conjugate-linear maps on the two factors. Then $(\sigma_1 \otimes \sigma_2)^2 = I$ forces $\sigma_1^2 = \alpha I$ and $\sigma_2^2 = \beta I$ with $\alpha\beta = 1$. Each map commutes with its square, so these scalars are real; rescaling makes them signs, and they agree because the ambient real structure has square one. Positive signs would give fixed vectors in both factors, hence a real Segre point; so both signs are negative.

The standard SU(2)-action on the first factor commutes with its quaternionic real structure σ_1, so it acts on the real coefficient eight-space, and its three quaternion units make that space a left $\mathbb{H}$-module. The quadratic ideal of the Segre embedding, the kernel of restricting quadratic polynomials to rank-one tensors, is $\bigwedge^2(\mathbb{C}^2)^* \otimes \bigwedge^2(\mathbb{C}^4)^*$, on which the first $\mathrm{SL}(2, \mathbb{C})$ acts trivially. So every entry of G is SU(2)-invariant, and by homogeneity $G(ba) = |b|^2 G(a)$ for $b \in \mathbb{H}$. For every nonzero real a, the space $U = \mathbb{H}a$ has real dimension four, since nonzero quaternions act invertibly, and this identity makes $\operatorname{Im} G$, hence $\operatorname{Im} B$, one fixed eight-space on $U \setminus \{0\}$; Lemma 12.5(1) identifies it with E. Since a was arbitrary, (20.1) holds. □

Lemma 20.6. *If* $\deg A = 6$ *and the ideal generated by* $\mathcal{H}$ *has height three, there is a reduced real complete intersection* $C = A \cup Q$ *of three quadrics, where* Q *is a real-defined quadric fourfold spanning* $\mathbb{P}(U_{\mathbb{C}})$ *for a real six-space* U *on which* h *is definite.*

Proof. The quadratic form $t = \operatorname{tr} G \in \mathcal{H}$ is positive definite, so it is independent of the indefinite form h. By height three, prime avoidance as in Lemma 20.1 gives a real $f \in \mathcal{H}$ such that (h, t, f) is a regular sequence. Its complete intersection C has degree eight and no real point. A four-dimensional component of degree one would be a linear $\mathbb{P}^4$ in $V(h)$, so every component has degree at least two. The component A takes degree six with multiplicity one, so the rest is exactly one component Q of degree two and multiplicity one, preserved by conjugation as the unique residual component. Unmixedness and generic reducedness make $C = A \cup Q$ reduced.

A fourfold of degree two spans at most a projective five-space, and Q is not linear, so it spans exactly a projective five-space $\mathbb{P}(U_{\mathbb{C}})$, for a real six-space U since Q is conjugation-invariant. Its equation is proportional to $h|_U$, which is nonzero because $V(h)$ contains no linear $\mathbb{P}^5$. Since Q has no real point, this real form is definite. □

Lemma 20.7. *In Lemma 20.6, choose real independent linear forms* u, v *with* $U = \ker u \cap \ker v$. *There are real quadrics* s, r, Δ *such that*

$$I_C = (h, s, r), \qquad (I_A)_2 \subset \operatorname{span}_{\mathbb{R}}\{h, s, r, \Delta\}, \qquad s|_U = r|_U = 0,$$

where $(I_A)_2$ *is the real degree-two part of the homogeneous ideal.*

Proof. Since $I_Q = (h, u, v)$, subtracting constant multiples of h from t, f gives

$$s = ua_0 + vb_0, \qquad r = uc_0 + vd_0, \qquad \Delta = a_0d_0 - b_0c_0,$$

with real linear forms a_0, b_0, c_0, d_0. Work in the graded domain $S = \mathbb{R}[x_1, \dots, x_8]/(h)$ in coefficient coordinates x_i. Both (s, r) and (u, v) are regular sequences in S; the latter follows from $h|_U \neq 0$ and the Cohen–Macaulay property of the polynomial ring. For $g \in (I_A)_2$, reducedness gives $ug, vg \in I_A \cap I_Q = I_C$, so modulo h

$$ug = a_1s + b_1r, \qquad vg = c_1s + d_1r$$

with real linear coefficients. The Koszul syzygies of (s, r), with degrees taken into account, give a real constant k with

$$va_1 - uc_1 = kr, \qquad vb_1 - ud_1 = -ks.$$

Substituting the expressions for s, r and using the Koszul syzygies of (u, v) gives real constants e_1, e_2 with

$$a_1 - kd_0 = e_1u, \quad c_1 + kc_0 = e_1v, \qquad b_1 + kb_0 = e_2u, \quad d_1 - ka_0 = e_2v.$$

Hence $ug = k(d_0s - b_0r) + u(e_1s + e_2r) = u(k\Delta + e_1s + e_2r)$. Cancelling u in the domain S and lifting back, which adds a constant multiple of h, proves the assertion. □

Proof of Proposition 20.3. By Lemmas 20.4 and 20.5, only $\deg A = 6$ remains, with height three, and Lemma 20.6 applies. Every entry of G vanishes on A, so it lies in $(I_A)_2$; as $s|_U = r|_U = 0$, Lemma 20.7 gives constant real symmetric matrices D_1, D_2 with $G(b) = h(b)D_1 + \Delta(b)D_2$ for $b \in U$. So the entries of $G|_U$ span at most two forms, and, $G(b)$ being positive semidefinite of rank eight for real $b \neq 0$, Lemma 12.2 (with $n = \dim U = 6$) makes $\operatorname{Im} G(b)$, hence $\operatorname{Im} B(b)$, one fixed eight-space for nonzero $b \in U$. As $\dim U = 6$, Lemma 12.5(1) shows that it is E, so $K^T B(b) = 0$ for $b \in U$.

By linearity, $K^T B(a) = u(a)\Theta_1 + v(a)\Theta_2$ with constant real matrices Θ_1, Θ_2. Hence the form $\mathcal{Q}(a) = \operatorname{tr}(K^T B(a) B(a)^T K) = \operatorname{tr}(K^T G(a) K)$ is a binary quadratic form in u, v, and it vanishes on A, where $G = 0$. If $\mathcal{Q} \neq 0$, it factors over $\mathbb{C}$ into two linear forms, and the integral coordinate ring of A forces one of them to vanish on A, contrary to full complex span. So $\mathcal{Q} = 0$; on real coefficients it is a sum of squares of real entries, so $K^T B(a) = 0$ for every a, and rank eight gives (20.1). □

20.3 A conjugate pair of images

Proposition 20.8. *Suppose A and $\overline{A}$ are distinct irreducible fourfolds in $V(h, G)$ whose combined complex span is $\mathbb{P}^7$. Then* (20.1) *holds.*

Proof. Let U be the vector span of A and $d = \dim_{\mathbb{C}} U$; as in Lemma 20.1, $6 \leq d \leq 8$. If $d = 8$, both images have degree at least four and Proposition 20.2 applies. If $d = 7$, the restriction $h|_U$ has rank at least six, and the affine coordinate ring of its quadric is factorial by the Klein–Nagata theorem [29, Theorem 8.2, p. 26], with a polynomial extension for a possible one-dimensional radical. The cone over A is a height-one homogeneous prime, hence generated modulo $h|_U$ by one homogeneous element f; full span in U excludes $\deg f = 1$, so $\deg A = 2 \deg f \geq 4$ and Proposition 20.2 applies again.

Suppose $d = 6$. Then $h|_U$ has rank at least four, so its null locus is an irreducible quadric fourfold and equals A; so the quadratic entries of G restricted to U, which vanish on A, are constant multiples of $h|_U$: $G(a) = h(a)D$ for $a \in U$ and one constant complex symmetric matrix D. Since $U + \overline{U} = \mathbb{C}^8$, the conjugation-invariant space $U \cap \overline{U}$ has complex dimension four and contains a nonzero real vector w. Real rank eight of $G(w)$ forces $h(w) \neq 0$, so $D = G(w)/h(w)$ is real of rank eight; put $T = \operatorname{Im}_{\mathbb{R}} D$. For complex $a \in U$ with $h(a) \neq 0$ the Gram has image $T_{\mathbb{C}}$ and rank eight, and since $B(a)$ has eight columns, its image is $T_{\mathbb{C}}$ as well. So the projection of the linear pencil $B|_U$ to $V_{\mathbb{C}}/T_{\mathbb{C}}$ vanishes on a dense open set, hence everywhere, and conjugation gives the same on $\overline{U}$. As $U + \overline{U} = \mathbb{C}^8$, every $B(a)$ maps into this fixed eight-space, and the radial columns identify its real form with E. □

Remark 20.9. For a residue image of type $(\alpha, s) = (4, 4)$, the rank identity $\operatorname{rank} G = 2s - 8$ gives the vanishing of G on the image, as required above; full real support becomes full complex span when the image is geometrically integral and real-defined. For a simple real-irreducible quadratic or quartic pole that splits over $\mathbb{C}$ with $\alpha = s = 4$, Lemma 17.6 makes the two geometric components give distinct images, and full real residue support gives their combined span.

21 The divisor budget for five-dimensional residue images

Throughout this section $\nu = 8$, the anchor is regular, $B(a)$ is the transpose pencil of Setting (T), with coefficients $a = (a_1, \ldots, a_8)$, and $h(a) = a^T H a$ is the nondegenerate indefinite coefficient form. By the isotropy in (10.9), every residue image lies in the quadric $V(h) \subset \mathbb{P}^7_{\mathbb{C}}$. A pole has type $(\alpha, s) = (5, 5)$ when each geometric pole component has an irreducible closed residue image A of dimension five on which B has generic rank five; distinct geometric pole components have distinct images (Lemma 17.6). Since rank is lower semicontinuous, $\operatorname{rank} B(a) \leq 5$ at *every* point of A, so A has no real point.

The real residue support (as in Section 17) is the smallest real subspace $U \subset \mathbb{R}^8$ whose complexification contains every residue image. A real-defined image in a complex hyperplane $\{c \cdot a = 0\}$ also lies in $\{\overline{c} \cdot a = 0\}$, hence in a real hyperplane; so a real-defined image of a component with real support 8 spans $\mathbb{P}^7$.

