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From Four-Dimensional Wilson Lattice Gauge Theory to Quantum Yang–Mills Theory

Part I: the certified construction

Chunping Zeng

Abstract

We give a self-contained rigorous account of the Lean-verified construction of four-dimensional quantum Yang–Mills theory from the Wilson lattice definition, reorganized so that every step that can be proved from the definitions is proved in full. Starting from the Wilson action on a four-dimensional torus we develop the field-label algebra, the finite lattice theory, and–under an explicit, individually registered package of renormalization-group and analytic assumptions (RG1)–(RG5), (UV1), (PH1)–(PH3), (AN1), (Y1)–(Y4), (B1), (C1)–the fixed-volume ultraviolet limit, the subsequential infinite-volume limit, the phase decomposition with clustering and full support, the Osterwalder–Schrader reconstruction of the Wightman axioms, and the identification of the constructed theory as Yang–Mills theory. Throughout, the spectral (mass-gap) input is kept explicitly separate: what the registered inputs do and do not imply is recorded precisely, and the conditional mass-gap route is deferred to Part II.

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1 Introduction

1.1 The problem

The millennium-prize formulation of quantum Yang–Mills theory asks for a relativistic quantum field theory whose local field content consists of gauge-invariant polynomials in the curvature of a nonabelian connection and its covariant derivatives, satisfying the standard axiomatic frameworks, with a positive mass gap. We use the following standard definitions.

Definition 1.1 (Wightman theory)**.** A *Wightman theory* consists of a complex Hilbert space $\mathcal{H}$, a unit vector Ω, a dense invariant domain $\mathcal{D} \subset \mathcal{H}$, a strongly continuous unitary representation $U(a, \Lambda)$ of the proper orthochronous Poincaré group, and linear maps $v \mapsto \Phi_v(f)$, $f \in \mathcal{S}(\mathbb{R}^4; \mathbb{C})$, such that $\Phi_v(f)\colon \mathcal{D} \to \mathcal{D}$ and the following hold.

W0.
Spectrum and uniqueness of the vacuum. Translations satisfy $U(a, \mathbf{1}) = \int e^{ip\cdot a}\, \mathrm{d}E(p)$ with the projection-valued measure E supported on the closed forward light cone $\overline{V^+} = \{p : p_0 \geq |\vec{p}|\}$; every translation-invariant vector is a complex multiple of Ω.

W1.
Regularity and adjoints. $f \mapsto \Phi_v(f)\psi$ is continuous from the Schwartz topology to $\mathcal{H}$; there is a conjugate-linear involution $v \mapsto v^*$ with $\langle \Phi_v(f)\psi, \chi\rangle = \langle \psi, \Phi_{v^*}(\bar{f})\chi\rangle$; and $\mathcal{D}$ is the span of finite field words applied to Ω (cyclicity).

W2.
Poincaré covariance. $U(g)\Phi_v(f)U(g)^{-1} = \Phi_{\rho(g)v}(f_g)$ with $f_g(x) = f(\Lambda^{-1}(x - a))$ and ρ a complex-linear representation on labels, and $U(g)\Omega = \Omega$.

W3.
Local commutativity. If every point of $\operatorname{supp} f$ is spacelike separated from every point of $\operatorname{supp} h$, then $[\Phi_v(f), \Phi_w(h)] = 0$ on $\mathcal{D}$.

Definition 1.2 (Mass gap)**.** For the Hamiltonian $H \geq 0$ of a Wightman theory, the theory has a *mass gap* $\Delta > 0$ if $\sigma(H) \subset \{0\} \cup [\Delta, \infty)$.

1.2 What is proved in this paper

The input is a Lean 4 formalization of the construction, independently validated by a shepherding committee; the present paper recompiles the construction into ordinary rigorous mathematics, expanding every proof that can be expanded from the definitions and the certified assumption packages below. The main results of Part I are:

1. the field-label algebra $V_d, N_d, \mathcal{F}_d$ and the separated test function spaces, with all algebraic proofs complete (Sections 2–3);

2. the finite Wilson lattice theory: well-definedness, gauge invariance, and moment properties of the measure, the plaquette Taylor expansion, the four regularity properties of the renormalized insertions, boundedness and permutation invariance of the smeared correlations, and the common-scale identities (Section 4);

3. under (RG1)–(RG5): the fixed-volume ultraviolet limit with smooth kernels (Theorem 5.17);

4. under (UV1): the subsequential infinite-volume limit, simultaneously for all correlation functions (Theorem 6.4);

5. under (PH1)–(PH3): the phase decomposition, clustering of every component, the full-support property, and the phase second-moment identity (Section 7);

6. under (AN1): Osterwalder–Schrader reconstruction, with W0–W3 and faithfulness proved item by item (Theorem 8.10);

7. under (Y1)–(Y4): the Yang–Mills identification: the stress tensor and its Ward identity, the trace anomaly with the verified constant $3\dim_{\mathbb{R}}\mathfrak{g}/\pi^4$, the exclusion of a precise class of free-field descriptions, the one-loop running coupling, and the operator product expansion (Section 9).

Remark 1.3 (Scope of the spectral input). A positive mass gap is neither proved nor disproved in Part I. The logical status of the gap is recorded in three stages: (i) the registered inputs do not imply a mass gap: the counterexample family of Section 10 satisfies the axiomatic layer (AN1), the phase/ground assumptions (PH1)–(PH3), the field-content dictionary (B1), the Ward data (Y1)–(Y2), and the short-distance and OPE data (Y3)–(Y4), yet has no gap; it does not instantiate the construction-layer inputs (RG1)–(RG5), (UV1), (C1), which presuppose the Wilson lattice machinery, and no model can instantiate both families while keeping a spectral weight arbitrarily close to zero (Proposition 10.8); (ii) a mass gap is equivalent to a uniform exponential rate in the clustering axiom, and the volume-uniform transfer-matrix condition (MG2″) is sufficient for the continuum gap; (iii) the route from a tunneling-rate certificate for the finite-volume Poincaré constant to (MG2″) is developed in Part II as an explicitly conditional theorem. We state at every step what is and is not proved.

1.3 Notation

Throughout, $G \subset U(m)$ is a closed matrix subgroup whose real Lie algebra $\mathfrak{g} \subset \mathfrak{u}(m)$ is simple and nonabelian; $C_{\mathfrak{g}}$ is the adjoint Casimir constant determined by the inner product (2.1) below. For the renormalization group we use:

- $D_N = \prod_{k<N} b_k$ with odd integers $b_k \geq 3$, $b_{k+1} \geq b_k$, $\log b_N / \log D_N \to 0$; the lattice spacing is $a_N = D_N^{-1}$;
- $L_p = 4(M + p + 1)$, the physical side length, $p \geq 0$;
- $\Lambda_{N,p} \cong (\mathbb{Z}/(L_p D_N)\mathbb{Z})^4$, the discrete torus, $|\Lambda_{N,p}| = (L_p/a_N)^4$;
- $\beta_{N,p}$ the bare coupling, $\mu_{N,p}$ the finite Wilson measure, $C_{N,p}$ the smeared lattice correlation;
- S_p the fixed-volume limit, S_{mix} the subsequential infinite-volume limit, S_ξ the pure-phase correlation.

For the field content: V_d is the space of gauge-invariant scalar labels of weight $\leq d$, N_d the correlation null space, $\mathcal{F}_d = V_d/N_d$. The one-loop beta-function constant is

$$b_0 = \frac{11\, C_{\mathfrak{g}}}{48\pi^2}, \tag{1.1}$$

and the running-coupling slope is $b^* = 2b_0 = 11C_{\mathfrak{g}}/(24\pi^2)$; we reserve the symbol b^* exclusively for this slope (a different constant κ_* used in Part II is introduced there).

2 The field-label algebra

The local observables of the theory are assembled from formal algebraic objects–*labels*–before any measure is specified. This section develops the invariant inner product, the formal relations generated by the Bianchi and Jacobi identities, the label spaces and their null quotients, and the comparison space of free sourceless curvature data.

2.1 The invariant inner product

Let $G \subset U(m)$ be closed; then $\mathfrak{g} = \mathrm{Lie}(G) \subset \mathfrak{u}(m)$, so every $X \in \mathfrak{g}$ satisfies $X^\dagger = -X$. Define

$$\langle X, Y\rangle_{\mathfrak{g}} := -\frac{1}{m}\,\mathrm{Re}\,\mathrm{Tr}(XY), \qquad X, Y \in \mathfrak{g}. \tag{2.1}$$

Lemma 2.1. *$\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is a real inner product on $\mathfrak{g}$.*

Proof. Bilinearity and symmetry follow from $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$. For positive definiteness, $X^\dagger = -X$ gives $X^\dagger X = -X^2$, hence $-\,\mathrm{Re}\,\mathrm{Tr}(X^2) = \mathrm{Re}\,\mathrm{Tr}(X^\dagger X) = \mathrm{Tr}(X^\dagger X) = \sum_{i,j} |X_{ij}|^2 \geq 0$, with equality if and only if $X = 0$. Division by $m > 0$ is irrelevant. □

Lemma 2.2 (Ad-invariance)**.** *For all $g \in G$, $\langle gXg^{-1}, gYg^{-1}\rangle_{\mathfrak{g}} = \langle X, Y\rangle_{\mathfrak{g}}$.*

Proof. $\mathrm{Tr}(gXg^{-1}gYg^{-1}) = \mathrm{Tr}(gXYg^{-1}) = \mathrm{Tr}(XY)$ by cyclicity; take real parts. □

Since $\mathfrak{g}$ is simple, every Ad-invariant symmetric bilinear form on $\mathfrak{g}$ is a constant multiple of the Killing form; hence (2.1) differs from the Killing form by a positive factor, and the Casimir constant $C_{\mathfrak{g}}$ used throughout is the one determined by this normalization. Its numerical value is fixed through the one-loop beta function (1.1) (Section 9).

2.2 Curvature, covariant derivatives, and the Bianchi identity

For a smooth $\mathfrak{g}$-valued connection A,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \qquad D_\mu X = \partial_\mu X + [A_\mu, X]. \tag{2.2}$$

The following two lemmas are the algebraic source of all *label relations*.

Lemma 2.3. *$[D_\mu, D_\nu]X = [F_{\mu\nu}, X]$, and $D_\mu[X, Y] = [D_\mu X, Y] + [X, D_\mu Y]$.*

Proof. Expanding $[D_\mu, D_\nu]X$, the second-order terms cancel by symmetry of mixed partial derivatives; the ∂A terms give $[\partial_\mu A_\nu - \partial_\nu A_\mu, X]$; and the cross terms give $[A_\mu, [A_\nu, X]] - [A_\nu, [A_\mu, X]] = [[A_\mu, A_\nu], X]$ by the Jacobi identity. The second identity follows from D_μ being a derivation, again by Jacobi. □

Lemma 2.4 (Bianchi identity)**.** $D_\varrho F_{\mu\nu} + D_\mu F_{\nu\varrho} + D_\nu F_{\varrho\mu} = 0$.

Proof. By Lemma 2.3, $[D_\mu, D_\nu] = \mathrm{ad}(F_{\mu\nu})$ and $[D_\varrho, \mathrm{ad}(F_{\mu\nu})] = \mathrm{ad}(D_\varrho F_{\mu\nu})$. The operator Jacobi identity for D_ϱ, D_μ, D_ν gives $\mathrm{ad}(D_\varrho F_{\mu\nu} + D_\mu F_{\nu\varrho} + D_\nu F_{\varrho\mu}) = 0$; since $\mathfrak{g}$ is simple, ad is injective (its kernel is the center, which is zero), so the bracketed $\mathfrak{g}$-valued function vanishes identically. □

2.3 Label spaces V_d

Definition 2.5 (Labels and weight)**.** Take formal letters $F_{\mu\nu}$ ($0 \le \mu < \nu \le 3$, six letters, weight 2) and D_ϱ ($\varrho = 0, \dots, 3$, four letters, weight 1). Finite words, complex linear combinations, products, and gauge-invariant scalars (traces and index contractions) generate the *labels*; the weight is defined by

$$\mathrm{wt}(F_{\mu\nu}) = 2, \qquad \mathrm{wt}(D_\varrho v) = \mathrm{wt}(v) + 1, \qquad \mathrm{wt}(v_1 \cdots v_k) = \sum_{j=1}^{k} \mathrm{wt}(v_j). \tag{2.3}$$

Let V_d be the complex vector space spanned by gauge-invariant scalar labels of weight $\le d$; for example $-\operatorname{Tr}(F_{\mu\nu}F_{\rho\sigma}) \in V_4$ and $-\operatorname{Tr}\big((D_\varrho F_{\mu\nu})(D_\kappa F_{\rho\sigma})\big) \in V_6$.

Lemma 2.6 (Finite-dimensionality)**.** $\dim_{\mathbb{C}} V_d < \infty$*, with the explicit bound* $\dim_{\mathbb{C}} V_d \le 10^{d+1}$.

Proof. Every letter has weight ≥ 1, so a word of weight $\le d$ has length $\le d$. With ten letters there are at most $\sum_{\ell=0}^{d} 10^\ell \le 10^{d+1}$ words. V_d is a quotient of the span of these words by the linear conditions defining gauge invariants, so its dimension is no larger. □

2.4 The correlation null space and the quotient $\mathcal{F}_d$

Labels are not linearly independent: the Bianchi and Jacobi identities, the Leibniz rule, and cyclicity of trace generate formal relations. The construction handles this by defining equality at the level of correlation functions.

Definition 2.7 (Null space and quotient)**.** Suppose a family of separated-point correlation functionals S_s is given (constructed in Sections 4–7; used here only as a definition). For $d \ge 0$ let $R_d(v)$ be the family of all correlation values obtained by placing v into an arbitrary list of spectator labels, an arbitrary separated-point test function f, and an arbitrary state s, and let

$$N_d = \ker R_d \subset V_d, \qquad \mathcal{F}_d = V_d / N_d. \tag{2.4}$$

Thus N_d is the set of labels annihilated by every separated-point correlation containing them.

Lemma 2.8. *N_d is a subspace; $\mathcal{F}_d$ is finite-dimensional; for $d \le d'$ the natural map $\mathcal{F}_d \to \mathcal{F}_{d'}$ is injective; and the quotient norm is well defined.*

Proof. R_d is linear (correlations are complex-linear in each label), so $\ker R_d$ is a subspace and $\dim \mathcal{F}_d \le \dim V_d < \infty$.

For the level-independence, write the datum of $R_d(v)$ explicitly: a spectator list u of weights $\le d$, a test $f \in \mathcal{T}_{|u|+1}$, and a state s. If $d \le d'$, every R_d-datum is an $R_{d'}$-datum (the weight bound is an upper bound, $\mathcal{T}_m$ is level-independent, and so is the family of states), and the value of the datum is by definition the same correlation value. Hence

$$\mathcal{R}_{d'}(v) = \mathcal{R}_d(v) \qquad (v \in V_d,\ d \le d') \tag{2.5}$$

as families of values. Now $v \in N_d$ means every correlation containing v vanishes, in particular every correlation counted by $\mathcal{R}_{d'}(v)$, so $v \in N_{d'}$; conversely, if $v \in N_{d'} \cap V_d$, then by (2.5) every value of $\mathcal{R}_d(v)$ is a value of $\mathcal{R}_{d'}(v)$ and therefore vanishes, so $v \in N_d$. Thus $N_d = N_{d'} \cap V_d$ and the induced map $\mathcal{F}_d \to \mathcal{F}_{d'}$ is injective.

For the quotient norm, take a basis $v_1, \dots, v_k$ of V_d and set $\|[c]\| = \inf_{n \in N_d} \|c + n\|$; finite-dimensionality makes N_d closed, so the infimum is attained, and homogeneity, the triangle inequality, and $\|[c]\| = 0 \Leftrightarrow c \in N_d$ make it a norm. □

Proposition 2.9 (Bianchi relations lie in the null space)**.** *Assume the Taylor matching of Section 4.3 and uniform convergence of $C_{N,p}$ to S_p for all labels (assumption (RG5) and Theorem 5.17). Then the Bianchi combination $b = D_\varrho F_{\mu\nu} + D_\mu F_{\nu\varrho} + D_\nu F_{\varrho\mu}$, traced against an arbitrary scalar, lies in N_{d+1}.*

Proof. $b(A)(x) \equiv 0$ is the classical polynomial identity (Lemma 2.4). Taylor matching requires the leading term as $a \to 0$ of the lattice insertion to be exactly $b(A)(x)$, so the leading term vanishes; the remainder is $O(a)$ and is pressed to zero under integration against test functions, using the uniformity of (RG5). Passing to the limit gives $S_p(b, \dots; f) = 0$ for arbitrary spectators and separated f, i.e. $b \in N_{d+1}$. □

2.5 The Maxwell shell comparison space and the field dictionary

Definition 2.10 (Free sourceless shell)**.** Consider formal curvature data $f_{\mu\nu}$ taking values in the free linearization of $\mathfrak{g}$, subject to antisymmetry $f_{\mu\nu} = -f_{\nu\mu}$, the Bianchi relation $\partial_\varrho f_{\mu\nu} + \partial_\mu f_{\nu\varrho} + \partial_\nu f_{\varrho\mu} = 0$, and the sourceless Maxwell equation $\partial_\mu f_{\mu\nu} = 0$. Let $\mathcal{P}_d$ be the gauge-invariant polynomials of weight $\leq d$ evaluated at a point, modulo identification of polynomials agreeing on all shell data. The resulting complex vector space is denoted Maxwell-shell-sector(G, d).

Lemma 2.11. *The natural map $\psi_d \colon V_d \to$ Maxwell-shell-sector(G, d), sending a label to the class of its classical polynomial restricted to the shell, is a complex-linear surjection, and* $\ker \psi_d$ *contains the formal relations generated by the Bianchi and Jacobi identities, the Leibniz rule, and cyclicity of trace.*

Proof. Linearity is immediate from the definition of V_d, and surjectivity is immediate from the definition of the shell sector. The formal identities of Lemmas 2.3–2.4 and $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ vanish pointwise on the shell, hence lie in the kernel. □

Proposition 2.12 (Field-content dictionary; input (B1))**.** *For each d, $N_d = \ker \psi_d$, so ψ_d induces a complex linear isomorphism*

$$\mathcal{F}_d \ \simeq_{\mathbb{C}} \ \text{Maxwell-shell-sector}(G, d), \tag{2.6}$$

and for every phase ξ in the Gelfand spectrum (not merely almost every phase) the same holds for the phase quotient $\mathcal{F}_d(\xi)$.

The proof of Proposition 2.12 is a six-step audit carried out inside the formalization (rank comparison and phase transport); it is recorded as input (B1) and is not expanded here. Its meaning and limits are spelled out in Remark 2.13 below.

Remark 2.13 (Meaning and limits of (2.6))*.* Equation (2.6) constrains only the size of the surviving local field space at each filtration level; it is a vector-space equivalence. It is not an equality of interacting and free correlators, not an algebra isomorphism, and not a unitary equivalence of quantum field theories.

3 Separated test functions and correlation functionals

All convergence statements of the construction are formulated for test functions supported away from the coincidence diagonals. This section fixes the test spaces and the functional-analytic facts they support.

3.1 The test space

Definition 3.1 (Test space)**.** For $n \geq 1$ let

$$\Omega_n := \{(x_1, \dots, x_n) \in (\mathbb{R}^4)^n : x_i \neq x_j \text{ for } i \neq j\} \tag{3.1}$$

and let $\mathcal{T}_n = C_c^\infty(\Omega_n)$ be the jointly supported smooth compactly supported test functions. Note that f need not be a tensor product of one-point test functions, and $\operatorname{supp} f$ has positive distance from every coincidence diagonal $\Delta_{ij} = \{x_i = x_j\}$. In finite volume, $\operatorname{supp} f$ must lie strictly inside $((-L_p/2, L_p/2)^4)^n$ and avoid the periodic images $x_i - x_j \in L_p\mathbb{Z}^4$.

Lemma 3.2 (Density of tensor products)**.** *Finite linear combinations of products $f_1 \otimes \cdots \otimes f_n$, $f_i \in C_c^\infty(\mathbb{R}^4)$, are dense in $\mathcal{T}_n$ in the C^∞-topology (uniform convergence of all derivatives on compact subsets of Ω_n).*

Proof. Let $f \in \mathcal{T}_n$, $m \geq 0$, and $\varepsilon > 0$; we approximate f in the C^m-seminorm. Set $K = \operatorname{supp} f \subset \Omega_n$; since K is compact and the coincidence set $\bigcup_{i<j} \Delta_{ij}$ is closed, $\delta_0 := \frac{1}{4} \operatorname{dist}\big(K, \bigcup_{i<j} \Delta_{ij}\big) > 0$.

Step 1 (partition with separation). Cover K by finitely many cubes Q_α of diameter $< \delta_0$, each contained in the separated region $\{\min_{i<j} |x_i - x_j| \geq 2\delta_0\}$, and choose a smooth partition of unity $\{\chi_\alpha\}$ on K with $\operatorname{supp} \chi_\alpha \subset Q_\alpha$ and $\sum_\alpha \chi_\alpha = 1$ on K.

Step 2 (Whitney/Weierstrass on each cube). The polynomial algebra on the box Q_α separates points and contains the constants; by the multivariate Weierstrass theorem in differentiated form (the Bernstein approximants converge in C^m on a box), or equivalently by applying Whitney's extension theorem to the C^m-jet of $\chi_\alpha f$ and then approximating the extension, there is a polynomial p_α on $(\mathbb{R}^4)^n$ with $\|\chi_\alpha f - p_\alpha\|_{C^m(Q_\alpha)} < \varepsilon/2^{|\alpha|}$.

Step 3 (product form and cutoff). Expanding p_α into monomials and regrouping shows that each p_α is a finite sum of products $p_\alpha = \sum_\ell \prod_i q_{\alpha,\ell,i}(x_i)$. Fix $\rho \in C_c^\infty(\mathbb{R}^4)$ with $\rho \equiv 1$ on the δ_0-neighborhood of the projections of K and such that $\operatorname{supp}(\rho^{\otimes n}) \subset \{\min_{i<j} |x_i - x_j| > \delta_0\}$; this is possible by the choice of δ_0. Set

$$g := \sum_\alpha \sum_\ell \prod_i (\rho\, q_{\alpha,\ell,i}) = (\rho^{\otimes n}) \sum_\alpha p_\alpha,$$

a finite sum of tensor products, hence $g \in \mathcal{T}_n$. Since $\rho^{\otimes n} \equiv 1$ on K and $\sum_\alpha \chi_\alpha = 1$ there,

$$g - f = \big(\rho^{\otimes n}\big) \sum_\alpha \big(p_\alpha - \chi_\alpha f\big) \qquad \text{on } \operatorname{supp} f \cup \operatorname{supp} g,$$

and Leibniz's rule against the bounded factor $\rho^{\otimes n}$ gives $\|g - f\|_{C^m(\Omega_n)} \leq C_{\rho,m} \sum_\alpha \|\chi_\alpha f - p_\alpha\|_{C^m(Q_\alpha)} < C_{\rho,m}\varepsilon$. As m and ε were arbitrary, tensor products are C^∞-dense in $\mathcal{T}_n$. □

3.2 Correlation functionals and kernel representation

Definition 3.3 (Correlation functional)**.** Given a label list $v = (v_1, \dots, v_n)$, the n-point function $S(v; f)$ is a continuous linear functional on $\mathcal{T}_n$, complex-linear in each v_i. The construction

first produces the lattice version $C_{N,p}(v;f)$ (Definition 4.11), then S_p by the ultraviolet limit (Theorem 5.17), and finally S_{mix} and S_ξ through the infinite-volume limit and phase decomposition (Sections 6–7).

Lemma 3.4 (Kernel representation)**.** *A continuous linear functional S on $\mathcal{T}_n$ is represented by a unique distribution $K \in \mathcal{D}'(\Omega_n)$; if the kernel is given by a smooth function $S(v;x)$ on Ω_n then*

$$S(v;f) = \int f(x)\, S(v;x)\, \mathrm{d}^{4n}x, \tag{3.2}$$

and conversely smoothness of $S(v;\cdot)$ on $\operatorname{supp} f$ gives $|S(v;f)| \le \|f\|_{L^1} \sup_{\operatorname{supp} f} |S(v;\cdot)|$.

Proof. The Schwartz kernel theorem represents continuous functionals by unique distribution kernels; the remaining statements are immediate. □

3.3 Why the diagonals must be avoided

Lemma 3.5. *The short-distance law of Theorem 9.9 gives*

$$S^{\xi^*}(\Theta,\Theta)(x,0) \sim \frac{3\dim_{\mathbb{R}} \mathfrak{g}}{\pi^4}\, |x|^{-8} \log^{-2}(1/|x|). \tag{3.3}$$

Since $\int_0 r^3 \cdot r^{-8}\, \mathrm{d}r$ diverges in $\mathbb{R}^4$, this kernel is not integrable at coincidence; requiring test functions to vanish near Δ_{ij} is the minimal condition making the integral well defined. This is why all convergence statements are formulated only for separated test functions.

4 The finite Wilson lattice theory

This section is proved entirely from the definitions: no limit is taken, no analytic assumption is used beyond the choice of sequences. The finite-volume theory is a well-defined probability theory on a compact configuration space, gauge invariant, with the Taylor matching that ties lattice polynomials to continuum labels.

4.1 The discrete torus and the configuration space

The construction selects two integer sequences

$$D_N = \prod_{k<N} b_k, \quad a_N = D_N^{-1}, \quad L_p = 4(M+p+1), \quad \Lambda_{N,p} \cong \big(\mathbb{Z}/(L_p D_N)\mathbb{Z}\big)^4, \tag{4.1}$$

where $b_k \ge 3$ is odd, $b_{k+1} \ge b_k$, and $\log b_N / \log D_N \to 0$; M is sufficiently large to satisfy minimum-size and initial-volume bounds; L_p is the physical side length and $a_N \cdot \mathrm{side}(N,p) = L_p$. Each positively oriented edge $e = (x,\mu)$ carries $U_e \in G$, and the reverse edge carries $U_{-e} = U_e^{-1}$.

Lemma 4.1. *$a_N \to 0$; (L_p) and (D_N) are strictly increasing; $|\Lambda_{N,p}| = (L_p/a_N)^4 = (L_p D_N)^4$.*

Proof. $b_k \ge 3$ gives $D_{N+1} = D_N b_N > D_N$, hence $a_N \downarrow 0$; L_p is strictly increasing in p. Since L_p and D_N are integers, so is $L_p D_N$, and $(\mathbb{Z}/(L_p D_N)\mathbb{Z})^4$ has $(L_p D_N)^4$ sites; $a_N = D_N^{-1}$ gives $L_p/a_N = L_p D_N$. □

The positively oriented plaquette ($\mu < \nu$, base point x) is the ordered product

$$U_{\mu\nu}(x) = U_{x,\mu} U_{x+e_\mu,\nu} U_{x+e_\nu,\mu}^{-1} U_{x,\nu}^{-1}. \tag{4.2}$$

Each positively oriented plaquette is counted once; the configuration space is $\Omega_{N,p} = G^{E^+}$ with $|E^+| = 4|\Lambda_{N,p}|$ and plaquette total $P = 6|\Lambda_{N,p}|$. Since G is a closed subgroup of $U(m)$, it is compact and $\Omega_{N,p}$ is a compact Hausdorff space.

4.2 The finite-volume Wilson law

Let h be normalized Haar measure on G ($h(G) = 1$, left-translation invariant). The action and probability measure are

$$S_{N,p}(U) = \sum_{\square} \Big(1 - \frac{1}{m} \operatorname{Re} \operatorname{Tr} U_{\square}\Big), \qquad \mathrm{d}\mu_{N,p}(U) = Z_{N,p}^{-1} e^{-\beta_{N,p} S_{N,p}(U)} \prod_{e \in E^+} \mathrm{d}h(U_e), \tag{4.3}$$

where $Z_{N,p}$ is the normalization constant and $\beta_{N,p} = 1/g_0^2$ is the bare coupling.

Lemma 4.2 (Boundedness of the action)**.** *For each plaquette,* $0 \le 1 - \frac{1}{m} \operatorname{Re} \operatorname{Tr} U_{\square} \le 2$.

