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WILLS' (1981) CONJECTURE HOLDS IF AND ONLY IF THE DIMENSION IS AT MOST 4

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Abstract. Let P be a lattice polytope in $\mathbb{R}^n$ that is symmetric about the origin and has the origin as its only interior lattice point. In 1981 Wills conjectured that each coefficient $g_i(P)$ of the Ehrhart polynomial of P is at most the corresponding coefficient of the cube $[-1,1]^n$, that is, $g_i(P) \le 2^i\binom{n}{i}$. Wills proved this for $n \le 3$, and Henze showed in 2014 that it fails for every $n \ge 7$; the dimensions 4, 5 and 6 remained open. We settle them. In dimension 4 the conjecture holds; the new bounds are $g_1(P) \le 8$ and $g_2(P) \le 24$, and, up to unimodular equivalence, $g_1(P) = 8$ only for the cube, one explicit 24-cell and one explicit cross-polytope. In dimension 5 we give a polytope with 50 vertices in $[-1,1]^5$ whose linear Ehrhart coefficient is $101/10 > 10$; products with segments then give counterexamples in every dimension $n \ge 6$. Hence Wills' conjecture holds in $\mathbb{R}^n$ if and only if $n \le 4$. The proof in dimension 4 combines Ehrhart reciprocity, the lower bound theorem for triangulated 3-spheres and an exhaustive computer enumeration of about three million sets of lattice points, listed up to symmetry.*

1. Introduction

A *lattice polytope* in $\mathbb{R}^n$ is the convex hull of finitely many points of $\mathbb{Z}^n$. Ehrhart [6] proved that for a lattice polytope P of dimension n the number of lattice points in the dilate kP is a polynomial in the integer $k \ge 0$,

$$L_P(k) = \#(kP \cap \mathbb{Z}^n) = g_0(P) + g_1(P)\,k + \cdots + g_n(P)\,k^n,$$

the *Ehrhart polynomial* of P. Here $g_0(P) = 1$, the leading coefficient $g_n(P)$ is the volume of P, and $g_{n-1}(P)$ is half the lattice surface area of P (Section 2); the remaining coefficients have no such simple description and may be negative [3].

Minkowski's first theorem [14] states that a convex body $K \subset \mathbb{R}^n$ that is symmetric about the origin and has no lattice point other than 0 in its interior has volume at most 2^n, the volume of the cube $C_n = [-1,1]^n$. Wills [20] looked for a discrete analog of this theorem, one that bounds every Ehrhart coefficient of a lattice polytope. Let $\mathcal{O}^n$ be the set

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*Additionally, Theorems 1.2, 1.3 and 7.1 and Corollaries 1.4 and 7.2 are derived in Lean 4 from six published results (Ehrhart's theorem, the properties (E1)–(E3) of Ehrhart polynomials, the boundary triangulation used in Lemma 4.3, and Minkowski's theorem on lattice tilings used in Lemma 5.1) and five computer checks (Computations 1, 3, 4 and 5, and the volume of W), each stated as an explicit hypothesis, and checked with `lake comparator`: https://github.com/david44hello/paper_wills-1981-conjecture/tree/v3.32

An expanded version of this paper, with further results, is [7], at https://doi.org/10.5281/zenodo.23955454

of lattice polytopes $P \subset \mathbb{R}^n$ with $P = -P$ whose only interior lattice point is the origin. The cube C_n belongs to $\mathcal{O}^n$, and $L_{C_n}(k) = (2k+1)^n$, so $g_i(C_n) = 2^i\binom{n}{i}$. Wills called these numbers the expected values of the coefficients [20, Remark (b)], and the following statement is known as Wills' conjecture. We state it as in [11, Conjecture 1.1]; see also [8].

Conjecture 1.1 (Wills' conjecture)**.** *For every $n \geq 1$, every $P \in \mathcal{O}^n$ and every $i \in \{0, 1, \ldots, n\}$,*

$$g_i(P) \leq g_i(C_n) = 2^i\binom{n}{i}.$$

For $i = 0$ there is equality, and for $i = n$ the inequality is Minkowski's theorem. Wills proved the case $i = n-1$ in every dimension [20, Theorem (i)]; hence the conjecture holds for $n \leq 2$. Wills also proved it for $n = 3$ [20, Theorem (i), (ii)], and, in dimension 4, the bounds $g_1(P) < 21$, $g_2(P) < 31$ and $g_2(P) \leq g_1(P) + 16$ [20, Theorem (iii)]. A discrete version of Minkowski's first theorem by Betke, Henk and Wills [4, Theorem 2.1] implies that $L_P(k) \leq L_{C_n}(k)$ for every $P \in \mathcal{O}^n$ and every integer $k \geq 0$, which is the inequality of the conjecture for the values of the two polynomials instead of their coefficients. For $i = n-1$, a bound in terms of the successive minima that implies Wills' bound is due to Henk, Schürmann and Wills [10]. Henze [11] constructed a family of polytopes P_n and showed that $P_7 \in \mathcal{O}^7$ has $g_1(P_7) = 1534/105 > 14$; taking products with segments, he obtained counterexamples in every dimension $n \geq 7$. On the positive side, Henze proved the conjecture for every $P \in \mathcal{O}^n$ whose Ehrhart roots all have real part $-1/2$ [11, Corollary 3.3]; by a theorem of Bey, Henk and Wills [5], every such polytope is reflexive. About the remaining dimensions, Henze wrote [11, Remark 2.2]: "In dimensions 4, 5 and 6 we tested Wills' conjecture on lattice polytopes that have a similar structure to P_n. [...] None of these examples turned out to be a counterexample." We settle these three dimensions.

TABLE 1. The best results before this paper and the results of this paper.

Best result before this paper	New result
$g_1(P) < 21$ [20, Theorem (iii)]	$g_1(P) \leq 8$ for every $P \in \mathcal{O}^4$, with equality exactly for C_4, a 24-cell T and a cross-polytope X, up to unimodular equivalence (Theorems 1.2 and 7.1)
$g_2(P) < 31$ and $g_2(P) \leq g_1(P) + 16$ [20, Theorem (iii)]	$g_2(P) \leq 24$ for every $P \in \mathcal{O}^4$, with equality exactly for C_4, up to unimodular equivalence (Theorem 1.2 and Corollary 7.2)
no counterexample known for $n \leq 6$; the tested families in dimensions 4, 5, 6 contain none [11, Remark 2.2]	the polytope W of Table 2 is in $\mathcal{O}^5$ and has $g_1(W) = 101/10 > 10$; counterexamples in every dimension $n \geq 5$ (Theorem 1.3)
true for $n \leq 3$ [20]; false for $n \geq 7$ [11]	Conjecture 1.1 holds in $\mathbb{R}^n$ if and only if $n \leq 4$ (Corollary 1.4)

Results. Our main results are Theorems 1.2 and 1.3 and Corollary 1.4 below, and Theorem 7.1 in Section 7, which determines the polytopes $P \in \mathcal{O}^4$ with $g_1(P) = 8$. Table 1 compares them with the best earlier results, and Figure 1 shows the status of the conjecture in each dimension. Two of the three polytopes with $g_1(P) = 8$ have Ehrhart roots off the line $\operatorname{Re} z = -\frac{1}{2}$, so Henze's theorem does not cover them (Remark 7.3).

