PHASE TRANSITION AND SPONTANEOUS TRANSLATION SYMMETRY BREAKING IN NEGATIVE TEMPERATURE POINT VORTEX ENSEMBLES

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AI USE STATEMENT. This work was largely generated and proof-checked by large language models through a series of interactions with the authors. It extends the study of negative-temperature point vortex ensembles initiated in the authors' previous works, retaining several of their main ideas. The authors intend to release an expanded version with a fuller exposition in the near future.

ABSTRACT. We establish a canonical phase transition and spontaneous translation symmetry breaking for balanced two-species point vortices on the square torus. A sharp inequality identifies the transition at $\beta = 4\pi^2$, below collapse at $\beta = 16\pi$. In the intervening regime, all global minimizers form one translation orbit of an explicit dipole. We prove a large deviation principle below collapse and identify the uniform random phase, phase-centered Gaussian fluctuations, and precise partition asymptotics in the ordered regime. For general prescribed bounded strengths, we distinguish global ordering, local instability, and collapse.

hexagon:2611.00538v1

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Date: November 16, 2026.
2020 *Mathematics Subject Classification.* Primary 60K35; Secondary 60F05, 60F10, 76B47.
Key words and phrases. Point vortices, negative inverse temperature, Gaussian fluctuations, sinh–Gordon equation, mean-field limit, cluster expansion.

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1. Introduction

We prove a phase transition and spontaneous breaking of translation symmetry in a canonical ensemble of two species of point vortices on the square torus. With equal populations of strengths $+1$ and -1, the equilibrium changes at an explicit negative temperature from a spatially uniform state to a dipole of oppositely signed vortex clouds. We identify every globally minimizing profile throughout the ordered regime and describe both its random position and the Gaussian fluctuations around it. These two-species results are the main focus of the paper. They are embedded in a theory for general prescribed bounded vortex strengths, which also explains how the transition depends on the strength distribution.

The spatial domain is the unit square torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, with Lebesgue measure dx of mass one and mean-zero Green kernel $G = (-\Delta)^{-1}$. For prescribed strengths ξ_i, the canonical law and partition function are

$$d\mu_{N,\beta} = Z_N(\beta)^{-1} e^{\beta E_N}\, dx^{\otimes N}, \qquad E_N = \frac{1}{N}\sum_{i<j} \xi_i \xi_j G(x_i - x_j).$$

Thus $\beta > 0$ corresponds to negative physical inverse temperature. Particles are labeled and self-interactions are omitted. In the balanced case we take $N = 2n$, with exactly n strengths of each sign, and write $Z_N^{\pm}$ for the partition function and $L_n^{\pm} = n^{-1}\sum_{\xi_i=\pm 1} \delta_{x_i}$ for the two empirical probability measures. The mean-field signed vorticity is $\mathfrak{q}_N = (L_n^+ - L_n^-)/2$.

1.1. The two-species transition and its equilibrium profiles.

For conditional probability densities ρ_+, ρ_-, let $\mathrm{Ent}(\rho) = \int \rho \log \rho$ and $Q(f) = \langle f, Gf\rangle$. The balanced density free energy is

$$\mathcal{A}_\beta(\rho_+, \rho_-) = \frac{1}{2}[\mathrm{Ent}(\rho_+) + \mathrm{Ent}(\rho_-)] - \frac{\beta}{8} Q(\rho_+ - \rho_-).$$

We extend this action by $+\infty$ outside the finite-entropy domain. Its minimizers correspond to those of the potential action

$$\mathcal{F}_\beta(u) = \frac{1}{2\beta}\int |\nabla u|^2 - \frac{1}{2}\log\int e^u - \frac{1}{2}\log\int e^{-u}, \qquad \int u = 0, \tag{1.1}$$

through $\rho_\pm = e^{\pm u}/\int e^{\pm u}$. The first nonzero eigenvalue of $-\Delta$ is $4\pi^2$.

Theorem. *For every real $u \in H^1(\mathbb{T}^2)$,*

$$\log\int e^u + \log\int e^{-u} \le \frac{1}{4\pi^2}\int |\nabla u|^2.$$

The coefficient is optimal, and equality holds only for constants. Consequently, the unique balanced minimizing density pair is $(1,1)$ for $0 < \beta \le 4\pi^2$. For every $4\pi^2 < \beta < 16\pi$, all minimizing potentials form one free translation orbit $\{U_\beta(\cdot - a) : a \in \mathbb{T}^2\}$ of an explicit nonconstant profile. The corresponding densities are $\rho_{\pm,\beta} = e^{\pm U_\beta}/A_\beta$, where $A_\beta = \int e^{U_\beta} = \int e^{-U_\beta}$. At these nonconstant profiles, the Hessian has precisely the two translation zero modes and is strictly positive transverse to them.

The sharp inequality, complete minimizing set, and Hessian assertion are proved in Theorems 3.8 and 3.10 and Proposition 3.12; the exact transition is stated in Corollary 3.9.

The profile is concrete. With $\mathrm{K}(m)$ and $\mathrm{E}(m)$ the complete elliptic integrals with parameter m, put

$$U_m(x,y) = 4\,\mathrm{artanh}\left[\sqrt{\frac{m}{1-m}}\,\mathrm{cn}\big(2\mathrm{K}(m)(x+y)\mid m\big)\,\mathrm{cn}\big(2\mathrm{K}(m)(x-y)\mid m\big)\right],$$
$$\mathfrak{b}(m) = 16\mathrm{K}(m)\big[4\mathrm{E}(m) - (3-2m)\mathrm{K}(m)\big].$$

The map $\mathfrak{b}$ is a strictly increasing bijection from $(0,1/2)$ onto $(4\pi^2, 16\pi)$, and $U_\beta = U_{m_\beta}$ with $\mathfrak{b}(m_\beta) = \beta$. Each conditional density has one maximum, and the two maxima are separated by the half-diagonal $(1/2,1/2)$ of the torus. Exchanging the species translates the same dipole and creates no additional orbit. The branch emerges continuously: for each fixed r, as $\beta \downarrow 4\pi^2$,

$$U_\beta(x,y) = \sqrt{\frac{2(\beta - 4\pi^2)}{\pi^2}}\big(\cos 2\pi x + \cos 2\pi y\big) + O_{C^r}\big((\beta - 4\pi^2)^{3/2}\big).$$

There is no nonconstant global minimizer coexisting with the flat state at the transition and no later exchange between inequivalent minimizing branches in $(4\pi^2, 16\pi)$.

The variational classification describes the finite-particle ensemble through a large deviation principle. Write $p(\beta) = -\min \mathcal{A}_\beta$ and $\mathcal{M}_\beta = \operatorname{argmin} \mathcal{A}_\beta$.

Theorem. *For every $0 < \beta < 16\pi$, the balanced empirical pair satisfies a speed-N large deviation principle in the weak topology, with good rate function $I_\beta = \mathcal{A}_\beta + p(\beta)$, and $N^{-1} \log Z_N^{\pm}(\beta) \to p(\beta)$. It concentrates exponentially in every fixed weak neighborhood of $\mathcal{M}_\beta$. In particular,*

$$\begin{cases} (L_n^+, L_n^-) \longrightarrow (dx, dx) & \text{in probability}, \qquad 0 < \beta \le 4\pi^2, \\ (L_n^+, L_n^-) \Longrightarrow (\rho_{+,\beta}(\cdot - a)dx, \rho_{-,\beta}(\cdot - a)dx), & 4\pi^2 < \beta < 16\pi, \end{cases}$$

where a is Haar-uniform on $\mathbb{T}^2$.

The full statements are Corollaries 5.11 and 5.12.

This gives the precise sense of spontaneous translation symmetry breaking. Every finite-N Gibbs law is translation invariant, and each one-particle marginal remains uniform. Above $4\pi^2$, a typical empirical configuration nevertheless has a nonconstant macroscopic profile with a uniformly random position. Conditioning on that position, or aligning the configuration with its dipole, selects one member of the minimizing orbit. Averaging over the position restores the symmetry and conceals the ordered profile.

The transition can also be expressed through a scalar order parameter. For $k \in \mathbb{Z}^2$, define

$$\widehat{\mathfrak{q}}_N(k) = \frac{1}{N} \sum_{i=1}^N \xi_i e^{-2\pi i k \cdot x_i}, \qquad M_N = \big(|\widehat{\mathfrak{q}}_N(1,0)|^2 + |\widehat{\mathfrak{q}}_N(0,1)|^2 \big)^{1/2}.$$

The empirical-pair limit implies $M_N \to M(\beta)$ in probability, where $M(\beta) = 0$ for $0 < \beta \le 4\pi^2$ and $M(\beta) > 0$ for $4\pi^2 < \beta < 16\pi$. Positivity follows from the strict coordinate monotonicity of the centered profile and $-\Delta U_\beta = \beta \mathfrak{q}_\beta$, where $\mathfrak{q}_\beta = (\rho_{+,\beta} - \rho_{-,\beta})/2$. The branch expansion above gives

$$M(\beta) = \frac{\sqrt{\beta - 4\pi^2}}{\pi} + O\big((\beta - 4\pi^2)^{3/2}\big) \qquad (\beta \downarrow 4\pi^2).$$

Thus the ordering has exponent $1/2$. Translation invariance forces the expectation of each nonzero Fourier coefficient to vanish at every N; its magnitude detects the ordered phase without selecting the dipole's position.

There is a thermodynamic singularity as well. The reduced quartic form (7.43), evaluated along the explicit minimizing branch, yields

$$p(\beta) = \begin{cases} 0, & 0 < \beta \le 4\pi^2, \\ \dfrac{(\beta - 4\pi^2)^2}{8\pi^4} + O\big((\beta - 4\pi^2)^3\big), & \beta \downarrow 4\pi^2 \text{ from above.} \end{cases}$$

The action along this branch has a one-sided analytic expansion: the profile is analytic in $\sqrt{m}$ near zero, its action is even in $\sqrt{m}$, and $\mathfrak{b}'(0) = 2\pi^2$. Consequently p is C^1 across the transition, while $p''(4\pi^2-) = 0$ and $p''(4\pi^2+) = 1/(4\pi^4)$. The energy conjugate to β is continuous, with no latent heat. This is therefore a continuous, second-order transition in the Ehrenfest sense; see [46, §1.1.2] for this classification and its relation to an order parameter. The singularity concerns the large-N mean-field limit on the fixed torus; finite-N partition functions are analytic near $4\pi^2$.

1.2. **Fluctuations and Onsager condensation.** The random phase is also essential at the fluctuation scale. Below the transition, the signed vorticity fluctuates around zero. Above it, one must first remove a random macroscopic dipole and its two translation directions. Let a_N be the equivariant maximum-likelihood phase defined later, and in the ordered regime set

$$\eta_N^\sigma = \frac{1}{\sqrt{N}} \left(\sum_{\xi_i = \sigma} \delta_{x_i - a_N} - \frac{N}{2} \rho_{\sigma,\beta}\, dx \right), \qquad \sigma = \pm 1.$$

All Gaussian fields below have zero spatial mean; their homogeneous covariance operators act on $L_0^2(\mathbb{T}^2)$. For a self-adjoint Hilbert–Schmidt operator K with eigenvalues $\kappa_j < 1$, our regularized determinant convention is $\det_2(I - K) = \prod_j (1 - \kappa_j) e^{\kappa_j}$.

Theorem. *For every $s > 1$, the balanced ensemble has the following fixed-parameter limits as $N = 2n \to \infty$. If $0 < \beta < 4\pi^2$, then*

$$\sqrt{N}\, \mathfrak{q}_N \Longrightarrow \mathcal{N}\big(0, (I - \beta G)^{-1}\big) \quad \text{in } H^{-s}(\mathbb{T}^2), \qquad Z_N^{\pm}(\beta) \longrightarrow \det{}_2(I - \beta G)^{-1/2}.$$

If $4\pi^2 < \beta < 16\pi$, *then*

$$(a_N, \eta_N^+, \eta_N^-) \Longrightarrow (a, \eta_\beta^+, \eta_\beta^-) \quad \text{in } \mathbb{T}^2 \times H^{-s}(\mathbb{T}^2)^2,$$
$$Z_N^\pm(\beta) = C_\beta N e^{Np(\beta)}(1 + o(1)).$$

where a *is Haar-uniform and independent of the centered Gaussian pair* η_β, *and* C_β *is explicit and positive. The Gaussian covariance is obtained by inverting the density Hessian on the mass-preserving orthogonal complement of the translation scores, in the weighted density space defined below.*

The flat result is given in Theorem 6.7. The phase-centered theorem is Theorem 7.1, with covariance and partition constant in (7.9) and (7.10). The two-dimensional minimizing orbit gives the factor N in the nonconstant-phase partition asymptotic. The unshifted field $\sqrt{N}\,\mathfrak{q}_N$ is not tight in that phase; the Gaussian law describes the remainder after subtracting and aligning the dipole. At $\beta = 4\pi^2$ the macroscopic law is still flat, but the first four Fourier directions have degenerate quadratic precision. We prove $Z_N^\pm(4\pi^2) \geq cN$ and $N^{-1}\log Z_N^\pm(4\pi^2) \to 0$ in Proposition 7.7. A rescaled critical fluctuation law and endpoint-uniform asymptotics are left open.

These results give a precise canonical realization of Onsager's negative-temperature condensation picture. The ordering transition occurs at $4\pi^2$, strictly before the collapse threshold 16π. Throughout the interval between them the two vortex clouds are smooth, have positive width, and minimize the global free energy. Their profiles, separation, random position, and fluctuations are all identified. In particular, the uniform averaged density is compatible with macroscopic clustering in a typical configuration. The phase-centered limit separates this positional randomness from internal Gaussian fluctuations of the ordered state.

There is also an exact link to concentration at the upper end of the branch. For centered mean-field minimizers, as $\beta \uparrow 16\pi$,

$$\rho_{+,\beta}\,dx \Rightarrow \delta_{(0,0)}, \qquad \rho_{-,\beta}\,dx \Rightarrow \delta_{(1/2,1/2)},$$
$$\ell_\beta \sim \frac{\sqrt{16\pi - \beta}}{4\pi}, \qquad \ell_\beta^2 \rho_{+,\beta}(\ell_\beta z) \longrightarrow \frac{1}{\pi(1+|z|^2)^2}.$$

The concentration scale and local limit are proved in Subsection 3.5. This is a separate limit of the mean-field profiles; the particle partition function is infinite at $\beta = 16\pi$ for $N \geq 4$. The resulting description connects the continuous onset of an ordered dipole to its eventual concentration, while distinguishing both from the dynamical formation and lifetime of Onsager vortices. Comparisons with earlier condensation theories, simulations, and experiments are given in the literature discussion below.

1.3. **General prescribed strengths and the partition estimates.** The balanced case fits into a common quenched framework. Assume $|\xi_i^{(N)}| \leq M$ and $\nu_N = N^{-1}\sum_i \delta_{\xi_i^{(N)}} \Rightarrow \nu$, with $q_j = \int \xi^j\,d\nu$ and $q_2 > 0$. The strength law is prescribed; no averaging over strengths occurs inside the Gibbs weight. For mean-zero $u \in H^1(\mathbb{T}^2)$, define

$$L_\nu(u) = \int \log\left(\int e^{\xi u}\,dx\right)\nu(d\xi), \qquad \beta_H = \inf_{\substack{u \neq 0 \\ \int u = 0}} \frac{\int |\nabla u|^2}{2L_\nu(u)}.$$

The global homogeneous threshold β_H, the homogeneous stability threshold $\beta_{\mathrm{stab}} = 4\pi^2/q_2$, and the collapse threshold $\beta_{\mathrm{coll}} = 8\pi/\mathcal{C}_\nu$ satisfy $8\pi/q_2 \leq \beta_H \leq \min\{\beta_{\mathrm{stab}}, \beta_{\mathrm{coll}}\}$. Here $\mathcal{C}_\nu$ is the supremum of $(\int_K \xi\,d\nu)^2/\nu(K)$ over positive-mass Borel sets of strengths contained in either strict half-line. For conditional densities r_ξ, write

$$F_{\beta,\nu}(r) = \int \mathrm{Ent}(r_\xi)\,\nu(d\xi) - \frac{\beta}{2} Q\left(\int \xi(r_\xi - 1)\,\nu(d\xi)\right).$$

Theorem. *For every* $0 < \beta < \beta_{\mathrm{coll}}$, *the marked empirical measure* $\widehat{\rho}_N = N^{-1}\sum_i \delta_{(x_i,\xi_i)}$ *satisfies a speed-*N *large deviation principle on* $\mathcal{P}(\mathbb{T}^2 \times [-M, M])$ *with the weak topology. Its good rate equals* $F_{\beta,\nu} - \min F_{\beta,\nu}$ *on finite-entropy measures with strength marginal* ν, *and is* $+\infty$ *otherwise. Moreover,* $N^{-1}\log Z_N(\beta) \to -\min F_{\beta,\nu}$. *For every* $0 < \beta < \beta_H$ *and every* $s > 1$,

$$Z_N(\beta) \longrightarrow \det{}_2(I - \beta q_2 G)^{-1/2},$$
$$\frac{1}{\sqrt{N}}\sum_i \xi_i^{(N)}(\delta_{x_i} - dx) \Longrightarrow \mathcal{N}\big(0, q_2(I - \beta q_2 G)^{-1}\big) \quad \text{in } H^{-s}(\mathbb{T}^2).$$

These conclusions are proved in Theorems 5.1, 6.1, and 6.2, together with Corollary 5.7. The thresholds are identified in Theorem 2.12. If $\beta_H < \beta_{\mathrm{coll}}$, every $\beta_H < \beta < \beta_{\mathrm{coll}}$ has nonhomogeneous global minimizers. Fourth-order examples show that β_H can lie strictly below both stability and collapse, even for neutral two-species laws with unequal strengths. For symmetric bounded strength laws, the balanced sharp inequality instead gives $\beta_H = \beta_{\mathrm{stab}} = 4\pi^2/q_2$, with uniqueness also at that parameter. Thus neutrality alone does not determine the transition mechanism.

The probabilistic input is partition-function control with a strict temperature margin. We give two proofs: a stopped cluster expansion combined with interaction splitting for one species, and a Boué–Dupuis stochastic-control argument for general prescribed strengths. The first covers the whole one-species range $\beta < 8\pi$. The second gives order-one bounds in the homogeneous regime and exponential-scale bounds throughout strict subcollapse. For the nonconstant balanced phase, a localized exponential estimate supplies the additional uniform integrability needed for the phase-centered Gaussian theorem.

1.4. **Sources of the arguments and comparison with the literature.** Our two immediate probabilistic starting points are the cluster expansion and the Gaussian representation of the authors' previous works [5, 12]. The former proves the determinant limit and Gaussian vorticity fluctuations for one species on the square torus at sufficiently small negative physical inverse temperature. We retain its canonical cancellations and unicyclic skeletons, but split off a fixed smooth interaction before estimating the singular remainder. The latter develops partition-function convergence to the energy–enstrophy Gaussian ensemble at positive physical temperature on the torus and in bounded domains. The real Gaussian transform used for attractive interactions here has a different sign from the oscillatory transform appropriate to repulsion. In both approaches, a uniform partition bound at a slightly larger parameter is the decisive input for the Gaussian change of measure; the independent-reference limit is a canonical U-statistic limit of the type studied by Dynkin–Mandelbaum [18].

