hexagon:2611.00541v2

WILLS' (1981) CONJECTURE HOLDS IF AND ONLY IF THE DIMENSION IS AT MOST 4, AND OTHER MISCELLANEA

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First version: November 3, 2026. This version (3.33): November 24, 2026.

This is the expanded version of the paper D. Goel, Wills' (1981) conjecture holds if and only if the dimension is at most 4 [8].

ABSTRACT. Let P be a lattice polytope in $\mathbb{R}^n$ that is symmetric about the origin and has the origin as its only interior lattice point. In 1981 Wills conjectured that each coefficient $g_i(P)$ of the Ehrhart polynomial of P is at most the corresponding coefficient of the cube $[-1,1]^n$, that is, $g_i(P) \le 2^i \binom{n}{i}$. Wills proved this for $n \le 3$, and Henze showed in 2014 that it fails for every $n \ge 7$; the dimensions 4, 5 and 6 remained open. We settle them. In dimension 4 the conjecture holds; the new bounds are $g_1(P) \le 8$ and $g_2(P) \le 24$, and, up to unimodular equivalence, $g_1(P) = 8$ only for the cube, one explicit 24-cell and one explicit cross-polytope. In dimension 5 we give a polytope with 50 vertices in $[-1,1]^5$ whose linear Ehrhart coefficient is $101/10 > 10$; products with segments then give counterexamples in every dimension $n \ge 6$. Hence Wills' conjecture holds in $\mathbb{R}^n$ if and only if $n \le 4$. The proof in dimension 4 combines Ehrhart reciprocity, the lower bound theorem for triangulated 3-spheres and an exhaustive computer enumeration of about three million sets of lattice points, listed up to symmetry. This expanded version also gives a second proof of the case in which the integer functionals bounded by 1 on P do not span $\mathbb{R}^4$, determines the second largest value of g_2 in dimension 4, discusses a width-two approach, describes the computer search for the counterexample, and describes all computations used in the proofs.*,†

1. INTRODUCTION

A *lattice polytope* in $\mathbb{R}^n$ is the convex hull of finitely many points of $\mathbb{Z}^n$. Ehrhart [7] proved that for a lattice polytope P of dimension n the number of lattice points in the dilate kP is a polynomial in the integer $k \ge 0$,

$$L_P(k) = \#(kP \cap \mathbb{Z}^n) = g_0(P) + g_1(P)\,k + \cdots + g_n(P)\,k^n,$$

the *Ehrhart polynomial* of P. Here $g_0(P) = 1$, the leading coefficient $g_n(P)$ is the volume of P, and $g_{n-1}(P)$ is half the lattice surface area of P (Section 2); the remaining coefficients have no such simple description and may be negative [3].

Minkowski's first theorem [16] states that a convex body $K \subset \mathbb{R}^n$ that is symmetric about the origin and has no lattice point other than 0 in its interior has volume at most 2^n, the volume of the cube $C_n = [-1,1]^n$. Wills [22] looked for a discrete analog of this theorem, one that bounds every Ehrhart coefficient of a lattice polytope. Let $\mathcal{O}^n$ be the set of lattice polytopes $P \subset \mathbb{R}^n$ with $P = -P$ whose only interior lattice point is the origin. The cube C_n belongs to $\mathcal{O}^n$, and

2020 *Mathematics Subject Classification.* Primary 52B20; Secondary 11H06, 52-08, 52C07.

Key words and phrases. Ehrhart polynomial, lattice polytope, centrally symmetric polytope, interior lattice point, Wills' conjecture.

*Additionally, Theorems 1.2, 1.3 and 7.1 and Corollaries 1.4 and 7.2 are derived in Lean 4 from six published results (Ehrhart's theorem, the properties (E1)–(E3) of Ehrhart polynomials, the boundary triangulation used in Lemma 4.3, and Minkowski's theorem on lattice tilings used in Lemma 5.1) and five computer checks (Computations 1, 3, 4 and 5, and the volume of W), each stated as an explicit hypothesis, and checked with `lake comparator`: https://github.com/david44hello/paper_wills-1981-conjecture/tree/v3.32

†This is an expanded version of the following focused version [8]: https://hexagonmath.org/2611.00540

$L_{C_n}(k) = (2k+1)^n$, so $g_i(C_n) = 2^i\binom{n}{i}$. Wills called these numbers the expected values of the coefficients [22, Remark (b)], and the following statement is known as Wills' conjecture. We state it as in [12, Conjecture 1.1]; see also [9].

Conjecture 1.1 (Wills' conjecture)**.** *For every $n \geq 1$, every $P \in \mathcal{O}^n$ and every $i \in \{0, 1, \ldots, n\}$,*

$$g_i(P) \leq g_i(C_n) = 2^i \binom{n}{i}.$$

For $i = 0$ there is equality, and for $i = n$ the inequality is Minkowski's theorem. Wills proved the case $i = n-1$ in every dimension [22, Theorem (i)]; hence the conjecture holds for $n \leq 2$. Wills also proved it for $n = 3$ [22, Theorem (i), (ii)], and, in dimension 4, the bounds $g_1(P) < 21$, $g_2(P) < 31$ and $g_2(P) \leq g_1(P) + 16$ [22, Theorem (iii)]. A discrete version of Minkowski's first theorem by Betke, Henk and Wills [4, Theorem 2.1] implies that $L_P(k) \leq L_{C_n}(k)$ for every $P \in \mathcal{O}^n$ and every integer $k \geq 0$, which is the inequality of the conjecture for the values of the two polynomials instead of their coefficients. For $i = n-1$, a bound in terms of the successive minima that implies Wills' bound is due to Henk, Schürmann and Wills [11]. Henze [12] constructed a family of polytopes P_n and showed that $P_7 \in \mathcal{O}^7$ has $g_1(P_7) = 1534/105 > 14$; taking products with segments, he obtained counterexamples in every dimension $n \geq 7$. On the positive side, Henze proved the conjecture for every $P \in \mathcal{O}^n$ whose Ehrhart roots all have real part $-1/2$ [12, Corollary 3.3]; by a theorem of Bey, Henk and Wills [5], every such polytope is reflexive. About the remaining dimensions, Henze wrote [12, Remark 2.2]: "In dimensions 4, 5 and 6 we tested Wills' conjecture on lattice polytopes that have a similar structure to P_n. [...] None of these examples turned out to be a counterexample." We settle these three dimensions.

TABLE 1. The best results before this paper and the results of this paper.

Best result before this paper	New result
$g_1(P) < 21$ [22, Theorem (iii)]	$g_1(P) \leq 8$ for every $P \in \mathcal{O}^4$, with equality exactly for C_4, a 24-cell T and a cross-polytope X, up to unimodular equivalence (Theorems 1.2 and 7.1)
$g_2(P) < 31$ and $g_2(P) \leq g_1(P) + 16$ [22, Theorem (iii)]	$g_2(P) \leq 24$ for every $P \in \mathcal{O}^4$, with equality exactly for C_4, up to unimodular equivalence (Theorem 1.2 and Corollary 7.2)
no counterexample known for $n \leq 6$; the tested families in dimensions 4, 5, 6 contain none [12, Remark 2.2]	the polytope W of Table 2 is in $\mathcal{O}^5$ and has $g_1(W) = 101/10 > 10$; counterexamples in every dimension $n \geq 5$ (Theorem 1.3)
true for $n \leq 3$ [22]; false for $n \geq 7$ [12]	Conjecture 1.1 holds in $\mathbb{R}^n$ if and only if $n \leq 4$ (Corollary 1.4)

Results. Our main results are Theorems 1.2 and 1.3 and Corollary 1.4 below, and Theorem 7.1 in Section 7, which determines the polytopes $P \in \mathcal{O}^4$ with $g_1(P) = 8$. Table 1 compares them with the best earlier results, and Figure 1 shows the status of the conjecture in each dimension. Two of the three polytopes with $g_1(P) = 8$ have Ehrhart roots off the line $\operatorname{Re} z = -\frac{1}{2}$, so Henze's theorem does not cover them (Remark 7.3).

Theorem 1.2. *Every $P \in \mathcal{O}^4$ satisfies $g_1(P) \leq 8$ and $g_2(P) \leq 24$.*

Theorem 1.3. *The polytope $W \subset \mathbb{R}^5$ of Table 2 belongs to $\mathcal{O}^5$, and*

$$L_W(k) = 1 + \tfrac{101}{10}\,k + \tfrac{127}{12}\,k^2 + \tfrac{139}{6}\,k^3 + \tfrac{557}{12}\,k^4 + \tfrac{386}{15}\,k^5.$$

In particular $g_1(W) = 101/10 > 10 = g_1(C_5)$. *For every* $n \geq 6$ *the polytope* $W \times C_{n-5}$ *belongs to* $\mathcal{O}^n$ *and has* $g_1 = 2n + 1/10 > 2n = g_1(C_n)$.

Corollary 1.4. *Wills' conjecture holds in* $\mathbb{R}^n$ *if and only if* $n \leq 4$.

Proof. For $n \leq 3$ this is Wills' theorem [22]. For $n = 4$, $g_0 = 1$ by (E1) of Section 2, $g_3 \leq 32$ by [22, Theorem (i)], $g_4 \leq 16$ by Minkowski's theorem, and g_1, g_2 are bounded by Theorem 1.2. For $n \geq 5$, Theorem 1.3 gives a counterexample. □

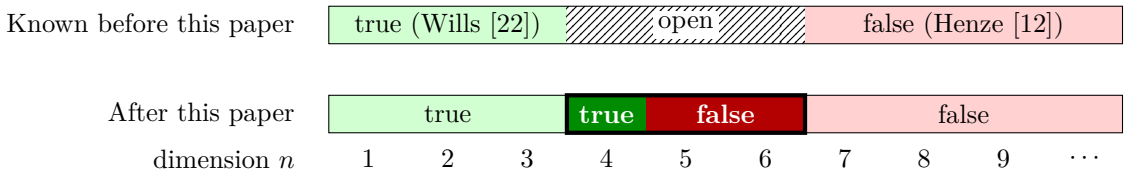


FIGURE 1. The status of Wills' conjecture in each dimension n. Before this paper the conjecture was known to hold for $n \leq 3$ (Wills [22]) and to fail for $n \geq 7$ (Henze [12]); the dimensions 4, 5 and 6 (hatched) were open. This paper proves it for $n = 4$ (Theorem 1.2 and Corollary 1.4) and disproves it for every $n \geq 5$ (Theorem 1.3); the dimensions settled here are drawn in full colour, inside the thick frame.

Idea of the proofs. Theorem 1.3 is a computation. The polytope W is given by its 50 vertices, and since W has only one interior lattice point, its linear coefficient is determined by four numbers: the numbers of lattice points in W, in $2W$ and in the interior of $2W$, and the volume of W (Proposition 3.1). Products with segments carry the example to every higher dimension.

For Theorem 1.2 we use Ehrhart reciprocity. Since $P \in \mathcal{O}^4$ has exactly one interior lattice point, $L_P(-1) = 1$; comparing this with $L_P(1) = \#(P \cap \mathbb{Z}^4)$ shows that $g_1(P) \leq 8$ is equivalent to the inequality

$$\#(P \cap \mathbb{Z}^4) \leq \sigma(P) + 17$$

between the number of lattice points of P and its lattice surface area $\sigma(P) = 2g_3(P)$, and that this inequality also gives $g_2(P) \leq 24$ (Proposition 4.2). A triangulation of the boundary of P that uses every boundary lattice point, together with Walkup's lower bound theorem for triangulated 3-spheres [21], proves the inequality when P has at most 29 lattice points (Lemma 4.3).

For P with at least 31 lattice points (their number is odd, since they come in pairs $\pm x$ together with 0) we look at the integer linear functionals that take values in $[-1, 1]$ on P. Either they contain a basis of $\mathbb{Z}^4$, and then P is unimodularly equivalent to a polytope in the cube C_4; or they span $\mathbb{R}^4$ without containing a basis, and then a linear map carries P, together with its lattice, onto a polytope in C_4 with vertices in a sublattice of index 2; or they span a proper subspace. In the last case we fix four lattice points of P whose determinant is as large as possible; in their coordinates the lattice points of P lie in $[-1, 1]^4$, in one of 39 lattices, up to signed permutations of the coordinates (Section 5). This reduction, by four lattice points of largest determinant, follows Averkov and Schymura [2], who used it to compute the largest number of columns of a Δ-modular integer matrix for small parameters; their enumeration builds on that of Averkov, Borger and Soprunov [1]. In each case only finitely many sets of lattice points remain, and they are enumerated by computer (Section 6).

Section 2 collects the facts we use from Ehrhart theory. Section 3 proves Theorem 1.3. Sections 4–6 prove Theorem 1.2, and Section 7 identifies the equality cases. Beyond the contents of the short paper [8], Section 8 gives a second proof of Case III, Section 9 studies the extremal polytopes and the next values of g_1 and g_2, Section 10 describes a width-two approach, Section 11 describes how the counterexample was found, and Section 12 describes the computations in detail.

2. Preliminaries

2.1. **Ehrhart polynomials.** Let $n \geq 1$, let $\Lambda \subset \mathbb{R}^n$ be a lattice (a discrete subgroup of rank n), and let P be a polytope of dimension n whose vertices lie in Λ. Then $L_{P,\Lambda}(k) = \#(kP \cap \Lambda) = \sum_i g_i(P)k^i$ is a polynomial in k of degree n [7]; its coefficients $g_i(P)$ depend on Λ, which will be clear from the context, and we write L_P when $\Lambda = \mathbb{Z}^n$. We call two pairs (P, Λ) and (P', Λ') *isomorphic* if a linear isomorphism ϕ of $\mathbb{R}^n$ satisfies $\phi(P) = P'$ and $\phi(\Lambda) = \Lambda'$; then $\phi(kP) \cap \Lambda' = \phi(kP \cap \Lambda)$ for every k, so $L_{P',\Lambda'} = L_{P,\Lambda}$. When $\Lambda = \Lambda' = \mathbb{Z}^n$, the map ϕ lies in $\mathrm{GL}_n(\mathbb{Z})$, and the two lattice polytopes are called *unimodularly equivalent*. We use four standard facts [3, Chapters 3–5].

(E1) $g_0(P) = 1$ and $g_n(P) = \mathrm{vol}(P)/\det\Lambda$.

(E2) *Reciprocity* (Ehrhart, Macdonald): $L_{P,\Lambda}(-k) = (-1)^n \#(\mathrm{int}(kP) \cap \Lambda)$ for every integer $k \geq 1$.

(E3) For a facet F of P let u_F be the primitive vector of the dual lattice Λ^* that is an outer normal of F, and let b_F be the integer with $F \subset \{x : \langle u_F, x\rangle = b_F\}$. Let $\mathrm{relvol}(F)$ be the $(n-1)$-dimensional volume of F, measured so that a fundamental domain of the lattice $\Lambda \cap \mathrm{aff}(F - F)$ has volume 1. Then $g_{n-1}(P) = \frac{1}{2}\sigma(P)$, where $\sigma(P) = \sum_F \mathrm{relvol}(F)$ is the *lattice surface area*. If $0 \in \mathrm{int}\, P$, then $\mathrm{vol}(P)/\det\Lambda = \frac{1}{n}\sum_F b_F\,\mathrm{relvol}(F)$, by cutting P into the pyramids with apex 0 over its facets.

(E4) $L_{P\times Q} = L_P\, L_Q$ for lattice polytopes $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$, with the lattice $\mathbb{Z}^{p+q}$, since $k(P \times Q) \cap \mathbb{Z}^{p+q} = (kP \cap \mathbb{Z}^p) \times (kQ \cap \mathbb{Z}^q)$.

In (E3), b_F is an integer because F contains points of Λ; when $0 \in \mathrm{int}\, P$ it is positive, and it is the lattice distance of F from the origin.