Theorem 1.2. *Every $P \in \mathcal{O}^4$ satisfies $g_1(P) \le 8$ and $g_2(P) \le 24$.*

Theorem 1.3. *The polytope $W \subset \mathbb{R}^5$ of Table 2 belongs to $\mathcal{O}^5$, and*

$$L_W(k) = 1 + \tfrac{101}{10}\,k + \tfrac{127}{12}\,k^2 + \tfrac{139}{6}\,k^3 + \tfrac{557}{12}\,k^4 + \tfrac{386}{15}\,k^5.$$

In particular $g_1(W) = 101/10 > 10 = g_1(C_5)$. For every $n \ge 6$ the polytope $W \times C_{n-5}$ belongs to $\mathcal{O}^n$ and has $g_1 = 2n + 1/10 > 2n = g_1(C_n)$.

Corollary 1.4. *Wills' conjecture holds in $\mathbb{R}^n$ if and only if $n \le 4$.*

Proof. For $n \le 3$ this is Wills' theorem [20]. For $n = 4$, $g_0 = 1$ by (E1) of Section 2, $g_3 \le 32$ by [20, Theorem (i)], $g_4 \le 16$ by Minkowski's theorem, and g_1, g_2 are bounded by Theorem 1.2. For $n \ge 5$, Theorem 1.3 gives a counterexample. □

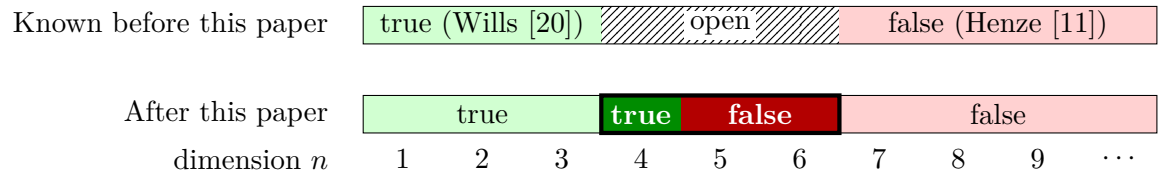


FIGURE 1. The status of Wills' conjecture in each dimension n. Before this paper the conjecture was known to hold for $n \le 3$ (Wills [20]) and to fail for $n \ge 7$ (Henze [11]); the dimensions 4, 5 and 6 (hatched) were open. This paper proves it for $n = 4$ (Theorem 1.2 and Corollary 1.4) and disproves it for every $n \ge 5$ (Theorem 1.3); the dimensions settled here are drawn in full colour, inside the thick frame.

Idea of the proofs. Theorem 1.3 is a computation. The polytope W is given by its 50 vertices, and since W has only one interior lattice point, its linear coefficient is determined by four numbers: the numbers of lattice points in W, in $2W$ and in the interior of $2W$, and the volume of W (Proposition 3.1). Products with segments carry the example to every higher dimension.

For Theorem 1.2 we use Ehrhart reciprocity. Since $P \in \mathcal{O}^4$ has exactly one interior lattice point, $L_P(-1) = 1$; comparing this with $L_P(1) = \#(P \cap \mathbb{Z}^4)$ shows that $g_1(P) \le 8$ is equivalent to the inequality

$$\#(P \cap \mathbb{Z}^4) \le \sigma(P) + 17$$

between the number of lattice points of P and its lattice surface area $\sigma(P) = 2g_3(P)$, and that this inequality also gives $g_2(P) \le 24$ (Proposition 4.2). A triangulation of the boundary of P that uses every boundary lattice point, together with Walkup's lower bound theorem for triangulated 3-spheres [19], proves the inequality when P has at most 29 lattice points (Lemma 4.3).

For P with at least 31 lattice points (their number is odd, since they come in pairs $\pm x$ together with 0) we look at the integer linear functionals that take values in $[-1,1]$ on P. Either they contain a basis of $\mathbb{Z}^4$, and then P is unimodularly equivalent to a polytope in the cube C_4; or they span $\mathbb{R}^4$ without containing a basis, and then a linear map carries P, together with its lattice, onto a polytope in C_4 with vertices in a sublattice of index 2; or they span a proper subspace. In the last case we fix four lattice points of P whose determinant is as large as possible; in their coordinates the lattice points of P lie in $[-1,1]^4$, in one of 39 lattices, up to signed permutations of the coordinates (Section 5). This reduction, by four lattice points of largest determinant, follows Averkov and Schymura [2], who used it to compute the largest number of columns of a Δ-modular integer matrix for small parameters; their enumeration builds on that of Averkov, Borger and Soprunov [1]. In each case only finitely many sets of lattice points remain, and they are enumerated by computer (Section 6).

Section 2 collects the facts we use from Ehrhart theory. Section 3 proves Theorem 1.3. Sections 4–6 prove Theorem 1.2, and Section 7 identifies the equality cases. The expanded version [7] of this paper also gives a second proof of Case III of Section 5, determines the second largest value of g_2 in dimension 4, and describes the computations in more detail.

2. Preliminaries

2.1. **Ehrhart polynomials.** Let $n \geq 1$, let $\Lambda \subset \mathbb{R}^n$ be a lattice (a discrete subgroup of rank n), and let P be a polytope of dimension n whose vertices lie in Λ. Then $L_{P,\Lambda}(k) = \#(kP \cap \Lambda) = \sum_i g_i(P) k^i$ is a polynomial in k of degree n [6]; its coefficients $g_i(P)$ depend on Λ, which will be clear from the context, and we write L_P when $\Lambda = \mathbb{Z}^n$. We call two pairs (P,Λ) and (P',Λ') *isomorphic* if a linear isomorphism ϕ of $\mathbb{R}^n$ satisfies $\phi(P) = P'$ and $\phi(\Lambda) = \Lambda'$; then $\phi(kP) \cap \Lambda' = \phi(kP \cap \Lambda)$ for every k, so $L_{P',\Lambda'} = L_{P,\Lambda}$. When $\Lambda = \Lambda' = \mathbb{Z}^n$, the map ϕ lies in $\mathrm{GL}_n(\mathbb{Z})$, and the two lattice polytopes are called *unimodularly equivalent*. We use four standard facts [3, Chapters 3–5].

(E1) $g_0(P) = 1$ and $g_n(P) = \mathrm{vol}(P)/\det\Lambda$.

(E2) *Reciprocity* (Ehrhart, Macdonald): $L_{P,\Lambda}(-k) = (-1)^n \#(\mathrm{int}(kP) \cap \Lambda)$ for every integer $k \geq 1$.