The full-subcritical one-species bound must also be compared directly with the recent preprint of Delgadino–Gvalani–Rosenzweig [19]. They prove an order-one partition bound for the periodic attractive logarithmic gas under a global log-HLS inequality, using a positive-definite smooth/singular splitting, centering inside cells, and an overcrowding estimate. In dimension two their kernel is $2\pi G$; their parameter is therefore $\beta/(2\pi)$ in our convention. The scalar torus inequality gives their threshold $B = 4$, so their result already supplies the bound for our one-species ensemble throughout $\beta < 8\pi$. For the one-species case our treatment provides two further alternative arguments, but our results extend to a more general setting treating prescribed signed and unequal intensities and fluctuations around a nonconstant translation orbit. The shared principle of restricting local particle counts also appears in our localized ultraviolet estimate, whose reference densities are nonuniform and whose enlarged moment parameter can exceed the unrestricted collapse threshold.

Our second partition-control argument comes from the stochastic variational approach to singular Gibbs measures. The representation formula is due to Boué–Dupuis [8], with the infinite-dimensional extensions in [9, 10]. Barashkov–Gubinelli [20] use stochastic control and renormalized coercivity to construct the Φ^4_3 measure; the Wick and covariance estimates here are also related to those in Barashkov–Gunaratnam–Hofstetter [11]. These works supply the variational method rather than a vortex partition theorem. The model-specific work below is the exact cancellation of the divergent variance, control of noise-dependent shifts in two size regimes, and the distinction between a zero-additive-constant global inequality, which gives an order-one bound, and an additive inequality, which gives only the exponential-scale bound needed for large deviations.

The variable-strength variational problem and its concentration threshold are grounded in Ricciardi–Suzuki [15]. Their average of logarithms is the appropriate quenched functional, and their strict-subcritical theorem supplies the boundedness input stated below. Conditional entropy duality then permits a common treatment of homogeneous stability, global minimization, and same-sign concentration. Our threshold comparison and the optimized fourth-order examples concern the separation of these mechanisms for a fixed strength law; neutrality and symmetry of that law have different consequences. The balanced sharp inequality makes the global threshold explicit for every symmetric bounded strength law.

The marked large-deviation argument draws on the logarithmic quasi-continuity mechanism of Bodineau–Guionnet [16] and the free-energy perspective of Berman [3]. Mariani [17] provides the equivalence with Γ-convergence of scaled relative entropies on laws of empirical measures. The additional points required by our deterministic triangular arrays are recovery laws with exactly the

actual empirical strength marginal and a local collision estimate compatible with the whole strict-subcollapse range. We prove the unnormalized variational limit before deriving pressure. An ordinary continuum free-energy Γ-limit, or convergence of mean energy under an arbitrary order-N entropy bound, would not replace these steps.

Onsager's negative-temperature mechanism [21] predicts that sufficiently energetic vortex configurations favor large-scale organization of like-signed vortices. Joyce–Montgomery [39] develop this picture through negative-temperature guiding-center plasma states. The rigorous canonical and microcanonical theories of Caglioti–Lions–Marchioro–Pulvirenti [22, 23], Kiessling [24], and Eyink–Spohn [25] establish the entropy–energy formulation and the role of nonuniform equilibria in logarithmic vortex systems. Our balanced results refine that picture by identifying the complete canonical minimizing orbit and its finite-particle fluctuations on the square torus. The ensemble and geometry are part of this conclusion; we do not infer equivalence with a microcanonical ensemble.

There are direct physical precedents for two-species Onsager condensation. Billam et al. [40] study neutral vortices in a periodic square through microcanonical sampling with a core-separation constraint and Gross–Pitaevskii simulations, obtaining opposite-sign macroscopic clusters and large-scale kinetic-energy accumulation. Simula–Davis–Helmerson [41] find the emergence of two coherent Onsager vortex clusters in simulations of an isolated planar superfluid and explain their formation through evaporative heating of vortices. These works address dynamical routes and quantum-fluid realizations. Our results give equilibrium statements for the singular canonical point-vortex measure, with a particle-limit theorem throughout its ordered regime.

Yu–Billam–Nian–Reeves–Bradley [42] give a particularly close mean-field comparison. For a neutral gas in a disk, their microcanonical theory describes a continuously emerging dipole, circulation-exchange and rotational symmetry breaking, and a clustering transition distinct from high-energy supercondensation. Our distinction between ordering and singular concentration therefore has precedents. The additional conclusions here are the exact canonical thresholds on the square torus, the classification of all global minimizers across the ordered interval, and the full empirical-measure and phase-centered fluctuation limits. In the present geometry species exchange is already realized by a translation, so the minimizing set has one orbit. Fluctuation descriptions also precede our work: Esler–Ashbee [43] develop microcanonical statistics and a transition from unimodal to bimodal leading-mode distributions. Their distinction between fluctuation and hydrodynamic energy scalings is relevant when comparing those laws with our speed-N variational limit and the Gaussian remainder around its nonconstant minimizers.

The experiments of Gauthier et al. [44] and Johnstone et al. [45] observe vortex clustering and the emergence of large-scale flow in planar superfluids, supporting the physical relevance of negative-temperature vortex ordering. They concern confined Bose–Einstein condensates driven away from global equilibrium, rather than the canonical torus limit proved here. Our equilibrium theory identifies a sharply defined onset, the ordered profiles that can be hidden by spatial averaging, and the fluctuations after their random position is removed; it does not establish an inverse cascade, a formation time, or a lifetime for these structures. More generally, the speed-N LDP locates minimizers but does not determine relative weights of inequivalent coexisting phases on subexponential scales.

For the scalar elliptic input, Lin–Lucia [14] identify periodic minimizers and give the sharp zero-additive-constant inequality on rectangular tori. Gui–Moradifam [6] and Gu–Gui–Hu–Li [26] develop symmetry and uniqueness for the single-exponential mean-field equation, the latter on arbitrary flat tori. The balanced functional here involves two separately normalized exponentials, so those scalar classifications do not establish its sharp inequality. The circle estimate used in its analysis follows from Lebedev–Milin in the form recalled by Chang–Gui [4].

Explicit elliptic sinh–Poisson vortex arrays are classical; early constructions include Ting–Chen–Lee [27]. The product family of Kuvshinov–Schep [29], recalled in Chow–Tsang–Mak [28, Eq. (4)], contains the family used here after rotation and period normalization. More precisely, set their moduli to $k = k_1 = \sqrt{m}$ and their wave numbers to $r = s = 2\sqrt{2}\mathrm{K}(m)$, and rotate to $\xi = (x+y)/\sqrt{2}$, $\eta = (x-y)/\sqrt{2}$. Gurarie–Chow [30] also study diagonal and elongated arrays and their dynamical stability numerically. Those stability observations do not establish a canonical entropy-Hessian kernel. We claim neither the first construction of that family nor a classification of every stationary sinh–Gordon solution. The argument below instead proves that every nonconstant global minimizer in the balanced square-torus problem belongs to the specified orbit, identifies the canonical parameter and its exact range, and proves the normal Hessian nondegeneracy needed for the fluctuation theorem. An explicit solution formula alone gives none of these variational or spectral conclusions.

Several recent results place the partition and fluctuation questions in a broader setting. Delgadino–Gvalani–Rosenzweig [31] obtain quantitative determinant normalization and linear-statistics limits for diffusive log/Riesz gases in the Hilbert–Schmidt regime, with an attractive logarithmic result in a perturbative range. Wang–Zhao [32] prove uniform partition estimates and determinant limits for repulsive modulated Coulomb-type energies at all positive temperatures on Euclidean space. These results treat more kernels and quantitative questions that we do not address because of our focus on the fluid dynamical vortex model, but do not identify the signed balanced translation orbit at negative temperature. Chodron de Courcel–Rosenzweig–Serfaty [33] distinguish stability and uniqueness thresholds and prove entropic propagation of chaos for the attractive periodic log gas using modulated free energy; their dynamics is overdamped and differs from the Hamiltonian vortex interpretation. Ren–Wang [34] give sharp quantitative chaos estimates for mean-field Gibbs measures under their convexity and regularity hypotheses. Our fixed-tag result is qualitative and uses deletion of finitely many interactions.

Positive-temperature generalized-Euler equilibria with random strengths and their fluctuations are studied by Geldhauser–Romito [35], with a jointly vanishing interaction cutoff for the more singular kernels. Grotto–Luongo–Romito [36] establish space-time equilibrium fluctuation limits for neutral vortices in bounded domains. The present work remains stationary; in particular its phase-centered Gaussian law is not a theorem about motion of the phase. For recent comparisons at other scales, Andreasson–Svensson [37] characterize finite-particle collapse and endpoint cluster measures for general logarithmic couplings on the sphere, whereas our β_{coll} describes boundedness of the limiting free energy and the strict range below it. Carreño-Donoso et al. [38] study dipolar and zonal states of an imbalanced gas in a disk using microcanonical theory, simulations, and annihilation–reinjection dynamics. Those results motivate related phase questions but do not replace the canonical torus classification or the centered Gaussian limit proved here.

1.5. **Organization of the paper.** Section 2 develops the common density–potential duality, compactness estimates, threshold comparisons, and examples for prescribed strengths. Section 3 treats the balanced equation: polarization and a third-order Jacobi identity classify global minimizers, the elliptic parameter identifies the transition and the complete orbit, and the linearized analysis gives normal nondegeneracy. It also gives the endpoint concentration scales and quantitative stability at the transition.

Section 4 contains the two alternative partition-control arguments in full. Section 5 proves the marked large deviation principle, deriving pressure from an unnormalized lifted free-energy limit before normalizing the relative entropy. Recovery laws retain the actual finite-N strength marginal, and local collision estimates handle the logarithmic singularity. The balanced empirical-pair LDP follows by specialization.

Section 6 proves the independent-reference limit and Gaussian change of measure once, and derives homogeneous partition asymptotics, marked chaos, and fixed-tag convergence in total variation. Section 7 proves the balanced phase-centered theorem using localized exponential integrability and an exact translation-pinning identity, and concludes with the critical behavior.

All distributional fluctuation fields lie in $H^{-s}(\mathbb{T}^2)$ for every $s > 1$ and have zero spatial average. All cutoffs, spatial partitions, and inverse temperatures are fixed before the particle limit unless stated otherwise. The general-array statements are asymptotic and allow a finite exceptional initial segment of nonnormalizable particle laws. The rectangular scalar example is explicitly identified as such.

2. General strengths: variational theory and thresholds

2.1. **The quenched ensemble and variational formulation.** Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, with normalized Lebesgue measure dx. All spatial potentials and fluctuation fields have zero spatial average. Let

$$A = -\Delta, \qquad G = A^{-1} \quad \text{on } L^2_0(\mathbb{T}^2), \qquad \lambda_1 = 4\pi^2, \qquad \lambda_k = 4\pi^2|k|^2 \quad (k \in \mathbb{Z}^2 \smallsetminus \{0\}). \tag{2.1}$$

We also write $G(x-y)$ for its kernel, normalized by $\int G = 0$. Its diagonal singularity is

$$G(x) = -\frac{1}{2\pi}\log|x| + O(1) \qquad (x \longrightarrow 0). \tag{2.2}$$

In particular, $G \in L^2(\mathbb{T}^2)$, and G is Hilbert–Schmidt but not trace class as an operator on L^2_0.

For each N, prescribe deterministic intensities $\xi_1^{(N)}, \dots, \xi_N^{(N)}$ satisfying

$$|\xi_i^{(N)}| \le M, \qquad \nu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_i^{(N)}} \Rightarrow \nu, \qquad q_2 := \int \xi^2 \, \nu(d\xi) > 0. \tag{2.3}$$

Superscripts (N) are suppressed when no ambiguity is possible. Write

$$m_N = \frac{1}{N}\sum_i \xi_i, \qquad q_N = \frac{1}{N}\sum_i \xi_i^2, \qquad q_j = \int \xi^j \, \nu(d\xi). \tag{2.4}$$

Then $q_N \to q_2$ and $m_N \to q_1$. Zero intensity is permitted. Neither neutrality nor symmetry of the intensity law is assumed.

With $P_N = dx^{\otimes N}$, define

$$E_N(x) = \frac{1}{N}\sum_{i<j} \xi_i \xi_j G(x_i - x_j), \qquad Z_N(\beta) = \mathbb{E}_{P_N} e^{\beta E_N}, \qquad d\mu_{N,\beta} = Z_N(\beta)^{-1} e^{\beta E_N} \, dP_N. \tag{2.5}$$

The sign convention is important: $\beta > 0$ corresponds to *negative physical inverse temperature*. Self-interactions are excluded. Nonzero mean intensity is compatible with this convention because G annihilates constants. The Gibbs law and its empirical law are defined whenever $Z_N(\beta) < \infty$. All general-array asymptotic assertions concern sufficiently large N and disregard any finite exceptional initial segment. A bound (P) at $b > \beta$ implies this eventual integrability by Hölder; it is proved in the ranges used below.

The marked empirical measure and centered vorticity fluctuation are

$$\widehat{\rho}_N = \frac{1}{N}\sum_i \delta_{(x_i,\xi_i)}, \qquad \eta_N = \frac{1}{\sqrt{N}}\sum_i \xi_i(\delta_{x_i} - dx). \tag{2.6}$$

The deterministic centering in η_N is $m_N dx$, not $q_1 dx$. Replacing it by $q_1 dx$ adds $\sqrt{N}(m_N - q_1)dx$, which need not vanish without a rate assumption. On mean-zero test functions the two centerings agree.

The two specializations developed below are the identical-strength one-species law $\nu = \delta_1$ and the balanced law $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$, with exactly equal finite populations. The general ensemble keeps the full prescribed intensity array; an intensity law is never averaged inside the Gibbs weight.

An admissible state is a measurable family of probability densities $r = (r_\xi)_{\xi \in [-M,M]}$ on $\mathbb{T}^2$, with fixed intensity marginal ν. Set

$$H_\nu(r) = \int \nu(d\xi) \int r_\xi \log r_\xi \, \mathrm{d}x, \tag{2.7}$$

$$w_r(x) = \int \xi(r_\xi(x) - 1)\, \nu(d\xi), \qquad \mathcal{E}_\nu(r) = \langle w_r, G w_r \rangle, \tag{2.8}$$

$$F_{\beta,\nu}(r) = H_\nu(r) - \frac{\beta}{2}\mathcal{E}_\nu(r). \tag{2.9}$$

We initially restrict to finite entropy; the energy is then finite, as proved below. The zero-temperature notation is not used here. At the homogeneous state $r_\xi \equiv 1$, both entropy and energy are zero.

For mean-zero $u \in H^1(\mathbb{T}^2)$ define

$$D(u) = \int |\nabla u|^2 \, \mathrm{d}x, \qquad L_\nu(u) = \int \log\left(\int e^{\xi u} \, \mathrm{d}x\right) \nu(d\xi), \qquad J_{\beta,\nu}(u) = \frac{D(u)}{2\beta} - L_\nu(u). \tag{2.10}$$

This is an *average of logarithms*, not the logarithm of an average over intensities. It is the quenched variational formulation; the density–potential duality is consistent with [15].

On a unit-area rectangular torus, set $\rho_0 = \min\{8\pi, \lambda_1\}$. Lin–Lucia [14, Theorem 1.1 and (1.4)] prove

$$\log \int e^u \, \mathrm{d}x \le \frac{D(u)}{2\rho_0}, \qquad \int u = 0, \tag{2.11}$$

with equality only for $u = 0$. In particular, on the unit square,

$$\log \int e^u \, \mathrm{d}x \le \frac{D(u)}{16\pi}. \tag{2.12}$$

This is a zero-additive-constant inequality, not merely the usual boundedness version of Moser–Trudinger. It implies, for every finite L, t,

$$\sup_{\substack{\int u=0 \\ D(u)\le L^2}} \int e^{t|u|}\,\mathrm{d}x \le 2\exp\left(\frac{t^2L^2}{16\pi}\right). \tag{2.13}$$

The last assertion follows by $e^{t|u|} \le e^{tu} + e^{-tu}$; it is a consequence rather than a separate imported theorem.

Lemma 2.1. *For each mean-zero $u \in H^1(\mathbb{T}^2)$,*

$$L_\nu(u) = \sup_r\{\langle w_r, u\rangle - H_\nu(r)\}, \tag{2.14}$$

where the supremum is attained at $r_\xi = e^{\xi u}/\int e^{\xi u}$. For every finite-entropy state r, $w_r \in H_0^{-1}(\mathbb{T}^2)$ and

$$\mathcal{E}_\nu(r) \le \frac{q_2}{4\pi} H_\nu(r). \tag{2.15}$$

Proof. The nonnegativity of conditional relative entropy gives, for each ξ,

$$\int r_\xi \log r_\xi\,\mathrm{d}x - \xi\int r_\xi u\,\mathrm{d}x + \log\int e^{\xi u}\,\mathrm{d}x \ge 0.$$

The identity follows after integrating in ν, with equality at the indicated conditional exponential density. Integrability is justified by the entropy inequality applied to $\pm 2\xi u$ and (2.13); alternatively begin with bounded u and truncate. The maximizing exponential density has finite entropy, since $\int e^{M|u|}(1+|u|)\,\mathrm{d}x < \infty$ and $\int e^{\xi u}\,\mathrm{d}x \ge 1$.

Averaging (2.12) applied to ξu gives

$$L_\nu(u) \le C_\nu D(u), \qquad C_\nu = \frac{q_2}{16\pi}. \tag{2.16}$$

For smooth mean-zero u and $t > 0$, (2.14) applied to $\pm tu$ yields

$$|\langle w_r, u\rangle| \le \frac{H_\nu(r)}{t} + C_\nu t D(u).$$

Optimizing in t shows $|\langle w_r, u\rangle| \le 2\sqrt{C_\nu H_\nu(r) D(u)}$. Thus w_r extends to an element of H_0^{-1} and $\|w_r\|_{H^{-1}}^2 \le 4C_\nu H_\nu(r)$, which is (2.15). If $H_\nu(r) = 0$, all $r_\xi = 1$ almost everywhere and the conclusion is immediate. □

Green energies and heat regularization. For a spatial density f, write $H(f) = \int f\log f$; for a mean-zero distribution g of finite Green energy write $Q(g) = \langle g, Gg\rangle$. The H^{-1} energy above agrees with the singular-kernel integral, and its mixed versions are compatible with heat approximation. Here are the details used in the density formulation. For a probability density f of finite entropy, the preceding estimate for the law δ_1 gives $Q(f-1) \le H(f)/(4\pi)$. Let $C_t = e^{t\Delta}G$. The heat-kernel formula gives

$$C_t \ge \min G, \qquad G - C_t = \int_0^t (p_s - 1)\,ds \ge -t, \qquad C_t \le G + t.$$

The Fourier quantities $\langle f, C_t * f\rangle$ increase to $Q(f-1)$ as $t \downarrow 0$. Since $C_t \to G$ off the diagonal, Fatou's lemma applied after subtracting $\min G$ shows that $\iint G(x-y)f(x)f(y)\,dx\,dy$ is finite and at most this Fourier limit. The upper bound $C_t \le G + t$ gives the reverse inequality in the limit. Thus the two definitions coincide.

For two such densities, Fourier Cauchy–Schwarz bounds the mixed heat energies by the product of their finite self-energy square roots. Applying the same Fatou and upper-bound argument to the positive product measure identifies the mixed integral with its Fourier limit as well. Constants disappear because G has zero mean. The argument extends to nonnegative finite-mass densities by normalization, and then to signed combinations by their positive and negative parts. In particular it applies to the marked vorticity: the nonnegative densities $v_\pm = \int \xi_\pm r_\xi\,\nu(d\xi)$ have masses $m_\pm = \int \xi_\pm\,\nu(d\xi)$, and, when $m_\pm > 0$, conditional Jensen gives

$$H(v_\pm/m_\pm) \le \frac{1}{m_\pm}\int \xi_\pm H(r_\xi)\,\nu(d\xi) \le \frac{M}{m_\pm} H_\nu(r).$$

Parts of zero mass are absent. This justifies all singular-kernel pairings in $\mathcal{E}_\nu$ on the finite-entropy domain.