2.2. **The class $\mathcal{O}^n$.** Let $\mathcal{O}^n_\Lambda$ be the set of polytopes P with vertices in Λ, $P = -P$ and $\mathrm{int}\, P \cap \Lambda = \{0\}$; so $\mathcal{O}^n = \mathcal{O}^n_{\mathbb{Z}^n}$. A polytope in $\mathcal{O}^n_\Lambda$ has dimension n, because 0 is an interior point. We write $G(P) = \#(P \cap \Lambda)$. Since the lattice points of P other than 0 come in pairs $\pm x$, $G(P)$ is odd.

Lemma 2.1. *Let $Y \subset \{-1, 0, 1\}^n$ be a set with $Y = -Y$ whose convex hull P has dimension n. Then $P \in \mathcal{O}^n$.*

Proof. The polytope P is a lattice polytope with $P = -P$. It has an interior point x, so $-x$ is interior too, and so is $0 = \frac{1}{2}x + \frac{1}{2}(-x)$. Finally $\mathrm{int}\, P \subset \mathrm{int}\, C_n = (-1, 1)^n$, whose only lattice point is 0. □

The same proof applies to a sublattice $\Gamma \subseteq \mathbb{Z}^n$ in place of $\mathbb{Z}^n$: if $S \subseteq \Gamma \cap C_n$ is symmetric and spans $\mathbb{R}^n$, then $\mathrm{conv}\, S \in \mathcal{O}^n_\Gamma$. We use this in Sections 5 and 6.

Lemma 2.2 (cf. [12, Proposition 2.3])**.** *If $P \in \mathcal{O}^p$ and $Q \in \mathcal{O}^q$, then $P \times Q \in \mathcal{O}^{p+q}$ and $g_1(P \times Q) = g_1(P) + g_1(Q)$.*

Proof. The product $P \times Q$ is a symmetric lattice polytope, and $\mathrm{int}(P \times Q) = \mathrm{int}\, P \times \mathrm{int}\, Q$ contains only the lattice point 0. By (E4) and (E1), the coefficient of k in $L_P L_Q$ is $g_1(P) \cdot 1 + 1 \cdot g_1(Q)$. □

3. A counterexample in dimension five

We compute four numbers for the polytope W of Table 2, three lattice-point counts and the volume; by Proposition 3.1 they determine $g_1(W)$.

Proposition 3.1. *Let $P \in \mathcal{O}^5$. Then*

$$g_1(P) = \frac{4(G(P) + 1) + 24\,\mathrm{vol}(P) - A(P)}{6}, \qquad A(P) = \tfrac{1}{2}(\#(2P \cap \mathbb{Z}^5) + \#(\mathrm{int}(2P) \cap \mathbb{Z}^5)).$$

TABLE 2. The polytope $W \subset \mathbb{R}^5$ is the convex hull of the 50 points $\pm p$, where p runs over the 25 points below ($\bar{1}$ stands for -1). All 50 points are vertices of W, and W has 92 facets.

$$\begin{array}{lllll}
(0,0,1,\bar{1},\bar{1}) & (0,0,1,1,\bar{1}) & (0,1,\bar{1},0,0) & (0,1,0,1,\bar{1}) & (0,1,0,1,1)\\
(0,1,1,\bar{1},\bar{1}) & (0,1,1,\bar{1},0) & (1,\bar{1},\bar{1},\bar{1},0) & (1,\bar{1},0,1,0) & (1,\bar{1},0,1,1)\\
(1,\bar{1},1,0,\bar{1}) & (1,0,\bar{1},0,\bar{1}) & (1,0,\bar{1},1,\bar{1}) & (1,0,\bar{1},1,1) & (1,0,0,\bar{1},1)\\
(1,0,0,1,1) & (1,0,1,\bar{1},0) & (1,0,1,\bar{1},1) & (1,0,1,0,\bar{1}) & (1,1,\bar{1},\bar{1},\bar{1})\\
(1,1,\bar{1},\bar{1},0) & (1,1,\bar{1},1,0) & (1,1,1,\bar{1},0) & (1,1,1,\bar{1},1) & (1,1,1,1,0)
\end{array}$$

Proof. Write $L = L_P$ and $g_i = g_i(P)$. By (E2) with $n = 5$, $L(-1) = -1$ and $L(-2) = -\#(\mathrm{int}(2P) \cap \mathbb{Z}^5)$. Hence

$$L(1) - L(-1) = 2(g_1 + g_3 + g_5) = G(P) + 1, \qquad L(2) - L(-2) = 2(2g_1 + 8g_3 + 32g_5) = 2A(P).$$

Eliminating g_3 and using $g_5 = \mathrm{vol}(P)$ (E1) gives $6g_1 = 4(G(P) + 1) + 24\,\mathrm{vol}(P) - A(P)$. □

Proof of Theorem 1.3. The 50 points of Table 2 lie in $\{-1, 0, 1\}^5$, form a symmetric set and span $\mathbb{R}^5$; by Lemma 2.1, $W \in \mathcal{O}^5$. Direct computation gives

$$G(W) = 117, \qquad \#(2W \cap \mathbb{Z}^5) = 1815, \qquad \#(\mathrm{int}(2W) \cap \mathbb{Z}^5) = 243, \qquad \mathrm{vol}(W) = \tfrac{386}{15},$$

so $A(W) = 1029$ and, by Proposition 3.1, $g_1(W) = \big(4 \cdot 118 + 24 \cdot \frac{386}{15} - 1029\big)/6 = \frac{101}{10}$. The whole polynomial is determined by $L_W(0) = 1$ and the values $L_W(k)$ for $k = 1, \ldots, 5$, namely 117, 1815, 10765, 39927 and 112639. All numbers in this proof can be computed with Normaliz [18] or LattE [14]; the input files are in [19].

For $n \geq 6$, Lemma 2.2 gives $W \times C_{n-5} \in \mathcal{O}^n$ and $g_1(W \times C_{n-5}) = \frac{101}{10} + 2(n-5) = 2n + \frac{1}{10}$, since $g_1(C_m) = 2m$. □

The polytope W was found by a computer search; the search and a first example, with 58 vertices, are described in Section 11.

4. Dimension four: the key inequality and polytopes with few lattice points

From now on $n = 4$. For $P \in \mathcal{O}^4_\Lambda$ we write $G = G(P)$, $g_i = g_i(P)$ (with respect to Λ) and $\sigma = \sigma(P)$.

Lemma 4.1. *For $P \in \mathcal{O}^4_\Lambda$, $g_1 + g_3 = g_2 + g_4 = \frac{G-1}{2}$.*

Proof. We have $L(1) = G = 1 + g_1 + g_2 + g_3 + g_4$, and by (E2) $L(-1) = \#(\mathrm{int}\, P \cap \Lambda) = 1 = 1 - g_1 + g_2 - g_3 + g_4$. Subtracting the second equation from the first, and adding the two, gives the two identities. □

Proposition 4.2. *Let $P \in \mathcal{O}^4_\Lambda$. Then $g_4 \leq 16$ and $g_3 \leq 2g_4 \leq 32$. Moreover*

$$g_1 \leq 8 \iff G \leq \sigma + 17,$$

and if $g_1 \leq 8$ then $g_2 \leq g_4 + 8 \leq 24$.

Proof. The bound $g_4 = \mathrm{vol}(P)/\det\Lambda \leq 16$ is Minkowski's theorem, applied in the coordinates of a basis of Λ. Since every $b_F \geq 1$, (E3) gives $2g_3 = \sigma \leq \sum_F b_F\,\mathrm{relvol}(F) = 4g_4$; this is Wills' argument for g_{n-1} [22, Theorem (i)]. By Lemma 4.1, $g_1 = \frac{G-1}{2} - g_3$, so $g_1 \leq 8$ if and only if $G \leq 2g_3 + 17 = \sigma + 17$. If this holds, then $G \leq 4g_4 + 17$, and Lemma 4.1 gives $g_2 = \frac{G-1}{2} - g_4 \leq g_4 + 8 \leq 24$. □

So Theorem 1.2 is equivalent to the key inequality $G \le \sigma + 17$ on $\mathcal{O}^4$. Let $\mathrm{NVol}(F) = 3!\,\mathrm{relvol}(F)$ be the normalized volume of a facet, so that a lattice tetrahedron in $\mathrm{aff}\, F$ has normalized volume a positive integer. The inequality reads

$$\sum_F \mathrm{NVol}(F) \;\ge\; 6G - 102. \tag{1}$$

Lemma 4.3. *If $P \in \mathcal{O}^4_\Lambda$ and $G \le 29$, then* (1) *holds with strict inequality; hence $g_1 < 8$.*

Proof. Take a triangulation Δ of the boundary of P whose vertex set is the set of all $G-1$ boundary lattice points and in which every tetrahedron lies in a facet. To construct one, take a triangulation of P whose vertices are exactly the points of $P \cap \Lambda$; one exists, for example by strong pulling ([10, Lemma 2.1(2)], applied in the coordinates of a basis of Λ). Its faces that lie in the boundary of P form Δ. Then Δ is a triangulated 3-sphere with $f_0 = G-1$ vertices. Each of its f_3 tetrahedra is a lattice tetrahedron in the affine hull of a facet, so it has normalized volume at least 1, and $\sum_F \mathrm{NVol}(F) \ge f_3$. Walkup's lower bound theorem [21, Theorem 1] gives $f_1 \ge 4f_0 - 10$ for every triangulated 3-sphere, and the Dehn–Sommerville relations $f_0 - f_1 + f_2 - f_3 = 0$, $2f_2 = 4f_3$ give $f_3 = f_1 - f_0$. Hence $\sum_F \mathrm{NVol}(F) \ge 3f_0 - 10 = 3G - 13$, and $3G - 13 > 6G - 102$ because $G \le 29$. □

The proof gives a bound on g_1 for every $P \in \mathcal{O}^4_\Lambda$: by Lemma 4.1 and (E3), $g_1 = \frac{G-1}{2} - \frac{\sigma}{2}$, and $\sigma = \frac{1}{6}\sum_F \mathrm{NVol}(F) \ge \frac{3G-13}{6}$, so $g_1 \le \frac{3G+7}{12}$ (the solid line in Figure 3).

Remark 4.4. A similar triangulation argument, with Euler's formula in place of Walkup's theorem, proves Wills' theorem in dimension 3. For $P \in \mathcal{O}^3$, $L(1) = G$ and $L(-1) = -1$ give $g_2 = \frac{G-3}{2}$ and $g_1 + g_3 = \frac{G+1}{2}$. A triangulation of the boundary by all $G-1$ boundary lattice points has $2G-6$ triangles, by Euler's formula, and the cones over them from 0 are lattice tetrahedra of normalized volume at least 1; hence $6g_3 \ge 2G-6$. By Minkowski's theorem, as in Proposition 4.2, $g_3 = \mathrm{vol}(P) \le 8$. This gives $G \le 27$, hence $g_2 \le 12$, and $g_1 = \frac{G+1}{2} - g_3 \le \frac{G+1}{2} - \frac{G-3}{3} = \frac{G+9}{6} \le 6$. In dimension 4 the lower bound theorem proves (1) only when $G \le 29$; this is why the computations of Section 6 are needed.

5. Dimension four: three cases

By Lemma 4.3 it remains to prove (1) for $P \in \mathcal{O}^4$ with $G \ge 31$. For such P let

$$M(P) = \{u \in \mathbb{Z}^4 :\ |\langle u, x\rangle| \le 1 \text{ for all } x \in P\}, \qquad r(P) = \dim \mathrm{span}\, M(P).$$

The set $M(P)$ is finite (since P contains a ball around 0) and symmetric, and $\langle u, x\rangle \in \{-1, 0, 1\}$ for $u \in M(P)$ and $x \in P \cap \mathbb{Z}^4$. We distinguish three cases: $M(P)$ contains a basis of $\mathbb{Z}^4$ (Case I); $r(P) = 4$ but $M(P)$ contains no basis (Case II); $r(P) \le 3$ (Case III). Figure 2 shows how each case is settled.

5.1. **Case I.** For a lattice Γ, a set $S \subseteq \Gamma$ is called Γ-*saturated* if $\mathrm{conv}\, S \cap \Gamma = S$; when the lattice is clear, we say *saturated*. If $M(P)$ contains a basis $u_1, \dots, u_4$ of $\mathbb{Z}^4$, let $U \in \mathrm{GL}_4(\mathbb{Z})$ have rows u_i. Then $UP \subset C_4$, so $UP = \mathrm{conv}\, S$ for the set $S = UP \cap \mathbb{Z}^4 \subset \{-1, 0, 1\}^4$, which is symmetric and saturated.

5.2. **Case II.** Let $r(P) = 4$, and let $u_1, \dots, u_4 \in M(P)$ be linearly independent, with U the matrix of rows u_i and $d = |\det U|$. Then $d \ge 2$, since otherwise $M(P)$ would contain a basis. The map $x \mapsto Ux$ carries $\mathbb{Z}^4$ onto the sublattice $\Gamma = U\mathbb{Z}^4$ of index d and carries P into C_4; so $(P, \mathbb{Z}^4)$ is isomorphic to (UP, Γ), $UP \in \mathcal{O}^4_\Gamma$, $L_{UP,\Gamma} = L_P$, and $UP = \mathrm{conv}\, S$ for the Γ-saturated symmetric set $S = UP \cap \Gamma \subset \Gamma \cap \{-1, 0, 1\}^4$. In particular

$$31 \le G(P) \le \#(\Gamma \cap \{-1, 0, 1\}^4).$$

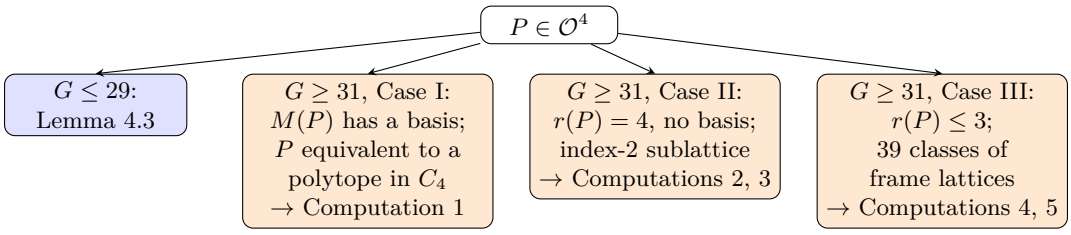


FIGURE 2. The proof of Theorem 1.2 splits $\mathcal{O}^4$ into four parts. Polytopes with at most 29 lattice points are handled by Lemma 4.3 (the blue box, on the left); for the others (the three orange boxes), the integer functionals bounded by 1 in absolute value on P decide which finite enumeration of Section 6 contains P, up to isomorphism (Section 2). The frame lattices are defined after Lemma 5.1.

Moreover $d \le 16$: the set S spans $\mathbb{R}^4$, so it contains four linearly independent points of $\Gamma \cap \{-1,0,1\}^4$; their determinant is a nonzero multiple of d, and it is at most 16 in absolute value by Hadamard's inequality.

The sublattices of index 2 are $\Gamma_I = \{x : \sum_{i\in I} x_i \text{ even}\}$ for nonempty $I \subseteq \{1,2,3,4\}$. A point of $\{-1,0,1\}^4$ lies in Γ_I exactly when an even number of its coordinates in I are nonzero, so

$$\#(\Gamma_I \cap \{-1,0,1\}^4) = 3^{4-m} \cdot \frac{3^m + (-1)^m}{2} = 27,\ 45,\ 39,\ 41 \qquad (m = |I| = 1,2,3,4).$$

Computation 2 shows that a sublattice of index $3 \le d \le 16$ contains at most 27 points of $\{-1,0,1\}^4$ whenever these points span $\mathbb{R}^4$. Since $27 < 31$, in Case II we have $d = 2$ and $\Gamma = \Gamma_I$ with $|I| \in \{2,3,4\}$.