(E3) For a facet F of P let u_F be the primitive vector of the dual lattice Λ^* that is an outer normal of F, and let b_F be the integer with $F \subset \{x : \langle u_F, x\rangle = b_F\}$. Let $\mathrm{relvol}(F)$ be the $(n-1)$-dimensional volume of F, measured so that a fundamental domain of the lattice $\Lambda \cap \mathrm{aff}(F-F)$ has volume 1. Then $g_{n-1}(P) = \frac{1}{2}\sigma(P)$, where $\sigma(P) = \sum_F \mathrm{relvol}(F)$ is the *lattice surface area*. If $0 \in \mathrm{int}\, P$, then $\mathrm{vol}(P)/\det\Lambda = \frac{1}{n}\sum_F b_F\, \mathrm{relvol}(F)$, by cutting P into the pyramids with apex 0 over its facets.

(E4) $L_{P\times Q} = L_P\, L_Q$ for lattice polytopes $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$, with the lattice $\mathbb{Z}^{p+q}$, since $k(P\times Q) \cap \mathbb{Z}^{p+q} = (kP \cap \mathbb{Z}^p) \times (kQ \cap \mathbb{Z}^q)$.

In (E3), b_F is an integer because F contains points of Λ; when $0 \in \mathrm{int}\, P$ it is positive, and it is the lattice distance of F from the origin.

2.2. **The class $\mathcal{O}^n$.** Let $\mathcal{O}^n_\Lambda$ be the set of polytopes P with vertices in Λ, $P = -P$ and $\mathrm{int}\, P \cap \Lambda = \{0\}$; so $\mathcal{O}^n = \mathcal{O}^n_{\mathbb{Z}^n}$. A polytope in $\mathcal{O}^n_\Lambda$ has dimension n, because 0 is an interior point. We write $G(P) = \#(P \cap \Lambda)$. Since the lattice points of P other than 0 come in pairs $\pm x$, $G(P)$ is odd.

Lemma 2.1. *Let $Y \subset \{-1,0,1\}^n$ be a set with $Y = -Y$ whose convex hull P has dimension n. Then $P \in \mathcal{O}^n$.*

Proof. The polytope P is a lattice polytope with $P = -P$. It has an interior point x, so $-x$ is interior too, and so is $0 = \frac{1}{2}x + \frac{1}{2}(-x)$. Finally $\operatorname{int} P \subset \operatorname{int} C_n = (-1,1)^n$, whose only lattice point is 0. □

The same proof applies to a sublattice $\Gamma \subseteq \mathbb{Z}^n$ in place of $\mathbb{Z}^n$: if $S \subseteq \Gamma \cap C_n$ is symmetric and spans $\mathbb{R}^n$, then $\operatorname{conv} S \in \mathcal{O}^n_\Gamma$. We use this in Sections 5 and 6.

Lemma 2.2 (cf. [11, Proposition 2.3])**.** *If $P \in \mathcal{O}^p$ and $Q \in \mathcal{O}^q$, then $P \times Q \in \mathcal{O}^{p+q}$ and $g_1(P \times Q) = g_1(P) + g_1(Q)$.*

Proof. The product $P \times Q$ is a symmetric lattice polytope, and $\operatorname{int}(P \times Q) = \operatorname{int} P \times \operatorname{int} Q$ contains only the lattice point 0. By (E4) and (E1), the coefficient of k in $L_P L_Q$ is $g_1(P) \cdot 1 + 1 \cdot g_1(Q)$. □

3. A counterexample in dimension five

We compute four numbers for the polytope W of Table 2, three lattice-point counts and the volume; by Proposition 3.1 they determine $g_1(W)$.

Table 2. The polytope $W \subset \mathbb{R}^5$ is the convex hull of the 50 points $\pm p$, where p runs over the 25 points below ($\bar{1}$ stands for -1). All 50 points are vertices of W, and W has 92 facets.

$(0,0,1,\bar{1},\bar{1})$	$(0,0,1,1,\bar{1})$	$(0,1,\bar{1},0,0)$	$(0,1,0,1,\bar{1})$	$(0,1,0,1,1)$
$(0,1,1,\bar{1},\bar{1})$	$(0,1,1,\bar{1},0)$	$(1,\bar{1},\bar{1},\bar{1},0)$	$(1,\bar{1},0,1,0)$	$(1,\bar{1},0,1,1)$
$(1,\bar{1},1,0,\bar{1})$	$(1,0,\bar{1},0,\bar{1})$	$(1,0,\bar{1},1,\bar{1})$	$(1,0,\bar{1},1,1)$	$(1,0,0,\bar{1},1)$
$(1,0,0,1,1)$	$(1,0,1,\bar{1},0)$	$(1,0,1,\bar{1},1)$	$(1,0,1,0,\bar{1})$	$(1,1,\bar{1},\bar{1},\bar{1})$
$(1,1,\bar{1},\bar{1},0)$	$(1,1,\bar{1},1,0)$	$(1,1,1,\bar{1},0)$	$(1,1,1,\bar{1},1)$	$(1,1,1,1,0)$

Proposition 3.1. *Let $P \in \mathcal{O}^5$. Then*

$$g_1(P) = \frac{4\big(G(P)+1\big) + 24\,\mathrm{vol}(P) - A(P)}{6}, \qquad A(P) = \tfrac{1}{2}\Big(\#(2P \cap \mathbb{Z}^5) + \#(\operatorname{int}(2P) \cap \mathbb{Z}^5)\Big).$$

Proof. Write $L = L_P$ and $g_i = g_i(P)$. By (E2) with $n = 5$, $L(-1) = -1$ and $L(-2) = -\#(\operatorname{int}(2P) \cap \mathbb{Z}^5)$. Hence

$$L(1) - L(-1) = 2(g_1 + g_3 + g_5) = G(P) + 1, \qquad L(2) - L(-2) = 2(2g_1 + 8g_3 + 32g_5) = 2A(P).$$

Eliminating g_3 and using $g_5 = \mathrm{vol}(P)$ (E1) gives $6g_1 = 4(G(P)+1) + 24\,\mathrm{vol}(P) - A(P)$. □

Proof of Theorem 1.3. The 50 points of Table 2 lie in $\{-1,0,1\}^5$, form a symmetric set and span $\mathbb{R}^5$; by Lemma 2.1, $W \in \mathcal{O}^5$. Direct computation gives

$$G(W) = 117, \qquad \#(2W \cap \mathbb{Z}^5) = 1815, \qquad \#(\operatorname{int}(2W) \cap \mathbb{Z}^5) = 243, \qquad \mathrm{vol}(W) = \tfrac{386}{15},$$

so $A(W) = 1029$ and, by Proposition 3.1, $g_1(W) = \Big(4 \cdot 118 + 24 \cdot \frac{386}{15} - 1029\Big)/6 = \frac{101}{10}$. The whole polynomial is determined by $L_W(0) = 1$ and the values $L_W(k)$ for $k = 1, \ldots, 5$, namely 117, 1815, 10765, 39927 and 112639. All numbers in this proof can be computed with Normaliz [16] or LattE [12]; the input files are in [17].