Heat regularization contracts conditional entropy. For $r_\xi^t = e^{t\Delta} r_\xi$, weak lower semicontinuity therefore gives $H_\nu(r^t) \to H_\nu(r)$, while Fourier convergence gives $\mathcal{E}_\nu(r^t) \to \mathcal{E}_\nu(r)$. The positive heat kernel supplies smooth strictly positive spatial densities. These are the continuum approximation properties used in the exact-mark recovery construction of Lemma 5.5.

Hypothesis 2.2. For the inverse temperature $\beta > 0$ under consideration, there exists $\gamma > \beta$ such that, for all sufficiently large N and every mean-zero $u \in H^1(\mathbb{T}^2)$,

$$L_{\nu_N}(u) = \frac{1}{N}\sum_{i=1}^{N} \log \int e^{\xi_i u}\,\mathrm{d}x \le \frac{D(u)}{2\gamma}. \tag{H}$$

There is no additive constant, and the same γ works for all these N.

Hypothesis 2.3. For some $\gamma > \beta$ and $A_\gamma < \infty$, uniformly for all sufficiently large N,

$$L_{\nu_N}(u) \le \frac{D(u)}{2\gamma} + A_\gamma \qquad \text{for every mean-zero } u \in H^1(\mathbb{T}^2). \tag{B}$$

Hypothesis 2.4. For the chosen $\beta > 0$, there exist $b > \beta$, $C_b < \infty$, and N_0 such that

$$Z_N(b) \le e^{C_b N} \qquad (N \ge N_0). \tag{P}$$

Hypothesis (H) is the case of zero additive constant. Hypothesis (B) permits negative nonhomogeneous minima and coexistence; its passage from the limiting law to the empirical laws is proved in Lemma 2.15.

Theorem 2.5. *The mean-field problems satisfy*

$$\inf_r F_{\beta,\nu}(r) = \inf_{\int u = 0} J_{\beta,\nu}(u). \tag{2.17}$$

Minimizing pairs are related by

$$r_\xi(x) = \frac{e^{\xi u(x)}}{\int e^{\xi u}\,\mathrm{d}x}, \qquad u = \beta G w_r, \qquad -\Delta u = \beta \int \xi \left(\frac{e^{\xi u}}{\int e^{\xi u}\,\mathrm{d}x} - 1 \right) \nu(d\xi). \tag{2.18}$$

Under Hypothesis 2.2, the limiting law also satisfies $L_\nu(u) \le D(u)/(2\gamma)$, and

$$\frac{\gamma}{2}\mathcal{E}_\nu(r) \le H_\nu(r), \qquad F_{\beta,\nu}(r) \ge \left(1 - \frac{\beta}{\gamma}\right) H_\nu(r). \tag{2.19}$$

Thus the homogeneous state is the unique minimizer. The homogeneous Hessian is strictly positive, with

$$I - \beta q_2 G \ge \left(1 - \frac{\beta}{\gamma}\right) I. \tag{2.20}$$

The corresponding finite-N inequalities hold with ν_N, q_N.

Remark 2.6. In this theorem, uniqueness and spectral stability are consequences of (H), not extra assumptions. Conversely, uniqueness alone on a noncompact variational space does not automatically imply (H). The precise converse below collapse, with strict homogeneous stability, is stated in Proposition 2.13.

Proof of the duality and coercivity assertions of Theorem 2.5. Introduce the joint functional

$$\mathscr{A}_\beta(r, u) = H_\nu(r) + \frac{D(u)}{2\beta} - \langle w_r, u \rangle. \tag{2.21}$$

For fixed u, Lemma 2.1 gives $\inf_r \mathscr{A}_\beta(r, u) = J_{\beta,\nu}(u)$. For fixed r, completion of the square gives

$$\mathscr{A}_\beta(r, u) = F_{\beta,\nu}(r) + \frac{1}{2\beta} D(u - \beta G w_r). \tag{2.22}$$

Taking the two infima in either order proves (2.17). Equality cases give (2.18); no minimax interchange is used, only two infima.

For a fixed u, the map $\xi \mapsto \log \int e^{\xi u}\,\mathrm{d}x$ is continuous on $[-M, M]$, by domination by $e^{M|u|}$. Hence (H) passes from ν_N to ν. The entropy inequality gives

$$H_\nu(r) \ge \langle w_r, u \rangle - \frac{D(u)}{2\gamma}.$$

Taking the supremum over $u \in H_0^{1\text{mean}}$ yields $H_\nu(r) \geq \gamma \mathcal{E}_\nu(r)/2$, and then (2.19). Equality in the final lower bound is possible only if $H_\nu(r) = 0$, because $\gamma > \beta$. This identifies the unique homogeneous minimizer. □

Lemma 2.7. *At $r_\xi = 1$, on the quenched tangent space $\int h_\xi \, \mathrm{d}x = 0$, the Hessian is*

$$\delta^2 F_{\beta,\nu}(1)[h,h] = \int \|h_\xi\|_2^2 \, \nu(d\xi) - \beta \langle \int \xi h_\xi \, \nu(d\xi), G \int \xi h_\xi \, \nu(d\xi) \rangle. \tag{2.23}$$

It is strictly coercive if and only if $\beta < \lambda_1/q_2$. If $L_\nu \leq D/(2\gamma)$, then $\gamma q_2 \leq \lambda_1$.

Proof. The second variation of $r \log r$ at $r = 1$ is h^2, and the energy is quadratic. For a real spatial eigenfunction with eigenvalue λ_k, the intensity-space quadratic form is

$$\|a\|^2_{L^2(\nu)} - \frac{\beta}{\lambda_k} \left| \int \xi a(\xi) \, \nu(d\xi) \right|^2 .$$

It has eigenvalue $1 - \beta q_2/\lambda_k$ in the direction ξ, and eigenvalue 1 in the orthogonal complement. Its smallest eigenvalue is $1 - \beta q_2/\lambda_1$. This proves the stability assertion and the formula for β_{stab}.

For smooth mean-zero φ, expand

$$L_\nu(t\varphi) = \frac{q_2 t^2}{2} \|\varphi\|_2^2 + O(t^3).$$

The assumed inequality gives $\gamma q_2 \|\varphi\|_2^2 \leq D(\varphi)$. Testing a first eigenfunction yields $\gamma q_2 \leq \lambda_1$. The same argument works for ν_N, so

$$I - b q_N G_\varepsilon \geq (1 - b/\gamma) I \qquad (0 < b < \gamma). \tag{2.24}$$

This also completes the spectral assertion of Theorem 2.5; here $G_\varepsilon = e^{-\varepsilon A} G$ denotes the heat regularization used in the partition estimates. □

2.2. **Global transition, local stability, and collapse.** For a fixed bounded intensity law ν with $q_2 > 0$, define

$$\beta_H(\nu) = \inf_{u \neq 0 \,,\, \int u = 0} \frac{D(u)}{2L_\nu(u)} = \inf_{\mathcal{E}_\nu(r) > 0} \frac{2H_\nu(r)}{\mathcal{E}_\nu(r)}. \tag{2.25}$$

The first infimum is over $H^1(\mathbb{T}^2)$. Here $L_\nu(u) > 0$ for $u \neq 0$, by strict Jensen and $q_2 > 0$. The threshold β_H is the upper endpoint of the strict (H) regime: a parameter $\gamma > \beta$ with $L_\nu \leq D/(2\gamma)$ exists exactly when $\beta < \beta_H$.

The homogeneous stability and collapse thresholds are

$$\begin{aligned} \beta_{\text{stab}} &= \sup\{\beta : \text{the homogeneous Hessian is strictly positive}\}, \\ \beta_{\text{coll}} &= \sup\{\beta : \inf F_{\beta,\nu} > -\infty\}. \end{aligned} \tag{2.26}$$

For clarity, define the *genuine pre-collapse phase threshold* by

$$\beta_{\text{phase}} = \inf\{\beta \in (0, \beta_{\text{coll}}) : F_{\beta,\nu} \text{ has multiple minimizers}\}, \qquad \inf \varnothing = +\infty. \tag{2.27}$$

The numerical boundary of the interval with exactly one minimizer is instead β_H. If $\beta_H = \beta_{\text{coll}}$, uniqueness can end because minimizers cease to exist, not because multiple minimizers appear. A critical minimizer exactly at β_{coll} is a separate endpoint question and is excluded from the definition (2.27).

Remark 2.8. The threshold β_{stab} concerns the homogeneous branch. Nonhomogeneous equilibria have translation zero modes, so strict positivity at such an equilibrium is not the same question unless translations are first factored out. Similarly, "one-sign" below means all nonzero intensities have the same sign; it is not a synonym for a thermodynamic pure phase.

Define

$$\mathcal{C}_\nu = \sup_{\substack{K \subset (0,M] \text{ or } K \subset [-M,0) \\ \nu(K) > 0}} \frac{\left(\int_K \xi \, \nu(d\xi) \right)^2}{\nu(K)}, \tag{2.28}$$

where K is Borel. The supremum is positive and at most q_2. Including zero intensities cannot improve it.

For a compactly supported probability law ν, let $\mathcal{C}_\nu$ be (2.28). Ricciardi–Suzuki [15, Theorem 4, with (6)] imply that

$$\inf_{\int u=0}\left\{\frac{D(u)}{2b}-L_\nu(u)\right\}>-\infty \quad \text{whenever} \quad b<\frac{8\pi}{\mathcal{C}_\nu}. \tag{2.29}$$

Their intensity interval $[-1,1]$ is converted to $[-M,M]$ by $\xi=M\alpha$ and $v=Mu$. We use only the strict subcritical theorem. Unboundedness above the displayed threshold is proved below by concentration; the general critical endpoint is left open here.

Lemma 2.9. *If $u_n \rightharpoonup u$ in mean-zero $H^1(\mathbb{T}^2)$, then*

$$L_\nu(u_n)\longrightarrow L_\nu(u). \tag{2.30}$$

As $D(u)\to 0$,

$$L_\nu(u)=\frac{q_2}{2}\,\|u\|_2^2+O(D(u)^{3/2}), \tag{2.31}$$

uniformly on sufficiently small H^1 balls for bounded intensity support.

Proof. Rellich compactness gives $u_n\to u$ in L^2. By the elementary exponential increment inequality,

$$\sup_{|\xi|\le M}\int |e^{\xi u_n}-e^{\xi u}|\,\mathrm{d}x\le M\,\|u_n-u\|_2\left(\left\|e^{M|u_n|}\right\|_2+\left\|e^{M|u|}\right\|_2\right)\longrightarrow 0,$$

where the bracket is uniformly bounded by (2.13). Every normalizing integral is at least one, so the logarithm is Lipschitz on their range with Lipschitz constant at most one. This proves (2.30).

For (2.31), Taylor's formula gives, uniformly for $|\xi|\le M$,

$$\int e^{\xi u}\,\mathrm{d}x=1+\frac{\xi^2}{2}\,\|u\|_2^2+O\left(\int |u|^3e^{M|u|}\,\mathrm{d}x\right).$$

The last integral is $O(D(u)^{3/2})$ by Sobolev, Hölder, and (2.13). Expanding the logarithm adds an $O(D(u)^2)$ error. Integrating in ν proves the assertion. □

Lemma 2.10. *For every $\gamma>0$, the following are equivalent:*

$$L_\nu(u)\le\frac{D(u)}{2\gamma}\quad \textit{for all mean-zero } u; \qquad H_\nu(r)\ge\frac{\gamma}{2}\mathcal{E}_\nu(r)\quad \textit{for all } r. \tag{2.32}$$

Consequently the two infima in (2.25) agree. The inequality with coefficient $1/(2\beta_H)$ holds, while a coefficient $1/(2\gamma)$ with $\gamma>\beta$ exists exactly for $\beta<\beta_H$.

Proof. The forward implication was proved by maximizing the entropy inequality over u in the duality argument above. Conversely, Cauchy–Schwarz in $\dot H^{-1}$–$\dot H^1$ gives

$$\langle w_r,u\rangle-H_\nu(r)\le\sqrt{\mathcal{E}_\nu(r)D(u)}-\frac{\gamma}{2}\mathcal{E}_\nu(r)\le\frac{D(u)}{2\gamma}.$$

Take the supremum over r and use (2.14). The remaining conclusions follow immediately by taking the largest admissible γ. Positivity and finiteness follow from (2.16) and a first-eigenfunction test, respectively. □

Lemma 2.11. *For every $0<\beta<8\pi/\mathcal{C}_\nu$, $J_{\beta,\nu}$ has a smooth minimizer. The associated conditional densities in (2.18) minimize $F_{\beta,\nu}$. On compact parameter intervals strictly below $8\pi/\mathcal{C}_\nu$, the minimizing potentials are precompact in every C^k topology, and their conditional densities are uniformly bounded in every spatial C^k norm and bounded away from zero uniformly in the intensity.*

Proof. Choose $\beta<b<8\pi/\mathcal{C}_\nu$. By (2.29), $J_{b,\nu}\ge -C_b$. Hence

$$J_{\beta,\nu}(u)=J_{b,\nu}(u)+\frac{1}{2}\left(\frac{1}{\beta}-\frac{1}{b}\right)D(u)\ge cD(u)-C_b.$$

A minimizing sequence is bounded in mean-zero H^1. Weak compactness, lower semicontinuity of D, and Lemma 2.9 yield a minimizer u.

The Euler–Lagrange equation is (2.18). Its right-hand side belongs to every finite L^p, since $\int e^{\xi u}\,\mathrm{d}x\ge 1$ and its absolute value is at most $\beta M(e^{M|u|}+1)$. Elliptic regularity first yields $u\in W^{2,p}$ for all finite p, hence boundedness and classical regularity. Bootstrapping, using bounded intensity

support, gives smoothness in x. The joint-functional identity (2.22) then gives the minimizing density family. No regularity in the intensity variable beyond measurability was required.

For compactness, fix $0 < \beta_- \le \beta \le \beta_+ < 8\pi/\mathcal{C}_\nu$ and choose b with $\beta_+ < b < 8\pi/\mathcal{C}_\nu$. Since a minimizer satisfies $J_{\beta,\nu}(u) \le J_{\beta,\nu}(0) = 0$, the preceding coercive bound gives a uniform H^1 bound. Equation (2.13) then gives uniform bounds on the right side of (2.18) in every finite L^p. Elliptic estimates and bootstrapping yield bounds in every C^k. The formula for r_ξ gives the same spatial bounds uniformly for $|\xi| \le M$, and $e^{-2M\|u\|_\infty} \le r_\xi \le e^{2M\|u\|_\infty}$ gives uniform positivity. Limits at converging parameters remain global minimizers by comparison with each fixed test potential and continuity of L_ν. □

Theorem 2.12. *On the unit square torus,*

$$\beta_{\mathrm{stab}} = \frac{4\pi^2}{q_2}, \qquad \beta_{\mathrm{coll}} = \frac{8\pi}{\mathcal{C}_\nu}, \tag{2.33}$$

and

$$\frac{8\pi}{q_2} \le \beta_H \le \min\{\beta_{\mathrm{stab}}, \beta_{\mathrm{coll}}\}. \tag{2.34}$$

For every $\beta < \beta_{\mathrm{coll}}$, the free energy has a minimizer. For every $\beta < \beta_H$, that minimizer is uniquely homogeneous. If $\beta_H < \beta_{\mathrm{coll}}$, then

$$\beta_{\mathrm{phase}} = \beta_H, \tag{2.35}$$

and all $\beta \in (\beta_H, \beta_{\mathrm{coll}})$ have nonhomogeneous global minimizers and hence multiple translates. If $\beta_H = \beta_{\mathrm{coll}}$, there is no pre-collapse nonhomogeneous minimizing phase, so (2.27) gives $\beta_{\mathrm{phase}} = +\infty$.

For general ν, boundedness or existence exactly at $\beta = \beta_{\mathrm{coll}}$ is not asserted. The subcritical formula for β_{coll} uses [15, Theorem 4 and formula (6)]*; the supercritical direction is proved below.*

There is no universal order between β_{stab} and β_{coll}. Their comparison is explicit:

$$\frac{\beta_{\mathrm{coll}}}{\beta_{\mathrm{stab}}} = \frac{2q_2}{\pi\mathcal{C}_\nu}. \tag{2.36}$$

Thus $\beta_{\mathrm{coll}} < \beta_{\mathrm{stab}}$, $\beta_{\mathrm{coll}} = \beta_{\mathrm{stab}}$, or $\beta_{\mathrm{coll}} > \beta_{\mathrm{stab}}$ according as $\mathcal{C}_\nu$ is greater than, equal to, or less than $2q_2/\pi$.

For finitely many distinct nonzero species, $\nu = \sum_{j=1}^m p_j\delta_{a_j}$, formula (2.28) becomes

$$\beta_{\mathrm{coll}} = 8\pi \min_{\substack{\varnothing \ne I \subset \{1,\dots,m\} \\ a_j \text{ have the same sign on } I}} \frac{\sum_{j\in I} p_j}{\left(\sum_{j\in I} p_j a_j\right)^2}. \tag{2.37}$$

It is essential to allow subsets containing several species of the same sign.

Proof of the collapse formula in Theorem 2.12. The subcritical direction is (2.29), together with the density–potential duality. For the supercritical direction, choose a positive-mass same-sign set K and put

$$m = \nu(K), \qquad Q_K = \int_K \xi\,\nu(d\xi).$$

Let $\rho \in C_c^\infty(B(0,1))$ be a nonnegative probability density and set $\rho_\delta(x) = \delta^{-2}\rho(x/\delta)$ in a small coordinate disk of the torus. Define $r_\xi = \rho_\delta$ on K and $r_\xi = 1$ off K. Then

$$H_\nu(r) = 2m\log(1/\delta) + O(1), \tag{2.38}$$

$$w_r = Q_K(\rho_\delta - 1),$$

$$\mathcal{E}_\nu(r) = \frac{Q_K^2}{2\pi}\log(1/\delta) + O(1), \tag{2.39}$$

where the last equality follows by scaling the logarithmic kernel (2.2); its bounded remainder contributes $O(1)$ and the logarithmic self-energy of the smooth bump is finite. Thus

$$F_{\beta,\nu}(r) = \left(2m - \frac{\beta Q_K^2}{4\pi}\right)\log(1/\delta) + O(1). \tag{2.40}$$

For $\beta > 8\pi/\mathcal{C}_\nu$, a set K can be chosen with $Q_K^2/m > 8\pi/\beta$, so (2.40) tends to $-\infty$. This proves the threshold formula. The finite-species formula follows by taking K to be a subset of the distinct species. Finally Cauchy–Schwarz gives $Q_K^2/m \le \int_K \xi^2\,\nu(d\xi) \le q_2$. □

Proof of the ordering and phase assertions of Theorem 2.12. The formula for β_{stab} was proved in Lemma 2.7. Averaging the scalar inequality gives $\beta_H \geq 8\pi/q_2$. The small first-eigenfunction test gives $\beta_H \leq \lambda_1/q_2$. Also $J_{\beta,\nu} \geq 0$ for $\beta \leq \beta_H$, so collapse cannot occur below β_H; equivalently (2.38)–(2.39) inserted in the entropy quotient give $\beta_H \leq \beta_{\mathrm{coll}}$. This proves (2.34).

For $\beta < \beta_H$, Lemma 2.10 and Theorem 2.5 show uniqueness. For $\beta > \beta_H$, there exists u with $J_{\beta,\nu}(u) < 0$. If $\beta < \beta_{\mathrm{coll}}$, Lemma 2.11 supplies a minimizer, and it cannot be zero. A unique minimizer on the torus would be invariant under every translation and hence constant; the mean-zero normalization would force it to be zero. Thus its distinct translates are multiple minimizers. This proves (2.35) when $\beta_H < \beta_{\mathrm{coll}}$, and proves absence of a pre-collapse phase when $\beta_H = \beta_{\mathrm{coll}}$. It also proves that no later return to a homogeneous globally minimizing phase is possible: the same negative competitor remains negative as β increases.

