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THE RELATIONS OF A LIE SUBALGEBRA CONSIDERED BY TAO: A COMPLETE PRESENTATION

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Abstract. Let $\mathfrak{g}$ be the Lie algebra over a field of characteristic 0 generated by D_i, E_i $(i \geq 1)$ subject to $[D_i, D_j] = [E_i, E_j] = 0$ and $[D_i, E_j] = [D_j, E_i]$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the subalgebra generated by the elements $[D_i, E_j]$ and E_i. On MathOverflow, T. Tao asked whether the obvious relations $[D_i, E_j] = [D_j, E_i]$, $[E_i, E_j] = 0$ and $[E_i, [D_j, E_k]] = [E_k, [D_j, E_i]]$ generate all relations of $\mathfrak{h}$. We prove that they do. The proof has four parts. First, a polarisation argument reduces the problem to multilinear components. Second, we give a presentation of the ideal of $\mathfrak{g}$ generated by the E_i. Third, a filtration argument shows that injectivity follows from an equality of dimensions; the associated graded algebra is computed explicitly. Fourth, the required dimensions are those of the multilinear components of $\mathfrak{g}$, which are given by Eulerian numbers. These dimensions are equivalent to known results: the multilinear components of the enveloping algebra of the ideal generated by the E_i coincide with the homology of the Losev–Manin spaces, whose Betti numbers were computed by Losev and Manin and whose algebra structure was presented by Dotsenko, Shadrin and Tamaroff. We explain this relation precisely, and we also include an independent, self-contained derivation, based on a Poincaré–Birkhoff–Witt basis of $U(\mathfrak{g})$ obtained from Bergman's diamond lemma and a sign-reversing involution. As by-products, $\mathfrak{h}$ is a semidirect product of a free Lie algebra and an abelian Lie algebra, and the elements $[D_i, E_j]$ $(i \leq j)$ freely generate a free Lie subalgebra of $\mathfrak{g}$. We were unable to find the affirmative answer to Tao's question or this freeness statement in the literature. We also report independent computer verifications.

1. Introduction

Throughout, $\mathbb{k}$ is a field of characteristic 0 (for instance $\mathbb{R}$), and $\mathbb{N} = \{1, 2, 3, \dots\}$. The following question was asked by T. Tao on MathOverflow in February 2024 [5].

Let $\mathfrak{g}$ be the Lie algebra over $\mathbb{k}$ generated by D_i, E_i, $i \in \mathbb{N}$, subject to

$$[D_i, D_j] = [E_i, E_j] = 0, \qquad [D_i, E_j] = [D_j, E_i] \qquad (i, j \in \mathbb{N}). \tag{1}$$

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie subalgebra generated by the elements $[D_i, E_j]$ and E_i. These elements satisfy

$$[D_i, E_j] = [D_j, E_i], \qquad [E_i, E_j] = 0, \tag{$*$}$$

$$[E_i, [D_j, E_k]] = [E_k, [D_j, E_i]], \tag{$**$}$$

the second line following from the Jacobi identity. Tao asked whether $(*)$ and $(**)$ generate all relations of $\mathfrak{h}$.

In a comment on the question [6], YCor reformulated it as follows. Let $\mathfrak{l}$ be the Lie algebra with generators F_{ij}, E_i $(i, j \in \mathbb{N})$ and relators

$$F_{ij} = F_{ji}, \qquad [E_i, E_j] = 0, \qquad [E_i, F_{jk}] = [E_k, F_{ij}]. \tag{2}$$

There is a natural homomorphism $\theta\colon \mathfrak{l} \to \mathfrak{g}$, $F_{ij} \mapsto [D_i, E_j]$, $E_i \mapsto E_i$, whose image is $\mathfrak{h}$. The question asks whether θ is injective.

Theorem 1.1. *The homomorphism $\theta\colon \mathfrak{l} \to \mathfrak{g}$ is injective. Equivalently, the relations $(*)$ and $(**)$, together with the axioms of a Lie algebra, generate all relations among the elements $[D_i, E_j]$ and E_i of $\mathfrak{g}$.*

Along the way we obtain the following structural information.

Corollary 1.2. (1) *The Lie algebra $\mathfrak{h} \cong \mathfrak{l}$ is a semidirect product $\mathcal{L} \rtimes \mathfrak{a}_E$. Here $\mathfrak{a}_E$ is the abelian Lie algebra with basis $(E_i)_{i\in\mathbb{N}}$. The algebra $\mathcal{L}$ is a free Lie algebra with free generators $Y_W = \mathrm{ad}(E_{j_1}) \cdots \mathrm{ad}(E_{j_r})[D_a, E_b]$, indexed by finite multisets $W = \{j_1, \dots, j_r, a, b\}$ of size at least 2; the element Y_W depends only on W. (Corollary 10.1.)*

(2) *In particular the elements $[D_i, E_j]$, $i \le j$, freely generate a free Lie subalgebra of $\mathfrak{g}$. This answers affirmatively a further question raised by YCor* [6]*. (Corollary 10.2.)*

(3) *Fix $n \ge 2$. Consider the span of the brackets in $\mathfrak{g}$ in which every index $1, \dots, n$ occurs exactly once and D's occur exactly a times. This subspace has dimension equal to the Eulerian number $A(n-1, a-1)$, the number of permutations of $n-1$ letters with $a-1$ descents. In total its dimension is $(n-1)!$ (summed over a). (Corollary 10.4.) This statement is equivalent to known results on the homology of Losev–Manin spaces* [4, 3]*; see Section 11.*

Status of the question and relation to prior work. As of November 16, 2026 the MathOverflow question has no answer. The comments on it contain the reformulation above and partial observations. We searched for a later resolution (MathOverflow, arXiv, zbMATH Open, Crossref, OpenAlex, OEIS and general web search; Google Scholar and MathSciNet were not searched directly). We were unable to find an answer to the question, nor the freeness statement of Corollary 1.2(2), in the literature.

Part of the machinery is, however, known in an equivalent form. Write the defining relators of the ideal of $\mathfrak{g}$ generated by the E_i (Section 4) in its enveloping algebra and restrict to multilinear components. One obtains exactly the relators of the twisted associative algebra tHyperCom of Dotsenko, Shadrin and Tamaroff, which they show to be the homology of the Losev–Manin spaces [3, Prop. 2.4.2]. The Betti numbers of these spaces are Eulerian numbers [4, Thm. 2.3]. Consequently, the Hilbert series of $U(\mathfrak{g})$ and of $\mathfrak{g}$ (Corollary 8.3, Proposition 9.1) and Corollary 1.2(3) follow from these known results. Section 11 gives the precise statement. Our proofs of these facts (Sections 7–9) are independent of [4, 3] and serve to keep the paper self-contained. The remaining results are Lemmas 5.4 and 6.1, Theorem 1.1, and Corollaries 10.1 and 10.2. We were unable to find them in the literature, but we make no stronger claim about them.

Outline of the proof.

- In Section 3 a polarisation argument reduces Theorem 1.1 to the multilinear components, where each index occurs at most once.
- In Section 4 we show that $\mathfrak{g} \cong \mathfrak{k} \rtimes \mathfrak{a}_D$, where $\mathfrak{k}$ is the ideal generated by the E_i. We give an explicit presentation of $\mathfrak{k}$ in which $\mathfrak{l}$ appears as the algebra defined by the "small" generators and relations.
- In Section 5 an *excess* filtration of $\mathfrak{k}$ is introduced. Its associated graded algebra is dominated by an explicit Lie algebra $\mathfrak{k}^{\mathrm{top}}$ (Lemma 5.4). If $\dim \mathfrak{k} = \dim \mathfrak{k}^{\mathrm{top}}$ in a multilinear degree, then θ is injective there.

- In Section 6 we determine $\mathfrak{k}^{\mathrm{top}}$ completely (Lemma 6.1) and compute its Hilbert series.
- In Sections 7–9 we compute the Hilbert series of $\mathfrak{g}$, hence of $\mathfrak{k}$, independently. We use a quadratic Gröbner basis of $U(\mathfrak{g})$, obtained from Bergman's diamond lemma [2], and count normal words by a sign-reversing involution. A generating-function identity shows that the two Hilbert series agree, which proves the theorem. Alternatively, the Hilbert series of $\mathfrak{g}$ can be deduced from [4, 3]; see Section 11.

The proof is self-contained apart from Bergman's diamond lemma [2, Theorem 1.2] and the Poincaré–Birkhoff–Witt (PBW) theorem (which is derived from the diamond lemma in [2, §3]). Section 11 discusses the relation to the work of Losev–Manin and Dotsenko–Shadrin–Tamaroff. Section 12 reports independent computer checks: exact rational arithmetic up to multilinear degree 8, and a modular computation, which is rigorous over $\mathbb{Q}$, in degree 9.

2. Conventions and standard facts

2.1. **Multisets and gradings.** A *multiset* T of indices is a finitely supported function $\mathbb{N} \to \mathbb{Z}_{\geq 0}$, and $|T|$ denotes its size. We write $T + i$ for T with one more copy of i, and $A + B$ for the sum of multisets. A multiset is a *set* if all multiplicities are ≤ 1.

All Lie algebras below are generated by elements carrying an *index content* (a multiset), and all defining relations are homogeneous for it. Hence they are graded by the monoid of multisets. A component of content S is called *multilinear* if S is a set. We also use two further gradings, the D-degree and the E-degree: D_i has bidegree $(1, 0)$ and E_i has bidegree $(0, 1)$.

Every permutation π of $\mathbb{N}$ induces automorphisms of all algebras considered ($D_i \mapsto D_{\pi(i)}$, and so on), because all defining relations are invariant under renaming indices. Such an automorphism maps the component of content S isomorphically onto that of content $\pi(S)$. Consequently, the dimension of a multilinear component depends only on $|S|$ and on the bidegree.

2.2. **Multilinear Hilbert series.** Let M be a graded vector space of this kind with finite-dimensional components. Its *multilinear Hilbert series* is the formal power series

$$\mathrm{H}_M(x) = \sum_{n \geq 0} \frac{x^n}{n!} \sum_{a,b \geq 0} \dim M_{[n]}^{(a,b)}\, y^a z^b \;\in\; \mathbb{Q}[y, z][[x]],$$

where $[n] = \{1, \dots, n\}$ and $M_{[n]}^{(a,b)}$ is the component of content $[n]$ and bidegree (a, b). We only use spaces in which the dimension of the content-S component depends only on $|S|$.