For $n \geq 6$, Lemma 2.2 gives $W \times C_{n-5} \in \mathcal{O}^n$ and $g_1(W \times C_{n-5}) = \frac{101}{10} + 2(n-5) = 2n + \frac{1}{10}$, since $g_1(C_m) = 2m$. □

4. Dimension four: the key inequality and polytopes with few lattice points

From now on $n = 4$. For $P \in \mathcal{O}^4_\Lambda$ we write $G = G(P)$, $g_i = g_i(P)$ (with respect to Λ) and $\sigma = \sigma(P)$.

Lemma 4.1. *For $P \in \mathcal{O}^4_\Lambda$, $g_1 + g_3 = g_2 + g_4 = \frac{G-1}{2}$.*

Proof. We have $L(1) = G = 1 + g_1 + g_2 + g_3 + g_4$, and by (E2) $L(-1) = \#(\operatorname{int} P \cap \Lambda) = 1 = 1 - g_1 + g_2 - g_3 + g_4$. Subtracting the second equation from the first, and adding the two, gives the two identities. □

Proposition 4.2. *Let $P \in \mathcal{O}^4_\Lambda$. Then $g_4 \leq 16$ and $g_3 \leq 2g_4 \leq 32$. Moreover*

$$g_1 \leq 8 \iff G \leq \sigma + 17,$$

and if $g_1 \leq 8$ then $g_2 \leq g_4 + 8 \leq 24$.

Proof. The bound $g_4 = \operatorname{vol}(P)/\det\Lambda \leq 16$ is Minkowski's theorem, applied in the coordinates of a basis of Λ. Since every $b_F \geq 1$, (E3) gives $2g_3 = \sigma \leq \sum_F b_F \operatorname{relvol}(F) = 4g_4$; this is Wills' argument for g_{n-1} [20, Theorem (i)]. By Lemma 4.1, $g_1 = \frac{G-1}{2} - g_3$, so $g_1 \leq 8$ if and only if $G \leq 2g_3 + 17 = \sigma + 17$. If this holds, then $G \leq 4g_4 + 17$, and Lemma 4.1 gives $g_2 = \frac{G-1}{2} - g_4 \leq g_4 + 8 \leq 24$. □

So Theorem 1.2 is equivalent to the key inequality $G \leq \sigma + 17$ on $\mathcal{O}^4$. Let $\operatorname{NVol}(F) = 3!\operatorname{relvol}(F)$ be the normalized volume of a facet, so that a lattice tetrahedron in $\operatorname{aff} F$ has normalized volume a positive integer. The inequality reads

$$\sum_F \operatorname{NVol}(F) \;\geq\; 6G - 102. \tag{1}$$

Lemma 4.3. *If $P \in \mathcal{O}^4_\Lambda$ and $G \leq 29$, then (1) holds with strict inequality; hence $g_1 < 8$.*

Proof. Take a triangulation Δ of the boundary of P whose vertex set is the set of all $G-1$ boundary lattice points and in which every tetrahedron lies in a facet. To construct one, take a triangulation of P whose vertices are exactly the points of $P \cap \Lambda$; one exists, for example by strong pulling ([9, Lemma 2.1(2)], applied in the coordinates of a basis of Λ). Its faces that lie in the boundary of P form Δ. Then Δ is a triangulated 3-sphere with $f_0 = G - 1$ vertices. Each of its f_3 tetrahedra is a lattice tetrahedron in the affine hull of a facet, so it has normalized volume at least 1, and $\sum_F \operatorname{NVol}(F) \geq f_3$. Walkup's lower bound theorem [19, Theorem 1] gives $f_1 \geq 4f_0 - 10$ for every triangulated 3-sphere, and the Dehn–Sommerville relations $f_0 - f_1 + f_2 - f_3 = 0$, $2f_2 = 4f_3$ give $f_3 = f_1 - f_0$. Hence $\sum_F \operatorname{NVol}(F) \geq 3f_0 - 10 = 3G - 13$, and $3G - 13 > 6G - 102$ because $G \leq 29$. □

The proof gives a bound on g_1 for every $P \in \mathcal{O}^4_\Lambda$: by Lemma 4.1 and (E3), $g_1 = \frac{G-1}{2} - \frac{\sigma}{2}$, and $\sigma = \frac{1}{6}\sum_F \operatorname{NVol}(F) \geq \frac{3G-13}{6}$, so $g_1 \leq \frac{3G+7}{12}$ (the solid line in Figure 3).

Remark 4.4. With Euler's formula in place of Walkup's theorem, a similar triangulation argument gives Wills' theorem in dimension 3 (see [7]). In dimension 4 the lower bound theorem proves (1) only when $G \leq 29$; this is why the computations of Section 6 are needed.

5. Dimension four: three cases

By Lemma 4.3 it remains to prove (1) for $P \in \mathcal{O}^4$ with $G \geq 31$. For such P let

$$M(P) = \{u \in \mathbb{Z}^4 : \ |\langle u, x\rangle| \leq 1 \text{ for all } x \in P\}, \qquad r(P) = \dim \operatorname{span} M(P).$$

The set $M(P)$ is finite (since P contains a ball around 0) and symmetric, and $\langle u, x\rangle \in \{-1, 0, 1\}$ for $u \in M(P)$ and $x \in P \cap \mathbb{Z}^4$. We distinguish three cases: $M(P)$ contains a basis of $\mathbb{Z}^4$ (Case I); $r(P) = 4$ but $M(P)$ contains no basis (Case II); $r(P) \leq 3$ (Case III). Figure 2 shows how each case is settled.

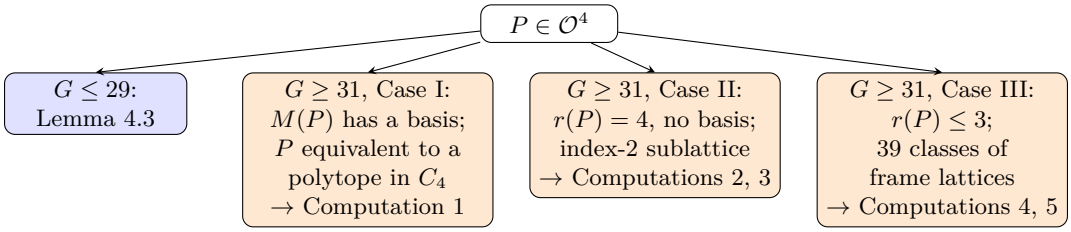


Figure 2. The proof of Theorem 1.2 splits $\mathcal{O}^4$ into four parts. Polytopes with at most 29 lattice points are handled by Lemma 4.3 (the blue box, on the left); for the others (the three orange boxes), the integer functionals bounded by 1 in absolute value on P decide which finite enumeration of Section 6 contains P, up to isomorphism (Section 2). The frame lattices are defined after Lemma 5.1.