Formula (2.36) is immediate from the explicit local and collapse thresholds. The equality criteria (2.42)–(2.43) follow from Lemma 2.10 and the corresponding upper bounds for β_H. □

Proposition 2.13. *If*

$$\beta_H < \min\{\beta_{\mathrm{stab}}, \beta_{\mathrm{coll}}\}, \tag{2.41}$$

then the quotient defining β_H is attained at a nonzero function. At $\beta = \beta_H$, the homogeneous state coexists with a nonhomogeneous minimizer. Above β_H and below both other thresholds, the homogeneous Hessian remains strictly positive, while the state is not globally minimizing. In the potential formulation it is a strict local minimum in the mean-zero H^1 topology.

Conversely, if $\beta < \beta_{\mathrm{coll}}$, the free energy has a unique minimizer, and the homogeneous Hessian is strictly positive, then $\beta < \beta_H$. Thus, below collapse, the interior of the unique strictly stable phase is exactly the strict (H) regime.

The exact global criteria for equality are

$$\beta_H = \beta_{\mathrm{stab}} \iff L_\nu(u) \leq \frac{q_2}{2\lambda_1} D(u) \quad \text{for every mean-zero } u, \tag{2.42}$$

$$\beta_H = \beta_{\mathrm{coll}} \iff L_\nu(u) \leq \frac{D(u)}{2\beta_{\mathrm{coll}}} \quad \text{for every mean-zero } u. \tag{2.43}$$

Equality with β_{stab} alone does not prove that the transition is continuous. It only identifies the numerical boundary; finite-amplitude minimizers at that boundary require separate analysis.

Proof of Proposition 2.13. Assume $\beta_H < \min\{\beta_{\mathrm{stab}}, \beta_{\mathrm{coll}}\}$ and choose $\beta_H < b < \beta_{\mathrm{coll}}$. Let $u_n \neq 0$ be a sequence with

$$r_n := \frac{D(u_n)}{2L_\nu(u_n)} \longrightarrow \beta_H.$$

Since $J_{b,\nu} \geq -C_b$, we have $L_\nu(u_n) \leq D(u_n)/(2b) + C_b$. Substitution of $D(u_n) = 2r_n L_\nu(u_n)$ gives

$$D(u_n)(1 - r_n/b) \leq 2r_n C_b.$$

Because $r_n < b$ eventually, the sequence is bounded in H^1. Extract $u_n \rightharpoonup u$. By Lemma 2.9, $L_\nu(u_n) \to L_\nu(u)$.

If $u = 0$, this convergence and $D(u_n) = 2r_n L_\nu(u_n)$ force $D(u_n) \to 0$. But (2.31) and Poincaré imply

$$L_\nu(u_n) \leq \frac{q_2}{2\lambda_1} D(u_n) + C D(u_n)^{3/2},$$

so $\liminf r_n \geq \lambda_1/q_2 = \beta_{\mathrm{stab}}$, a contradiction. Thus $u \neq 0$. Weak lower semicontinuity gives

$$\frac{D(u)}{2L_\nu(u)} \leq \liminf r_n = \beta_H.$$

By definition of β_H, equality holds. In particular $J_{\beta_H,\nu}(u) = 0 = J_{\beta_H,\nu}(0)$, giving coexistence. The corresponding density is nonhomogeneous: otherwise the minimizing-pair equation $u = \beta_H G w_r$ would force $u = 0$.

For the converse, uniqueness below collapse forces the minimizer to be homogeneous by translation invariance and therefore $\beta \leq \beta_H$. Strict homogeneous stability means $\beta < \beta_{\mathrm{stab}}$. If $\beta = \beta_H$, the assumptions give $\beta_H < \min\{\beta_{\mathrm{stab}}, \beta_{\mathrm{coll}}\}$, and the just-proved attainment would contradict uniqueness. Hence $\beta < \beta_H$. For the asserted local minimum, (2.31) and Poincaré give

$$J_{\beta,\nu}(u) \geq \frac{1}{2}\left(\frac{1}{\beta} - \frac{q_2}{\lambda_1}\right) D(u) - C D(u)^{3/2}.$$

When $\beta < \beta_{\mathrm{stab}}$, this is positive for every sufficiently small nonzero mean-zero H^1 potential. No claim of local minimality in an unspecified topology on density measures is needed. □

2.3. **Dependence on strengths and a fourth-order transition criterion.**

Proposition 2.14. *If ν_n, ν are supported in $[-M, M]$, $\nu_n \Rightarrow \nu$, and $q_2(\nu) > 0$, then*

$$\beta_H(\nu_n) \longrightarrow \beta_H(\nu). \tag{2.44}$$

More precisely, with $S(\nu) = 1/(2\beta_H(\nu))$,

$$|S(\nu_n) - S(\nu)| \le \frac{M}{4\pi} W_1(\nu_n, \nu). \tag{2.45}$$

Consequently, for every $\beta < \beta_H(\nu)$, a common $\gamma > \beta$ in (H) holds for all sufficiently large empirical laws ν_N. Hence Theorems 4.4, 6.2 and 6.5 apply throughout $0 < \beta < \beta_H(\nu)$ under (2.3).

Proof of Proposition 2.14. For nonzero mean-zero u, define

$$f_u(\xi) = \frac{\log \int e^{\xi u}\, \mathrm{d}x}{D(u)}.$$

It is convex, nonnegative, vanishes at zero, and satisfies $f_u(\xi) \le \xi^2/(16\pi)$ for every real ξ. For $|\xi| \le M$, convexity and the values at $\pm 2M$ bound every subgradient between $-M/(4\pi)$ and $M/(4\pi)$. For example the right derivative is at most

$$\frac{f_u(2M) - f_u(\xi)}{2M - \xi} \le \frac{(2M)^2}{16\pi M} = \frac{M}{4\pi},$$

and the left bound follows symmetrically. The entire family (f_u) is therefore uniformly Lipschitz on $[-M, M]$.

Now $S(\nu) = \sup_{u \ne 0} \int f_u \, d\nu$. The Kantorovich–Rubinstein bound yields

$$|S(\nu_n) - S(\nu)| \le \sup_{u \ne 0} \left| \int f_u \, d(\nu_n - \nu) \right| \le \frac{M}{4\pi} W_1(\nu_n, \nu).$$

Weak convergence on a common compact interval implies $W_1 \to 0$. The first-eigenfunction expansion gives $S(\nu) \ge q_2/(2\lambda_1) > 0$, so inversion proves continuity of β_H. For $\beta < \gamma < \beta_H(\nu)$, eventually $\gamma < \beta_H(\nu_N)$, which is exactly the uniform finite-N inequality (H). □

Lemma 2.15. *Under (2.3), every $\beta < \beta_{\mathrm{coll}}(\nu)$ satisfies (B) for some $\gamma > \beta$.*

Proof. Choose $\beta < \gamma < \gamma' < \beta_{\mathrm{coll}}$. By (2.29),

$$L_\nu(u) \le D(u)/(2\gamma') + A_{\gamma'}.$$

The argument proving Proposition 2.14 actually gives the stronger square-torus estimate

$$|L_{\nu_N}(u) - L_\nu(u)| \le \frac{M}{4\pi} W_1(\nu_N, \nu) D(u). \tag{2.46}$$

For $u = 0$ this is immediate; otherwise divide by $D(u)$ and use the uniform Lipschitz bound in the intensity variable. Eventually the coefficient on the right is smaller than $1/(2\gamma) - 1/(2\gamma')$. This yields (B), with the same finite additive constant. This proves the first implication in (5.22). Proposition 4.14 proves the second implication and (5.23). □

Proposition 2.16. *For a bounded law ν, if*

$$\frac{q_3^2}{q_2 q_4} > \frac{3}{13}, \tag{2.47}$$

then $\beta_H < \beta_{\mathrm{stab}}$. If also $\beta_{\mathrm{stab}} < \beta_{\mathrm{coll}}$, then

$$\beta_{\mathrm{phase}} = \beta_H < \beta_{\mathrm{stab}} < \beta_{\mathrm{coll}}. \tag{2.48}$$

The condition is sufficient, not necessary.

The test underlying this proposition is explicit. Put $\beta_0 = 4\pi^2/q_2$ and

$$f(x,y) = \cos(2\pi x) + \cos(2\pi y), \tag{2.49}$$

$$g(x,y) = \frac{q_3}{12q_2}\big(\cos(4\pi x) + \cos(4\pi y)\big) + \frac{q_3}{q_2}\cos(2\pi x)\cos(2\pi y). \tag{2.50}$$

Then

$$J_{\beta_0,\nu}(tf + t^2 g) = \frac{1}{96}\left(3q_4 - \frac{13q_3^2}{q_2}\right)t^4 + O(t^6). \tag{2.51}$$

The full computation, including the optimization over second harmonics, is in the calculation below.

Proof of Proposition 2.16. For a small bounded mean-zero u, the cumulant expansion is

$$L_\nu(u) = \frac{q_2}{2}\int u^2\,\mathrm{d}x + \frac{q_3}{6}\int u^3\,\mathrm{d}x + \frac{q_4}{24}\left[\int u^4\,\mathrm{d}x - 3\left(\int u^2\,\mathrm{d}x\right)^2\right] + O(\|u\|_\infty^5). \tag{2.52}$$

This expansion is uniform in ξ because the intensity support is bounded. Let f be (2.49), and first take arbitrary real coefficients A, B in

$$g = A(\cos(4\pi x) + \cos(4\pi y)) + B\cos(2\pi x)\cos(2\pi y).$$

Orthogonality of the trigonometric modes gives

$$\int f^2\,\mathrm{d}x = 1, \quad \int f^3\,\mathrm{d}x = 0, \quad \int f^4\,\mathrm{d}x = \frac{9}{4}, \quad \int fg\,\mathrm{d}x = 0, \tag{2.53}$$

$$\int f^2 g\,\mathrm{d}x = \frac{A+B}{2}, \qquad \|g\|_2^2 = A^2 + \frac{B^2}{4}, \qquad \frac{D(g)}{\lambda_1} = 4A^2 + \frac{B^2}{2}. \tag{2.54}$$

For $u_t = tf + t^2 g$ and $\beta_0 = \lambda_1/q_2$, the t^2 term cancels and the t^3 term vanishes. The t^4 coefficient in $J_{\beta_0,\nu}(u_t)$ is

$$\frac{q_2}{2}\left(3A^2 + \frac{B^2}{4}\right) - \frac{q_3}{4}(A+B) + \frac{q_4}{32}. \tag{2.55}$$

This is minimized at $A = q_3/(12q_2)$ and $B = q_3/q_2$, giving the coefficient in (2.51).

The map $t \mapsto J_{\beta_0,\nu}(u_t)$ is analytic near zero. Translation by $(1/2, 1/2)$ sends f to $-f$ and leaves g unchanged; hence it sends u_t to u_{-t}. Translation invariance makes the function of t even. Thus the error after the quartic term is $O(t^6)$, without any symmetry assumption on ν.

Under (2.47), the coefficient is negative. Fix a sufficiently small nonzero t with $J_{\beta_0,\nu}(u_t) < 0$. By continuity in β, the same strict inequality holds for some $\beta < \beta_0$. Therefore $\beta_H < \beta_{\mathrm{stab}}$. If $\beta_{\mathrm{stab}} < \beta_{\mathrm{coll}}$, Theorem 2.12 identifies the genuine phase threshold and proves (2.48). □

2.4. **Examples, rescaling, and symmetric strength laws.**

Example 2.17. *Identical positive intensities.* For $\nu = \delta_a$, $a > 0$,

$$\beta_H = \beta_{\mathrm{coll}} = \frac{8\pi}{a^2} < \frac{4\pi^2}{a^2} = \beta_{\mathrm{stab}}. \tag{2.56}$$

The homogeneous minimizer is unique even at $\beta = 8\pi/a^2$, by [14, Theorem 1.1]. Above that value the functional is unbounded below. Thus there is no intervening phase of multiple global minimizers. With definition (2.27), $\beta_{\mathrm{phase}} = +\infty$; if one instead calls the endpoint of uniqueness the phase boundary, its value is $8\pi/a^2$.

For one positive species δ_a on a unit-area rectangular torus with first eigenvalue λ_1, the Lin–Lucia theorem gives

$$\beta_H = \frac{\min\{8\pi, \lambda_1\}}{a^2}, \qquad \beta_{\mathrm{stab}} = \frac{\lambda_1}{a^2}, \qquad \beta_{\mathrm{coll}} = \frac{8\pi}{a^2}. \tag{2.57}$$

Thus sufficiently elongated rectangles have $\beta_H = \beta_{\mathrm{phase}} = \beta_{\mathrm{stab}} < \beta_{\mathrm{coll}}$. When $\lambda_1 = 8\pi$, the three numerical boundaries $\beta_H, \beta_{\mathrm{stab}}, \beta_{\mathrm{coll}}$ coincide, but there is still no pre-collapse interval of multiple minimizers. The numerical results in the preceding subsections refer to the square torus unless stated otherwise.

Verification. For $\nu = \delta_a$, put $v = au$. Then

$$J_{\beta,\delta_a}(u) = \frac{D(v)}{2\beta a^2} - \log\int e^v\,\mathrm{d}x.$$

By (2.11), zero is the unique minimizer for $\beta a^2 \le \min\{8\pi, \lambda_1\}$. The local upper bound and collapse upper bound give the opposite inequality for β_H, proving (2.57). The collapse constant is a^2, since the only nonempty active species set contains intensity a. On the square, $\lambda_1 = 4\pi^2 > 8\pi$, which gives (2.56). Equality in (2.11) gives uniqueness at the critical scalar endpoint, and (2.40) gives unboundedness above it.

Example 2.18. *Unequal positive intensities.* For

$$\nu = \frac{1}{101}\delta_{10} + \frac{100}{101}\delta_1, \tag{2.58}$$

we have

$$q_2 = \frac{200}{101}, \quad q_3 = \frac{1100}{101}, \quad q_4 = 100, \quad \frac{q_3^2}{q_2 q_4} = \frac{121}{202} > \frac{3}{13}. \tag{2.59}$$

Here $\mathcal{C}_\nu = (110/101)^2$, and

$$\beta_{\mathrm{phase}} = \beta_H < \beta_{\mathrm{stab}} = \frac{101\pi^2}{50} < \frac{20402\pi}{3025} = \beta_{\mathrm{coll}}. \tag{2.60}$$

Therefore the identical-strength one-sign picture does not extend to arbitrary positive intensity distributions.

Verification. For (2.58), direct arithmetic gives

$$q_2 = \frac{100+100}{101}, \quad q_3 = \frac{1000+100}{101}, \quad q_4 = \frac{10000+100}{101} = 100.$$

The two singleton values of $Q_K^2/\nu(K)$ are both $100/101$, whereas the full two-species value is $(110/101)^2$. The latter is larger, giving $\mathcal{C}_\nu = 12100/10201$. The moment ratio is $121/202 > 3/13$, so $\beta_H < \beta_{\mathrm{stab}}$. Finally

$$\frac{\beta_{\mathrm{coll}}}{\beta_{\mathrm{stab}}} = \frac{404}{121\pi} > 1.$$

This verifies every inequality in (2.60); it does not evaluate β_H itself.

Example 2.19. *An asymmetric non-neutral signed example.* For

$$\nu = \frac{1}{17}\delta_4 + \frac{16}{17}\delta_{-1}, \tag{2.61}$$

we have $q_1 = -12/17$, $q_2 = 32/17$, $q_3 = 48/17$, and $q_4 = 16$. The fourth-order coefficient in (2.51) is $-5/68$, and

$$\beta_{\mathrm{phase}} = \beta_H < \beta_{\mathrm{stab}} = \frac{17\pi^2}{8} < \frac{17\pi}{2} = \beta_{\mathrm{coll}}. \tag{2.62}$$

These examples establish strict separation, but do not compute the exact numerical value of β_H for the displayed mixtures.

Verification. For (2.61), the two same-sign subsets are singletons, both giving $\mathcal{C}_\nu = 16/17$. The moments are

$$q_2 = \frac{32}{17}, \quad q_3 = \frac{48}{17}, \quad q_4 = 16,$$

and the quartic coefficient is

$$\frac{1}{96}\left(48 - 13\frac{(48/17)^2}{32/17}\right) = -\frac{5}{68}.$$

Thus $\beta_H < \beta_{\mathrm{stab}}$. The comparison $17\pi^2/8 < 17\pi/2$ is equivalent to $\pi < 4$. This proves (2.62).

Example 2.20. *Neutral two-species laws.* Neutrality means $q_1 = 0$. It neither implies symmetry of ν nor forces $q_3 = 0$.

For

$$\nu = \frac{b}{a+b}\delta_a + \frac{a}{a+b}\delta_{-b}, \qquad a, b > 0, \tag{2.63}$$

we have

$$q_2 = ab, \quad q_3 = ab(a-b), \quad q_4 = ab(a^2 - ab + b^2), \tag{2.64}$$

and

$$\beta_{\mathrm{stab}} = \frac{4\pi^2}{ab}, \qquad \beta_{\mathrm{coll}} = \frac{8\pi(a+b)}{ab\max\{a,b\}}. \tag{2.65}$$

Let $r = \max\{a,b\}/\min\{a,b\} \geq 1$ and $r_0 = 2/(\pi - 2)$. Then

$$\frac{\beta_{\mathrm{coll}}}{\beta_{\mathrm{stab}}} = \frac{2}{\pi}\left(1 + \frac{1}{r}\right), \qquad \begin{cases} \beta_{\mathrm{stab}} < \beta_{\mathrm{coll}}, & 1 \leq r < r_0, \\ \beta_{\mathrm{stab}} = \beta_{\mathrm{coll}}, & r = r_0, \\ \beta_{\mathrm{coll}} < \beta_{\mathrm{stab}}, & r > r_0. \end{cases} \tag{2.66}$$

Both orders occur even under neutrality. Same-sign subpopulations can concentrate while the opposite sign remains diffuse.

The fourth-order condition is equivalent to

$$r > r_1 := \frac{23 + \sqrt{129}}{20}. \tag{2.67}$$

In particular, throughout $r_1 < r < r_0$ there is a genuine transition with $\beta_{\mathrm{phase}} = \beta_H < \beta_{\mathrm{stab}} < \beta_{\mathrm{coll}}$.

Example 2.21. *A concrete neutral strict-separation example.* For

$$\nu = \frac{4}{11}\delta_7 + \frac{7}{11}\delta_{-4}, \tag{2.68}$$

we have $q_2 = 28$, $q_3 = 84$, $q_4 = 1036$, and $q_3^2/(q_2 q_4) = 9/37 > 3/13$. The quartic coefficient is $-7/4$, so

$$\beta_{\mathrm{phase}} = \beta_H < \beta_{\mathrm{stab}} = \frac{\pi^2}{7} < \frac{22\pi}{49} = \beta_{\mathrm{coll}}. \tag{2.69}$$

This disproves equality of phase and stability thresholds for general neutral laws.

Verification. Neutrality gives the proportions in (2.63). Direct calculation yields (2.64). The two possible concentration constants are

$$\frac{a^2 b}{a+b}, \qquad \frac{ab^2}{a+b},$$

so $\mathcal{C}_\nu = ab\max\{a,b\}/(a+b)$, proving (2.65) and (2.66).