Lemma 2.1 (exponential formula)**.** *Let $M, M_1, \dots, M_k$ be graded spaces as above, with $M_\emptyset = 0$.*

(1) $\mathrm{H}_{M_1 \otimes \cdots \otimes M_k} = \mathrm{H}_{M_1} \cdots \mathrm{H}_{M_k}$*, where content and bidegree are additive on tensor products.*
(2) $\mathrm{H}_{T(M)} = 1/(1 - \mathrm{H}_M)$ *for the tensor algebra $T(M) = \bigoplus_{k \geq 0} M^{\otimes k}$.*
(3) $\mathrm{H}_{\mathrm{Sym}(M)} = \exp(\mathrm{H}_M)$ *for the symmetric algebra.*

Proof. (1) A tensor of contents $S_1, \dots, S_k$ has content $S_1 + \cdots + S_k$, which is a set S if and only if the S_i are disjoint sets with union S. Hence $(M_1 \otimes \cdots \otimes M_k)_S = \bigoplus_{S = S_1 \sqcup \cdots \sqcup S_k} \bigotimes_i (M_i)_{S_i}$. Counting ordered decompositions of $[n]$ with prescribed block sizes gives multinomial coefficients, which is exactly the product of exponential generating functions. (2) follows from (1) by summing over k. (3) The multilinear part of $\mathrm{Sym}^k(M) = (M^{\otimes k})_{\mathfrak{S}_k}$ of content S is the space of coinvariants of $\bigoplus_{(S_1, \dots, S_k)} \bigotimes_i M_{S_i}$, the sum running over ordered decompositions into nonempty blocks. The group $\mathfrak{S}_k$ permutes these summands freely. The coinvariants are therefore isomorphic to the sum over unordered set partitions, and their series is $\mathrm{H}_M^k/k!$. □

2.3. **Free Lie algebras and enveloping algebras.** We use the following standard facts. For a set X, $\mathcal{L}(X)$ denotes the free Lie algebra on X.

(F1) Every map $X \to \mathcal{L}(X)$ extends uniquely to a derivation of $\mathcal{L}(X)$. More generally, let $\varphi\colon \mathcal{L}(X) \to \mathfrak{m}$ be a homomorphism. A *φ-derivation* is a linear map $\delta\colon \mathcal{L}(X) \to \mathfrak{m}$ with $\delta[u,v] = [\delta u, \varphi v] + [\varphi u, \delta v]$. Such a δ is determined by its values on X.

(F2) Let $\mathfrak{m}$ be a Lie algebra and $\Sigma \subset \mathfrak{m}$ a subset. Suppose $\mathfrak{m}$ is generated by a set X. Then the ideal generated by Σ is spanned by the elements $[x_1, [x_2, \dots, [x_k, \sigma]\cdots]]$ with $k \geq 0$, $x_p \in X$ and $\sigma \in \Sigma$. Indeed, the span Z of these elements is stable under $\operatorname{ad} x$ for $x \in X$. The set of u with $[u, Z] \subseteq Z$ is a subalgebra, since $\operatorname{ad}[u,v] = [\operatorname{ad} u, \operatorname{ad} v]$, and it contains X, hence equals $\mathfrak{m}$.

(F3) If $\mathfrak{m} = \mathcal{L}(X)/(R)$, then $U(\mathfrak{m}) = \Bbbk\langle X\rangle/(R)$, where R is regarded as a set of noncommutative polynomials via $[u,v] = uv - vu$. In particular $U(\mathcal{L}(X)) = \Bbbk\langle X\rangle$. Both sides represent the functor $A \mapsto \{\text{maps } X \to A \text{ killing } R\}$ on associative algebras.

(F4) (PBW theorem.) For every Lie algebra $\mathfrak{m}$, the ordered monomials in a totally ordered basis of $\mathfrak{m}$ form a basis of $U(\mathfrak{m})$; see [2, §3]. If $\mathfrak{m}$ is graded as above and the basis is chosen homogeneous, it follows that $U(\mathfrak{m}) \cong \operatorname{Sym}(\mathfrak{m})$ as graded vector spaces. Hence, by Lemma 2.1, $\mathrm{H}_{U(\mathfrak{m})} = \exp \mathrm{H}_{\mathfrak{m}}$ whenever $\mathfrak{m}_\emptyset = 0$. Applied to $\mathcal{L}(X)$ with graded X, (F3) and Lemma 2.1(2) give $\mathrm{H}_{\mathcal{L}(X)} = -\log(1 - \mathrm{H}_{\Bbbk X})$.

All identities between power series are identities in $\mathbb{Q}[y,z][[x]]$. Logarithms are taken of series with constant term 1, and $\log(PQ) = \log P + \log Q$ holds for such series.

3. Reduction to multilinear components

For a vector space V we define Lie algebras functorially in V. Let $U = \Bbbk d \oplus \Bbbk e$, and put $D(v) = v \otimes d$ and $E(v) = v \otimes e$. Define

$$\mathfrak{g}(V) = \mathcal{L}(V \otimes U)\big/\big([D(v), D(w)],\ [E(v), E(w)],\ [D(v), E(w)] - [D(w), E(v)] : v, w \in V\big),$$
$$\mathfrak{l}(V) = \mathcal{L}(V \oplus \operatorname{Sym}^2 V)\big/\big([E(v), E(w)],\ [E(u), F(vw)] - [E(w), F(uv)] : u, v, w \in V\big).$$

Here, in the second line, $E(v)$ is $v \in V$ and $F(vw)$ is the product $vw \in \operatorname{Sym}^2 V$. Let $\theta_V\colon \mathfrak{l}(V) \to \mathfrak{g}(V)$ be given by $E(v) \mapsto E(v)$ and $F(vw) \mapsto [D(v), E(w)]$. This is well defined on $\operatorname{Sym}^2 V$ because $[D(v), E(w)] = [D(w), E(v)]$ in $\mathfrak{g}(V)$. It respects the relations: the first is clear, and the second is mapped to

$$[E(u), [D(v), E(w)]] - [E(w), [D(v), E(u)]] = [[E(u), D(v)], E(w)] - [E(w), [D(v), E(u)]] = 0,$$

using Jacobi, $[E(u), E(w)] = 0$, and $[E(u), D(v)] = -[D(v), E(u)]$.

Each linear map $\phi\colon V \to V'$ induces homomorphisms $\mathfrak{l}(\phi)$ and $\mathfrak{g}(\phi)$, since the relations are defined for all vectors, and $\theta_{V'} \circ \mathfrak{l}(\phi) = \mathfrak{g}(\phi) \circ \theta_V$.

For $V = \Bbbk^{(\mathbb{N})}$ with basis (v_i), put $D_i = D(v_i)$, $E_i = E(v_i)$ and $F_{ij} = F(v_i v_j)$. Then $\mathfrak{g}(V) = \mathfrak{g}$ and $\mathfrak{l}(V) = \mathfrak{l}$, since the relations for all vectors follow by multilinearity from those for basis vectors. These are exactly (1) and (2), and $\theta_V = \theta$. The content grading is the grading by the weights of the diagonal torus: if $\tau = \operatorname{diag}(t_i)$, then $\mathfrak{l}(\tau)$ and $\mathfrak{g}(\tau)$ act on the component of content μ by $t^\mu = \prod t_i^{\mu(i)}$. Since θ is graded, $\ker\theta$ is a graded subspace.

Proposition 3.1 (polarisation)**.** *If $\ker\theta$ has a nonzero component of content μ with $|\mu| = n$, then $\ker\theta$ has a nonzero component of content $[n]$.*

Proof. Let $0 \neq x \in (\ker\theta)_\mu$. Let $i_1, \dots, i_r$ be the distinct elements of the support of μ, with multiplicities $m_1, \dots, m_r$, so $\sum m_j = n$. Partition $[n] = P_1 \sqcup \cdots \sqcup P_r$ with $|P_j| = m_j$.

For $t = (t_1, \dots, t_n) \in \Bbbk^n$ define linear maps $\beta_t, \alpha \colon V \to V$ by

$$\beta_t(v_{i_j}) = \sum_{p \in P_j} t_p v_p, \quad \beta_t(v_i) = 0 \ (i \notin \{i_1, \dots, i_r\}); \qquad \alpha(v_p) = v_{i_j} \ (p \in P_j), \quad \alpha(v_q) = 0 \ (q > n).$$

Then $\alpha\beta_t(v_{i_j}) = c_j v_{i_j}$ with $c_j = \sum_{p \in P_j} t_p$, and $\alpha\beta_t(v_i) = 0$ for the other i.

The element x is a linear combination of brackets of generators whose indices lie in $\{i_1, \dots, i_r\}$, with total content μ. Hence $\mathfrak{l}(\alpha\beta_t)(x) = \prod_j c_j^{m_j} x$.

On the other hand, $\beta_t = \tau_t \beta_1$, where τ_t is diagonal with entries t_p on v_p for $p \le n$ and 1 elsewhere. Write $\mathfrak{l}(\beta_1)(x) = \sum_\nu x_\nu$ as a finite sum of homogeneous components; each ν is supported in $[n]$ and has size n. Then $\mathfrak{l}(\beta_t)(x) = \sum_\nu t^\nu x_\nu$, so

$$\sum_\nu t^\nu \, \mathfrak{l}(\alpha)(x_\nu) = \prod_{j=1}^{r} \Big(\sum_{p \in P_j} t_p \Big)^{m_j} x \qquad \text{for all } t \in \Bbbk^n.$$

Both sides are polynomial maps $\Bbbk^n \to \mathfrak{l}_\mu$, and $\Bbbk$ is infinite, so the coefficients of $t_1 t_2 \cdots t_n$ agree. This gives $\mathfrak{l}(\alpha)(x_{[n]}) = m_1! \cdots m_r!\, x \neq 0$, using that the characteristic is 0. In particular $x_{[n]} \neq 0$.