5.1. **Case I.** For a lattice Γ, a set $S \subseteq \Gamma$ is called Γ-*saturated* if $\operatorname{conv} S \cap \Gamma = S$; when the lattice is clear, we say *saturated*. If $M(P)$ contains a basis $u_1, \ldots, u_4$ of $\mathbb{Z}^4$, let $U \in \mathrm{GL}_4(\mathbb{Z})$ have rows u_i. Then $UP \subset C_4$, so $UP = \operatorname{conv} S$ for the set $S = UP \cap \mathbb{Z}^4 \subset \{-1, 0, 1\}^4$, which is symmetric and saturated.

5.2. **Case II.** Let $r(P) = 4$, and let $u_1, \ldots, u_4 \in M(P)$ be linearly independent, with U the matrix of rows u_i and $d = |\det U|$. Then $d \geq 2$, since otherwise $M(P)$ would contain a basis. The map $x \mapsto Ux$ carries $\mathbb{Z}^4$ onto the sublattice $\Gamma = U\mathbb{Z}^4$ of index d and carries P into C_4; so $(P, \mathbb{Z}^4)$ is isomorphic to (UP, Γ), $UP \in \mathcal{O}^4_\Gamma$, $L_{UP,\Gamma} = L_P$, and $UP = \operatorname{conv} S$ for the Γ-saturated symmetric set $S = UP \cap \Gamma \subset \Gamma \cap \{-1, 0, 1\}^4$. In particular

$$31 \leq G(P) \leq \#(\Gamma \cap \{-1, 0, 1\}^4).$$

Moreover $d \leq 16$: the set S spans $\mathbb{R}^4$, so it contains four linearly independent points of $\Gamma \cap \{-1, 0, 1\}^4$; their determinant is a nonzero multiple of d, and it is at most 16 in absolute value by Hadamard's inequality.

The sublattices of index 2 are $\Gamma_I = \{x : \sum_{i \in I} x_i \text{ even}\}$ for nonempty $I \subseteq \{1, 2, 3, 4\}$. A point of $\{-1, 0, 1\}^4$ lies in Γ_I exactly when an even number of its coordinates in I are nonzero, so

$$\#(\Gamma_I \cap \{-1, 0, 1\}^4) = 3^{4-m} \cdot \frac{3^m + (-1)^m}{2} = 27,\ 45,\ 39,\ 41 \qquad (m = |I| = 1, 2, 3, 4).$$

Computation 2 shows that a sublattice of index $3 \leq d \leq 16$ contains at most 27 points of $\{-1, 0, 1\}^4$ whenever these points span $\mathbb{R}^4$. Since $27 < 31$, in Case II we have $d = 2$ and $\Gamma = \Gamma_I$ with $|I| \in \{2, 3, 4\}$.

5.3. **Case III: the frame.** Now let $r(P) \le 3$. Choose lattice points $v_1, \dots, v_4$ of P for which $D = |\det(v_1, \dots, v_4)|$ is as large as possible, and let V be the matrix with columns v_i. We call $(v_1, \dots, v_4)$ a *frame* of P. Since P spans $\mathbb{R}^4$, $D \ge 1$. The map $x \mapsto V^{-1}x$ carries $\mathbb{Z}^4$ onto the lattice $\Lambda = V^{-1}\mathbb{Z}^4 \supseteq \mathbb{Z}^4$, of index D over $\mathbb{Z}^4$, and carries $P \cap \mathbb{Z}^4$ onto a set S.

Lemma 5.1. *In this situation:*

(i) $S \subset \Lambda \cap C_4$, $S = -S$, *and* S *contains the unit vectors* $\pm e_1, \dots, \pm e_4$;
(ii) $|\det(s_1, \dots, s_4)| \le 1$ *for all* $s_1, \dots, s_4 \in S$;
(iii) $\operatorname{conv} S \cap \Lambda = S$ *and* $\operatorname{int}(\operatorname{conv} S) \cap \Lambda = \{0\}$;
(iv) Λ *contains no point* $c \ne 0$ *with* $|c_1| + \cdots + |c_4| < 1$;
(v) $D \le 23$.

Proof. For $x_1, \dots, x_4 \in P \cap \mathbb{Z}^4$ we have $|\det(V^{-1}x_1, \dots, V^{-1}x_4)| = |\det(x_1, \dots, x_4)|/D \le 1$ by the choice of $v_1, \dots, v_4$, which gives (ii). For (i), write a lattice point x of P as $x = \sum c_i v_i$; by Cramer's rule $|c_i| = |\det(v_1, \dots, x, \dots, v_4)|/D \le 1$. Property (iii) holds because $x \mapsto V^{-1}x$ carries $\operatorname{conv}(P \cap \mathbb{Z}^4) \cap \mathbb{Z}^4 = P \cap \mathbb{Z}^4$ and $\operatorname{int} P \cap \mathbb{Z}^4 = \{0\}$ to the corresponding statements for S and Λ. For (iv), the open cross-polytope $\{c : |c_1| + \cdots + |c_4| < 1\}$ lies in the interior of $\operatorname{conv}\{\pm e_i\} \subseteq \operatorname{conv} S$, so by (iii) its only point of Λ is 0.

For (v), P contains the cross-polytope $\operatorname{conv}\{\pm v_i\}$ of volume $\frac{2^4}{4!}D = \frac{2}{3}D$, and $\operatorname{vol}(P) \le 16$, so $D \le 24$. If $D = 24$, then $P = \operatorname{conv}\{\pm v_i\}$ and $\operatorname{vol} P = 16$. The translates $\frac{1}{2}P + z$, $z \in \mathbb{Z}^4$, have disjoint interiors (an overlap would put a nonzero lattice point in $\operatorname{int} P$) and total density $\operatorname{vol}(\frac{1}{2}P) = 1$, so they tile $\mathbb{R}^4$: the union of the translates is closed, and its complement is open, invariant under $\mathbb{Z}^4$ and of density 0; a nonempty such set would contain a ball and all its $\mathbb{Z}^4$-translates, hence have positive density. A convex polytope that tiles space by lattice translations has centrally symmetric facets (Minkowski [15]; see Venkov [18] and McMullen [13]), but the facets of a cross-polytope are tetrahedra. So $D \le 23$. □