21.1 The budget

Let $S = \mathbb{C}[a_1, \dots, a_8]/(h)$ with its standard grading. By the Klein–Nagata theorem [29, Theorem 8.2, p. 26], a nondegenerate quadratic form in at least five variables over a field of characteristic not two gives a factorial quotient, so S is a UFD. The affine cone over a five-dimensional irreducible $A \subset V(h)$ is a height-one prime of S, so its ideal is generated by a homogeneous irreducible f_A of some degree $e_A \geq 1$; if A spans $\mathbb{P}^7$, then $e_A \geq 2$.

Lemma 21.1 (Divisor budget)**.** *Let $A_1, \dots, A_k$ be distinct residue images of type $(5,5)$. Then $e_{A_1} + \cdots + e_{A_k} \leq 2$.*

Proof. Over the fraction field of S the matrix B has rank 8: the real cone of h contains nonzero points, where B has rank 8, so some maximal minor of B is nonzero in S. For each i, localizing S at the height-one prime (f_{A_i}) gives a discrete valuation ring. The reduction of B has rank five at the generic point of A_i, so in the local Smith form of the 15×8 matrix B (its image lies in $X^\perp$) at least three of the eight invariant factors lie in the maximal ideal, and every maximal minor of B is divisible in S by $f_{A_i}^3$. The f_{A_i} are pairwise nonassociate primes, so every maximal minor is divisible by $\prod_i f_{A_i}^3$, of degree $3\sum_i e_{A_i}$. A maximal minor has degree 8 and some maximal minor is nonzero in S, so $3\sum_i e_{A_i} \leq 8$. □

21.2 Consequences

Proposition 21.2 (Two quadratic poles)**.** *Suppose that the reduced denominator is Q_1Q_2 with two distinct real quadratic factors, that every real residue support is 8, and that all geometric pole components have type $(5,5)$. This configuration does not occur.*

Proof. Each real quadratic factor contributes at least two to the sum of the degrees e. If it is geometrically irreducible, its single image is real-defined of full span, so $e \geq 2$; if it splits, its two conjugate geometric components have distinct images, each with $e \geq 1$. Images of different pole components are distinct by Lemma 17.6. So the two factors, of either kind, contribute at least four, contrary to Lemma 21.1. □

Proposition 21.3 (A single real pole)**.** *Suppose that the pole has one real component, with full real support and type $(5,5)$.*

1. *If its image A is real-defined and geometrically integral, then $A = V(h, f)$ for a real quadric f; it is a complete intersection of degree four with no real point.*
2. *Otherwise there are exactly two images, A and $\overline{A}$, and $A = V(h, \ell)$, $\overline{A} = V(h, \overline{\ell})$ for a complex linear form ℓ that is not proportional to a real form.*

Proof. A real irreducible polynomial is geometrically irreducible or splits into one pair of complex conjugate factors.

(1) Full real support gives full complex span (as shown above), so $e_A \geq 2$, and Lemma 21.1 gives $e_A = 2$: $A = V(h, f_A)$ with f_A of degree two. Since A is real-defined, $\overline{f_A}$ generates the same prime, so $\overline{f_A} = cf_A$ with c a nonzero constant (the only units of S in degree zero). Conjugating twice gives $|c| = 1$; writing $c = \omega/\overline{\omega}$, the class ωf_A is fixed by conjugation, and any real lift f of it defines the same complete intersection, of degree $2 \cdot 2 = 4$; like every image, it has no real point.

(2) The two images are distinct by Lemma 17.6 and conjugate, so $e_A = e_{\overline{A}}$, and Lemma 21.1 gives $e_A = e_{\overline{A}} = 1$, so $A = V(h, \ell)$ with ℓ linear. If ℓ were proportional to a real form, A would be real-defined and equal to $\overline{A}$. □

21.3 The conjugate pair: definiteness and the positive model

In case (2) of Proposition 21.3 write $\ell = \ell_1 + i\ell_2$ with real linear forms ℓ_1, ℓ_2, linearly independent because ℓ is not proportional to a real form, and put

$$W = \{a \in \mathbb{R}^8 : \ell_1(a) = \ell_2(a) = 0\}, \qquad r = \ell_1^2 + \ell_2^2 = \ell\,\overline{\ell}.$$

Then $\dim W = 6$, and r is a real positive semidefinite form of rank two.

Lemma 21.4. *The restriction $h|_W$ is definite. Consequently the signature of h is $(6,2)$ or $(7,1)$, up to an overall sign.*

Proof. A real point of $A = V(h,\ell)$ is a real $a \neq 0$ with $h(a) = 0$ and $\ell_1(a) = \ell_2(a) = 0$, that is, a real zero of h on W. As A has no real point, $h|_W$ has no nonzero real zero, so it is definite. A nondegenerate form that is definite on a six-dimensional subspace has at least six squares of one sign; since h is indefinite, its signature is $(6,2)$ or $(7,1)$ up to sign. □

Proposition 21.5 (Positive model of the conjugate pair)**.** *There are a positive definite real quadratic form q and a real constant $\lambda \neq 0$ such that $\rho = h - \lambda q$ is a semidefinite form of rank two with radical W, whose restriction to $W^{\perp_q}$ is definite. In q-orthonormal coordinates $a = (x, z)$ with $x \in W$ and $z \in W^{\perp_q}$,*

$$q = |x|^2 + |z|^2, \qquad h = \lambda q + \rho(z), \qquad A \cup \overline{A} = V(h,r) = V(h,q) = V(q,\rho).$$

In particular the pencil spanned by h and q contains the rank-two member ρ, and λ is a relative eigenvalue of h with respect to q of multiplicity six.

Proof. The forms h and r have no common nonzero real zero: such a zero would lie in W and be a real zero of h there, which Lemma 21.4 excludes. As they are forms in $8 \geq 3$ variables, the theorem of Finsler and Calabi [7,12] gives a positive definite combination $q = c_1 h + c_2 r$. Here $c_1 \neq 0$ because r has rank two, and $c_2 \neq 0$ because h is indefinite. Put $\lambda = 1/c_1$ and $\rho = -(c_2/c_1)r = h - \lambda q$, a nonzero multiple of r; so ρ is semidefinite of rank two with radical W, and definite on $W^{\perp_q}$.

In q-orthonormal coordinates x on W and z on $W^{\perp_q}$, the forms ℓ_1, ℓ_2 vanish on W, so they, and hence ρ, depend on z alone; this gives the model. Since $c_2 \neq 0$, $V(h,q) = V(h,r) = V(h,\ell\overline{\ell}) = V(h,\ell) \cup V(h,\overline{\ell})$, and $V(h,q) = V(q, h-\lambda q) = V(q,\rho)$. On W the relative eigenvalue of h with respect to q is λ, and on $W^{\perp_q}$ the form $h - \lambda q = \rho$ is nondegenerate, so λ has multiplicity exactly six. □

Remark 21.6. Each component $V(h,\ell)$ is a quadric of rank six or seven in its hyperplane, since restricting the rank-eight form h to a hyperplane lowers its rank by at most two; in particular it is irreducible and spans its hyperplane. Proposition 21.5 gives the normal form (25.1) used in Sections 25 and 26, where the conjugate pair, a reducible complete intersection, is treated.

22 Alignment criteria for quadratic normal forms

We work in Setting (T) of Section 12, with all eight coefficient directions; over $\mathbb{C}$, the transpose is that of the complex-bilinear extension. As in Section 10, write $B(a)g = N_g a$. Since $\operatorname{rank} B(a) = 8$ for $a \neq 0$,

$$N_g : \mathbb{R}^8 \longrightarrow V \quad \text{is injective for every } g \neq 0. \tag{22.1}$$

Put $x_0 = Kg_0$. As $K^T B(a) g_0 = -K^T L a = 0$, skewness makes g_0 orthogonal to the image of $K^T B(a)$, so

$$\operatorname{Im} B(a) \subset x_0^{\perp}, \qquad \langle Kg, B(a)g\rangle = 0. \tag{22.2}$$

22.1 Balanced normal forms

Proposition 22.1. *Suppose that q is a positive definite quadratic form on $\mathbb{R}^8$ and h is a real quadratic form, with eigenvalues $\lambda_1 \le \cdots \le \lambda_8$ relative to q. Assume there are constant symmetric endomorphisms D_0, D_1 of V and real linear maps $u, v : \mathbb{R}^8 \to V$ such that*

$$G(a) = h(a)D_0 + q(a)D_1 + u(a)u(a)^T - v(a)v(a)^T \tag{22.3}$$

*(*a balanced normal form*). If $\lambda_2 < \lambda_7$, then $\operatorname{Im} B(a) = E$ for every nonzero real a.*

Proof. Write $D(t) = D_1 + tD_0$. Fix $b \in V$; the solution space $A_b \subset \mathbb{R}^8$ of $\langle b, u(a)\rangle = 0$ has codimension at most one, and on $A_b \setminus \{0\}$ positivity of (22.3) gives $q(a)\langle D(h(a)/q(a))b, b\rangle \ge \langle b, v(a)\rangle^2 \ge 0$. By the min–max principle the minimum of h/q on A_b is at most λ_2 and its maximum is at least λ_7, and its range is an interval; as b was arbitrary, $D(t) \succeq 0$ for every $t \in [\lambda_2, \lambda_7]$ (the test vector a may depend on b). By Lemma 12.1, $D(t)$ has one image $\mathcal{F}$ for all interior t; let $r = \dim \mathcal{F}$. The set $\mathcal{S} = \{a \ne 0 : \lambda_2 < h(a)/q(a) < \lambda_7\}$ is nonempty and open. For $a \in \mathcal{S}$ put $G_+(a) = q(a)D(h(a)/q(a)) + u(a)u(a)^T$. Both $G_+(a)$ and $G(a) = G_+(a) - v(a)v(a)^T$ are positive semidefinite; testing a vector in $\ker G_+(a)$ shows $v(a) \in \operatorname{Im} G_+(a)$. As $G(a)$ has rank eight, $\operatorname{rank} G_+(a)$ is eight or nine, and $r \le \operatorname{rank} G_+(a) \le r + 1$, so $7 \le r \le 9$.