The quartic ratio simplifies to

$$\frac{q_3^2}{q_2 q_4} = \frac{(a-b)^2}{a^2 - ab + b^2} = \frac{(r-1)^2}{r^2 - r + 1}.$$

The condition that this exceed $3/13$ is $10r^2 - 23r + 10 > 0$. For $r \geq 1$ it is equivalent to (2.67). Since $r_1 < r_0$, this gives the asserted interval of strict separation.

For $a = 7, b = 4$, the neutral proportions are $4/11, 7/11$, and

$$q_2 = 28, \quad q_3 = 84, \quad q_4 = 1036, \quad \frac{q_3^2}{q_2 q_4} = \frac{9}{37} > \frac{3}{13}.$$

The quartic coefficient is

$$\frac{1}{96}\left(3 \cdot 1036 - 13\frac{84^2}{28}\right) = -\frac{7}{4}.$$

Moreover, $\mathcal{C}_\nu = 196/11$, giving $\beta_{\mathrm{coll}} = 22\pi/49$, while $\beta_{\mathrm{stab}} = \pi^2/7$. Their strict order is equivalent to the classical inequality $\pi < 22/7$. This proves (2.69).

Rescaling intensities. If $\nu^{(a)}$ is the law of $a\xi$ with $a \neq 0$, then $L_{\nu^{(a)}}(u) = L_\nu(au)$ and $D(au) = a^2 D(u)$. Consequently $\beta_H, \beta_{\mathrm{stab}}, \beta_{\mathrm{coll}}$, and any genuine finite β_{phase} are all divided by a^2. Thus normalizing q_2 to one preserves every strict ordering and every equality of the thresholds.

Symmetric intensity laws. The balanced sharp inequality proved in Theorem 3.8 has a consequence for every symmetric strength distribution, not only for two unit species.

Corollary 2.22. *Let ν be symmetric about zero, boundedly supported, and satisfy $q_2 > 0$. Then*

$$L_\nu(u) \leq \frac{q_2}{8\pi^2} D(u), \qquad \beta_H = \beta_{\mathrm{phase}} = \beta_{\mathrm{stab}} = \frac{4\pi^2}{q_2} < \frac{16\pi}{q_2} \leq \beta_{\mathrm{coll}}. \tag{2.70}$$

Equality in the potential inequality holds only for constant u. The homogeneous state is the unique global minimizer for $0 < \beta \leq 4\pi^2/q_2$, and nonhomogeneous global minimizers exist for every $4\pi^2/q_2 < \beta < \beta_{\mathrm{coll}}$.

Proof. Symmetry and Theorem 3.8, applied to ξu, give

$$L_\nu(u) = \frac{1}{2}\int \left[\log \int e^{\xi u}\,\mathrm{d}x + \log \int e^{-\xi u}\,\mathrm{d}x\right] \nu(d\xi) \le \frac{1}{2}\int \frac{\xi^2}{4\pi^2} D(u)\,\nu(d\xi).$$

If u is nonconstant, the integrand inequality is strict for every $\xi \neq 0$, a set of positive ν mass because $q_2 > 0$. A first-eigenfunction test gives the opposite bound $\beta_H \le 4\pi^2/q_2$, proving equality with β_{stab} and uniqueness at that parameter. For any same-sign set K, Cauchy–Schwarz and symmetry give

$$\frac{\left(\int_K \xi\,\nu(d\xi)\right)^2}{\nu(K)} \le \int_K \xi^2\,\nu(d\xi) \le \frac{q_2}{2}.$$

Consequently $\mathcal{C}_\nu \le q_2/2$ and $\beta_{\mathrm{coll}} \ge 16\pi/q_2 > 4\pi^2/q_2$. The phase assertions follow from Theorem 2.12. □

For the balanced unit law, (2.70) specializes to

$$\beta_H = \beta_{\mathrm{phase}} = \beta_{\mathrm{stab}} = 4\pi^2 < 16\pi = \beta_{\mathrm{coll}}. \tag{2.71}$$

The fourth-order coefficient in (2.51) is $1/32 > 0$. The global conclusion here comes from the sharp inequality; a positive local quartic coefficient by itself would not exclude a finite-amplitude competitor.

3. The balanced neutral equation and its minimizers

3.1. **Variational formulation and symmetry of minimizers.** Take $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$. The conditional densities are denoted by ρ_+, ρ_-. The general functionals become

$$\mathcal{A}_\beta(\rho_+,\rho_-) = \frac{1}{2}[\mathrm{Ent}(\rho_+) + \mathrm{Ent}(\rho_-)] - \frac{\beta}{8} Q(\rho_+ - \rho_-), \qquad Q(f) = \langle f, G * f\rangle, \tag{3.1}$$

where $\mathrm{Ent}(\rho) = \int \rho\log\rho$ and $\int \rho_\pm = 1$. Set $C(\rho,\eta) = \langle \rho, G*\eta\rangle$, $\mathcal{S} = \frac{1}{2}[\mathrm{Ent}(\rho_+) + \mathrm{Ent}(\rho_-)]$, $\mathcal{E}_\pm = \frac{1}{8}Q(\rho_+ - \rho_-)$, $p(\beta) = -\inf \mathcal{A}_\beta$, and $\mathcal{M}_\beta = \operatorname{argmin} \mathcal{A}_\beta$. For potentials write $\mathcal{E}(u) = D(u) = \int |\nabla u|^2$, $A_\pm(u) = \int e^{\pm u}$, and $\mathcal{S}(u) = \log A_+(u) + \log A_-(u)$. Thus the potential action is (1.1), invariant under adding constants; $\mathcal{U}_\beta$ denotes its mean-zero minimizing set.

The following records the specialization of the common variational theory, rather than repeating its proof.

Corollary 3.1. *For $0 < \beta < 16\pi$, the density and potential infima agree and are attained. Their minimizers correspond by*

$$\rho_\pm = \frac{e^{\pm u}}{A_\pm(u)}, \qquad u = \frac{\beta}{2} G * (\rho_+ - \rho_-), \tag{3.2}$$

and the smooth potential solves

$$-\Delta u = \frac{\beta}{2}\left(\frac{e^u}{A_+(u)} - \frac{e^{-u}}{A_-(u)}\right). \tag{3.3}$$

On compact subintervals of $(0, 16\pi)$ the minimizing potentials and densities are precompact in every C^k topology, and the densities are uniformly bounded away from zero.

Proof. The concentration constant is $\mathcal{C}_\nu = 1/2$, so $\beta_{\mathrm{coll}} = 16\pi$. Apply the general duality and subcritical compactness results to this law. The factors $1/2$ in the strength marginal give exactly (3.1) and (3.2). □

For later endpoint estimates it is useful to retain an explicit balanced coercivity bound. The scalar inequality, after subtracting the mean, implies the ordinary Moser–Trudinger estimate (also recorded in [2, Eq. (2.1)])

$$\log \int e^w \le \int w + \frac{1}{16\pi}\mathcal{E}(w) + C. \tag{3.4}$$

Apply it to $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$. The positive and negative gradient energies add, so $\mathcal{S}(u) \le \mathcal{E}(u)/(16\pi) + \int |u| + 2C$. For mean-zero u, Poincaré gives $\int |u| \le \sqrt{\mathcal{E}(u)}/(2\pi)$ and therefore

$$\mathcal{F}_\beta(u) \ge \left(\frac{1}{2\beta} - \frac{1}{32\pi}\right)\mathcal{E}(u) - \frac{1}{4\pi}\sqrt{\mathcal{E}(u)} - C. \tag{3.5}$$

The entropy–energy estimate from the general duality also gives $\mathrm{Ent}(\rho) \geq 4\pi Q(\rho)$ and hence, since $C(\rho_+, \rho_-) \geq \min G$,

$$\mathcal{A}_\beta(\rho_+, \rho_-) \geq \frac{1}{2}\left(1 - \frac{\beta}{16\pi}\right)[\mathrm{Ent}(\rho_+) + \mathrm{Ent}(\rho_-)] - C_\beta. \tag{3.6}$$

The action is extended by $+\infty$ outside the finite-entropy domain.

Lemma 3.2. *For a nonconstant real $v \in H^1(\mathbb{T})$,*

$$\log \int_{\mathbb{T}} e^v + \log \int_{\mathbb{T}} e^{-v} < \frac{1}{4\pi^2} \int_{\mathbb{T}} |v'|^2.$$

In particular a nonconstant function of x alone, or of y alone, cannot minimize $\mathcal{F}_{4\pi^2}$.

Proof. Subtract the mean and write $v = \sum_{n\neq 0} \widehat{v}(n) e^{2\pi i n x}$. The classical Lebedev–Milin inequality, in the form recalled in [4, Eq. (2.1)], gives

$$\log \int e^{\pm v} \leq \sum_{n\geq 1} n|\widehat{v}(n)|^2 \leq \sum_{n\geq 1} n^2 |\widehat{v}(n)|^2 = \frac{1}{8\pi^2} \int |v'|^2.$$

If any frequency $n \geq 2$ is present, the second inequality is strict. Otherwise $v = r\cos(2\pi x - \theta)$. For $r \neq 0$,

$$\int e^v = I_0(r) = \sum_{j=0}^{\infty} \frac{(r^2/4)^j}{(j!)^2} < \exp(r^2/4).$$

This proves strictness also in the remaining case. Since $\mathcal{F}_{4\pi^2}(0) = 0$, a nonconstant stripe, whose action is positive, is not a global minimizer. □

Lemma 3.3. *Let u be a global minimizer of $\mathcal{F}_\beta$ which depends on both coordinates. After translation,*

$$u(-x, y) = u(x, y), \qquad u(x, -y) = u(x, y), \tag{3.7}$$

$$u_x(x, y) < 0 \quad (0 < x < \tfrac{1}{2},\ y \in \mathbb{T}), \tag{3.8}$$

$$u_y(x, y) < 0 \quad (0 < y < \tfrac{1}{2},\ x \in \mathbb{T}). \tag{3.9}$$

The maximum at $(0, 0)$ is nondegenerate. Each coordinate has fundamental period exactly one.

Proof. For $R_a(x, y) = (2a - x, y)$ and $H_a = (a, a + 1/2) \times \mathbb{T}$, form the polarization $P_a u$: the maximum of $u, u \circ R_a$ on H_a and their minimum on the other half-torus. Pairing reflected points preserves $\int \Phi(u)$ for every continuous Φ. The weak-gradient formulas for maximum and minimum also give

$$\int |\nabla P_a u|^2 = \int |\nabla u|^2.$$

Thus $P_a u$ is a global minimizer, is smooth, and solves the same local semilinear equation as u:

$$-\Delta w = f(w), \qquad f(s) = \frac{\beta}{2}\left(\frac{e^s}{A_+(u)} - \frac{e^{-s}}{A_-(u)}\right), \qquad f'(s) > 0.$$

On H_a, $z = P_a u - u \geq 0$ satisfies $-\Delta z = c(x, y) z$ with bounded positive c. The strong minimum principle implies either $z = 0$ or $z > 0$ throughout H_a. Applying it also to $u - u \circ R_a$ in the ordered case yields the trichotomy

$$u - u \circ R_a > 0 \text{ on } H_a, \quad u - u \circ R_a < 0 \text{ on } H_a, \quad \text{or} \quad u = u \circ R_a. \tag{3.10}$$

In a strict case, the Hopf lemma at $x = a$ gives $u_x(a, y) \neq 0$ for every y. Therefore any zero $u_x(a, y_0) = 0$ forces reflection symmetry about $x = a$.

For completeness, this order property implies one common maximum axis and no smaller coordinate period. The subgroup of x-translations preserving u is closed. Since u depends on x, it is a finite subgroup, with a least positive period $1/n$. If $n \geq 2$, each reflection difference $u - u \circ R_a$ has period $1/n \leq 1/2$ and integral zero over one such period on every horizontal fiber. It cannot have a strict single sign throughout H_a. By (3.10) every reflection would then preserve u, forcing independence of x, a contradiction. Hence $n = 1$.

Choose a maximum of a nonconstant horizontal fiber, at $x = a$. It gives a reflection axis for the whole function. Two reflection axes a, b produce the translation period $2(a - b)$; consequently the only axes are a and $a + 1/2$ modulo one. At every other x, $u_x(x, y)$ is nowhere zero, by the Hopf conclusion above. Its sign is constant on the connected half-cylinder and is negative on $(a, a + 1/2)$, since a was a maximum. Translate a to zero. Repeating in y proves (3.7)–(3.9).

The function $-u_x > 0$ on $(0,1/2)\times\mathbb{T}$ vanishes on its boundary and solves $(-\Delta - f'(u))(-u_x) = 0$. Hopf gives $u_{xx}(0,y) < 0$ and $u_{xx}(1/2,y) > 0$. Likewise $u_{yy}(x,0) < 0$ and $u_{yy}(x,1/2) > 0$. At $(0,0)$, parity gives $u_{xy} = 0$, proving nondegeneracy of the maximum. □

3.2. **Elliptic classification of minimizing profiles.**

Proposition 3.4. *Let $U \in C^\infty(\mathbb{T}^2)$ solve*

$$-\Delta U = \kappa \sinh U, \qquad \kappa > 0, \tag{3.11}$$

and satisfy

$$U(-x,y) = U(x,y), \qquad U(x,-y) = U(x,y),$$
$$U_x(x,y) < 0 \quad (0 < x < \tfrac{1}{2},\ y \in \mathbb{T}), \quad U_y(x,y) < 0 \quad (0 < y < \tfrac{1}{2},\ x \in \mathbb{T}).$$

Then there exists $0 < m < 1/2$ such that

$$\kappa = 16\mathrm{K}(m)^2(1-2m), \quad \mathrm{K}(m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{-1/2}\, d\theta. \tag{3.12}$$

and

$$U(x,y) = 4\tanh^{-1}\left[\sqrt{\frac{m}{1-m}}\ \mathrm{cn}(2\mathrm{K}(m)(x+y)\mid m)\ \mathrm{cn}(2\mathrm{K}(m)(x-y)\mid m)\right]. \tag{3.13}$$

The proof is divided into the following four steps.

Step 1: the third-order Jacobi field. Set $\partial_z = (\partial_x - i\partial_y)/2$, $\partial_{\bar z} = (\partial_x + i\partial_y)/2$, and $L = -\Delta - \kappa\cosh U$. Direct differentiation gives

$$\Phi = U_{zzz} - \tfrac{1}{2}U_z^3, \qquad L\Phi = 0.$$

Indeed, $U_{z\bar z} = -\kappa\sinh U/4$ and

$$\partial_{\bar z}\Phi = -\frac{\kappa}{4}\cosh U\, U_{zz} + \frac{\kappa}{8}\sinh U\, U_z^2, \qquad \partial_z\partial_{\bar z}\Phi = -\frac{\kappa}{4}\cosh U\,\Phi.$$

The real part of Φ is odd in x and even in y. On the x half-cylinder, $-U_x$ is a strictly positive Dirichlet eigenfunction of L at eigenvalue zero. The principal Dirichlet eigenvalue is therefore zero and is simple. Since $\Re\Phi$ is a Dirichlet null vector, it is a constant multiple of U_x. Similarly $\Im\Phi$ is a constant multiple of U_y. Equivalently, there are real constants a, b such that

$$U_{zzz} - \tfrac{1}{2}U_z^3 = aU_z + bU_{\bar z}. \tag{3.14}$$

Differentiating in $\bar z$ and setting $c = 4b/\kappa$ gives $\cosh U\, U_{zz} + cU_{\bar z\bar z} = \sinh U\left(a + \tfrac{1}{2}U_z^2\right)$. Its imaginary part is

$$(\cosh U - c)U_{xy} = \tfrac{1}{2}\sinh U\, U_x U_y. \tag{3.15}$$

Step 2: determination of the separation coefficient. We show that necessarily $c = -1$. Work in $Q = (0,1/2)^2$, where $U_xU_y \neq 0$. On a patch with $U \neq 0$ and $\cosh U \neq c$, put

$$f(U) = \frac{\sinh U}{2(\cosh U - c)}, \qquad q(U) = -\kappa\sinh U.$$

Then $U_{xy} = f(U)U_xU_y$ and $\Delta U = q(U)$. The compatibility identity $\Delta(U_{xy}) = (\Delta U)_{xy}$ gives

$$(f'' + 2ff')|\nabla U|^2 + 3qf' + 2qf^2 + q'f - q'' = 0. \tag{3.16}$$

For clarity, $\Delta(f(U)U_xU_y)$ divided by U_xU_y is $(f'' + 2ff')|\nabla U|^2 + 3qf' + 2q'f + 2qf^2$, while the other side divided by U_xU_y is $q'' + q'f$. The coefficients in (3.16) simplify to

$$f'' + 2ff' = \frac{(c^2-1)\sinh U}{2(\cosh U - c)^3}, \qquad 3qf' + 2qf^2 + q'f - q'' = \frac{\kappa(c^2-1)\sinh U}{(\cosh U - c)^2}.$$

If $c^2 \neq 1$, this yields

$$|\nabla U|^2 = 2\kappa(c - \cosh U). \tag{3.17}$$

The excluded level sets have empty interior because U_x, U_y are strictly negative in Q. Thus (3.17) extends throughout Q and to its closure. Bochner's identity, compared with the Laplacian of (3.17), gives

$$|D^2U|^2 = \kappa^2\sinh^2 U = (\Delta U)^2, \quad \text{hence} \quad \det D^2U = 0.$$

Continuity contradicts the negative-definite Hessian at $(0,0)$, which follows by Hopf exactly as in Lemma 3.3. Therefore $c = \pm 1$.

Integrating (3.11) shows $\int \sinh U = 0$. The strict monotonicity implies that the maximum is positive, the minimum is negative, and the diagonal through Q crosses the level $U = 0$ at an interior point. If $c = 1$, differentiating (3.15) in x at that point gives $0 = \frac{1}{2}U_x^2 U_y$, a contradiction. Consequently

$$c = -1, \qquad b = -\kappa/4, \qquad U_{xy} = \tfrac{1}{2}\tanh(U/2)U_x U_y. \tag{3.18}$$

Step 3: separated variables and quartic oscillators. Define

$$v = \arctan(\sinh(U/2)) \in (-\pi/2, \pi/2).$$

Since $dv/dU = \frac{1}{2}\operatorname{sech}(U/2)$, (3.18) says $v_{xy} = 0$. Thus, globally on the torus,

$$v = f(x) + g(y).$$

The inverse is $U = 2\operatorname{artanh}(\sin v)$, with $dU/dv = 2\sec v$. Put $p = f'$, $q = g'$. The equation for v is

$$p' + q' + \tan v\,(p^2 + q^2 + \kappa) = 0. \tag{3.19}$$

The functions p, q vanish exactly at $0, 1/2$ modulo one and are strictly negative between these points. In particular each has fundamental period one, and $p'(0), q'(0) < 0$.