Let $\mathcal{L}$ be the set of lattices $\Lambda \supseteq \mathbb{Z}^4$ of index at most 23 over $\mathbb{Z}^4$ that satisfy (iv); a lattice of index D over $\mathbb{Z}^4$ lies in $\frac{1}{D}\mathbb{Z}^4$, so $\mathcal{L}$ is finite. We call the lattices in $\mathcal{L}$ the *frame lattices*. For $\Lambda \in \mathcal{L}$ let $\mathcal{S}(\Lambda)$ be the family of sets S satisfying (i)–(iii). The signed permutation matrices act on $\mathcal{L}$, and Computation 4 shows that $\mathcal{L}$ consists of 39 orbits; we fix one representative in each. Integer functionals transfer as well. For a set $S \subset \Lambda$ let $M(S) = \{w \in \Lambda^* : |\langle w, s\rangle| \le 1 \text{ for } s \in S\}$ and $r(S) = \dim \operatorname{span} M(S)$. Since $\langle u, Vs\rangle = \langle V^{\mathsf{T}}u, s\rangle$ and $V^{\mathsf{T}}\mathbb{Z}^4 = \Lambda^*$, a vector $u \in \mathbb{Z}^4$ lies in $M(P)$ if and only if $V^{\mathsf{T}}u$ lies in $M(S)$; so $r(P) = r(S)$. In Case III we therefore have a lattice Λ among the 39 representatives and a set $S \in \mathcal{S}(\Lambda)$ with $\#S \ge 31$ and $r(S) \le 3$ such that $(P, \mathbb{Z}^4)$ is isomorphic to $(\operatorname{conv} S, \Lambda)$ (by $x \mapsto V^{-1}x$, followed by a signed permutation of the coordinates); in particular $L_{\operatorname{conv} S, \Lambda} = L_P$. Lemma 5.1 does not use $r(P) \le 3$, so this reduction applies to every $P \in \mathcal{O}^4$. We use it only in Case III, where the condition $r(S) \le 3$ shortens the search of Computation 5; the polytopes with $r(P) = 4$ are listed directly, in Cases I and II.

6. The computations

All enumerations below are exhaustive and use exact integer and rational arithmetic; the repository [17] contains the programs and, for each computation, its output. Computations 1, 3 and 5 list sets of lattice points up to the symmetries named. For each set S the Ehrhart polynomial of $\operatorname{conv} S$ is computed exactly: in Computations 1 and 3 from $G = \#S$

and the volume and the lattice surface area of $\operatorname{conv} S$ (taken from a triangulation of its boundary), by Lemma 4.1, (E1) and (E3); in Computation 5 from the numbers of lattice points of $k \operatorname{conv} S$ for $k = 1, 2, 3$ and of the interior of $k \operatorname{conv} S$ for $k = 1, 2$, which give its values at $k = -2, \dots, 3$ by (E1) and (E2) and so determine it. Figure 3 shows, for each $G \geq 31$, the largest value of g_1 in Case I (Computation 1) and the three equality cases of Theorem 7.1, and, for $G \leq 29$, the bound on g_1 stated after the proof of Lemma 4.3.

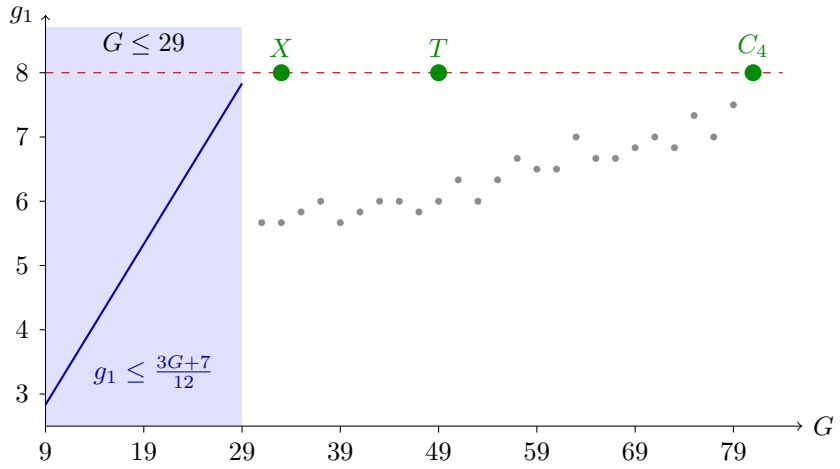


FIGURE 3. The largest values of g_1 in Case I, and the three equality cases of Theorem 7.1, against the number G of lattice points. The small grey dots show, for each odd $G \geq 31$, the largest value of g_1 in Case I (Computation 1). The three larger green dots mark the polytopes C_4, T and X of Theorem 7.1: C_4 belongs to Case I, and T and X belong to Case III. For Cases II and III only the largest value of g_1 over all G is recorded: at most 6 in Case II (Computation 3), and at most 8 in Case III, attained only by T and X (Computation 5). The dashed line is $g_1 = 8$. In the shaded region $G \leq 29$, the solid line is the bound $g_1 \leq \frac{3G+7}{12}$ stated after the proof of Lemma 4.3.

Computations 1 and 3 walk downward from the full set, $\{-1, 0, 1\}^4$ or $\Gamma_I \cap \{-1, 0, 1\}^4$, deleting one pair of opposite vertices at a time. By the next lemma, such a walk reaches every saturated symmetric subset of the full set. Sets that a symmetry carries onto each other form a class, and the walk keeps one set of each class; classes are recognized by a canonical form, the least image of the set under the symmetry group in a fixed total order. This loses no class: a symmetry γ maps saturated symmetric sets to saturated symmetric sets and the vertices of $\operatorname{conv} Q$ to those of $\operatorname{conv} \gamma Q$, so the sets obtained from γQ by deleting a pair of opposite vertices are the images under γ of those obtained from Q. Computation 5 uses a different method, a sandwich search, described after its statement.

Lemma 6.1. *Let Λ be a lattice and $Q \subset \Lambda$ a finite symmetric set with $\operatorname{conv} Q \cap \Lambda = Q$.*

(i) *If v is a vertex of $\operatorname{conv} Q$, then $Q' = Q \setminus \{v, -v\}$ is symmetric and satisfies $\operatorname{conv} Q' \cap \Lambda = Q'$.*

(ii) *If $S \subsetneq Q$ is symmetric with $\operatorname{conv} S \cap \Lambda = S$, then some vertex v of $\operatorname{conv} Q$ is not in S, and $S \subseteq Q \setminus \{v, -v\}$.*

Proof. (i) Since v is a vertex, $v \notin \operatorname{conv}(Q \setminus \{v\}) \supseteq \operatorname{conv} Q'$, and by symmetry $-v \notin \operatorname{conv} Q'$; as $\operatorname{conv} Q' \cap \Lambda \subseteq Q$, it equals Q'. (ii) If every vertex of $\operatorname{conv} Q$ were in S, then $Q = \operatorname{conv} Q \cap \Lambda \subseteq \operatorname{conv} S \cap \Lambda = S$. A vertex $v \notin S$ has $-v \notin S$ by symmetry. □

So every saturated symmetric $S \subseteq Q$ is reached from Q through saturated symmetric sets that contain S; in particular a search that stops at 31 points misses no set with at least 31 points.