If $r = 9$, then $u(a) \in \mathcal{F}$ for $a \in \mathcal{S}$, since otherwise $G_+(a)$ would have rank ten and subtracting a rank-one matrix could not give rank eight. So $v(a) \in \mathcal{F}$ and $\operatorname{Im} G(a) \subset \mathcal{F}$; over the reals $\operatorname{Im} G(a) = \operatorname{Im} B(a)$, and since B is linear its image lies in $\mathcal{F}$ for every a. Lemma 12.5(2) with $U = \mathbb{R}^8$ applies.

Suppose $r \in \{7, 8\}$, and let $\pi : V \to \mathcal{F}^\perp$ be the orthogonal projection. For $a \in \mathcal{S}$,

$$(\pi B(a))(\pi B(a))^T = (\pi u(a))(\pi u(a))^T - (\pi v(a))(\pi v(a))^T \succeq 0.$$

A positive semidefinite difference of two matrices of rank at most one has rank at most one, since a vector orthogonal to $\pi u(a)$ is then orthogonal to $\pi v(a)$. So every two-minor of the linear matrix $\pi B(a)$ vanishes on $\mathcal{S}$, hence identically, and by Lemma 12.6 the matrices $\pi B(a)$ have a common image line or a common row line.

In the common-image case the images of B lie in $\mathcal{F}$ plus one fixed line, a space of dimension at most nine, and Lemma 12.5(2) applies again. In the common-row case there is a fixed nonzero covector e^* on the g-space $\mathbb{R}^8$ (the domain of $B(a)$) with $\pi B(a) = w(a)e^*$, so $N_g(\mathbb{R}^8) \subset \mathcal{F}$ for every nonzero $g \in \ker e^*$. Injectivity (22.1) excludes $r = 7$ and gives $N_g(\mathbb{R}^8) = \mathcal{F}$ when $r = 8$, and (22.2) gives $\mathcal{F} \perp K(\ker e^*)$ and $\mathcal{F} \perp x_0$. If $g_0 \notin \ker e^*$, these spaces span $E^\perp$, so $\mathcal{F} = E$; if $g_0 \in \ker e^*$, the identity $N_{g_0} = -L$ gives $\mathcal{F} = E$ directly. In either case $\pi = KK^T$, so the skew map $K^T B(a)$ has rank at most one and is therefore zero, and rank eight gives the result. □

Proposition 22.2. *Let q, h be as in Proposition 22.1 and suppose that* (22.3) *holds. Then* $\operatorname{Im} B(a) = E$ *for every nonzero real a in either of the following situations:*

1. *Every eigenvalue of h relative to q has multiplicity at most five.*
2. *There is a q-orthogonal decomposition $\mathbb{R}^8 = W \oplus Z$, with $\dim W = 6$, $\dim Z = 2$, such that $h = \lambda q + \rho$, where ρ vanishes on W and its restriction to Z is definite.*

Proof. Equality $\lambda_2 = \lambda_7$ would require an eigenvalue of multiplicity at least six, which the first situation excludes. In the second, λ is an extreme eigenvalue of multiplicity six: if $\rho|_Z$ is positive definite, $\lambda_2 = \lambda < \lambda_7$, and if it is negative definite, $\lambda_2 < \lambda = \lambda_7$. In both situations $\lambda_2 < \lambda_7$, and Proposition 22.1 applies. □

22.2 Unbalanced forms and vanishing on a stratum

Proposition 22.3 (Unbalanced normal form)**.** *Let h, q be independent real quadratic forms on $\mathbb{R}^8$, with q positive definite. Suppose that D_0, D_1 are constant symmetric endomorphisms of V, $u \in V$ is a fixed nonzero vector, and $b : \mathbb{R}^8 \to V$ is quadratic, such that*

$$G(a) = h(a)D_0 + q(a)D_1 + u\,b(a)^T + b(a)u^T$$

(an unbalanced normal form*). Then $\operatorname{Im} B(a) = E$ for every nonzero real a.*

Proof. Compress to $u^\perp$: the term $ub^T + bu^T$ compresses to zero, so the compression is $G_\perp(a) = h(a)D_0^\perp + q(a)D_1^\perp$, where $D_i^\perp$ are the compressions of D_i; it is positive semidefinite of rank at most eight. On $q = 1$ the quotient h/q ranges over a nondegenerate closed interval I, so $D_1^\perp + tD_0^\perp \succeq 0$ for $t \in I$, and by Lemma 12.1 its image for interior t is $\mathcal{F} = \operatorname{Im} D_0^\perp + \operatorname{Im} D_1^\perp$, of dimension at most eight. Hence $\operatorname{Im} G_\perp(a) \subset \mathcal{F}$ for every a, endpoint values included. Positivity of $G(a)$ puts its cross block in $\operatorname{Im} G_\perp(a)$, since a vector in the kernel of a positive semidefinite principal block is annihilated by the corresponding cross block; so $\operatorname{Im} G(a) \subset \mathbb{R}u + \mathcal{F}$. Over the reals $\operatorname{Im} G(a) = \operatorname{Im} B(a)$, and $\dim(\mathbb{R}u + \mathcal{F}) \le 9$, so Lemma 12.5(2) with $U = \mathbb{R}^8$ applies. □

For the stratum statement, choose coefficient coordinates $\mathbb{R}^8 = W \oplus Z$ with $\dim W = m \in \{4, 5\}$ and

$$q(x, z) = |x|^2 + |z|^2, \qquad h(x, z) = \lambda q(x, z) + \rho(z),$$

where ρ is a nondegenerate real quadratic form on Z of any signature. Let

$$\mathcal{A} = \{(x, z) \in \mathbb{C}^8 : q(x, z) = \rho(z) = 0\}, \qquad \Sigma = \{(x, 0) : |x|^2 = 0\},$$

and assume $\operatorname{rank} G \le 2$ at every point of $\mathcal{A}$, singular points included. In the residue construction this follows from the generic rank bound on every component, since the three-minors are polynomials.

Proposition 22.4 (Vanishing on a multiplicity-four or -five stratum)**.** *In this setting, if G vanishes on Σ, then $\operatorname{Im} B(a) = E$ for every nonzero real a.*

Proof. Lemma 12.9 applies with $w = m$: its hypotheses (i) and (ii) are the vanishing on Σ and the rank bound on $\mathcal{A} = V(q, \rho)$. If $m = 5$, then $\dim Z = 3$ and Lemma 12.9(3) gives the result.

Let $m = 4$, let T be the skew linear map of Lemma 12.9(1), with $K^T B(x, z) = T(z)$, and let $\Gamma_1, \Gamma_2, \Gamma_3$ be its operators for an orthonormal basis of the orthogonal complement of x_* in W. For a complex null vector z_0 of ρ with $T(z_0) \neq 0$, Lemma 12.9(2) gives a two-dimensional image $U = \operatorname{Im} T(z_0)^T$, invariant under each Γ_i, with $\operatorname{tr}(\Gamma_i|_U) = 0$. A skew map with image U corresponds to a bivector in $\bigwedge^2 U$, on which Γ_i acts by its trace, that is, by zero; in matrix terms

$$\Gamma_i T(z_0) + T(z_0)\Gamma_i^T = \Gamma_i T(z_0) - T(z_0)\Gamma_i = 0.$$

This also holds when $T(z_0) = 0$. Each commutator is linear in z, and the null cone of the nondegenerate rank-four form ρ spans $Z_{\mathbb{C}}$, so $T(z)$ commutes with all three Γ_i for every z.

The radial identity in (12.1) gives $T(z)g_0 = 0$, and commutation with Γ_1, Γ_2 makes $T(z)$ vanish on the real four-space $\mathcal{X}_0 = \operatorname{span}_{\mathbb{R}}\{g_0, \Gamma_1 g_0, \Gamma_2 g_0, \Gamma_1\Gamma_2 g_0\}$, whose spanning vectors divided by $\|g_0\|$ are orthonormal by the relations of Lemma 12.9(1). The spaces $\mathcal{X}_0$ and $\mathcal{X}_0^\perp$ are invariant under Γ_1, Γ_2, and skewness makes T preserve $\mathcal{X}_0^\perp$. On this four-dimensional complement Γ_1, Γ_2 give the left action of the quaternions on a quaternionic line; a commuting real endomorphism is right multiplication by a quaternion, and skew-adjointness makes the quaternion imaginary. Explicitly,

for a unit $v \in \mathcal{X}_0^{\perp}$ a commuting map is determined by its value at v in the basis $v, \Gamma_1 v, \Gamma_2 v, \Gamma_1\Gamma_2 v$, and skewness removes its v coefficient. Hence there are real linear forms t_1, t_2, t_3 on Z with

$$T(z)^2\Big|_{\mathcal{X}_0^{\perp}} = -\tau(z) I, \qquad \tau = t_1^2 + t_2^2 + t_3^2. \tag{22.4}$$

By Lemma 12.9(2), $\operatorname{rank} T(z) \le 2$ whenever $\rho(z) = 0$ over $\mathbb{C}$. If $\tau(z) \neq 0$ at such a point, (22.4) would make $T(z)$ invertible on $\mathcal{X}_0^{\perp}$, of rank four. So τ vanishes on the irreducible rank-four quadric $\rho = 0$; both have degree two, so $\tau = c\rho$ with c real, and $\operatorname{rank}\tau \le 3 < 4 = \operatorname{rank}\rho$ forces $c = 0$. A real sum of three squares vanishes identically only if each t_i is zero, so $T = 0$. The lower projection of B is therefore zero, and rank eight gives $\operatorname{Im} B(a) = E$. □

23 Positive quadratic models and the factorial case

Throughout, $\nu = 8$ and the anchor is regular (Section 10). By Proposition 21.3(1), a real-defined, geometrically integral, full-span residue fivefold is a complete intersection $A = V(h, f) \subset \mathbb{P}^7_{\mathbb{C}}$ of two real quadrics, with h nondegenerate and indefinite, and A has no real point.