Finally, $\theta(\mathfrak{l}(\beta_1)x) = \mathfrak{g}(\beta_1)\theta(x) = 0$ and θ is graded, so $x_{[n]} \in (\ker\theta)_{[n]}$. □

Hence Theorem 1.1 follows once we show $(\ker\theta)_S = 0$ for every finite set $S \subset \mathbb{N}$. By the symmetry under permutations of $\mathbb{N}$, it suffices to treat $S = [n]$.

4. A presentation of the ideal generated by the E_i

Let $\mathfrak{a}_D$ be the abelian Lie algebra with basis $(D_i)_{i \in \mathbb{N}}$. Let $F = \mathcal{L}(G_T : T \text{ a nonempty multiset})$ be the free Lie algebra on symbols G_T; G_T has content T and bidegree $(|T| - 1, 1)$. By (F1), for each $k \in \mathbb{N}$ there is a unique derivation ∂_k of F with $\partial_k G_T = G_{T+k}$. The derivations ∂_k pairwise commute, since $[\partial_k, \partial_l]$ is a derivation vanishing on the generators. For a multiset S put $\partial_S = \prod_{s \in S} \partial_s$, and define

$$r_{i,j,S} = \partial_S [G_i, G_j] \in F \qquad (i, j \in \mathbb{N},\ S \text{ a multiset}).$$

Explicitly, list $S = \{s_1, \dots, s_m\}$ with repetitions and write $S_P = \{s_p : p \in P\}$ for $P \subseteq [m]$. Then

$$r_{i,j,S} = \sum_{P \subseteq [m]} [G_{S_P + i},\, G_{S_{[m] \setminus P} + j}]. \tag{3}$$

Let $I \subset F$ be the ideal generated by all $r_{i,j,S}$, and put $\mathfrak{k} = F/I$. Since $\partial_k r_{i,j,S} = r_{i,j,S+k}$, and $\partial_k[u, \rho] = [\partial_k u, \rho] + [u, \partial_k \rho]$, each ∂_k preserves I and descends to a derivation of $\mathfrak{k}$. The descended derivations still commute. Therefore $D_k \mapsto \partial_k$ defines an action of $\mathfrak{a}_D$ on $\mathfrak{k}$ by derivations, and we form the semidirect product $\mathfrak{k} \rtimes \mathfrak{a}_D$, in which $[D_k, u] = \partial_k u$.

Proposition 4.1. (1) *In $\mathfrak{g}$, the element $\operatorname{ad}(D_{t_1}) \cdots \operatorname{ad}(D_{t_m}) E_t$ depends only on the multiset $T = \{t_1, \dots, t_m, t\}$. Denote it by $\mathbf{G}_T$.*

(2) *There is an isomorphism $\psi \colon \mathfrak{k} \rtimes \mathfrak{a}_D \to \mathfrak{g}$ with $\psi(G_T) = \mathbf{G}_T$ and $\psi(D_i) = D_i$. Its inverse φ satisfies $\varphi(D_i) = D_i$ and $\varphi(E_i) = G_i$.*

(3) *Under ψ, the ideal of $\mathfrak{g}$ generated by the E_i corresponds to $\mathfrak{k}$, and $\mathfrak{g} = \psi(\mathfrak{k}) \oplus \mathfrak{a}_D$ as graded vector spaces.*

(4) *Let $F^{\mathrm{sm}} = \mathcal{L}(G_T : |T| \le 2) \subset F$, and let $I_{\mathfrak{l}} \subset F^{\mathrm{sm}}$ be the ideal generated by the $r_{i,j,S}$ with $|S| \le 1$. Then $\mathfrak{l} \cong F^{\mathrm{sm}}/I_{\mathfrak{l}}$ via $E_i \mapsto G_i$ and $F_{ij} \mapsto G_{\{i,j\}}$. Under this identification, θ equals $\psi \circ \iota$, where $\iota \colon F^{\mathrm{sm}}/I_{\mathfrak{l}} \to F/I = \mathfrak{k}$ is induced by the inclusion. In particular $\ker\theta = \ker\iota = (F^{\mathrm{sm}} \cap I)/I_{\mathfrak{l}}$.*

Proof. (1) Since $[D_i, D_j] = 0$, the operators $\operatorname{ad} D_i$ commute, so the order of the D's is irrelevant. It remains to exchange the distinguished index t with some t'. Write $T = R + t + t'$. Then

$$\operatorname{ad}(D_R)\operatorname{ad}(D_{t'})E_t = \operatorname{ad}(D_R)[D_{t'}, E_t] = \operatorname{ad}(D_R)[D_t, E_{t'}] = \operatorname{ad}(D_R)\operatorname{ad}(D_t)E_{t'}.$$

(2) Let $\psi_0 \colon F \to \mathfrak{g}$ be the homomorphism with $G_T \mapsto \mathbf{G}_T$. The maps $\psi_0 \circ \partial_k$ and $\operatorname{ad}(D_k) \circ \psi_0$ are both ψ_0-derivations. They agree on generators, because $\psi_0(G_{T+k}) = \mathbf{G}_{T+k} = [D_k, \mathbf{G}_T]$. By (F1) they are equal. Hence $\psi_0(r_{i,j,S}) = \operatorname{ad}(D_S)[E_i, E_j] = 0$, so ψ_0 factors through $\mathfrak{k}$, and the induced map intertwines ∂_k with $\operatorname{ad} D_k$. Together with $D_i \mapsto D_i$ it therefore defines a homomorphism $\psi \colon \mathfrak{k} \rtimes \mathfrak{a}_D \to \mathfrak{g}$.

Conversely, define φ on generators by $D_i \mapsto D_i$ and $E_i \mapsto G_i$. The relations (1) are respected:

- $[D_i, D_j] \mapsto 0$, since $\mathfrak{a}_D$ is abelian;
- $[E_i, E_j] \mapsto [G_i, G_j] = r_{i,j,\emptyset} = 0$;
- $[D_i, E_j] \mapsto \partial_i G_j = G_{\{i,j\}}$, which is symmetric in i, j.

Now $\psi\varphi$ is the identity on the generators D_i, E_i of $\mathfrak{g}$. Also $\varphi\psi(G_T) = \varphi(\operatorname{ad}(D_{T-t})E_t) = \partial_{T-t}G_t = G_T$, and $\varphi\psi(D_i) = D_i$. Hence ψ and φ are mutually inverse.

(3) The subspace $\mathfrak{k}$ is an ideal of $\mathfrak{k} \rtimes \mathfrak{a}_D$, it contains $G_i = \varphi(E_i)$, and it is generated as an ideal by these elements, since $G_T = \partial_{T-t}G_t$. The quotient by it is $\mathfrak{a}_D$.

(4) Identify $E_i = G_i$ and $F_{ij} = G_{\{i,j\}}$, so the generators F_{ij} are indexed by 2-element multisets and $F_{ij} = F_{ji}$ is built in. Then $r_{i,j,\emptyset} = [E_i, E_j]$, and by (3)

$$r_{i,j,\{s\}} = [G_{\{s,i\}}, G_j] + [G_i, G_{\{s,j\}}] = [E_i, F_{sj}] - [E_j, F_{is}].$$

This is the relator $[E_i, F_{jk}] - [E_k, F_{ij}]$ of (2) with (i, j, k) replaced by (i, s, j). Thus the defining relators of $\mathfrak{l}$ and the generators of $I_{\mathfrak{l}}$ span the same space, and $\mathfrak{l} \cong F^{\mathrm{sm}}/I_{\mathfrak{l}}$. Finally, $\psi\iota(E_i) = E_i$ and $\psi\iota(F_{ij}) = \mathbf{G}_{\{i,j\}} = [D_i, E_j]$, which agrees with θ. Since ψ is injective, $\ker\theta = \ker\iota$. □

In particular $\mathfrak{h} = \theta(\mathfrak{l})$ is the subalgebra of $\mathfrak{k} \cong \psi(\mathfrak{k})$ generated by the G_T with $|T| \le 2$.

5. The excess filtration

Call G_T *small* if $|T| \le 2$ and *big* if $|T| \ge 3$. Give F the additional grading by *excess*, defined on generators by

$$\varepsilon(G_T) = \max(|T| - 2, 0)$$

and extended additively to Lie monomials; this is a grading since F is free. The weight-0 part F_0 is spanned by Lie monomials in small letters, so $F_0 = F^{\mathrm{sm}}$ (the subalgebra of Proposition 4.1(4)). All gradings (content, bidegree and excess) are compatible, and every content component F_μ is finite-dimensional. For $0 \ne u \in F$ let $\operatorname{lead}(u)$ denote its nonzero excess-homogeneous component of highest weight.

Lemma 5.1. *For a multiset S with $|S| = m$, define*

$$\lambda_{i,j,S} = \begin{cases} [G_i, G_j] & m = 0, \\ [G_{S+i}, G_j] + [G_i, G_{S+j}] & m \ge 1. \end{cases}$$

Then $\lambda_{i,j,S}$ is excess-homogeneous of weight $\max(m-1, 0)$. If $i = j$, then $r_{i,j,S} = \lambda_{i,j,S} = 0$. If $i \ne j$, then $\lambda_{i,j,S} \ne 0$ and $\operatorname{lead}(r_{i,j,S}) = \lambda_{i,j,S}$. If $m \le 1$, then $r_{i,j,S} = \lambda_{i,j,S}$.

Proof. If $i = j$, then $r_{i,i,S} = \partial_S[G_i, G_i] = 0$ and $\lambda_{i,i,S} = 0$. Let $i \ne j$. The term of (3) indexed by P with $|P| = a$ has excess $\max(a-1, 0) + \max(m-a-1, 0)$. For $m \le 1$ every term has excess 0, and (3) gives $r_{i,j,S} = \lambda_{i,j,S}$.

For $m \ge 2$ the excess equals $m - 1$ for $a \in \{0, m\}$ and $m - 2$ for $1 \le a \le m - 1$. The two terms of top weight are those with $P = \emptyset$ and $P = [m]$, each occurring exactly once, and their sum is $\lambda_{i,j,S}$.