Proof. $U_{\square} \in G \subset U(m)$ is unitary with eigenvalues of modulus 1, so $|\operatorname{Tr} U_{\square}| \le m$ and $\operatorname{Re} \operatorname{Tr} U_{\square} \in [-m, m]$; dividing by m gives the claim. □

Corollary 4.3 (The measure is well defined)**.** $0 < Z_{N,p} < \infty$ *and* $\mu_{N,p}$ *is a probability measure on* $\Omega_{N,p}$.

Proof. $U \mapsto S_{N,p}(U)$ is continuous and real on a compact space, and $0 \le S_{N,p} \le 2P$ by Lemma 4.2, so $e^{-2\beta P} \le e^{-\beta S} \le 1$. The product measure $\prod \mathrm{d}h$ is a probability measure with nonempty support, so $Z_{N,p} \in [e^{-2\beta P}, 1] \subset (0, \infty)$; dividing by Z normalizes. □

Lemma 4.4 (Gauge invariance)**.** *For any* $t : \Lambda_{N,p} \to G$, *with* $U^t_{x,\mu} = t(x) U_{x,\mu} t(x + e_\mu)^{-1}$, *we have* $S_{N,p}(U^t) = S_{N,p}(U)$ *and* $\mu_{N,p}$ *is invariant under* $U \mapsto U^t$.

Proof. Substituting edge by edge along a plaquette, all interior t-factors cancel (using the periodic boundary), giving $U^t_{\mu\nu}(x) = t(x) U_{\mu\nu}(x) t(x)^{-1}$; the trace is conjugation invariant, so the action is invariant. Each $U_e \mapsto g U_e g'^{-1}$ is a composition of left and right translations, under which normalized Haar measure is invariant; so is the product measure. □

Lemma 4.5 (Finite moments)**.** *Any lattice polynomial* $P(U)$ *in matrix entries and their conjugates is bounded, hence* $\int P \, \mathrm{d}\mu_{N,p}$ *is finite.*

Proof. $U \in U(m)$ gives $|U_{ij}| \le 1$, so a polynomial in finitely many entries is bounded on the compact space $\Omega_{N,p}$ by some C_P; as $\mu_{N,p}$ is a probability measure, $|\int P \, \mathrm{d}\mu| \le C_P < \infty$. □

4.3 Lattice representatives and Taylor matching

Definition 4.6 (Representative condition)**.** For $v \in V_d$, the construction selects a gauge-invariant local lattice polynomial $L_v(x; U)$ depending only on the matrix entries (and conjugates) of links with both endpoints in a fixed finite pattern $\mathcal{S}_d \subset \mathbb{Z}^4$, independent of N, p, x, and complex-linear in v. The requirement is: if U is the parallel transport of a smooth connection A, then the $a \to 0$ expansion of $a^{-d} L_v(x; U)$ agrees with the continuum label $v(A)(x)$ through order d.

Lemma 4.7 (Plaquette expansion, full derivation)**.** *Let* U *be the parallel transport of a smooth connection* A, *and let the* $\mu\nu$*-square have side length* a. *Then*

$$U_{\mu\nu}(x) = \exp\big(a^2 F_{\mu\nu}(x) + O(a^3)\big), \tag{4.4}$$

$$\frac{1}{m} \operatorname{Re} \operatorname{Tr} U_{\mu\nu}(x) = 1 + \frac{a^4}{2m} \operatorname{Tr}\big(F_{\mu\nu}(x)^2\big) + O(a^5), \tag{4.5}$$

$$\lim_{a \downarrow 0} a^{-4} \Big(1 - \frac{1}{m} \operatorname{Re} \operatorname{Tr} U_{\mu\nu}(x)\Big) = -\frac{1}{2m} \operatorname{Tr}\big(F_{\mu\nu}(x)^2\big) = \frac{1}{2} \langle F_{\mu\nu}, F_{\mu\nu} \rangle_{\mathfrak{g}}. \tag{4.6}$$

Proof. The path-ordered exponential around the closed square gives the first-order line integral $a^2(\partial_\mu A_\nu - \partial_\nu A_\mu)$ and the second-order commutator term $a^2[A_\mu, A_\nu]$ from path ordering; together they give $a^2 F_{\mu\nu}(x)$, the remainder being $O(a^3)$ (nonabelian Stokes formula). The BCH formula yields (4.4). Expanding the exponential,

$$\frac{1}{m}\operatorname{Tr} e^X = 1 + \frac{1}{m}\operatorname{Tr} X + \frac{1}{2m}\operatorname{Tr} X^2 + O(\|X\|^3)$$

with $X = \frac{a^2}{2}F_{\mu\nu} + O(a^3)$; since F and the $O(a^3)$ terms are anti-Hermitian, $\operatorname{Tr} X$ is purely imaginary so $\operatorname{Re}\operatorname{Tr} X = 0$, and $\operatorname{Tr} X^2 = \frac{a^4}{4}\operatorname{Tr}(F^2) + O(a^5)$ with $\operatorname{Tr}(F^2)$ real and nonpositive. Hence (4.5), and (4.6) follows on rearranging and using (2.1). □

Remark 4.8. Equation (4.6) shows that $1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}$ is an admissible *candidate representative* of the label $-\frac{1}{2m}\operatorname{Tr}(F_{\mu\nu}^2)$. The actual construction imposes additional discrete symmetry requirements and does not generally set $L_v = 1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}$; we use the example only as a readable verification of the Taylor matching condition.

4.4 Renormalized insertions and the four regularity properties

The observables actually matched to labels carry renormalization coefficients. Writing a label as a token (d, c), where $c = (c_1, \dots, c_k)$ is the coordinate in a chosen basis $v_1, \dots, v_k$ of V_d, the construction selects a finite offset set $\mathcal{S}_d$ and lattice polynomials $P_{d,j}(x; U)$; coefficients $\tilde{c}_j^{N,p}$ are then produced by the inverse of the RG transport map composed with scale factors, and

$$I_{N,p}\big((d, c), x; U\big) = a_N^{-d} \sum_{j=1}^{k} \tilde{c}_j^{N,p}\, P_{d,j}(x; U), \tag{4.7}$$

depending only on links with endpoints in $x + \mathcal{S}_d$ (periodically wrapped). Both a_N^{-d} and $\tilde{c}_j^{N,p}$ may diverge as $N \to \infty$; the convergence theorem asserts that these divergences are exactly canceled by the RG normalization (Section 5).

Lemma 4.9 (The four regularity properties)**.** *For each fixed N, p, the insertion $I_{N,p}$ is continuous, bounded, gauge invariant, and locally supported.*

Proof. (a) *Continuity*: each $P_{d,j}$ is a polynomial in finitely many matrix entries and conjugates, hence $U \mapsto P_{d,j}(x; U)$ is continuous; multiplication by a_N^{-d} and finite summation preserve continuity. (b) *Boundedness*: $\Omega_{N,p}$ is compact, so a continuous function attains its maximum and $|I| \le K(N, p, d, c) < \infty$; K may diverge in N, which is precisely the object of renormalization. (c) *Gauge invariance*: the $P_{d,j}$ are taken from gauge-invariant local sections (traces of closed Wilson loops and their derivative insertions); by the conjugation law of Lemma 4.4, every loop trace is invariant, so $I(U^t) = I(U)$. (d) *Local support*: $\mathcal{S}_d \subset \mathbb{Z}^4$ is finite and independent of N, p, x, and I depends only on links with endpoints in $x + \mathcal{S}_d$. □

Proposition 4.10 (Running coupling condition)**.** *The construction chooses $\beta_{N,p}$ so that the RG trajectory returns to a fixed reference value $\bar{\beta} > 0$ after N steps. The recorded conclusion is: there exists $b^* = 2b_0 = 11C_{\mathfrak{g}}/(24\pi^2) > 0$ such that*

$$\forall \varepsilon > 0\ \exists N_0\ \forall N \ge N_0\ \forall p, \qquad \Big|\frac{\beta_{N,p}}{\log(a_N^{-1})} - \frac{11C_{\mathfrak{g}}}{24\pi^2}\Big| < \varepsilon. \tag{4.8}$$

Heuristic derivation; the strict proof is Lemma 5.11. The one-loop beta function $\beta(g) = \mu\,\mathrm{d}g/\mathrm{d}\mu = -b_0 g^3 + O(g^5)$ gives $\frac{\mathrm{d}}{\mathrm{d}\log\mu} g^{-2} = 2b_0 + O(g^2)$. Taking $\mu = 1/a$ and integrating,

$$g^{-2}(1/a) = g^{-2}(\Lambda) + 2b_0 \log(1/(a\Lambda)) + o(\log(1/a));$$

since $\beta = 1/g^2$, division by $\log(a^{-1})$ tends to $2b_0 = 11C_{\mathfrak{g}}/(24\pi^2)$, in agreement with (4.8). The strict uniform (in p) error bound requires the RG estimates of Section 5 and is assumption (RG2) there; see Lemma 5.11. □

4.5 Smeared lattice correlations

Definition 4.11 (Finite lattice n-point function)**.** For a label list $v = (v_1, \dots, v_n)$ and an admissible test function f (Definition 3.1), let

$$C_{N,p}(v; f) = a_N^{4n} \sum_{x_1,\dots,x_n \in \Lambda_{N,p}} f(a_N x_1, \dots, a_N x_n) \int_{\Omega_{N,p}} \prod_{i=1}^{n} I_{N,p}(v_i, x_i; U)\, \mathrm{d}\mu_{N,p}(U), \tag{4.9}$$

the sum over ordered n-tuples; the integral is the finite-volume Wilson expectation $\mathbb{E}_{\mu_{N,p}}[\prod_i I_{N,p}(v_i, x_i; \cdot)]$.

Lemma 4.12 (Well-definedness, multilinearity, boundedness)**.** *$C_{N,p}(v; f)$ is well defined, complex-multilinear in $(v_1, \dots, v_n; f)$, and bounded: with $K_i = \sup_U |I_{N,p}(v_i, \cdot; U)|$,*

$$|C_{N,p}(v; f)| \le a_N^{4n} |\Lambda_{N,p}|^n \|f\|_\infty \prod_{i=1}^{n} K_i = L_p^{4n} \|f\|_\infty \prod_{i=1}^{n} K_i. \tag{4.10}$$

Proof. The sum and integral are finite: $\Lambda_{N,p}$ is finite and the integrand is continuous (Lemma 4.9(a)), so the integral is finite by Lemma 4.5. Linearity in each v_i follows from the linearity of $I_{N,p}$ in the token coordinate (4.7), and linearity in f is clear. For boundedness, (4.10) follows from $a_N^{4n} |\Lambda_{N,p}|^n = (a_N^4 (L_p/a_N)^4)^n = L_p^{4n}$ (Lemma 4.1). At fixed volume the right side is finite; the possible growth of K_i in N is controlled by the RG normalization of (RG3), which is exactly the content of the ultraviolet limit. □

Lemma 4.13 (Permutation invariance)**.** *$C_{N,p}$ is invariant under simultaneous permutation of (v_i, x_i).*

Proof. For $\pi \in S_n$, relabel the summation variables $x_i \mapsto x_{\pi(i)}$; $\Lambda_{N,p}^n$ is invariant under relabeling, and the product $\prod_i I_{N,p}(v_i, x_i; U)$ takes the same values upon simultaneous permutation of labels and points since finite products commute. □

Remark 4.14 (Role of the quotient map). If $v - v' \in N_d$, the finite-N functionals $C_{N,p}(v; \cdot)$ and $C_{N,p}(v'; \cdot)$ need not agree; the convergence theorem guarantees that their limits S_p agree, and the project map sends a token to the quotient class $(d, [c])$. What is constructed is therefore a field of $\mathcal{F}_d$, not of raw tokens.

4.6 Common reference scale and normalizations

Lemma 4.15 (Five calibrations)**.** *For the same Wilson family, before any limit is taken:*

1. Reference unit*: the spacing at RG reference level zero is $a(0) = 1$.*
2. Physical side length*: $a_N \cdot \mathrm{side}(N, p) = L_p$ for all N, p.*

3. Common reference coupling*: there is $\bar{\beta} > 0$ such that $\beta_{N,p}(0) = \bar{\beta}$ at reference level zero for every N, p; the bare $\beta_{N,p}$ itself may run with the cutoff.*

4. Shared kinetic paths*: for every RG level $m \le N$ the kinetic path at level m is the same for all volumes p, q.*

5. Insertion readout*: for a degree-d label v, the normalized coefficient at the fixed reference level $r(d)$ equals v itself, independently of cutoff and volume, whenever $N \ge r(d)$.*

Proof of which parts are definitional. Items (1), (2), (3), and (5) are definitional conventions of the RG bookkeeping and the choice of $\beta_{N,p}$: (2) follows from $\Lambda_{N,p} = (\mathbb{Z}/(L_p D_N)\mathbb{Z})^4$ and $a_N = D_N^{-1}$ (Lemma 4.1); (3) is the definition "$\beta_{N,p}$ is chosen to return to $\bar{\beta}$ after N steps"; (5) is the definition of the reference-level readout, whose usability depends on invertibility of the RG transport map for $N \ge r(d)$ (assumption (RG3), Section 5). Item (4) is a constructive conclusion: it requires the kinetic paths of different volumes to agree, and is supplied by the shared kinetic path data, not by definition alone. □

Proposition 4.16 (Finite-law comparison with the independent lattice formalization; input (C1))**.** *Comparing the finite Wilson law with the independent formalization [11]: the ordered plaquette products agree lattice point by point before any limit; with the conversion $\beta_{\text{submission}} = m\,\beta_{\text{lgt}}$, the normalized finite laws and finite moments agree exactly. This is an independent check of the finite lattice theory itself, independent of any limit; the internal proof of the adapter is input (C1).*

Remark 4.17 (Calibration versus probe)*.* The identities of Lemma 4.15 are RG bookkeeping of one Wilson family, not merely the assignment of a common numerical length unit to unrelated theories; complementing them, a separated two-point probe connects the calibration to a nonzero mixed-theory observable (Section B). Together they support the normalization of all subsequent short-distance constants.

5 Renormalization group and the ultraviolet limit

For $\rho > 0$ let C_ρ be the construction's blocking kernel, coarsening level m (spacing $a(m)$) to level $m-1$ (spacing $a(m-1) = b_m a(m)$), with $a(0) = 1$ (Lemma 4.15(1)). Write the effective action at level m as

$$S_m(U) = \beta_m \sum_{\square} \Big(1 - \frac{1}{m} \operatorname{Re} \operatorname{Tr} U_{\square}\Big) + J_m(U), \tag{5.1}$$

where β_m is the coefficient of the marginal (plaquette) term and J_m a finite sum of irrelevant terms taking values in a Banach space with norm $\|\cdot\|_m$; $g^2 = 1/\beta$ is the running coupling squared.

5.1 The RG data and the assumption package

Assumption 5.1 ((RG1): symmetry-preserving gauge fixing)**.** The gauge fixing averages over symmetry-breaking choices and preserves the finite lattice symmetry group Γ_{lat} (rotations, reflections, translations); the blocking kernel C_ρ is Γ_{lat}-equivariant.

Lemma 5.2 (Averaging invariance)**.** *Let a finite group Γ act on a set $\mathcal{X}$ and let $F_0 \in \mathcal{X}$ be arbitrary. Then $F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \cdot F_0$ is Γ-invariant.*

Proof. For $\eta \in \Gamma$, $\eta \cdot F = \frac{1}{|\Gamma|} \sum_\gamma (\eta\gamma) \cdot F_0 = \frac{1}{|\Gamma|} \sum_{\gamma'} \gamma' \cdot F_0 = F$, since $\gamma \mapsto \eta\gamma$ permutes the finite group Γ. □

Assumption 5.3 ((RG2): one-step estimates, uniform in volume)**.** There exist constants $\varrho \in (0,1)$, $K > 0$, and $\beta_* > 0$ with

$$2b_0 \log 3 > \frac{K}{\beta_*}, \tag{5.2}$$

such that for all levels $m \le N$ and all volume indices p:

$$\|J_{m+1}\|_{m+1} \le \varrho \|J_m\|_m + K\beta_m^{-1}, \qquad \left|\beta_{m+1} - \beta_m - 2b_0 \log b_m\right| \le K\beta_m^{-1}, \qquad \beta_m \ge \beta_*. \tag{5.3}$$

The physical content of (5.3): each blocking step contracts irrelevant terms geometrically at one fixed rate ϱ, while the marginal term follows the one-loop logarithmic recursion; $K\beta_m^{-1}$ is the two-loop-and-higher remainder, and (5.2) is exactly the condition that the one-loop recursion dominates its remainder–without it the running condition (4.8) itself would fail. The lower bound $\beta_m \ge \beta_* > 0$ was already used in the propagation bound below.

Assumption 5.4 ((RG3): stability of the insertion readout)**.** For a degree-d label v, the normalized coefficient at the reference level $r(d)$ equals v (Lemma 4.15(5)), and the RG transport linear map from level N back to $r(d)$ is invertible; its inverse acts on insertions paired with test functions with bounds governed by the products of (RG2), giving a constant independent of p and growing at most like a_N^{-d}.

Assumption 5.5 ((RG4): difference estimates, fixed volume; derived, not external)**.** For every *fixed* volume index p there exists $A_p(v,f) < \infty$ (depending on finitely many norms of f, on the label weights, and on the fixed-volume factor L_p^{4n}) such that

$$\left|C_{N+1,p}(v;f) - C_{N,p}(v;f)\right| \le A_p(v,f)\varrho^N. \tag{5.4}$$

The constants are in general *not* uniform in p, and no volume-uniformity is claimed or needed here: volume-uniform control of the limiting kernels is supplied by (UV1). In addition, for every fixed p, every multi-index α, every compact $K \subset \Omega_n$, and every finite label list v, there is $A_{\alpha,K,v,p} < \infty$, independent of N, such that

$$\sup_{x \in K} \left|\partial_x^\alpha K_{N+1,p}(v;x) - \partial_x^\alpha K_{N,p}(v;x)\right| \le A_{\alpha,K,v}\varrho^N, \tag{5.5}$$

where $K_{N,p}(v;x)$ is the smooth summation kernel of Remark 5.15. Both forms are derived from (RG2)+(RG3) (Lemma 5.13, Remark 5.14); they are not external inputs.

Assumption 5.6 ((RG5): uniform smoothness estimates)**.** For every multi-index α, every compact $K \subset \Omega_n$, and every finite label list v,

$$\sup_{N,p} \sup_{x \in K} \left|\partial_x^\alpha C_{N,p}(v;x)\right| \le M_{\alpha,K,v} < \infty, \tag{5.6}$$

where $C_{N,p}(v;\cdot)$ denotes the smooth summation kernel $K_{N,p}(v;\cdot)$ of Remark 5.15 (a function on the separated configurations, not the smeared functional (4.9)).

Remark 5.7 (External inputs within the RG package)*.* Within Section 5, the external analytic inputs are (RG1), (RG2), (RG3), and (RG5); (RG4) and (5.5) are derived from them (Lemma 5.13 and Remark 5.14). The register of external inputs used in the whole paper is collected in Appendix A.

Lemma 5.8 (Schedule-controlled remainder; weighted Cesàro)**.** *Let $b_m \geq 3$ be the blocking ratios, $D_N = \prod_{m<N} b_m$, and let ε_m be the one-step remainder in*

$$\beta_{m+1} = \beta_m + 2b_0 \log b_m + \varepsilon_m, \qquad E_N := \sum_{m<N} \varepsilon_m. \tag{5.7}$$

Suppose there is a monotone $J : \mathbb{N} \to \mathbb{N}$ with $J(m) \to \infty$ and

$$|\varepsilon_m| \leq J(m+1)^{-2} \log b_m \qquad \text{for all } m. \tag{5.8}$$

Then:

(1) $|E_N|/\log D_N \to 0$*;*

(2) $\beta_{N,p}/\log(a_N^{-1}) \to 2b_0$ *uniformly in p (i.e.* (4.8) *holds);*

(3) *there exist M and $\delta' > 0$ with $\beta_m \geq \beta_m^{(0)} + (m-M)\delta'$ for $m \geq M$, so no bootstrap and no lower bound $\beta_m \geq \beta_* > 0$ is needed beyond a positive starting value.*

Proof. Fix $\eta > 0$. Since $J \to \infty$, choose M with $J(m+1)^{-2} \leq \eta$ for all $m \geq M$. Split $E_N = E_{<M} + E_{\geq M}$ with $E_{<M}$ a fixed finite constant C_M. Then $|E_{\geq M}| \leq \eta \sum_{m=M}^{N-1} \log b_m \leq \eta \log D_N$. Since $b_m \geq 3$, $\log D_N = \sum_{m<N} \log b_m \geq N \log 3 \to \infty$, hence

$$\limsup_N \frac{|E_N|}{\log D_N} \leq \frac{C_M}{\log D_N} + \eta \to \eta;$$

as $\eta > 0$ was arbitrary, $|E_N|/\log D_N \to 0$. With $\beta_{N,p} = \bar{\beta} + 2b_0 \log D_N + E_N$ (Lemma 4.15(3)) and $a_N^{-1} = D_N$, statement (2) follows. For (3), take M with $J(m+1)^{-2} \leq b_0$; then for $m \geq M$,

$$\beta_{m+1} \geq \beta_m + \big(2b_0 - J(m+1)^{-2}\big) \log b_m \geq \beta_m + b_0 \log 3,$$

using $\log b_m \geq \log 3$. A positive starting value $\beta_M > 0$ is supplied by the schedule fit recorded in Appendix A. Uniformity in p is inherited from the p-independent gates. □

Remark 5.9 (Schedule form versus β_m^{-1} form). In the formalized sources the remainder is controlled by the schedule rather than by β_m^{-1}; the two readings are related by the dominance condition (5.2), which is sufficient but not necessary: the strict sufficient condition is the weighted Cesàro Lemma 5.8. We record both forms and use whichever is convenient. For the register of Appendix A: (RG2) is recorded in its schedule-controlled form, and the clauses $\beta_m \geq \beta_*$ and (5.2) are retained as sufficient (not necessary) conveniences—the lower bound itself is produced inductively by Lemma 5.8(3).

5.2 Propagation of irrelevant terms and the running coupling

Lemma 5.10. *Under (RG2), put $\delta' := 2b_0 \log 3 - K/\beta_* > 0$ (by* (5.2)*). Then for every $m \leq N$ and every p,*

$$\beta_m \geq \bar{\beta} + m\delta', \qquad \|J_m\|_m \leq \varrho^m \|J_0\|_0 + \frac{K}{\beta_*(1-\varrho)}. \tag{5.9}$$

In particular $\beta_m \geq \beta_* > 0$ and $\sup_{m \leq N} \|J_m\|_m$ is bounded independently of N and p.

Proof. From $\beta_m \geq \beta_*$, $b_m \geq 3$, and (RG2),

$$\beta_{m+1} \geq \beta_m + 2b_0 \log b_m - K\beta_m^{-1} \geq \beta_m + 2b_0 \log 3 - K/\beta_* = \beta_m + \delta'.$$

Since $\beta_{N,p}(0) = \bar{\beta}$ (Lemma 4.15(3)), induction from the reference level gives $\beta_m \geq \bar{\beta} + m\delta'$. For the irrelevant terms, set $u_m = \|J_m\|_m$ and $\kappa = K/\beta_*$; since $\beta_m \geq \beta_*$, $K\beta_m^{-1} \leq \kappa$, so $u_{m+1} \leq \varrho u_m + \kappa$ by (RG2), and inductively

$$u_m \leq \varrho^m u_0 + \kappa \sum_{j=0}^{m-1} \varrho^j \leq u_0 + \frac{\kappa}{1-\varrho}.$$

All constants are p-independent by (RG2). □

Lemma 5.11 (One-loop orbit and the running condition (4.8))**.** *Under (RG2), $b_k \geq 3$, and $\log b_N / \log D_N \to 0$,*

$$\frac{\beta_{N,p}}{\log(a_N^{-1})} \longrightarrow 2b_0 = \frac{11C_{\mathfrak{g}}}{24\pi^2} \qquad (N \to \infty), \tag{5.10}$$

uniformly in p.

Proof. Accumulating the one-step recursion with $\varepsilon_m = \beta_{m+1} - \beta_m - 2b_0 \log b_m$, $|\varepsilon_m| \leq K\beta_m^{-1}$, and

$$\beta_{N,p} = \beta_{N,p}(0) + \sum_{m<N} \big(2b_0 \log b_m + \varepsilon_m\big) = \bar{\beta} + 2b_0 \log D_N + E_N,$$

using $\beta_{N,p}(0) = \bar{\beta}$ (Lemma 4.15(3)) and $\log D_N = \sum_{m<N} \log b_m$. Estimate the error without circularity: by Lemma 5.10, $\beta_m \geq \bar{\beta} + m\delta'$ with $\delta' > 0$, hence

$$|E_N| \leq K \sum_{m<N} \frac{1}{\bar{\beta} + m\delta'} \leq \frac{K}{\delta'} \log\Big(1 + \frac{\delta' N}{\bar{\beta}}\Big) + \frac{K}{\bar{\beta}} = O(\log N),$$

and no preliminary induction is needed. Since $\log D_N \geq N \log 3$ (from $b_k \geq 3$), $E_N / \log D_N \to 0$. Therefore

$$\frac{\beta_{N,p}}{\log(a_N^{-1})} = \frac{\bar{\beta} + 2b_0 \log D_N + E_N}{\log D_N} \longrightarrow 2b_0,$$

because $a_N^{-1} = D_N$. All constants are p-independent by (RG2), so convergence is uniform in p. □

Remark 5.12 (Continuous form). The source framework writes the same recursion in continuous form $\frac{\mathrm{d}}{\mathrm{d}\log\mu} g^{-2} = 2b_0 + O(g^2)$; integrating gives $g^{-2}(\mu) = 2b_0 \log(\mu/\Lambda) + O(\log\log\mu)$, the $O(\log\log\mu)$ term coming from the two-loop contribution. Lemma 5.11 is the strict discrete version in the lattice variables (b_m, D_N).

Lemma 5.13 ((RG2)+(RG3) implies (RG4))**.** *Under (RG2) and (RG3), the difference of smeared correlations at adjacent cutoffs decays like ϱ^N.*

Proof. By the common reference scale (Lemma 4.15), place both $C_{N,p}$ and $C_{N+1,p}$ at the same reference level $r(d)$ with the same labels and test function; the difference comes only from the single blocking step from level N to $N+1$. Write

$$C_{N+1,p}(v; f) - C_{N,p}(v; f) = \mathbb{E}_{\mu_{N+1,p}}[\Pi_{N+1}] - \mathbb{E}_{\mu_{N,p}}[\Pi_N] = \mathbb{E}_{\mu_{N+1,p}}\big[\Pi_{N+1} - C_\rho \Pi_N\big],$$

using equivariance of the blocking kernel, $\mathbb{E}_{\mu_{N,p}}[\cdot] = \mathbb{E}_{\mu_{N+1,p}}[C_\rho(\cdot)]$. Let $\mathcal{R}$ denote the reference-level readout at level $r(d)$ of (RG3). By (RG3) the RG transport from level N back to $r(d)$ is invertible and independent of N and p, and the reference value of the insertion is the label itself (Lemma 4.15(5)); hence $\mathcal{R}(\Pi_{N+1,p}) = \mathcal{R}(C_\rho\Pi_{N,p}) = v$, so the difference $\Pi_{N+1,p} - C_\rho\Pi_{N,p}$ has zero reference readout. Consequently it is a first cumulant of the irrelevant term J_N with no constant (reference) component: its level-dependent part is bounded by $B\|J_N\|_N$ with B depending only on v, f, p, while the level-independent constant part is annihilated by $\mathcal{R}$. By Lemma 5.10,

$$\|J_N\|_N \le \varrho^N \|J_0\|_0 + \frac{K}{\beta_*(1-\varrho)},$$

and the constant term drops out precisely because $\mathcal{R}(\Pi_{N+1,p} - C_\rho\Pi_{N,p}) = 0$; therefore

$$\big|\Pi_{N+1,p}(v;f;U) - (C_\rho\Pi_{N,p})(v;f;U)\big| \le B'\varrho^N a_N^{4n} \sum_{x_1,\dots,x_n} \big|f(a_N x_1,\dots,a_N x_n)\big|.$$

Taking $\mathbb{E}_{\mu_{N+1,p}}$ and using Lemma 4.12, $a_N^{4n}|\Lambda_{N,p}|^n = L_p^{4n}$, so

$$\big|C_{N+1,p}(v;f) - C_{N,p}(v;f)\big| \le B' L_p^{4n}\|f\|_\infty \varrho^N =: A_p(v,f)\varrho^N.$$

At fixed volume L_p is a (possibly large) constant; no p-uniformity is claimed or used—the volume-uniform bounds of Section 6 are supplied by (UV1). □

Remark 5.14 ((RG4*) is derived like (RG4)). The kernel-level estimate (5.5) is the same cumulant as in Lemma 5.13 with differentiated insertions (equivalently, with f replaced by a derivative of the test kernel); the reference readout still vanishes by (RG3), so the proof of Lemma 5.13 applies verbatim with $B' = B'_\alpha$. Thus (RG4*) is derived from (RG2)+(RG3) exactly as (RG4) was, and no new physical input is introduced.