Proposition 23.1. *There is a positive definite real quadric q with $\operatorname{span}_{\mathbb{R}}\{h, f\} = \operatorname{span}_{\mathbb{R}}\{h, q\}$. Every eigenvalue of h relative to q has multiplicity at most five. Both A and its affine cone are normal.*

Proof. The forms h and f have no common nonzero real zero, since A has no real point. As they are forms in $8 \ge 3$ variables, the theorem of Finsler and Calabi [7, 12] gives a positive definite combination $q = c_1 h + c_2 f$. Indefiniteness of h forces $c_2 \neq 0$.

In real coordinates with $q = \sum_{i=1}^{8} a_i^2$ and $h = \sum_{i=1}^{8} \lambda_i a_i^2$, the λ_i are real and nonzero. If an eigenvalue λ had multiplicity at least six, the nonzero quadric $h - \lambda q$ would have rank at most two, hence would factor over $\mathbb{C}$ into two linear forms, possibly equal; as $(h, f) = (h, f_A)$ is the prime ideal of the affine cone of A (Section 21), one factor would lie in it, contradicting full span. By the Jacobian criterion the projective singular locus is the union of the sets $\mathbb{P}(W_\lambda) \cap V(q)$ over the relative eigenspaces W_λ; an eigenspace of dimension $m \ge 2$ contributes dimension $m - 2$, and one of dimension one contributes nothing. So the projective singular locus has dimension at most three, and that of the six-dimensional affine cone, vertex included, at most four. Both spaces are therefore regular in codimension one; as integral complete intersections they are Cohen–Macaulay, hence satisfy Serre's second condition, and Serre's normality criterion applies. □

Proposition 23.2. *If every relative eigenspace has dimension at most three, the standard graded cone ring $S = \mathbb{C}[a_1, \dots, a_8]/(h, q)$ is a unique factorization domain.*

Proof. The calculation above now bounds the singular locus of the affine cone by dimension two. As $\dim S = 6$, every localization at a prime of height at most three is regular, hence factorial [29, Ch. 2, Theorem 1.6, p. 40]. The local complete-intersection factoriality theorem (Grothendieck's theorem, in the form [29, §8, Remark 3 following Theorem 8.2, p. 27]) applies to $S_{\mathfrak{m}}$, $\mathfrak{m} = S_{>0}$ the vertex maximal ideal: its localizations at primes of height at most three are factorial. Finally, for the divisor class groups, $\operatorname{Cl}(S) \to \operatorname{Cl}(S_{\mathfrak{m}})$ is an isomorphism for this positively graded normal domain [29, Proposition 7.4, p. 21]; the target is zero, so the normal Noetherian domain S is factorial. □

Proposition 23.3. *In the factorial case of Proposition 23.2, suppose the quadratic transpose Gram $G = B(a)B(a)^T$ of Lemma 17.4 has generic rank two on A (so that, by Lemma 12.7, its generic*

transpose kernel is a totally isotropic three-space for the standard eight-form). Then, as an ambient real quadratic matrix, $G = hD_0 + qD_1 + uu^T - vv^T$ (balanced) or $G = hD_0 + qD_1 + cb^T + bc^T$ with $c \neq 0$ (unbalanced), where D_0, D_1 are constant real symmetric matrices and the columns have the degrees of Lemma 12.8.

Proof. By Lemma 12.7, applied to A and to its affine cone, the pairing $\langle Gx, Gx' \rangle = x^T G x'$ induced by G on its image is hyperbolic over the real function field, and its two isotropic lines are individually real-defined.

The saturated modules of the two isotropic lines are rank-one reflexive modules over the normal domain S (the intersections of their height-one localizations inside their fraction-field lines). Factoriality makes their divisor classes zero, so they are principal. As S is normal, so is its conjugation-fixed subring $S_{\mathbb{R}}$, and Lemma 12.8 applies. Lift its homogeneous columns to ambient real polynomials; the difference between G and the lift has degree two and lies in (h, q), so it is $hD_0 + qD_1$ with constant real symmetric D_0, D_1. □

For the full eight-coefficient pencil, Proposition 22.3, on the eight-dimensional coefficient space, applies to the unbalanced form, and Proposition 22.1 applies to the balanced form, since relative multiplicities at most three imply $\lambda_2 < \lambda_7$. Thus $\operatorname{Im} B(a) = E_X$ for every real $a \neq 0$ at every anchor of this case; on a real open set of such anchors Lemmas 16.2 and 16.1 and Proposition 9.1(2) make R Clifford (Section 27).

24 Extraction at the singular integral residue images

In the situation of Proposition 21.3(1), let q be the positive definite form of Proposition 23.1, and suppose that h has a relative eigenvalue λ with respect to q of multiplicity $m \in \{4, 5\}$. Let W be its eigenspace and Z the q-orthogonal complement, so that $\dim W = m$ and $\dim Z = 8 - m$; write $a = (w, z)$ and $q(w, z) = q_W(w) + q_Z(z)$, and put $\rho = h - \lambda q$, a form in z alone that is nondegenerate on Z. The residue image is the real geometrically integral variety $A = V(h, q) = V(q, \rho) \subset \mathbb{P}^7$, and Proposition 23.1 gives normality of the standard graded complex cone ring $S = \mathbb{C}[w, z]/(q_W + q_Z, \rho)$. Let G be the real symmetric quadratic transpose Gram matrix. For type $(\alpha, s) = (5, 5)$, Lemmas 17.4 and 10.3 give it rank zero or two on A and generic rank two. Hence, by Lemma 12.7, its generic transpose kernel is a totally isotropic three-space in the standard eight-form, and the binary pairing induced by G on its image is hyperbolic over the real function field, with its two isotropic lines individually real-defined. The singular locus of A contains the stratum $\{z = 0,\ q_W(w) = 0\}$, and, when the relative spectrum is $(4, 4)$, also the stratum $\{w = 0,\ q_Z(z) = 0\}$; these are the singular strata used below (Remark 24.7). If G vanishes on one of these strata, Proposition 22.4 applies; we extract a real normal form in the remaining case, where G has rank two at the generic points of those strata.

We use divisor classes of affine normal rings. Localization of a normal domain induces a surjection on divisor class groups whose kernel is generated by the height-one primes removed [29, Theorem 6.3 and formula (6.4), p. 17]; its factoriality corollary is Nagata's criterion [29, Corollary to Theorem 6.3′, p. 18].

Lemma 24.1. *Let F be a field of characteristic zero, $\mathcal{N} = F[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$, $\mathfrak{m} = (z_1, z_2, z_3, z_4)$ and $P = (z_1, z_3)$. The ring $\mathcal{N} \cong F[su, sv, tu, tv]$ is a normal domain (a quadric of rank four), so its divisor classes are defined, and the class of $P\mathcal{N}_{\mathfrak{m}}$ has infinite order in* $\operatorname{Cl}(\mathcal{N}_{\mathfrak{m}})$.

Proof. In the Segre presentation $z_1 = su$, $z_3 = sv$, $z_4 = tu$, $z_2 = tv$, the valuation at P is the exponent of s. A homogeneous element of standard degree d has bidegree (d, d) in (s, t) and (u, v),

so its s-order is at most d. Hence the symbolic power $P^{(n)}$ has no nonzero homogeneous element of degree less than n, and its degree-n part contains the $n+1$ independent monomials $z_1^{n-i}z_3^i$, $0 \le i \le n$. Localization at $\mathfrak{m}$ only introduces denominators with nonzero constant term, so it preserves initial degree, and every element of $\mathfrak{m}P^{(n)}\mathcal{N}_{\mathfrak{m}}$ has initial degree at least $n+1$. So these monomials remain independent modulo $\mathfrak{m}P^{(n)}\mathcal{N}_{\mathfrak{m}}$, and $P^{(n)}\mathcal{N}_{\mathfrak{m}}$ needs at least $n+1$ generators; it is not principal for any $n \ge 1$. In a normal domain, $n[P] = 0$ would make this divisorial symbolic power principal. Hence $[P]$ has infinite order. □

Lemma 24.2. *For $m = 5$, let η, the singular-stratum prime, be the prime of S generated by the coordinates of z. Then $\mathrm{Cl}(S) \cong \mathbb{Z}/2$, and $\mathrm{Cl}(S) \to \mathrm{Cl}(S_\eta)$ is injective.*

Proof. Choose complex coordinates with $\rho = z_1z_2 - z_3^2$ and $q_W = w_1w_2 + w_3^2 + w_4^2 + w_5^2$. Inverting z_1 and eliminating z_2 gives $S[z_1^{-1}] = \mathcal{R}[w_1, \dots, w_5]/(w_1w_2 + g)$ with $\mathcal{R} = \mathbb{C}[z_1, z_1^{-1}, z_3]$ and $g = w_3^2 + w_4^2 + w_5^2 + q_Z(z_1, z_3^2/z_1, z_3)$. Over $\mathrm{Frac}(\mathcal{R})$ the quadratic leading part of g in the three w variables has rank three, whereas a product of affine linear factors would have leading rank at most two; so g is irreducible, and it is primitive over the factorial ring $\mathcal{R}$. Thus w_1 is prime in $S[z_1^{-1}]$. Inverting w_1 also eliminates w_2 and gives a Laurent polynomial ring, so $S[z_1^{-1}]$ is factorial by Nagata's criterion.

The only height-one prime above z_1 is $P = (z_1, z_3)$: the relation modulo z_1 is $z_3^2 = 0$, and the reduced quotient has a rank-five w-quadratic part and is a domain. At P, z_2 is a unit and $z_1 = z_3^2/z_2$. Hence $\mathrm{div}(z_1) = 2P$, and localization shows that $\mathrm{Cl}(S)$ is generated by $[P]$ with $2[P] = 0$.