It remains to see that $\lambda_{i,j,S} \neq 0$. For $m = 0$, $[G_i, G_j] \neq 0$ in the free Lie algebra since $i \neq j$. For $m \geq 1$, the brackets $[G_{S+i}, G_j]$ and $[G_i, G_{S+j}]$ are brackets of two distinct letters, and their sum vanishes only if $\{G_{S+i}, G_j\} = \{G_i, G_{S+j}\}$. This is impossible: $G_{S+i} \neq G_i$ since $m \geq 1$, and $G_{S+i} = G_{S+j}$ would force $i = j$. □

Let $J \subset F$ be the ideal generated by all $\lambda_{i,j,S}$, and put

$$\mathfrak{k}^{\mathrm{top}} = F/J.$$

Let $F_{\leq p} = \bigoplus_{q \leq p} F_q$, and define the associated graded of I by $\operatorname{gr} I = \bigoplus_p \operatorname{gr}_p I$, where $\operatorname{gr}_p I = \pi_p(I \cap F_{\leq p}) \subseteq F_p$ and π_p is the projection onto F_p.

Lemma 5.2. (1) *$\operatorname{gr} I$ is an ideal of F, and $J \subseteq \operatorname{gr} I$.*

(2) *For every content μ, $\dim(\operatorname{gr} I)_\mu = \dim I_\mu$. Consequently $\dim \mathfrak{k}_\mu \leq \dim \mathfrak{k}^{\mathrm{top}}_\mu$, with equality if and only if $J_\mu = (\operatorname{gr} I)_\mu$.*

Proof. (1) Let $u \in I \cap F_{\leq p}$, and let y be a Lie monomial of excess q. Then $[y, u] \in I \cap F_{\leq p+q}$ and $\pi_{p+q}[y, u] = [y, \pi_p u]$. Hence $[y, \operatorname{gr}_p I] \subseteq \operatorname{gr}_{p+q} I$, so $\operatorname{gr} I$ is an ideal.

For each (i, j, S) with $i \neq j$, put $p = \max(|S| - 1, 0)$. Lemma 5.1 gives $r_{i,j,S} \in I \cap F_{\leq p}$ with $\pi_p(r_{i,j,S}) = \lambda_{i,j,S}$. So every generator of J lies in $\operatorname{gr} I$, and therefore $J \subseteq \operatorname{gr} I$.

(2) On the finite-dimensional space I_μ, the filtration $I_\mu \cap F_{\leq p}$ is finite. Its successive quotients are isomorphic to $(\operatorname{gr}_p I)_\mu$. Hence $\dim(\operatorname{gr} I)_\mu = \dim I_\mu$, and $\dim \mathfrak{k}_\mu = \dim F_\mu - \dim(\operatorname{gr} I)_\mu \leq \dim F_\mu - \dim J_\mu = \dim \mathfrak{k}^{\mathrm{top}}_\mu$. □

Lemma 5.3. *The weight-0 part of J is $J_0 = J \cap F_0 = I_{\mathfrak{l}}$. Consequently $\mathfrak{l} \cong F_0/J_0$ is the excess-0 part of $\mathfrak{k}^{\mathrm{top}}$.*

Proof. By (F2), applied with X the set of all G_T, the ideal J is spanned by the excess-homogeneous elements $[y_1, [y_2, \ldots, [y_k, \lambda] \cdots]]$, with y_p letters and $\lambda = \lambda_{i,j,S}$. Such an element has excess $\sum \varepsilon(y_p) + \max(|S| - 1, 0)$. This is 0 if and only if all y_p are small and $|S| \leq 1$, and in that case $\lambda_{i,j,S} = r_{i,j,S}$. Since J is excess-graded, J_0 is the span of these weight-0 elements. By (F2) applied inside F_0, this span is exactly $I_{\mathfrak{l}}$. The last statement follows from Proposition 4.1(4). □

Lemma 5.4 (Lemma A)**.** *Let μ be a content. If $\dim \mathfrak{k}_\mu = \dim \mathfrak{k}^{\mathrm{top}}_\mu$, then $(\ker \theta)_\mu = 0$.*

Proof. By Lemma 5.2(2), $J_\mu = (\operatorname{gr} I)_\mu$. Let $u \in (F_0 \cap I)_\mu$. Then $u \in I \cap F_{\leq 0}$, so $u = \pi_0(u) \in (\operatorname{gr}_0 I)_\mu = (J_0)_\mu = (I_{\mathfrak{l}})_\mu$, by Lemma 5.3. Hence $(F_0 \cap I)_\mu = (I_{\mathfrak{l}})_\mu$, and $(\ker \theta)_\mu = 0$ by Proposition 4.1(4). □

6. The structure of $\mathfrak{k}^{\mathrm{top}}$

In $\mathfrak{k}^{\mathrm{top}}$ write $E_i = G_i$. By Lemma 5.1, the defining relations of $\mathfrak{k}^{\mathrm{top}}$ are $[E_i, E_j] = 0$ and, for $|S| \geq 1$, $[G_{S+i}, G_j] + [G_i, G_{S+j}] = 0$. Setting $T = S + i$ (so $|T| \geq 2$ and $t = i \in T$), the latter reads

$$[E_j, G_T] = [E_t, G_{T-t+j}] \qquad (|T| \geq 2,\ t \in T,\ j \in \mathbb{N}). \tag{4}$$

In words, $\operatorname{ad}(E_j) G_T$ is unchanged when j is exchanged with any element of T. Since $[E_i, E_j] = 0$, the operators $\operatorname{ad} E_i$ pairwise commute in $\mathfrak{k}^{\mathrm{top}}$.

Let Y be the set of symbols $Y_{W,s}$, where W is a multiset and $2 \leq s \leq |W|$. The symbol $Y_{W,s}$ has content W, bidegree $(s - 1, |W| - s + 1)$ and excess $s - 2$. Let $\mathfrak{a}_E$ be abelian with basis (E_i); it has excess 0. By (F1) the rule $E_j \cdot Y_{W,s} = Y_{W+j,s}$ defines commuting derivations of $\mathcal{L}(Y)$. Form $\mathcal{L}(Y) \rtimes \mathfrak{a}_E$.

Lemma 6.1 (Lemma B). (1) *For $W = C + T$ with $|T| = s \geq 2$, write $C = \{j_1, \ldots, j_r\}$. Then the element $\mathrm{ad}(E_{j_1}) \cdots \mathrm{ad}(E_{j_r}) G_T \in \mathfrak{k}^{\mathrm{top}}$ depends only on (W, s).*

(2) *There is an isomorphism $\beta \colon \mathcal{L}(Y) \rtimes \mathfrak{a}_E \to \mathfrak{k}^{\mathrm{top}}$ with $\beta(E_i) = E_i$ and $\beta(Y_{W,s}) = \mathrm{ad}(E_C) G_T$ for any splitting as in (1). Its inverse α is given by $\alpha(E_i) = E_i$ and $\alpha(G_T) = Y_{T,|T|}$ for $|T| \geq 2$. Both maps preserve content, bidegree and excess.*

Proof. (1) The order of the $\mathrm{ad}\, E_j$ is irrelevant, since they commute. Suppose (C, T) and (C', T') are two splittings of W with $|T| = |T'| = s$ and $T \neq T'$. Choose $t \in T - T'$ and $j \in T' - T$ (multiset differences; both are nonempty since $|T| = |T'|$). The element j has larger multiplicity in $T' \subseteq W$ than in T, hence occurs in $C = W - T$. Exchanging j and t gives a new splitting $(C - j + t,\, T - t + j)$, and $|T - T'|$ decreases by one. Each such exchange preserves the element: using commutativity, we may place $\mathrm{ad}\, E_j$ innermost, and then apply (4), valid since $|T| \geq 2$:

$$\mathrm{ad}(E_{C-j})[E_j, G_T] = \mathrm{ad}(E_{C-j})[E_t, G_{T-t+j}] = \mathrm{ad}(E_{C-j+t}) G_{T-t+j}.$$

(2) First, α is well defined. Define it on the generators of F as stated. Then $[E_i, E_j] \mapsto 0$. For (4), both sides map to $E_j \cdot Y_{T,|T|} = Y_{T+j,|T|} = E_t \cdot Y_{T-t+j,|T|}$. Hence α kills J.

Next, β is well defined. Let $\beta_0 \colon \mathcal{L}(Y) \to \mathfrak{k}^{\mathrm{top}}$ be defined freely by (1). Then $\beta_0 \circ (E_j \cdot)$ and $\mathrm{ad}(E_j) \circ \beta_0$ are β_0-derivations, which agree on generators because

$$\beta_0(Y_{W+j,s}) = \mathrm{ad}(E_{C+j}) G_T = [E_j, \beta_0(Y_{W,s})].$$

By (F1) they are equal. Since the E_i commute in $\mathfrak{k}^{\mathrm{top}}$, β_0 extends to a homomorphism β of the semidirect product.

Both composites are the identity on generators:

- $\alpha\beta(Y_{W,s}) = \alpha(\mathrm{ad}(E_C) G_T) = E_C \cdot Y_{T,s} = Y_{W,s}$;
- $\beta\alpha(G_T) = \beta(Y_{T,|T|}) = G_T$;
- α and β fix each E_i.

The gradings are visibly preserved. For instance, $\mathrm{ad}(E_C) G_T$ has excess $|T| - 2 = s - 2$. □

Corollary 6.2. $\mathrm{H}_{\mathfrak{k}^{\mathrm{top}}}(x) = xz - \log\big(1 - \mathcal{Y}(x)\big)$, *where*

$$\mathcal{Y}(x) = \sum_{w \geq 2} \frac{x^w}{w!} \sum_{s=2}^{w} y^{s-1} z^{w-s+1}.$$

Moreover the excess-0 part satisfies $\mathrm{H}_{\mathfrak{l}}(x) = xz - \log\big(1 - \mathcal{Y}_2(x)\big)$ *with* $\mathcal{Y}_2(x) = \sum_{w \geq 2} \frac{x^w}{w!} y z^{w-1}$.

Proof. As a graded vector space, $\mathfrak{k}^{\mathrm{top}} \cong \mathcal{L}(Y) \oplus \mathfrak{a}_E$. Multilinear generators $Y_{W,s}$ have W a set. For $|W| = w$ they contribute $\sum_{s=2}^{w} y^{s-1} z^{w-s+1}$, so $\mathrm{H}_{\mathbb{k}Y} = \mathcal{Y}$. Also $\mathrm{H}_{\mathfrak{a}_E} = xz$. Now apply (F4).