5.3. **Case III: the frame.** Now let $r(P) \le 3$. Choose lattice points $v_1, \dots, v_4$ of P for which $D = |\det(v_1, \dots, v_4)|$ is as large as possible, and let V be the matrix with columns v_i. We call $(v_1, \dots, v_4)$ a *frame* of P. Since P spans $\mathbb{R}^4$, $D \ge 1$. The map $x \mapsto V^{-1}x$ carries $\mathbb{Z}^4$ onto the lattice $\Lambda = V^{-1}\mathbb{Z}^4 \supseteq \mathbb{Z}^4$, of index D over $\mathbb{Z}^4$, and carries $P \cap \mathbb{Z}^4$ onto a set S.

Lemma 5.1. *In this situation:*

(i) $S \subset \Lambda \cap C_4$, $S = -S$, *and* S *contains the unit vectors* $\pm e_1, \dots, \pm e_4$;
(ii) $|\det(s_1, \dots, s_4)| \le 1$ *for all* $s_1, \dots, s_4 \in S$;
(iii) $\operatorname{conv} S \cap \Lambda = S$ *and* $\operatorname{int}(\operatorname{conv} S) \cap \Lambda = \{0\}$;
(iv) Λ *contains no point* $c \ne 0$ *with* $|c_1| + \dots + |c_4| < 1$;
(v) $D \le 23$.

Proof. For $x_1, \dots, x_4 \in P \cap \mathbb{Z}^4$ we have $|\det(V^{-1}x_1, \dots, V^{-1}x_4)| = |\det(x_1, \dots, x_4)|/D \le 1$ by the choice of $v_1, \dots, v_4$, which gives (ii). For (i), write a lattice point x of P as $x = \sum c_i v_i$; by Cramer's rule $|c_i| = |\det(v_1, \dots, x, \dots, v_4)|/D \le 1$. Property (iii) holds because $x \mapsto V^{-1}x$ carries $\operatorname{conv}(P \cap \mathbb{Z}^4) \cap \mathbb{Z}^4 = P \cap \mathbb{Z}^4$ and $\operatorname{int} P \cap \mathbb{Z}^4 = \{0\}$ to the corresponding statements for S and Λ. For (iv), the open cross-polytope $\{c : |c_1| + \dots + |c_4| < 1\}$ lies in the interior of $\operatorname{conv}\{\pm e_i\} \subseteq \operatorname{conv} S$, so by (iii) its only point of Λ is 0.

For (v), P contains the cross-polytope $\operatorname{conv}\{\pm v_i\}$ of volume $\frac{2^4}{4!}D = \frac{2}{3}D$, and $\operatorname{vol}(P) \le 16$, so $D \le 24$. If $D = 24$, then $P = \operatorname{conv}\{\pm v_i\}$ and $\operatorname{vol} P = 16$. The translates $\frac{1}{2}P + z$, $z \in \mathbb{Z}^4$, have disjoint interiors (an overlap would put a nonzero lattice point in $\operatorname{int} P$) and total density $\operatorname{vol}(\frac{1}{2}P) = 1$, so they tile $\mathbb{R}^4$: the union of the translates is closed, and its complement is open, invariant under $\mathbb{Z}^4$ and of density 0; a nonempty such set would contain a ball and all its $\mathbb{Z}^4$-translates, hence have positive density. A convex polytope that tiles space by lattice translations has centrally symmetric facets (Minkowski [17]; see Venkov [20] and McMullen [15]), but the facets of a cross-polytope are tetrahedra. So $D \le 23$. □

Let $\mathcal{L}$ be the set of lattices $\Lambda \supseteq \mathbb{Z}^4$ of index at most 23 over $\mathbb{Z}^4$ that satisfy (iv); a lattice of index D over $\mathbb{Z}^4$ lies in $\frac{1}{D}\mathbb{Z}^4$, so $\mathcal{L}$ is finite. We call the lattices in $\mathcal{L}$ the *frame lattices*. For $\Lambda \in \mathcal{L}$ let $\mathcal{S}(\Lambda)$ be the family of sets S satisfying (i)–(iii). The signed permutation matrices act on $\mathcal{L}$, and Computation 4 shows that $\mathcal{L}$ consists of 39 orbits; we fix one representative in each. Integer functionals transfer as well. For a set $S \subset \Lambda$ let $M(S) = \{w \in \Lambda^* : |\langle w, s\rangle| \le 1 \text{ for } s \in S\}$ and $r(S) = \dim \operatorname{span} M(S)$. Since $\langle u, Vs\rangle = \langle V^{\mathsf{T}}u, s\rangle$ and $V^{\mathsf{T}}\mathbb{Z}^4 = \Lambda^*$, a vector $u \in \mathbb{Z}^4$ lies in $M(P)$ if and only if $V^{\mathsf{T}}u$ lies in $M(S)$; so $r(P) = r(S)$. In Case III we therefore have a lattice Λ among the 39 representatives and a set $S \in \mathcal{S}(\Lambda)$ with $\#S \ge 31$ and $r(S) \le 3$ such that $(P, \mathbb{Z}^4)$ is isomorphic to $(\operatorname{conv} S, \Lambda)$ (by $x \mapsto V^{-1}x$, followed by a signed permutation of the coordinates); in particular $L_{\operatorname{conv} S,\Lambda} = L_P$. Lemma 5.1 does not use $r(P) \le 3$, so this reduction applies to every $P \in \mathcal{O}^4$. We use it only in Case III, where the condition $r(S) \le 3$ shortens the search of Computation 5; the polytopes with $r(P) = 4$ are listed directly, in Cases I and II.

6. The computations

All enumerations below are exhaustive and use exact integer and rational arithmetic; the repository [19] contains the programs and, for each computation, its output. Computations 1, 3 and 5 list sets of lattice points up to the symmetries named. For each set S the Ehrhart polynomial of $\operatorname{conv} S$ is computed exactly: in Computations 1 and 3 from $G = \#S$ and the volume and the lattice surface area of $\operatorname{conv} S$ (taken from a triangulation of its boundary), by Lemma 4.1, (E1) and (E3); in Computation 5 from the numbers of lattice points of $k \operatorname{conv} S$ for $k = 1, 2, 3$ and of the interior of $k \operatorname{conv} S$ for $k = 1, 2$, which give its values at $k = -2, \dots, 3$ by (E1) and (E2) and so determine it. Figure 3 shows, for each $G \ge 31$, the largest value of g_1 in Case I (Computation 1) and the three equality cases of Theorem 7.1, and, for $G \le 29$, the bound on g_1 stated after the proof of Lemma 4.3.

Computations 1 and 3 walk downward from the full set, $\{-1, 0, 1\}^4$ or $\Gamma_I \cap \{-1, 0, 1\}^4$, deleting one pair of opposite vertices at a time. By the next lemma, such a walk reaches every saturated symmetric subset of the full set. Sets that a symmetry carries onto each other form a class, and the walk keeps one set of each class; classes are recognized by a canonical form, the least image of the set under the symmetry group in a fixed total order. This loses no class: a symmetry γ maps saturated symmetric sets to saturated symmetric sets and the vertices of $\operatorname{conv} Q$ to those of $\operatorname{conv} \gamma Q$, so the sets obtained from γQ by deleting a pair of opposite vertices are the images under γ of those obtained from Q. Computation 5 uses a different method, a sandwich search, described after its statement; Section 12 describes a second search for it, which first finds the maximal members of $\mathcal{S}(\Lambda)$ and then walks downward from them.

Lemma 6.1. *Let Λ be a lattice and $Q \subset \Lambda$ a finite symmetric set with $\operatorname{conv} Q \cap \Lambda = Q$.*

(i) *If v is a vertex of $\operatorname{conv} Q$, then $Q' = Q \setminus \{v, -v\}$ is symmetric and satisfies $\operatorname{conv} Q' \cap \Lambda = Q'$.*
(ii) *If $S \subsetneq Q$ is symmetric with $\operatorname{conv} S \cap \Lambda = S$, then some vertex v of $\operatorname{conv} Q$ is not in S, and $S \subseteq Q \setminus \{v, -v\}$.*

Proof. (i) Since v is a vertex, $v \notin \operatorname{conv}(Q \setminus \{v\}) \supseteq \operatorname{conv} Q'$, and by symmetry $-v \notin \operatorname{conv} Q'$; as $\operatorname{conv} Q' \cap \Lambda \subseteq Q$, it equals Q'. (ii) If every vertex of $\operatorname{conv} Q$ were in S, then $Q = \operatorname{conv} Q \cap \Lambda \subseteq \operatorname{conv} S \cap \Lambda = S$. A vertex $v \notin S$ has $-v \notin S$ by symmetry. □

So every saturated symmetric $S \subseteq Q$ is reached from Q through saturated symmetric sets that contain S; in particular a search that stops at 31 points misses no set with at least 31 points.

Computation 1. *Case I.* We list all symmetric sets $S \subseteq \{-1, 0, 1\}^4$ with $\operatorname{conv} S \cap \mathbb{Z}^4 = S$ and $\#S \ge 31$, up to the 384 signed permutations of the coordinates. Each such S spans $\mathbb{R}^4$ (a hyperplane through 0 contains at most 27 points of $\{-1, 0, 1\}^4$, since on it one coordinate is

determined by the other three), so $\operatorname{conv} S \in \mathcal{O}^4$ by Lemma 2.1. There are 2,547,694 classes; for all of them the linear coefficient g_1 of $\operatorname{conv} S$ is at most 8, with equality only for $S = \{-1, 0, 1\}^4$.

Computation 2. *Index bound.* We list every sublattice $\Gamma \subset \mathbb{Z}^4$ of index d with $2 \le d \le 16$ once, as $\Gamma = B\mathbb{Z}^4$ with B in Hermite normal form under column operations ($B \mapsto BU$); there are 40,406 of them. For each Γ whose points in $\{-1, 0, 1\}^4$ span $\mathbb{R}^4$ we count these points. The maximum is 45 for $d = 2$, and it is at most 27 for every $d \ge 3$.

Computation 3. *Case II.* A permutation of coordinates carries Γ_I onto Γ_J when $|I| = |J|$, so it suffices to take $\Gamma_m = \Gamma_{\{1,\dots,m\}} = \{x : x_1 + \cdots + x_m \text{ even}\}$ for $m = 2, 3, 4$. For each of these three lattices we list all Γ_m-saturated symmetric sets $S \subseteq \Gamma_m \cap \{-1, 0, 1\}^4$ that span $\mathbb{R}^4$ and have $\#S \ge 31$, up to the signed permutations that preserve Γ_m; each such $\operatorname{conv} S$ lies in $\mathcal{O}^4_{\Gamma_m}$ (see the paragraph after Lemma 2.1). There are 531, 58 and 27 classes for $m = 2$, 3, 4, and for all of them the linear coefficient g_1 of $\operatorname{conv} S$ with respect to Γ_m is at most 6.

Computation 4. *The frame lattices.* We list all lattices in $\mathcal{L}$. For $U \in \mathrm{GL}_4(\mathbb{Z})$ the frame matrices V and UV give the same lattice $\Lambda = V^{-1}\mathbb{Z}^4$, so it suffices to take V in Hermite normal form under row operations ($V \mapsto UV$), with determinant $D \le 23$, and to keep the lattices Λ that satisfy (iv). There are 1,040 frame lattices, in 39 orbits under signed permutations, with indices $D \in \{1, \dots, 10, 12, 16, 18\}$.

Computation 5. *Case III.* For each of the 39 representatives Λ we list all $S \in \mathcal{S}(\Lambda)$ with $\#S \ge 31$ and $r(S) \le 3$, up to the signed permutations that preserve Λ. There are 588,592 classes; for all of them the linear coefficient g_1 of $\operatorname{conv} S$ with respect to Λ is at most 8, with equality for exactly two of them, one with $G = 49$ and one with $G = 33$.

The search follows [2, Algorithm 1]. It keeps a lower set $S^- \in \mathcal{S}(\Lambda)$ and an upper set $S^+ \supseteq S^-$, and it starts from $S^- = \operatorname{conv}\{\pm e_1, \dots, \pm e_4\} \cap \Lambda$ and $S^+ = \Lambda \cap C_4$. Every member of $\mathcal{S}(\Lambda)$ lies between these two sets: it lies in S^+ by (i), and it contains S^- by (i) and (iii). The set S^- is itself a member: its points c have $|c_1| + \cdots + |c_4| \le 1$, so (ii) holds by Hadamard's inequality and (iii) by (iv).

For a pair $\pm v$ in $S^+ \setminus S^-$ the search branches: either $\pm v$ is excluded and S^+ is replaced by $S^+ \setminus \{\pm v\}$, or $\pm v$ is included and S^- is replaced by $\operatorname{conv}(S^- \cup \{\pm v\}) \cap \Lambda$; every saturated set between S^- and S^+ lies in exactly one branch. An include branch is abandoned when the new S^- is not contained in S^+, violates (ii), or has a nonzero point of Λ in the interior of its convex hull, since then no set in the branch satisfies (i)–(iii). A pair $\pm y$ in S^+ for which $S^- \cup \{\pm y\}$ violates (ii) is removed from S^+, since no member contains it together with S^-. A branch is abandoned when $\#S^+ < 31$ or $r(S^+) = 4$; the second rule uses that M reverses inclusion, that is, $M(Q') \subseteq M(Q)$ whenever $Q \subseteq Q'$, so that every subset of a set Q' with $r(Q') = 4$ also has $r = 4$. The search stops at the pairs (S^-, S^+) with $S^- = S^+$. Each include step keeps S^- in $\mathcal{S}(\Lambda)$, so these sets are members, and they are the sets found. A pair (S^-, S^+) that a signed permutation preserving Λ carries onto a pair met earlier in the search is discarded, since the permutation carries the sets between the two pairs onto each other; the sets found are then reduced to one of each class by the canonical form described before Lemma 6.1.

Proof of Theorem 1.2. Let $P \in \mathcal{O}^4$. If $G(P) \le 29$, Lemma 4.3 gives $g_1(P) < 8$. If $G(P) \ge 31$, then by Section 5 the pair $(P, \mathbb{Z}^4)$ is isomorphic to $(\operatorname{conv} S, \Lambda_S)$ for a set S listed in Computation 1 (Case I), in Computation 3 (Case II, using Computation 2), or in Computation 5 (Case III, using Computation 4), where Λ_S is the lattice of that computation ($\mathbb{Z}^4$, Γ_m or a frame lattice); so $L_P = L_{\operatorname{conv} S, \Lambda_S}$. In each list $g_1 \le 8$. Hence $g_1(P) \le 8$, and Proposition 4.2 gives $g_2(P) \le 24$. $\square$

7. The equality cases

Let $T \subset \mathbb{R}^4$ be the convex hull of the 24 points $\pm f_a \pm f_b$, $1 \le a < b \le 4$, where $f_i = e_i$ for $i \le 3$ and $f_4 = (-1,-1,-1,2)$. This is the 24-cell $\operatorname{conv}\{\pm e_a \pm e_b\}$ written in the basis $e_1, e_2, e_3, \frac{1}{2}(1,1,1,1)$ of the lattice $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$; in other words, $(T, \mathbb{Z}^4)$ is isomorphic to the pair formed by this 24-cell and this lattice. Let

$$X = \operatorname{conv}\{\pm(1,0,0,0),\ \pm(0,1,0,0),\ \pm(1,1,3,0),\ \pm(2,4,3,6)\}.$$

Theorem 7.1. *A polytope $P \in \mathcal{O}^4$ has $g_1(P) = 8$ if and only if it is unimodularly equivalent to C_4, T or X.*

We first check that T and X lie in $\mathcal{O}^4$, that C_4, T and X have $g_1 = 8$, and that $M(T) = M(X) = \{0\}$.