At η, w_2 is invertible, and eliminating w_1 identifies S_η exactly with the vertex local ring of $z_1z_2 = z_3^2$ over $F = \mathbb{C}(w_2, w_3, w_4, w_5)$. The three z variables are independent in its cotangent space, where (z_1, z_3) has a two-dimensional image, so (z_1, z_3) is not principal. Being a height-one prime of a normal local domain, it is divisorial, so its class is nonzero locally. A cyclic group of order dividing two whose generator has nonzero image maps injectively. This proves both assertions. □

Lemma 24.3. *Suppose $m = 4$ and q_Z is not proportional to ρ. For the prime η of S generated by the coordinates of z, the map $\mathrm{Cl}(S) \to \mathrm{Cl}(S_\eta)$ is injective, and $\mathrm{Cl}(S) \cong \mathbb{Z}$.*

Proof. In complex coordinates with $\rho = z_1z_2 - z_3z_4$ and $q_W = w_1w_2 + w_3w_4$, inverting z_1 gives $S[z_1^{-1}] = \mathcal{R}[w_1, \dots, w_4]/(w_1w_2 + w_3w_4 + \psi)$ with $\mathcal{R} = \mathbb{C}[z_1, z_1^{-1}, z_3, z_4]$ and $\psi = q_Z(z_1, z_3z_4/z_1, z_3, z_4)$. Our hypothesis gives $\psi \neq 0$: otherwise q_Z would vanish on the irreducible ρ-quadric and, being quadratic, would be proportional to ρ. The polynomial $w_3w_4 + \psi$ is irreducible over $\mathrm{Frac}(\mathcal{R})$ and primitive over $\mathcal{R}$, so w_1 is prime; inverting it produces a Laurent polynomial ring, and as before $S[z_1^{-1}]$ is factorial.

The two height-one primes above z_1 are $P = (z_1, z_3)$ and $P' = (z_1, z_4)$; their quotients are domains because the w-quadratic has rank four. The equation $z_1z_2 = z_3z_4$ gives z_1 valuation one at each, so $\mathrm{div}(z_1) = P + P'$, and consequently $\mathrm{Cl}(S)$ is cyclic, generated by $[P]$. At η, invert w_2 and eliminate $w_1 = -(w_3w_4 + q_Z(z))/w_2$: the local ring is exactly $\left(F[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4)\right)_{(z_1,z_2,z_3,z_4)}$ with $F = \mathbb{C}(w_2, w_3, w_4)$. There P maps to the plane prime of Lemma 24.1, of infinite order, so the map from the cyclic global class group is injective, which proves the claim. □

When q_Z is proportional to ρ, the relative spectrum has two four-dimensional eigenspaces, and the two defining equations give a product of two rank-four quadric cones.

Lemma 24.4. *Let $S = \mathbb{C}[w_1, \dots, w_4, z_1, \dots, z_4]/(w_1w_2 - w_3w_4,\ z_1z_2 - z_3z_4)$, $\eta_W = (w_1, \dots, w_4)$ and $\eta_Z = (z_1, \dots, z_4)$. Then $\mathrm{Cl}(S) \cong \mathbb{Z}^2$, and $\mathrm{Cl}(S) \to \mathrm{Cl}(S_{\eta_W}) \oplus \mathrm{Cl}(S_{\eta_Z})$ is injective.*

Proof. Inverting w_1 and z_1 eliminates w_2, z_2 and gives a Laurent polynomial ring. The removed height-one primes are $P_W = (w_1, w_3)$, $P'_W = (w_1, w_4)$, $P_Z = (z_1, z_3)$ and $P'_Z = (z_1, z_4)$, and the principal divisors of w_1, z_1 give $[P'_W] = -[P_W]$ and $[P'_Z] = -[P_Z]$; so $[P_W], [P_Z]$ generate the class group. At η_W the local ring is that of the cone $w_1w_2 = w_3w_4$ at its vertex, over the function field of the z-quadric; the class of P_W has infinite order by Lemma 24.1, while P_Z becomes the unit ideal since $z_1 \notin \eta_W$. At η_Z the roles reverse. So $n[P_W] + k[P_Z]$ vanishes in both local groups only if $n = k = 0$, which proves injectivity and excludes every relation between the two global generators. □

Lemma 24.5 (Local splitting)**.** *Let T be a normal local domain in which 2 is invertible. A unimodular symmetric bilinear form on a free T-module of rank two which is hyperbolic over* Frac(T) *splits over T into two free isotropic lines.*

Proof. There is a vector whose norm is a unit of T: otherwise all diagonal entries and, by evaluation on sums of basis vectors, all matrix entries would lie in the maximal ideal. Splitting it off gives $\mathrm{diag}(a, b)$ with units a, b. Generic hyperbolicity gives $r^2 = -b/a$ in the fraction field; r is integral over T, hence in T by normality, and a unit. The isotropic vectors $(r, 1), (-r, 1)$ form a basis, since $2r$ is a unit. □

Proposition 24.6. *In the setup of this section, suppose G has rank two at the generic singular-stratum prime in Lemma 24.2 or Lemma 24.3, or at* both *primes in Lemma 24.4, as applicable. Then, in ambient real coefficient coordinates, $G = qD_0 + \rho D_1 + uu^T - vv^T$ (balanced) or $G = qD_0 + \rho D_1 + cb^T + bc^T$ (unbalanced), where D_0, D_1 are constant real symmetric matrices, u, v are real linear columns, c is a nonzero constant real column, and b is a real quadratic column. No primitivity of G is assumed: common divisorial factors are allowed.*

Proof. Over Frac(S) the hyperbolic binary pairing defines two isotropic image lines of G. For each such line L let $\mathcal{L} = L \cap S^N \subset \mathrm{Frac}(S)^N$ be its saturated closure, N the matrix size. It is a rank-one reflexive module: its height-one localizations are saturated line lattices, and normality identifies their intersection with $\mathcal{L}$.

At each prime η of Lemmas 24.2–24.4 at which G has rank two by hypothesis, a unit two-minor and vanishing three-minors of G make its image a free rank-two direct summand, and the induced symmetric pairing is unimodular: viewing G as a symmetric map from the dual free module, it is an isomorphism onto its image after quotienting its kernel, and its inverse defines the pairing. By the isometry $K_0^\perp/K_0 \to \mathrm{Im}\, G$ induced by B (Lemma 12.7), this is the pairing whose generic isotropic lines were used above. Lemma 24.5 makes each line closure free at η. The injectivity in Lemmas 24.2–24.4 therefore makes each global class zero; in the case of Lemma 24.4 this uses both primes η_W and η_Z.

A class-zero rank-one reflexive module is free, so both closures are principal. As S is normal, so is its conjugation-fixed subring $S_{\mathbb{R}} = \mathbb{R}[w, z]/(q, \rho)$, and the pairing is hyperbolic over Frac($S_{\mathbb{R}}$); so Lemma 12.8 applies and gives, modulo the defining ideal, the balanced form with real linear columns u, v or the unbalanced form with a nonzero constant real column c and a real quadratic column b. Real homogeneous elements lift to real ambient polynomials, and the degree-two part of the ideal of A is precisely $\mathrm{span}_{\mathbb{R}}\{q, \rho\}$, so the remainders are $qD_0 + \rho D_1$ with real symmetric D_0, D_1. □

Remark 24.7. For a single multiplicity-four or multiplicity-five block the relevant stratum is $z = 0$, $q_W(w) = 0$. For relative spectrum $(4, 4)$ there are two strata, one for each cone vertex; for the stratum $w = 0$, $q_Z(z) = 0$, apply Proposition 22.4 with W and Z exchanged, with $h = \lambda' q + \rho'$, where λ' is the relative eigenvalue on Z and $\rho' = (\lambda - \lambda')q_W$ is nondegenerate on W. If the Gram

vanishes identically on a required stratum, Proposition 22.4 applies. Otherwise its pointwise ranks zero or two give rank two at every required generic point, and Proposition 24.6 applies. The balanced and unbalanced alignment results, Propositions 22.2 and 22.3, then give $\operatorname{Im} B(a) = E_X$ for every real $a \neq 0$ at this anchor, and Section 27 concludes (Lemmas 16.2 and 16.1 on a real open set of anchors).

25 The conjugate five-dimensional images: first reductions

We work in Setting (T) of Section 12, with the transpose pencil $B(a)$ at a regular anchor with $\nu = 8$ and $G = BB^T$. Proposition 21.5 gives real coordinates $a = (x, z) \in W \oplus Z$, $\dim W = 6$, $\dim Z = 2$, with

$$q(x, z) = |x|^2 + |z|^2, \qquad h(x, z) = \lambda q(x, z) + \rho(z), \tag{25.1}$$

where the definite binary form ρ is a nonzero real multiple of $\ell\overline{\ell}$ for the complex linear form ℓ of Proposition 21.3(2). Over $\mathbb{C}$ the two null lines of ρ give distinct conjugate components $A_+ = V(h, \ell)$, $A_- = V(h, \overline{\ell})$, and $A_+ \cup A_- = V(h, q) = V(q, \rho)$ is a reduced complete intersection with ideal $(q, \rho) = (h, q)$: (q, ℓ) and $(q, \overline{\ell})$ are prime, and in the factorial ring $\mathbb{C}[a]/(q)$ [29, Theorem 8.2, p. 26] the nonassociate primes $\ell, \overline{\ell}$ satisfy $(\ell) \cap (\overline{\ell}) = (\ell\overline{\ell})$. By Proposition 21.3, Remark 21.6 and Lemmas 17.4, 17.6 and 10.3 in type $(\alpha, s) = (5, 5)$, each component spans its nonreal hyperplane and is a quadric of rank six or seven, and the Gram has rank at most two at every point of the union, generically two on each component; by Lemma 10.3 its rank is even at every point. On complex vectors $|\cdot|^2$ denotes the complex-bilinear extension of the real form. Set $\Sigma = A_+ \cap A_- = \{z = 0,\ |x|^2 = 0\}$.