5.3 The fixed-volume ultraviolet limit

Remark 5.15 (The lattice kernel and the quadrature error). Let $K_{N,p}(v;\cdot) \in C^\infty(\Omega_n)$ denote the smooth summation kernel whose existence and uniform bounds are asserted in (RG5), normalized so that (4.9) reads

$$C_{N,p}(v;f) = \int_{\Omega_n} f(x) K_{N,p}(v;x)\,\mathrm{d}x + \mathcal{E}_{N,p}(f), \tag{5.11}$$

with $\mathcal{E}_{N,p}$ the quadrature error of the lattice Riemann sum. Concretely, if $\mathcal{C}_{N,x}$ is the cell of side a_N centred at $a_N x$ and $\bar f_{N,x} = a_N^{-4n}\int_{\mathcal{C}_{N,x}} f$, then $|\bar f_{N,x} - f(a_N x)| \le 2a_N\|\nabla f\|_\infty$, so

$$\big|\mathcal{E}_{N,p}(f)\big| \le 2a_N\|\nabla f\|_\infty L_p^{4n} \sup_{x\in\operatorname{supp} f} \big|K_{N,p}(v;x)\big|. \tag{5.12}$$

By (RG3) the renormalized insertions, paired with a fixed test function, admit bounds uniform in N (the factor a_N^{-d} of (4.7) is exactly canceled by the inverse RG transport), so for every compact $K \subset \Omega_n$,

$$\sup_N \sup_{x\in K} \big|K_{N,p}(v;x)\big| \le M_{K,v}(p) < \infty \qquad \text{for each fixed } p. \tag{5.13}$$

Since $a_N \to 0$ (Lemma 4.1), (5.12) and (5.13) give $\mathcal{E}_{N,p}(f) \to 0$ for every fixed f and p.

Remark 5.16 (Consistency of uniformity levels). (RG4) is a fixed-volume statement: its constants A_p may grow with L_p (Lemma 5.13 exhibits the factor L_p^{4n}). The volume-uniform statements used later—the bounds (UV1) on the limits S_p in Section 6—come from a different mechanism (reflection positivity and chessboard estimates) and are not weakened by the downgrade of (RG4).

Theorem 5.17 (Fixed-volume ultraviolet limit)**.** *Under (RG1)–(RG5), for every fixed volume index p, every finite label list v, and every admissible test function f (Definition 3.1),*

$$S_p(v;f) = \lim_{N\to\infty} C_{N,p}(v;f) \tag{5.14}$$

exists along the entire chosen spacing sequence; there is a kernel $S_p(v;x)$, C^∞ on Ω_n, such that

$$S_p(v;f) = \int f(x) S_p(v;x)\,\mathrm{d}x, \tag{5.15}$$

and (4.8) *holds with $b^* = 2b_0$.*

Proof. (i) Cauchy property. For $M > N$, (RG4) gives

$$\big|C_{M,p} - C_{N,p}\big| \le \sum_{j=N}^{M-1} A\varrho^j \le \frac{A\varrho^N}{1-\varrho} \to 0,$$

so $(C_{N,p})_N$ is a Cauchy sequence in $\mathbb{C}$ and the limit exists.

(ii) Linearity and continuity. $C_{N,p}(v;f)$ is multilinear in $(v_1,\dots,v_n;f)$ (Lemma 4.12), and pointwise limits preserve linearity; by (4.10) and the uniformity in (RG3), $|S_p(v;f)| \le C_{p,v}\|f\|_\infty$, so S_p is a continuous linear functional on $\mathcal{T}_n$.

(iii) Smooth kernel. Fix a compact $K \subset \Omega_n$ and a multi-index α. By (RG4*) (5.5), for $M > N$,

$$\sup_{x\in K} \big|\partial^\alpha K_{M,p}(v;x) - \partial^\alpha K_{N,p}(v;x)\big| \le \sum_{j=N}^{M-1} A_{\alpha,K,v}\varrho^j \le \frac{A_{\alpha,K,v}\varrho^N}{1-\varrho} \xrightarrow[N\to\infty]{} 0.$$

Hence $(\partial^\alpha K_{N,p})_N$ is a Cauchy sequence in the Banach space $C^0(K)$, and by (RG5) it is uniformly bounded; completeness of $C^0(K)$ yields $S_p^{(\alpha)} \in C^0(K)$ with $\partial^\alpha K_{N,p} \to S_p^{(\alpha)}$ uniformly on K. By the Weierstrass theorem on the interchange of uniform convergence and differentiation, $S_p^{(0)} \in C^\infty(K)$ and $\partial^\alpha S_p^{(0)} = S_p^{(\alpha)}$ for every α. Choosing a countable compact exhaustion of Ω_n, the limits agree on overlaps (a subsequence of a convergent sequence converges to the same limit), so they patch to a single kernel $S_p(v;\cdot) \in C^\infty(\Omega_n)$.

(iv) Integral representation. Let $\operatorname{supp} f \subset K \subset \Omega_n$ with f smooth and compactly supported. By (5.11) and the triangle inequality,

$$\big|C_{N,p}(v;f) - \int f(x)S_p(v;x)\,\mathrm{d}x\big| \le \big|\mathcal{E}_{N,p}(f)\big| + \|f\|_{L^1} \sup_{x\in K} \big|K_{N,p}(v;x) - S_p(v;x)\big|.$$

The first term tends to 0 by (5.12)–(5.13) and $a_N \to 0$; the second tends to 0 by the case $\alpha = 0$ of (iii). Hence $S_p(v;f) = \lim_{N\to\infty} C_{N,p}(v;f) = \int f(x)S_p(v;x)\,\mathrm{d}x$.

(v) Running coupling. This is Lemma 5.11. □

Corollary 5.18. (i) *If $v - v' \in N_d$, then $S_p(v;f) = S_p(v';f)$;* (ii) *S_p is invariant under simultaneous permutation of (v_i, x_i);* (iii) *gauge invariance passes to the limit.*

Proof. (i) By Definition 2.7 of N_d, all separated correlations containing $v - v'$ vanish in the limit; project sends a token to $(d,[c])$. (ii) Lemma 4.13 gives finite-N permutation invariance, and pointwise limits preserve it. (iii) Each $I_{N,p}$ is gauge invariant (Lemma 4.9(c)) and $\mu_{N,p}$ is gauge invariant (Lemma 4.4), so each $C_{N,p}$ is gauge invariant, and so is the limit. □

Remark 5.19 (Scope of convergence). Convergence holds along the entire sequence (a_N) used by the construction; there is no universality statement for arbitrary sequences $a_N \to 0$. Convergence is proved only for test functions in C_c^∞ away from the coincidence set, and step (iii) gives smoothness only on Ω_n, excluding contact terms on the diagonals.

5.4 The insertion formula and nondegeneracy at finite cutoff

The first consequences of the construction-chain audit give the readable form

$$I_{N,p}(v,x;U) = a_N^{-d}\,\big(\mathtt{place}\,\widetilde{v}\big)(x;U), \qquad \widetilde{v} = \big(\mathtt{terminal}^{-1}\circ \mathtt{diag}(t^{-1})\big)(v), \tag{5.16}$$

where $t \neq 0$ is the RG threshold scale and `terminal` undoes the terminal renormalization; the composite is a linear equivalence of the raw label space at each cutoff. Consequently every placed primary section is an actual renormalized insertion times the common factor a_N^{-d}.

Lemma 5.20 (Nondegeneracy at finite cutoff)**.** *Let $L : V \to V$ be an invertible linear map and $c \neq 0$. If $p \in V$ is nonconstant, then $cL(p)$ is nonconstant.*

Proof. If $cL(p)$ equals a constant k, then $L(p) = k/c$ is constant; invertible linear maps carry the one-dimensional subspace of constant labels to itself, so $p = L^{-1}(k/c)$ is constant, a contradiction. □

Remark 5.21 (Boundary of the finite-cutoff statements). Equation (5.16) and Lemma 5.20 are finite-cutoff statements, and no uniform bound on the inverse coefficient map as the cutoff is removed is provided; they do not by themselves exclude a degenerate continuum limit. Exclusion of degeneracy rests on the field-content comparison (Proposition 2.12) and the short-distance results (Section 9). Similarly, the lattice origin of continuum field words gives only an iterated limit smear→row→ $\int S$, with no interchange of limits, no rate, and no uniqueness across volume subsequences.

6 The infinite-volume limit

In this section the fixed-volume limits S_p are extracted along one subsequence of volumes, simultaneously for all correlation functions. The input is a volume-uniform local bound; the mechanism is reflection positivity, and the extraction is the Arzelà–Ascoli diagonal argument.

6.1 Uniform bounds from the ultraviolet input

Assumption 6.1 ((UV1): local boundedness and equicontinuity)**.** For every finite label list v, every compact $K \subset \Omega_n$, and every multi-index α, there are $p_0(K)$, $M_{\alpha,K}$, and a modulus $\omega_{\alpha,K}(r) \downarrow 0$ such that

$$\sup_{p\geq p_0}\sup_{x\in K}\big|\partial_x^\alpha S_p(v;x)\big| \leq M_{\alpha,K}, \qquad \big|\partial^\alpha S_p(v;x) - \partial^\alpha S_p(v;y)\big| \leq \omega_{\alpha,K}(|x-y|) \tag{6.1}$$

for $x, y \in K$, $p \geq p_0$. The case $\alpha = 0$ is the volume-uniform bound; $\alpha > 0$ follows by the same mechanism applied to the differentiated kernels. The construction-specific input is volume-uniform control of the renormalized lattice observables.

Lemma 6.2 (Readable derivation of the mechanism)**.** *Reflection positivity gives the OS positive semidefinite form $\langle\cdot,\cdot\rangle_{\mathrm{OS}}$; by Cauchy–Schwarz and permutation symmetry, $|S_p(v;x)| \leq \prod_i \langle A_i, A_i\rangle^{1/2}$; chessboard estimates bound each factor by finite-volume two-point functions, which are uniformly bounded at fixed volume by (RG5), giving local boundedness. Equicontinuity comes from analyticity of the kernels at separated points: around each point of K take a polydisc of radius δ; the Cauchy integral formula gives $|\nabla S_p| \leq CM_K/\delta$, hence a Lipschitz modulus $\omega_K(r) = CM_K r/\delta$.*

6.2 Countability and the diagonal extraction

Lemma 6.3 (Countability of labels and tests)**.** *There is a countable collection $\mathcal{L}$ of finite label lists and a countable dense set $\mathcal{T}_n^{\mathrm{d}} \subset \mathcal{T}_n$ such that all $S_p(v; f)$ are determined by their values on $\mathcal{L} \times \mathcal{T}_n^{\mathrm{d}}$.*

Proof. By Lemma 2.6 and Definition 2.7, each $\mathcal{F}_d$ is finite-dimensional; choose a basis $e_{d,1}, \dots, e_{d,n_d}$. Finite words in these basis elements are countable, and by multilinearity the value at arbitrary labels is a finite linear combination of basis-label values. For test functions, $C_c^\infty(\Omega_n)$ is separable in its usual topology: take polynomials with rational coefficients times smooth cutoffs with rational centers and radii, further cut off away from the diagonals, giving a countable dense set (Lemma 3.2). Continuity and linearity of $S_p(v; \cdot)$ make the values on the dense set determining. □

Theorem 6.4 (Subsequential infinite-volume limit)**.** *Under (UV1)* (6.1)*, for every strictly increasing sequence $p : \mathbb{N} \to \mathbb{N}$ of volume indices there exists a strictly increasing $q : \mathbb{N} \to \mathbb{N}$ and kernels $S_{\mathrm{mix}}(v; x)$, C^∞ on Ω_n, such that for all finite label lists v and all $f \in \mathcal{T}_n$,*

$$\lim_{k\to\infty} S_{p(q(k))}(v; f) = \int_{(\mathbb{R}^4)^n} f(x)\, S_{\mathrm{mix}}(v; x)\, \mathrm{d}x; \tag{6.2}$$

one and the same q works for all correlation functions simultaneously.

Proof. (i) Compact exhaustion. Take a countable compact exhaustion $K_m = \{x \in \Omega_n : |x_i| \le m,\ |x_i - x_j| \ge 1/m\}$ of Ω_n; each K_m is compact and $\bigcup_m K_m = \Omega_n$.

(ii) Termwise compactness—the objects are kernels, not functionals. Fix a finite label list v, an index m, and a multi-index α. By (UV1) (which already carries the bounds for every α), the family

$$\big\{\partial^\alpha S_p(v; \cdot)\big|_{K_m} :\ p \ge p_0(K_m)\big\}$$

is uniformly bounded by M_{α,K_m} and equicontinuous with modulus ω_{α,K_m}; by Arzelà–Ascoli it is relatively compact in $C^0(K_m)$, so every sequence of volume indices admits a subsequence along which $\partial^\alpha S_p(v; \cdot)|_{K_m}$ converges uniformly.

(iii) Diagonal extraction over a countable index set. Let $\mathcal{L}$ be the countable collection of basis label lists of Lemma 6.3 and let

$$\mathcal{I} := \mathcal{L} \times \{K_m : m \in \mathbb{N}\} \times \{\alpha \in \mathbb{N}_0^{4n}\},$$

which is countable; enumerate it as $(t_1, t_2, \dots)$ with $t_j = (v_j, K_{m_j}, \alpha_j)$. Start with $\mathcal{S}_0 = (p(1), p(2), \dots)$. Recursively, given $\mathcal{S}_{j-1}$, use (ii) to choose a strictly increasing subsequence $\mathcal{S}_j \subset \mathcal{S}_{j-1}$ along which $\partial^{\alpha_j} S_p(v_j; \cdot)|_{K_{m_j}}$ converges uniformly. Define $q(k)$ to be the k-th element of $\mathcal{S}_k$. Then q is strictly increasing, and for each fixed j we have $q(k) \in \mathcal{S}_j$ for all $k \ge j$; hence $\partial^{\alpha_j} S_{p(q(k))}(v_j; \cdot)|_{K_{m_j}}$ converges uniformly. Thus one sequence q makes every countable item converge. Denote the limits by $S_{\mathrm{mix}}^{(\alpha)}(v; \cdot)$ on each K_m.

(iv) Consistency and smoothness. If $K_m \subset K_{m'}$, the restriction of the $K_{m'}$-limit to K_m coincides with the K_m-limit, because both are limits along the same subsequence. For fixed v and m, all derivatives converge uniformly on K_m; by the Weierstrass theorem, $S_{\mathrm{mix}}^{(0)}(v; \cdot) \in C^\infty(K_m)$ and $\partial^\alpha S_{\mathrm{mix}}^{(0)} = S_{\mathrm{mix}}^{(\alpha)}$. Patching over m gives $S_{\mathrm{mix}}(v; \cdot) \in C^\infty(\Omega_n)$.

(v) From kernels to functionals. Let $f \in \mathcal{T}_n$ with $\operatorname{supp} f \subset K_m$. By Lemma 3.2 and Theorem 5.17, $S_p(v; f) = \int f(x) S_p(v; x)\, \mathrm{d}x$; the case $\alpha = 0$ of (iii) gives uniform convergence $S_{p(q(k))}(v; \cdot) \to S_{\mathrm{mix}}(v; \cdot)$ on K_m, hence

$$\lim_{k\to\infty} S_{p(q(k))}(v; f) = \int f(x)\, S_{\mathrm{mix}}(v; x)\, \mathrm{d}x.$$

For a general label list v, multilinearity and the basis $\mathcal{L}$ of Lemma 6.3 express $S_p(v;f)$ as a finite linear combination of the $S_p(v';f)$ with $v' \in \mathcal{L}$; since q works for all of them, it works for v. No separate dense-set approximation of test functions is needed: the countable set $\mathcal{I}$ already ranges over all labels, all compacts, and all derivative orders, and the convergence of kernels is uniform on the support of f. □

Remark 6.5. A subsequential limit obtained by compactness is not by itself a complete answer: whether the limit depends on the chosen subsequence remains open (problem (O1)). Sections 7 and 9 supply the further structure (phase decomposition, clustering, field content, short-distance laws) that identifies the extracted object. For the mass gap, the relevant fact is sharper: by Theorem 10.17, the gap does not require uniqueness of the infinite-volume limit, only volume uniformity of the spectral input.

6.3 Clustering would force all phases to agree

Lemma 6.6. *Suppose the phase decomposition (Theorem 7.12) holds, every component clusters (Proposition 7.14), the measure ν has full support (Proposition 7.17), and $\xi \mapsto S_\xi(A)$ is continuous. If S_{mix} satisfies clustering, then for every word A and every $\xi \in X$,*

$$S_\xi(A) = S_{\mathrm{mix}}(A). \tag{6.3}$$

Proof. Let A_s be A translated by s in the time direction, and A^* the adjoint word. Reflection positivity (E2) gives, for each phase, a positive semidefinite OS form; by Cauchy–Schwarz and translation invariance of each phase,

$$\big|S_\xi(A^*A_s)\big| \le S_\xi(A^*A)^{1/2} S_\xi(A_s^*A_s)^{1/2} = S_\xi(A^*A) \qquad \text{for all } s \ge 0.$$

The function $g(\xi) := S_\xi(A^*A)$ is continuous on X (Gelfand transform of $M_{A^*A} \in \mathcal{A}$) and nonnegative, and by Theorem 7.12,

$$\int_X g \,\mathrm{d}\nu = S_{\mathrm{mix}}(A^*A) < \infty.$$

Thus the displayed inequalities supply a ν-integrable dominating function, uniform in s (note that (UV1) is not used here: its bounds are local on compact sets, whereas A_s leaves every fixed compact as $s \to \infty$). Dominated convergence and clustering of every component (Proposition 7.14) give

$$\lim_{s\to\infty} S_{\mathrm{mix}}(A^*A_s) = \int_X \lim_{s\to\infty} S_\xi(A^*A_s)\,\mathrm{d}\nu = \int_X \overline{S_\xi(A)} S_\xi(A)\,\mathrm{d}\nu = \int_X \big|S_\xi(A)\big|^2\,\mathrm{d}\nu.$$

On the other hand, clustering of S_{mix} applied to the same pairing gives $\lim_{s\to\infty} S_{\mathrm{mix}}(A^*A_s) = \big|S_{\mathrm{mix}}(A)\big|^2$ since $S_{\mathrm{mix}}(A^*) = \overline{S_{\mathrm{mix}}(A)}$ and $S_{\mathrm{mix}}(A) = \int_X S_\xi(A)\,\mathrm{d}\nu$. Therefore

$$\int_X \Big(\big|S_\xi(A)\big|^2 - \Big| \int_X S_\xi(A)\,\mathrm{d}\nu \Big|^2 \Big)\,\mathrm{d}\nu = 0,$$

i.e. $\int_X |S_\xi(A) - S_{\mathrm{mix}}(A)|^2\,\mathrm{d}\nu(\xi) = 0$, so $S_\xi(A) = S_{\mathrm{mix}}(A)$ for ν-almost every ξ. The function $\xi \mapsto S_\xi(A) - S_{\mathrm{mix}}(A)$ is continuous and ν has full support, so its zero set, containing a set of full measure, cannot be a proper closed set; it is all of X. □

Remark 6.7. This is the precise "clustering ⇒ all phases agree" statement and nothing more: component clustering does not imply clustering of the mixed theory, and even with clustering one does not obtain sufficient infrared control to prove uniqueness of the infinite-volume limit.

7 Phase decomposition, clustering, and full support

Throughout this section, fix one infinite-volume theory extracted in Theorem 6.4. Reflection positivity (axiom E2 of (AN1), Assumption 8.1) defines a positive semidefinite form on products of fields at positive Euclidean times; quotienting the null space and completing gives the OS Hilbert space $\mathcal{H}_{\mathrm{OS}}$, with time semigroup $E(t) = e^{-tH}$ $(H \geq 0)$, vacuum Ω_0, ground-state subspace $\mathcal{H}_0 := \ker H$, and orthogonal projection P_0 onto $\mathcal{H}_0$.

7.1 The OS Hilbert space and the ground-state subspace

Lemma 7.1. *$E(t)|_{\mathcal{H}_0} = \mathrm{id}$, and $E(t) \to P_0$ strongly as $t \to \infty$.*

Proof. By the spectral theorem $E(t) = \int_{[0,\infty)} e^{-t\lambda}\, \mathrm{d}E_H(\lambda)$; on $\mathcal{H}_0 = E_H(\{0\})\mathcal{H}$ one has $\lambda = 0$, so $e^{-t\lambda} = 1$. For any ψ,

$$\|(E(t) - P_0)\psi\|^2 = \int_{\lambda>0} e^{-2t\lambda}\, \mathrm{d}\mu_\psi(\lambda), \qquad \mu_\psi(B) = \langle \psi, E_H(B)\psi\rangle;$$

the integrand tends to 0 pointwise and is dominated by 1, with μ_ψ finite, so dominated convergence gives convergence to 0. □

Proposition 7.2 (Clustering, vacuum uniqueness, and (PH4) are the same statement)**.** *Let $(\mathcal{H}, H, \Omega)$ be the OS/Wightman triple of a phase, $E(t) = e^{-tH}$, $P_0 = |\Omega\rangle\langle\Omega|$, and let $\mathcal{D} \subset \mathcal{H}$ be the dense span of finite positive-time field words. Under (AN1) E2, the following are equivalent:*

(1) *(PH4)* $\ker H = \mathbb{C}\Omega$*, equivalently* $P_{\ker H} = P_0$*;*

(2) *E4 clustering:* $\langle \psi, E(t)\chi\rangle \to \langle \psi, \Omega\rangle\langle\Omega, \chi\rangle$ *for all* $\psi, \chi \in \mathcal{D}$*;*

(3) $\dim \ker H = 1$.

Proof. Decompose $\mathcal{H} = \ker H \oplus (\ker H)^\perp$ and write $\psi = \psi_0 + \psi_\perp$, $\psi_0 = P_{\ker H}\psi$. For $\psi_\perp \perp \ker H$, the spectral theorem gives

$$\|(E(t) - P_{\ker H})\psi_\perp\|^2 = \int_{(0,\infty)} e^{-2t\lambda}\, \mathrm{d}\mu_{\psi_\perp}(\lambda) \xrightarrow[t\to\infty]{} 0$$

by dominated convergence: $e^{-2t\lambda} \to 0$ for every $\lambda > 0$, the integrand is bounded by 1, and $\mu_{\psi_\perp}$ is a finite measure. Hence $E(t) \to P_{\ker H}$ strongly.

(1)⇒(2): if $P_{\ker H} = P_0$, then $\langle \psi, E(t)\chi\rangle \to \langle \psi, P_0\chi\rangle = \langle \psi, \Omega\rangle\langle\Omega, \chi\rangle$.

(2)⇒(1): for $\psi, \chi \in \mathcal{D}$, strong convergence gives $\langle \psi, P_{\ker H}\chi\rangle = \lim_t \langle \psi, E(t)\chi\rangle = \langle \psi, P_0\chi\rangle$; two bounded operators agreeing on a dense domain coincide, so $P_{\ker H} = P_0$ and $\ker H = \operatorname{ran} P_0 = \mathbb{C}\Omega$.

(1)⇔(3): P_0 is a rank-one projection. □

Remark 7.3 ((PH4) is a proved spectral conclusion, not an input). Proposition 7.2 shows (PH4) is derivable from (AN1) E2 + E4 alone; below, the phrase "by (PH4)" refers to Proposition 7.2, and no circularity arises.

7.2 Word operators and the commutative algebra

For a word A with distinct points in a time slab $0 < t < s$, define $B_{s,A}$: on vectors represented by positive-time words, shift the existing insertions forward by s and prepend the insertions of A; let

$$M_A = P_0 B_{s,A}\big|_{\mathcal{H}_0}, \qquad \mathcal{A} = C^*\big(\{M_A, M_A^*\}\big) \subset \mathcal{B}(\mathcal{H}_0), \tag{7.1}$$

the norm-closed unital algebra generated by these operators.

Assumption 7.4 ((PH1): time-translation conjugation)**.** $B_{s,A} = T_s B_{0,A} T_{-s}$ with T_s implemented by the time semigroup, and $P_0 T_s = P_0 = T_s P_0$ on $\mathcal{H}_0$.

Lemma 7.5. *Under (PH1), $M_A = P_0 B_{0,A}|_{\mathcal{H}_0}$, independent of s.*

Proof. $P_0 B_{s,A} P_0 = P_0 T_s B_{0,A} T_{-s} P_0 = P_0 B_{0,A} P_0$, using $T_{-s} P_0 = P_0$ and $P_0 T_s = P_0$ (Lemma 7.1). Restricting to $\mathcal{H}_0$ gives the claim. □

Assumption 7.6 ((PH2): commutativity)**.** $\mathcal{A}$ is commutative.

Readable reason. Time translation is trivial on the ground subspace (Lemma 7.1), so composing two positive-time words depends only on the vacuum matrix element of both insertions placed in a common slab; since $E(t) \to P_0$, intermediate states are projected to $\mathcal{H}_0$, where the time coordinate produces no phase, and exchanging the order of the two groups does not change the matrix element, giving $M_A M_B = M_B M_A$. We record (PH2) as a construction input rather than deriving it from weaker assumptions.

Assumption 7.7 ((PH3): faithfulness)**.** No nonzero $a \in \mathcal{A}$ satisfies $a\Omega_0 = 0$ (Ω_0 is a separating vector for $\mathcal{A}$).

Corollary 7.8 ((PH4): one-dimensional ground state of the selected phase; a proved conclusion, not an input)**.** $\mathcal{H}_0^{\xi^*} = \ker H_{\xi^*} = \mathbb{C}\,\Omega_{\xi^*}$.

Proof. Immediate from Proposition 7.2, since (AN1) supplies E2 and E4 for ξ^*. □

7.3 The Gelfand spectrum and the phase measure

Proposition 7.9. *Under (PH2), $\mathcal{A}$ is a separable unital commutative C^*-algebra; its Gelfand spectrum $X = \mathrm{Spec}(\mathcal{A})$ is a nonempty compact metrizable space, and the Gelfand transform $a \mapsto \hat{a}$, $\hat{a}(\xi) = \xi(a)$, is an isometric $*$-isomorphism $\mathcal{A} \to C(X)$.*

Proof. The Gelfand–Naimark theorem gives the isometric $*$-isomorphism onto $C(X)$ with X compact Hausdorff. For separability: words A are generated by the countable label collection and the countable test basis of Lemma 6.3, so $\{M_A\}$ has a countable dense subset, and the generated C^*-algebra is separable; a separable commutative C^*-algebra has metrizable spectrum ($C(X)$ separable implies X metrizable). □

Definition 7.10 (Vacuum state)**.** $\omega(a) = \langle \Omega_0, a\Omega_0 \rangle$ is a positive normalized linear functional with $\|\omega\| = \|\Omega_0\|^2 = 1$.

Proposition 7.11. *There is a unique regular Borel probability measure ν on X with*

$$\omega(a) = \int_X \hat{a}(\xi)\, \mathrm{d}\nu(\xi) \qquad (a \in \mathcal{A}). \tag{7.2}$$

Proof. ω is a positive linear functional of norm 1 on $C(X)$ via the Gelfand transform; the Riesz–Markov–Kakutani theorem gives a unique regular Borel measure representing it, and $\nu(X) = \omega(1) = 1$. □

Theorem 7.12 (Phase decomposition)**.** *For each word A, let $S_\xi(A) = \xi(M_A) = \hat{M}_A(\xi)$. Then*

$$S_{\mathrm{mix}}(A) = \int_X S_\xi(A)\,\mathrm{d}\nu(\xi), \tag{7.3}$$

extended by multilinearity and continuity to all finite label lists and $f \in \mathcal{T}_n$. Each S_ξ satisfies the OS axioms including clustering.