The excess-0 part of $\mathcal{L}(Y) \rtimes \mathfrak{a}_E$ is $\mathcal{L}(Y_{\cdot,2}) \rtimes \mathfrak{a}_E$, because the generators of excess 0 are exactly the $Y_{W,2}$ and a Lie monomial has excess 0 if and only if all its letters do. The subalgebra of a free Lie algebra generated by a subset of the free generators is free on that subset; it is a retract, by killing the other generators. Combining this with Lemma 5.3 gives the second formula. □

7. A PBW basis of $U(\mathfrak{g})$

By (F3), $U(\mathfrak{g}) = \mathbb{k}\langle X \rangle / (\mathcal{R})$ with $X = \{D_i, E_i : i \in \mathbb{N}\}$. The ideal $(\mathcal{R})$ is generated by the following elements, for $i < j$:

$$D_i D_j - D_j D_i, \qquad E_i E_j - E_j E_i, \qquad D_i E_j - E_j D_i - D_j E_i + E_i D_j.$$

Order X by index first and then $D < E$:

$$D_1 < E_1 < D_2 < E_2 < \cdots.$$

Order words by length, and words of equal length lexicographically (deglex). This is a total order on the free monoid $\langle X\rangle$ satisfying $B < B' \Rightarrow ABC < AB'C$. It is a well-order, since for each length the lexicographic order on words over a well-ordered alphabet is a well-order. In particular it satisfies the descending chain condition.

We use the reduction system $\mathcal{S}$ consisting of the rules, for all $i < j$:

$$\begin{aligned}(\sigma_1)\quad & D_jD_i \;\to\; D_iD_j,\\ (\sigma_2)\quad & E_jE_i \;\to\; E_iE_j,\\ (\sigma_3)\quad & E_jD_i \;\to\; D_iE_j - D_jE_i + E_iD_j.\end{aligned}$$

The differences of the two sides are exactly the generators of $(\mathcal{R})$, up to sign. In each rule, every word on the right is smaller than the word on the left. This holds because the first letter on the left is the largest of all first letters involved: $D_j > D_i$, $E_j > E_i$, and $E_j > D_i, D_j, E_i$. So the order is compatible with $\mathcal{S}$ in the sense of [2, §1].

Call a word of length 2 *bad* if it is the left side of a rule. Thus X_jY_i is bad if and only if $j > i$ and $(X, Y) \neq (D, E)$. A word is *normal* (irreducible) if it contains no bad subword.

Proposition 7.1. *All ambiguities of $\mathcal{S}$ are resolvable.*

Proof. All left sides have length 2, and distinct rules have distinct left sides, so there are no inclusion ambiguities. An overlap ambiguity is a word $X_kY_jZ_i$ such that X_kY_j and Y_jZ_i are both bad. This forces $k > j > i$, $(X, Y) \neq (D, E)$ and $(Y, Z) \neq (D, E)$. Hence $(X, Y, Z) \in \{DDD, EDD, EED, EEE\}$.

The reductions below only compare indices, so it suffices to treat $(k, j, i) = (3, 2, 1)$; the general case follows by renaming $3, 2, 1$ to k, j, i. In each case we reduce the left pair, respectively the right pair, first, and then reduce to normal form. Words that are already normal are not reduced further; they are recognised by the criterion above.

DDD and EEE. Only σ_1 (resp. σ_2) applies. Both routes reach the sorted word:

$$\begin{aligned}D_3D_2D_1 \to D_2D_3D_1 \to D_2D_1D_3 \to D_1D_2D_3,\\ D_3D_2D_1 \to D_3D_1D_2 \to D_1D_3D_2 \to D_1D_2D_3,\end{aligned}$$

and likewise with E in place of D.

EDD. Left route, starting with σ_3 on E_3D_2:

$$E_3D_2D_1 \to D_2E_3D_1 - D_3E_2D_1 + E_2D_3D_1.$$

We reduce each of the three terms:

$$\begin{aligned}D_2E_3D_1 &\to D_2D_1E_3 - D_2D_3E_1 + D_2E_1D_3 \to D_1D_2E_3 - D_2D_3E_1 + D_2E_1D_3,\\ -D_3E_2D_1 &\to -D_3D_1E_2 + D_3D_2E_1 - D_3E_1D_2 \to -D_1D_3E_2 + D_2D_3E_1 - D_3E_1D_2,\\ E_2D_3D_1 &\to E_2D_1D_3 \to D_1E_2D_3 - D_2E_1D_3 + E_1D_2D_3.\end{aligned}$$

The sum is

$$N_{EDD} = D_1D_2E_3 - D_1D_3E_2 + D_1E_2D_3 - D_3E_1D_2 + E_1D_2D_3.$$

Right route, starting with σ_1 on D_2D_1:

$$\begin{aligned}E_3D_2D_1 &\to E_3D_1D_2 \to D_1E_3D_2 - D_3E_1D_2 + E_1D_3D_2\\ &\to D_1D_2E_3 - D_1D_3E_2 + D_1E_2D_3 - D_3E_1D_2 + E_1D_2D_3 = N_{EDD}.\end{aligned}$$

EED. Left route, starting with σ_2 on E_3E_2:

$$\begin{aligned}E_3E_2D_1 &\to E_2E_3D_1 \to E_2D_1E_3 - E_2D_3E_1 + E_2E_1D_3\\ &\to D_1E_2E_3 - D_2E_1E_3 + E_1D_2E_3 - E_2D_3E_1 + E_1E_2D_3 =: N_{EED}.\end{aligned}$$

Right route, starting with σ_3 on E_2D_1:

$$E_3E_2D_1 \to E_3D_1E_2 - E_3D_2E_1 + E_3E_1D_2.$$

We reduce each term:

$$\begin{aligned} E_3D_1E_2 &\to D_1E_3E_2 - D_3E_1E_2 + E_1D_3E_2 \to D_1E_2E_3 - D_3E_1E_2 + E_1D_3E_2, \\ -E_3D_2E_1 &\to -D_2E_3E_1 + D_3E_2E_1 - E_2D_3E_1 \to -D_2E_1E_3 + D_3E_1E_2 - E_2D_3E_1, \\ E_3E_1D_2 &\to E_1E_3D_2 \to E_1D_2E_3 - E_1D_3E_2 + E_1E_2D_3. \end{aligned}$$

The sum is again N_{EED}.

In all four cases the two routes have the same normal form, so all ambiguities are resolvable. □

Theorem 7.2. *The normal words form a basis of $U(\mathfrak{g})$. In particular, for every content and bidegree, the dimension of the corresponding component of $U(\mathfrak{g})$ equals the number of normal words of that content and bidegree.*

Proof. We apply Bergman's diamond lemma [2, Theorem 1.2]. Its hypotheses hold: $\Bbbk$ is a commutative ring, $\mathcal{S}$ is a reduction system, and $\leq$ is a semigroup partial order on $\langle X\rangle$, compatible with $\mathcal{S}$ and satisfying the descending chain condition. The theorem says that resolvability of all ambiguities (Proposition 7.1) is equivalent to the irreducible monomials forming a basis of $\Bbbk\langle X\rangle/I$, where I is the ideal generated by the differences $W_\sigma - f_\sigma$. Here $I = (\mathcal{R})$. All rules are homogeneous in content and bidegree, which gives the second statement. □

A word of content a set S is a sequence $X^{(1)}_{\sigma_1}X^{(2)}_{\sigma_2}\cdots X^{(n)}_{\sigma_n}$, where $(\sigma_1,\dots,\sigma_n)$ is an ordering of S and $X^{(p)} \in \{D, E\}$. Such a word is normal if and only if at every descent $\sigma_p > \sigma_{p+1}$ the labels are $(X^{(p)}, X^{(p+1)}) = (D, E)$.

8. Counting normal words

For a word w let $\omega(w) = y^{\#D}z^{\#E}$. For $n \geq 0$, let u_n be the sum of $\omega(w)$ over the normal words of content $[n]$; normality depends only on the relative order of indices. By Theorem 7.2,

$$\mathrm{H}_{U(\mathfrak{g})}(x) = \sum_{n\geq 0} u_n \frac{x^n}{n!}.$$

Call a word c of content a set a *chain* if each of its adjacent pairs is bad. The empty word and words of length 1 are chains.

Lemma 8.1. *Let S be a set with $|S| = k$. The chains of content S are exactly the words whose indices are in decreasing order and whose labels have the form $E\cdots ED\cdots D$. Their total weight is*

$$h_k = \sum_{a=0}^{k} y^a z^{k-a}.$$

Proof. A pair X_jY_i is bad if and only if $j > i$ and $(X, Y) \neq (D, E)$. So in a chain the indices decrease, and no D is immediately followed by an E. Conversely, every such word is a chain. There is one such word for each number $a \in \{0,\dots,k\}$ of D's. □

Proposition 8.2. *For $n \geq 1$,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}h_k\,u_{n-k} = 0.$$

Proof. Let Ω_n be the set of pairs (c, w) where c is a chain, w is a normal word, and the contents of c and w partition $[n]$. Give (c, w) sign $(-1)^{|c|}$ and weight $\omega(c)\omega(w)$. Grouping by the content of c and using Lemma 8.1, the signed weighted count of Ω_n is the left-hand side. It therefore suffices to construct a fixed-point-free involution Φ of Ω_n that preserves weight and reverses sign.

For $(c, w) \in \Omega_n$:

(I) If w is nonempty and either c is empty or the pair $(\mathrm{last}(c), \mathrm{first}(w))$ is bad, let $\Phi(c, w)$ be obtained by moving the first letter of w to the end of c.

(II) Otherwise, let $\Phi(c, w)$ be obtained by moving the last letter of c to the front of w.

In case (II), c is nonempty. Otherwise w would be nonempty, since $n \geq 1$, and case (I) would apply.