Proposition 25.1. *If G vanishes identically on Σ, then $\operatorname{Im} B(a) = E$ for every nonzero real a.*

Proof. Lemma 12.9 applies with $\dim W = 6$: its hypothesis (i) is our assumption, and (ii) holds because the Gram has rank at most two at every point of $V(q, \rho) = A_+ \cup A_-$. As $\dim Z = 2$ and ρ is definite, hence nondegenerate, part (3) of that lemma gives the claim. □

Proposition 25.2. *Suppose that on A_+ one has $G = ub^T + bu^T$, where $u = u_1 + iu_2$, with $u_1, u_2 \in V$, is a fixed nonzero complex vector and b is a homogeneous quadratic vector in the component coordinate ring. Then $\operatorname{Im} B(a) = E$ for every nonzero real a.*

Proof. Let $\mathcal{F} = \operatorname{span}_{\mathbb{R}}\{u_1, u_2\}$, so $\dim \mathcal{F} \leq 2$, and let $\mathcal{P}$ be real orthogonal projection to $\mathcal{F}^{\perp}$. Then $\mathcal{P}G\mathcal{P}$ vanishes on A_+, and by reality on A_-. As the reduced union has degree-two ideal $\operatorname{span}\{h, q\}$, $\mathcal{P}G\mathcal{P} = hD_0 + qD_1$ with constant real symmetric D_0, D_1; it is positive semidefinite of rank at most eight at every real point. On $q = 1$ the Rayleigh range of h is a nontrivial interval $[t_0, t_1]$, and every $D_1 + tD_0$ with $t \in [t_0, t_1]$ is positive semidefinite, so by Lemma 12.1 all of them with interior t have one image V_0, of dimension at most eight. On the nonempty real open set of interior Rayleigh values, $(\mathcal{P}B)(\mathcal{P}B)^T$ has image V_0, so the projection of the linear pencil $\mathcal{P}B$ to $V_0^{\perp}$ vanishes there, hence identically. Thus all images of B lie in the fixed real space $\mathcal{F} + V_0$ of dimension at most ten, and Lemma 16.3 applies to the reduced union. For its component hypothesis, a nonzero real linear form vanishing on A_+ would vanish on its span $\ker \ell$, hence would be a complex multiple of ℓ and, being real, also of $\overline{\ell}$, which is impossible as $\ell, \overline{\ell}$ are independent; likewise for A_-. □

By Lemma 12.7, applied to A_+, the rank-two symmetric pairing induced by G on its image is hyperbolic over the function field of A_+.

Proposition 25.3. *If $\operatorname{Im} B(a) \neq E$ for some nonzero real a, then $G = pr^T + rp^T$ on A_+ with complex homogeneous linear columns p, r; on A_- the factorization is their conjugate.*

Proof. The ring $S_+ = \mathbb{C}[a_1, \dots, a_8]/(h, \ell)$ is a UFD by Klein–Nagata [29, Theorem 8.2, p. 26], as the restricted quadric has rank six or seven (with a polynomial extension if there is a radical variable). So the saturated rank-one reflexive modules of the two isotropic lines of the generic rank-two pairing are principal over the UFD, and Lemma 12.8 applies: $G = t(pr^T + rp^T)$ on A_+ with homogeneous $t \in S_+$ and homogeneous columns p, r of nonnegative degrees, and $\deg t + \deg p + \deg r = 2$. If a column has degree zero, absorb t and the other column into the quadratic vector of Proposition 25.2; that proposition then gives $\operatorname{Im} B(a) = E$ for every nonzero real a, contrary to the hypothesis. Otherwise both columns are linear and t is a nonzero complex constant, which we absorb into one column. □

26 Gluing the conjugate residue components

We keep Setting (T) of Section 12 and the notation of Section 25; the radial isomorphism L need not be an isometry.

Lemma 26.1. *Let $W \subset \mathbb{R}^8$ have dimension six, and let q_W be positive definite on W. There do not exist constant real symmetric operators D, D' on V and a nonzero real quadratic form s on W such that*

$$G(x) = q_W(x)D + s(x)D' \quad (x \in W), \tag{26.1}$$

where $\operatorname{rank} s \le 2$ *and D' has rank two and signature* $(1, 1)$.

Proof. As s, q_W are independent, the range of s on the ellipsoid $q_W = 1$ is a nondegenerate closed interval, and every $D + tD'$ on it is positive semidefinite of rank eight. By Lemma 12.1, equality of the ranks gives these matrices a common kernel, endpoints included, so $B(x)$ has a fixed image $\mathcal{F}$ of dimension eight for every real $x \in W \setminus 0$, and Lemma 12.5(1) gives $\mathcal{F} = E$.

By independence of q_W, s, the operators D, D' are supported on E. The radical of s has dimension at least four, and at a nonzero vector in it (26.1) shows that $D|_E$ is positive definite. So the two nonzero eigenvalues of $D^{-1/2}D'D^{-1/2}|_E$ are $\alpha > 0 > \beta$, and determinants on E give

$$\det(G(x)|_E) = \det(D|_E)\, q_W(x)^6 \big(q_W(x) + \alpha s(x)\big)\big(q_W(x) + \beta s(x)\big).$$

The last two factors are positive on the radical of s, and neither vanishes on the connected set $W \setminus 0$, because their product is positive there; so they are positive definite, irreducible real quadratics of rank six. Independence of q_W, s and $\alpha \ne \beta$ make the three quadratic factors pairwise nonassociate, so the last two occur to odd multiplicity. But in orthonormal coordinates on E the matrix $B(x)$ is square and $\det(G(x)|_E) = (\det B(x))^2$, contradicting unique factorization in the real polynomial ring on W. □

Keep the coordinates $a = (x, z) \in W \oplus Z$, the forms q, ρ, ℓ and the sets $A_\pm, \Sigma$ of Section 25, and put $q_W(x) = |x|^2$, $U_+ = \ker \ell$, $U_- = \ker \bar{\ell}$. As ρ vanishes on $U_\pm$, $A_\pm = V(q) \cap \mathbb{P}(U_\pm)$, and $\Sigma = V(q_W) \subset \mathbb{P}(W_{\mathbb{C}})$. These quadrics have rank at least six, so their homogeneous coordinate rings are factorial [29, Theorem 8.2, p. 26], also with the additional transverse variable on either component.

Proposition 26.2. *Assume that on A_+ there are complex linear columns p, r with $G = pr^T + rp^T$, that G has generic rank two on Σ, and that its induced binary pairing there is hyperbolic over $\mathbb{R}(\Sigma)$. Then there exist real linear columns $\hat{p}, \hat{r}$ on $\mathbb{R}^8$ and constant real symmetric operators D_0, D_1 such that*

$$G = qD_0 + \rho D_1 + \hat{p}\hat{r}^T + \hat{r}\hat{p}^T. \tag{26.2}$$

Proof. Write $p_0 = p|_W$, $r_0 = r|_W$. As G has rank two on Σ, they are nonzero and independent over $\mathbb{C}(\Sigma)$ and span the two isotropic lines of the induced binary pairing; hyperbolicity over $\mathbb{R}(\Sigma)$ makes each line invariant under conjugation.

Let $S_0 = \mathbb{C}[W]/(q_W)$, and suppose first that p_0 is primitive over this factorial ring. Its conjugate is primitive and spans the same fraction-field line; the proportionality factor and its inverse have nonnegative valuation at every height-one prime, so the factor is a unit of S_0, hence a nonzero constant, of modulus one by conjugating twice. Rescale p by a constant and r inversely so that p_0 is real. Reality of G on Σ gives $p_0(r_0 - \bar{r}_0)^T + (r_0 - \bar{r}_0)p_0^T = 0$, and $v \mapsto p_0 v^T + v p_0^T$ is injective (a nonzero coordinate of p_0, its diagonal entry and its row force $v = 0$, as in the proof of Lemma 12.8); so $r_0 = \bar{r}_0$ in S_0. As the ideal (q_W) has no degree-one part, both restrictions are real linear columns on W. The argument is symmetric if r_0 is primitive.

If neither restriction is primitive, homogeneity gives $p_0 = fc$ and $r_0 = gd$ with nonzero complex linear forms f, g and independent constant vectors c, d, which conjugation invariance of each constant line lets us take real by rescaling f, g. Then $D' = cd^T + dc^T$ is real symmetric of rank two and signature $(1, 1)$. Reality of G on Σ implies $fg - \overline{fg} \in \mathbb{C}q_W$; the left side has rank at most four and q_W has rank six, so $s = fg$ is a real polynomial, nonzero and of rank at most two. Lifting the quadratic identity on Σ gives $G(x, 0) = q_W(x)D + s(x)D'$ with D constant real symmetric, contrary to Lemma 26.1. So at least one restriction is primitive, and both are real.

Choose a nonzero $\zeta \in U_+ \cap Z_{\mathbb{C}}$. As $\zeta, \bar{\zeta}$ are independent and the prescriptions commute with conjugation, there is a unique real linear column $\hat{p}$ with $\hat{p}|_W = p_0$, $\hat{p}(\zeta) = p(\zeta)$ and $\hat{p}(\bar{\zeta}) = \overline{p(\zeta)}$; define $\hat{r}$ similarly. These restrict to p, r on U_+ and to their conjugates on U_-, so $G - \hat{p}\hat{r}^T - \hat{r}\hat{p}^T$ vanishes on both component quadrics. In the factorial ring $\mathbb{C}[a]/(q)$ the distinct primes $\ell, \bar{\ell}$ satisfy $(\ell) \cap (\bar{\ell}) = (\ell\bar{\ell})$, so the reduced union has quadratic ideal $\operatorname{span}_{\mathbb{C}}\{q, \rho\}$, and reality and symmetry give (26.2) with real symmetric coefficients. □

For the transpose pencil the first hypothesis is Proposition 25.3, and the last two hold unless G vanishes identically on Σ. By Lemma 10.3, $\operatorname{rank} G = 2 \operatorname{rank} B - 8$ is even and at most two at every point of $\Sigma \subset A_+$. As Σ is irreducible, either G vanishes identically on Σ, the case of Proposition 25.1, or G has generic rank two on Σ. In the latter case Lemma 12.7, applied to the real-defined quadric Σ, makes the induced binary pairing hyperbolic over $\mathbb{R}(\Sigma)$.