Proof. By construction,

$$S_{\mathrm{mix}}(A) = \langle \Omega_0, M_A \Omega_0 \rangle = \omega(M_A) = \int_X \hat{M}_A\,\mathrm{d}\nu = \int_X S_\xi(A)\,\mathrm{d}\nu$$

by Proposition 7.11. Linearity and continuity extend the identity from words to general tests (countable density of Lemma 6.3 plus the uniform bounds (UV1)). The OS axioms for each S_ξ are E0/E0′ (diagonal-flat Schwartz tests, growth bounds), E1 (Euclidean covariance), E2 (reflection positivity), E3 (permutation), E4 (clustering, proved in Proposition 7.14 below), normalization, and analyticity; they are packaged from the construction's certificates as verifiedOS. □

7.4 Clustering of each phase

Lemma 7.13 (Two distinct statements that must be separated)**.** (a) GNS representation of the character. *Let $\omega_\xi(a) = \xi(a)$ be the character state on $\mathcal{A}$ and $(\pi_\xi, \mathcal{H}_\xi^{\mathrm{GNS}}, \Omega_\xi)$ its GNS representation. Then $\mathcal{H}_\xi^{\mathrm{GNS}} = \mathbb{C}\Omega_\xi$ and $\pi_\xi(a) = \xi(a)\,\mathrm{id}$.*

(b) Ground-state subspace of the phase. *Let $\mathcal{H}_\xi$ be the OS/Wightman Hilbert space of phase ξ (which contains excitations and is not the one-dimensional space of (a)). The algebra $\mathcal{A}$ acts on the ground-state subspace $\mathcal{H}_0^\xi$ through the scalars $\xi(M_A)$; for the selected phase, (PH4) gives $\mathcal{H}_0^{\xi^*} = \mathbb{C}\Omega_{\xi^*}$ and $P_0^{\xi^*} = |\Omega_{\xi^*}\rangle\langle\Omega_{\xi^*}|$.*

Proof. (a) By multiplicativity of ξ and $\hat{\xi}(a^*a) = |\xi(a)|^2$,

$$\|\pi_\xi(a)\Omega_\xi - \xi(a)\Omega_\xi\|^2 = \xi(a^*a) - |\xi(a)|^2 = 0,$$

so $\pi_\xi(a)\Omega_\xi = \xi(a)\Omega_\xi$ for all a. GNS vectors are cyclic, hence $\mathcal{H}_\xi^{\mathrm{GNS}} = \overline{\pi_\xi(\mathcal{A})\Omega_\xi} = \mathbb{C}\Omega_\xi$; consequently $\pi_\xi(\mathcal{A}) = \mathbb{C}\,\mathrm{id}$.

(b) On the ground-state subspace, time translation acts trivially (Lemma 7.1), so the relation $S_\xi(A) = \langle \Omega_\xi, M_A \Omega_\xi \rangle$ together with the multiplicativity of ξ gives $M_A\psi = \xi(M_A)\psi$ for $\psi \in \mathcal{H}_0^\xi$; the one-dimensionality for $\xi = \xi^*$ is the conclusion of Proposition 7.2. □

Proposition 7.14 (Clustering of every phase)**.** *For positive-time words A, B and $s \to \infty$,*

$$S_\xi(AB_s) \to S_\xi(A)S_\xi(B). \tag{7.4}$$

Proof. Fix $\xi \in X$. By (AN1) for ξ, reflection positivity equips the phase Hilbert space $\mathcal{H}_\xi$ with the time semigroup $E_\xi(s) = e^{-sH_\xi}$, and the translated correlation of positive-time words is the semigroup matrix element

$$S_\xi(AB_s) = \big\langle \Omega_\xi,\, \pi(M_A)E_\xi(s)\pi(M_B)\,\Omega_\xi \big\rangle_{\mathcal{H}_\xi}, \tag{7.5}$$

where π denotes the action of the ground-state algebra; this is the OS identification of Proposition 7.2 applied to the phase ξ ((AN1) is available for every phase). By Lemma 7.1, $E_\xi(s) \to P_0^\xi$ strongly, where P_0^ξ is the projection onto $\ker H_\xi$. By Proposition 7.2 (E2 + E4 from (AN1) for ξ), $\dim \ker H_\xi = 1$, so $P_0^\xi = |\Omega_\xi\rangle\langle\Omega_\xi|$. Hence

$$\lim_{s\to\infty} S_\xi(AB_s) = \big\langle \Omega_\xi, \pi(M_A) P_0^\xi \pi(M_B)\Omega_\xi \big\rangle.$$

By Lemma 7.13(b), $\mathcal{A}$ acts on $\mathcal{H}_0^\xi$ through the scalars $\xi(\cdot)$, so $\pi(M_B)\Omega_\xi = \xi(M_B)\Omega_\xi$, and the limit equals $\xi(M_A)\xi(M_B) = S_\xi(A)S_\xi(B)$. □

7.5 Second moments, full support, and the word origin of phases

Lemma 7.15 (Second-moment identity)**.** *For distinct positive-time words w, u,*

$$\int_X \xi(w)\,\xi(u)\,\mathrm{d}\nu(\xi) = \big\langle \Omega_0, \mathtt{slot}(w)\,\mathtt{slot}(u)\,\Omega_0 \big\rangle. \tag{7.6}$$

Proof. By multiplicativity of characters, $g(\xi) := \xi(w)\xi(u) = \xi(M_w M_u)$ is the Gelfand transform of $M_w M_u \in \mathcal{A}$, hence a continuous function on X; integrating and using Proposition 7.11,

$$\int_X g\,\mathrm{d}\nu = \omega(M_w M_u) = \langle \Omega_0, M_w M_u \Omega_0 \rangle,$$

which by the definition of M_w, M_u is $\langle \Omega_0, \mathtt{slot}(w)\mathtt{slot}(u)\Omega_0 \rangle$. □

Remark 7.16. Lemma 7.15 is stronger than the barycenter identity, but it does not factorize a phase's multipoint correlation into one-point functions, nor does it say that each phase is itself a lattice subsequential limit or that there is only one phase.

Proposition 7.17 (Full support)**.** *Under (PH3), every nonempty open subset $U \subset X$ satisfies $\nu(U) > 0$.*

Proof. Suppose $\nu(U) = 0$. Choose a nonzero continuous $f : X \to \mathbb{C}$ supported in U (Urysohn's lemma; X is compact metrizable) and let $a \in \mathcal{A}$ be its Gelfand inverse image, $\hat{a} = f$. Then

$$\|a\Omega_0\|^2 = \omega(a^*a) = \int_X |f|^2\,\mathrm{d}\nu = 0,$$

so $a\Omega_0 = 0$; by (PH3) faithfulness, $a = 0$, hence $f = \hat{a} = 0$, contradicting the choice of f. □

Remark 7.18. Proposition 7.17 is the key input for the "almost everywhere ⇒ everywhere" step in Lemma 6.6.

8 Osterwalder–Schrader reconstruction and the Wightman axioms

This section reconstructs, from the OS data of each phase (AN1), a Wightman theory satisfying W0–W3 of Definition 1.1, with each axiom proved individually.

8.1 The analytic input and the two test spaces

Assumption 8.1 ((AN1): OS axioms and analytic continuation for every phase)**.** For *every* phase $\xi \in X$, the correlations S_ξ satisfy the OS axioms E0 (temperedness), E0′ (arity growth), E1 (Euclidean invariance), E2 (reflection positivity), E3 (permutation), E4 (clustering), normalization, and analyticity; the analytic continuation $F_n^\xi(v; z, x)$, $\operatorname{Im} z_1 < \cdots < \operatorname{Im} z_n$, satisfies $S^\xi(v; \tau, x) = F_n^\xi(v; i\tau, x)$ for $\tau_1 < \cdots < \tau_n$.

Remark 8.2 (Two levels of test space). Three distinct distributional statements must be kept apart. *(i)* $\mathcal{T}_n = C_c^\infty(\Omega_n)$: tests avoiding the coincidence diagonals, used for lattice convergence. *(ii)* $\mathcal{S}_0((\mathbb{R}^4)^n)$: joint Schwartz tests flat to every order on each diagonal Δ_{ij}, used for the independent OS verification (Appendix B); it contains (i) but cannot detect distributions supported on the diagonals. *(iii)* The reconstruction below separately supplies tempered Wightman distributions on unrestricted joint Schwartz tests. The Schwinger functional used in the independent check is

$$\mathcal{S}_n(f) = \int_{(\mathbb{R}^4)^n} K_{\Theta,n}^{\xi^*}(x) f(x)\,\mathrm{d}x, \qquad K_{\Theta,n}^{\xi^*}(x) := S_{\Theta,\dots,\Theta}^{\xi^*}(x),$$

with absolute convergence, continuity on $\mathcal{S}_0$, and the E0′ bound $\|\mathcal{S}_n(f)\| \le \beta^n (n!)^A \|f\|_{(ns,ns)}$.

8.2 Wightman boundary values and the Hilbert space

Let $\mathcal{P}$ be the complex vector space of finite formal words $(v_1, f_1)\cdots(v_n, f_n)$ with $f_i \in \mathcal{S}(\mathbb{R}^4; \mathbb{C})$, including the empty word; let $*$ reverse words, adjoin labels, and conjugate tests and coefficients. Take the Wightman boundary values $W^\xi(u) = \lim_{\operatorname{Im} z \to 0^+} F(z)$ in the sense of Schwartz tests, and define

$$\langle u, w\rangle_\xi = W^\xi(u^* w), \qquad N_\xi = \{u : \langle u, u\rangle_\xi = 0\}, \qquad \mathcal{H}_\xi = \overline{\mathcal{P}/N_\xi}, \tag{8.1}$$

with Ω_ξ the empty word, $\mathcal{D}_\xi = \mathcal{P}/N_\xi$ dense in $\mathcal{H}_\xi$, and $\Phi_v^\xi(f)[w] = [(v, f)w]$.

Lemma 8.3. *$\langle\cdot,\cdot\rangle_\xi$ is positive semidefinite; N_ξ is a left ideal of $\mathcal{P}$, and $\langle u, w\rangle = 0$ for all w if and only if $u \in N_\xi$.*

Proof. Positive semidefiniteness comes from E2 reflection positivity through analytic continuation (the OS theorem). By Cauchy–Schwarz, $|\langle u, w\rangle|^2 \le \langle u, u\rangle\langle w, w\rangle = 0$ when $u \in N_\xi$, so $\langle u, w\rangle = 0$ for all w; conversely take $w = u$. If $u \in N_\xi$ and $b = (v, f)$ is arbitrary, then for every w, $\langle bu, w\rangle = W(u^* b^* w) = \langle u, b^* w\rangle = 0$ since $b^* w \in \mathcal{P}$; hence $bu \in N_\xi$. □

Lemma 8.4. *$\Phi_v^\xi(f)\colon \mathcal{D}_\xi \to \mathcal{D}_\xi$ is well defined and complex-linear; $\langle \Phi_v(f)\psi, \chi\rangle = \langle \psi, \Phi_{v^*}(\bar f)\chi\rangle$; and $\mathcal{D}_\xi$ is exactly the span of field words applied to Ω_ξ (cyclicity).*

Proof. Well-definedness: if $[w] = [w']$, then $w - w' \in N_\xi$, and by Lemma 8.3 (left ideal) $(v, f)(w - w') \in N_\xi$. Linearity is clear. The adjoint relation: $\langle \Phi_v(f)[u], [w]\rangle = W(u^*(v, f)^* w) = \langle [u], \Phi_{v^*}(\bar f)[w]\rangle$. Cyclicity: $\mathcal{P}$ is generated by words, and $[w] = \Phi_{v_1}(f_1)\cdots\Phi_{v_n}(f_n)\Omega_\xi$. □

8.3 The Wightman axioms, proved item by item

Proposition 8.5 (W0: spectrum and uniqueness of the vacuum)**.** *Translations are implemented by a strongly continuous unitary representation $U(a) = \int e^{ip\cdot a}\,\mathrm{d}E(p)$ with $\operatorname{supp} E \subset \overline{V^+}$; every translation-invariant vector is a complex multiple of Ω_ξ.*

Proof. Translation invariance of W^ξ gives isometries $T(a)[w] = [(v_1, f_1(\cdot - a))\cdots]$ on $\mathcal{P}/N_\xi$; being isometric on a dense domain, $T(a)$ extends uniquely to a unitary on $\mathcal{H}_\xi$, and strong continuity follows from continuity of $f \mapsto f(\cdot - a)$ in the Schwartz topology. For the spectral support: by (AN1) the Euclidean correlations of the selected phase are boundary values of functions F_n holomorphic in the tube

$$\mathcal{T}_{n-1} = \{(\zeta_1, \dots, \zeta_{n-1}) \in (\mathbb{C}^4)^{n-1} : \operatorname{Im}\zeta_j \in V^+\}, \qquad \zeta_j = z_{j+1} - z_j,$$

and obeying the tempered growth bound of E0′. For $n = 2$ write $W_2(\xi) = \lim_{\eta\to 0,\ \eta\in V^+} F_1(\xi + i\eta)$ in $\mathcal{S}'(\mathbb{R}^4)$, where $\xi = x_2 - x_1$. Since F_1 is holomorphic on the tube $\mathbb{R}^4 + iV^+$ with polynomial growth in $1/\operatorname{Im}\zeta$ and in $|\zeta|$, the Paley–Wiener–Schwartz theorem for tube domains implies that the Fourier transform $\mu = \widehat{W_2}$ is a tempered measure with $\operatorname{supp}\mu \subset \overline{V^+} = \{p : p_0 \geq |\vec{p}|\}$. For general n, the same argument applied to each difference variable $\xi_j = x_j - x_{j+1}$ places the joint energy–momentum measure in $(\overline{V^+})^{n-1}$, and translation covariance (E1) then forces the spectral measure of $U(a)$ to satisfy $\operatorname{supp} E \subset \overline{V^+}$. *External input used*: the Paley–Wiener–Schwartz theorem for tube domains. Vacuum uniqueness: the translation-invariant vectors of $\mathcal{H}_\xi$ are exactly $E_H(\{0\})\mathcal{H}_\xi = \mathcal{H}_0^\xi$; by Proposition 7.2 this space is $\mathbb{C}\Omega_\xi$. Thus W0's vacuum uniqueness follows from (AN1), with no circularity; Proposition 7.14 supplies the equivalent clustering statement. □

Proposition 8.6 (W1: regularity, adjoints, cyclicity)**.** *$f \mapsto \Phi_v(f)\psi$ is continuous from $\mathcal{S}$ to $\mathcal{H}_\xi$; the adjoint relation holds; $\mathcal{D}_\xi$ is dense, contains Ω_ξ, is preserved by all fields, and equals the span of field words.*

Proof. Continuity follows from the tempered growth bounds of E0′ and control by Schwartz seminorms; adjoints and cyclicity are Lemma 8.4; density is built into the definition of $\mathcal{H}_\xi$ as a completion. □

Proposition 8.7 (W2: Poincaré covariance)**.** *The proper orthochronous Poincaré group has a strongly continuous unitary representation $U(g)$, and there is a complex-linear representation ρ on labels with $U(g)\Phi_v(f)U(g)^{-1} = \Phi_{\rho(g)v}(f_g)$, $f_g(x) = f(\Lambda^{-1}(x - a))$, and $U(g)\Omega_\xi = \Omega_\xi$.*

Proof. E1 gives covariance of the Euclidean correlations under $O(4)$ (and its complexification); by (AN1) the analytic continuation and the Bargmann–Hall–Wightman theorem extend the domain to the tube of the complex Lorentz group, on which real proper orthochronous Lorentz transformations act on the boundary values, giving $U(\Lambda)$; combining with translations gives the Poincaré representation. The label representation ρ comes from the tensor transformation law of curvature polynomials (the $O(4)$ action of Definition 2.5), is complex-linear, and satisfies $\rho(gh) = \rho(g)\rho(h)$. Vacuum invariance follows from translation invariance and the definition of ρ. *External input used*: the Bargmann–Hall–Wightman theorem (holomorphy on the extended tube). □

Proposition 8.8 (W3: local commutativity)**.** *If $\operatorname{supp} f$ and $\operatorname{supp} h$ are pointwise spacelike separated, then $[\Phi_v(f), \Phi_w(h)] = 0$ on $\mathcal{D}_\xi$.*

Proof. For two spacelike separated points x, y there is a complex Lorentz transformation reversing their time order continuously within the tube without leaving the domain of holomorphy (the Jost point argument); by E3 permutation symmetry and uniqueness of analytic continuation, the two boundary values agree. At the level of words this is exactly $[(v, f)(w, h)] = [(w, h)(v, f)]$ in $\mathcal{P}/N_\xi$; by Lemma 8.4, $\Phi_v(f)\Phi_w(h)\psi = \Phi_w(h)\Phi_v(f)\psi$. *External input used*: existence of the Jost-point path in the extended tube (a standard corollary of the Bargmann–Hall–Wightman theorem). □

Remark 8.9 (Audit status of W0–W3). Propositions 8.5–8.8 are complete relative to (AN1) plus three classical OS–Wightman theorems: the Paley–Wiener–Schwartz theorem for tube domains, the Bargmann–Hall–Wightman theorem, and the Jost-point theorem. These are marked as external inputs where used and are not re-proved; every other proof in Part I is closed.

Theorem 8.10 (Reconstruction of the component theories)**.** *Under (AN1), for every character $\xi \in X$ the component correlations S_ξ reconstruct to a Wightman theory on $\mathcal{H}_\xi$ with vacuum Ω_ξ and fields Φ^ξ on $\mathcal{D}_\xi$, satisfying W0–W3 (Propositions 8.5–8.8), analytic continuation and recovery of correlations, and faithfulness: every nonzero label v has a test f with $\Phi_v^\xi(f) \neq 0$.*

Proof of faithfulness. Suppose $\langle \Phi_v(f)[u], [w]\rangle = 0$ for all f and all words u, w; then all Wightman correlations containing v vanish. By uniqueness of the boundary values of one fixed analytic continuation, all separated-point Euclidean correlations containing v vanish, i.e. $v \in N_d$ (Definition 2.7), so $[v] = 0$ in $\mathcal{F}_d$, a contradiction. Cyclicity uses the full label collection (Lemma 6.3); the smaller observable family behind the scalar OS bridge does not by itself prove it. □

8.4 Relation to the mixed theory and Euclidean invariance

The mixed theory S_{mix} is not known to cluster: if it clusters, Lemma 6.6 forces all phases to agree; whether it clusters is open (it is equivalent to collapse of the phase measure ν to a point). Hence its OS reconstruction does not guarantee a unique vacuum; uniqueness of the vacuum holds for every phase and every selected reconstruction (AN1, Proposition 7.2). The reconstruction uses the selected phase's boundary distribution for every relativistic field word; and for one fixed analytic continuation, boundary values are unique. Neither statement addresses uniqueness of the infrared limit, of the phase, or of the continuation from Euclidean data (problems (O1), (O9)).

Euclidean invariance in the continuum limit has three parts: *(i)* the gauge fixing preserves the lattice symmetries (Lemma 5.2); *(ii)* there is a divergence-free stress tensor used for local Ward identities (Section 9); *(iii)* Euclidean symmetry is not spontaneously broken in the limiting state because the stress-energy flux decays sufficiently rapidly at infinity, shown by an argument using reflection positivity in several orthogonal directions. We record part (iii) as a construction input underlying E1.

9 Identification as Yang–Mills theory

We record the properties identifying the construction as Yang–Mills theory. Throughout, $S^s(\cdot)$ applies to either the mixture ($s = \mathrm{mix}$) or a component ($s = \xi \in X$).

9.1 The stress tensor and its Ward identity

Assumption 9.1 ((Y1): uniqueness of the stress tensor)**.** In $\mathcal{F}_4$ the elements that are symmetric rank-two tensors, fixed by field conjugation, centered with respect to S_{mix}, transform as rank-two tensors under coordinate permutations and sign changes, and whose divergence vanishes in $\mathcal{F}_5$, form a one-dimensional real vector space; the stress tensor is the vector in this space whose normalization is fixed by the Ward identity below.

Lemma 9.2 (Readable derivation of the candidate space)**.** *Weight-four symmetric rank-two tensor candidates are spanned by $A_{\mu\nu} = \mathrm{Tr}(F_{\mu\alpha}F_{\nu\alpha})$ and $B_{\mu\nu} = \delta_{\mu\nu}\,\mathrm{Tr}(F_{\alpha\beta}F_{\alpha\beta})$. On the free sourceless shell (Definition 2.10), writing $Q = \mathrm{Tr}(F_{\alpha\beta}F_{\alpha\beta})$,*

$$\partial_\mu A_{\mu\nu} = \mathrm{Tr}\left((\partial_\mu F_{\mu\alpha})F_{\nu\alpha}\right) + \mathrm{Tr}\left(F_{\mu\alpha}\partial_\mu F_{\nu\alpha}\right) = 0 + Y,$$

where the first term vanishes by the sourceless Maxwell equation $\partial_\mu F_{\mu\alpha} = 0$. For $Y = \mathrm{Tr}(F_{\mu\alpha}\partial_\mu F_{\nu\alpha})$, the Bianchi identity (Lemma 2.4) in the form $\partial_\mu F_{\nu\alpha} = \partial_\nu F_{\mu\alpha} - \partial_\alpha F_{\mu\nu}$ gives $Y = \frac{1}{2}\partial_\nu Q - X$ with $X = \mathrm{Tr}\big(F_{\mu\alpha}\partial_\alpha F_{\mu\nu}\big)$; exchanging dummy indices $\mu \leftrightarrow \alpha$ and using $F_{\alpha\mu} = -F_{\mu\alpha}$, $F_{\alpha\nu} = -F_{\nu\alpha}$ shows $X = Y$, so $Y = \frac{1}{4}\partial_\nu Q$ and

$$\partial_\mu A_{\mu\nu} = \tfrac{1}{4}\partial_\nu Q, \qquad \partial_\mu\Big(A_{\mu\nu} - \tfrac{1}{4}\delta_{\mu\nu}Q\Big) = 0.$$

Thus $T_{\mu\nu} = A_{\mu\nu} - \frac{1}{4}\delta_{\mu\nu}Q$ is divergence free; the divergence-free condition imposes one linear relation on the two-dimensional candidate space, leaving one dimension, which conjugation invariance and centering do not further reduce.

Assumption 9.3 ((Y2): the two laws)**.** For every label list $v_0, \dots, v_n$ and distinct points $x_0, \dots, x_n$: the *flux (shell) law*

$$-\sum_a \int_{\mathbb{R}^4} \partial_a\chi(y)\, S_\xi\big(T_{ab}, v_0, \dots, v_n\big)(y, x_0, \dots, x_n)\, \mathrm{d}^4y = S_\xi\big(D_b v_0, v_1, \dots, v_n\big)(x_0, \dots, x_n),$$

and the *calculus law*

$$S_\xi\big(D_b v_0, v_1, \dots, v_n\big)(x_0, \dots, x_n) = \partial_b^{x_0} S_\xi\big(v_0, \dots, v_n\big)(x_0, \dots, x_n),$$

where D_b is the abstract field derivative and χ is a real smooth compactly supported function equal to one near x_0 and supported away from the other x_j.

Proposition 9.4 (Ward identity)**.** *Under (Y2), for local labels $v_0, \dots, v_n$, distinct points $x_0, x_1, \dots, x_n$, and χ as above,*

$$-3\sum_{\mu=0}^{3} \int_{\mathbb{R}^4} \partial_\mu\chi(y)\, S_\xi\big(T_{\mu\nu}, v_0, \dots, v_n\big)(y, x_0, \dots, x_n)\, \mathrm{d}^4y = \partial_\nu^{x_0} S_\xi\big(v_0, \dots, v_n\big)(x_0, \dots, x_n). \tag{9.1}$$

Proof. Composing the two laws of (Y2), the right side of the flux law is $\partial_\nu^{x_0} S_\xi(v_0, \dots, v_n)$; the coefficient -3 is fixed by the contraction of $\partial_a\chi$ with the T_{ab} components relative to the construction's Euclidean coordinate and derivative conventions (the tensor normalization is fixed by this very identity). Three accompanying facts are proved separately. *Integrability*: $\partial\chi$ vanishes near every insertion, so the integrand avoids all coincidence diagonals; by Theorem 5.17(iii) the kernel is C^∞ on Ω_{n+1}, and with $\partial\chi$ compactly supported the integral is a genuine integral rather than a totalized value. *Window independence*: if χ_1, χ_2 are both admissible and $\psi := \chi_1 - \chi_2$ vanishes near all insertions, then $\int \partial_a\psi\, S(T_{ab}, \dots)\, \mathrm{d}y = -\int \psi\, \partial_{y^a} S(T_{ab}, \dots)\, \mathrm{d}y = 0$ by integration by parts (no boundary terms since ψ is compactly supported), the divergence-free condition, and the vanishing of ψ at the insertions. *Coverage*: for every separated configuration there is an admissible window. □

Remark 9.5. The check of Proposition 9.4 is sharper than stress-tensor conservation or the trace two-point function alone, but it does not prove that all possible stress-tensor improvements vanish; it is a separated-insertion shell identity and does not construct an unrestricted contact-term extension, a dilation Ward identity, or a comparison with an independently chosen Minkowski sign convention.

9.2 A nonzero trace and its short-distance law

In the selected component theory indexed by ξ^*, define

$$\Theta := \sum_{\mu=0}^{3} T_{\mu\mu} - S_{\xi^*}\Big(\sum_{\mu=0}^{3} T_{\mu\mu}\Big)\mathbf{1}, \tag{9.2}$$

that is, for any list of labels $v_1,\dots,v_n$ and points $x_0,\dots,x_n$, with $v_0 := \sum_\mu T_{\mu\mu}$,

$$S_{\xi^*}\big(\Theta, v_1,\dots,v_n\big)(x_0,x_1,\dots,x_n) = S_{\xi^*}\big(v_0,v_1,\dots,v_n\big)(x_0,x_1,\dots,x_n) - S_{\xi^*}\big(v_0\big)(x_0)\, S_{\xi^*}\big(v_1,\dots,v_n\big)(x_1,\dots,x_n).$$

Thus $\Theta \in \mathcal{F}_4$ and $S_{\xi^*}(\Theta) = 0$.

Assumption 9.6 ((Y3): short-distance input)**.** The RG analysis of the construction gives

$$K_\Theta(r) = \Big(\frac{\beta(g_r)}{2g_r}\Big)^2 \big\langle Q(\tfrac{r}{2}e_0)\, Q(-\tfrac{r}{2}e_0)\big\rangle_{\mathrm{free},c}\,(1+o(1)), \qquad g_r = g(1/r),$$

as $r \downarrow 0$, where $\beta(g) = \mu\, \mathrm{d}g/\mathrm{d}\mu$ and $\langle\cdot\rangle_{\mathrm{free},c}$ is the connected two-point function of the free Gaussian curvature; the marginal character of the curvature-squared operator transmits the one-loop running into a logarithmic enhancement of the singularity, while the residue is the free-field value.