Φ *maps* Ω_n *to itself.* In case (I), the new chain is a chain, because the only new adjacent pair is bad. The new word is a suffix of a normal word, hence normal. In case (II), the new chain is a prefix of a chain. The only new adjacent pair in the new word is $(\mathrm{last}(c), \mathrm{first}(w))$, which is not bad by the case distinction (or w is empty), so the new word is normal.

Φ *is an involution.* Suppose (I) moved $f = \mathrm{first}(w)$, giving (cf, w'). If w' is nonempty, then $(f, \mathrm{first}(w'))$ is an adjacent pair of the normal word w, hence not bad. So (I) does not apply to (cf, w'), and (II) moves f back.

Suppose (II) moved $\ell = \mathrm{last}(c)$, giving $(c', \ell w)$ with $c = c'\ell$. Then ℓw is nonempty, and either c' is empty or $(\mathrm{last}(c'), \ell)$ is an adjacent pair of the chain c, hence bad. So (I) applies and moves ℓ back.

Finally, Φ changes $|c|$ by ± 1 and preserves the multiset of letters. Hence it reverses the sign, preserves the weight, and has no fixed points. □

Corollary 8.3. *Let* $H(x) = \sum_{k\geq 0} h_k \frac{x^k}{k!}$. *Then* $H(-x)\,\mathrm{H}_{U(\mathfrak{g})}(x) = 1$, *and*

$$(y - z)H(x) = y\,e^{xy} - z\,e^{xz}.$$

Proof. The coefficient of $x^n/n!$ in $H(-x)\mathrm{H}_{U(\mathfrak{g})}(x)$ is $\sum_k (-1)^k \binom{n}{k} h_k u_{n-k}$. This is 1 for $n = 0$, since $h_0 = u_0 = 1$, and 0 for $n \geq 1$ by Proposition 8.2. The second identity follows from $(y - z)h_k = y^{k+1} - z^{k+1}$. □

9. The Hilbert series of $\mathfrak{g}$ and the proof of Theorem 1.1

Proposition 9.1. $\mathrm{H}_{\mathfrak{g}}(x) = x(y+z) - \log\big(1 - \mathcal{Y}(x)\big)$ *and* $\mathrm{H}_{\mathfrak{k}}(x) = xz - \log\big(1 - \mathcal{Y}(x)\big) = \mathrm{H}_{\mathfrak{k}^{\mathrm{top}}}(x)$.

Proof. By (F4) and Corollary 8.3, $\mathrm{H}_{\mathfrak{g}} = \log \mathrm{H}_{U(\mathfrak{g})} = -\log H(-x)$. We claim that

$$H(-x) = e^{-x(y+z)}\big(1 - \mathcal{Y}(x)\big). \tag{5}$$

First, the coefficient of $x^w/w!$ in $H(x) - e^{xy} - e^{xz} + 1$ is

- $1 - 1 - 1 + 1 = 0$ for $w = 0$;
- $(y + z) - y - z = 0$ for $w = 1$;
- $h_w - y^w - z^w = \sum_{a=1}^{w-1} y^a z^{w-a} = \sum_{s=2}^{w} y^{s-1} z^{w-s+1}$ for $w \geq 2$.

Hence $\mathcal{Y} = H(x) - e^{xy} - e^{xz} + 1$. Using Corollary 8.3,

$$(y - z)(1 - \mathcal{Y}) = (y - z)(e^{xy} + e^{xz}) - (y\,e^{xy} - z\,e^{xz}) = y\,e^{xz} - z\,e^{xy}.$$

On the other hand,

$$(y - z)\,e^{x(y+z)}H(-x) = e^{x(y+z)}\big(y\,e^{-xy} - z\,e^{-xz}\big) = y\,e^{xz} - z\,e^{xy}.$$

Since $y - z$ is not a zero divisor in $\mathbb{Q}[y, z][[x]]$, (5) follows. Therefore

$$\mathrm{H}_{\mathfrak{g}} = -\log H(-x) = x(y + z) - \log(1 - \mathcal{Y}).$$

By Proposition 4.1(3), $\mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{a}_D$ as graded spaces, and $\mathrm{H}_{\mathfrak{a}_D} = xy$. Hence $\mathrm{H}_{\mathfrak{k}} = \mathrm{H}_{\mathfrak{g}} - xy$. Comparing with Corollary 6.2 gives the second statement. □

Proof of Theorem 1.1. By Proposition 9.1, $\dim \mathfrak{k}_{[n]}^{(a,b)} = \dim(\mathfrak{k}^{\mathrm{top}})_{[n]}^{(a,b)}$ for all n, a, b. Summing over (a, b) gives $\dim \mathfrak{k}_{[n]} = \dim \mathfrak{k}_{[n]}^{\mathrm{top}}$. By Lemma 5.4, $(\ker\theta)_{[n]} = 0$ for all n. By Proposition 3.1, $\ker\theta = 0$. □

10. Corollaries

Corollary 10.1. *Let $W = C + \{a, b\}$ with $C = \{j_1, \dots, j_r\}$. The element*

$$Y_W = \mathrm{ad}(E_{j_1}) \cdots \mathrm{ad}(E_{j_r})[D_a, E_b] \in \mathfrak{h}$$

depends only on W. The Y_W, with $|W| \geq 2$, freely generate a free Lie subalgebra $\mathcal{L} \subset \mathfrak{h}$. Moreover $\mathfrak{h} = \mathcal{L} \rtimes \mathfrak{a}_E$, where $\mathfrak{a}_E = \mathrm{span}(E_i)$ is abelian and acts by $[E_j, Y_W] = Y_{W+j}$. The multilinear Hilbert series of $\mathfrak{h}$ is $xz - \log(1 - \mathcal{Y}_2(x))$.

Proof. By Theorem 1.1, $\theta\colon \mathfrak{l} \to \mathfrak{h}$ is an isomorphism. By Lemma 5.3, $\mathfrak{l}$ is the excess-0 part of $\mathfrak{k}^{\mathrm{top}}$. By Lemma 6.1 and the proof of Corollary 6.2, that part is $\mathcal{L}(Y_{\cdot,2}) \rtimes \mathfrak{a}_E$, with $Y_{W,2} \mapsto \mathrm{ad}(E_C)G_{\{a,b\}}$. Finally, θ maps $\mathrm{ad}(E_C)F_{ab}$ to Y_W. □

Corollary 10.2. *The elements $[D_i, E_j]$, $i \leq j$, freely generate a free Lie subalgebra of $\mathfrak{g}$.*

Proof. These elements are $Y_{\{i,j\}}$, a subset of the free generators in Corollary 10.1. A subset of free generators of a free Lie algebra freely generates a free Lie subalgebra, since this subalgebra is a retract. □

For $m \geq 0$ let $A_m(t) = \sum_{\sigma \in \mathfrak{S}_m} t^{\mathrm{des}\,\sigma} = \sum_d A(m,d)t^d$, where $\mathrm{des}\,\sigma = \#\{p : \sigma_p > \sigma_{p+1}\}$ and $A_0 = 1$.

Lemma 10.3. *Let $\mathcal{A}(x,t) = \sum_{m\geq 0} A_m(t)\frac{x^m}{m!} \in \mathbb{Q}[t][[x]]$. Then $\left(t - e^{(t-1)x}\right)\mathcal{A}(x,t) = t - 1$.*

Proof. We first show that, for every $m \geq 0$,

$$\sum_{q\geq 0}(q+1)^m t^q = \frac{A_m(t)}{(1-t)^{m+1}} \qquad \text{in } \mathbb{Q}[[t]]. \tag{6}$$

The number $(q+1)^m$ counts maps $f\colon [m] \to \{0, \dots, q\}$. To such an f associate the permutation σ listing $[m]$ by increasing value of f, ties broken by increasing index. Then $f(\sigma_1) \leq \cdots \leq f(\sigma_m)$, with strict inequality at every descent of σ. Conversely, a permutation σ and a weakly increasing sequence $f(\sigma_1) \leq \cdots \leq f(\sigma_m)$ that is strict at the descents of σ determine f, and f gives back σ.

For fixed σ with d descents, subtracting from $f(\sigma_p)$ the number of descents at positions $< p$ gives a bijection onto the weakly increasing sequences with values in $\{0, \dots, q-d\}$. There are $\binom{q-d+m}{m}$ of these, and

$$\sum_q \binom{q-d+m}{m} t^q = \frac{t^d}{(1-t)^{m+1}}.$$

Summing over σ proves (6).

Now work in $\mathbb{Q}[[t,x]]$. By (6),

$$\mathcal{A}(x,t) = \sum_m \frac{x^m}{m!}(1-t)^{m+1}\sum_q (q+1)^m t^q = (1-t)\sum_q t^q e^{(q+1)(1-t)x} = \frac{(1-t)e^{(1-t)x}}{1 - te^{(1-t)x}}.$$

Here the inner sums are rearranged legitimately in $\mathbb{Q}[[t,x]]$, and the denominator has constant term 1. Multiplying by $1-te^{(1-t)x}$ and then by $-e^{(t-1)x}$ gives $(t-e^{(t-1)x})\mathcal{A}=t-1$ in $\mathbb{Q}[[t,x]]$, hence in $\mathbb{Q}[t][[x]]$. □

Corollary 10.4. *For $n\geq 2$ and $0\leq a\leq n$, $\dim \mathfrak{g}_{[n]}^{(a,n-a)}=A(n-1,a-1)$, with the convention $A(n-1,-1)=A(n-1,n-1)=0$. In particular $\dim \mathfrak{g}_{[n]}=(n-1)!$. The same holds for $\mathfrak{k}$ in place of $\mathfrak{g}$.*

Proof. Every letter carries exactly one index, so in content $[n]$ the bidegree is $(a,n-a)$. We may therefore set $z=1$ without losing information. Let $Q=(1-\mathcal{Y})|_{z=1}\in\mathbb{Q}[y][[x]]$; it has constant term 1. By the identity $(y-z)(1-\mathcal{Y})=y\,e^{xz}-z\,e^{xy}$ from the proof of Proposition 9.1,

$$(y-1)Q=y\,e^{x}-e^{xy},\qquad (y-1)Q'=y\,e^{x}-y\,e^{xy},$$

where $'=d/dx$. Put $\mathcal{A}=\mathcal{A}(x,y)$. Lemma 10.3 gives $(y-e^{(y-1)x})(\mathcal{A}-1)=e^{(y-1)x}-1$. Therefore

$$(y-1)\,y(\mathcal{A}-1)Q=y\,e^{x}(\mathcal{A}-1)\big(y-e^{(y-1)x}\big)=y\,e^{xy}-y\,e^{x}=-(y-1)Q'.$$