To extract their ordinary differential equations without any independence assumption on p^2 and q^2, retain the full identity (3.14). Substitute $U = 2\operatorname{artanh}(\sin v)$ into its real and imaginary parts. On Q this gives

$$\frac{p''}{p} + 3\tan v(p' - q') - p^2 + 3q^2 = 4a - \kappa, \tag{3.20}$$

$$\frac{q''}{q} + 3\tan v(q' - p') - q^2 + 3p^2 = -4a - \kappa. \tag{3.21}$$

For example,

$$\Re\Phi = \frac{\sec v}{4}\left[p'' + 3\tan v\, p(p' - q') - p^3 + 3pq^2\right],$$

which verifies (3.20) directly. Differentiating (3.19) in x and y, dividing respectively by p, q, and subtracting, yields

$$\frac{p''}{p} - \frac{q''}{q} = -2\tan v(p' - q').$$

Elimination in (3.20)–(3.21) therefore gives

$$p'' = c_1 p - 2p^3, \qquad q'' = c_2 q - 2q^3, \qquad c_1 = -\kappa - 2a, \quad c_2 = -\kappa + 2a.$$

These equations extend to the endpoints by smoothness. Their first integrals are

$$(p')^2 = -p^4 + c_1 p^2 + d_1, \qquad (q')^2 = -q^4 + c_2 q^2 + d_2, \quad d_1, d_2 > 0.$$

There is a second relation between the constants. Let $P = p'(0) < 0$, $Q_0 = q'(0) < 0$, $v_0 = v(0,0)$. At this corner (3.19) gives $P + Q_0 = -\kappa \tan v_0$. Differentiate (3.19) twice in x, evaluate at the corner, and divide by P. Using $p'''(0) = c_1 P$ gives $c_1 + \kappa \sec^2 v_0 + 2P \tan v_0 = 0$. The corresponding y identity has c_2, Q_0 in place of c_1, P. Since $d_1 = P^2, d_2 = Q_0^2$, subtraction gives

$$d_1 - d_2 = \frac{\kappa}{2}(c_1 - c_2).$$

Together with $c_1 + c_2 = -2\kappa$, this is equivalent to

$$d_1 + c_1^2/4 = d_2 + c_2^2/4. \tag{3.22}$$

Step 4: the square periods fix the elliptic family. A crossing periodic oscillator $h'' = ch - 2h^3$, with first-integral constant $d > 0$, has the parametrization

$$h(t) = \omega\sqrt{m}\ \operatorname{cn}(\omega t + \theta \mid m), \quad c = \omega^2(2m - 1), \quad d = \omega^4 m(1 - m), \quad 0 < m < 1.$$

This follows either from the defining differential equation of cn or from the positive and negative roots of $-h^4 + ch^2 + d$. In particular

$$d + c^2/4 = \omega^4/4, \qquad \text{fundamental period} = 4\mathrm{K}(m)/\omega.$$

For p, q, (3.22) gives $\omega_1 = \omega_2$. Their fundamental periods are both one, so $K(m_1) = K(m_2)$. Since K is strictly increasing, $m_1 = m_2 = m$. Consequently $c_1 = c_2 = -\kappa$, $\quad a = 0$, $\quad \omega = 4\mathrm{K}(m)$, $\quad \kappa = 16\mathrm{K}(m)^2(1 - 2m)$. The positivity of κ gives $m < 1/2$. Their identical initial data give $p = q$.

Write $K = \mathrm{K}(m)$ and

$$F_m(t) = -\arcsin\bigl(\sqrt{m}\,\operatorname{sn}(4Kt - K \mid m)\bigr) = \arctan\left(\sqrt{\frac{m}{1-m}}\,\operatorname{cn}(4Kt \mid m)\right). \tag{3.23}$$

Its derivative is $-4K\sqrt{m}\,\operatorname{cn}(4Kt - K \mid m)$, so $v = F_m(x) + F_m(y) + C$. At $(0, 1/2)$ the terms p', q' cancel while $p = q = 0$. Equation (3.19) gives $\kappa \tan C = 0$. Here $C = v(0, 1/2) \in (-\pi/2, \pi/2)$; hence $C = 0$.

The Jacobi addition identities, detailed below, now give

$$\tan\frac{F_m(x) + F_m(y)}{2} = \sqrt{\frac{m}{1-m}}\,\operatorname{cn}(2K(x+y) \mid m)\operatorname{cn}(2K(x-y) \mid m).$$

Since $\tanh(U/4) = \tan(v/2)$, this proves (3.13) and completes the proof of Proposition 3.4.

For checking the remaining identities, we use Jacobi functions with parameter $m = k^2$. The standard addition formulas for cn and dn are given in [7, §22.8(i)]. Here is the algebra needed for the particular product identity in the proof. Write $k = \sqrt{m}$, $k' = \sqrt{1-m}$, $a = k/k'$, and $B(t) = \arctan(a\,\operatorname{cn}(t \mid m))$. The identities $\operatorname{dn}^2 t = k'^2 + k^2 \operatorname{cn}^2 t$ give

$$\sin B(t) = \frac{k \operatorname{cn} t}{\operatorname{dn} t}, \qquad \cos B(t) = \frac{k'}{\operatorname{dn} t}.$$

Thus

$$\tan\frac{B(2P) + B(2Q)}{2} = a\frac{\operatorname{cn}(2P)\operatorname{dn}(2Q) + \operatorname{cn}(2Q)\operatorname{dn}(2P)}{\operatorname{dn}(2P) + \operatorname{dn}(2Q)}. \tag{3.24}$$

To simplify the fraction, set $R = P + Q$, $S = P - Q$ and

$$D = 1 - m\operatorname{sn}^2 R \operatorname{sn}^2 S, \quad A = \operatorname{cn} R \operatorname{cn} S, \quad B_1 = \operatorname{sn} R \operatorname{sn} S \operatorname{dn} R \operatorname{dn} S,$$

$C = \operatorname{dn} R \operatorname{dn} S$, $\quad D_1 = m \operatorname{sn} R \operatorname{sn} S \operatorname{cn} R \operatorname{cn} S$. The addition formulas give

$$\operatorname{cn}(R \pm S) = \frac{A \mp B_1}{D}, \qquad \operatorname{dn}(R \pm S) = \frac{C \mp D_1}{D}.$$

The numerator of the fraction in (3.24) is $2(AC - B_1 D_1)/D^2 = 2AC/D$, and its denominator is $2C/D$. The fraction is therefore $\operatorname{cn} R \operatorname{cn} S$. With $P = 2Kx$, $Q = 2Ky$, and $F_m(t) = B(4Kt)$, we obtain

$$\tan\frac{F_m(x) + F_m(y)}{2} = \sqrt{\frac{m}{1-m}}\,\operatorname{cn}(2K(x+y) \mid m)\operatorname{cn}(2K(x-y) \mid m).$$

All half-angle branches are unambiguous: $|F_m| \le \arcsin\sqrt{m} < \pi/4$ when $m < 1/2$. The phase-shift identity in (3.23) follows from $\operatorname{sn}(t - K) = -\operatorname{cn} t/\operatorname{dn} t$ and the same relations for sine and cosine.

Remark 3.5. Conversely, the formula (3.13) defines a smooth periodic solution of (3.11) for every $0 < m < 1/2$, with κ given by (3.12). Here is a direct check, which also justifies applying the normalization calculation to every member of the family. In orthogonal coordinates $\xi = (x+y)/\sqrt{2}$, $\eta = (x-y)/\sqrt{2}$, set $\omega = 2\sqrt{2}\mathrm{K}(m)$ and $w = \sqrt{m/(1-m)}\,\operatorname{cn}(\omega\xi \mid m)\operatorname{cn}(\omega\eta \mid m)$. The defining Jacobi identities are

$$(\operatorname{cn}')^2 = (1-m) + (2m-1)\operatorname{cn}^2 - m\operatorname{cn}^4, \qquad \operatorname{cn}'' = (2m-1)\operatorname{cn} - 2m\operatorname{cn}^3.$$

Substitution, with $\kappa = 2\omega^2(1-2m)$, gives

$$(1-w^2)\Delta w + 2w|\nabla w|^2 + \kappa w(1+w^2) = 0.$$

For $U = 4\operatorname{artanh} w$, this identity is exactly $-\Delta U = \kappa \sinh U$. The bound $|w| < 1$ gives smoothness, and the half-period signs of cn give one-periodicity in x, y.

3.3. **The canonical parameter and the complete minimizing set.** The coefficient κ in (3.11) is not the parameter β in (3.3). Starting from a canonical critical point u, set

$$U = u + \tfrac{1}{2}\log\frac{A_-(u)}{A_+(u)}, \qquad \kappa = \frac{\beta}{\sqrt{A_+(u)A_-(u)}}. \tag{3.25}$$

Then

$$-\Delta U = \kappa \sinh U, \qquad \beta = \kappa \int \cosh U. \tag{3.26}$$

The shift preserves all the geometry in Lemma 3.3.

The half-period identity

$$U_m(x+\tfrac12, y+\tfrac12) = -U_m(x,y) \tag{3.27}$$

shows that $\int U_m = 0$ and $A_m := \int e^{U_m} = \int e^{-U_m} = \int \cosh U_m$. The canonical parameter will be denoted by $\mathfrak{b}(m) = \kappa_m A_m$, to distinguish it from the finite self-interaction correction introduced in Section 7. We use elliptic parameter m, not modulus k; thus $m = k^2$ relative to the convention in [7]. Define

$$\mathrm{K}(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \qquad \mathrm{E}(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\, d\theta.$$

Here K in the classification is the same integral K.

Proposition 3.6. *The map $\mathfrak{b} : (0, 1/2) \to (4\pi^2, 16\pi)$ is a real-analytic strictly increasing bijection, and*

$$\mathfrak{b}(m) = 16\mathrm{K}(m)\big[4\mathrm{E}(m) - (3-2m)\mathrm{K}(m)\big]. \tag{3.28}$$

The calculation follows from a normalization identity which is useful independently of the explicit profile.

Lemma 3.7. *If a smooth real periodic function satisfies $-\Delta U = \kappa \sinh U$, with $\kappa > 0$, define $v = \arctan(\sinh(U/2))$. Then*

$$\kappa \int \cosh U = \kappa + 2\int |\nabla v|^2.$$

Proof. The range of v is contained in $(-\pi/2, \pi/2)$ and $U = 2\,\mathrm{artanh}(\sin v)$, $U_v = 2\sec v$, and $\sinh U = 2\sec v \tan v$. The equation becomes

$$\Delta v + \tan v\big(|\nabla v|^2 + \kappa\big) = 0.$$

Since U is bounded, $\log\cos v$ is a smooth periodic function. Direct differentiation gives $\Delta \log\cos v = -\tan v\, \Delta v - \sec^2 v\, |\nabla v|^2 = \kappa \tan^2 v - |\nabla v|^2$. Integration yields $\kappa \int \tan^2 v = \int |\nabla v|^2$. Finally $\cosh U = 1 + 2\tan^2 v$, and the torus has mass one. □

For U_m, the separated representation obtained in the rigidity proof is

$$v_m = f_m(x) + f_m(y), \qquad f_m(t) = -\arcsin\big(\sqrt{m}\,\mathrm{sn}(4\mathrm{K}t - \mathrm{K} \mid m)\big), \quad \mathrm{K} = \mathrm{K}(m). \tag{3.29}$$

Equivalently,

$$f_m(t) = \arctan\left(\sqrt{\frac{m}{1-m}}\,\mathrm{cn}(4\mathrm{K}t \mid m)\right).$$

The Jacobi addition formulas verify that (3.29) corresponds to (3.13); the branch choices are unambiguous because $|f_m| \le \arcsin\sqrt{m} < \pi/4$ [7, §22.8].

Differentiating the arcsine representation, using $1 - m\,\mathrm{sn}^2 = \mathrm{dn}^2$ and $\mathrm{dn} > 0$ on the real axis, gives $f_m'(t) = -4\mathrm{K}\sqrt{m}\,\mathrm{cn}(4\mathrm{K}t - \mathrm{K} \mid m)$. The standard period integral is

$$\int_0^{4\mathrm{K}} \mathrm{cn}^2(s \mid m)\, ds = \frac{4}{m}\big[\mathrm{E}(m) - (1-m)\mathrm{K}(m)\big]. \tag{3.30}$$

For completeness, the substitution $s = F(\theta \mid m)$ gives $\int_0^{4\mathrm{K}} \mathrm{dn}^2(s \mid m)\, ds = 4\mathrm{E}(m)$. Now apply $\mathrm{dn}^2 = 1 - m\,\mathrm{sn}^2$ and $\mathrm{cn}^2 = 1 - \mathrm{sn}^2$ to obtain (3.30). Therefore

$$\int_0^1 (f_m')^2 = 16\mathrm{K}(m)\big[\mathrm{E}(m) - (1-m)\mathrm{K}(m)\big].$$

Since $\int |\nabla v_m|^2 = 2\int_0^1 (f_m')^2$, Lemma 3.7 gives

$$\begin{aligned}\mathfrak{b}(m) &= 16\mathrm{K}^2(1-2m) + 64\mathrm{K}\big[\mathrm{E} - (1-m)\mathrm{K}\big] \\ &= 16\mathrm{K}\big[4\mathrm{E} - (3-2m)\mathrm{K}\big],\end{aligned}$$

proving (3.28). In particular,

$$A_m = \frac{4\mathrm{E}(m) - (3-2m)\mathrm{K}(m)}{(1-2m)\mathrm{K}(m)}.$$

The derivative identities, in parameter convention, are

$$\frac{d\mathrm{K}}{dm} = \frac{\mathrm{E} - (1-m)\mathrm{K}}{2m(1-m)}, \qquad \frac{d\mathrm{E}}{dm} = \frac{\mathrm{E} - \mathrm{K}}{2m}.$$

They follow by differentiating the defining integrals; they are also DLMF §19.4 after $m = k^2$ [7]. Differentiating (3.28) and simplifying gives the factorization

$$\mathfrak{b}'(m) = \frac{16}{m(1-m)}\big(2\mathrm{E}(m) - \mathrm{K}(m)\big)\big(\mathrm{E}(m) - (1-m)\mathrm{K}(m)\big) > 0. \tag{3.31}$$

Both factors are strictly positive, as their integral expressions show:

$$\mathrm{E} - (1-m)\mathrm{K} = m\int_0^{\pi/2} \frac{\cos^2\theta}{\sqrt{1-m\sin^2\theta}}\,d\theta > 0,$$

$$2\mathrm{E} - \mathrm{K} = \int_0^{\pi/2} \frac{1-2m\sin^2\theta}{\sqrt{1-m\sin^2\theta}}\,d\theta > 0 \qquad (0 < m < 1/2).$$

At $m = 0$, $\mathrm{K}(0) = \mathrm{E}(0) = \pi/2$, so $\mathfrak{b}(0) = 4\pi^2$. Let $\mathrm{K}_* = \mathrm{K}(1/2)$ and $\mathrm{E}_* = \mathrm{E}(1/2)$. Legendre's relation is

$$\mathrm{K}(m)\mathrm{E}(1-m) + \mathrm{E}(m)\mathrm{K}(1-m) - \mathrm{K}(m)\mathrm{K}(1-m) = \frac{\pi}{2};$$

see DLMF §19.7(i) [7]. At the self-complementary parameter $1/2$ it gives

$$2\mathrm{K}_*\mathrm{E}_* - \mathrm{K}_*^2 = \frac{\pi}{2}.$$

Consequently,

$$\mathfrak{b}(1/2) = 32\mathrm{K}_*(2\mathrm{E}_* - \mathrm{K}_*) = 16\pi. \tag{3.32}$$

All these endpoint evaluations refer to the analytic expression (3.28); the profile itself becomes unbounded at $m = 1/2$. Equations (3.31)–(3.32) prove that $\mathfrak{b}$ is an increasing bijection onto the stated interval. Its inverse is real-analytic because $\mathfrak{b}'$ never vanishes.

This proves Proposition 3.6.

Theorem 3.8. *For every real $u \in H^1(\mathbb{T}^2)$,*

$$\mathcal{S}(u) \le \frac{1}{4\pi^2}\mathcal{E}(u). \tag{3.33}$$

Equality holds if and only if u is constant almost everywhere. The coefficient $1/(4\pi^2)$ cannot be decreased.

Proof of Theorem 3.8. Let u minimize $\mathcal{F}_{4\pi^2}$. Such a smooth minimizer exists by Corollary 3.1, since $4\pi^2 < 16\pi$. A nonconstant coordinate stripe is excluded by Lemma 3.2. If u depends on both coordinates, Lemma 3.3 provides its strict coordinate monotonicity after translation. Apply the balancing shift (3.25). Proposition 3.4 then gives $U = U_m$ for some $0 < m < 1/2$. Equation (3.26) and Proposition 3.6 give

$$4\pi^2 = \beta = \kappa_m \int \cosh U_m > 4\pi^2,$$

a contradiction. Thus every minimizer is constant, and mean zero forces $u = 0$. Hence $\inf \mathcal{F}_{4\pi^2} = 0$ with that unique minimizer.

It follows that (3.33) holds for every H^1 function. Any equality function, after subtracting its mean, is a global minimizer and therefore is zero. This also handles the equality case without a separate smoothness assumption.

Finally, for $u_\varepsilon = \varepsilon\cos 2\pi x$,

$$\mathcal{E}(u_\varepsilon) = 2\pi^2\varepsilon^2, \qquad \mathcal{S}(u_\varepsilon) = 2\log I_0(\varepsilon) = \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{32} + O(\varepsilon^6).$$

Thus $\mathcal{S}(u_\varepsilon)/\mathcal{E}(u_\varepsilon) \to 1/(4\pi^2)$, proving optimality. □

By the entropy–energy equivalence in Lemma 2.10, the sharp inequality also has the density formulation

$$\mathrm{Ent}(r_+) + \mathrm{Ent}(r_-) \ge \pi^2\langle r_+ - r_-, G(r_+ - r_-)\rangle$$

for probability densities r_+, r_- of finite entropy. Equality is possible only for $(r_+, r_-) = (1, 1)$, by the minimizing-pair correspondence.

Corollary 3.9. *The flat profile is the unique minimizer of $\mathcal{F}_\beta$ for $0 < \beta \le 4\pi^2$. For $4\pi^2 < \beta < 16\pi$, the minimum is strictly negative. Hence the global transition parameter in this canonical normalization is exactly $\beta_{\mathrm{tr}} = 4\pi^2$, and there is no nonflat global minimizer coexisting at the transition.*

Proof. For $\beta \le 4\pi^2$,

$$\mathcal{F}_\beta(u) = \mathcal{F}_{4\pi^2}(u) + \left(\frac{1}{2\beta} - \frac{1}{8\pi^2}\right)\mathcal{E}(u),$$

so the first claim follows from Theorem 3.8. For $\beta > 4\pi^2$,

$$\mathcal{F}_\beta(\varepsilon\cos 2\pi x) = \frac{\varepsilon^2}{4}\left(\frac{4\pi^2}{\beta} - 1\right) + O(\varepsilon^4) < 0$$

for small nonzero ε. □

Theorem 3.10. *For each $4\pi^2 < \beta < 16\pi$, let m_β be the unique solution of $\mathfrak{b}(m_\beta) = \beta$. Then*

$$\mathcal{U}_\beta = \{U_{m_\beta}(\,\cdot - a) : a \in \mathbb{T}^2\}.$$

The translation action is free. The centered profile depends real-analytically on β locally into every C^r space.

Proof. For $\beta > 4\pi^2$, a minimizing profile exists and is nonflat by Corollary 3.9. A coordinate stripe cannot minimize, since for $v = v(x)$ and $w = w(y)$,

$$\mathcal{F}_\beta(v + w) = \mathcal{F}_\beta^{(1)}(v) + \mathcal{F}_\beta^{(1)}(w),$$

and a small nonzero first harmonic in w has negative one-dimensional action. Lemma 3.3 and Proposition 3.4 therefore put each minimizer, after balancing and translation, on U_m with $\mathfrak{b}(m) = \beta$. Strict monotonicity forces $m = m_\beta$. The balancing shift is zero because both the original minimizer and U_m have mean zero. Existence gives one minimizing member of the orbit, and translation invariance makes all its members minimizing.