Lemma 9.7 (Free Wick contraction coefficient, complete verification)**.** *With*

$$\big\langle F^a_{\mu\nu}(x) F^b_{\rho\sigma}(0)\big\rangle_{\mathrm{free}} = \frac{\delta^{ab}}{\pi^2|x|^4}\big(I_{\mu\rho}(x) I_{\nu\sigma}(x) - I_{\mu\sigma}(x) I_{\nu\rho}(x)\big), \qquad I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{|x|^2},$$

one has

$$\big\langle Q(x)Q(0)\big\rangle_{\mathrm{free},c} = \frac{48\dim_{\mathbb{R}}\mathfrak{g}}{\pi^4|x|^8}. \tag{9.3}$$

Proof. Write $P_{\mu\nu} = x_\mu x_\nu/|x|^2$; then $P^2 = P$ (rank-one projection), $I = \mathbf{1} - 2P$, so

$$I^2 = \mathbf{1} - 4P + 4P^2 = \mathbf{1}, \qquad \mathrm{Tr}(I^2) = 4, \qquad \mathrm{Tr}(I^4) = 4.$$

Let $A_{\mu\nu,\rho\sigma} = I_{\mu\rho}I_{\nu\sigma} - I_{\mu\sigma}I_{\nu\rho}$. The connected two-point function receives two (equal) Wick pairings and the color contraction $\sum_{a,b}\delta^{ab}\delta^{ab} = \dim_{\mathbb{R}}\mathfrak{g}$; the index contraction is

$$\sum_{\mu\nu\rho\sigma} A^2_{\mu\nu,\rho\sigma} = \underbrace{\Big(\sum I^2_{\mu\rho}\Big)\Big(\sum I^2_{\nu\sigma}\Big)}_{4\cdot 4=16} + \underbrace{16}_{\text{second term}} - 2\underbrace{\mathrm{Tr}(I^4)}_{4} = 16 + 16 - 8 = 24.$$

Hence $\langle Q(x)Q(0)\rangle_{\mathrm{free},c} = 2\cdot\dim_{\mathbb{R}}\mathfrak{g}\cdot\frac{24}{\pi^4|x|^8} = \frac{48\dim_{\mathbb{R}}\mathfrak{g}}{\pi^4|x|^8}$. □

Lemma 9.8 (One-loop and anomaly coefficients, complete derivation)**.** *With* $\beta(g) = -b_0 g^3 + O(g^5)$, $b_0 = 11C_{\mathfrak{g}}/(48\pi^2)$, *and* $\ell = \log\frac{1}{\Lambda r}$,

$$g^2(1/r) = \frac{1}{2b_0\ell}\big(1+o(1)\big), \qquad \frac{\beta(g_r)}{2g_r} = -\frac{1}{4\ell}\big(1+o(1)\big). \tag{9.4}$$

Proof. From $\frac{\mathrm{d}(g^{-2})}{\mathrm{d}\log\mu} = -\frac{2\beta(g)}{g^3} = 2b_0 + O(g^2)$, integration gives $g^{-2}(\mu) = 2b_0\log(\mu/\Lambda) + O(\log\log\mu) = 2b_0\ell(1+o(1))$ at $\mu = 1/r$. Hence $g_r^2 = (2b_0\ell)^{-1}(1+o(1))$ and

$$\frac{\beta(g)}{2g} = \frac{-b_0 g^3 + O(g^5)}{2g} = -\frac{b_0}{2}g^2 + O(g^4) = -\frac{b_0}{2}\cdot\frac{1}{2b_0\ell}\big(1+o(1)\big) = -\frac{1}{4\ell}\big(1+o(1)\big).$$ □

Theorem 9.9 (Trace asymptotic)**.** *With the centered trace* Θ *of* (9.2)*,*

$$\lim_{r\downarrow 0} r^8 \log^2(1/r)\, S_{\xi^*}\big(\Theta,\Theta\big)(-\tfrac{r}{2}e_0, \tfrac{r}{2}e_0) = \frac{3\dim_{\mathbb{R}}\mathfrak{g}}{\pi^4} =: A_G > 0, \tag{9.5}$$

and Euclidean covariance upgrades this to the all-directions law: with $K_\Theta(x) = \operatorname{Re} S_{\xi^*}(\Theta,\Theta)(x,0)$*,*

$$|x|^8 \log^2(1/|x|)\, K_\Theta(x) \longrightarrow \frac{3\dim_{\mathbb{R}}\mathfrak{g}}{\pi^4} \qquad (x\to 0,\ x\neq 0). \tag{9.6}$$

Proof. Algebraic assembly. By (Y3), Lemmas 9.7 and 9.8, writing $c(r) = r^8\log^2(1/r)$ and $\log(1/r) = \ell + O(1)$,

$$c(r)K_\Theta(r) = r^8\ell^2\cdot\frac{1}{16\ell^2}\cdot\frac{48\dim_{\mathbb{R}}\mathfrak{g}}{\pi^4 r^8}\big(1+o(1)\big) = \frac{3\dim_{\mathbb{R}}\mathfrak{g}}{\pi^4}\big(1+o(1)\big).$$

Note that b_0^2 from the two anomaly insertions cancels against $g_r^4 \sim (2b_0\ell)^{-2}$ exactly, so the amplitude depends only on $\dim_{\mathbb{R}}\mathfrak{g}$.

From lattice-approximated points to the full limit (squeeze). The formalized statement first proves the limit at approximated positions: for s near 1,

$$c(r_n)G_\Theta\big(r_n\delta_n(s)\big) \longrightarrow \frac{A_G}{s^8}, \qquad \delta_n(s)\to s,$$

where $\delta_n(s)$ corrects for approximating physical points separated by $r_n s$ by nearby lattice points. By reflection positivity and the spectral representation, G_Θ is real-valued and nonincreasing, $G(r) = \int_0^\infty e^{-r\lambda}\,\mathrm{d}\rho(\lambda)$ with $\rho\geq 0$. Choosing $s_- < 1 < s_+$, for all sufficiently large n, $r_n\delta_n(s_+) \geq r_n \geq r_n\delta_n(s_-)$; so monotonicity gives

$$G_\Theta\big(r_n\delta_n(s_+)\big) \leq G_\Theta(r_n) \leq G_\Theta\big(r_n\delta_n(s_-)\big);$$

multiplying by $c(r_n) > 0$ and taking $\liminf$ / $\limsup$,

$$\frac{A_G}{s_+^8} \leq \liminf_n c(r_n)G_\Theta(r_n) \leq \limsup_n c(r_n)G_\Theta(r_n) \leq \frac{A_G}{s_-^8},$$

and $s_\pm\to 1$ squeezes the limit to A_G. E1 (Euclidean covariance) then promotes the time-direction result to all directions, giving (9.6). □

Remark 9.10. Although a classical smooth connection has traceless stress tensor, Θ is in general nonzero as a result of renormalization–the trace anomaly. In particular the two-point function is nonzero for sufficiently small r, so some smeared $\Phi_\Theta(f)$ is nonzero. The choice $\Lambda = 1$ records the construction's reference units; it does not define an intrinsic physical scale.

9.3 Excluded free-field descriptions

For $m\geq 0$, an integer $q\geq 0$, and a continuous real-valued function h on $\mathbb{R}^3$, homogeneous of nonnegative integral degree, consider

$$K_{m,q,h}(r) = \int_{\mathbb{R}^3} \frac{E_m(p)^q\, h(p)^2}{2E_m(p)}\, e^{-rE_m(p)}\,\mathrm{d}p, \qquad E_m(p) = \sqrt{m^2+|p|^2}, \tag{9.7}$$

and the class of finite complex linear combinations of finite products

$$W(r) = \sum_{a=1}^{N} c_a \prod_{b=1}^{n_a} K_{ab}(r), \tag{9.8}$$

with $K_{ab} = K_{m_{ab},q_{ab},h_{ab}}$. This class includes the suitable polynomial observables and derivatives in free fields with finitely many masses.

Lemma 9.11 (Finite asymptotic expansion)**.** *Each $K_{m,q,h}$ has, as $r \downarrow 0$, a finite expansion $\sum_{\nu=1}^{M} a_\nu r^{\alpha_\nu}(\log r)^{k_\nu} + O(1)$ with $a_\nu \in \mathbb{R}$, $\alpha_\nu \in \mathbb{Z}$, $k_\nu \in \mathbb{N}_0$; the class is closed under finite products and finite linear combinations.*

Proof. Homogeneity of h of degree j makes the radial part of the integrand $\propto p^{2j}E^{q-1}\varphi(\hat{p})$ with φ smooth on the sphere; the radial integral is $\int_0^\infty p^{2j}E^{q-1}\varphi\, e^{-rE}\,\mathrm{d}p$. If $m = 0$, then $E = p$, the integrand is $\propto p^{2j+q+1}e^{-rp}$, and $\int_0^\infty p^s e^{-rp}\,\mathrm{d}p = \Gamma(s+1)r^{-(s+1)}$ gives integer powers. If $m > 0$, the large-p region $E = p + \frac{m^2}{2p} + \cdots$ gives integer powers $r^{-(2j+q+2)}, r^{-(2j+q)}, \ldots$, and the small-$p$ region $E = m + \frac{p^2}{2m} + \cdots$ with $e^{-rE} = e^{-rm}(1 + O(rp^2))$ gives $r^0, r^1, \ldots$. When powers from the two regions resonate, the critical radial integral of type $\int p^{-1}\,\mathrm{d}p$ produces $\log r$; each resonance raises the logarithmic power by one, and the number of resonances is finite, so k_ν is finite and ≥ 0. Products of two finite expansions are finite expansions of the same type ($r^{\alpha+\alpha'}(\log r)^{k+k'}$), and finite linear combinations are clear. □

Lemma 9.12 (Finite limit forces zero, at the correct weight)**.** *Let $c(r) = r^8 \log^2(1/r)$. If W belongs to the class of Lemma 9.11 and $\lim_{r\downarrow 0} c(r)\,\mathrm{Re}\,W(r)$ exists and is finite, then the limit is 0.*

Proof. Write $\mathrm{Re}\,W(r) = \sum_\nu a_\nu r^{\alpha_\nu}(\log r)^{k_\nu} + O(1)$ (taking real parts affects only the coefficients, and $(\log r)^k$ differs from $\log^k(1/r)$ by a sign, irrelevant to powers and divergence). If $\alpha_\nu < -8$ then $r^{\alpha_\nu+8}\log^{2+k_\nu}(1/r) \to \infty$, forcing $a_\nu = 0$; if $\alpha_\nu = -8$ the term contributes $a_\nu \log^{2+k_\nu}(1/r)$, which diverges unless $a_\nu = 0$ (note $k_\nu \geq 0$); if $\alpha_\nu > -8$ the term tends to 0; and the $O(1)$ term is killed by $c(r) \to 0$. Hence a finite limit forces every surviving coefficient to vanish, and the limit is 0. □

Proposition 9.13 (Free-field exclusion)**.** *There is no W of the form* (9.8) *and no $\delta > 0$ such that $K_\Theta(r) = W(r)$ on $(0, \delta)$.*

Proof. By Theorem 9.9, $\lim_{r\downarrow 0} c(r)K_\Theta(r) = A_G = 3\dim_{\mathbb{R}} \mathfrak{g}/\pi^4 > 0$. If $K_\Theta = W$ on $(0,\delta)$, then $\lim_{r\downarrow 0} c(r)\,\mathrm{Re}\,W(r) = A_G > 0$, contradicting Lemma 9.12. □

Remark 9.14 (Weight of the exclusion). We use the weight $c(r) = r^8\log^2(1/r)$ of Theorem 9.9: under the $\log^2$-weight every term with $\alpha = -8$, $k \geq 0$ diverges, so the obstruction argument is more direct than under the $\log^1$-weight and does not require a classification of which resonant terms occur. The conclusion is unchanged.

Remark 9.15. The exclusion covers only the finite class (9.8): finitely many masses, finitely many factors, finite sums. It does not address descriptions by infinite composites of Gaussian fields; and it does not use reflection positivity or the squeeze argument–two-point positivity enters only through $A_G > 0$. The general exclusion of continuous-mass generalized free fields is outside the scope of the construction.

9.4 Asymptotic freedom and the running coupling

Definition 9.16 (g_W)**.** Starting from the bare couplings supplied by Theorem 5.17, perform RG blocking down to the first lattice spacing below r, and define $g_W(r)$ as the inverse square root of the resulting normalized coefficient of the marginal (curvature-squared) term in the effective action, after removal of the initial cutoff and extraction of a subsequential infinite-volume limit; the normalization requires the quadratic term of the blocked effective action to have identity coefficients. Independence of $g_W(r)$ from the subsequential limit is not proved.

Assumption 9.17 ((Y4): the one-loop value)**.** At one loop, the coefficient of the marginal operator receives the gauge-boson loop, the Faddeev–Popov ghost loop, and the measure (BRS) contribution, which for a simple algebra with adjoint Casimir $C_{\mathfrak{g}}$ sum to $b_0 = \frac{11C_{\mathfrak{g}}}{48\pi^2}$, with no scalar or fermion loops.

Theorem 9.18 (Effective coupling at short distances)**.** *For each fixed selected subsequential infinite-volume limit and all sufficiently small r (depending on the limit), $g_W(r) > 0$, and*

$$\log(1/r)\, g_W(r)^2 \longrightarrow \frac{1}{2b_0} \qquad (r \downarrow 0). \tag{9.9}$$

Proof. Step 1 (recursion). The recursion of Lemma 5.11, $\frac{\mathrm{d}}{\mathrm{d}\log\mu} g^{-2} = 2b_0 + O(g^2)$, is equivalent to $\beta(g) = -b_0 g^3 + O(g^5)$; integrating (Lemma 9.8) gives $g^{-2}(\mu) = 2b_0 \log(\mu/\Lambda) + O(\log\log\mu)$. *Step 2 (value).* This is (Y4). *Step 3 (from the lattice trajectory to g_W).* By construction g_W lies on the same RG trajectory as the bare couplings $\beta_{N,p}$, so $g_W^{-2}(r) = 2b_0 \log(1/r) + O(\log\log(1/r))$; taking reciprocals and multiplying by $\log(1/r)$ gives (9.9). □

Remark 9.19. Two convention dependencies are explicit: Λ is scheme-dependent (a change of scheme adds a constant to $\log(1/r)$ and does not affect (9.9)), and the normalization of g_W is fixed by the kinetic normalization. The argument is perturbative in the marginal coupling only: it controls $g_W(r)$ for small r, where $g_W(r) \to 0$, and does not address the infrared side of the flow.

9.5 Operator product expansion and contact extensions

Theorem 9.20 (Operator product expansion)**.** *Let v_i and u_j be local labels of degrees d_i and $w = \sum_i d_i$. There exist labels $o_\alpha \in \mathcal{F}_{d_\alpha}$, finitely many per degree, such that for every $D \geq 0$ and $\varepsilon > 0$,*

$$S^s\Big(\prod_i A_i(x_i)\, B\Big) = \sum_{d_\alpha \leq D} c_\alpha(A; x, o)\, S^s\big(O_\alpha(o) B\big) + R_{D,s}, \qquad |R_{D,s}| \leq r^{D+1-w-\varepsilon}, \tag{9.10}$$

where s denotes the mixed theory or a component theory; the inputs form a cluster near the expansion point o while spectators stay separated. More precisely, for fixed $M \geq 0$, $\eta, c > 0$, the allowed configurations at scale $r > 0$ satisfy

$$\|x_i - o\| \leq Mr \ \ \forall i, \qquad \|x_i - x_j\| \geq \eta r \ \ i \neq j, \qquad \|y_\ell - o\| \geq c \ \ \forall \ell, \qquad \|y_\ell - y_m\| \geq c \ \ \ell \neq m, \tag{9.11}$$

and the bounds are

$$|c_\alpha(A; x, o)| \leq r^{d_\alpha - w - \varepsilon}, \qquad |R_{D,s}| \leq r^{D+1-w-\varepsilon}, \tag{9.12}$$

holding for all sufficiently small $r > 0$ for every $\varepsilon > 0$; the coefficient bound requires only the first two conditions of (9.11)*, the remainder bound requires all four and holds uniformly over all states s. The coefficients are independent of spectators, state, and truncation degree.*

Proof: extraction of the coefficients and bounds. The coefficients $c_\alpha(A; x, o)$ are produced by a fixed algebraic prescription on the label spaces before any correlation is evaluated; their independence of spectators, phases, and truncation orders is automatic rather than dynamical. *(1) Degree decomposition*: for each d fix a splitting $\mathcal{F}_d = \mathcal{F}_{d-1} \oplus E_d$; the channels $\alpha = (d, i)$ index a conjugation-invariant basis of the new layer E_d, with $n_d = \dim E_d$ fields per layer. *(2) Test-field formation*: form a degree-d field from finitely many correlations of the inputs against additional test insertions supported on a fixed "stratum" configuration; the separation hypotheses (9.11) keep the stratum tests

away from both the input cluster and the spectators. *(3) Symmetrization and projection*: average over $O(4)$, symmetrize under label conjugation, and strip the $\mathcal{F}_{d-1}$ components; the coordinates in the basis $O_{(d,i)}$ define $c_\alpha(A; x, o)$. *(4) Bounds*: because the prescription is fixed, the coefficient bound in (9.12) requires only the first two conditions of (9.11); the remainder bound requires all four, uniformly over s. The analytic estimates entering Step (4) are those of the RG analysis of Section 5; no perturbative computation or comparison of coefficient values is involved. □

Proposition 9.21 (Compact-set uniform bounds)**.** *Fix labels, truncation degree D and $\varepsilon > 0$, and let $\mathcal{K}$ be a compact set of configurations satisfying* (9.11)*, with $N_D = \#\{\alpha : d_\alpha \le D\} < \infty$. Suppose the pointwise bounds* (9.12) *hold on $\mathcal{K}$ for all s. Then any norm of the complete coefficient vector $c = (c_\alpha)_\alpha$ is uniformly bounded on $\mathcal{K}$, and the remainder bound holds uniformly in $(x, o, y) \in \mathcal{K}$ and s.*

Proof. From the pointwise bound, $\|c\|_\infty \le \max_\alpha r^{d_\alpha - w - \varepsilon} =: B_D$, a finite constant; in the finite-dimensional space $\mathbb{C}^{N_D}$ all norms are equivalent, so there is C_D with $\|c\| \le C_D B_D$ independent of the configuration. The remainder bound's right side contains no configuration or state, so $\sup_{\mathcal{K},s} |R_{D,s}| \le r^{D+1-w-\varepsilon}$. □

Remark 9.22 (Contact extensions). The contact-term extension is isolated behind an explicit predicate: for a hierarchy T, a state option, and insertions with labels $v : \operatorname{Fin} n \to T.\mathrm{Tag}$, the predicate asserts the existence of a distribution $\hat{D}$ on the full configuration space (continuous and linear) such that every test function whose tubular support lies in the separated insertion region satisfies $\hat{D}(f) = \int f(x)\,\mathrm{correlation}(s, v, x)\,\mathrm{d}x$. The bridge theorem proves this predicate for the constructed hierarchy, transporting the OPE-based multiple-contact development (harmonic extension around the collision locus). Three limitations remain: the predicate does not assert uniqueness of the extending distribution; it does not prove reflection positivity for the extended correlations on unrestricted Euclidean test spaces; and lattice convergence of the extended functionals is not established.

9.6 The field-content audit

The dictionary between the constructed operator spaces and classical gauge-invariant polynomials is checked in six steps, recorded here in the notation of the main text ($\mathcal{F}_d = V_d/N_d$ of Definition 2.7; X the phase space of Proposition 7.9).

1. *Exhaustiveness of the register*: every invariant polynomial appears at some register level (exhaustiveness relative to the submitted algebraic dictionary; human review must still identify that dictionary with the intended local gauge-invariant fields, and the theorem does not address nonlocal observables).

2. *Free on-shell comparison space*: at filtration d, the free on-shell polynomial comparison space is exactly the invariant part of the free lower-weight cap, and the separated free correlations leave no further residual null space.

3. *Interacting quotient dictionary*: the rank comparison yields a complex linear equivalence $\mathcal{F}_d \simeq_{\mathbb{C}} \text{Maxwell-shell-sector}(G, d)$ for every d (this is Proposition 2.12, input (B1)).

4. *No field is lost in a phase*: for every character phase $\xi \in X$–not merely almost every phase under ν–the phase and mixed null spaces agree, and the preceding equivalence transports to every phase.

5. *Cyclicity uses the full collection*: the field-operator map assigns a Hilbert-space vector to every registered field word applied to the vacuum, and the linear span of its range is dense. Cyclicity is partly a feature of reconstructing a Hilbert space from field words; its physical force comes from combining it with the field-content comparison and the faithful representation.

6. *Surviving fields act nontrivially*: every nonzero local quotient field has some nonzero smeared operator; passing to the reconstructed operator representation therefore kills no surviving local field class.

The audit concerns the full registered invariant collection; it does not supply an independently checked charge-conjugation-equivariant dictionary or a classification of charge-conjugation odd particles.

10 What is proved: rate theorems, the counterexample family, and the single residual predicate

This section records the logical boundary of the construction with respect to the mass gap: what the registered inputs imply, what they cannot imply, and the exact shape of the missing input. The fifteen external inputs are (RG1)–(RG3), (RG5), (UV1), (PH1)–(PH3), (AN1), (Y1)–(Y4), (B1), (C1) (see Appendix A); (RG4) is derived (Lemma 5.13) and (PH4) is a proved spectral conclusion (Proposition 7.2).

10.1 A spectral representation toolkit

Lemma 10.1 (Exponential decay versus spectral support)**.** *Let ρ be a finite positive measure on $[0,\infty)$ and $G(\tau) = \int_{[0,\infty)} e^{-\tau\lambda}\,\mathrm{d}\rho(\lambda)$. For any $m > 0$ the following are equivalent:*

(i) $\rho\big((0,m)\big) = 0$*;*

(ii) *there exists $K < \infty$ such that $\tilde{G}(\tau) := \int_{(0,\infty)} e^{-\tau\lambda}\,\mathrm{d}\rho(\lambda) \le K e^{-m\tau}$ for all $\tau \ge 0$.*

Proof. (i)⇒(ii): if $\operatorname{supp}\tilde{\rho} \subset [m,\infty)$, then

$$\tilde{G}(\tau) = e^{-m\tau}\int_{[m,\infty)} e^{-\tau(\lambda-m)}\,\mathrm{d}\tilde{\rho}(\lambda) \le e^{-m\tau}\|\rho\|,$$

so $K = \|\rho\|$ works. (ii)⇒(i): suppose $a := \tilde{\rho}\big((0,m-\varepsilon)\big) > 0$ for some $\varepsilon > 0$; then $\tilde{G}(\tau) \ge \int_{(0,m-\varepsilon)} e^{-\tau\lambda}\,\mathrm{d}\tilde{\rho} \ge a e^{-(m-\varepsilon)\tau}$, so $a e^{\varepsilon\tau} \le K$ for all τ, which is impossible. Hence $\tilde{\rho}\big((0,m-\varepsilon)\big) = 0$ for all $\varepsilon > 0$, i.e. $\rho((0,m)) = 0$. □

Lemma 10.2 (Gap implies uniform exponential clustering)**.** *If $\sigma(H) \subset \{0\} \cup [\Delta,\infty)$, then for all $\psi,\chi \in \mathcal{H}$,*

$$\big|\langle\psi, E(s)\chi\rangle - \langle\psi,\Omega\rangle\langle\Omega,\chi\rangle\big| \le e^{-\Delta s}\|\psi - \langle\Omega,\psi\rangle\Omega\|\,\|\chi - \langle\Omega,\chi\rangle\Omega\|.$$

Proof. Let $\chi_\perp = \chi - P_0\chi$ and $\psi_\perp = \psi - P_0\psi$; by Proposition 7.2 $P_0 = |\Omega\rangle\langle\Omega|$, so $\langle\psi,E(s)\chi\rangle - \langle\psi,P_0\chi\rangle = \langle\psi_\perp, E(s)\chi_\perp\rangle$. By the spectral theorem this equals $\int_{[\Delta,\infty)} e^{-s\lambda}\,\mathrm{d}\mu_{\psi_\perp,\chi_\perp}(\lambda)$, and $\|\mu_{\psi_\perp,\chi_\perp}\| \le \|\psi_\perp\|\|\chi_\perp\|$ by polarization and Cauchy–Schwarz; the claim follows. □

Proposition 10.3 (Dense channel coverage gives the full gap)**.** *Let $\mathcal{D}_0 \subset \Omega^\perp$ be dense, and suppose there exist $\mu > 0$ and $K(\psi) < \infty$ such that*

$$\big|\langle \psi, E(s)\psi\rangle\big| \le K(\psi) e^{-\mu s} \qquad (\psi \in \mathcal{D}_0,\ s \ge 0).$$

Then $\sigma(H) \subset \{0\} \cup [\mu, \infty)$.

Proof. For $\psi \in \mathcal{D}_0$ we have $\rho_\psi(\{0\}) = \|P_0\psi\|^2 = 0$, so Lemma 10.1 gives $\rho_\psi((0,\mu)) = 0$. Now suppose $(0,\mu) \cap \sigma(H) \neq \varnothing$, pick $\lambda_0 \in (0,\mu) \cap \sigma(H)$ and $\varepsilon > 0$ with $I := [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset (0,\mu)$. Then $E_H(I) \neq 0$; take a nonzero ϕ in its range, so that $\rho_\phi(I) = \|\phi\|^2 > 0$. Choose a continuous $h : [0,\infty) \to [0,1]$ with $h \equiv 1$ on I and $\operatorname{supp} h \subset (0,\mu)$ (Urysohn's lemma). By the functional calculus $h(H)$ is bounded with $\|h(H)\| \le 1$. Since $\mathcal{D}_0$ is dense in $\Omega^\perp$, choose $\psi_j \in \mathcal{D}_0$ with $\psi_j \to \phi$. Then

$$\int h \,\mathrm{d}\rho_{\psi_j} = \langle \psi_j, h(H)\psi_j\rangle \longrightarrow \langle \phi, h(H)\phi\rangle = \int h \,\mathrm{d}\rho_\phi \ge \rho_\phi(I) = \|\phi\|^2 > 0,$$

while $\rho_{\psi_j}((0,\mu)) = 0$ and $\operatorname{supp} h \subset (0,\mu)$ force $\int h \,\mathrm{d}\rho_{\psi_j} = 0$ for every j–a contradiction. Hence $(0,\mu) \cap \sigma(H) = \varnothing$. □

Remark 10.4. The dense subspace $\mathcal{D}_0$ is supplied by the field-word cyclicity (Section 9, audit step 5); the smaller observable family behind the scalar OS bridge does not suffice. An exponential rate in a single channel therefore yields only a channel gap, not the full gap of Definition 1.2. Moreover the rate must be common to all registered label words: Proposition 10.23 below shows (Y2) supplies no degree-uniformity, and Proposition 10.25 exhibits a model with per-tower exponential rates but no common rate.

10.2 A model within the registered inputs without a mass gap

Let

$$\rho := \delta_0 + \sum_{k \ge 1} 2^{-k} \delta_{2^{-k}} \quad \text{on } [0,\infty), \tag{10.1}$$

a finite positive measure with $\|\rho\| = 2$ and $\operatorname{supp}\rho \subset [0, \frac{1}{2}]$. Let $\mathcal{H} = L^2([0,\infty), \mathrm{d}\rho)$, let H be the multiplication operator $(H\psi)(\lambda) = \lambda\psi(\lambda)$ ($H \ge 0$), $\Omega = \mathbf{1}/\sqrt{2}$, and let P_0 be the projection onto the one-dimensional atom at $\lambda = 0$. Take the generalized free field with Källen–Lehmann spectral measure ρ: its two-point function is

$$G(\tau) = \int_{(0,\infty)} e^{-\tau\lambda} \,\mathrm{d}\tilde\rho(\lambda) = \sum_{k \ge 1} 2^{-k} e^{-\tau 2^{-k}}, \qquad \tilde\rho = \rho - \delta_0, \tag{10.2}$$

and the full hierarchy is given by the Wick formula.

Proposition 10.5 (Logical independence of the mass gap)**.** *There is an OS hierarchy satisfying every clause of (AN1) (E0, E0′, E1, E2, E3, E4 clustering, normalization, analyticity) and (PH1)–(PH3), whose reconstructed Hamiltonian satisfies $\sigma(H) = \{0\} \cup \{2^{-k} : k \ge 1\}$, so that 0 is an accumulation point of the spectrum and there is no mass gap; yet E4 clustering and vacuum uniqueness hold.*

Proof. The OS axioms. E1 and E3 are inherited from the construction of the generalized free field; E0 holds because the compactly supported ρ gives finite moments, holomorphy of the Laplace transform for $\operatorname{Re}\tau > 0$, and tempered kernels at separated points; E0′ holds by Wick counting: the $2n$-point function is a sum of $(2n-1)!!$ products of two-point functions, and compact support gives uniform bounds on separated sets; E2 holds because G is the Laplace transform of a positive measure, hence

completely monotone, and reflection positivity of the two-point function is inherited by the Gaussian hierarchy; analyticity follows from holomorphy of the Laplace transform on $\operatorname{Re}\tau > 0$. (PH1)–(PH3): the model is time-translation invariant with trivial ground-state algebra action, commutativity, faithfulness of the vacuum vector for the scalars, and P_0 one-dimensional by construction.

(a) Clustering. By dominated convergence, $\tilde{G}(\tau) \to 0$ as $\tau \to \infty$; all connected functions of the Wick hierarchy are products of connected two-point functions, so E4 holds.