The polytope $\operatorname{conv}\{\pm e_a \pm e_b\}$ is the set of points x with $|x_i| \le 1$ for all i and $|x_1| + \cdots + |x_4| \le 2$, so its lattice points in $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$ other than 0 are the 24 vertices, the 8 points $\pm e_i$ and the 16 points $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$; each has a coordinate ± 1 or all four coordinates $\pm\frac{1}{2}$, so it lies on one of the facets $\{x_i = \pm 1\}$ or $\{\pm x_1 \pm x_2 \pm x_3 \pm x_4 = 2\}$. Hence $T \in \mathcal{O}^4$ and $G(T) = 49$. The dual lattice of $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$ is $\{y \in \mathbb{Z}^4 : y_1 + \cdots + y_4 \text{ even}\}$; its primitive vectors normal to these facets are $\pm 2e_i$ and $(\pm 1, \pm 1, \pm 1, \pm 1)$, and they take the value 2 on them, so every facet of T has lattice distance 2 from the origin. Moreover $\operatorname{vol}(T) = 16$, since $\operatorname{conv}\{\pm e_a \pm e_b\}$ has volume 8 and the lattice has determinant $\frac{1}{2}$.

For X, a direct computation (see [19]) shows that 0 is its only interior lattice point, that $G(X) = 33$, and that each of its 16 facets has lattice distance 3 from the origin; and $\operatorname{vol}(X) = \frac{2^4}{4!} \cdot 18 = 12$, since the vectors $(1,0,0,0)$, $(0,1,0,0)$, $(1,1,3,0)$, $(2,4,3,6)$ have determinant 18.

If every facet of $P \in \mathcal{O}^4$ has lattice distance ℓ, then (E3) gives $g_3 = \frac{1}{2}\sigma = 2g_4/\ell$, and Lemma 4.1 gives $g_1 = \frac{G-1}{2} - g_3$ and $g_2 = \frac{G-1}{2} - g_4$. For C_4 we have $G = 81$ and $\operatorname{vol}(C_4) = 16$, and its facets $\{x_i = \pm 1\}$ have lattice distance 1. With $(G, g_4, \ell) = (81, 16, 1)$, $(49, 16, 2)$ and $(33, 12, 3)$ this gives

$$L_{C_4} = 1 + 8k + 24k^2 + 32k^3 + 16k^4, \qquad L_T = 1 + 8k + 8k^2 + 16k^3 + 16k^4,$$
$$L_X = 1 + 8k + 4k^2 + 8k^3 + 12k^4.$$

For T, let $u \in M(T)$ and $w_a = \langle u, f_a \rangle \in \mathbb{Z}$. Then $|w_a \pm w_b| \le 1$ for $a \ne b$, so at most one w_a is nonzero, and it is ± 1. Here $w_a = u_a$ for $a \le 3$ and $w_4 = 2u_4 - u_1 - u_2 - u_3$. If $w_1 = w_2 = w_3 = 0$, then $u_1 = u_2 = u_3 = 0$ and $w_4 = 2u_4$ is even, so $u_4 = 0$. Otherwise $w_a = u_a = \pm 1$ for one $a \le 3$ and $w_4 = 0$, so $2u_4 = u_a = \pm 1$, which is impossible. So $M(T) = \{0\}$.

For X, the conditions $|\langle u, v \rangle| \le 1$ for the first three vertices, namely $|u_1| \le 1$, $|u_2| \le 1$ and $|u_1 + u_2 + 3u_3| \le 1$, leave nine triples (u_1, u_2, u_3), one for each (u_1, u_2). The fourth condition, $|2u_1 + 4u_2 + 3u_3 + 6u_4| \le 1$, requires $2u_1 + 4u_2 + 3u_3 \equiv -1$, 0 or 1 (mod 6). Of the nine triples only $(0,0,0)$ satisfies this, and then $u_4 = 0$. So $M(X) = \{0\}$.

Proof of Theorem 7.1. The three polytopes have $g_1 = 8$, and they are pairwise inequivalent because they have 81, 49 and 33 lattice points. Conversely, let $g_1(P) = 8$. By Lemma 4.3, $G(P) \ge 31$, so, as in the proof of Theorem 1.2, $(P, \mathbb{Z}^4)$ is isomorphic to $(\operatorname{conv} S, \Lambda_S)$ for a set S with $g_1 = 8$ in one of Computations 1, 3, 5. In Computation 1 this is the cube, and the isomorphism lies in $\mathrm{GL}_4(\mathbb{Z})$; Computation 3 has none; in Computation 5 it is one of the two sets with $G = 49$ and $G = 33$. Now T and X have $M(T) = M(X) = \{0\}$, so they fall under Case III, and $(T, \mathbb{Z}^4)$ and $(X, \mathbb{Z}^4)$ are isomorphic to pairs $(\operatorname{conv} S, \Lambda_S)$ with S in Computation 5 and $g_1 = 8$; comparing G, these are the two sets found there. If $(P, \mathbb{Z}^4)$ and $(T, \mathbb{Z}^4)$ are both isomorphic to the same pair (after a signed permutation fixing its lattice, if needed), then composing the two isomorphisms gives a linear map that carries P onto T and $\mathbb{Z}^4$ onto $\mathbb{Z}^4$, that is, a map in $\mathrm{GL}_4(\mathbb{Z})$, and in the same way for X. Hence, in this case, P is unimodularly equivalent to T or to X. □

Corollary 7.2. *A polytope $P \in \mathcal{O}^4$ has $g_2(P) = 24$ if and only if it is unimodularly equivalent to C_4.*

Proof. The cube has $g_2(C_4) = 24$. Conversely, let $g_2(P) = 24$. Since $g_1(P) \le 8$ by Theorem 1.2, Proposition 4.2 gives $24 = g_2 \le g_4 + 8 \le 24$, so $g_4 = 16$, and then $G = 2(g_2 + g_4) + 1 = 81$ by Lemma 4.1. The same lemma gives $g_1 + g_3 = 40$, and $g_1 \le 8$, $g_3 \le 32$ force $g_1 = 8$. By Theorem 7.1, P is unimodularly equivalent to C_4, T or X, and of these only C_4 has 81 lattice points. □

Remark 7.3. Henze's theorem [12, Corollary 3.3] covers C_4 but neither T nor X. All roots of $L_{C_4} = (1 + 2k)^4$ have real part $-\frac{1}{2}$, but those of L_T and L_X do not: if all four roots of L_P, for $P \in \mathcal{O}^4$, have real part $-\frac{1}{2}$, they sum to -2, so $g_3(P) = 2g_4(P)$ by Vieta's formulas; but L_T has $g_3 = 16 \ne 2 \cdot 16$ and L_X has $g_3 = 8 \ne 2 \cdot 12$. Alternatively, by [5] such a P would be reflexive, that is, all its facets would have lattice distance 1 from the origin, while the facets of T and X have lattice distance 2 and 3.

8. Case III by the rank of M

Computation 5 closes Case III by one enumeration over the 39 frame lattices. In this section we sort Case III by the number $r(P) = \dim \operatorname{span} M(P) \in \{0, 1, 2, 3\}$ and prove it a second time. For $r \le 2$ we use the frame reduction of Section 5 again, with a stronger cut: for a bound $\rho \le 2$ on r, the programs abandon every branch whose sets all have $r > \rho$. For $r = 3$ we use a different reduction: every $P \in \mathcal{O}^4$ with $r(P) = 3$ is isomorphic to the convex hull of a set of lattice points of one of six explicit parallelepipeds, which we call boxes (Theorem 8.2). Each of the two proofs of Case III is complete by itself.

8.1. **Ranks zero, one and two.** The rank is monotone: as noted in Section 6, $M(Q') \subseteq M(Q)$ whenever $Q \subseteq Q'$; hence $r(Q') \le r(Q)$. So for every ρ, a set of $\mathcal{S}(\Lambda)$ that contains a set S with $r(S) \le \rho$ also has $r \le \rho$. Computation 5 uses this property for $\rho = 3$, and its two programs (Section 12) remain exhaustive, by the same argument, when the bound 3 is replaced by any ρ.

Computation 6. *Case III with $r \le 2$.* For each of the 39 representatives Λ of Computation 4 and for $\rho = 0, 1, 2$: all $S \in \mathcal{S}(\Lambda)$ with $\#S \ge 31$ and $r(S) \le \rho$, up to the signed permutations that preserve Λ. The two programs are those of Computation 5, described in Section 12; the first abandons a branch as soon as its current set together with all its remaining candidates has $r > \rho$, and the second abandons a branch when its upper set S^+ has $r(S^+) > \rho$. Results:

(a) $\rho = 0$: exactly three sets. Two of them are T ($G = 49$, in a frame lattice of index $D = 8$) and X ($G = 33$, $D = 18$). The third, which we call R_{41}, has $G = 41$, lies in a lattice of index 16, and has $L = 1 + \frac{20}{3}k + \frac{20}{3}k^2 + \frac{40}{3}k^3 + \frac{40}{3}k^4$.

(b) $\rho = 1$: the three sets of (a) and exactly two sets with $r = 1$. One has $G = 47$, lies in the lattice of T, and has $L = 1 + 7k + 9k^2 + 16k^3 + 14k^4$; the other has $G = 39$, lies in the lattice of R_{41}, and has $L = 1 + \frac{17}{3}k + 7k^2 + \frac{40}{3}k^3 + 12k^4$. The set R_{41} and the set with $G = 39$ are listed in [19].

(c) $\rho = 2$: the five sets of (b) and 7,527 sets with $r = 2$. These 7,527 sets satisfy $g_1 \le \frac{19}{3}$ and $g_2 \le 14$.

The set with $G = 47$ in (b) is obtained from T by deleting one pair of opposite vertices. The 384 signed permutations of the coordinates preserve the 24-cell $\operatorname{conv}\{\pm e_a \pm e_b\}$ and the lattice $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$, and they act transitively on the 24 vertices; so it does not matter, up to isomorphism, which pair is deleted. We put

$$T' = \operatorname{conv}\big(\{\pm f_a \pm f_b : 1 \le a < b \le 4\} \setminus \{\pm(f_1 + f_2)\}\big),$$

with $f_1, \dots, f_4$ as in Section 7. Then $T' \in \mathcal{O}^4$, because T' is symmetric and 4-dimensional, and $\operatorname{int} T' \subseteq \operatorname{int} T$. Each lattice point of T that is not a vertex is the midpoint of two vertices of T other than $\pm(f_1+f_2)$. So T' is the convex hull of the 47 lattice points of T other than $\pm(f_1+f_2)$. By Lemma 6.1(i) these are all its lattice points, and its vertices are the remaining 22 vertices of T.

Lemma 8.1. *$M(T') = \{0, \pm(1,1,0,1)\}$; in particular $r(T') = 1$, and T' is the set with $G = 47$ of Computation 6*(b).

Proof. Let $u \in M(T')$ and $w_a = \langle u, f_a\rangle$, so that $w_a = u_a$ for $a \le 3$ and $w_4 = 2u_4 - u_1 - u_2 - u_3$. The vertices of T' give $|w_a \pm w_b| \le 1$ for every pair $\{a,b\} \ne \{1,2\}$, and $|w_1 - w_2| \le 1$. Hence $|w_a| + |w_b| \le 1$ whenever $\{a,b\} \ne \{1,2\}$. If $w_3 \ne 0$, then $w_1 = w_2 = w_4 = 0$ and $w_3 = \pm 1$, so $u_1 = u_2 = 0$, $u_3 = \pm 1$ and $2u_4 = u_3 = \pm 1$, which is impossible. If $w_4 \ne 0$, then $u_1 = u_2 = u_3 = 0$ and $w_4 = 2u_4$ is even, which is impossible as $w_4 = \pm 1$. So $w_3 = w_4 = 0$, that is, $u_3 = 0$ and $2u_4 = u_1 + u_2$. Now $|u_1|, |u_2| \le 1$, $|u_1 - u_2| \le 1$ and $u_1 + u_2$ is even, so $u_1 = u_2 = u_4 \in \{-1, 0, 1\}$. Conversely $u = (1,1,0,1)$ has $w = (1,1,0,0)$, so $|w_a \pm w_b| \le 1$ for all pairs other than $\{1,2\}$ and $|w_1 - w_2| = 0$; hence $u \in M(T')$. Thus $r(T') = 1$. As in the proof of Theorem 7.1, $(T', \mathbb{Z}^4)$ is isomorphic to a pair $(\operatorname{conv} S, \Lambda)$ with S in the list of Computation 6(b) and $\#S = 47$, and there is only one such set. □

Table 3 sorts the 588,592 sets of Computation 5 by their rank. The rows $r = 0, 1, 2$ are those of Computation 6; the row $r = 3$ is the rest of Computation 5. Both programs of Computation 5 report this histogram, and they agree on it for every lattice.

TABLE 3. The sets S of Computation 5 ($\#S \ge 31$, $r(S) \le 3$), sorted by $r(S)$. Counts are numbers of sets up to the symmetries of each frame lattice, summed over the 39 lattices.

$r(S)$	sets	largest g_1	largest g_2	largest g_1 attained by
0	3	8	8	T and X
1	2	7	9	T'
2	7,527	19/3	14	
3	581,060	19/3	18	
all	588,592	8	18	T and X

8.2. **Rank three: six boxes.** For a sublattice $\Gamma \subseteq \mathbb{Z}^3$ let $\Gamma^* = \{c \in \mathbb{R}^3 : \langle c, z\rangle \in \mathbb{Z} \text{ for all } z \in \Gamma\}$ be its dual lattice; then $\mathbb{Z}^3 \subseteq \Gamma^*$. For $a \in \mathbb{R}^3$ let

$$\Pi(\Gamma, a) = \{(z,t) \in \mathbb{R}^3 \times \mathbb{R} : \ |z_1| \le 1,\ |z_2| \le 1,\ |z_3| \le 1,\ |t - \langle a, z\rangle| \le 1\},$$

with the lattice $\Gamma \times \mathbb{Z}$. This is a parallelepiped, and we call it a *box*.

Theorem 8.2. *Let $P \in \mathcal{O}^4$ with $r(P) = 3$. There are a sublattice $\Gamma \subseteq \mathbb{Z}^3$ of index $d \le 6$, a vector $a \in \mathbb{Q}^3$, and a linear isomorphism ϕ of $\mathbb{R}^4$ with $\phi(\mathbb{Z}^4) = \Gamma \times \mathbb{Z}$ and $\phi(P) \subseteq \Pi(\Gamma, a)$, such that:*

(i) *the points of $\Gamma \cap \{-1,0,1\}^3$ span $\mathbb{R}^3$;*
(ii) *$\langle a, z_j\rangle \in \mathbb{Z}$ for three linearly independent points z_1, z_2, z_3 of $\Gamma \cap \{-1,0,1\}^3$, and $a \notin \Gamma^*$;*
(iii) *the only point of $\Gamma \times \mathbb{Z}$ in the interior of $\Pi(\Gamma, a)$ is 0.*

In particular $(P, \mathbb{Z}^4)$ is isomorphic to $(\operatorname{conv} S, \Gamma \times \mathbb{Z})$, where $S = \phi(P) \cap (\Gamma \times \mathbb{Z})$ is a symmetric set of lattice points of $\Pi(\Gamma, a)$ that spans $\mathbb{R}^4$, satisfies $\operatorname{conv} S \cap (\Gamma \times \mathbb{Z}) = S$, and has $r(S) = 3$.

Proof. The idea is to use three linearly independent functionals of $M(P)$ as the first three coordinates. They confine P to a cube times a line. The fourth coordinate is then bounded above by an affine function of the first three, because the upper boundary of P is concave, and a vertex of the polytope of all such affine bounds gives the vector a.