Computation 1. *Case I.* We list all symmetric sets $S \subseteq \{-1,0,1\}^4$ with $\operatorname{conv} S \cap \mathbb{Z}^4 = S$ and $\#S \geq 31$, up to the 384 signed permutations of the coordinates. Each such S spans $\mathbb{R}^4$ (a hyperplane through 0 contains at most 27 points of $\{-1,0,1\}^4$, since on it one coordinate is determined by the other three), so $\operatorname{conv} S \in \mathcal{O}^4$ by Lemma 2.1. There are 2,547,694 classes; for all of them the linear coefficient g_1 of $\operatorname{conv} S$ is at most 8, with equality only for $S = \{-1,0,1\}^4$.

Computation 2. *Index bound.* We list every sublattice $\Gamma \subset \mathbb{Z}^4$ of index d with $2 \leq d \leq 16$ once, as $\Gamma = B\mathbb{Z}^4$ with B in Hermite normal form under column operations ($B \mapsto BU$); there are 40,406 of them. For each Γ whose points in $\{-1,0,1\}^4$ span $\mathbb{R}^4$ we count these points. The maximum is 45 for $d = 2$, and it is at most 27 for every $d \geq 3$.

Computation 3. *Case II.* A permutation of coordinates carries Γ_I onto Γ_J when $|I| = |J|$, so it suffices to take $\Gamma_m = \Gamma_{\{1,\dots,m\}} = \{x : x_1 + \cdots + x_m \text{ even}\}$ for $m = 2, 3, 4$. For each of these three lattices we list all Γ_m-saturated symmetric sets $S \subseteq \Gamma_m \cap \{-1,0,1\}^4$ that span $\mathbb{R}^4$ and have $\#S \geq 31$, up to the signed permutations that preserve Γ_m; each such $\operatorname{conv} S$ lies in $\mathcal{O}^4_{\Gamma_m}$ (see the paragraph after Lemma 2.1). There are 531, 58 and 27 classes for $m = 2$, 3, 4, and for all of them the linear coefficient g_1 of $\operatorname{conv} S$ with respect to Γ_m is at most 6.

Computation 4. *The frame lattices.* We list all lattices in $\mathcal{L}$. For $U \in \mathrm{GL}_4(\mathbb{Z})$ the frame matrices V and UV give the same lattice $\Lambda = V^{-1}\mathbb{Z}^4$, so it suffices to take V in Hermite normal form under row operations ($V \mapsto UV$), with determinant $D \leq 23$, and to keep the lattices Λ that satisfy (iv). There are 1,040 frame lattices, in 39 orbits under signed permutations, with indices $D \in \{1,\dots,10,12,16,18\}$.

Computation 5. *Case III.* For each of the 39 representatives Λ we list all $S \in \mathcal{S}(\Lambda)$ with $\#S \geq 31$ and $r(S) \leq 3$, up to the signed permutations that preserve Λ. There are 588,592 classes; for all of them the linear coefficient g_1 of $\operatorname{conv} S$ with respect to Λ is at most 8, with equality for exactly two of them, one with $G = 49$ and one with $G = 33$.

The search follows [2, Algorithm 1]. It keeps a lower set $S^- \in \mathcal{S}(\Lambda)$ and an upper set $S^+ \supseteq S^-$, and it starts from $S^- = \operatorname{conv}\{\pm e_1,\dots,\pm e_4\} \cap \Lambda$ and $S^+ = \Lambda \cap C_4$. Every member of $\mathcal{S}(\Lambda)$ lies between these two sets: it lies in S^+ by (i), and it contains S^- by (i) and (iii). The set S^- is itself a member: its points c have $|c_1| + \cdots + |c_4| \leq 1$, so (ii) holds by Hadamard's inequality and (iii) by (iv).

For a pair $\pm v$ in $S^+ \setminus S^-$ the search branches: either $\pm v$ is excluded and S^+ is replaced by $S^+ \setminus \{\pm v\}$, or $\pm v$ is included and S^- is replaced by $\operatorname{conv}(S^- \cup \{\pm v\}) \cap \Lambda$; every saturated set between S^- and S^+ lies in exactly one branch. An include branch is abandoned when the new S^- is not contained in S^+, violates (ii), or has a nonzero point of Λ in the interior of its convex hull, since then no set in the branch satisfies (i)–(iii). A pair $\pm y$ in S^+ for which $S^- \cup \{\pm y\}$ violates (ii) is removed from S^+, since no member contains it together

with S^-. A branch is abandoned when $\#S^+ < 31$ or $r(S^+) = 4$; the second rule uses that M reverses inclusion, that is, $M(Q') \subseteq M(Q)$ whenever $Q \subseteq Q'$, so that every subset of a set Q' with $r(Q') = 4$ also has $r = 4$. The search stops at the pairs (S^-, S^+) with $S^- = S^+$. Each include step keeps S^- in $\mathcal{S}(\Lambda)$, so these sets are members, and they are the sets found. A pair (S^-, S^+) that a signed permutation preserving Λ carries onto a pair met earlier in the search is discarded, since the permutation carries the sets between the two pairs onto each other; the sets found are then reduced to one of each class by the canonical form described before Lemma 6.1.

Proof of Theorem 1.2. Let $P \in \mathcal{O}^4$. If $G(P) \leq 29$, Lemma 4.3 gives $g_1(P) < 8$. If $G(P) \geq 31$, then by Section 5 the pair $(P, \mathbb{Z}^4)$ is isomorphic to $(\operatorname{conv} S, \Lambda_S)$ for a set S listed in Computation 1 (Case I), in Computation 3 (Case II, using Computation 2), or in Computation 5 (Case III, using Computation 4), where Λ_S is the lattice of that computation ($\mathbb{Z}^4$, Γ_m or a frame lattice); so $L_P = L_{\operatorname{conv} S, \Lambda_S}$. In each list $g_1 \leq 8$. Hence $g_1(P) \leq 8$, and Proposition 4.2 gives $g_2(P) \leq 24$. □

7. The equality cases

Let $T \subset \mathbb{R}^4$ be the convex hull of the 24 points $\pm f_a \pm f_b$, $1 \leq a < b \leq 4$, where $f_i = e_i$ for $i \leq 3$ and $f_4 = (-1, -1, -1, 2)$. This is the 24-cell $\operatorname{conv}\{\pm e_a \pm e_b\}$ written in the basis $e_1, e_2, e_3, \frac{1}{2}(1,1,1,1)$ of the lattice $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$; in other words, $(T, \mathbb{Z}^4)$ is isomorphic to the pair formed by this 24-cell and this lattice. Let

$$X = \operatorname{conv}\{\pm(1,0,0,0),\ \pm(0,1,0,0),\ \pm(1,1,3,0),\ \pm(2,4,3,6)\}.$$

Theorem 7.1. *A polytope $P \in \mathcal{O}^4$ has $g_1(P) = 8$ if and only if it is unimodularly equivalent to C_4, T or X.*

We first check that T and X lie in $\mathcal{O}^4$, that C_4, T and X have $g_1 = 8$, and that $M(T) = M(X) = \{0\}$.