(b) The clustering rate is only power law. Let $k_0 = \lfloor \log_2 \tau \rfloor$ ($\tau \geq 2$). Lower bound: taking $k = k_0$, $2^{k_0} \leq \tau < 2^{k_0+1}$, so $2^{-k_0} \geq 1/\tau$ and $1 \leq \tau 2^{-k_0} < 2$, hence $G(\tau) \geq 2^{-k_0} e^{-\tau 2^{-k_0}} \geq e^{-2}/\tau$. Upper bound: for $k > k_0$, $\sum_{k>k_0} 2^{-k} = 2^{-k_0} \leq 2/\tau$; for $k \leq k_0$, writing $j = k_0 - k \geq 0$, $2^{-k} = 2^j 2^{-k_0} \leq 2^j/\tau$ and $\tau 2^{-k} \geq 2^{j-1}$, so this part is at most $\frac{1}{\tau}\sum_{j\geq 0} 2^j e^{-2^{j-1}} = C/\tau$. Therefore

$$\frac{e^{-2}}{\tau} \leq G(\tau) \leq \frac{C+2}{\tau} \qquad (\tau \geq 2),$$

i.e. $G(\tau) \asymp \tau^{-1}$; in particular $e^{m\tau} G(\tau) \to \infty$ for every $m > 0$.

(c) No spectral gap. $\operatorname{supp}\tilde{\rho} = \{2^{-k}\}$ accumulates at 0, so $\inf(\sigma(H) \setminus \{0\}) = 0$ and $\rho((0,\varepsilon)) > 0$ for every $\varepsilon > 0$; by Lemma 10.1, no spectral gap holds.

(d) Logical conclusion. If "the theory has a mass gap" were derivable from (AN1), (PH1)–(PH3), and the standard OS theorems, it would hold in every model of those axioms; (c) exhibits a counterexample. □

Remark 10.6 (What the model does and does not say). The model does not claim that the constructed Yang–Mills theory has no mass gap; it proves that the axioms together with the registered inputs do not force one. It lies outside the exclusion of Proposition 9.13 (which covers only the finite class (9.8)); the general exclusion of continuous-mass generalized free fields remains open (problem (O6) below) and is a necessary, not sufficient, prerequisite for a gap: even after such exclusion, (MG) below must still supply a positive lower bound. Ultraviolet data–the trace anomaly, asymptotic freedom, the OPE error bounds–constrain the infrared spectral weights of such a model not at all.

Proposition 10.7 (Ultraviolet data do not constrain the infrared spectrum)**.** *There exists a tempered generalized-free-field hierarchy satisfying (AN1) E0–E4, vacuum uniqueness, the field-content dictionary (B1), the stress-tensor candidate calculation (Y1), the calculus and flux laws (Y2), the exact short-distance constant A_G of Theorem 9.9, and the OPE coefficients and remainder bound of Theorem 9.20, whose Hamiltonian has no spectral gap.*

Proof outline. Work with a Källen–Lehmann weight ρ on $[0,\infty)$ which is locally finite and of polynomial growth (tempered), so that for every Schwartz test f the induced vector spectral measure $\rho_f = |\hat{f}|^2\rho$ is finite and Lemma 10.1 applies. Set

$$\rho = \delta_0 + \rho_{\mathrm{UV}} + \rho_{\mathrm{IR}}, \qquad \rho_{\mathrm{IR}} = \sum_{k\geq 1} 2^{-k}\delta_{2^{-k}}, \qquad \rho_{\mathrm{UV}}(\mathrm{d}E) = c\frac{E^5}{\log^2(E+e)}\mathbf{1}_{E\geq 1}\,\mathrm{d}E,$$

with $\operatorname{supp}\rho_{\mathrm{IR}} \subset (0, \frac{1}{2}]$ and $\operatorname{supp}\rho_{\mathrm{UV}} = [1,\infty)$ disjoint. The full hierarchy is Gaussian/Wick with two-point kernel G and the label Gram matrix of Section 2.

Short distance. The four-dimensional Euclidean propagator of mass E is $\Delta_E(r) = \frac{E}{4\pi^2 r}K_1(Er)$, with $\Delta_E(r) \asymp \frac{1}{4\pi^2 r^2}\mathbf{1}_{Er\lesssim 1}$ and exponential suppression for $Er \gg 1$. Hence

$$K(r) \asymp \frac{1}{4\pi^2 r^2}\int_0^{1/r} \mathrm{d}\rho(E) \sim \frac{c}{24\pi^2}\, r^{-8}\log^{-2}(1/r),$$

because $\int_1^{1/r} E^5 \log^{-2} E \, \mathrm{d}E \sim \frac{1}{6}(1/r)^6 \log^{-2}(1/r)$. The infrared piece has finite total mass and bounded support, contributing at most $\|\rho_{\mathrm{IR}}\|/(4\pi^2 r^2) = O(r^{-2})$, a strictly lower-order singularity. Choosing c fixes the leading coefficient to A_G; the assembly of Theorem 9.9 involves only the large-E Taylor coefficients and never touches the spectrum below $E = 1$, so A_G is unchanged.

Long distance. The ultraviolet piece gives $O(e^{-\tau})$ as $\tau \to \infty$; the infrared piece gives $\sum_k 2^{-k} e^{-\tau 2^{-k}} \asymp \tau^{-1}$ by the computation of Proposition 10.5(b). Hence E4 holds with a power-law rate, $e^{m\tau} G(\tau) \to \infty$ for every $m > 0$, and $\inf(\sigma(H) \setminus \{0\}) = 0$.

The axioms and the dictionary. $\rho \geq 0$ gives E2; E0/E0′ follow from polynomial growth plus Wick counting as in Proposition 10.5; E1/E3 and analyticity are inherited; ρ has no atom at 0 other than δ_0, so by Proposition 7.2 E4 is equivalent to vacuum uniqueness, which holds. Because the infrared modification multiplies the scalar two-point factor and leaves the label Gram matrix untouched, the null spaces N_d are unchanged, hence (B1) is unchanged. (Y1) is the purely algebraic computation of Lemma 9.2 on the free shell. For (Y2) we verify both laws for the translation-invariant Gaussian hierarchy. All insertions share the scalar two-point factor G, so by Wick $S_\xi(X) = \sum_\pi \prod_{\{a,b\}\in\pi} G(a-b) \cdot (\text{label contractions})$. *Calculus law*: translation invariance of G gives $S_\xi(D_b v_0, v_1, \ldots, v_n) = \partial_b^{x_0} S_\xi(v_0, v_1, \ldots, v_n)$, since D_b differentiates the scalar factor at x_0. *Flux law*: let $X = (v_0, \ldots, v_n)$ and let χ be a bump with $\chi \equiv 1$ near x_0 and supported away from $x_1, \ldots, x_n$. The component T_{ab} transforms as a fixed rank-two tensor under translations, so the full correlation $C_{ab}(y) := S_\xi(T_{ab}(y) v_0(x_0) \cdots v_n(x_n))$ obeys the tensor translation identity

$$\Big(\partial_{y^a} + \sum_{j=0}^{n} \partial_{x_j^a}\Big) C_{ab}(y) = 0 \tag{10.3}$$

(equivalently, the Gaussian Schwinger–Dyson identity for the improved stress current; away from the insertions the equation-of-motion term vanishes by Gaussian integration by parts in the y-variable). Since $\operatorname{supp} \partial_a \chi$ is disjoint from every insertion, integration by parts in y gives

$$-\int \partial_a \chi(y)\, C_{ab}(y)\, \mathrm{d}y = \int \chi(y)\, \partial_{y^a} C_{ab}(y)\, \mathrm{d}y = -\sum_{j=0}^{n} \int \chi(y)\, \partial_{x_j^a} C_{ab}(y)\, \mathrm{d}y.$$

For $j \geq 1$, χ vanishes near x_j, and $\partial_{x_j^a}$ commutes with the y-integral on the support of χ, giving zero. For $j = 0$, $\chi \equiv 1$ near x_0, so the remaining term is the x_0-translation of the scalar factor contracted with the T_{ab} index structure—exactly $\partial_b^{x_0} S_\xi(v_0, \ldots, v_n) = S_\xi(D_b v_0, v_1, \ldots, v_n)$ by the calculus law. Thus (10.3) integrates to the flux law, and both clauses of (Y2) hold. Finally, the OPE coefficients are produced by the fixed algebraic prescription of Theorem 9.20 from the short-distance Taylor coefficients; the $O(r^{-2})$ infrared contribution does not enter the $r^{-8}\log^{-2}$ layer. □

Proposition 10.8 (The gap is not continuous in ultraviolet data)**.** *Let T be a generalized free field with a gap $\Delta > 0$ and let T_ε be obtained by adjoining $\varepsilon\rho_{\mathrm{IR}}$, $\varepsilon > 0$. Then T_ε and T have identical full formal short-distance asymptotics, identical OPE coefficients, identical Ward identities and identical field-content ranks; all separated-point correlations differ by at most $C\varepsilon$; yet* $\operatorname{gap}(T) = \Delta$ *while* $\operatorname{gap}(T_\varepsilon) = 0$ *for every $\varepsilon > 0$. Consequently no criterion built from finitely many–or from all formal orders of–short-distance data can decide the mass gap.*

Proof. Immediate from the support separation in Proposition 10.7: the modification is $O(r^{-2})$ at short distance and carries a positive spectral weight arbitrarily close to 0. The bound on separated correlations follows from $\|\varepsilon\rho_{\mathrm{IR}}\| = \varepsilon$; the gap statement follows from Lemma 10.1. □

Proposition 10.9 (Dimensional transmutation produces no physical mass in this construction)**.** *All registered scale-bearing objects are ultraviolet or conventional: the short-distance theorems concern $r \to 0$, $a_N \to 0$; the Ward identity is a separated-insertion shell law and no dilation Ward identity is constructed (Remark 9.5); the calibrations are conventions. Hence no proved object links the ultraviolet scale to an infrared spectral lower bound: the inference "trace anomaly → broken scale invariance → Λ_{QCD} → $m \sim \Lambda_{\mathrm{QCD}}$" lacks its middle term in this construction. A proof would require a full-scale dilation Ward identity / Callan–Symanzik relation, which is absent.*

Proposition 10.10 (Boundary: what cannot yield a gap)**.** *No argument using only ultraviolet data (Theorems 9.9, 9.18, 9.20), rateless clustering (Proposition 7.14), and pointwise finite-volume gaps can produce a mass gap.*

Proof. The model of Proposition 10.5 satisfies (AN1) and (PH1)–(PH3) with rateless clustering, and its ultraviolet behavior is configurable independently of its infrared weights; every finite-volume truncation of any model has a pointwise gap. If a gap were derivable from such premises, it would hold in every model satisfying them; the counterexample shows otherwise. □

10.3 Spectral-gap bridges

Lemma 10.11 (Doeblin spectral gap)**.** *Let (X, μ) be a probability space and $P(x, \mathrm{d}y)$ a Markov kernel with stationary measure ν. If $P(x, \mathrm{d}y) \geq \varepsilon\, \nu(\mathrm{d}y)$ for some $\varepsilon > 0$ and all x, then for every f with $\int f \,\mathrm{d}\nu = 0$,*

$$\|Pf\|_{L^2(\nu)} \leq (1-\varepsilon)\|f\|_{L^2(\nu)}.$$

If moreover $T \geq 0$ is the self-adjoint reflection-positive transfer operator with continuous kernel $c_1 \leq K \leq c_2$ and Perron root λ_1, then with $R = \log(c_2/c_1)$ the h-transform gives a Doeblin constant of order e^{-2R} and

$$\Delta \geq -a_t^{-1} \log\left(1 - e^{-2R}\right) > 0.$$

Proof. Write $P = \varepsilon\nu + (1-\varepsilon)Q$ with Q a Markov kernel with the same stationary measure ν. For $\int f \,\mathrm{d}\nu = 0$, $Pf = (1-\varepsilon)Qf$. By Jensen $|Qf|^2 \leq Q(f^2)$, and stationarity gives $\int Q(f^2)\,\mathrm{d}\nu = \int f^2 \,\mathrm{d}\nu$; hence the L^2 bound. For the second part, let $Th = \lambda_1 h$, $h > 0$, and set $P(x, \mathrm{d}y) = K(x,y)h(y)(\lambda_1 h(x))^{-1}\,\mathrm{d}\mu(y)$; then $\int P(x, \mathrm{d}y) = 1$ and $\nu = h^2\mu / \int h^2 \mathrm{d}\mu$ is stationary by symmetry of K. From $c_1 \int h \,\mathrm{d}\mu \leq \lambda_1 h(x) \leq c_2 \int h \,\mathrm{d}\mu$ one gets $h(y)/h(x) \geq c_1/c_2$ and $\lambda_1 \leq c_2$, hence $P(x, \mathrm{d}y) \geq (c_1/c_2)^2\,\mathrm{d}\mu(y) = e^{-2R}\,\mathrm{d}\mu(y)$. The spectral radius on the zero-mean subspace is then $\leq 1 - \varepsilon$ and $\Delta = -a_t^{-1}\log(\lambda_2/\lambda_1) \geq -a_t^{-1}\log(1-\varepsilon)$. □

Proposition 10.12 (Change of layer: the gap is scale-invariant)**.** *Assume exact blocking, i.e. the spatial transfer operator commutes with the one-step transfer matrix $\mathcal{T}_m$. Then blocking one step $a^{(m)} \mapsto b_m a^{(m)}$ replaces $\mathcal{T}_m$ by $\mathcal{T}_m^{b_m}$ and leaves the physical gap unchanged:*

$$\Delta_{m+1} = -\frac{1}{b_m a^{(m)}} \log \lambda_2\big(\mathcal{T}_m^{b_m}\big) = -\frac{b_m}{b_m a^{(m)}} \log \lambda_2(\mathcal{T}_m) = -\frac{1}{a^{(m)}} \log \lambda_2(\mathcal{T}_m) = \Delta_m.$$

Consequently a uniform gap may equivalently be read at the reference layer $a(0) = 1$.

Proof. Under exact blocking the coarse-grained transfer matrix is the b_m-fold time composition of the fine one, so its eigenvalues are $\lambda_j(\mathcal{T}_m)^{b_m}$ (the Perron root stays 1 after normalization). Substituting into $\Delta_m = -a^{(m)^{-1}} \log \lambda_2$ gives the displayed identity. At the reference level Lemma 4.15(3) gives $\beta_{N,p}(0) = \bar{\beta}$, a fixed constant; the running (5.10) is a statement about the bare layer only. Hence a strong-coupling/Dobrushin-type analysis at the reference layer is not excluded by asymptotic freedom. □

Remark 10.13. Actual (RG2) truncations break exact blocking; the resulting remainder is controlled at the schedule level by Lemma 5.8, not by a Kato perturbation series.

Proposition 10.14 (The global positivity bound degenerates with the volume)**.** *At the reference layer the one-step kernel is $K_0 = e^{-L_E}$ with $L_E \geq 0$ the Wilson action on a whole spatial slice. Then the Doeblin lower bound of Lemma 10.11 degenerates as $p \to \infty$.*

Proof. Take $U = \mathbf{1}$: every plaquette term vanishes, so $L_E(U) = 0$. Take $U_\bullet$ obtained by setting a fixed fraction of the spatial links to -1 (possible for odd side length); at least cL_p^3 plaquettes then have $1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr}(U_\square) \geq c' > 0$, so $L_E(U_\bullet) \geq c_0\bar{\beta}L_p^3$. Hence

$$R_0(p) = \sup L_E - \inf L_E \geq c_0\bar{\beta}L_p^3 \to \infty.$$

With $K_0 \in [e^{-R_0(p)}, 1]$, Lemma 10.11 gives $\varepsilon_p \lesssim e^{-2R_0(p)} \to 0$, so the lower bound $\Delta(p) \gtrsim -a_t^{-1}\log(1-\varepsilon_p) \approx e^{-2c_0\bar{\beta}L_p^3}$ degenerates. This is a degeneration of a *bound*, not of the gap: Lüscher-type finite-volume corrections $\Delta m(L) \sim CL^{-1}e^{-\alpha m_0 L}$ show that on the massive side the true gap is volume-uniform. The corollary is that (MG2″) cannot be proved by a single global Doeblin estimate; a local mixing estimate whose rate is independent of L_p is required. □

Proposition 10.15 (The kernel family fixes the finite-volume spectral measure)**.** *Fix a volume p. Suppose two Wilson families satisfying (RG1)–(RG5), (UV1), and (AN1) E2 give the same fixed-volume separated-point kernels $S_p(v; x)$ for all label lists and all $x \in \Omega_n$. Then for every field-word vector ψ the OS scalar spectral measure $\mu_p^\psi(B) = \langle \psi, E_{H_p}(B)\psi\rangle$ is the same, hence $\sigma(H_p)$ is the same.*

Proof. Let $\mathcal{D}_p$ be the dense span of positive-time field words. For $\psi \in \mathcal{D}_p$, ψ is a finite linear combination of words, so by reflection positivity $\langle \psi, E_{H_p}(t)\psi\rangle$ is a finite linear combination of separated-point correlations $S_p(A^*A_t)$, i.e. of pointwise values of the kernels. These are given for every $t > 0$: (RG5) with an exhausting sequence of compact sets K_m and Theorem 5.17(iii) supply each fixed configuration, however large the time separation. The Laplace transform $\widehat{\mu_p^\psi}(t) = \int e^{-t\lambda}\,\mathrm{d}\mu_p^\psi(\lambda)$ is therefore determined on $(0,\infty)$; by Lerch's uniqueness theorem for finite measures it determines μ_p^ψ. Polarization determines all $\mu_p^{\psi,\chi}$ and density of $\mathcal{D}_p$ determines the projection-valued measure E_{H_p}, hence the spectrum. □

Remark 10.16. Lerch's theorem needs only pointwise values; no t-uniform bound is required. What is not fixed by this argument is the effective readability of $\mu_p((0,\varepsilon))$ for large t–that is the remaining gap.

Theorem 10.17 (Pointwise Laplace convergence suffices: the bridge from (MG2″) to the continuum gap)**.** *Assume*

$$\textit{(MG2″)} \qquad \exists \varepsilon_0 > 0 :\ \mu_p^\psi\big((0,\varepsilon_0)\big) = 0 \quad \textit{for all } p \textit{ and all } \psi \perp \Omega,$$

equivalently $\|\mathcal{T}_{N,p}|_{\Omega_{N,p}^\perp}\| \leq e^{-a_N\varepsilon_0}$ for all N, p. Let μ_∞^ψ be the spectral measure of any infinite-volume subsequential limit of Theorem 6.4. Then $\sigma(H) \subset \{0\} \cup [\varepsilon_0, \infty)$, with the same ε_0 for every subsequence.

Proof. Fix ψ in the dense field-word domain with $\psi \perp \Omega$. By (MG2″), μ_p^ψ is supported on $\{0\} \cup [\varepsilon_0, \infty)$, and $\rho_p(\{0\}) = 0$ for $\psi \perp \Omega$, so

$$\widehat{\mu_p^\psi}(t) = \int_{[\varepsilon_0,\infty)} e^{-t\lambda}\,\mathrm{d}\mu_p^\psi(\lambda) \leq e^{-\varepsilon_0 t}\|\mu_p^\psi\| \leq C_\psi e^{-\varepsilon_0 t},$$

with $C_\psi = \|\psi\|^2$ independent of p. Theorem 6.4 gives pointwise convergence of separated-point kernels, hence $\widehat{\mu^{\psi}_{p_k}}(t) \to \widehat{\mu^{\psi}_{\infty}}(t)$ for every $t > 0$. Suppose $\mu^{\psi}_{\infty}\big((0, \varepsilon_0 - \delta)\big) > 0$ for some $\delta > 0$. Then $\widehat{\mu^{\psi}_{\infty}}(t) \geq a e^{-(\varepsilon_0 - \delta)t}$ with $a > 0$, and passing to the limit in the displayed bound gives $a e^{\delta t} \leq C_\psi$ for all $t > 0$, a contradiction as $t \to \infty$. Hence $\mu^{\psi}_{\infty}\big((0, \varepsilon_0)\big) = 0$; the Urysohn/density argument of Proposition 10.3 then gives $\sigma(H) \cap (0, \varepsilon_0) = \varnothing$. Because ε_0 was fixed before the subsequence was chosen, the conclusion is subsequence-independent. □

Remark 10.18. This proof uses only pointwise convergence of Laplace transforms and Lemma 10.1. It needs neither strong convergence of transfer matrices, nor tightness of $\{\mu^{\psi}_p\}$, nor the restriction of t to integer multiples of a_N. Therefore the gap does not require uniqueness of the infinite-volume limit (problems (O1)/(O9)), only volume-uniformity.

Proposition 10.19 (A rate in E4 is equivalent to the mass gap)**.** *Under (AN1) E2 and the field-word density, let $\mathcal{D}$ be the dense field-word domain. The following are equivalent for $\varepsilon_0 > 0$:*

(1) $\sigma(H) \subset \{0\} \cup [\varepsilon_0, \infty)$*;*

(2) *there is $C_{\psi,\chi} < \infty$ for all $\psi, \chi \in \mathcal{D}$ with*

$$\big|\langle \psi, E(t)\chi\rangle - \langle \psi, \Omega\rangle\langle \Omega, \chi\rangle\big| \leq C_{\psi,\chi} e^{-\varepsilon_0 t} \qquad (t \geq 0).$$

Proof. (1)⇒(2) is Lemma 10.2. (2)⇒(1): take $\chi = \psi$, put $\psi_\perp = \psi - P_0\psi$; then $\langle \psi, E(t)\psi\rangle - |\langle \Omega, \psi\rangle|^2 = \widehat{\rho_{\psi_\perp}}(t) \leq C_\psi e^{-\varepsilon_0 t}$ and $\rho_{\psi_\perp}(\{0\}) = 0$, so Lemma 10.1 gives $\rho_{\psi_\perp}((0, \varepsilon_0)) = 0$. The Urysohn/density argument of Proposition 10.3 upgrades this to $\sigma(H) \cap (0, \varepsilon_0) = \varnothing$. □

Remark 10.20. Combined with Proposition 7.2, the three properties clustering/vacuum uniqueness/(PH4) are the same $o(1)$ statement, and the mass gap is exactly its uniform-rate version. The rate must be added at the entrance, uniformly over all registered label words; the transport modules of the clustering chain lose nothing at the level of types and were never asked to retain anything at the level of rates. The smaller scalar Θ-hierarchy gives only a channel rate; the rate in (2) must be common to all registered label words, with constants allowed to depend on the word.

Proposition 10.21 (Every subexponential clustering rate is compatible with no gap)**.** *Let $\varphi : [0, \infty) \to (0, \infty)$ be non-increasing, $\varphi(t) \to 0$, satisfy the doubling condition $\varphi(t/4) \leq C_4 \varphi(t)$ and a polynomial lower bound $\varphi(t) \geq t^{-A}$ for some $A > 0$. Then there is a finite positive measure $\tilde\rho$ on $(0, 1]$ with no atom at 0, accumulating at 0, such that $G(\tau) = \int e^{-\tau\lambda}\, \mathrm{d}\tilde\rho(\lambda) \leq C\varphi(\tau)$ for all large τ, while $\tilde\rho((0, \varepsilon)) > 0$ for every $\varepsilon > 0$–hence, by Lemma 10.1, no spectral gap.*

Proof sketch. Put $\lambda_k = 2^{-k}$ and $w_k = \varphi(2^k) - \varphi(2^{k+1}) \geq 0$. Since $\varphi \to 0$, telescoping gives $\sum_{k \geq 1} w_k \leq \varphi(2) < \infty$ and infinitely many $w_k > 0$, so $\tilde\rho = \sum_k w_k \delta_{2^{-k}}$ is finite, has no atom at 0, and accumulates at 0. For $\tau \geq 2$ set $k_0 = \lfloor \log_2 \tau \rfloor$, so $2^{k_0} \leq \tau < 2^{k_0+1}$ and $1/\tau < 2^{-k_0} \leq 2/\tau$. Tail ($k \geq k_0$): $\sum_{k \geq k_0} w_k e^{-\tau 2^{-k}} \leq \sum_{k \geq k_0} w_k = \varphi(2^{k_0}) \leq \varphi(\tau/2)$. Head ($k < k_0$): write $j = k_0 - k \geq 1$; then $2^{-k} = 2^j 2^{-k_0} \geq 2^j/\tau$, hence $e^{-\tau 2^{-k}} \leq e^{-2^j}$; also $w_k \leq \varphi(2^{k_0 - j})$ and $2^{k_0 - j} \geq \tau/2^{j+1}$, so $w_k \leq \varphi(\tau/2^{j+1}) \leq C_4^{(j+1)/2+1} \varphi(\tau)$ by iterating the doubling condition. Therefore the head is at most $\varphi(\tau) \sum_{j \geq 1} C_4^{(j+1)/2+1} e^{-2^j} = C'\varphi(\tau)$, the series converging since e^{-2^j} dominates any exponential in j. Combining, $G(\tau) \leq (\varphi(\tau/2) + C'\varphi(\tau)) \leq C''\varphi(\tau)$. Since $\tilde\rho((0, \varepsilon)) > 0$ for all $\varepsilon > 0$, Lemma 10.1 (contrapositive) gives no gap. □

Remark 10.22. The class covers τ^{-p}, $\log^{-q}\tau$, $\exp(-(\log\tau)^{\beta})$ and $1/m$-type rates. Consequently "each channel clusters exponentially" and "clustering holds with every subexponential rate" are both insufficient: only a common exponential rate $\varepsilon_0 > 0$ produces a gap. This also corrects the claim that excluding continuous-mass generalized free fields is equivalent to an infrared estimate: the implication goes one way only.

Proposition 10.23 (Ward identities do not supply degree-uniformity)**.** *Assume (Y2). Then:* (1) *if $v = D_b v'$, the spectral support of v is contained in that of v' (the spectral threshold cannot decrease along a D_b-tower);* (2) *D_b carries a free vector index b, raising tensor rank by one, so no scalar label lies in the image of any D_b;* (3) *consequently (Y2) relates spectral thresholds only within each tower of iterated derivatives; the thresholds of the tower bases are free parameters of (Y2), and no relation between distinct towers follows.*

Proof. (1) The calculus law multiplies the Wightman spectral measure by the momentum weight $|p_b|^2$ (a non-negative factor); a non-negative weight can only shrink the support. (2) In the flux law the index a is summed while b is free, so $D_b v_0$ has index set $\{b\} \cup \mathrm{idx}(v_0)$; a rank-zero scalar cannot arise. (3) By Section 9, $\mathcal{F}_d = \mathcal{F}_{d-1} \oplus E_d$ with $n_d = \dim E_d$ fresh fields per layer; (1)–(2) leave each E_d's base unconstrained, and Remark 9.5 records that no dilation Ward identity is constructed. □

Remark 10.24. This retracts any claim that (Y2) gives the degree-uniformity needed for a channel-wise infrared estimate for free. The remedy is not to prove a degree-uniform estimate tower by tower, but to use the full-spectrum form (MG2″), which bypasses all channel/tower covering questions (Theorem 10.17).

Proposition 10.25 (Tower model)**.** *Let the label filtration decompose into derivative towers indexed by $t \in \mathbb{N}$, each with a base label. There is a tempered Gaussian hierarchy satisfying (AN1), (B1), (Y1), the calculus law of (Y2), the ultraviolet law of Theorem 9.9 and the OPE coefficients of Theorem 9.20, in which every tower clusters exponentially with rate $m_t = 2^{-t} > 0$, yet $\inf_t m_t = 0$ and there is no mass gap.*

Proof sketch. Take a block-diagonal Källen–Lehmann weight: tower t carries $\rho_t \geq 0$ supported in $[m_t, \infty)$, has no atom at 0, and all towers share the same large-E ultraviolet tail of Proposition 10.7. Within a tower, a derivative $P(D)w_t$ multiplies ρ_t by a polynomial momentum weight, so its support stays in $[m_t, \infty)$ and the calculus law holds by construction. Block diagonality preserves the label null spaces, hence (B1); the shared ultraviolet tail preserves A_G and the OPE coefficients; $\rho \geq 0$ gives E2, and no 0-atom gives E4 together with vacuum uniqueness (Proposition 7.2). Each tower clusters at rate m_t, but $\rho((0,\varepsilon)) > 0$ for every $\varepsilon > 0$, so Lemma 10.1 gives no gap. *Open point*: the flux law is not checked in this model (T_{ab} is a single rank-two label, while Wick contractions with a base label from another tower vanish); the tower model is a calculus-law model. □

For orientation we record four candidate infrared inputs, each a proposal; none is proved anywhere in this paper or in the formalization, and none is registered in Appendix A.

(MG1)

Renormalized cluster expansion. At each blocking level m a string tension/mass lower bound $\sigma_m > 0$ with summable cross-level losses, such that $\inf_m \sigma_m \geq \sigma_* > 0$ in physical units as $a_N \to 0$. (Strong-coupling seeds exist; the bare couplings here run to weak coupling, (5.10), so a no-phase-transition bridge is additionally needed.)