Coordinates. We identify $(\mathbb{Z}^4)^*$ with $\mathbb{Z}^4$. The subspace $\operatorname{span} M(P)$ has dimension 3, so $\bar{M} = \operatorname{span} M(P) \cap \mathbb{Z}^4$ is a sublattice of rank 3 with $\mathbb{Z}^4 \cap \operatorname{span} \bar{M} = \bar{M}$. Choose a basis $u_1, \dots, u_4$ of $\mathbb{Z}^4$ such that u_1, u_2, u_3 is a basis of $\bar{M}$, and let $\psi(x) = (y, t)$ with $y_i = \langle u_i, x\rangle$ for $i \le 3$ and $t = \langle u_4, x\rangle$. Then $\psi \in \mathrm{GL}_4(\mathbb{Z})$, and the functionals in $\bar{M}$ are the maps $x \mapsto \langle w, y\rangle$ with $w \in \mathbb{Z}^3$. Choose linearly independent $\lambda_1, \lambda_2, \lambda_3 \in M(P)$ and write $\lambda_i(x) = \langle w_i, y\rangle$ with $w_i \in \mathbb{Z}^3$. Let U be the integer matrix with rows w_1, w_2, w_3, let $d = |\det U| \ge 1$, and put $\phi(x) = (Uy, t)$, where $(y, t) = \psi(x)$. Then $\phi(\mathbb{Z}^4) = \Gamma \times \mathbb{Z}$ with $\Gamma = U\mathbb{Z}^3$, a sublattice of index d in $\mathbb{Z}^3$, and the first three coordinates of $\phi(x)$ are $z_i = \lambda_i(x)$.

The projection, and $d \le 6$. Let $Q = \phi(P)$ and $\pi(z, t) = z$. Since $|\lambda_i| \le 1$ on P, we have $\pi(Q) \subseteq [-1, 1]^3$, and the lattice points of Q project into $\Gamma \cap \{-1, 0, 1\}^3$, because λ_i takes integer values on $\mathbb{Z}^4$. As Q spans $\mathbb{R}^4$, these projections span $\mathbb{R}^3$; this is (i). The polytope $\pi(Q)$ is symmetric, has dimension 3, and has its vertices in Γ, so it contains three linearly independent points $p_1, p_2, p_3 \in \Gamma$. The cross-polytope $\operatorname{conv}\{\pm p_1, \pm p_2, \pm p_3\}$ has volume $\frac{8}{6}|\det(p_1, p_2, p_3)| \ge \frac{4}{3}d$, since the determinant is a nonzero multiple of $\det \Gamma = d$; it lies in $[-1, 1]^3$, of volume 8, so $d \le 6$.

The fourth coordinate. For $z \in \pi(Q)$ let $\tau(z) = \max\{t : (z, t) \in Q\}$. If $\tau(0) > 1$, the lattice point $(0, 1)$ lies on the segment from the interior point 0 of Q to the point $(0, \tau(0))$ of Q, and is different from its endpoint $(0, \tau(0))$, so it is an interior point of Q; this is impossible. Hence $\tau(0) \le 1$. The point $(0, \tau(0))$ lies on the boundary of Q, so Q has a supporting hyperplane at it, $\{(z, t) : \langle c, z\rangle + \kappa t = \kappa\tau(0)\}$ with $\langle c, z\rangle + \kappa t \le \kappa\tau(0)$ on Q. If $\kappa = 0$, then $c \ne 0$ and $\langle c, z\rangle \le 0$ on $\pi(Q)$, which is impossible because 0 is an interior point of $\pi(Q)$. If $\kappa < 0$, the inequality fails at the origin, because $\tau(0) > 0$. So $\kappa > 0$, and with $a = -c/\kappa$ we get $t \le \tau(0) + \langle a, z\rangle \le \langle a, z\rangle + 1$ on Q. Since $Q = -Q$, also $t \ge \langle a, z\rangle - 1$ on Q. Therefore the set

$$\Omega = \{a \in \mathbb{R}^3 : \ |t - \langle a, z\rangle| \le 1 \text{ for all lattice points } (z, t) \text{ of } Q\}$$

is not empty. It is a polyhedron, and it is bounded: if $a + se \in \Omega$ for all $s \ge 0$, then $\langle e, z\rangle = 0$ for the projections z of all lattice points of Q, which span $\mathbb{R}^3$, so $e = 0$. Hence Ω is a polytope.

A vertex. Let a be a vertex of Ω. Then a satisfies three linearly independent equations $\langle a, z_j\rangle = t_j \mp 1$, where (z_j, t_j) are lattice points of Q; so z_1, z_2, z_3 are linearly independent points of $\Gamma \cap \{-1, 0, 1\}^3$ and $\langle a, z_j\rangle \in \mathbb{Z}$. In particular $a \in \mathbb{Q}^3$. Suppose that $a \in \Gamma^*$. Then the functional $h(z, t) = t - \langle a, z\rangle$ takes integer values on $\Gamma \times \mathbb{Z}$, so $h \circ \phi$ is an integer functional on $\mathbb{Z}^4$; it satisfies $|h \circ \phi| \le 1$ on the lattice points of P, hence on P, so $h \circ \phi \in M(P)$. It does not lie in the span of $\lambda_1, \lambda_2, \lambda_3$, because it depends on t. Then $r(P) = 4$, a contradiction. So $a \notin \Gamma^*$, which proves (ii). Since Q is the convex hull of its lattice points and $a \in \Omega$, we have $Q \subseteq \Pi(\Gamma, a)$.

The interior. If $(z, t) \in \Gamma \times \mathbb{Z}$ is an interior point of $\Pi(\Gamma, a)$, then $|z_i| < 1$ and $z_i \in \mathbb{Z}$, so $z = 0$; then $|t| < 1$, so $t = 0$. This is (iii).

Finally ϕ carries P onto $Q = \operatorname{conv} S$ and $\mathbb{Z}^4$ onto $\Gamma \times \mathbb{Z}$, so the two pairs are isomorphic; S is symmetric and saturated because $P \cap \mathbb{Z}^4$ is, and $r(S) = r(P) = 3$ because ϕ carries integer functionals to integer functionals. □

Three kinds of maps carry boxes to boxes and are isomorphisms of the pairs $(\Pi(\Gamma, a), \Gamma \times \mathbb{Z})$: a signed permutation of the coordinates z (which replaces Γ by its image and applies the same signed permutation to a), the map $(z, t) \mapsto (z, -t)$ (which replaces a by $-a$), and, for $c \in \Gamma^*$, the map $(z, t) \mapsto (z, t + \langle c, z\rangle)$, which is unimodular on $\Gamma \times \mathbb{Z}$ and replaces a by $a + c$. So only the class of (Γ, a) under these maps matters. Conversely, by (iii) and the argument of Lemma 2.1, every symmetric $(\Gamma \times \mathbb{Z})$-saturated set of lattice points of a box that spans $\mathbb{R}^4$ gives a member of $\mathcal{O}^4_{\Gamma\times\mathbb{Z}}$; such a set may, however, have $r = 4$.

Computation 7. *The boxes.* All pairs (Γ, a) with $\Gamma \subseteq \mathbb{Z}^3$ of index $d \le 6$ that satisfy (i) and (ii) of Theorem 8.2, up to the three kinds of maps above. Each Γ is listed once, by its Hermite normal form. For each triple of linearly independent points z_1, z_2, z_3 of $\Gamma \cap \{-1,0,1\}^3$, let Z be the matrix with rows z_j; the vectors $a = Z^{-1}c$ with $c \in \{0,1,\dots,|\det Z|-1\}^3$ represent every class of $\{a : Za \in \mathbb{Z}^3\}$ modulo Γ^*, because $|\det Z|\, Z^{-1}\mathbb{Z}^3 \subseteq \mathbb{Z}^3 \subseteq \Gamma^*$; the class of 0 is dropped. Result: exactly six classes, listed in Table 4. Four have $d = 1$ and two have $d = 2$; no class has $3 \le d \le 6$.

The list was computed by two programs. The first identifies equivalent pairs by a canonical form, the least image under the signed permutations and $t \mapsto -t$ with a reduced modulo Γ^*. The second uses no canonical form: it finds 33 boxes before identification and identifies them by explicit unimodular maps between them, which gives the same six classes. The second program also reports the numbers of sublattices of $\mathbb{Z}^3$ of index $1,\dots,6$, namely 1, 7, 13, 35, 31, 91, of which 1, 4, 4, 7, 0, 0 have points in $\{-1,0,1\}^3$ that span $\mathbb{R}^3$. The six classes have different numbers of lattice points, so they are pairwise inequivalent. Both programs use exact rational arithmetic.

TABLE 4. The six boxes $\Pi(\Gamma, a)$ of Computation 7 and the sets of Computation 8. For $d = 1$, $\Gamma = \mathbb{Z}^3$; for $d = 2$, $\Gamma = \{z \in \mathbb{Z}^3 : z_2 \equiv z_3 \pmod 2\}$. G is the number of lattice points of the box, and (g_1, g_2, g_3, g_4) are the Ehrhart coefficients of their convex hull. The last three columns concern the sets S with $\#S \ge 31$ and $r(S) = 3$: their number as point sets (in parentheses, up to the symmetries of the box), and the largest values of g_1 and g_2.

d	a	G	(g_1, g_2, g_3, g_4)	sets S with $r(S)=3$	largest g_1	largest g_2
1	$(0, \frac{1}{2}, \frac{1}{2})$	69	$(6, 18, 28, 16)$	2,495,033 (160,502)	6	18
1	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	67	$(\frac{19}{3}, \frac{52}{3}, \frac{80}{3}, \frac{47}{3})$	5,691,415 (122,559)	$\frac{19}{3}$	$\frac{52}{3}$
1	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	63	$(6, \frac{31}{2}, 25, \frac{31}{2})$	389,891 (66,050)	$\frac{37}{6}$	$\frac{31}{2}$
1	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$	61	$(6, \frac{44}{3}, 24, \frac{46}{3})$	75,603 (19,804)	6	$\frac{44}{3}$
2	$(\frac{1}{2}, 0, \frac{1}{2})$	39	$(5, 11, 14, 8)$	209	5	11
2	$(\frac{1}{2}, \frac{1}{4}, -\frac{1}{4})$	37	$(\frac{16}{3}, \frac{31}{3}, \frac{38}{3}, \frac{23}{3})$	42	$\frac{16}{3}$	$\frac{31}{3}$

Computation 8. *Rank three.* For each of the six boxes $\Pi = \Pi(\Gamma, a)$ of Table 4: all symmetric $(\Gamma \times \mathbb{Z})$-saturated sets S of lattice points of Π that span $\mathbb{R}^4$ and have $\#S \ge 31$ and $r(S) = 3$. Results in Table 4. In particular $g_1 \le \frac{19}{3}$ for all these sets, and the value $\frac{19}{3}$ is attained by the convex hull of all lattice points of the box with $G = 67$.

All six boxes have $r = 3$. The searches walk downwards from the set of all lattice points of the box by Lemma 6.1. A set with $r = 4$ is recorded but not expanded: by the monotonicity of r, all its subsets have $r = 4$, and they belong to Cases I and II, which are closed by Computations 1–3. No set with $r = 3$ is lost by this cut, because by Lemma 6.1 it is reached through sets that contain it, and these have $r \le 3$.

For the four large boxes two programs walk downwards and reduce each level (the sets with a given G) by the symmetries of the box: the maps of the three kinds above that carry the box onto itself, which form groups of order 32, 96, 12 and 8 for $G = 69$, 67, 63 and 61. They compute g_1 by different formulas. By Lemma 4.1 and (E2), every $P \in \mathcal{O}^4_\Lambda$ satisfies

$$\#\big(\partial(2P) \cap \Lambda\big) = L_P(2) - L_P(-2) = 8G - 8 - 12g_1; \tag{2}$$

the first program obtains g_1 from (2) by counting the lattice points of $2 \operatorname{conv} S$ and of its interior, while the second computes g_3 from a triangulation of the boundary of $\operatorname{conv} S$ and sets $g_1 = \frac{G-1}{2} - g_3$. For every set, both programs verify exactly that the facet inequalities hold on S, that S is saturated, that 0 is the only interior lattice point, and that the computed vertex list is the

exact one. Both report, for each level G, six numbers: the numbers of classes and of point sets with $r = 3$, the largest g_1 and g_2 among them, and the numbers of classes and of point sets with $r = 4$ that were cut. The two outputs agree in all six numbers on all 72 levels of the four large boxes. For the two small boxes ($G = 39$ and $G = 37$), both programs and a third exact program that uses no symmetry find 209 and 42 sets with $r = 3$; without the cut at $r = 4$, both programs find all 483 and 220 symmetric saturated subsets with $\#S \geq 31$. The largest values of g_1 on the small boxes are 5 and $\frac{16}{3}$, attained by the boxes' own polytopes.

Corollary 8.3. *Let $P \in \mathcal{O}^4$ with $r(P) \leq 3$ and $G(P) \geq 31$. Then $g_1(P) \leq 8$, with equality only if P is unimodularly equivalent to T or X. Moreover $g_1(P) \leq 7$ if $r(P) \in \{1, 2\}$, and $g_1(P) \leq \frac{19}{3}$ if $r(P) = 3$. This gives a proof of Case III that does not use Computation 5 itself (for $r \leq 2$ it reuses its programs with a different cut; see Remark 8.4).*

Proof. If $r(P) \leq 2$, Section 5 gives a pair $(\operatorname{conv} S, \Lambda)$ isomorphic to $(P, \mathbb{Z}^4)$ with $S \in \mathcal{S}(\Lambda)$, $\#S = G(P)$ and $r(S) = r(P)$, and S is one of the sets of Computation 6. If $r(P) = 3$, Theorem 8.2 gives a pair $(\operatorname{conv} S, \Gamma \times \mathbb{Z})$ isomorphic to $(P, \mathbb{Z}^4)$ with S one of the sets of Computation 8. The bounds are read off from the two computations. The only sets with $g_1 = 8$ are T and X in Computation 6(a), and the argument at the end of the proof of Theorem 7.1 shows that P is then unimodularly equivalent to T or to X. □

Remark 8.4. For $r \leq 2$ the two proofs of Case III use the same frame reduction and the same two programs, with different cuts; for $r = 3$ they use different reductions. Their counts are not comparable. Computation 5 counts 581,060 sets with $r = 3$, up to the symmetries of each frame lattice; Computation 8 counts 8,651,942 point sets in the four large boxes (368,915 up to the symmetries of the boxes) and $209 + 42$ in the two small ones. A polytope may have several frames and lie in several boxes. Both routes give the same largest value $g_1 = \frac{19}{3}$ for $r = 3$.

9. The extremal polytopes and the next values of g_1 and g_2

This section studies the three polytopes with $g_1 = 8$ and the values of g_1 and g_2 just below 8 and 24. Each of C_4, T and X has all its facets at the same lattice distance from the origin (Proposition 9.1), and no larger member of $\mathcal{O}^4$ contains X (Proposition 9.2). The cube with one pair of opposite vertices deleted has $g_1 = \frac{15}{2}$ and $g_2 = \frac{277}{12}$. The value $\frac{277}{12}$ is the second largest value of g_2 on $\mathcal{O}^4$, and $\frac{15}{2}$ is the second largest value of g_1 unless some $P \in \mathcal{O}^4$ with 29 lattice points has $\frac{15}{2} < g_1(P) < 8$ (Proposition 9.3 and the paragraph after it).

9.1. **Lattice distances.** Kasprzyk and Nill [13] introduced ℓ-reflexive polytopes. We use Henze's version [12, Definition 3.4 and Remark 3.5], which drops their condition that all vertices be primitive: we call $P \in \mathcal{O}^n$ *ℓ-reflexive* if every facet of P has lattice distance $b_F = \ell$ from the origin (Section 2). The vertices of C_4, T and X are primitive, so for these three polytopes the two definitions agree.

Proposition 9.1. *The polytopes C_4, T and X are 1-, 2- and 3-reflexive, with 8, 24 and 16 facets. For every ℓ-reflexive $P \in \mathcal{O}^4$, $\sigma(P) = 4\operatorname{vol}(P)/\ell$, and inequality (1) reads*

$$G(P) - 1 \leq 16 + \frac{4\operatorname{vol}(P)}{\ell}.$$

For C_4, T and X it holds with equality: $80 = 16 + 64$, $48 = 16 + 32$ and $32 = 16 + 16$.