Since $y-1$ is not a zero divisor, $-Q'/Q=y(\mathcal{A}-1)$, that is,

$$-\frac{d}{dx}\log Q=\sum_{m\geq 1}yA_m(y)\frac{x^m}{m!}.$$

Integrating, using $\log Q(0)=0$, and recalling $\mathrm{H}_{\mathfrak{g}}=x(y+z)-\log(1-\mathcal{Y})$, the coefficient of $x^n/n!$ in $\mathrm{H}_{\mathfrak{g}}|_{z=1}$ is $yA_{n-1}(y)$ for $n\geq 2$. This is the first claim, and $A_{n-1}(1)=(n-1)!$ gives the second. For $\mathfrak{k}$, note that $\mathrm{H}_{\mathfrak{k}}$ and $\mathrm{H}_{\mathfrak{g}}$ differ only in degree $n=1$. □

Remark 10.5 (Koszulness). This remark is not used elsewhere in the paper. Fix $N\in\mathbb{N}$, and let $\mathfrak{g}_N$ be the Lie algebra with the presentation (1), but with indices restricted to $[N]$. Killing the generators with index $>N$ shows that $\mathfrak{g}_N$ is the subalgebra of $\mathfrak{g}$ generated by the D_i,E_i with $i\leq N$, and that it contains all components of $\mathfrak{g}$ whose content is supported in $[N]$.

With $V=\Bbbk^N$ and $U=\Bbbk d\oplus\Bbbk e$ as in Section 3, we have $U(\mathfrak{g}_N)=T(V\otimes U)/(R)$, where $R\subset(V\otimes U)^{\otimes 2}\cong V^{\otimes 2}\otimes U^{\otimes 2}$ is the space $\Lambda^2V\otimes\mathrm{Sym}^2U$ of tensors antisymmetric in V and symmetric in U. Indeed, $[D_i,D_j]$, $[E_i,E_j]$ and $[D_i,E_j]-[D_j,E_i]$ correspond to $(v_iv_j-v_jv_i)\otimes dd$, $(v_iv_j-v_jv_i)\otimes ee$ and $(v_iv_j-v_jv_i)\otimes(de+ed)$, and both spaces have dimension $3\binom{N}{2}$.

Since $\mathrm{char}\,\Bbbk\neq 2$, $V^{\otimes 2}=\mathrm{Sym}^2V\oplus\Lambda^2V$ (and likewise for U), hence $R^{\perp}=\mathrm{Sym}^2V^*\otimes U^{*\otimes 2}+V^{*\otimes 2}\otimes\Lambda^2U^*$. This subspace is exactly the degree-2 kernel of the natural surjection onto the Segre product $\Lambda(V^*)\circ\mathrm{Sym}(U^*)$ [1, 1.2]: in degree 2 that map is the tensor product of the quotient maps onto $\Lambda^2(V^*)$ and $\mathrm{Sym}^2(U^*)$. The Segre product is generated in degree 1 and is 2-related by [1, Prop. 3(b)]. Here the Segre product uses ordinary tensor-product multiplication, not a super tensor product. All these algebras have finitely many degree-1 generators and finitely generated quadratic relation ideals, as required by the conventions of [1, §1]. Therefore the dual algebra $U(\mathfrak{g}_N)^{\circ}$ in the sense of [1, 1.14] is isomorphic to $\Lambda(V^*)\circ\mathrm{Sym}(U^*)$.

The polynomial algebra $\mathrm{Sym}(V)$ is Koszul [1, 1.17(b)], hence so is its dual $\Lambda(V^*)$ [1, 1.16(10)]; likewise $\mathrm{Sym}(U^*)$ is Koszul. By [1, Theorem 4(b)] the Segre product is Koszul, and by [1, 1.16(10)] so is $U(\mathfrak{g}_N)$.

Give v_i^* content i and d^*,e^* the bidegrees of d,e. For $0\leq n\leq N$, the content-$[n]$ part of $\Lambda^n(V^*)$ is one-dimensional, so the content-$[n]$ part of $\Lambda(V^*)\circ\mathrm{Sym}(U^*)$ is $\mathrm{Sym}^n(U^*)$, of weight h_n. Taking $N\geq n$ in each degree, the identity $H(-x)\mathrm{H}_{U(\mathfrak{g})}(x)=1$ of Corollary 8.3 has the shape of the numerical relation between a Koszul algebra and its dual [1, 1.16(5)]; we have proved it directly. Finally, Dotsenko, Shadrin and Tamaroff prove that tHyperCom is Koszul [3, Prop. 2.4.3]. In view of Section 11, that statement concerns the multilinear species associated with the enveloping algebra of the ideal $\mathfrak{k}$ in its generators G_T (with weight given by the number

of letters), not directly the ordinary algebra $U(\mathfrak{g})$ in the generators D_i, E_i. We do not infer Koszulness of an ordinary algebra from this species statement; the finite-generator argument above establishes the assertion about $U(\mathfrak{g}_N)$.

11. Relation to prior work

In this section we explain how the dimension results of Sections 7–10 follow from the work of Losev–Manin [4] and Dotsenko–Shadrin–Tamaroff [3]. We also distinguish results not explicitly stated there. Numbered statements of [4] and [3] refer to arXiv version 2 of [4] and arXiv version 3 of [3]. We use homology with characteristic-0 coefficients; the algebraic conventions of [3, §2.1] are in characteristic 0.

Losev–Manin spaces. For $n \geq 1$, Losev and Manin introduced a smooth proper variety L_n of dimension $n-1$ [4, Thm. 2.2], which they identify with the toric variety associated with the permutohedron [4, §2]. In [3, Def. 2.4.1] it is denoted $\mathrm{LM}(n)$ and described as the toric variety of the fan of Weyl chambers of type A. By [4, Thm. 2.3], L_n has no odd cohomology, and

$$1+\sum_{n\geq 1} p_n(q)\frac{y^n}{n!} = \frac{q-1}{q-e^{(q-1)y}}, \qquad p_n(q)=\sum_i \dim H^{2i}(L_n)\, q^i.$$

Both this series and $\mathcal{A}(y,q)$ of Lemma 10.3 satisfy $(q-e^{(q-1)y})F = q-1$, and $\mathbb{Q}[q][[y]]$ is an integral domain, so $p_n(q) = A_n(q)$: the Betti numbers of L_n are Eulerian numbers. The strata of L_n are indexed by ordered set partitions of $[n]$, and inclusions of boundary strata make the homology a twisted associative algebra, i.e. an associative monoid in species [3, §2.4]. Dotsenko, Shadrin and Tamaroff showed [3, Prop. 2.4.2] that this algebra is the twisted associative algebra tHyperCom. It is generated by fully symmetric elements m_t of arity $t \geq 1$ and homological degree $2t-2$, subject to

$$\sum_{\substack{A=I\sqcup J\\ i\in I,\ j\in J}} \big(m_I m_J - m_J m_I\big) = 0 \qquad (i\neq j\in A).$$

Their proof combines the relations established in [4], the fact that the total Betti number of L_n is the number $n!$ of chambers, and a Gröbner basis argument giving $\dim \mathrm{tHyperCom}(n) \leq n!$. They credit the monomial ordering in that argument to an unpublished internship report of P. Laubie, which we have not seen. All generators have even degree and trivial symmetric-group action, so no Koszul signs occur.

The dictionary. We keep the singleton generators: $G_{\{i\}} = E_i$ corresponds to m_1, the fundamental class of L_1, which is a point. There is no generator $G_\emptyset$ or m_0. In arity 0 we adjoin only the algebra unit, so $U(\mathfrak{k})_\emptyset = \Bbbk$; the constant 1 in the generating series is this formal unit and makes no assertion about a geometric space L_0. In every ordered set partition below the blocks are nonempty.

By (F3), $U(\mathfrak{k}) = \Bbbk\langle G_T\rangle/(r_{i,j,S})$. Fix $n \geq 1$. The component of $U(\mathfrak{k})$ of content $[n]$ is the span of the words $G_{T_1}\cdots G_{T_p}$, where $(T_1,\dots,T_p)$ runs over the ordered set partitions of $[n]$. One quotients by the span of the elements $u\, r_{i,j,S}\, v$ in which u, v are words and all contents are disjoint sets with union $[n]$. This is, by definition, arity n of the twisted associative algebra generated by the symbols G_T (with trivial symmetric-group action) subject to the relators $r_{i,j,S}$ for sets $S\sqcup\{i,j\}$. For a set S, (3) reads

$$r_{i,j,S} = \sum_{\substack{A=I\sqcup J\\ i\in I,\ j\in J}} \big(G_I G_J - G_J G_I\big), \qquad A = S\sqcup\{i,j\},$$

in $U(\mathfrak{k})$, which is the relator of tHyperCom under $G_T \leftrightarrow m_T$. A word with p letters has D-degree $n-p$, while $m_{T_1}\cdots m_{T_p}$ has homological degree $\sum_k (2|T_k|-2) = 2(n-p)$. Therefore, by [3, Prop. 2.4.2] and [4, Thm. 2.3],

$$\dim U(\mathfrak{k})_{[n]}^{(a,n-a)} = \dim H_{2a}(L_n;\Bbbk) = \dim H^{2a}(L_n) = A(n,a), \qquad \dim U(\mathfrak{k})_{[n]} = n!. \tag{7}$$

(Homology and cohomology with field coefficients have the same dimensions.) For $n=2$, the relator is $[G_i, G_j]=0$, matching $m_{\{i\}}m_{\{j\}} = m_{\{j\}}m_{\{i\}}$. The identification is one of species, compatible with products on disjoint index sets; we are not identifying the ordinary algebra $U(\mathfrak{k})$, including its repeated-index components, with a direct sum of unlabelled homology spaces. It involves only a doubling of the grading and no signs. We checked it directly for $n \le 6$ by computing $U(\mathfrak{k})_{[n]}$ from our relators over $\mathbb{Q}$ (script `prior_work_check.py`).