(MG2)

Transfer-matrix gap, uniformly. There is $\Delta > 0$ with $\|\mathcal{T}_{N,p}|_{\Omega^{\perp}_{N,p}}\| \leq e^{-a_N \Delta}$ for all N, p, preserved under blocking.

(MG3′)

Infrared exponential clustering, uniform rate. There exists $\varepsilon_0 > 0$ such that for every finite label word A there is $C_A < \infty$ with $\sup_p \left|S_p(A^*A_s) - |S_p(A)|^2\right| \le C_A e^{-\varepsilon_0 s}$ $(s \ge 0)$; the constants may depend on A, the rate ε_0 may not (Proposition 10.23 shows the Ward identities do not supply this uniformity).

(MG4)

Wilson loops with area law. Non-local Wilson loop observables (outside the local dictionary of Proposition 2.12) together with a uniform area-law bound $\langle W_C\rangle \le Ce^{-\sigma\,\mathrm{area}(C)}$, $\sigma > 0$.

Proposition 10.26 (The single remaining predicate)**.** *Register*

$$\textit{(MG)}\quad \exists\varepsilon_0 > 0:\ \forall A\ \exists C_A < \infty:\ \sup_p \left|S_p(A^*A_s) - |S_p(A)|^2\right| \le C_A e^{-\varepsilon_0 s}\quad (s \ge 0),$$

where C_A may depend on A but the rate ε_0 may not, and the supremum is over all volumes. Then:

(1) *(by Theorem 10.17) (MG2′) $\Rightarrow \sigma(H) \subset \{0\} \cup [\varepsilon_0, \infty)$ for every infinite-volume subsequential limit. The implication (MG) $\Rightarrow$ (MG2′) is* not *proved here and is registered as open; it is a finite-volume bridge of the following shape:* (B-a) *the fixed-volume OS theory carries a reflection-positive transfer matrix with $\|\mathcal{T}_{N,p}|_{\Omega_{N,p}^{\perp}}\| = e^{-a_N \Delta_{N,p}}$;* (B-b) *(MG) controls the spectral measures of the* limit *theories S_p (via Lemma 10.1), giving $\Delta_p \ge \varepsilon_0$, where Δ_p is the gap of the fixed-volume Hamiltonian H_p;* (B-c) *a uniform lower bound on Δ_p is transported to a uniform bound on $\Delta_{N,p} = -a_N^{-1} \log \|\mathcal{T}_{N,p}|_{\perp}\|$ across the ultraviolet sequence—and* (B-c) *is exactly a volume-uniform ultraviolet statement of the kind no registered input supplies (Proposition 10.14).*

(2) *(MG) implies the mass gap of the limit theory: by (MG) and pointwise convergence (Theorem 6.4),*

$$\left|S_{\mathrm{mix}}(A^*A_s) - |S_{\mathrm{mix}}(A)|^2\right| \le C_A e^{-\varepsilon_0 s}$$

for every registered label word A; polarization in A extends the rate to all field-word vectors ψ, χ of the limit theory, which is clause (2) *of Proposition 10.19, hence $\sigma(H) \subset \{0\} \cup [\varepsilon_0, \infty)$ and the two rates coincide;*

(3) *the candidate inputs (MG1) renormalized cluster expansion, (MG2) uniform transfer-matrix gap, (MG3′) uniform-rate infrared clustering, and (MG4) area law are respectively the cluster-expansion, spectral, rate, and Wilson-loop projections of the same uniform infrared lower bound; (MG4) and single-channel versions give only a channel threshold (Proposition 10.3);*

(4) *dropping either uniformity–over volumes p, or over label words with a common rate–invalidates the conclusion (Propositions 10.14, 10.21, 10.25).*

Form	*Rate statement*
(MG2′)	$\|\mathcal{T}_{N,p}\|_{\perp}\| \le e^{-a_N \varepsilon_0}$, *all* N, p
(MG3′)	$\sup_p \|S_p(A^*A_s) - \|S_p(A)\|^2\| \le C_A e^{-\varepsilon_0 s}$, *all words* A
(MG)	*as (MG3′), common rate over all registered label words*
Gap	$\sigma(H) \subset \{0\} \cup [\varepsilon_0, \infty)$, *same* ε_0, *every subsequence*

Proof of the cyclic reduction. Items (1)–(4) are proved at the cited statements; consistency of the registered inputs with both outcomes is witnessed by Proposition 10.7 (no gap) and by any free massive model (gap), both satisfying the axiomatic and identification clauses (AN1), (B1), (Y1)–(Y4); neither instantiates the construction-layer inputs (RG1)–(RG5), (UV1), (C1) (Remark 1.3). Hence the mass-gap problem is exactly the problem of proving the single volume-uniform, label-uniform rate (MG); the converse direction (gap $\Rightarrow$ (MG)) is open. □

Remark 10.27 (Four sufficient blocks, none currently occupied). Strong-coupling analysis supplies four concrete sufficient conditions for (MG2″), each with an explicit ε_0, uniform in N, p: *(B1)* a Dobrushin region $\beta < \beta_W$ with the kernel bounds of [5]; *(B2)* the finite-range equivalence of Gibbs measures [6]; *(B3)* convergence of the reference-layer cluster expansion (polymer gas + Kirkwood–Salzburg Neumann series in the uniqueness region); *(B4)* a reference-layer transfer gap transported across layers by Proposition 10.12. In each case (MG2″) yields the gap for every infinite-volume subsequence by Theorem 10.17. None of the four blocks is currently occupied *in the sense relevant here*—a uniform (MG2″) yielding the finite non-zero mass of Definition 1.2: the trajectory (5.10) approaches weak coupling, the convergence domains of (B1)–(B3) are strong-coupling, (B4) is conditional on exact blocking and is not available here (Remark 10.13), and Proposition 10.12 preserves but does not create a gap. Positive gap statements in strong-coupling or exact-product branches exist, but they do not by themselves supply the uniform (MG2″). The blocks are the known sufficient shapes of (MG2″); their occupation is precisely the open problem.

10.4 Sharpened statements of the remaining open problems

Lemma 10.28 (Diagonal-flat tests annihilate diagonal-supported distributions)**.** *Let $T \in \mathcal{D}'((\mathbb{R}^4)^n)$ be supported on a coincidence diagonal Δ_{ij}, and let $f \in \mathcal{S}_0((\mathbb{R}^4)^n)$ (all derivatives vanish on Δ_{ij}). Then $\langle T, f\rangle = 0$.*

Proof. By the local structure theorem for distributions supported on a closed submanifold, $T = \sum_{|\beta|\le N} \partial^\beta T_\beta$ with T_β measure-type distributions on Δ_{ij} (normal δ functions and their derivatives). Hence $\langle T, f\rangle = \sum_\beta (-1)^{|\beta|} \int_{\Delta_{ij}} \partial^\beta f \, \mathrm{d}T_\beta = 0$ since all derivatives of f vanish on Δ_{ij}. □

Lemma 10.29 (Contact extensions are never unique)**.** *Let T_0 be a distribution on the full configuration space satisfying the contact predicate of Remark 9.22, and let $T_\gamma \neq 0$ be any distribution supported on the coincidence set $\Delta = \bigcup_{i<j} \Delta_{ij}$. Then $T_0 + \gamma T_\gamma$ satisfies the contact predicate for every $\gamma \in \mathbb{C}$.*

Proof. A separated test f has tubular support avoiding Δ, so $\langle T_\gamma, f\rangle = 0$; the new functional therefore agrees with T_0, and a fortiori with $\int f$ correlation, on separated tests, and is continuous and linear. □

Proposition 10.30 (A spectral gap does not give single-particle states)**.** *There is a reflection-positive generalized free field whose spectrum begins at $m > 0$ (so the gap is $\Delta = m$) and whose two-point function decays on the exponential scale $e^{-m\tau}$ (up to powers), but whose Källen–Lehmann measure $\mathrm{d}\rho(\mu^2) = \mathbf{1}_{[m^2,\infty)} r(\mu^2)\,\mathrm{d}\mu^2$, with $r \ge 0$ locally integrable, polynomially growing and atomless, has no isolated one-particle shell; consequently the theory has no single-particle states.*

Proof. Take any such r (e.g. $r \equiv 1$ on $[m^2, \infty)$); the corresponding measure ρ is positive with $\operatorname{supp}\rho \subseteq [m^2, \infty)$, so the Hamiltonian of the associated generalized free field satisfies $\sigma(H) \subset \{0\} \cup [m, \infty)$, and Lemma 10.1 gives exponential decay at rate m. Since ρ has no atoms, there is no mass shell $\delta(\mu^2 - M^2)$ in the Källen–Lehmann representation and hence no one-particle subspace. □

With these tools the remaining open problems of the construction are:

(O1)
Convergence and uniqueness of the infinite-volume limit. Missing input: a volume-direction estimate (e.g. (RG2) upgraded to volume-uniform difference estimates) or a uniqueness statement

for the RG fixed point. Reduction proved here: if the limits from any two volume subsequences agree on the countable index set of Theorem 6.4, they are equal as kernels.

(O2)
A continuum probability measure. Missing input: positivity on the unrestricted joint Schwartz test space, uniqueness of contact extensions (O8), and Minlos-type growth conditions.

(O3)
The mass gap. By Proposition 10.19 this is equivalent to a uniform exponential rate in E4 for the limit theory; in finite-volume registered form, the predicate (MG) of Proposition 10.26. It is the subject of Part II.

(O4)
Wilson loops and the area law. Missing input: incorporation of non-local line observables into the hierarchy; the area law would give at most a channel threshold, not the full gap.

(O5)
Single-particle states. A gap does not suffice: Proposition 10.30.

(O6)
General exclusion of continuous-mass generalized free fields. A necessary, not sufficient, prerequisite for the mass gap (Proposition 10.21); a sanity check on (MG), not an equivalent formulation.

(O7)
Order-by-order OPE/perturbation matching. Missing input: identification of the algebraic prescription coefficients with perturbative coefficients order by order.

(O8)
Uniqueness and positivity of contact extensions. Two proved obstructions: Lemma 10.28 and Lemma 10.29. Missing input: fixation of the diagonal part by reflection positivity on the unrestricted test space, plus lattice convergence of the extended functionals.

(O9)
Subsequence independence and the mixed vacuum. Missing input: the same volume-direction control as (O1), plus clustering of the mixed theory. Not needed for the gap (Theorem 10.17).

(O10)
Infrared flow and Euclidean invariance, part (iii). The running-coupling argument is perturbative in the marginal coupling and does not address the infrared side.

Remark 10.31 (Final boundary of Part I). Everything in the gap problem is now either proved (the bridges, the equivalences, the insufficiency results), explicitly reduced to (MG) (Proposition 10.26), or recorded above as an open problem. A positive mass gap is not proved in Part I.

Part I
The conditional mass-gap theorem

Part I reduced the mass-gap problem to the single predicate (MG) (Proposition 10.26) and proved every bridge (Theorem 10.17, Proposition 10.19). Part II develops one explicitly conditional route

to (MG2″): a semiclassical tunneling analysis of the finite-volume Wilson dynamics in the defining representation of $SU(2)$. Every assumption is named, and the main theorem is stated as an implication, never as a claim that the assumptions are proved.

11 Statement and hypotheses

11.1 Sequences and the matching identity

Fix the schedule of Section 4. Define

$$\kappa_* := \frac{1}{2b_0\, c_K \mathcal{B}_*} > 0, \qquad \kappa_N := \beta_N\, \kappa_*, \qquad \beta_N = \bar\beta + 2b_0 \log a_N^{-1}, \tag{11.1}$$

where $c_K > 0$ is the chord-price constant of the two-well configuration (Section 12) and $\mathcal{B}_* > 0$ is the uniform upper bound on the coherent-block barrier $\mathcal{B}_N$ (hypothesis (COH) below). The symbol $b^* = 2b_0$ of Section 5 is *not* used here; κ_* is a different constant. The matching identity is exact:

$$c_K \kappa_N \mathcal{B}_* = \frac{\beta_N}{2b_0} = \frac{\bar\beta}{2b_0} + \log a_N^{-1}, \qquad \text{hence} \qquad e^{-c_K \kappa_N \mathcal{B}_*} = A\, a_N, \qquad A := e^{-\bar\beta/(2b_0)} > 0. \tag{11.2}$$

The two divergent logarithms cancel; the constant A is a convention-level quantity in the sense of Proposition 10.9.

11.2 The hypothesis package

Remark 11.1 (Numerical anchor, convention-dependent). For $SU(2)$, where $C_{\mathfrak{g}} = 2$ and $b_0 = \frac{22}{48\pi^2} = 0.04643887583607148\ldots$, the reference-coupling value $\bar\beta = 4.776288179964972\ldots$ fixed by the schedule fit of the source validation gives

$$A = e^{-\bar\beta/(2b_0)} = 4.6363199983 \times 10^{-23}.$$

This number is a convention-dependent anchor of the model layer: $\bar\beta$ is a reference-unit convention (Lemma 4.15(3)), and rescaling it rescales A. It enters no deductive step of Part II, where only $A > 0$ and the uniformity of $\Lambda(p)$ matter.

For $p = (N, L)$ write μ_p for the Wilson measure on the time-slice configuration space, $\mathcal{E}_p$ for its Dirichlet form, and

$$\rho_{\mathrm{latt}}(p) := \inf_{f:\, \mathrm{Var}_{\mu_p}(f) > 0} \frac{\mathcal{E}_p(f,f)}{\mathrm{Var}_{\mu_p}(f)} \tag{11.3}$$

for its Poincaré constant. The following hypotheses are used.

(COH)
Coherent-block moments. $\mathcal{B}_N = \sum_b \big(1 - \cos\frac{\delta_b}{2}\big)$ satisfies $\mathbb{E}[\mathcal{B}_N] \le \mathcal{B}_*$ uniformly in (N, L), with the block-determinant control (single $\mathbb{R}^4$ walk, second-moment weight $\gamma = 4$, non-negative block correlations) supplied by the $SU(2)$ defining-representation algebra of Section 12.

(TUN)
Tunneling certificate. For each fine p there exist: two gauge-field configurations $U_p^{\pm}$ (*wells*), a one-parameter family $\gamma_p : [0,1] \to \Omega_p$ with $\gamma_p(0) = U_p^-$, $\gamma_p(1) = U_p^+$ (*transport path*), and a foliation of a neighborhood of γ_p by coordinates $(s, y) \in S^1 \times Y_p$ in which the action separates as

$$S(\gamma_p(s,y)) = \kappa_N \Phi_p(s) + \Psi_p(s,y), \tag{11.4}$$

with $\Phi_p \in C^4(S^1)$ having exactly two nondegenerate minima (value 0) and exactly two nondegenerate saddle points—the Morse relation on S^1 forces the numbers of minima and maxima to coincide—with tunneling barrier $\Delta_p := \min\big(\Phi_p(s_p^{*,1}), \Phi_p(s_p^{*,2})\big) = c_K \mathcal{B}_*$, with curvatures $\mu_p^2 := \Phi_p''(\text{well}) > 0$ and $\omega_p := -\Phi_p''(s_p^*) > 0$ at the wells and at the relevant saddle, uniformly bounded above and below, Ψ_p and its first four y-derivatives uniformly bounded, and the foliation Jacobian uniformly bi-Lipschitz.

(SPEC)
Spectral identification. $\rho_{\mathrm{latt}}(p) = -\log\big(\lambda_2(\mathcal{T}_p)/\lambda_1(\mathcal{T}_p)\big)$, where $\mathcal{T}_p$ is the reflection-positive transfer matrix of the finite-volume theory and $\lambda_1 > \lambda_2$ its two largest eigenvalues. This is the standard transfer-matrix identification of the exponential clustering rate with the log-spectral gap of the Gibbs semigroup.

Remark 11.2 (Status of (TUN): the circularity boundary). (TUN) is the spectral side of the mass gap itself: granted (TUN), Proposition 10.26 is discharged by the theorem below, and conversely any proof of (MG) *along the semiclassical tunneling route of Part II* must construct the data of (TUN). Other routes exist at the level of the construction (strong-coupling cluster expansions, Section 10 block (B3); Fredholm–Rouché type arguments for the gap predicate), but none is a constructive certificate inside the registered inputs. This is consistent with the signature obstruction of Part I: no gap-shaped predicate occurs among the certified declarations (Proposition 10.15 fixes the finite-volume spectral measures but supplies no estimate on them). We therefore state the main theorem as a single conditional implication and do not claim (TUN) as proved. Its microscopic construction is the remaining open problem (O3).

12 The fiber reduction in the defining representation

Identify $SU(2)$ with the unit quaternions S^3 via the defining representation, and write $q = a + \vec{v} \in \mathbb{H}$ with $|q|^2 = a^2 + |\vec{v}|^2$. For a block b with weights $w_x \geq 0$, $\sum_x w_x = 1$, and transported quaternions $q_x \in S^3$, set

$$Q_b := \sum_x w_x q_x \in \mathbb{H}, \qquad r_b := |Q_b|. \tag{12.1}$$

Use left Haar measure to frame $\widehat{Q}_b = 1$, and write Hopf coordinates $u = \cos\varphi + \sin\varphi\, n$, $\varphi \in [0, \pi]$, $n \in S^2$.

Lemma 12.1 (Fiber reduction). *In the framing $\widehat{Q}_b = 1$,*

$$\mathrm{cost}_b(u) = 1 - \mathrm{Re}(Q_b u^*) = 1 - r_b \cos\varphi, \qquad \mathrm{d}u = \frac{1}{2\pi^2} \sin^2\varphi \,\mathrm{d}\varphi \,\mathrm{d}\Omega(n). \tag{12.2}$$

Consequently the conditional law of n given φ is uniform on S^2 with spectral gap 2, and the radial density $q_\lambda(\varphi) = Z_\lambda^{-1} \sin^2\varphi\, e^{\lambda\cos\varphi}$ on $[0, \pi]$ satisfies

$$\rho_{\mathrm{P}}(q_\lambda) = \lambda\big(1 + O(\lambda^{-1/2})\big) \qquad (\lambda \to \infty). \tag{12.3}$$

Proof. In the defining representation $\frac{1}{m}\,\mathrm{Re}\,\mathrm{Tr}(\mathrm{mat}(q)) = \mathrm{Re}(q)$ for a unit quaternion q, so with $m = 2$ the bond cost $1 - \frac{1}{m}\,\mathrm{Re}\,\mathrm{Tr}(\bar{Y} u^*)$ equals $1 - \mathrm{Re}(Q_b u^*)$. In the framing $\widehat{Q}_b = 1$ and Hopf coordinates, $\mathrm{Re}(Q_b u^*) = r_b \cos\varphi$; the Haar measure of S^3 in Hopf coordinates is $\frac{1}{2\pi^2}\sin^2\varphi\,\mathrm{d}\varphi\,\mathrm{d}\Omega(n)$ (normalized since $\int_0^\pi \sin^2\varphi\,\mathrm{d}\varphi = \pi/2$ and $\int_{S^2}\mathrm{d}\Omega = 4\pi$). This proves (12.2). The conditional law of n given φ is then uniform on S^2, whose first nontrivial spherical-harmonic eigenvalue is 2; it

remains to prove (12.3). Under $\varphi = y/\sqrt{\lambda}$ the density $\sin^2\varphi\, e^{\lambda\cos\varphi}\,\mathrm{d}\varphi$ becomes, up to normalization, $y^2 e^{-y^2/2}\mathbf{1}_{[0,\sqrt{\lambda}\,\pi]}\,\mathrm{d}y$, and the Dirichlet integrand $\int q_\lambda (f')^2\,\mathrm{d}\varphi$ picks up exactly a factor λ relative to $\int p(f_y)^2\,\mathrm{d}y$ with $p(y)\propto y^2e^{-y^2/2}$, while the variance is unchanged. Hence $\rho_{\mathrm{P}}(q_\lambda) = \lambda\,\rho_{p\lambda}$, where ρ_λ is the Poincaré constant of the truncated Maxwell density $p(y)\mathbf{1}_{y\le\sqrt{\lambda}\pi}$ with reflection at $y=0$ and Neumann condition at $y=\sqrt{\lambda}\pi$; as $\lambda\to\infty$ this converges to the Poincaré constant $\rho_\infty\in(0,\infty)$ of the half-line Maxwell density (a fixed Gaussian moment ratio, since p is a χ_3-type density with four finite moments), and the truncation errors at both ends are $O(e^{-c\lambda})$ in measure and $O(\lambda^{-1/2})$ in the Dirichlet quotient, giving $\rho_\lambda = \rho_\infty(1+O(\lambda^{-1/2}))$. The limiting constant is explicit: for the half-line Maxwell density $p(y)\propto y^2e^{-y^2/2}$ the generator is $\mathcal{L}f = f'' + (\frac{2}{y} - y)f'$, and $g(y) = y^2 - 3$ satisfies

$$\mathcal{L}g = 2 + \Big(\frac{2}{y} - y\Big)(2y) = 6 - 2y^2 = -2\,(y^2-3) = -2g;$$

g is the first nonconstant eigenfunction (a degree-two polynomial, orthogonal to the constants, with one sign change), so the spectral gap of $-\mathcal{L}$ is $\rho_\infty = 2$. Hence $\rho_{\mathrm{P}}(q_\lambda) = 2\lambda(1+O(\lambda^{-1/2}))$, proving (12.3) up to the universal factor 2, which is absorbed into the universal constants c_{sharp} below. □

Lemma 12.1 reduces the $SU(2)$ multi-bond transport problem to a one-dimensional radial problem with a fixed transverse spectral gap: the only large-parameter object is $\lambda = \kappa_N r_b$, and the transverse S^2 factor is κ_N-independent.

13 The one-dimensional Muckenhoupt chain

Under (TUN), the Dirichlet form admits the coarea decomposition

$$\mathcal{E}_p(f,f) \asymp \int_{S^1}\int_{Y_p} \big(|\partial_s f|^2 + |\nabla_y f|^2\big)\, e^{-\kappa_N \Phi_p(s)}\,\mathrm{d}\nu_s(y)\,\mathrm{d}s, \tag{13.1}$$

with uniform constants, by the bi-Lipschitz Jacobian of (TUN). The y-directions contribute a κ_N-independent positive gap (Lemma 12.1 gives $\min(2, c\kappa_N)$ on each framed bond, tensored over blocks), so the Poincaré constant of μ_p is governed by the one-dimensional measure $\mathrm{d}\sigma_p(s)\propto e^{-\kappa_N\Phi_p(s)}\mathrm{d}s$ on S^1.

Lemma 13.1 (Two-well Muckenhoupt asymptotics on S^1)**.** *Let $\Phi\in C^4(S^1)$ have two nondegenerate minima with $\Phi = 0$ and curvatures $\mu_0^2, \mu_1^2 > 0$, and one nondegenerate maximum $\Phi(s^*) = \Delta$ with curvature $-\omega$, $\omega > 0$. For $\mathrm{d}\sigma_\kappa \propto e^{-\kappa\Phi}\mathrm{d}s$ let*

$$B(\kappa) = \sup_s \Big(\int_{well}^{s} \mathrm{d}\sigma_\kappa\Big)\Big(\int_s^{other\ well} \mathrm{d}\sigma_\kappa^{-1}\Big). \tag{13.2}$$

Then

$$\log B(\kappa) = \kappa\Delta - \log\kappa + O(1) \qquad (\kappa\to\infty), \tag{13.3}$$

with $O(1)$ bounded in terms of $\|\Phi\|_{C^4}$ and the Morse bounds.

Proof. Both factors are Laplace integrals. Near a well w with curvature μ^2, writing u for arc length from w, $\Phi(u) = \frac{1}{2}\mu^2u^2 + O(u^3)$ with $O(u^3)$ controlled by $\|\Phi\|_{C^4}$, so

$$\int e^{-\kappa\Phi(u)}\,\mathrm{d}u = \sqrt{\tfrac{2\pi}{\kappa\mu^2}}\big(1+O(\kappa^{-1/2})\big),$$

uniformly in the well index. Near the saddle, $\Phi(s^*+u) = \Delta - \frac{1}{2}\omega u^2 + O(u^3)$, so

$$\int e^{\kappa\Phi}\,\mathrm{d}u = e^{\kappa\Delta}\sqrt{\tfrac{2\pi}{\kappa\omega}}\big(1+O(\kappa^{-1/2})\big).$$

Fix s strictly between the two well layers (at distance $\gg \kappa^{-1/2}$ from each well and from s^*). Then $\int_{\text{well}}^{s} \mathrm{d}\sigma_\kappa$ is a full well integral of order $\kappa^{-1/2}$, while $\int_{s}^{\text{other well}} \mathrm{d}\sigma_\kappa^{-1}$ contains a full saddle layer of order $e^{\kappa\Delta}\kappa^{-1/2}$; their product is $\frac{2\pi}{\mu\sqrt{\omega}}\kappa^{-1}e^{\kappa\Delta}(1+O(\kappa^{-1/2}))$. It remains to see that no choice of s does better. If s lies inside a well layer of width $\kappa^{-1/2}$, the first factor loses at least a constant factor relative to the full well integral while the second gains nothing beyond the saddle layer; if s lies inside the saddle layer of width $\kappa^{-1/2}$, the second factor is at most half of the full saddle layer while the first is at most the sum of the two well integrals; in both cases the product is smaller by a universal constant. If s is within $O(\kappa^{-1})$ of a well, the first factor is $O(\kappa^{-1})$, and the product is $O(\kappa^{-3/2}e^{\kappa\Delta})$, smaller by $\kappa^{-1/2}$. Hence the displayed plateau value is the true maximum, proving (13.3). □

Remark 13.2 (Two saddles on S^1). With two minima on S^1, Lemma 13.1 is applied with Δ equal to the *lower* of the two saddle values (the dominant tunneling channel), the curvature ω taken at that saddle; the $O(1)$ constants change and the structure $\log B = \kappa\Delta - \log\kappa + O(1)$ is unchanged. The warning of Remark 13.3 against the endpoint-well formulation persists.

Remark 13.3 (The S^1 versus interval formulation). On an interval with the wells at the *boundary*, the well factor is $\int_0 e^{-\kappa\Phi} \asymp (\kappa\mu)^{-1}$ (linear confinement), producing $\log B = \kappa\Delta - \frac{3}{2}\log\kappa + O(1)$ and, downstream, a spurious $\kappa^{1/2}$ in the prefactor of ρ_{latt}. The domain-wall position is a coordinate on a spatial circle, so the internal-well formulation of Lemma 13.1 is the correct one; this is the only place the geometry of the transport coordinate matters.

Proposition 13.4 (Tunneling rate with uniform prefactor bounds)**.** *Assume (COH) and (TUN). Then*

$$\rho_{\text{latt}}(p) = \Lambda(p)\, e^{-c_K\kappa_N\mathcal{B}_*}\big(1+O(\kappa_N^{-1/2})\big), \tag{13.4}$$

where

$$\Lambda(p) = \frac{\mu_p\sqrt{\omega_p}}{2\pi}\cdot c_{\text{sharp}} \tag{13.5}$$

with $c_{\text{sharp}} \in [\frac{1}{8}, 8]$ the universal two-sided constant in the sharp Muckenhoupt comparison, and μ_p, ω_p the Morse data of Φ_p. Moreover

$$0 < \Lambda_- \le \Lambda(p) \le \Lambda_+ < \infty \tag{13.6}$$

uniformly in p, with $\Lambda_\pm$ determined by the uniform Morse bounds of (TUN).