Proof. For C_4 and T the facets and their lattice distances are found in Section 7. The 8 facets $\{x_i = \pm 1\}$ of C_4 have lattice distance 1. Section 7 treats T in the model $\operatorname{conv}\{\pm e_a \pm e_b\}$ with the lattice $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$, whose $8 + 16 = 24$ facets lie in the hyperplanes $\{x_i = \pm 1\}$ and $\{\pm x_1 \pm x_2 \pm x_3 \pm x_4 = 2\}$, all at lattice distance 2. Isomorphisms of pairs preserve facets and lattice distances, so the same holds for T.

For $X = \operatorname{conv}\{\pm v_1, \dots, \pm v_4\}$ with $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, $v_3 = (1,1,3,0)$, $v_4 = (2,4,3,6)$, the 16 facets are $\operatorname{conv}\{\epsilon_1 v_1, \dots, \epsilon_4 v_4\}$ for $\epsilon \in \{\pm 1\}^4$. The functional u with $\langle u, \epsilon_i v_i\rangle = 1$ for all i is

$$u = \Big(\epsilon_1,\ \epsilon_2,\ \frac{\epsilon_3 - \epsilon_1 - \epsilon_2}{3},\ \frac{\epsilon_4 - \epsilon_1 - 3\epsilon_2 - \epsilon_3}{6}\Big),$$

and the lattice distance of the facet is the least $b > 0$ with $bu \in \mathbb{Z}^4$ (then bu is primitive: if $bu = cw$ with $w \in \mathbb{Z}^4$ and an integer $c \geq 2$, then $(b/c)u = w \in \mathbb{Z}^4$, against the choice of b). The third coordinate has denominator 1 or 3. The number $\epsilon_4 - \epsilon_1 - 3\epsilon_2 - \epsilon_3$ is even, so the fourth coordinate is $m/3$ with $m = (\epsilon_4 - \epsilon_1 - 3\epsilon_2 - \epsilon_3)/2 \in \mathbb{Z}$. Thus $b \in \{1, 3\}$, and $b = 1$ would need $3 \mid \epsilon_3 - \epsilon_1 - \epsilon_2$, that is $\epsilon_1 = \epsilon_2 = -\epsilon_3 = s$, and $3 \mid m = (\epsilon_4 - 3s)/2$; but $(\epsilon_4 - 3s)/2 \in \{\pm 1, \pm 2\}$. So $b = 3$ for every facet.

If $P \in \mathcal{O}^4$ is ℓ-reflexive, (E3) gives $\operatorname{vol}(P) = \frac{1}{4}\sum_F b_F \operatorname{relvol}(F) = \frac{\ell}{4}\sigma(P)$, and by Proposition 4.2 inequality (1) is $G \leq \sigma + 17$. Finally $(G, \operatorname{vol}) = (81, 16)$, $(49, 16)$ and $(33, 12)$ for C_4, T and X. □

Equivalently, $g_3(P) = 2\operatorname{vol}(P)/\ell$ for ℓ-reflexive $P \in \mathcal{O}^4$, which is the case $n = 4$ of [12, Proposition 3.6].

9.2. **The cross-polytope X is maximal.**

Proposition 9.2. *If $P \in \mathcal{O}^4$ and $X \subseteq P$, then $P = X$.*

The proof rests on the following computation.

Computation 9. *Enlargements of X.* Let V be the matrix with columns $v_1, \dots, v_4$. The parallelepiped $V[-1,1]^4$ contains 417 points of $\mathbb{Z}^4$: the 33 lattice points of X and 192 further pairs $\pm x$. For each of these 192 pairs, $\operatorname{conv}(X \cup \{\pm x\})$ has a lattice point other than 0 in its interior.

Proof of Proposition 9.2. Since X already realizes the largest possible frame index, every lattice point of P lies in the frame parallelepiped of X, where Computation 9 applies. Since $X \subseteq P$, we have $M(P) \subseteq M(X) = \{0\}$, so $r(P) = 0$, and $G(P) \geq G(X) = 33$; so P is in Case III, and the frame construction of Section 5 applies to it. Let D be the largest value of $|\det(x_1, \dots, x_4)|$ over lattice points $x_1, \dots, x_4$ of P. The vertices $v_1, \dots, v_4$ of X give $D \geq |\det V| = 18$. By Lemma 5.1 the frame lattice of a maximizing quadruple lies in $\mathcal{L}$ and has index D, and by Computation 4 every lattice in $\mathcal{L}$ has index at most 18. Hence $D = 18$, and $v_1, \dots, v_4$ is a maximizing quadruple. By Lemma 5.1(i) every lattice point of P lies in $V[-1,1]^4$. If $P \neq X$, some vertex x of P is not in X; then $\operatorname{conv}(X \cup \{\pm x\}) \subseteq P$, and by Computation 9 it has a nonzero interior lattice point, which is interior to P as well. This contradicts $P \in \mathcal{O}^4$. □

9.3. **The next values.** Table 5 gives the result of Computation 1 level by level, and Figure 3 shows its largest values of g_1 together with the bound for $G \leq 29$.

Proposition 9.3. *Let $P \in \mathcal{O}^4$ with $G(P) \geq 31$.*

(i) *In Case I, if P is not unimodularly equivalent to C_4, then $g_1(P) \leq \frac{15}{2}$ and $g_2(P) \leq \frac{277}{12}$. Both bounds are attained only by the cube with one pair of opposite vertices deleted, which has $G = 79$ and $L = 1 + \frac{15}{2}k + \frac{277}{12}k^2 + \frac{63}{2}k^3 + \frac{191}{12}k^4$.*

(ii) *In Case II, $g_1(P) \leq 6$ and $g_2(P) \leq 14$.*

(iii) *In Case III, if P is not unimodularly equivalent to T or X, then $g_1(P) \leq 7$, with equality for T'; if moreover P is not equivalent to T', then $g_1(P) \leq \frac{20}{3}$, with equality for R_{41}. In Case III, $g_2(P) \leq 18$, with equality for the convex hull of all lattice points of the box with $G = 69$ in Table 4, which has $L = 1 + 6k + 18k^2 + 28k^3 + 16k^4$.*

Proof. (i) is read off from Table 5: the level $G = 79$ consists of one class, and every other level below 81 has smaller maxima. All vertices of C_4 are equivalent under signed permutations, so this class is C_4 with a pair of opposite vertices deleted; its volume is $16 - 2\cdot\frac{1}{24} = \frac{191}{12}$, and with $g_1 = \frac{15}{2}$

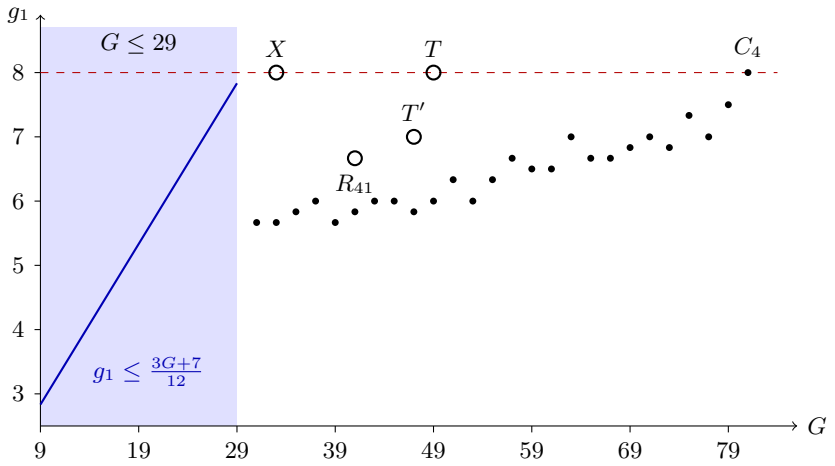


FIGURE 3. Large values of g_1 on $\mathcal{O}^4$. The dots show, for each $G \ge 31$, the largest g_1 in Case I (Table 5); the dot at $G = 81$ is the cube C_4. The circles show the polytopes of Case III with $g_1 > \frac{19}{3}$, namely T, X, T' and R_{41}; all other polytopes of Cases II and III have $g_1 \le 6$ and $g_1 \le \frac{19}{3}$, respectively (Proposition 9.3, Table 3). The dashed line is $g_1 = 8$. In the shaded region $G \le 29$, the solid line is the bound $g_1 \le \frac{3G+7}{12}$ stated after Lemma 4.3, which comes from the triangulation in its proof. Every polytope with $G \ge 31$ other than C_4, T and X has $g_1 \le \frac{15}{2}$ (Proposition 9.3).

from Table 5, Lemma 4.1 gives $g_2 = \frac{277}{12}$ and $g_3 = \frac{63}{2}$. (ii) For g_1 this is Computation 3; for g_2, the largest value over the classes of Computation 3 is 14. (iii) follows from Table 3, Computation 6, Lemma 8.1 and Table 4: the values of g_1 above $\frac{19}{3}$ in Case III are 8 (T, X), 7 (T') and $\frac{20}{3}$ (R_{41}), and the largest g_2 is 18. □

Since $g_2 = \frac{G-1}{2} - g_4 < 14$ when $G \le 29$, Proposition 9.3 shows that $\frac{277}{12}$ is the second largest value of g_2 on $\mathcal{O}^4$, attained only by the cube with a pair of opposite vertices deleted. For g_1 we can say less: the bound $g_1 \le \frac{3G+7}{12}$ stated after Lemma 4.3 gives $g_1 \le \frac{22}{3} < \frac{15}{2}$ for $G \le 27$. So every $P \in \mathcal{O}^4$ other than C_4, T and X with $G(P) \ne 29$ satisfies $g_1(P) \le \frac{15}{2}$. We do not know whether some $P \in \mathcal{O}^4$ with $G(P) = 29$ has $\frac{15}{2} < g_1(P) < 8$; the bound above gives only $g_1 \le \frac{47}{6}$ there.

10. A WIDTH-TWO APPROACH

The proof of Theorem 1.2 rests on large enumerations. In this section we describe a reduction that might replace part of them. Every $P \in \mathcal{O}^4$ with $r(P) \ge 1$ has a direction of lattice width 2, and in such a direction P is determined by two sets of lattice points in $\mathbb{R}^3$. Inequality (1) then becomes an inequality about pairs of 3-dimensional polytopes. Lemma 10.1, Proposition 10.2, Lemmas 10.3, 10.4, 10.6 and Proposition 10.5 are proved; inequality (3) below is open; the stronger inequality (4), which the reduction needs in one case, is false. None of this is used in the proof of Theorem 1.2.

10.1. **Layers.** For a polytope $Q \subset \mathbb{R}^3$ of dimension 3 we write $\#\partial Q = \#(\partial Q \cap \mathbb{Z}^3)$ for the number of lattice points on its boundary.

Lemma 10.1. *Let $P \in \mathcal{O}^4$ and $u \in M(P) \setminus \{0\}$. Then u is primitive and $\max_P \langle u, x\rangle = 1$. After a unimodular change of coordinates with $\langle u, x\rangle = x_4$, let $S_0, S_1 \subset \mathbb{Z}^3$ be the sets with $P \cap \mathbb{Z}^4 =$*

TABLE 5. Computation 1 by level: for each G, the number of classes of symmetric saturated sets $S \subseteq \{-1,0,1\}^4$ with $\#S = G$ up to the 384 signed permutations, and the largest values of g_1 and g_2 among them. In total 2,547,694 classes, of which 2,853 have $G \geq 63$. Both programs of Computation 1 give every entry.

G	classes	g_1	g_2	G	classes	g_1	g_2
81	1	8	24	55	24,940	19/3	33/2
79	1	15/2	277/12	53	42,740	6	47/3
77	3	7	133/6	51	68,657	19/3	47/3
75	6	22/3	65/3	49	103,684	6	15
73	19	41/6	83/4	47	146,480	35/6	169/12
71	45	7	61/3	45	193,898	6	14
69	123	41/6	235/12	43	239,893	6	27/2
67	299	20/3	113/6	41	278,087	35/6	51/4
65	728	20/3	217/12	39	302,042	17/3	37/3
63	1,628	7	19	37	308,825	6	35/3
61	3,517	13/2	217/12	35	298,228	35/6	34/3
59	7,134	13/2	209/12	33	273,637	17/3	32/3
57	13,760	20/3	52/3	31	239,319	17/3	31/3

$S_0 \times \{0\} \cup S_1 \times \{1\} \cup (-S_1) \times \{-1\}$, put $b = \#S_1$, $K_1 = \operatorname{conv} S_1$ and $K_0 = \operatorname{conv}\big(S_0 \cup \frac{1}{2}(S_1 - S_1)\big)$. Then:

(i) *$P \cap \{x_4 = s\} = \big((1-s)K_0 + sK_1\big) \times \{s\}$ for $0 \leq s \leq 1$, and $G(P) = \#S_0 + 2b$;*
(ii) *K_0 is symmetric and has dimension 3, $K_0 \cap \mathbb{Z}^3 = S_0$, and 0 is the only lattice point in the interior of K_0;*
(iii) *$K_1 - K_1 \subseteq 2K_0$.*

Proof. We show that the lattice points of P lie in the three hyperplanes $x_4 \in \{-1,0,1\}$ and that $x_4 = 0$ cuts P into two halves, each the convex hull of two parallel slices. If $u = mu'$ with $u' \in \mathbb{Z}^4$ and $m \geq 2$, then $|\langle u', x\rangle| < 1$, so $\langle u', x\rangle = 0$, at every lattice point x of P, which is impossible because P is the convex hull of its lattice points and has dimension 4. So u is primitive and can be completed to a basis of $\mathbb{Z}^4$. The maximum of u on P is attained at a vertex, so it is an integer; it is at most 1, and it is not 0, since otherwise $P = -P$ would lie in the hyperplane $\langle u, x\rangle = 0$; so it is 1. Every vertex of P lies at height -1, 0 or 1. Hence every vertex of $P \cap \{0 \leq x_4 \leq 1\}$ lies at height 0 or 1, so this polytope is the convex hull of its two slices at these heights, and its slice at height s is $(1-s)$ times the first plus s times the second. The slice at height 1 is the face $K_1 \times \{1\}$. A point of P at height 0 is a convex combination of lattice points of P in which the total weight μ at height 1 equals the total weight at height -1; if $\mu > 0$, the part at heights ± 1 equals 2μ times a point $\frac{1}{2}(y - y')$ with $y, y' \in K_1$, and $\frac{1}{2}(K_1 - K_1) = \operatorname{conv} \frac{1}{2}(S_1 - S_1)$. Since also $\frac{1}{2}(y, 1) + \frac{1}{2}(-y', -1) \in P$ for $y, y' \in S_1$, the slice at height 0 is $\operatorname{conv}\big(S_0 \cup \frac{1}{2}(S_1 - S_1)\big) = K_0$. This gives (i), and (iii) follows from $\frac{1}{2}(S_1 - S_1) \subseteq K_0$. Since $K_0 \times \{0\} = P \cap \{x_4 = 0\}$, the lattice points of K_0 are the points of S_0. The slice K_0 is symmetric because P is. Since 0 is an interior point of P, the hyperplane $x_4 = 0$ meets the interior of P, so K_0 has dimension 3 and its interior points are the interior points of P at height 0; the interior lattice points of P lie at height 0 because the heights ± 1 are faces. This gives (ii). □

10.2. **An exact identity and the shapes of K_1.**

Proposition 10.2. *In the situation of Lemma 10.1,*

$$12\big(8 - g_1(P)\big) = \delta_0 + 2\delta_1 + 2\delta_2,$$

where

$$\delta_0 = \#\partial(2K_0) - (4\#S_0 - 10), \qquad \delta_1 = \#\partial(K_0+K_1) - (2\#S_0 + 3b - 37), \qquad \delta_2 = L_{K_1}(2) - (5b-10).$$

Proof. The idea is to count the lattice points on the boundary of $2P$ by their heights. By (2), $12(8-g_1) = \#(\partial(2P)\cap\mathbb{Z}^4) - 8G + 104$. By Lemma 10.1(i), the slices of $2P$ at heights 0, ±1, ±2 are $2K_0$, $\pm(K_0+K_1)$ and $\pm2K_1$. The slices at heights ±2 are faces of $2P$, so all their $L_{K_1}(2)$ lattice points lie on $\partial(2P)$. The hyperplanes at heights 0 and ±1 meet the interior of $2P$, so a lattice point at these heights lies on $\partial(2P)$ if and only if it lies on the boundary of the slice. Hence $\#(\partial(2P)\cap\mathbb{Z}^4) = \#\partial(2K_0) + 2\,\#\partial(K_0+K_1) + 2L_{K_1}(2)$. Substituting this and $G = \#S_0 + 2b$ gives the identity. □

The number $12(8-g_1)$ does not depend on the direction u, so a proof may choose the direction.