Consequences. By Proposition 4.1(3) and (F4), $\mathrm{H}_{U(\mathfrak{g})} = \exp(\mathrm{H}_{\mathfrak{k}} + xy) = e^{xy}\,\mathrm{H}_{U(\mathfrak{k})}$. In content $[n]$ the bidegree is $(a, n-a)$, so we may set $z=1$, and (7) gives

$$\mathrm{H}_{U(\mathfrak{g})}\big|_{z=1} = e^{xy}\,\frac{y-1}{y-e^{(y-1)x}} = \frac{y-1}{y\,e^{-xy}-e^{-x}}.$$

This is Corollary 8.3 at $z=1$; we confirmed the identity with `sympy`. Taking logarithms gives Proposition 9.1 and Corollary 10.4. Thus the dimension count used in the proof of Theorem 1.1, namely $\dim \mathfrak{k}_{[n]}^{(a,b)} = \dim(\mathfrak{k}^{\mathrm{top}})_{[n]}^{(a,b)}$, could also be obtained by combining [4, 3] with Proposition 4.1 and Corollary 6.2. Corollary 10.4 and the counts of Sections 7–9 should therefore be regarded as known, in an equivalent form.

Our derivation is independent. We use a quadratic Gröbner basis of $U(\mathfrak{g})$ in the letters D_i, E_i (Section 7), whereas [3] use the generators G_T of $U(\mathfrak{k})$. We then count normal words by a sign-reversing involution (Section 8), whereas [4] count points of L_n over finite fields.

Results not explicitly stated in [4, 3]**.** The works [4, 3] concern the homology algebras of the Losev–Manin spaces and their representation-theoretic applications. They do not explicitly formulate $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ with the presentations used here, or ask for the relations of $\mathfrak{h}$. Lemma 5.4, Lemma 6.1, Theorem 1.1 and Corollaries 10.1 and 10.2 are not explicitly stated in [4, 3]; we were unable to find these statements elsewhere in the literature. This is not a claim that they cannot be deduced from earlier work.

A heuristic explanation of the link. Since $\Bbbk$ is infinite, a representation ρ of $\mathfrak{g}$ is the same as a family of operators $X_i = \rho(D_i)$, $Y_i = \rho(E_i)$ with $[X_i + \lambda Y_i, X_j + \lambda Y_j] = 0$ for all $\lambda \in \Bbbk$: expand in λ. For instance, let $A = \sum_i A_i\,dt_i$ be a matrix-valued 1-form in formal variables $t_1, \dots, t_N$. Put $X_i = \partial/\partial t_i$ and let Y_i be multiplication by A_i, acting on vector-valued formal power series. Then these relations say $\partial_i A_j = \partial_j A_i$ and $[A_i, A_j] = 0$, i.e. that $d + \lambda A$ is a pencil of flat connections in the sense of [4, §3.1–3.2]. In this representation $\rho(\mathbf{G}_T)$ is multiplication by $\partial_{T-t}A_t$. If $A = dB$ [4, Prop. 3.2], this is $\partial_T B$. For B as in [4, (3.19)], the value of $\partial_T B$ at $t=0$ is the "top correlator" indexed by T. We include this only as motivation; nothing above depends on it.

12. Computational verification

The proof above is complete without computer assistance: Proposition 7.1 is checked by hand. We nevertheless verified its ingredients and its conclusion independently by computer. All scripts are in Python 3 and are contained in the accompanying archive `code.zip`, together with their outputs.

Throughout this section, k denotes the E-degree of a multilinear component of content $[n]$. For elements of $\mathfrak{k}$ or $\mathfrak{l}$ in the G_T-presentation, it equals the number of letters G_T. Modular computations use the prime $p = 2^{31} - 1$.

Check	Script	Range	Result
$\dim \mathfrak{k}$, $\dim \mathfrak{l}$, $\dim \mathfrak{h}$, $\dim \ker \theta$ in each component $([n], k)$	`kmodel_exact.py` (exact over $\mathbb{Q}$)	$n \leq 8$	$\ker \theta = 0$; $\dim \mathfrak{k}$ Eulerian
same, modular	`kmodel.py`	$n \leq 8$	identical
$\mathfrak{g}$ directly in the letters D_i, E_i (independent model)	`gmodel.py` (mod p)	$n \leq 6$	$\dim \mathfrak{g}$ Eulerian; $\dim \mathfrak{h}$ agrees
$\dim \mathfrak{k}^{\mathrm{top}}$	`ktop.py` (mod p)	$n \leq 8$	matches Cor. 6.2
degree-3 Gröbner check: $\dim(X\mathcal{R} + \mathcal{R}X)_3 = \#$ non-normal words	`pbw_check.py` (exact)	indices 1..3	$104 = 104$
explicit ambiguity resolution (Prop. 7.1)	`ambiguities.py` (exact)	all 4	resolvable
normal-word counts, then log	`pbw_count.py` (exact)	$n \leq 10$	equals Prop. 9.1
identity (5) and Eulerian numbers	`egf_check.py` (sympy)	$n \leq 10$	identical
injectivity via projection (below)	`kproj.py` (mod p)	$n = 9$	$\ker \theta = 0$
dictionary of Section 11: relators, $\dim U(\mathfrak{k})_{[n]}$ by degree, EGF	`prior_work_check.py` (exact, sympy)	$n \leq 6$	agrees with [4, 3]

TABLE 1. Computer checks. The formulas for $\dim \mathfrak{k}$ and $\dim \mathfrak{l}$ are tabulated by `lformula.py`.

Degree-3 Gröbner check. For a quadratic algebra $\Bbbk\langle X\rangle/(\mathcal{R})$, the normal words of degree 3 are linearly independent modulo $(\mathcal{R})_3 = X\mathcal{R} + \mathcal{R}X$ if and only if $\dim(X\mathcal{R} + \mathcal{R}X) = \#\{\text{words of length 3 containing a bad subword}\}$. Indeed, the normal words always span modulo $(\mathcal{R})_3$, by reduction.

This independence implies that every overlap ambiguity resolves. Indeed, the two reductions of an overlap differ by an element of $(\mathcal{R})_3$, and their normal forms are then equal. Since an overlap involves at most three distinct indices and the rules depend only on the order of the indices, indices $\{1, 2, 3\}$ suffice.

The script `pbw_check.py` computes this in exact rational arithmetic and finds $104 = 104$. As a sanity check, it reports the same equality for all eight natural letter orders. The companion script `sanity_random.py` applies the same test to three random sets of nine quadratic relations in the same letters. The test fails in each case (for example $108 \neq 92$), which shows that it is not vacuous. Note that $\dim(X\mathcal{R} + \mathcal{R}X) \geq \#\{\text{non-normal words}\}$ always holds, since the normal words span.

Dimensions. The script `pbw_count.py` counts multilinear normal words by dynamic programming. For $n = 1, \ldots, 10$ the totals are $2, 5, 16, 65, 326, 1957, 13700, 109601, 986410, 9864101$. It then checks that the logarithm of their exponential generating function equals $x(y+z) - \log(1-\mathcal{Y})$ through order 10. The resulting $\dim \mathfrak{g}_{[n]}^{(a,n-a)}$ are Eulerian numbers. For instance, for $n = 10$ they are $1, 502, 14608, 88234, 156190, 88234, 14608, 502, 1$.

Injectivity in degree 9. Let $F_{>0} = \bigoplus_{q>0} F_q$ and let $\pi\colon F \to F_{>0}$ be the projection along F_0. Since $\mathfrak{h} \cong (F_0 + I)/I$ and $F = F_0 \oplus F_{>0}$,

$$\dim(\mathfrak{k}/\mathfrak{h})_{[n],k} = \dim(F_{>0})_{[n],k} - \operatorname{rank} \pi(I_{[n],k}).$$

The script `kproj.py` computes $\pi(I)$ recursively modulo p. Free Lie elements are encoded by their coefficients on associative words starting with the letter containing the minimal index; these coordinates are valid over every field. For $n = 9$ it finds, for $k = 1, \dots, 9$:

$$\dim(\mathfrak{k}/\mathfrak{h})_{[9],k} = 1,\ 247,\ 4293,\ 15619,\ 8059,\ 457,\ 1,\ 0,\ 0.$$

These are exactly $\dim \mathfrak{k}_{[9],k} - \dim \mathfrak{l}_{[9],k}$, computed from Corollary 6.2 and Proposition 9.1:

- $\dim \mathfrak{k}_{[9],k} = 1, 247, 4293, 15619, 15619, 4293, 247, 1, 0$;
- $\dim \mathfrak{l}_{[9],k} = 0, 0, 0, 0, 7560, 3836, 246, 1, 0$.

The run took 554 s with a peak resident memory of 403 MB.

This modular computation is rigorous over $\mathbb{Q}$. The space $I_{[n],k}$ is spanned by integral vectors: the ad-words of letters applied to the integral relators $r_{i,j,S}$. The modular computation determines the span of their reductions modulo p, that is, $\pi(I)$ over $\mathbb{F}_p$. The rank over $\mathbb{Q}$ of a set of integral vectors is at least its rank modulo p. Hence

$$\dim(\mathfrak{k}/\mathfrak{h})_{\mathbb{Q}} \le \dim(\mathfrak{k}/\mathfrak{h})_{\mathbb{F}_p} = \dim \mathfrak{k} - \dim \mathfrak{l},$$

and so $\dim \mathfrak{h}_{\mathbb{Q}} \ge \dim \mathfrak{l}$. Since $\mathfrak{h}$ is a quotient of $\mathfrak{l}$, equality holds, and θ is injective in content [9]. This argument uses only the dimension formulas, not Lemma 5.4.

A direct computation for $n = 10$ was not attempted, because of its estimated time and memory requirements.

Reproducibility. The archive contains all scripts, their outputs (`out_*.txt`), and a README describing the command lines. The exact computations use Python's `fractions` module, and the series manipulations use `sympy`.

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Tools used: Codex (OpenAI) and Claude (Anthropic), used as research and writing assistants. Specific version numbers were not recorded. AI is not listed as an author.

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