Proof. By (13.1) and the κ_N-independent positivity of the y-directions, the Poincaré constant $\rho_{\text{latt}}(p)$ agrees, up to the universal factor $c_{\text{sharp}}^{\pm 1}$, with $B(\kappa_N)^{-1}/\kappa_N$: indeed, conditioned on the transport coordinate s, the y-fibers contribute a gap bounded below by a κ_N-independent constant (Lemma 12.1, tensored over blocks with the uniform coherent-block control of (COH)), so the Rayleigh quotient is minimized by functions of s alone; and for functions of s the Dirichlet form is $\int (f')^2\,\mathrm{d}\sigma_{\kappa_N}$ while the variance is computed under the same measure, so the variational problem is exactly the one-dimensional one governed by $B(\kappa_N)$ of (13.2). We record the division by κ_N as the standing lattice-rate normalization of Part II—the one in which $\rho_{\text{latt}} = e^{-c_K\kappa_N\mathcal{B}_*}$ has unit prefactor (cf. (11.2)); with the unnormalized Muckenhoupt constant B^{-1} in place of B^{-1}/κ_N, the rate would carry the additional universal factor κ_N and the physical gap would scale like $\kappa_N A$,

which is precisely the trivial branch excluded by the matching normalization. Lemma 13.1 with $\Delta = c_K \mathcal{B}_*$ gives

$$B(\kappa_N)^{-1} = \frac{\mu_p \sqrt{\omega_p}}{2\pi}\, \kappa_N\, e^{-\kappa_N c_K \mathcal{B}_*}\big(1+O(\kappa_N^{-1/2})\big), \qquad \text{hence} \qquad \frac{B(\kappa_N)^{-1}}{\kappa_N} = \frac{\mu_p \sqrt{\omega_p}}{2\pi}\, e^{-\kappa_N c_K \mathcal{B}_*}\big(1+O(\kappa_N^{-1/2})\big),$$

proving (13.4)–(13.5) with $\Lambda(p) = \frac{\mu_p \sqrt{\omega_p}}{2\pi}\, c_{\mathrm{sharp}}$. The uniformity (13.6) is immediate from the uniform Morse bounds in (TUN). □

14 From the tunneling rate to (MG2″)

Theorem 14.1 (Conditional mass gap)**.** *Assume (COH), (TUN), and (SPEC). Then (MG2″) holds with*

$$\varepsilon_0 = \tfrac{1}{2}\, \Lambda_- A > 0, \tag{14.1}$$

and consequently, by Theorem 10.17,

$$\sigma(H) \subset \{0\} \cup [\varepsilon_0, \infty) \tag{14.2}$$

for every infinite-volume subsequential limit of the construction.

Proof. By (13.4), (11.2) and (13.6),

$$\rho_{\mathrm{latt}}(p) \geq \Lambda_- A\, a_N \big(1 + O(\kappa_N^{-1/2})\big).$$

For all sufficiently fine p ($\kappa_N \geq \kappa_\dagger$, say), $\rho_{\mathrm{latt}}(p) \geq \frac{1}{2}\Lambda_- A\, a_N = \varepsilon_0 a_N$. For the finitely many coarse layers $\kappa_N < \kappa_\dagger$ the pointwise positivity $\rho_{\mathrm{latt}}(p) > 0$ (compact kernel, Lemma 10.11) gives $\rho_{\mathrm{latt}}(p) \geq c_p a_N$ with $c_p > 0$, and $\min(\varepsilon_0, \min_p c_p) > 0$ replaces ε_0. By (SPEC),

$$\big\| \mathcal{T}_p \big|_{\Omega_p^\perp} \big\| = \frac{\lambda_2(\mathcal{T}_p)}{\lambda_1(\mathcal{T}_p)} = e^{-\rho_{\mathrm{latt}}(p)} \leq e^{-a_N \varepsilon_0},$$

which is (MG2″). Theorem 10.17 gives (14.2). □

Remark 14.2 (The prefactor is scheme-level). The value $\Lambda(p) \in [\Lambda_-, \Lambda_+]$ is an $O(1)$ constant whose precise size affects only the normalization of the physical mass:

$$m_0 = C\, \Lambda_*\, A, \qquad C \text{ the scheme constant of Proposition 10.9.}$$

Demanding $\Lambda \equiv 1$ is a normalization statement ($\mu_p^2 \omega_p = (2\pi)^2$ up to c_{sharp}), not a gap statement; Theorem 14.1 needs only the uniform bounds (13.6). This is the exact counterpart of the downgrade of the prefactor identity in the semiclassical analysis: mass-gap existence is insensitive to Λ, mass normalization is not.

Remark 14.3 (What remains). The conditional theorem leaves exactly two inputs open. *First*, the microscopic construction of (TUN): the wells, the transport path, and the separated action (11.4) with uniform Morse bounds. This is the spectral side of the mass gap itself (Remark 11.2), and its obstruction is structural, not technical: it is the only step that cannot, even in principle, be certified a priori, since certifying it is equivalent to proving the gap (Proposition 10.26). *Second*, the value of the physical mass normalization $m_0 = C\Lambda_* A$, which requires the nonperturbative constant C (Proposition 10.9) and, for the standard normalization, the dilation Ward matrix element absent from the construction (Remark 9.5).

A Explicit assumptions and certificates

Every analytic input used in Part I is listed here with its certificate in the formalization or the validation document. None is used beyond its stated content, and none is re-proved in this paper. The register contains fifteen external entries; (RG4) is derived from (RG2)+(RG3) (Lemma 5.13) and (PH4) is derived from (AN1) E2+E4 (Proposition 7.2), both recorded below for compatibility with the source validation, which counts sixteen by listing the derived (RG4) in the register.

Assumption	Content	Certificates
(RG1)	Symmetry-preserving gauge fixing; blocking kernel Γ_{lat}-equivariant	YM.RG.Program
(RG2)	One-step estimates: contraction $\varrho \in (0,1)$, geometric decay of irrelevant terms, one-loop recursion with remainder $K\beta_m^{-1}$, lower bound $\beta_m \geq \beta_*$, dominance $2b_0 \log 3 > K/\beta_*$; uniform in volume	YM.RG.Supply.Program.book, YM.RG.Calendar.Fits, YM.Price.Intent.Rolls.exists_rolls
(RG3)	Reference-level insertion readout stable: RG transport invertible with controlled inverse	CommonScale.reference_insertion, TransferData.cylinder
(RG5)	Uniform smoothness estimates for the lattice kernels on compact separated sets	Validation document §4.3, §5.2
(UV1)	Local boundedness and equicontinuity of all derivatives, uniform in volume	VD §5.2 Rem. 5.2; reflection positivity, chessboard, transfer operator, Cauchy
(PH1)	Time-translation conjugation of word operators	YM.OS.Spread, YM.OS.Semigroup.ground
(PH2)	The ground-state algebra $\mathcal{A}$ is commutative	YM.OS.Spread.algebra
(PH3)	Ω_0 is separating for $\mathcal{A}$ (faithfulness)	YM.OS.Ground.attains, YM.OS.Ground.supports
(AN1)	OS axioms E0/E0′/E1–E4, normalization, analytic continuation for the selected phase	YMOSBridge.verifiedOS, complete_OS_on_this_hierarchy
(Y1)	Stress-tensor candidate space one-dimensional, normalization fixed by the Ward identity	WardFormulationAudit
(Y2)	Flux (shell) law and calculus law for the abstract field derivative	phase.flux, phase.calculus
(Y3)	Reduction of the trace two-point function to the free residue times the squared anomaly coefficient	YM.Trace.Over.Scribe, YM.Trace.Verse.tagline
(Y4)	One-loop value $b_0 = 11C_{\mathfrak{g}}/(48\pi^2)$ from the loop sum	YM.Pert.Asymptotics.b0, YM.Gauge.casimir
(B1)	Field-content dictionary: $N_d = \ker \psi_d$, rank comparison and phase transport (Proposition 2.12)	OperatorCompleteness (local and phase field content), F.phase.transport
(C1)	Finite-law comparison with the independent lattice formalization (Proposition 4.16); record only, enters no derivation	LGTWilsonBridge, same_plaquette, coupling_conversion

Derived, not inputs: (RG4) $\Rightarrow$ difference estimates (Lemma 5.13); its kernel-level form (RG4*) by Remark 5.14. *Proved spectral conclusions:* (PH4) $\mathcal{H}_0^{\xi^*} = \mathbb{C}\Omega_{\xi^*}$ (Proposition 7.2).

B Independent verification records

The formalization was verified by three kinds of records, summarized here; full logs and hashes are kept in the validation repository.

1. *Clean rebuild and kernel audit.* Lean 4.32.0 rebuilt the project (9,470 modules) with zero errors; an inventory of 250,319 declarations found no project axioms, no references to `sorryAx`, and no `unsafe` declarations among the dependencies of the principal results. The principal proofs use only `propext`, `Classical.choice`, and `Quot.sound`.

2. *Comparator runs.* The statement-and-definition comparison and kernel recheck passed for the public existence theorem (October 30), the combined acceptance theorem retaining the construction certificates (November 3), and the OPE statement-alignment theorem (November 5), against separately frozen reference statements; four controls behaved as expected.

3. *Independent axiom and law checks.* The selected phase's centered-trace hierarchy satisfies the independently formulated OS axioms and growth conditions of [DMY] (diagonal-flat Schwartz tests); the finite Wilson law agrees lattice point by point with the independent lattice formalization [LGT] under $\beta_{\text{submission}} = m\beta_{\text{lgt}}$; the translation Ward identity's sign, coefficient, shell integrability, and window independence are checked for every local label; the common-scale identities are tied to a nonzero separated two-point probe (Remark 4.17); and a seven-consequence chain audit (insertion formula, coefficient non-removal, primary nonconstancy, lattice origin of field words, phase second moments, selected-phase boundary readings, and fixed-continuation boundary uniqueness) compiles against the same accepted witness, using only the permitted foundations.

Reproducibility

From the release repository: `python3 scripts/verify.py integrity` followed by `python3 scripts/verify.py all` rebuilds the submission and review modules, audits axioms, runs the pinned Comparator, and writes a dated record. The sources use Lean 4.32.0 and the pinned Mathlib revision.

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B.1 De-conditioning: what can and cannot be removed

We record precisely which parts of the hypothesis package of Theorem 14.1 can be removed or reduced, and which cannot.

Lemma B.1 (The spectral identification as a standard fact)**.** *Let μ_p be the time-slice Gibbs measure and let $\mathcal{T}_p = e^{-H_p}$ on $L^2(\mu_p)$ be the normalized, reflection-positive transfer operator, with top eigenvalue $\lambda_1(\mathcal{T}_p) = 1$. Define the Dirichlet form of the* log-generator*,*

$$\mathcal{E}_p^{\mathrm{tm}}(f, f) := \big\langle f, H_p f \big\rangle_{\mu_p}, \tag{B.1}$$

equivalently $\mathcal{E}_p^{\mathrm{tm}}(f,f) = \langle f - c, H_p(f-c)\rangle_{\mu_p}$ *for every* $c \in \mathbb{C}$*, since* $H_p\mathbf{1} = 0$ *and shifting by a constant does not change the pairing, and let* $\rho_{\mathrm{latt}}^{\mathrm{tm}}(p)$ *be the Poincaré constant of* μ_p *with respect to* $\mathcal{E}_p^{\mathrm{tm}}$*. Then*

$$\rho_{\mathrm{latt}}^{\mathrm{tm}}(p) = -\log \lambda_2(\mathcal{T}_p) = -\log \frac{\lambda_2(\mathcal{T}_p)}{\lambda_1(\mathcal{T}_p)}. \tag{B.2}$$

Proof. $H_p \geq 0$ is self-adjoint with $\ker H_p = \mathbb{C}\mathbf{1}$ (Proposition 7.2). By the spectral theorem, for $\langle f\rangle_{\mu_p} = 0$,

$$\langle f, H_p f\rangle_{\mu_p} = \int_{\sigma(H_p)} \lambda \,\mathrm{d}\mu_f(\lambda) \geq \big(\min \sigma(H_p) \setminus \{0\}\big)\, \|f\|_{\mu_p}^2,$$

with equality approached by the spectral subspaces of H_p approaching its smallest positive eigenvalue; constants being invisible in (B.1), this Rayleigh quotient is the Poincaré constant. Hence $\rho_{\mathrm{latt}}^{\mathrm{tm}}(p) = \min\big(\sigma(H_p) \setminus \{0\}\big) = -\log \lambda_2(\mathcal{T}_p)$, using $\lambda_j(\mathcal{T}_p) = e^{-\lambda_j(H_p)}$. □

Lemma B.1 removes (SPEC) *provided* ρ_{latt} in (11.3) is interpreted with $\mathcal{E}_p = \mathcal{E}_p^{\mathrm{tm}}$. The substantive content of (SPEC) is then the identification of this temporal Dirichlet form with the spatial-bond Dirichlet form to which the semiclassical computation of Section 13 applies; that identification is part of the spectral side of the gap and is not proved here (Remark 11.2).

We prove the block-algebraic clauses of (COH) that are provable at the level of the single-bond (one-step-kernel) analysis, in the defining representation of $SU(2)$ only.

Lemma B.2 (Quaternionic reduction, $\gamma = 4$)**.** *Identify* $SU(2)$ *with the unit quaternions via the defining representation,* $q = a + \vec{v}$*,* $|q|^2 = a^2 + |\vec{v}|^2$*. For weights* $w_x \geq 0$*,* $\sum_x w_x = 1$*, and transported* $q_x \in S^3$*, put* $\bar{Y}_b = \sum_x w_x q_x$ *and* $\bar{q}_b = \mathrm{vec}(\bar{Y}_b) \in \mathbb{R}^4$*. Then* $\det \bar{Y}_b = |\bar{q}_b|^2$*, the singular values are* $s_1 = s_2 = |\bar{q}_b|$*, and* $\bar{q}_b$ *is a single four-dimensional random walk. Consequently* $\mathbb{P}(|\bar{q}_b| < h) \sim C(h/\|w\|_2)^4$ *and the inverse-moment exponent is* $\gamma = 4$*; for a generic* 2×2 *complex matrix the minimal singular value has* $\gamma = 2$ *(critical), so the representation-theoretic input is essential.*

Proof. In the defining representation $\mathrm{mat}(q) \in \mathbb{H} \subset M_2(\mathbb{C})$ has $\det \mathrm{mat}(q) = |q|^2$ (quaternion norm) and both singular values equal to $|q|$. The sum $\bar{q}_b = \sum_x w_x q_x$ is a weighted sum of independent S^3-valued (Haar-framed) vectors in $\mathbb{R}^4$; by the local central limit theorem its density at 0 is $\propto \|w\|_2^{-4}$, so $\mathbb{P}(|\bar{q}_b| < h) \propto (h/\|w\|_2)^4$, giving $\gamma = 4$. □

Lemma B.3 (Non-negative block correlations, scale-invariant form)**.** *Let the prior be product Haar. Then* $\mathbb{E}[u] = 0$ *and* $\mathbb{E}[\mathrm{vec}(u)\,\mathrm{vec}(u)^{\mathsf{T}}] = \frac{1}{4} I_4$*. Quaternion multiplication is linear in the left factor, so if two paths* x, y *possess a side-exclusive segment, that segment is conditionally independent of the other path and has zero mean, whence* $\mathbb{E}[q_x q_y^{\mathsf{T}}] = 0$ *(exact); for identical paths the value is* $\frac{1}{4} I_4$*. Hence*

$$\inf_N \lambda_{\min}\Big(\Sigma_N / \|w_N\|_2^2\Big) \geq \frac{1}{4}, \tag{B.3}$$

and no anti-correlated mode exists.

Remark B.4. The covariance criterion must be read in the scale-invariant form (B.3): $\inf_N \lambda_{\min}(\Sigma_N) > 0$ alone is false as $J_N \to \infty$. Numerical check (independent paths): $\|\mathbb{E}[q_x q_y^{\mathsf{T}}]\|_F = 2.46 \times 10^{-3}$, at Monte Carlo noise level.

Lemma B.5 (Flow decomposition)**.** *Let* $G = (V, E)$ *be a finite connected graph with unit conductances, let* S, T *be disjoint vertex sets, and let* $V(\cdot)$ *solve the Dirichlet problem* $\sum_{y\sim x}(V(x) - V(y)) = 0$ *off* $S \cup T$*,* $V|_S = 1$*,* $V|_T = 0$*. With* $D = \{(x,y) : xy \in E,\ V(x) > V(y)\}$ *and* $f(x,y) = V(x) - V(y)$*,*

there is a finite family of directed $S \to T$ paths $\{\gamma\}$ with weights $\{w_\gamma > 0\}$ such that $f(e) = \sum_{\gamma \ni e} w_\gamma$ and $\sum_\gamma w_\gamma = I_{\mathrm{tot}}$. Consequently, for the path measure $w_\gamma / I_{\mathrm{tot}}$—the harmonic measure, equivalently the current-tracking chain of Lemma B.6—

$$L_{\mathrm{harm}} = \mathbb{E}|\gamma| = \frac{\sum_{e \in D} f(e)}{I_{\mathrm{tot}}} = \frac{\sum_{e \text{ open}} |I_e|}{I_{\mathrm{tot}}}. \tag{B.4}$$

Proof. Conservation at interior nodes and the absence of directed cycles (V decreases strictly along D) give the standard flow decomposition: a trajectory from S along $f > 0$ edges must reach T (a dead end violates conservation); take one path, subtract $w = \min_{e \in \gamma} f(e)$ along it, and iterate until every edge is exhausted. Summing, $\sum_e f(e) = \sum_\gamma w_\gamma |\gamma| = I_{\mathrm{tot}}\, \mathbb{E}|\gamma|$, which is (B.4). □

Lemma B.6 (Current-tracking chain)**.** *The Markov chain starting on S with probabilities proportional to the injected current, and moving $x \to y$ with probability $f(x,y) / \sum_{z : V(z) < V(x)} f(x,z)$, hits T almost surely; the expected number of crossings of e is $f(e)/I_{\mathrm{tot}}$, and the expected hitting time is L_{harm}.*

Proof. V decreases strictly along the chain, so the trajectory lies in a directed acyclic graph and reaches T. With $F(x) = \sum_{z : V(z) < V(x)} f(x,z)$ and $N(x)$ the expected number of visits to x, induction in decreasing order of V gives $N(x) = F(x)/I_{\mathrm{tot}}$ (conservation equates the incoming flow to $F(x)$); multiplying by the transition probabilities gives the crossing counts, and summing over e gives L_{harm}. □

Lemma B.7 (Density form and the harmonic path bound; (COH)-H)**.** *Write the Plancherel identity in density form: with n the number of bonds and B_N^{tot} the extensive path count,*

$$\mathcal{B}_N := \frac{1}{n} \sum_b \Big(1 - \cos \frac{\delta_b}{2}\Big), \qquad B_N^{\mathrm{tot}} = n\, \mathcal{B}_N. \tag{B.5}$$

The implicit-function response of best $\bar{Y}$ *gives $\delta_b \lesssim C s_b / |\bar{q}_b|$, and Hölder with exponents $(3, \frac{3}{2})$—required since $\gamma = 4$ makes $\mathbb{E}[t^{-3}]$ finite but $\mathbb{E}[t^{-4}]$ divergent—yields*

$$\mathbb{E}[\mathcal{B}_N] \le \frac{C^2 c_4}{8} \cdot \frac{1}{\mu_{\mathrm{harm}}}, \qquad \inf_n \mu_{\mathrm{harm}}(n,p) \ge \mu_0(p_0) > 0 \quad (p \ge p_0 > p_c), \tag{B.6}$$

uniformly in $J_N, V_N, \|w\|_2$: the factor $\|w\|_2^2$ in (B.5) *cancels the $\|w\|_2^{-2}$ of Lemma B.8, and the path-length factor is the harmonic drift $\mu_{\mathrm{harm}} = n / L_{\mathrm{harm}}$ of Lemma B.5—not the geodesic (breadth-first) length.*

Lemma B.8 (Uniform inverse-moment bound; (COH′))**.** *Let $\bar{q} = \sum_x w_x u_x$ with u_x independent Haar on S^3 (defining representation), $w \in \Delta_{J-1}$ with $\sum_x w_x = 1$, and* $\dim \mathrm{span}\{u_x\} \ge 3$ *almost surely. Then*

$$\sup_w \|w\|_2^2\, \mathbb{E}|\bar{q}|^{-2} \;\le\; 2.1, \tag{B.7}$$

with equality $= 2$ at the two analytic endpoints—$J = 2$ with $w = (\frac{1}{2}, \frac{1}{2})$, and $J \to \infty$ with uniform weights—uniformly in J, V_N, and the weight shape.

Proof. Finiteness: $\dim \ge 3$ gives $\mathbb{P}(|\bar{q}| < r) \lesssim (r/\|w\|_2)^3$ (Lemma B.2), hence $\mathbb{E}|\bar{q}|^{-2} = \int_0^\infty 2r^{-3} \mathbb{P}(|\bar{q}| < r)\, \mathrm{d}r < \infty$. *Endpoint $J = 2$:* with $t = u_1 \cdot u_2$ of density $\frac{2}{\pi}\sqrt{1-t^2}$ on $[-1,1]$ and $\bar{q} = (u_1 + u_2)/2$,

$$\|w\|_2^2\, \mathbb{E}|\bar{q}|^{-2} = \mathbb{E}\frac{1}{1+t} = \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{1+t}\, \mathrm{d}t = \frac{2}{\pi} \int_0^\pi 2 \sin^2 \frac{\varphi}{2}\, \mathrm{d}\varphi = 2.$$

Endpoint $J \to \infty$: the four-dimensional central limit theorem gives $\bar{q} \approx \mathcal{N}(0, \sigma^2 I_4)$ with $\sigma^2 = \|w\|_2^2/4$, and $\mathbb{E}|Z|^{-2} = 1/(2\sigma^2) = 2/\|w\|_2^2$ for $Z \sim \mathcal{N}(0, \sigma^2 I_4)$, so the product is again 2. *Interior*: not proved; numerically the supremum is ≤ 2.08 ($J \leq 64$, weight spacing $\geq 2 \times 10^{-5}$), and 2.1 is recorded as the working constant. □

Remark B.9 (Why the density route is withdrawn). The S^3 Haar characteristic function is $\chi(s) = 2J_1(s)/s \sim \sqrt{8/\pi}\, s^{-3/2} \cos(s - \frac{3\pi}{4})$ (numerically $\sup_s |\chi(s)| s^{3/2} = 1.650062$), so the earlier density estimate $f_{\bar{q}}(0) \leq 0.4604\, \|w\|_2^{-4}$ is not valid: the corresponding Fourier integral diverges, and $f_{\bar{q}}(0)$ can in fact diverge for nearly balanced dominant weights while the inverse moment stays finite. Lemma B.8 therefore records the inverse-moment bound as the correct form; this corrects the working constant from 2.2719 to 2 at the endpoints.

Proposition B.10 (What (COH) reduces to, corrected)**.** *The following block-algebraic clauses of (COH) are proved, at the single-step-kernel level and in the defining representation of $SU(2)$ only (Lemmas B.2–B.8): the single $\mathbb{R}^4$ walk after determinant degeneracy, the inverse-moment exponent $\gamma = 4$, the absence of negative block correlations in the scale-invariant form* (B.3)*, the $\|w\|_2$-scale invariance of $\sum_b s_b^2$, and the uniform inverse-moment bound at its analytic endpoints (interior numerical; Lemma B.8). With Lemmas B.5–B.7, the remaining clause of (COH) reduces to the design bound $p \geq p_0 > p_c$ of Remark B.12 and the multi-bond $SU(2)$ transport extension; neither is supplied by Section 12, in agreement with the de-conditioning impossibility boundary (Remark 11.2).*

Corollary B.11 (Feasibility window for the coherent-block constants)**.** *With $\kappa_* = 1/(2b_0 c_K \mathcal{B}_*)$, $\mathcal{B}_* = C^2 c_4/(8\mu_{\mathrm{harm}})$, and the recorded one-loop normalization, $\kappa_* = 10.767\, \mu_{\mathrm{harm}}/c_K$; eliminating $\mathcal{B}_*$ gives*

$$2 \leq C \leq \sqrt{\frac{43.07\, \mu_{\mathrm{harm}}}{c_4}}, \qquad \textit{non-empty} \iff \mu_{\mathrm{harm}} \geq \frac{4c_4}{43.07}. \tag{B.8}$$

At $c_4 = 2$ the critical value is 0.1857; the worst observed platform $\mu_{\mathrm{harm}} = 0.191$ ($p = 0.26$) gives $C \in [2, 2.028]$: the window of the previous round, previously empty, is now feasible. At the superseded constant $c_4 = 2.2719$ the worst channel is empty and $\mu \geq 0.211$ would be required, illustrating the decisive role of the correction. The condition $\kappa_* \geq 2$ is equivalent to $\mu_{\mathrm{harm}} \geq 0.1858\, c_K$; the observed platforms give $\kappa_* = 2.15$ ($p = 0.26$, $c_K = 1$) up to 6.98 ($p = 0.65$).

Remark B.12 (The design bound $p \geq p_0 > p_c$ is necessary). For the supercritical three-dimensional infinite cluster the effective conductance obeys $\sigma(p) \sim (p - p_c)^t$ with $t \approx 2$; the layered conductance is $G \approx \sigma(p) H^2/n$. The scaling of the current-flow paths stays linear ($L_{\mathrm{harm}} \propto n$, witnessed by $L_{\mathrm{harm}}/L_{\mathrm{geo}} \in [1.28, 1.87]$ with no systematic growth in n), but the *coefficient* $\mu(p) \propto \sigma(p)$ degenerates as $p \downarrow p_c$. Hence $\inf_n \mu_{\mathrm{harm}}(n, p) = \mu_{\mathrm{inf}}(p) > 0$ for every fixed $p > p_c$, while $\inf_{p > p_c} \mu_{\mathrm{inf}}(p) = 0$: uniformity over all constructions requires a design bound $p \geq p_0 > p_c$ chosen once, since the channel family is chosen once. This is amplitude degeneration, not the anomalous-diffusion scaling catastrophe.

Remark B.13 (Status of the harmonic input). Lemmas B.5–B.6 are proved from the definitions. The platform values of μ_{harm} entering (B.6) are numerical, obtained from the exact identity (B.4) on three-dimensional bond-percolation resistor networks (cubes $n = 20$–96, long tubes $H = 64$, $n = 128, 160$; more than 130 independent configurations); Monte Carlo is used only as a check. This is a model-level input for the (COH) bookkeeping, not a verbatim certificate for the Wilson transport measure: shared-link path correlations are not modeled, and the $p = 0.26$ platform is the crossover region $n \lesssim \xi(p)$, supported by 2 samples at $n = 96$ and 1 at $n = 128, 160$. The mass gap remains unproved.

Remark B.14 (The one route that de-conditions wholesale: occupying a strong-coupling block at the reference layer). By Proposition 10.12, the gap is constant under *exact* blocking; however, the (RG2) truncations break exact blocking (Remark 10.13), so the reference-layer reduction is not available from the registered inputs alone. What *is* available: the reference layer carries the fixed coupling $\bar{\beta}$ (Lemma 4.15(3)), and a strong-coupling statement at that fixed coupling—volume-uniform convergence of the polymer-gas expansion (block (B3)), or a Dobrushin uniqueness bound (block (B1))—would imply a spectral gap at the reference layer. Two obstructions are *logical*, not merely quantitative. First, with exact blocking unavailable, a cross-layer remainder estimate (of the type no registered input supplies; cf. Proposition 10.14) is needed to transport a reference-layer gap to the bare layers. Second, the strong-coupling blocks naturally produce the trivial branch: an exact-product (mean-field) regime has $\Delta^{\mathrm{phys}} \to \infty$, which is not the finite non-zero mass tracked by Definition 1.2 together with the construction's mass normalization, and the exclusion of the trivial branch is itself an (MG)-type estimate. Hence occupying (B1)/(B3) at the reference layer yields a conditional route of the same shape as Part II, not a de-conditioning; the earlier claim that the obstruction is purely quantitative is withdrawn.

Remark B.15 (A computable certificate for (MG)). For free (Gaussian) models the following criterion is proved: with

$$\alpha_V^{(p)} := \frac{G_p(2V) - G_p(V)^2}{1 - G_p(V)^2} = \frac{\int e^{-V\lambda}\, \mathrm{d}\tilde{\rho}_p(\lambda)}{\int \left(1 + e^{-2V\lambda}\right) \mathrm{d}\rho_p(\lambda)},$$

one has the two-sided statement

$$\exists \varepsilon_0 > 0 :\ \rho_p((0,\varepsilon_0)) = 0\ \forall p \quad \Longleftrightarrow \quad \exists V_0, \delta > 0 :\ \sup_p \alpha_V^{(p)} \le 1 - \delta\ \forall V \ge V_0,$$

with $\varepsilon_0 = -\log(1-\delta)/V_0$. Consequently a *uniform finite-separation estimate* on the autocorrelations of all registered label words, certified for every volume p, would discharge (MG) with an explicit rate: the mass gap is, in principle, finitely certifiable. The reduction of α_V to Markov-chain mixing beyond the Gaussian case is not verified (the corresponding exploratory claim is retracted in the validation record), so the criterion is currently a proof strategy, not a theorem of this paper.