Lemma 10.3. (i) *If K_0 is a lattice polytope, that is, $K_0 = \operatorname{conv} S_0$, then $\delta_0 = 0$.*

(ii) *If K_1 has dimension 3, with b_∂ lattice points on its boundary and $b_\circ \geq 1$ in its interior, then $\operatorname{NVol}(K_1) = 6\operatorname{vol}(K_1) \geq 2b_\partial + 3b_\circ - 7$, and $\delta_2 \geq 0$.*

Proof. (i) For a lattice polytope $Q \subset \mathbb{R}^3$ of dimension 3, reciprocity (E2) gives $\#\partial(kQ) = L_Q(k) + L_Q(-k) = 2 + 2g_2(Q)k^2$. For $Q = K_0$, $\#\partial K_0 = \#S_0 - 1$ by Lemma 10.1(ii), so $2g_2(K_0) = \#S_0 - 3$ and $\#\partial(2K_0) = 4\#S_0 - 10$.

(ii) We count the tetrahedra of a triangulation of K_1 that uses all its lattice points. Triangulate ∂K_1 with vertex set the b_∂ boundary lattice points, as in the proof of Lemma 4.3; by Euler's formula it has $2b_\partial - 4$ triangles. Coning them from one interior lattice point triangulates K_1 into $2b_\partial - 4$ tetrahedra. Insert the remaining $b_\circ - 1$ interior lattice points one at a time, each time subdividing the cells of the current triangulation that contain the new point in their relative interior: a tetrahedron is replaced by 4, two tetrahedra sharing a triangle by 6, and the $k \geq 3$ tetrahedra around an interior edge by $2k$. Each insertion adds at least 3 tetrahedra, so the final triangulation has at least $2b_\partial + 3b_\circ - 7$ lattice tetrahedra, each of normalized volume at least 1. For a lattice polytope of dimension 3, $L_{K_1}(1) = b$ and $L_{K_1}(-1) = -b_\circ$ give $L_{K_1}(2) = 6\operatorname{vol}(K_1) + 3b_\partial + 2b_\circ - 3$, so $\delta_2 = 6\operatorname{vol}(K_1) - 2b_\partial - 3b_\circ + 7 \geq 0$. □

Lemma 10.4. *If K_1 has dimension 3 and no interior lattice point, then $6\operatorname{vol}(K_1) \geq b - 3$, $L_{K_1}(2) = 6\operatorname{vol}(K_1) + 3b - 3$, and $\delta_2 \geq 4 - b$.*

Proof. Take a triangulation of K_1 whose vertex set is S_1; all its $f_0 = b$ vertices lie on ∂K_1, and its boundary is a triangulated 2-sphere with $3b-6$ edges and $2b-4$ triangles. Let f_1, f_2, f_3 be the numbers of edges, triangles and tetrahedra, and e, t the numbers of interior edges and triangles. Euler's formula for the ball, $f_0 - f_1 + f_2 - f_3 = 1$, gives $t = f_3 + e - 1$, and counting pairs (triangle, tetrahedron) gives $4f_3 = 2t + 2b - 4$. Hence $f_3 = b - 3 + e \geq b - 3$, and $6\operatorname{vol}(K_1) \geq f_3$. The formula for $L_{K_1}(2)$ is that of Lemma 10.3(ii) with $b_\circ = 0$, and $\delta_2 = 6\operatorname{vol}(K_1) - 2b + 7 \geq 4 - b$. □

Proposition 10.5. *In the situation of Lemma 10.1, $g_1(P) \leq 8$ holds whenever $\delta_0 + 2\delta_1$ is at least the number in the last column of Table 6, for the row that describes K_1.*

Proof. By Proposition 10.2, $g_1 \leq 8$ if and only if $\delta_0 + 2\delta_1 \geq -2\delta_2$, so it suffices that $\delta_0 + 2\delta_1$ is at least -2 times the lower bound for δ_2. The first two rows are Lemmas 10.3(ii) and 10.4. For a lattice polygon with b lattice points, β of them on the boundary, Pick's theorem [3] gives $L(k) = (b - \frac{\beta}{2} - 1)k^2 + \frac{\beta}{2}k + 1$, hence $L(2) = 4b - \beta - 3$. A lattice segment with b lattice points has $L(2) = 2b - 1$, and a point has $L(2) = 1$. □

So inequality (1) for all P with $r(P) \geq 1$ would follow from the inequality

$$\#\partial(2K_0) + 2\,\#\partial(K_0+K_1) \;\geq\; 8\#S_0 + 6b - 84 \qquad (\iff \delta_0 + 2\delta_1 \geq 0) \tag{3}$$

TABLE 6. The bound for δ_2 by the shape of $K_1 = \operatorname{conv} S_1$, $b = \#S_1$, and the resulting sufficient condition for $g_1 \le 8$. For a polygon, β is the number of lattice points on its boundary.

K_1	$L_{K_1}(2)$	δ_2	sufficient: $\delta_0 + 2\delta_1 \ge$
dimension 3, $b_\circ \ge 1$	$6\operatorname{vol}(K_1) + 3b_\partial + 2b_\circ - 3$	≥ 0	0
dimension 3, $b_\circ = 0$	$6\operatorname{vol}(K_1) + 3b - 3$	$\ge 4 - b$	$2b - 8$
polygon	$4b - \beta - 3$	$= 7 - b - \beta$	$2(b + \beta - 7)$
segment	$2b - 1$	$= 9 - 3b$	$6b - 18$
point ($b = 1$)	1	$= 6$	-12

when K_1 has interior lattice points, from

$$\#\partial(2K_0) + 2\,\#\partial(K_0 + K_1) \;\ge\; 8\#S_0 + 8b - 92 \qquad (\iff \delta_0 + 2\delta_1 \ge 2b - 8) \tag{4}$$

when K_1 is 3-dimensional without interior lattice points, and from the inequalities of the last three rows of Table 6 when K_1 is flat (of dimension at most 2). Inequality (3) is sharp: if $K_0 = \operatorname{conv} S_0$ and $K_1 = K_0 + v$ for some $v \in \mathbb{Z}^3$, then $\delta_0 = 0$ and $\delta_1 = 27 - \#S_0$ by Lemma 10.3(i), so (3) reads $\#S_0 \le 27$. This holds, because the lattice points of K_0 are pairwise incongruent modulo 3: if $x \ne y$ were congruent, then $\frac{1}{3}(x - y) = \frac{2}{3} \cdot \frac{1}{2}(x - y) + \frac{1}{3} \cdot 0$ would be a nonzero lattice point in the interior of K_0. Equality holds for $K_0 = K_1 - v = [-1, 1]^3$, that is, for $P = C_4$.

The next lemma shows which shapes of K_1 can occur.

Lemma 10.6. *In the situation of Lemma 10.1, let* $M_0 = \{w \in \mathbb{Z}^3 : |\langle w, y\rangle| \le 1 \text{ for all } y \in K_0\}$ *and* $r_0 = \dim \operatorname{span} M_0$.

(i) $r(P) = 1 + r_0$.
(ii) *If* $r(P) = 4$, *then* K_1 *has at most one interior lattice point.*
(iii) *The interior lattice points of* K_1 *are pairwise incongruent modulo* 2*; hence there are at most* 8 *of them.*

Proof. Write integer functionals on $\mathbb{Z}^4$ as $(w, s) \in \mathbb{Z}^3 \times \mathbb{Z}$, acting by $(y, t) \mapsto \langle w, y\rangle + st$. Then $(w, s) \in M(P)$ if and only if $|\langle w, y\rangle| \le 1$ for $y \in S_0$ and $|\langle w, y\rangle + s| \le 1$ for $y \in S_1$.

(i) If $(w, s) \in M(P)$, then $w \in M_0$: the bound holds on S_0, and for $y, y' \in S_1$, $\langle w, \frac{1}{2}(y - y')\rangle = \frac{1}{2}((\langle w, y\rangle + s) - (\langle w, y'\rangle + s)) \in [-1, 1]$. Conversely, let $w \in M_0$. By Lemma 10.1(iii), $\langle w, y - y'\rangle \le 2$ for $y, y' \in S_1$, so the values $\langle w, y\rangle$, $y \in S_1$, lie in an interval $[m, m + 2]$ with $m \in \mathbb{Z}$, and $(w, -m - 1) \in M(P)$. Since also $(0, \pm 1) \in M(P)$, the span of $M(P)$ is spanned by $(0, 1)$ and the vectors $(w, -m_w - 1)$, $w \in M_0$, and has dimension $1 + r_0$.

(ii) If $r(P) = 4$, then M_0 spans $\mathbb{R}^3$ by (i). Suppose that $p \ne q$ are interior lattice points of K_1, and choose $w \in M_0$ with $\langle w, p\rangle > \langle w, q\rangle$, so $\langle w, p\rangle \ge \langle w, q\rangle + 1$. Since p and q are interior and the vertices of K_1 are lattice points, the maximum of w on K_1 is an integer larger than $\langle w, p\rangle$, and the minimum is an integer smaller than $\langle w, q\rangle$; so the width of K_1 in the direction w is at least 3. But $K_1 - K_1 \subseteq 2K_0$ and $|w| \le 1$ on K_0 give width at most 2.

(iii) If $p \ne q$ are interior lattice points of K_1 with $p \equiv q \pmod 2$, then $\frac{1}{2}(p - q)$ is a nonzero lattice point in the interior of $\frac{1}{2}(K_1 - K_1) \subseteq K_0$, which contradicts Lemma 10.1(ii). There are 8 classes modulo 2. □

10.3. **What is known about the two inequalities.** Inequality (3) is not proved. It is an equality for the cube.

Inequality (4) is false. For T' the direction $u = (1, 1, 0, 1)$ of Lemma 8.1 gives $\#S_0 = 19$ and $b = 14$; K_1 has dimension 3 and no interior lattice point, and K_0 is not a lattice polytope. Here

$\#\partial(2K_0) = 50$, $\#\partial(K_0 + K_1) = 54$, $L_{K_1}(2) = 63$ and $6\,\mathrm{vol}(K_1) = 24$, so

$$\delta_0 = -16, \qquad \delta_1 = 11, \qquad \delta_2 = 3, \qquad \delta_0 + 2\delta_1 + 2\delta_2 = 12 = 12\big(8 - g_1(T')\big).$$

Thus $\delta_0 + 2\delta_1 = 6 < 20 = 2b - 8$. The loss comes from Lemma 10.4: the bound $6\,\mathrm{vol}(K_1) \geq b - 3 = 11$ is far from the true value 24. The bound $6\,\mathrm{vol}(K_1) \geq b - 3$ of Lemma 10.4 is attained for every $b \geq 4$, by $\mathrm{conv}\{0, (b-3)e_1, e_2, e_3\}$, so a proof along these lines needs a lower bound for $\mathrm{vol}(K_1)$ that uses more than the absence of interior lattice points, for example the inclusion $K_1 - K_1 \subseteq 2K_0$ of Lemma 10.1(iii).

By Lemma 10.6(ii), in Cases I and II the first row of Table 6 occurs only with $b_\circ = 1$.

In summary: the identity of Proposition 10.2, the bounds of Table 6 and Lemma 10.6 are proved; (3) is open; (4) is false, so the row of Table 6 for 3-dimensional K_1 without interior lattice points needs a different argument. Computations 6 and 8 close the cases $r \leq 3$ anyway. A proof of (3), of a correct replacement for (4) and of the flat rows would prove (1) for every P with $r(P) \geq 1$ without enumeration; the remaining case $r(P) = 0$ is Computation 6(a).

11. The counterexample in dimension five and how it was found

The polytope W of Theorem 1.3 was obtained in two steps. A randomized local search over a class of layered polytopes found a polytope W_5 with 58 vertices and $g_1(W_5) = \frac{301}{30} > 10$; a greedy reduction then deleted four pairs of opposite vertices. This section describes both steps and shows that no single further pair of vertices can be deleted from W without losing the property $g_1 > 10$.

11.1. **Layered polytopes.** For $n \geq 2$, a symmetric set $Y_0 \subseteq \{-1, 0, 1\}^{n-1}$ and an arbitrary set $Y_1 \subseteq \{-1, 0, 1\}^{n-1}$ let

$$P(Y_0, Y_1) = \mathrm{conv}\big(Y_0 \times \{0\} \;\cup\; Y_1 \times \{1\} \;\cup\; (-Y_1) \times \{-1\}\big) \subset \mathbb{R}^n.$$

This is a lattice polytope in C_n, and $P(Y_0, Y_1) = -P(Y_0, Y_1)$ whether or not Y_1 is symmetric. If it has dimension n, it belongs to $\mathcal{O}^n$ by Lemma 2.1. Henze's polytopes are of this form: $P_n = P(\{-1, 1\}^{n-1}, \{\pm e_1, \ldots, \pm e_{n-1}\})$ is the convex hull of an $(n-1)$-dimensional cube at height 0 and two $(n-1)$-dimensional cross-polytopes at heights 1 and -1 [12, proof of Theorem 2.1]. Henze showed $g_1(P_7) = \frac{1534}{105} > 14$; in dimension 5, a direct count gives $g_1(P_5) = \frac{22}{3} < 10$.

11.2. **A randomized local search.** Fix n and the index i of a coefficient, and let $m = n - 1$. The search keeps a pair (Y_0, Y_1) as above and changes it by one point or one pair at a time.

(i) *Start.* $Y_0 = \{-1, 1\}^m$ and $Y_1 = \{\pm e_1, \ldots, \pm e_m\}$, so that the first polytope is P_n.

(ii) *Move.* With probability $\frac{3}{4}$, choose a point $y \in \{-1, 0, 1\}^m$ uniformly at random and add it to Y_1 if it is not in Y_1, or remove it if it is. Otherwise choose a pair $\{y, -y\}$ with $y \in \{-1, 0, 1\}^m \setminus \{0\}$ uniformly at random and add it to Y_0 or remove it from Y_0 in the same way.

(iii) *Evaluation.* If the new polytope $P(Y_0, Y_1)$ does not span $\mathbb{R}^n$, the move is rejected. Otherwise its Ehrhart polynomial is computed exactly: the program finds the facets with integer normals, counts the lattice points of kP and of $\mathrm{int}(kP)$ for $1 \leq k \leq \lceil n/2 \rceil + 1$, checks that 0 is the only interior lattice point of P, and interpolates by (E2); the extra values serve as a check.

(iv) *Acceptance.* The move is accepted if the new value of g_i is at least the current one, and otherwise with probability 0.03. The best polytope seen is recorded.

With $n = 5$ and $i = 1$ the search produced the polytope W_5 below; it is a heuristic and need not reach $g_1 > 10$.

11.3. **The polytope W_5.** Let

$$q_1 = (0,1,-1,-1,0), \quad q_2 = (0,1,-1,0,1), \quad q_3 = (1,-1,0,-1,0), \quad q_4 = (1,0,1,0,1),$$

and let $W_5 = \operatorname{conv}\big(W \cup \{\pm q_1, \pm q_2, \pm q_3, \pm q_4\}\big)$, the convex hull of the 50 points of Table 2 and the 8 points $\pm q_j$. In the notation of the search, $W_5 = P(Y_0, Y_1)$ for a set Y_0 of 18 pairs and a set Y_1 of 25 points; of these 86 points, 58 are vertices, 22 at height 0 and 18 at each of the heights ± 1.