The argument of $4\operatorname{artanh}$ in (3.13) is maximized only when the two cn factors equal 1 or both equal -1. Equivalently $x + y$ and $x - y$ are integers of the same parity. Thus $(0, 0)$ is the unique maximum on the torus. Any stabilizing translation must fix it, proving freeness. Analyticity follows from the explicit profile and $\mathfrak{b}'(m) > 0$. □

The profile is invariant under coordinate reflections and interchange. By (3.27), changing its sign is a translation by $(1/2, 1/2)$, so species exchange gives no extra minimizing orbit. By Corollary 3.1, the density minimizing set is

$$\mathcal{M}_\beta = \left\{\left(\frac{e^{U_{m_\beta}(\cdot - a)}}{A_{m_\beta}}, \frac{e^{-U_{m_\beta}(\cdot - a)}}{A_{m_\beta}}\right) : a \in \mathbb{T}^2\right\}.$$

For $0 < \beta \le 4\pi^2$ it consists only of $(1, 1)$. Consequently there is no further exchange between inequivalent globally minimizing branches in $(4\pi^2, 16\pi)$. The pressure is real-analytic in this open interval. This conclusion by itself does not imply Hessian nondegeneracy; that is proved in the next subsection.

3.4. **Nondegeneracy transverse to translations.** Fix $4\pi^2 < \beta < 16\pi$ and write $u = U_{m_\beta}$, $A = \int e^u = \int e^{-u}$ and $\rho_\pm = e^{\pm u}/A$. The canonical stream Hessian is

$$\mathscr{H}[h, h] = \frac{1}{\beta}\int |\nabla h|^2 - \frac{1}{2}\int (\rho_+ + \rho_-)h^2 + \frac{1}{2}\left(\int \rho_+ h\right)^2 + \frac{1}{2}\left(\int \rho_- h\right)^2, \qquad \int h = 0.$$

It is nonnegative at the global minimizer.

Lemma 3.11. *Fix $m \in (0, 1/2)$ and write $U = U_m$, $\kappa = \kappa(m)$ and $L = -\Delta - \kappa\cosh U$. If a smooth real function w, even in both coordinates, and a scalar h satisfy*

$$Lw = h\sinh U,$$

then, for some real τ,

$$w = \tau\partial_m U_m, \qquad h = \tau\kappa'(m).$$

Proof. We give the linearized argument explicitly; it does not assume that an arbitrary Jacobi field integrates to a family of nonlinear solutions. Set $U_\varepsilon = U + \varepsilon w$, $\kappa_\varepsilon = \kappa + \varepsilon h$. Then $-\Delta U_\varepsilon - \kappa_\varepsilon \sinh U_\varepsilon = O(\varepsilon^2)$ in every C^r norm. All identities below can equivalently be read in the algebra $\mathbb{R}[\varepsilon]/(\varepsilon^2)$.

For $\partial_z = (\partial_x - i\partial_y)/2$, the third-order expression $\Phi(U) = U_{zzz} - \frac{1}{2}U_z^3$ satisfies $L\Phi(U) = 0$. For the square profile the rigidity identity is

$$\Phi(U) = -\frac{\kappa}{4} U_{\bar z}.$$

Let $b = -\kappa/4$. Linearizing the equations for Φ and $U_{\bar z}$, their common inhomogeneous term cancels in $\Psi = w_{zzz} - \frac{3}{2}U_z^2 w_z - b w_{\bar z}, \qquad L\Psi = 0$. Its real part is odd in x and even in y, and its imaginary part has the opposite parities. On $(0,1/2)\times\mathbb{T}$, $-U_x > 0$ is a Dirichlet zero eigenfunction of L; hence zero is the simple principal Dirichlet eigenvalue. The corresponding assertion holds for $-U_y$. Thus $\Psi = a_1 U_z + b_1 U_{\bar z}$ for real a_1, b_1. Consequently the formal profiles satisfy

$$\Phi(U_\varepsilon) = a_\varepsilon (U_\varepsilon)_z + b_\varepsilon (U_\varepsilon)_{\bar z} + O(\varepsilon^2), \quad a_\varepsilon = \varepsilon a_1, \quad b_\varepsilon = -\kappa/4 + \varepsilon b_1. \tag{3.34}$$

Put $c_\varepsilon = 4b_\varepsilon/\kappa_\varepsilon = -1 + \varepsilon\dot c + O(\varepsilon^2)$. Differentiating (3.34) in $\bar z$ and taking the imaginary part gives

$$(\cosh U_\varepsilon - c_\varepsilon)(U_\varepsilon)_{xy} = \tfrac{1}{2}\sinh U_\varepsilon (U_\varepsilon)_x (U_\varepsilon)_y + O(\varepsilon^2). \tag{3.35}$$

On an interior patch where $U \neq 0$, compatibility with the sinh–Gordon equation gives

$$\frac{(c_\varepsilon^2 - 1)\sinh U_\varepsilon}{2(\cosh U_\varepsilon - c_\varepsilon)^3}\ \big[|\nabla U_\varepsilon|^2 + 2\kappa_\varepsilon(\cosh U_\varepsilon - c_\varepsilon)\big] = O(\varepsilon^2). \tag{3.36}$$

For completeness this is obtained from $U_{xy} = f(U)U_xU_y$, $\Delta U = q(U)$ using

$$(f'' + 2ff')|\nabla U|^2 + 3qf' + 2qf^2 + q'f - q'' = 0, \quad f = \frac{\sinh U}{2(\cosh U - c)}, \quad q = -\kappa\sinh U.$$

At $c = -1$ the square bracket in (3.36) is strictly positive. Its coefficient of ε therefore forces $\dot c = 0$.

Let $v_\varepsilon = \arctan(\sinh(U_\varepsilon/2))$. Equation (3.35), now with $c_\varepsilon = -1 + O(\varepsilon^2)$, yields $(v_\varepsilon)_{xy} = O(\varepsilon^2)$. Hence $v_\varepsilon = f_\varepsilon(x) + g_\varepsilon(y) + O(\varepsilon^2)$ with periodic even $f_\varepsilon, g_\varepsilon$. Write $p_\varepsilon = f_\varepsilon'$, $q_\varepsilon = g_\varepsilon'$. Retaining the real and imaginary parts of (3.34) gives, to first order,

$$p_\varepsilon'' = c_{1,\varepsilon} p_\varepsilon - 2p_\varepsilon^3, \qquad c_{1,\varepsilon} = -\kappa_\varepsilon - 2a_\varepsilon, \tag{3.37}$$

$$q_\varepsilon'' = c_{2,\varepsilon} q_\varepsilon - 2q_\varepsilon^3, \qquad c_{2,\varepsilon} = -\kappa_\varepsilon + 2a_\varepsilon. \tag{3.38}$$

Here and until the end of this paragraph equality means modulo $O(\varepsilon^2)$. One way to verify these equations is to use $p' + q' + \tan v(p^2 + q^2 + \kappa) = 0$ and

$$p''/p + 3\tan v(p' - q') - p^2 + 3q^2 = 4a - \kappa,$$
$$q''/q + 3\tan v(q' - p') - q^2 + 3p^2 = -4a - \kappa.$$

Differentiate the first equation in x, y, divide by p, q on the interior, and subtract; the tangent terms then eliminate. The resulting equations extend to the boundary by smoothness. Their first-integral constants $d_{1,\varepsilon}, d_{2,\varepsilon}$ satisfy

$$d_{1,\varepsilon} + \tfrac{1}{4}c_{1,\varepsilon}^2 = d_{2,\varepsilon} + \tfrac{1}{4}c_{2,\varepsilon}^2. \tag{3.39}$$

To check this last relation, put $P_\varepsilon = p_\varepsilon'(0)$, $Q_\varepsilon = q_\varepsilon'(0)$ and $v_{0,\varepsilon} = v_\varepsilon(0,0)$. The separated equation and its second x and y derivatives at the corner give

$$P + Q = -\kappa\tan v_0, \quad c_1 + \kappa\sec^2 v_0 + 2P\tan v_0 = 0, \quad c_2 + \kappa\sec^2 v_0 + 2Q\tan v_0 = 0.$$

Since $d_1 = P^2, d_2 = Q^2$, subtraction gives $d_1 - d_2 = \kappa(c_1 - c_2)/2$; using $c_1 + c_2 = -2\kappa$ proves (3.39).

The crossing oscillator $p'' = cp - 2p^3$, started at $p(0) = 0$, $p'(0) = -\sqrt{d}$, is analytic in (c,d) near the present data $d > 0$. Its elliptic variables obey $c = \omega^2(2m-1), \qquad d = \omega^4 m(1-m), \qquad d + c^2/4 = \omega^4/4, \qquad \mathcal{P}(c,d) = 4\mathrm{K}(m)/\omega$. The first-order perturbations in (3.37)–(3.38) are derivatives of these scalar oscillator families by uniqueness of the linearized initial-value problem. Their periodicity forces $\dot{\mathcal{P}}_1 = \dot{\mathcal{P}}_2 = 0$: differentiate the period identity and use $p'(0) \neq 0$. Equation (3.39) gives $\dot\omega_1 = \dot\omega_2$, and the zero period derivatives then give $\dot m_1 = \dot m_2 = \tau$, because $\mathrm{K}'(m) > 0$. It follows that $\dot a = 0$, $h = \kappa'(m)\tau$, and both oscillator variations equal their derivatives along the square family. The remaining additive constant in $\dot v$ vanishes by evaluating the separated equation at $(0,1/2)$, where $v = 0$, $p = q = 0$, and the two oscillator second-derivative contributions cancel. Therefore $\dot v = \tau\partial_m v_m$, whence $w = \tau\partial_m U_m$. □

Proposition 3.12. *The mean-zero stream Hessian has precisely the translation kernel:*

$$\ker \mathscr{H} = \operatorname{span}\{u_x, u_y\}. \tag{3.40}$$

Proof. Regard $\mathscr{H}$ as a nonnegative bilinear form on mean-zero H^1. If $\mathscr{H}[h_0, h_0] = 0$, Cauchy–Schwarz for this form gives $\mathscr{H}[h_0, h] = 0$ for every mean-zero h. With $c_\pm = \int \rho_\pm h_0$, its weak equation is

$$-\Delta h_0 = \frac{\beta}{2}\big[(\rho_+ + \rho_-)h_0 - c_+\rho_+ - c_-\rho_-\big].$$

The possible constant multiplier vanishes because the right side has integral zero. Smoothness of $\rho_\pm$ first gives $h_0 \in H^2$ and then, by elliptic bootstrapping, $h_0 \in C^\infty$. Thus the smooth rigidity lemma applies to every weak kernel vector. Reflection parity commutes with the Hessian. In the sector odd in x, its two nonlocal rank-one terms vanish and $\beta\mathscr{H}$ is $L = -\Delta - \kappa\cosh u$. Principal Dirichlet simplicity on the half-cylinder gives only u_x. The odd-in-y sector similarly gives only u_y, and their odd–odd intersection contains no nonzero null vector.

Let h_0 be an even–even mean-zero null vector. Set

$$c_\pm = \int \rho_\pm h_0, \quad c = \tfrac{1}{2}(c_+ + c_-), \quad d = \tfrac{1}{2}(c_+ - c_-), \quad w = h_0 - c, \quad \dot\kappa = -\kappa d.$$

Its Hessian equation becomes

$$Lw = \dot\kappa \sinh u, \qquad \dot\kappa A + \kappa \int \sinh u\, w = 0.$$

The second identity is the fixed-canonical-parameter condition; it follows directly from the definitions and $\int \sinh u = 0$. Lemma 3.11 gives $w = \tau\partial_m U_m$, $\dot\kappa = \tau\kappa'(m)$. The normalization identity therefore reads $\tau\mathfrak{b}'(m) = 0$. By (3.31), $\tau = 0$, and then $h_0 = c$ is zero by its mean-zero constraint. This proves (3.40).

□

3.5. **Transition asymptotics and endpoint concentration.** The closed parameter formula also provides precise endpoint scales. Expanding the defining elliptic integrals at zero yields

$$\mathfrak{b}(m) = 4\pi^2 + 2\pi^2 m + \frac{3\pi^2}{8}m^2 + O(m^3).$$

Thus, as $\beta \downarrow 4\pi^2$,

$$m_\beta = \frac{\beta - 4\pi^2}{2\pi^2} + O((\beta - 4\pi^2)^2).$$

For every fixed r,

$$U_{m_\beta}(x, y) = \sqrt{\frac{2(\beta - 4\pi^2)}{\pi^2}}\big(\cos 2\pi x + \cos 2\pi y\big) + O_{C^r}((\beta - 4\pi^2)^{3/2}).$$

This describes the globally minimizing, not merely stationary, small-amplitude branch.

The derivative formula extends to the endpoint and, using Legendre's relation, gives

$$\mathfrak{b}'(1/2) = \frac{8\pi^2}{\mathrm{K}_*^2}.$$

Writing $\varepsilon = 16\pi - \beta$ and $\delta = 1 - 2m_\beta$,

$$\delta = \frac{\mathrm{K}_*^2}{4\pi^2}\varepsilon + O(\varepsilon^2), \qquad A_{m_\beta} \sim \frac{\pi}{\mathrm{K}_*^2\delta}.$$

Only the positive maximum and the negative minimum become singular. In a fixed small coordinate neighborhood of the origin, the argument w_m of $4\,\mathrm{artanh}$ satisfies $1 - w_m(x, y) = \delta + 4\mathrm{K}_*^2(x^2 + y^2) + o(\delta + x^2 + y^2)$. With $\ell_\beta = \sqrt{\delta}/(2\mathrm{K}_*)$, the centered positive density satisfies, locally uniformly for $z \in \mathbb{R}^2$,

$$\ell_\beta^2 \rho_{+,\beta}(\ell_\beta z) \longrightarrow \frac{1}{\pi(1 + |z|^2)^2}, \qquad \ell_\beta \sim \frac{\sqrt{16\pi - \beta}}{4\pi}.$$

The limit density on $\mathbb{R}^2$ integrates to one. Away from the origin e^{U_m} stays uniformly bounded while $A_m \to \infty$; this proves weak convergence of $\rho_{+,\beta}dx$ to $\delta_{(0,0)}$. The negative density is its half-diagonal translate and converges to $\delta_{(1/2,1/2)}$.

The global sharp inequality has a stronger local form. Let E_1 be the real first eigenspace and define $\mathscr{D}(u) = \mathcal{E}(u)/(4\pi^2) - \mathcal{S}(u) = 2\mathcal{F}_{4\pi^2}(u)$.

Lemma 3.13. *Write $u = z + w$ with $z \in E_1$ and $w \perp (\mathbb{R} \oplus E_1)$. There are $c, r_0 > 0$ such that*

$$\mathscr{D}(u) \geq c\big(\|w\|_{H^1}^2 + \|z\|_2^4\big) \quad \text{if } \|u\|_{H^1} \leq r_0. \tag{3.41}$$

Proof. For $\int u = 0$ and $\|u\|_{H^1}$ small, Taylor expansion and (2.13) give the uniform estimate

$$\mathscr{D}(u) = \frac{\mathcal{E}(u)}{4\pi^2} - \|u\|_2^2 - \frac{1}{12}\left[\int u^4 \,\mathrm{d}x - 3\|u\|_2^4\right] + O(\|u\|_{H^1}^6). \tag{3.42}$$

To justify the uniform remainder, expand each $\int e^{\pm u}$ through degree five; the error is at most $C\int |u|^6 e^{|u|}\,\mathrm{d}x \leq C\|u\|_{H^1}^6$ by Hölder, Sobolev embedding into every finite L^p, and (2.13). The integrals are close to one, so the logarithms can also be expanded uniformly. Odd total degrees cancel exactly.

Let $u = z + w$ with $z \in E_1$, $w \perp (\mathbb{R} \oplus E_1)$. The next eigenvalue is $8\pi^2$. Orthogonality therefore gives

$$\frac{\mathcal{E}(u)}{4\pi^2} - \|u\|_2^2 = \frac{\mathcal{E}(w)}{4\pi^2} - \|w\|_2^2 \geq \frac{\mathcal{E}(w)}{8\pi^2} \geq c_0\|w\|_{H^1}^2.$$

After independent translations in the coordinates, $z = a\cos(2\pi x) + b\cos(2\pi y)$. Direct integration yields

$$\int z^4 \,\mathrm{d}x - 3\|z\|_2^4 = -\frac{3}{8}(a^4 + b^4),$$

so the fourth-order term of the deficit is $(a^4 + b^4)/32 \geq c_1\|z\|_2^4$.

Put $r = \|w\|_{H^1}$ and $s = \|z\|_{H^1}$. Sobolev embedding bounds the difference of the quartic expressions at $z + w$ and at z by

$$C(s^3 r + s^2 r^2 + s r^3 + r^4).$$

The first term satisfies $s^3 r \leq \delta r^2 + C_\delta s^6$; the other terms are absorbed into δr^2 by restricting $r + s$ to a small ball. The sixth-order remainder is similarly absorbed into $r^2 + s^4$ with an arbitrarily small coefficient. Choosing the radius after δ proves (3.41). This gives quadratic control of the higher modes and quartic control of the first eigenspace, uniformly over the full small H^1 ball.

□

4. Partition control: two alternative arguments

4.1. **Interaction splitting and Gaussian resummation.** In the first approach all strengths equal one. Write $H_N = \sum_{i<j} G(x_i - x_j)$ and $U_N = H_N/N$. The one-species partition function is $Z_N(\beta) = \int e^{\beta U_N}\, dx^N$. For $\omega_N = N^{-1/2}(\sum_i \delta_{x_i} - N dx)$, the shared independent-reference limit in Lemma 6.3 gives

$$(\omega_N, U_N) \Longrightarrow (\xi, Y), \qquad Y = \tfrac{1}{2} :\langle \xi, G * \xi\rangle :, \tag{4.1}$$

where ξ is mean-zero white noise. The general change-of-measure argument in Section 6 reduces the fluctuation theorem to the uniform bound

$$\sup_{N\geq 2} Z_N(b) < \infty \qquad (0 \leq b < 8\pi). \tag{4.2}$$

We now prove this bound by the cluster-expansion route, independently of the Boué–Dupuis control argument that follows.

We use two established subcritical inputs:

(EI) For every $b_0 < 8\pi$ there is $C_{b_0} < \infty$ such that, for $0 \leq b \leq b_0$ and $2 \leq v \leq N$,

$$\int_{(\mathbb{T}^2)^v} \exp\left(\frac{b}{N}\sum_{i<j} |G(x_i - x_j)|\right) \mathrm{d}x^v \leq C_{b_0}^v. \tag{4.3}$$

(MF) For every $0 \leq \alpha < 8\pi$,

$$\log Z_N(\alpha) = o(N). \tag{4.4}$$

Estimate (4.3) is [5, Lemma 2]. The mean-field variational formula and flat-profile uniqueness give (4.4); see [3, 6] and the discussion in [5]. These two previously established inputs are the only nonperturbative one-species estimates used in this argument. In fact the first bound also follows from the second and finite-particle integrability. If $G \geq -M$, then $|G| \leq G + 2M$. Put $\alpha = bv/N$. The integral in (4.3) is at most $e^{Mbv} Z_v(\alpha)$. Log-convexity, $Z_v(0) = 1$, and $0 \leq \alpha \leq b_0$ bound $Z_v(\alpha)$ by $\max\{1, Z_v(b_0)\}$; (4.4) makes this at most C^v, including the finitely many initial values.