Finally, $u = (\hat{p} + \hat{r})/\sqrt{2}$ and $v = (\hat{p} - \hat{r})/\sqrt{2}$ turn the last term of (26.2) into $uu^T - vv^T$. As $\operatorname{span}\{q, \rho\} = \operatorname{span}\{q, h\}$, this is the real balanced form required by Proposition 22.1, whose Setting (T) survives the real change of coefficient coordinates to (x, z) (Section 12). The sixfold relative eigenvalue is extreme in this conjugate configuration, so Proposition 22.2 supplies $\lambda_2 < \lambda_7$, and Proposition 22.1 gives $\operatorname{Im} B(a) = E$ for every nonzero real a.

27 Assembly of the proof of Theorem A

27.1 The anchor set and the pole alternative

Fix R as in Theorem A. If R has two reduced eigenvalues, Theorem C applies. Theorems C, E and F classify twenty of the thirty-six multiplicity patterns on $\mathbb{R}^{16}$ that contain a multiplicity 7 or 8 (the two-eigenvalue pattern $(8, 7)$ and nineteen with three or more eigenvalues), leaving exactly sixteen:

$$(8,4,3),\ (8,3,3,1),\ (8,3,2,2),\ (8,3,2,1,1),\ (8,3,1^4);$$
$$(7,5,3),\ (7,4,4),\ (7,4,3,1),\ (7,4,2,2),\ (7,4,2,1,1),\ (7,4,1^4),$$
$$(7,3,3,2),\ (7,3,3,1,1),\ (7,3,2,2,1),\ (7,3,2,1^3),\ (7,3,1^5).$$

Shift the eigenvalue of multiplicity 8 or 7 to zero. The shifted family has constant rank $\nu = 7$ or $\nu = 8$ on $\mathbb{R}^{16} \setminus 0$, and its nonzero spectral matrix Λ is nonscalar. If Λ is definite, Theorem G shows that R is Clifford; from now on Λ is indefinite.

The construction of Lemma 9.3 applies at an anchor x whenever the generic-pair conditions of [23, Section 4, Lemmas 2(1) and 3, pp. 7–9 of arXiv:math/0204258v2] hold for (x, y) with y in a dense subset of $x^{\perp}$; the condition $\operatorname{Im} \mathcal{J}_x \cap \operatorname{Im} \mathcal{J}_y = 0$ is the nonvanishing of the sum of squares of the 2ν-minors of the 16×32 matrix $[\mathcal{J}_x \ \mathcal{J}_y]$, that is, $\operatorname{rank}[\mathcal{J}_x \ \mathcal{J}_y] = 2\nu$, a polynomial condition $g(x, y) \neq 0$. Substituting $y = (\|x\|^2 I - xx^T)z$ and expanding in z, the anchors at which $g(x, \cdot)$ vanishes identically form a real algebraic set, proper because a good generic pair exists. Remove it, together with the proper algebraic exceptional loci used in the preceding sections: the non-generic anchors of Lemma 11.1 and Propositions 11.3 and 11.4 (a semialgebraic set of dimension less than sixteen, hence inside a proper algebraic set) and the loci used for spreading out geometric factors (Remark 27.2). The result is a nonempty Euclidean-open cone $U_0 \subset \mathbb{R}^{16} \setminus 0$; its points are the *regular anchors* of Section 10 (the construction at x is understood after normalizing to $x/\|x\|$). For $x \in U_0$ we write F_x for the rational upper block and P_x for the linear lower block, of size $(16 - \nu) \times \nu$.

By Lemma 10.1, if F_x has no pole for one $x \in U_0$, then R is Clifford. So we may assume that F_x has a pole for every $x \in U_0$; the rest of the proof shows that this again leads to a Clifford tensor.

27.2 A finite semialgebraic case partition

At each $x \in U_0$ the pole data of F_x consist of finitely many discrete invariants: the real factorization type of the reduced denominator and its pole orders; the ranks of the skew residues; the real residue support d; the residue-variety dimension α and the generic transpose rank s; the number, degrees and dimensions of the complex spans of the residue images; the relative multiplicities of the coefficient pencil and the signatures involved; and whether the transpose Gram vanishes on the singular stratum. They do not depend on the choice of orthonormal eigenframes at x: two choices differ by constant orthogonal block transformations commuting with Λ, which preserve every rank, dimension, support, degree and image inclusion.

Lemma 27.1 (Real-open branch)**.** *There is a finite partition $U_0 = \mathcal{U}_1 \sqcup \cdots \sqcup \mathcal{U}_N$ into semialgebraic sets on each of which every invariant above is constant. At least one $\mathcal{U}_j$ has nonempty Euclidean interior.*

Proof. All objects have bounded degree in (x, y), so each invariant above takes only finitely many values. The geometric invariants (number, degrees, dimensions and spans of the residue images, α and s) are constructible functions of x [18, §9.7], as in Remark 27.2, and a constructible set defined over $\mathbb{R}$ has a semialgebraic set of real points. Include the image and kernel frames at x as auxiliary real variables, constrained by the polynomial orthonormality and spectral equations; by frame independence, existential and universal quantification over the frames give the same predicate. Ranks and signatures are sign conditions on minors; a factorization of bounded degree is a system of coefficient equations with auxiliary real and imaginary coefficients; the absence of real points, the positive pencil member and its relative multiplicities are quantified polynomial conditions. So each value of each invariant is a first-order formula over $\mathbb{R}$ in x, and by Tarski–Seidenberg elimination [4, Chapter 2] each such set is semialgebraic and the common refinement is finite. Outside the union of the zero sets of the finitely many nonzero polynomials in quantifier-free descriptions of the $\mathcal{U}_j$, membership in each $\mathcal{U}_j$ is locally constant; a finite union of proper real zero sets contains no nonempty open set, so some point of U_0 has a neighbourhood inside one $\mathcal{U}_j$. □

The branches need not be Zariski open, and signature boundaries may form lower-dimensional strata; the proof never uses them.

Remark 27.2 (Galois arguments). Several branch results use the action of a Galois group on the geometric factors of the pole data over $\mathbb{R}(x)$. Irreducibility over $\mathbb{R}(x)$, or a particular orbit, need not survive at a real anchor ($y^2 - x_1$ is irreducible over $\mathbb{R}(x)$ but splits over $\mathbb{R}$ when $x_1 > 0$). What persists, after a finite algebraic cover and the removal of a proper algebraic subset, are the geometric components and their algebraic invariants (ranks, degrees, spans and (α, s)): the number of geometric irreducible components of the fibres of a morphism of finite type, and these invariants of the components, are constructible functions on the base [18, §9.7]. So if Galois transitivity forces such an invariant to be equal on all conjugate geometric factors, the equality persists at the specialized anchors, even where real conjugation orbits split; real splittings and signatures are recorded by Lemma 27.1. This is the only form in which Galois transitivity is used.

27.3 The cases of the partition

Every branch is covered by one of the results below. Each either classifies R directly, contradicts the branch, or gives *full transpose alignment* throughout the real-open region of the branch: the full transpose pencil has image $E_x = \operatorname{Im} \mathcal{J}_x$ for every real coefficient $a \neq 0$, at every anchor x of the region.

(L1) $\nu = 7$: there is no pole at any anchor (Proposition 13.1).

(L2) $\nu = 8$, pole orders. The reduced denominator D satisfies $D^2 \mid \det P_x$ and has no real linear or odd-degree factor, so it is a quadratic Q, its square, a product $Q_1 Q_2$ of distinct quadratics, or a real-irreducible quartic f (Proposition 11.3). Double quadratic poles are excluded (Proposition 15.1), as is a simple quadratic pole with residue rank four (Proposition 14.1). A quartic residue has rank two by Lemma 10.4, and for $Q_1 Q_2$ the same lemma, $Q_1^{r_1} Q_2^{r_2} \mid \det P_x$, gives $r_1 = r_2 = 2$. So every remaining pole is simple, with residue rank two.

(L3) Proper real support $d \leq 7$. For a quadratic component, Q^2 divides the maximal d-minors, so $d \geq 4$; $d = 4, 5$ are excluded componentwise (Lemma 17.1). For a sole quadratic, $d = 6$ is excluded by Proposition 19.2, and both $d = 7$ alternatives, $s = 4$ and $s = 5$, by Proposition 19.9. For two quadratics, a component with $d = 6$ is excluded by Proposition 19.2, and $d = 7$ by Proposition 19.5. A real-irreducible quartic pole always has $d = 8$ (Lemma 17.3).

(L4) Full support. Incidence gives $4 \leq s \leq \alpha \leq 5$; the pair $(\alpha, s) = (5, 4)$ gives full transpose alignment (Proposition 17.5). In the remaining cases $\alpha = s$, and Lemma 17.6 makes distinct pole components have distinct residue images.

(L5) Two quadratic poles. Proposition 18.4 leaves two geometric quadratics or four geometric lines in one Galois orbit, so all geometric components have one type (α, s) (Remark 27.2). Type $(5, 5)$ is excluded by the divisor budget (Proposition 21.2), and type $(4, 4)$ is aligned by Proposition 20.2, whose hypotheses hold by Remark 20.9.

(L6) A single real pole of type $(4, 4)$. By Remark 20.9, G vanishes on every image of type $(4, 4)$, and full real support gives full complex span. A real geometrically integral image of full span has degree 4, 6 or 8, and each is treated in Proposition 20.3. A conjugate pair of images aligns (Proposition 20.8).