Proposition 11.1. *The polytope W_5 belongs to $\mathcal{O}^5$. All 58 points $\pm p$, $\pm q_j$ are vertices of W_5, and W_5 has 102 facets, at lattice distances 1, 3, 4, 5, 6, 7, 8, 9, 10, 12 and 17 from the origin. Its Ehrhart polynomial is*

$$L_{W_5}(k) = 1 + \tfrac{301}{30}\,k + \tfrac{47}{4}\,k^2 + \tfrac{53}{2}\,k^3 + \tfrac{197}{4}\,k^4 + \tfrac{397}{15}\,k^5;$$

in particular $g_1(W_5) = \frac{301}{30} > 10$.

Proof. As for W: the 58 points lie in $\{-1,0,1\}^5$, form a symmetric set and span $\mathbb{R}^5$, so $W_5 \in \mathcal{O}^5$ by Lemma 2.1. Counting lattice points gives $G(W_5) = 125$, $\#(2W_5 \cap \mathbb{Z}^5) = 1915$, $\#(\operatorname{int}(2W_5) \cap \mathbb{Z}^5) = 243$ and $\operatorname{vol}(W_5) = \frac{397}{15}$, so $A(W_5) = 1079$ and, by Proposition 3.1, $g_1(W_5) = \big(4 \cdot 126 + 24 \cdot \frac{397}{15} - 1079\big)/6 = \frac{301}{30}$. The values $L_{W_5}(k)$ for $k = 0, \dots, 6$ are 1, 125, 1915, 11273, 41635, 117147, 275841; those for $k \le 5$ determine the polynomial, and the value for $k = 6$ agrees with it. The polynomial can be computed with Normaliz [18] or LattE [14]. □

Remark 11.2. By Proposition 3.1, g_1 grows with the volume when $G(P)$ and $A(P)$ are fixed. Given $G(W_5) = 125$ and $A(W_5) = 1079$, the condition $g_1(W_5) > 10$ is equivalent to $\operatorname{vol}(W_5) > \frac{635}{24}$, and $\operatorname{vol}(W_5) = \frac{397}{15}$ exceeds $\frac{635}{24}$ by exactly $\frac{1}{120} = \frac{1}{5!}$. Since the volume of a lattice polytope in $\mathbb{R}^5$ is an integer multiple of $\frac{1}{5!}$, this is the smallest possible excess. For W the excess is $\frac{3}{120} = \frac{1}{40}$, and $g_1(W) - 10 = \frac{1}{10}$, three times $g_1(W_5) - 10 = \frac{1}{30}$.

Remark 11.3. Henze tested polytopes of two kinds in dimensions 4, 5 and 6 [12, Remark 2.2]: convex hulls of lower-dimensional cubes or cross-polytopes at heights 0, -1 and 1, and Hanner polytopes. In dimension 5 the former have at most $3 \cdot 16 = 48$ vertices and the latter at most $2^5 = 32$. Since W_5 and W have 58 and 50 vertices, neither belongs to these families. Among the 3840 signed permutations of the coordinates, only the identity and $x \mapsto -x$ map W_5 to itself, and the same holds for W.

11.4. **From W_5 to W.** Starting from W_5, a greedy reduction deletes pairs of opposite vertices one at a time. At each step it considers all pairs $\pm v$ of vertices whose deletion leaves a polytope with $g_1 > 10$, and deletes one for which this value of g_1 is largest; it stops when no deletion keeps $g_1 > 10$. It uses the exact Ehrhart computation of the search. It deleted the four pairs $\pm q_1, \dots, \pm q_4$ and stopped at W. Two of the deleted pairs lie at height 0 and two at heights ± 1 (the height is the last coordinate); W has 9 vertex pairs at height 0 and 16 at heights ± 1.

Proposition 11.4. *For each of the 25 points p of Table 2, the polytope $W^{(p)}$, the convex hull of the vertices of W other than $\pm p$, belongs to $\mathcal{O}^5$, has 48 vertices, and satisfies $g_1(W^{(p)}) \le 10$. Equality holds exactly for $p = (1,0,-1,0,-1)$ and $p = (1,0,0,1,1)$, and the smallest value, $\frac{43}{6}$, occurs for $p = (1,1,-1,-1,-1)$. Table 7 lists all 25 values.*

Proof. Each $W^{(p)}$ spans $\mathbb{R}^5$, so it belongs to $\mathcal{O}^5$ by Lemma 2.1. The values of g_1 were computed by an exact program that finds the facets by testing every set of five vertices and verifying the resulting inequalities in integers, counts the lattice points of $kW^{(p)}$ directly for $k = 0, \dots, 6$, interpolates from $k \le 5$, checks the value at $k = 6$, and checks (E2) for $k = 1, 2$. The inequality $g_1(W^{(p)}) \le 10$ for all p is also the stopping condition of the greedy reduction, which used a different exact program. Normaliz gives the same 25 values. □

TABLE 7. The 25 polytopes $W^{(p)}$ obtained from W by deleting one pair $\pm p$ of opposite vertices ($\bar{1}$ stands for -1), and their linear Ehrhart coefficients. Each $W^{(p)}$ belongs to $\mathcal{O}^5$ and has 48 vertices.

p	$g_1(W^{(p)})$	p	$g_1(W^{(p)})$
$(0,0,1,\bar{1},\bar{1})$	99/10	$(1,0,\bar{1},1,1)$	134/15
$(0,0,1,1,\bar{1})$	19/2	$(1,0,0,\bar{1},1)$	149/15
$(0,1,\bar{1},0,0)$	49/5	$(1,0,0,1,1)$	10
$(0,1,0,1,\bar{1})$	271/30	$(1,0,1,\bar{1},0)$	299/30
$(0,1,0,1,1)$	281/30	$(1,0,1,\bar{1},1)$	47/5
$(0,1,1,\bar{1},\bar{1})$	131/15	$(1,0,1,0,\bar{1})$	97/10
$(0,1,1,\bar{1},0)$	49/5	$(1,1,\bar{1},\bar{1},\bar{1})$	43/6
$(1,\bar{1},\bar{1},\bar{1},0)$	127/15	$(1,1,\bar{1},\bar{1},0)$	146/15
$(1,\bar{1},0,1,0)$	49/5	$(1,1,\bar{1},1,0)$	48/5
$(1,\bar{1},0,1,1)$	83/10	$(1,1,1,\bar{1},0)$	143/15
$(1,\bar{1},1,0,\bar{1})$	259/30	$(1,1,1,\bar{1},1)$	271/30
$(1,0,\bar{1},0,\bar{1})$	10	$(1,1,1,1,0)$	87/10
$(1,0,\bar{1},1,\bar{1})$	93/10		

The two polytopes with $g_1 = 10$ are members of $\mathcal{O}^5$, different from the cube, whose linear coefficient equals that of the cube. Proposition 11.4 shows that W is minimal for the deletion of one vertex pair. We do not know the least number of vertices of a polytope $P \in \mathcal{O}^5$ with $g_1(P) > 10$.

12. The computations in detail

Table 8 lists, for each computational claim that a proof in this paper uses, the two programs that establish it and how their outputs were compared. This section describes what the enumerations of Computations 1–6 and 8 have in common (exact arithmetic, walks, frame searches and lattice lists) and why each is exact. The programs for W, W_5 and the deletions from W are described in Sections 3 and 11, and those for the boxes in Section 8. The programs, their outputs and one command per claim are in the repository [19].

12.1. **Exact arithmetic.** All programs compute facets with integer normal vectors and integer right-hand sides, so that membership, interiority and saturation are decided by comparisons of integers, and they compute volumes and Ehrhart coefficients as rational numbers. Each enumeration program stops with an error if a set fails a check: in particular, if $\#(\operatorname{conv} S \cap \Lambda) \neq \#S$ or if $\operatorname{conv} S$ has an interior lattice point other than 0. The Ehrhart coefficients are computed in two ways: as described at the beginning of Section 6, and by counting the lattice points of $k \operatorname{conv} S$ and of its interior for small k and interpolating by (E2).

12.2. **Walks.** Computations 1, 3 and 8 walk downwards from a largest set. A level is the set of classes with a given G; the children of a set Q are the sets $Q \setminus \{v, -v\}$ for the vertices v of $\operatorname{conv} Q$, and each level is reduced by a canonical form under a group of symmetries before the next level is formed. By Lemma 6.1 this reaches every saturated symmetric subset, through sets that contain it. Reducing a level by the group loses nothing: a symmetry γ carries the vertices of $\operatorname{conv} Q$ to those of $\operatorname{conv} \gamma(Q)$, so it carries the children of Q onto the children of $\gamma(Q)$; hence every set reached from Q is, up to symmetry, reached from the representative of the class of Q, and since $\#S$ is invariant, each class lives on one level. Stopping below $G = 31$ loses no set with $G \geq 31$. The vertices must be found exactly: a missed vertex would lose children, and a non-vertex would produce an unsaturated child. In the first program of Computation 1, every class is checked

TABLE 8. The computations, the two programs for each, and the comparison of their outputs.

Claim	First program	Second program	Compared
$W \in \mathcal{O}^5$ and L_W (Theorem 1.3)	facets from all sets of 5 vertices, verified in integers; direct counts of kW, $k \leq 6$	Normaliz 3.10.2; LattE integrale 1.7.6; the exact counter of the search	all six coefficients; the 92 facets
W_5 and L_{W_5} (Proposition 11.1)	the exact counter of the search	Normaliz; LattE; two further exact counters; the first program of the row above	all coefficients; 102 facets
deletions of W (Proposition 11.4)	the first program of the first row	the greedy reduction (bound $g_1 \leq 10$ only); Normaliz (all 25 values)	the bound $g_1 \leq 10$
Case I (Computation 1)	top-down walk; canonical form under 384 signed permutations; exact check of every class	top-down walk; canonical bit masks; Ehrhart polynomial by counting	per level $G = 81, \ldots, 31$: classes, largest g_1, g_2 (26 levels)
index bound (Computation 2)	all sublattices of index 2 to 16 by Hermite normal forms	an independent enumeration of the same	40,406 sublattices; maximum per index
Case II (Computation 3)	top-down walk in all 11 lattices Γ_I with $\lvert I\rvert \geq 2$	the walker of Case I, for $\Gamma_2, \Gamma_3, \Gamma_4$	classes 531, 58, 27; largest g_1, g_2
frame lattices (Computation 4)	all lattices of index $D \leq 24$ (one more than Lemma 5.1(v) needs) by Hermite normal forms	the same, independently written	number of lattices per D; 39 orbits
Case III (Computation 5)	maximal sets by branching, then a top-down walk; cut $r = 4$	sandwich search with include/exclude branching; cut $r(S^+) = 4$	per lattice: counts, ranks, largest g_1, g_2, sets with $g_1 \geq 8$
$r \leq \rho$, $\rho = 0, 1, 2$ (Computation 6)	as above, cut $r > \rho$	as above, cut $r(S^+) > \rho$	the lists for $\rho = 0, 1$; per lattice, the counts for $\rho = 2$
boxes (Computation 7)	canonical form	explicit unimodular maps	the six classes and their G
rank three (Computation 8)	top-down walk reduced by the symmetries of the box; g_1 by (2)	independent top-down walk with its own symmetry computation; g_3 by triangulation	large boxes: six numbers per level (72 levels); small boxes: 209 and 42 sets
T, X; X maximal (Section 7, Proposition 9.2)	two exact counters; growth of X by all 192 pairs	Normaliz; the walker of Case I; an independent check of the 192 pairs	polynomials; none of the 192 pairs gives a member of $\mathcal{O}^4$

exactly: the facet inequalities hold on S, the cube points satisfying them are exactly S, the origin is the only interior lattice point, and the exact vertex list equals the list found by the convex hull routine. The second program determines the facets by an exact double description, declares a point a vertex when the normals of the facets through it have rank 4, and asserts $L(1) = \#S$ and a single interior lattice point. Lower-dimensional children are skipped, since their subsets are lower-dimensional too; in Computation 1 no set with $G \geq 31$ is lower-dimensional.

12.3. **Frame searches.** The first program of Computations 5 and 6 finds the maximal members of $\mathcal{S}(\Lambda)$ by a branching search. A member S is maximal if and only if no single pair $\pm x$ can be added: if $S \subsetneq S' \in \mathcal{S}(\Lambda)$ and $\pm x \in S' \setminus S$, then $\mathrm{conv}(S \cup \{\pm x\}) \cap \Lambda \subseteq S'$ satisfies (i)–(iii) of Lemma 5.1, because S' does.

The search adds one pair at a time, prunes by a determinant test (a point x with $|\det(x, s_1, s_2, s_3)| > 1$ for points s_j of the current set cannot be added), and breaks symmetry as follows. As in the Bron–Kerbosch algorithm [6], it keeps, besides the current set S, a set of candidates and a set of excluded points; at each node the candidates are taken in a fixed order, and a candidate that is the image, under the stabilizer of S, of an earlier candidate that was branched on is not branched on but goes to the excluded set. No maximal set is lost. Without symmetry breaking, the search finds a maximal set R along one path: at each node it enters the branch of the first candidate in R. (A set R is found below a node only if R contains the current set of that node and none of its excluded points; a set that contains an excluded point is found in an earlier sibling branch, at a smaller position.) Compare two such paths by the positions of these candidates, lexicographically. Among the images of a maximal set under the symmetries of Λ, take the one, Q^*, whose path is least. If the branch of this path at some node were skipped because its point x_i equals $\gamma(x_j)$ with $j < i$ and γ in the stabilizer of the current set, then $\gamma^{-1}(Q^*)$ would be an image whose path is smaller: it contains the current set, because γ fixes it, and hence every point chosen above the node, so its path either leaves the path of Q^* earlier, at a smaller position, or reaches the node and there takes a candidate at position at most j. This contradicts the choice of Q^*. So every maximal set is found, up to the symmetries of Λ. From the maximal sets the program walks downwards by Lemma 6.1, never deleting $\pm e_i$.

The second program is the sandwich search of Section 6; in Computation 6 it also cuts the nodes with $r(S^+) > \rho$, by the monotonicity of r. Equivalent nodes are skipped by a canonical form under the symmetries of Λ; this loses no set, because the sets found below a node depend only on the node. The order of branching changes only the running time.

12.4. **Lattice lists.** In Computations 2 and 4 every lattice is listed exactly once by its Hermite normal form. Computation 2 considers 40,406 sublattices of index 2 to 16; the largest numbers of points of $\{-1, 0, 1\}^4$ in a sublattice of index d whose cube points span $\mathbb{R}^4$ are 45, 27, 27, 17, 15, 11, 17 for $d = 2, \ldots, 8$ and 9 for $d = 9, 10, 12, 16$, and no sublattice of index 11, 13, 14 or 15 has cube points that span $\mathbb{R}^4$. Computation 4 finds 1, 11, 24, 73, 56, 160, 96, 229, 72, 96, 128, 86, 8 lattices of index $D = 1, \ldots, 10, 12, 16, 18$ (in total 1,040), forming 1, 3, 2, 7, 2, 4, 1, 9, 2, 1, 2, 4, 1 orbits under signed permutations, and none of any other index $D \leq 24$. Both programs give these numbers.

12.5. **Comparison.** For each claim the two outputs agree: level by level in the walks, lattice by lattice in the frame searches (the lattices being matched by their Hermite normal forms), and, for the boxes, level by level in six numbers, including the cut sets. In Computation 1 both programs flag the cube, and only the cube, as a set with $g_1 = 8$; in Computation 5 both flag exactly T and X.

12.6. **Counting conventions.** The counts in this paper are numbers of classes under the symmetry groups named in each computation: the 384 signed permutations in Computation 1, the stabilizer of Γ_m in Computation 3, the stabilizer of each frame lattice, summed over the 39 lattices, in Computations 5 and 6, and point sets or classes under the symmetries of each box in Computation 8. The same polytope may be counted in more than one computation, or under more than one frame or box. None of these totals is a number of unimodular equivalence classes of polytopes.

Acknowledgments

This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Google's Gemini, Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

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