The polytope $\operatorname{conv}\{\pm e_a \pm e_b\}$ is the set of points x with $|x_i| \leq 1$ for all i and $|x_1| + \cdots + |x_4| \leq 2$, so its lattice points in $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$ other than 0 are the 24 vertices, the 8 points $\pm e_i$ and the 16 points $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$; each has a coordinate ± 1 or all four coordinates $\pm\frac{1}{2}$, so it lies on one of the facets $\{x_i = \pm 1\}$ or $\{\pm x_1 \pm x_2 \pm x_3 \pm x_4 = 2\}$. Hence $T \in \mathcal{O}^4$ and $G(T) = 49$. The dual lattice of $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$ is $\{y \in \mathbb{Z}^4 : y_1 + \cdots + y_4 \text{ even}\}$; its primitive vectors normal to these facets are $\pm 2e_i$ and $(\pm 1, \pm 1, \pm 1, \pm 1)$, and they take the value 2 on them, so every facet of T has lattice distance 2 from the origin. Moreover $\operatorname{vol}(T) = 16$, since $\operatorname{conv}\{\pm e_a \pm e_b\}$ has volume 8 and the lattice has determinant $\frac{1}{2}$.

For X, a direct computation (see [17]) shows that 0 is its only interior lattice point, that $G(X) = 33$, and that each of its 16 facets has lattice distance 3 from the origin; and $\operatorname{vol}(X) = \frac{2^4}{4!} \cdot 18 = 12$, since the vectors $(1,0,0,0)$, $(0,1,0,0)$, $(1,1,3,0)$, $(2,4,3,6)$ have determinant 18.

If every facet of $P \in \mathcal{O}^4$ has lattice distance ℓ, then (E3) gives $g_3 = \frac{1}{2}\sigma = 2g_4/\ell$, and Lemma 4.1 gives $g_1 = \frac{G-1}{2} - g_3$ and $g_2 = \frac{G-1}{2} - g_4$. For C_4 we have $G = 81$ and $\operatorname{vol}(C_4) = 16$, and its facets $\{x_i = \pm 1\}$ have lattice distance 1. With $(G, g_4, \ell) = (81, 16, 1)$, $(49, 16, 2)$

and $(33, 12, 3)$ this gives

$$L_{C_4} = 1 + 8k + 24k^2 + 32k^3 + 16k^4, \qquad L_T = 1 + 8k + 8k^2 + 16k^3 + 16k^4,$$
$$L_X = 1 + 8k + 4k^2 + 8k^3 + 12k^4.$$

For T, let $u \in M(T)$ and $w_a = \langle u, f_a \rangle \in \mathbb{Z}$. Then $|w_a \pm w_b| \leq 1$ for $a \neq b$, so at most one w_a is nonzero, and it is ± 1. Here $w_a = u_a$ for $a \leq 3$ and $w_4 = 2u_4 - u_1 - u_2 - u_3$. If $w_1 = w_2 = w_3 = 0$, then $u_1 = u_2 = u_3 = 0$ and $w_4 = 2u_4$ is even, so $u_4 = 0$. Otherwise $w_a = u_a = \pm 1$ for one $a \leq 3$ and $w_4 = 0$, so $2u_4 = u_a = \pm 1$, which is impossible. So $M(T) = \{0\}$.

For X, the conditions $|\langle u, v \rangle| \leq 1$ for the first three vertices, namely $|u_1| \leq 1$, $|u_2| \leq 1$ and $|u_1 + u_2 + 3u_3| \leq 1$, leave nine triples (u_1, u_2, u_3), one for each (u_1, u_2). The fourth condition, $|2u_1 + 4u_2 + 3u_3 + 6u_4| \leq 1$, requires $2u_1 + 4u_2 + 3u_3 \equiv -1$, 0 or 1 (mod 6). Of the nine triples only $(0, 0, 0)$ satisfies this, and then $u_4 = 0$. So $M(X) = \{0\}$.

Proof of Theorem 7.1. The three polytopes have $g_1 = 8$, and they are pairwise inequivalent because they have 81, 49 and 33 lattice points. Conversely, let $g_1(P) = 8$. By Lemma 4.3, $G(P) \geq 31$, so, as in the proof of Theorem 1.2, $(P, \mathbb{Z}^4)$ is isomorphic to $(\operatorname{conv} S, \Lambda_S)$ for a set S with $g_1 = 8$ in one of Computations 1, 3, 5. In Computation 1 this is the cube, and the isomorphism lies in $\mathrm{GL}_4(\mathbb{Z})$; Computation 3 has none; in Computation 5 it is one of the two sets with $G = 49$ and $G = 33$. Now T and X have $M(T) = M(X) = \{0\}$, so they fall under Case III, and $(T, \mathbb{Z}^4)$ and $(X, \mathbb{Z}^4)$ are isomorphic to pairs $(\operatorname{conv} S, \Lambda_S)$ with S in Computation 5 and $g_1 = 8$; comparing G, these are the two sets found there. If $(P, \mathbb{Z}^4)$ and $(T, \mathbb{Z}^4)$ are both isomorphic to the same pair (after a signed permutation fixing its lattice, if needed), then composing the two isomorphisms gives a linear map that carries P onto T and $\mathbb{Z}^4$ onto $\mathbb{Z}^4$, that is, a map in $\mathrm{GL}_4(\mathbb{Z})$, and in the same way for X. Hence, in this case, P is unimodularly equivalent to T or to X. □

Corollary 7.2. *A polytope $P \in \mathcal{O}^4$ has $g_2(P) = 24$ if and only if it is unimodularly equivalent to C_4.*

Proof. The cube has $g_2(C_4) = 24$. Conversely, let $g_2(P) = 24$. Since $g_1(P) \leq 8$ by Theorem 1.2, Proposition 4.2 gives $24 = g_2 \leq g_4 + 8 \leq 24$, so $g_4 = 16$, and then $G = 2(g_2 + g_4) + 1 = 81$ by Lemma 4.1. The same lemma gives $g_1 + g_3 = 40$, and $g_1 \leq 8$, $g_3 \leq 32$ force $g_1 = 8$. By Theorem 7.1, P is unimodularly equivalent to C_4, T or X, and of these only C_4 has 81 lattice points. □

Remark 7.3. Henze's theorem [11, Corollary 3.3] covers C_4 but neither T nor X. All roots of $L_{C_4} = (1 + 2k)^4$ have real part $-\frac{1}{2}$, but those of L_T and L_X do not: if all four roots of L_P, for $P \in \mathcal{O}^4$, have real part $-\frac{1}{2}$, they sum to -2, so $g_3(P) = 2g_4(P)$ by Vieta's formulas; but L_T has $g_3 = 16 \neq 2 \cdot 16$ and L_X has $g_3 = 8 \neq 2 \cdot 12$. Alternatively, by [5] such a P would be reflexive, that is, all its facets would have lattice distance 1 from the origin, while the facets of T and X have lattice distance 2 and 3.

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