(L7) A single real pole of type $(5, 5)$ with one real integral image $V(h, q)$, where q is positive and the maximal relative multiplicity of the pencil (h, q) is at most five (Propositions 21.3 and 23.1). The maximal multiplicity is at most three (Propositions 23.2 and 23.3), five (Propositions 22.4 and 24.6, using Lemma 24.2), or four (Propositions 22.4 and 24.6, using Lemmas 24.3 and 24.4).

The normal forms produced there align: unbalanced forms by Proposition 22.3, balanced forms by Proposition 22.2, which covers every relative multiplicity at most five; the extraction uses Lemma 12.8, Proposition 22.4 and Lemma 24.1.

(L8) A single real pole of type $(5,5)$ with a conjugate pair of images $V(h,\ell)$, $V(h,\overline{\ell})$ (Proposition 21.3, Lemma 21.4 and Proposition 21.5). A transpose Gram vanishing on $\Sigma = V(h,\ell) \cap V(h,\overline{\ell})$ gives alignment (Proposition 25.1), and so does a constant column (Proposition 25.2, using Lemma 16.3). Otherwise Proposition 25.3 supplies complex linear columns, and Proposition 26.2 gives a global real balanced form, which aligns by Proposition 22.2.

The list is exhaustive. Items (L2)–(L4) reduce every pole at $\nu = 8$ to simple poles with rank-two residues, full real support, and type $(\alpha, s) = (4,4)$ or $(5,5)$. The denominator has one real pole component (Q or f) or two (Q_1Q_2), the latter being (L5). A single real pole component is geometrically irreducible or splits over $\mathbb{C}$ into one conjugate pair, so it has one geometrically integral real image or a conjugate pair of images; the two conjugate components of a split real pole have the same (α, s), since B is real. With the two types this gives four alternatives: the two of (L6), and (L7) and (L8); in (L7) the maximal relative multiplicity takes one of the three listed values.

27.4 Proof of Theorem A

Proof. By the reductions above we may assume that Λ is indefinite. Suppose that R is not Clifford. Then F_x has a pole at every $x \in U_0$, and $\nu = 8$ by (L1); the cases that conclude "Clifford" now give contradictions. By Lemma 27.1 some branch $\mathcal{U}_j$ contains a nonempty open set $\mathcal{U}'$ of anchors. If the result of (L1)–(L8) covering $\mathcal{U}_j$ shows that R is Clifford, or that no anchor has the invariants of $\mathcal{U}_j$, we have a contradiction. Otherwise it gives full transpose alignment: at every $x \in \mathcal{U}'$ the full transpose pencil has image E_x for every real $a \neq 0$. By Lemma 16.2, this gives $\mathcal{J}_y z = 0$ for all $y, z \in \ker \mathcal{J}_x$, $x \in \mathcal{U}'$. Lemma 16.1 applies on the open set $\mathcal{U}'$ and makes the full Gram matrix of the coherent partner radial, hence polynomial. Since Λ is nonscalar, Proposition 9.1(2) shows that R is Clifford, a contradiction. □

Acknowledgments

This work was done overwhelmingly by AI models and supervised by the author. The models were Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

Statements and Declarations

Funding. No funding was received for conducting this study.

Competing interests. The author has no competing interests to declare that are relevant to the content of this article.

Data availability. No datasets were generated or analysed during the current study. The Lean files are available at the GitHub link in the footnote on the first page; the expanded version is [17].

References

[1] V. Andrejić, *The proportionality principle for Osserman manifolds*, J. Geom. Phys. **176** (2022), 104516. doi:10.1016/j.geomphys.2022.104516.

[2] V. Andrejić, *Two-root Riemannian manifolds*, Mediterr. J. Math. **20** (2023), article 100. doi:10.1007/s00009-023-02340-2; arXiv:2009.12834v4.

[3] V. Andrejić and K. Lukić, *On the existence of a curvature tensor for given Jacobi operators*, Filomat **37**:25 (2023), 8465–8471. doi:10.2298/FIL2325465A. `https://www.pmf.ni.ac.rs/filomat-content/2023/37-25/37-25-6-19416.pdf`.

[4] J. Bochnak, M. Coste and M.-F. Roy, *Real Algebraic Geometry*, Ergeb. Math. Grenzgeb. (3) **36**, Springer, Berlin, 1998. doi:10.1007/978-3-662-03718-8.

[5] J. Bochnak and W. Kucharz, *On homology classes represented by real algebraic varieties*, Banach Center Publ. **44** (1998), 21–35. `https://eudml.org/doc/208886`.

[6] R. B. Brown and A. Gray, *Riemannian manifolds with holonomy group* Spin(9), in *Differential Geometry (in Honor of Kentaro Yano)*, Kinokuniya, Tokyo, 1972, 41–59.

[7] E. Calabi, *Linear systems of real quadratic forms*, Proc. Amer. Math. Soc. **15** (1964), 844–846. doi:10.1090/S0002-9939-1964-0166203-7.

[8] Q.-S. Chi, *A curvature characterization of certain locally rank-one symmetric spaces*, J. Differential Geom. **28** (1988), 187–202. doi:10.4310/jdg/1214442277.

[9] Q.-S. Chi, *Isoparametric hypersurfaces with four principal curvatures, IV*, J. Differential Geom. **115** (2020), 225–301. doi:10.4310/JDG/1589853626; arXiv:1605.00976.

[10] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Small schemes and varieties of minimal degree*, Amer. J. Math. **128** (2006), 1363–1389. doi:10.1353/ajm.2006.0043; page numbers refer to arXiv:math/0404517.

[11] D. Ferus, H. Karcher and H.-F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502. doi:10.1007/BF01219082. English translation by T. E. Cecil, *Clifford algebras and new isoparametric hypersurfaces*, arXiv:1112.2780.

[12] P. Finsler, *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Comment. Math. Helv. **9** (1937), 188–192. doi:10.1007/BF01258188.

[13] E. García-Río, D. N. Kupeli and R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry*, Lecture Notes in Math. **1777**, Springer, Berlin, 2002. doi:10.1007/b83213.

[14] P. B. Gilkey, *Manifolds whose curvature operator has constant eigenvalues at the basepoint*, J. Geom. Anal. **4** (1994), no. 2, 155–158. doi:10.1007/BF02921544.

[15] P. B. Gilkey, *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*, World Scientific, River Edge, NJ, 2001.

[16] P. Gilkey, A. Swann and L. Vanhecke, *Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator*, Quart. J. Math. Oxford Ser. (2) **46** (1995), no. 3, 299–320. doi:10.1093/qmath/46.3.299.

[17] D. Goel, *The Osserman (1990) conjecture is fully resolved, and other miscellanea*, expanded version, Zenodo, 2026. doi:10.5281/zenodo.23603278.

[18] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas, troisième partie*, Publ. Math. IHÉS **28** (1966), 5–255. doi:10.1007/BF02684343.

[19] W. Kryszewski, *Remarks to the Vietoris theorem*, Topol. Methods Nonlinear Anal. **8** (1996), 383–405. doi:10.12775/TMNA.1996.041; `https://projecteuclid.org/journals/topological-methods-in-nonlinear-analysis/volume-8/issue-2/Remarks-to-the-Vietoris-theorem/tmna/1479265248.full`.

[20] J. E. Martínez-Legaz, *On Brickman's theorem*, J. Convex Anal. **12** (2005), 139–143. `https://www.heldermann-verlag.de/jca/jca12/jca0474_b.pdf`.

[21] H.-F. Münzner, *Isoparametrische Hyperflächen in Sphären. I*, Math. Ann. **251** (1980), 57–71. doi:10.1007/BF01420281.

[22] Y. Nikolayevsky, *Osserman manifolds of dimension 8*, Manuscripta Math. **115** (2004), 31–53. doi:10.1007/s00229-004-0480-y; arXiv:math/0310387.

[23] Y. Nikolayevsky, *Osserman conjecture in dimension $n \neq 8, 16$*, Math. Ann. **331** (2005), 505–522. doi:10.1007/s00208-004-0580-8; arXiv:math/0204258v2; page numbers cited from the arXiv version refer to v2.

[24] Y. Nikolayevsky, *On Osserman manifolds of dimension 16*, in *Contemporary Geometry and Related Topics* (N. Bokan et al., eds.), Proceedings of the Conference, Belgrade, June 26–July 2, 2005, Faculty of Mathematics, University of Belgrade, 2006, 379–398. `https://www.maths.tcd.ie/EMIS/proceedings/CGRT2005/Articles/cgrt24.pdf`.

[25] Y. Nikolayevsky, *Conformally Osserman manifolds*, Pacific J. Math. **245** (2010), 315–358. doi:10.2140/pjm.2010.245.315; arXiv:0810.5621.

[26] Y. Nikolayevsky, *Conformally Osserman manifolds of dimension 16 and a Weyl–Schouten theorem for rank-one symmetric spaces*, Ann. Mat. Pura Appl. **191** (2012), 677–709 (published online 2011). doi:10.1007/s10231-011-0201-6. Preprint: *Osserman manifolds and the Weyl–Schouten theorem for rank-one symmetric spaces*, arXiv:0910.1645v1.

[27] Y. Nikolayevsky and Z. Rakić, *A note on Rakić duality principle for Osserman manifolds*, Publ. Inst. Math. (Beograd) (N.S.) **94**(108) (2013), 43–45. doi:10.2298/PIM1308043N; arXiv:1204.1600.

[28] R. Osserman, *Curvature in the eighties*, Amer. Math. Monthly **97** (1990), 731–756. doi:10.1080/00029890.1990.11995659.

[29] P. Samuel, *Lectures on Unique Factorization Domains*, Tata Institute of Fundamental Research, Bombay, 1964. `https://mathweb.tifr.res.in/Documents/Publications/Lectures/tifr30.pdf`.

[30] D. B. Shapiro, *Compositions of Quadratic Forms*, de Gruyter Expositions in Mathematics **33**, Walter de Gruyter, Berlin–New York, 2000.

[31] R. Wood, *Polynomial maps from spheres to spheres*, Invent. Math. **5** (1968), 163–168. doi:10.1007/BF01425547.