For $\Lambda \geq 2$, let

$$C_\Lambda = e^{\Lambda^{-2}\Delta} G, \qquad R_\Lambda = G - C_\Lambda.$$

The nonzero Fourier multipliers of R_Λ are

$$r_{\Lambda,k} = \frac{1 - e^{-4\pi^2|k|^2/\Lambda^2}}{4\pi^2|k|^2}.$$

Hence, for every $s > 2$,

$$0 \leq R_\Lambda \leq \Lambda^{-2} I \quad \text{on } L^2_0(\mathbb{T}^2), \tag{4.5}$$

$$\|R_\Lambda\|_2 \leq C\Lambda^{-1}, \qquad \|R_\Lambda\|_s \leq C_s \Lambda^{-2/s}. \tag{4.6}$$

Indeed $0 \leq r_{\Lambda,k} \leq \min\{\Lambda^{-2}, (4\pi^2|k|^2)^{-1}\}$. Splitting the sum at $|k| = \Lambda$ proves the L^2 estimate by Parseval. For $s > 2$, Hausdorff–Young, with $s' = s/(s-1)$, gives

$$\|R_\Lambda\|_s \leq \left(\sum_{k \neq 0} \min\{\Lambda^{-2}, C|k|^{-2}\}^{s'} \right)^{1/s'} \leq C_s \Lambda^{-2/s}.$$

Moreover,

$$R_\Lambda(x) \geq -\Lambda^{-2}, \qquad |R_\Lambda(x)| \leq G_+(x) + C. \tag{4.7}$$

Indeed

$$R_\Lambda(x) = \int_0^{\Lambda^{-2}} (p_t(x) - 1)\, \mathrm{d}t.$$

The lower bound follows from $p_t \geq 0$. If $d = \operatorname{dist}(x, 0)$, the nearest heat-kernel image gives

$$\int_0^{\Lambda^{-2}} p_t(x)\, \mathrm{d}t \leq \frac{1}{2\pi} \log_+ \frac{1}{\Lambda d} + C,$$

Comparison with the logarithmic expansion in (2.2) proves (4.7), with singular coefficient exactly one. For the nearest image one uses

$$\int_0^{\Lambda^{-2}} \frac{e^{-d^2/(4t)}}{4\pi t}\, dt = \frac{1}{4\pi} E_1(\Lambda^2 d^2/4).$$

The other lattice images have uniformly bounded integrals.

Let γ_Λ be the centered smooth Gaussian field with covariance C_Λ. Put

$$t = \sqrt{\frac{\beta}{N}}, \qquad A(u) = \int_{\mathbb{T}^2} e^{u(x)}\, \mathrm{d}x, \qquad \rho_u = \frac{e^u}{A(u)}.$$

Gaussian integration gives the exact identity

$$Z_N(\beta) = e^{-\beta C_\Lambda(0)/2} \mathbb{E}_{\gamma_\Lambda} \left[A(t\phi)^N Q_{N,\Lambda}(\rho_{t\phi}) \right],$$

where

$$Q_{N,\Lambda}(\rho) = \int_{(\mathbb{T}^2)^N} e^{\beta H_N^{R_\Lambda}/N} \rho^{\otimes N}(\,\mathrm{d}x^N).$$

Since $\beta\|C_\Lambda\| \leq \beta/(4\pi^2) < 1$, tilting by $e^{\beta\|\phi\|_2^2/2}$ yields a centered Gaussian law $\widehat{\gamma}_{\beta,\Lambda}$ with covariance $\widehat{C}_{\beta,\Lambda} = C_\Lambda(I - \beta C_\Lambda)^{-1}$ and

$$Z_N(\beta) = D_\Lambda(\beta) \widehat{\mathbb{E}}_{\beta,\Lambda} \left[e^{\mathcal{L}_N(\phi)} Q_{N,\Lambda}(\rho_{t\phi}) \right], \tag{4.8}$$

where

$$\mathcal{L}_N(\phi) = N \log A(t\phi) - \frac{\beta}{2}\|\phi\|_2^2, \qquad D_\Lambda(\beta) = \det{}_2(I - \beta C_\Lambda)^{-1/2}.$$

For fixed Λ, $D_\Lambda(\beta) < \infty$ and $\widehat{C}_{\beta,\Lambda}$ is trace class.

Fix $\eta > 0$ and define $\mathcal{G}_N = \{\|t\phi\|_\infty \leq \eta\}$, $\mathcal{B}_N = \mathcal{G}_N^c$. If η is small, Taylor expansion, using $\int \phi = 0$, gives on $\mathcal{G}_N$

$$|\mathcal{L}_N(\phi)| \leq C\beta\eta\|\phi\|_2^2. \tag{4.9}$$

To see the Taylor estimate, use $e^u = 1 + u + u^2/2 + O(\eta u^2)$ for $\|u\|_\infty \leq \eta$, then $\int u = 0$ and $\int u^2 \leq \eta^2$ when replacing $A(u) - 1$ by $\log A(u)$. Also $e^{-2\eta} \leq \rho_{t\phi} \leq e^{2\eta} \leq 2$ and

$$\|\rho_{t\phi} - 1\|_2 \leq C\|t\phi\|_2. \tag{4.10}$$

4.2. **Stopped expansion and the one-species partition bound.** For a probability density ρ, set

$$r_\rho = R_\Lambda * \rho, \quad m_\rho = \int r_\rho \rho, \quad h_\rho = r_\rho - m_\rho,$$

and

$$k_\rho(x,y) = R_\Lambda(x-y) - h_\rho(x) - h_\rho(y) - m_\rho.$$

Then

$$\int k_\rho(x,y)\rho(y)\,\mathrm{d}y = 0 \tag{4.11}$$

and

$$H_N^{R_\Lambda} = \sum_{i<j} k_\rho(x_i,x_j) + (N-1)\sum_i h_\rho(x_i) + \binom{N}{2} m_\rho. \tag{4.12}$$

On $\mathcal{G}_N$, writing $u = t\phi$ and $f = \rho_u - 1$, equations (4.5)–(4.10) imply

$$0 \le m_{\rho_u} = \langle f, R_\Lambda f\rangle \le C\Lambda^{-2}\|u\|_2^2, \tag{4.13}$$

$$\|h_{\rho_u}\|_{L^2(\rho_u)} \le C\Lambda^{-2}\|u\|_2, \tag{4.14}$$

$$\|h_{\rho_u}\|_\infty \le C\Lambda^{-1}\|u\|_2. \tag{4.15}$$

Fix $\sigma > 1$ and $q > 1$ so close to one that

$$q\sigma\beta < 8\pi, \qquad p = \frac{q}{q-1} > 2. \tag{4.16}$$

By (4.6), (4.13)–(4.15), and $\rho_u \le 2$,

$$\sup_x \|k_{\rho_u}(x,\cdot)\|_{L^p(\rho_u)} \le C\Lambda^{-2/p}, \tag{4.17}$$

$$\|k_{\rho_u}\|_{L^{2p}(\rho_u\otimes\rho_u)} \le C\Lambda^{-1/p}. \tag{4.18}$$

Furthermore, by (4.7) and (4.15),

$$|k_{\rho_u}(x,y)| \le G_+(x-y) + C. \tag{4.19}$$

Lemma 4.1. *There are $\Lambda_0(\beta) < \infty$ and $C_\beta < \infty$ such that, for $\Lambda \ge \Lambda_0$, every N, and every $\phi \in \mathcal{G}_N$,*

$$\log \mathbb{E}_{\rho_{t\phi}^{\otimes N}} \exp\left(\frac{\sigma\beta}{N}\sum_{i<j} k_{\rho_{t\phi}}(X_i,X_j)\right) \le C_\beta \Lambda^{-2/p}. \tag{4.20}$$

Proof. Put $z = \sigma\beta/N$, $\mu = \rho_{t\phi}\,\mathrm{d}x$ and $k = k_{\rho_{t\phi}}$. By (4.19), (4.3) and (4.16), the direct power series in the pair kernels is absolutely integrable. Its terms are loopless multigraphs. By (4.11), a connected component containing a vertex of total degree one has zero amplitude. Every nonzero connected component on v vertices therefore has at least v edge copies. Choose a spanning tree in its support and one remaining edge copy; the latter may duplicate a tree edge. This gives a spanning unicyclic multigraph U with exactly v edge copies.

Reserve the factors belonging to U and sum every unreserved multiplicity. Since $m! \ge (m-u)!$ for $0 \le u \le m$, the total absolute connected amplitude on a fixed set of v labels is at most

$$\sum_U z^v \int \prod_{e\in U} |k_e| \exp\left(z\sum_{i<j}|k_{ij}|\right)\,\mathrm{d}\mu^v.$$

Apply Hölder with exponents p, q. The exponential factor is at most C_β^v by (4.3), (4.19), and $q\sigma\beta < 8\pi$. For the skeleton, put $H = |k|^p$,

$$A = \sup_x \int H(x,y)\,\mathrm{d}\mu(y), \qquad B = \iint H(x,y)^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

Peeling all trees attached to the unique cycle gives one factor A per peeled edge. If the cycle has length $r \ge 3$, its integral is $\mathrm{Tr}(T_H^r)$ and $|\mathrm{Tr}(T_H^r)| \le \|T_H\|_{\mathrm{HS}}^2 \|T_H\|_{\mathrm{op}}^{r-2} \le BA^{r-2}$; for a doubled-edge cycle the integral is exactly B. Thus, by (4.17)–(4.18),

$$\left(\int \prod_{e\in U} |k_e|^p\,\mathrm{d}\mu^v\right)^{1/p} \le \|k\|_{2p}^2 \left(\sup_x \|k(x,\cdot)\|_p\right)^{v-2} \le C^v \Lambda^{-2(v-1)/p}.$$

There are at most $v^{v-2}\binom{v}{2} \le v^v/2$ such labeled skeletons. After choosing the labels among N, the total absolute connected activity of size v is therefore bounded by $Cv^{-1/2}(C_\beta)^v\Lambda^{-2(v-1)/p}$. For Λ sufficiently large these activities are summable in v. Factoring a multigraph into connected components, dropping the disjointness constraint and exponentiating the sum of connected activities proves (4.20). □

Lemma 4.2. *For Λ sufficiently large and $\phi \in \mathcal{G}_N$,*

$$Q_{N,\Lambda}(\rho_{t\phi}) \le \exp\Big(C_\beta\Lambda^{-2/p} + C_\beta\Lambda^{-2}\|\phi\|_2^2\Big). \tag{4.21}$$

Proof. Use (4.12) and Hölder with conjugate exponents σ and $s = \sigma/(\sigma-1)$. Lemma 4.1 treats the canonical term. Since $\int h_\rho\rho = 0$ and $\|h_\rho\|_\infty$ is small, the elementary centered mgf estimate

$$\log \int e^{\theta h_\rho}\rho \le C\theta^2\|h_\rho\|^2_{L^2(\rho)}$$

applies to the fixed value $\theta = s\beta(N-1)/N$. Equations (4.13)–(4.14) and $Nt^2 = \beta$ then give

$$\frac{\beta(N-1)}{2}m_\rho \le C_\beta\Lambda^{-2}\|\phi\|_2^2, \qquad \frac{N}{s}\log\int e^{s\beta(N-1)h_\rho/N}\rho \le C_\beta\Lambda^{-4}\|\phi\|_2^2.$$

Combining the three terms proves (4.21). □

Fix $0 < \beta < 8\pi$. First choose Λ so large that Lemmas 4.1–4.2 hold and $C_\beta\Lambda^{-2} < (\lambda_1 - \beta)/4$. Here the constants are uniform for η in a fixed small interval. Since $\|\widehat{C}_{\beta,\Lambda}\| \le (\lambda_1-\beta)^{-1}$, we can then choose $\eta > 0$ with $C_\beta\eta < (\lambda_1-\beta)/4$ so that

$$c := C_\beta(\eta + \Lambda^{-2}) < \frac{1}{2\|\widehat{C}_{\beta,\Lambda}\|}.$$

From (4.8), (4.9) and (4.21), the good-field contribution satisfies

$$Z^{\mathrm{good}}_{N,\Lambda} \le D_\Lambda(\beta)e^{C_\beta\Lambda^{-2/p}}\widehat{\mathbb{E}}_{\beta,\Lambda}e^{c\|\phi\|_2^2} \le C_{\beta,\Lambda,\eta}, \tag{4.22}$$

independently of N.

We next control the bad fields. For the fixed smooth Gaussian field γ_Λ, the Borell–TIS inequality [1] gives constants $c, C > 0$ such that

$$\gamma_\Lambda(\mathcal{B}_N) = \gamma_\Lambda\Big(\|\phi\|_\infty > \eta\sqrt{N/\beta}\Big) \le Ce^{-cN}. \tag{4.23}$$

We need one elementary perturbative fact about the fixed smooth kernel C_Λ.

Lemma 4.3. *For the fixed Λ there exists $\delta_0(\Lambda) > 0$ such that*

$$\sup_{N\ge 2}\int_{(\mathbb{T}^2)^N}\exp\left(\frac{\delta}{N}H_N^{C_\Lambda}\right)\mathrm{d}x^N < \infty, \qquad 0 \le \delta \le \delta_0(\Lambda).$$

Proof. This is the Mayer–unicyclic argument of [5], simplified by the boundedness of the fixed kernel C_Λ. Normalize

$$K_N(x) = I_N^{-1}e^{\delta C_\Lambda(x)/N} - 1, \qquad I_N = \int e^{\delta C_\Lambda/N}.$$

Since $\int C_\Lambda = 0$ and $C_\Lambda \in L^\infty$,

$$|\log I_N| \le C(\Lambda)\delta^2N^{-2}, \qquad |K_N| \le C(\Lambda)\delta N^{-1}, \qquad 1 + |K_N| \le e^{C(\Lambda)\delta/N}.$$

A connected simple graph containing a bridge has zero amplitude because $\int K_N = 0$. Hence every nonzero connected graph on v vertices contains a spanning unicyclic graph. Reserving that skeleton and summing all remaining simple edges gives, after choosing labels,

$$\binom{N}{v}\sum_{\Gamma\ \text{connected}}|A_N(\Gamma)| \le Cv^{-1/2}(C(\Lambda)\delta)^v.$$

For δ sufficiently small, the connected activities are summable and the exponential formula bounds the full graph sum. The prefactor $I_N^{N(N-1)/2}$ is bounded by $e^{C(\Lambda)\delta^2}$. □

Choose $\theta > 1$ and $a > 1$ so close to one that, with $a' = a/(a-1)$,

$$\alpha = a\theta\beta < 8\pi, \qquad \delta = a'\beta\theta(\theta-1) < \delta_0(\Lambda).$$

This is possible, for example, by taking $\theta = 1+\varepsilon$ and $a = 1+\sqrt{\varepsilon}$. Hölder's inequality in the Gaussian and particle variables gives

$$\begin{aligned} Z_{N,\Lambda}^{\mathrm{bad}} &\le \gamma_\Lambda(\mathcal{B}_N)^{1-1/\theta} e^{\beta(\theta-1)C_\Lambda(0)/2} \\ &\quad \times Z_N(\alpha)^{1/(a\theta)} \left[\int_{(\mathbb{T}^2)^N} e^{\delta H_N^{C_\Lambda}/N} \, \mathrm{d}x^N \right]^{1/(a'\theta)}. \end{aligned} \tag{4.24}$$

Indeed, after the first Hölder inequality, Gaussian integration produces the mixed interaction $\theta^2 H_N^{C_\Lambda} + \theta H_N^{R_\Lambda} = \theta H_N^G + \theta(\theta-1) H_N^{C_\Lambda}$, and the second Hölder inequality separates the two terms.

By (4.23), Lemma 4.3, (4.4), and (4.24),

$$\log Z_{N,\Lambda}^{\mathrm{bad}} \le -c_{\beta,\Lambda,\eta} N + o(N) + O_{\beta,\Lambda,\eta}(1) \longrightarrow -\infty.$$

Together with (4.22), this proves $\sup_{N\ge 2} Z_N(\beta) < \infty$. The finitely many initial values are finite by (4.3). The one-species fluctuation result is deduced from this bound in Section 6.

4.3. **The Boué–Dupuis representation and analytic estimates.** The second partition-control argument applies to all prescribed bounded strengths. Its strict version uses (H) and gives an order-one bound; its additive version uses (B) and gives an exponential-scale bound. The real Gaussian transform preserves the signs of the strengths.

Theorem 4.4. *Assume (2.3) and (H). For every fixed $0 < b < \gamma$, the singular partition functions are finite for all sufficiently large N, and*

$$\sup_{N \ge N_b} Z_N(b) < \infty. \tag{4.25}$$

All cutoff estimates below are uniform for $0 < \varepsilon \le 1$; no relation between N and the cutoff is imposed.

For $0 < \varepsilon \le 1$, let

$$G_\varepsilon = e^{-\varepsilon A} A^{-1}, \qquad E_{N,\varepsilon} = \frac{1}{N} \sum_{i<j} \xi_i \xi_j G_\varepsilon(x_i - x_j), \qquad Z_{N,\varepsilon}(b) = \mathbb{E}_{P_N} e^{bE_{N,\varepsilon}}. \tag{4.26}$$

Fix $0 < b < \gamma$ and set

$$t_i = \sqrt{b/N}\, \xi_i, \qquad a_N = b q_N, \tag{4.27}$$

$$R_{N,b}(f) = \sum_i \log \int e^{t_i f} \, \mathrm{d}x - \frac{b q_N}{2} \|f\|_2^2, \tag{4.28}$$

$$Q_{N,b,\varepsilon} = G_\varepsilon (I - b q_N G_\varepsilon)^{-1}, \qquad Y_{N,b,\varepsilon} \sim \mathcal{N}(0, Q_{N,b,\varepsilon}). \tag{4.29}$$

The real Gaussian representation and a quadratic tilt give the exact identity

$$Z_{N,\varepsilon}(b) = \det{}_2(I - b q_N G_\varepsilon)^{-1/2} \mathbb{E} e^{R_{N,b}(Y_{N,b,\varepsilon})}. \tag{4.30}$$

Theorem 4.5. *Assume (H). For any*

$$\frac{b}{2\gamma} < c < \frac{1}{2}, \tag{4.31}$$

there are N_* and $C < \infty$, independent of ε, and nonnegative random variables $K_{N,b,\varepsilon}$ with

$$\sup_{N \ge N_*,\, 0<\varepsilon\le 1} \mathbb{E} K_{N,b,\varepsilon}^2 < \infty, \tag{4.32}$$

such that, almost surely and simultaneously for every mean-zero $h \in H^1(\mathbb{T}^2)$,

$$R_{N,b}(Y_{N,b,\varepsilon} + h) \le c\big(D(h) - b q_N \|h\|_2^2\big) + C(1 + K_{N,b,\varepsilon}^2). \tag{4.33}$$

Realize $Y_{N,b,\varepsilon} = Q_{N,b,\varepsilon}^{1/2} W_1$ using cylindrical Brownian motion on L_0^2. For every progressively measurable, finite-energy control v, set

$$h(v) = Q_{N,b,\varepsilon}^{1/2} \int_0^1 v_t \, dt. \tag{4.34}$$

Then

$$\mathbb{E}R_{N,b}(Y_{N,b,\varepsilon}+h(v)) \le c\,\mathbb{E}\int_0^1 \|v_t\|_2^2\,dt + C. \tag{BD}$$

There is no independence assumption between $h(v)$ and $Y_{N,b,\varepsilon}$, and no relation between N and ε is imposed.

If X lies in the kth homogeneous Wiener chaos, then for every $m \ge 2$,

$$\|X\|_{L^m(\Omega)} \le (m-1)^{k/2}\,\|X\|_{L^2(\Omega)}\,. \tag{4.35}$$

This standard Gaussian theorem can be found in [13]. The covariance, stationarity, and hypercontractivity method for cutoff-uniform Wick estimates is also used in [11, Lemma 2.8]. We track the dependence on k explicitly below, rather than assuming a Wick-exponential estimate from a fixed-degree result.

Let W be cylindrical Brownian motion on a separable real Hilbert space. For a bounded measurable functional $\mathscr{F}$ of its path,

$$\log \mathbb{E}e^{\mathscr{F}(W)} = \sup_v \mathbb{E}\left[\mathscr{F}\left(W + \int_0^{\cdot} v_t\,dt\right) - \frac{1}{2}\int_0^1 \|v_t\|^2\,dt\right], \tag{4.36}$$

where v ranges over progressively measurable finite-energy controls. The finite-dimensional formula is [8]; the infinite-dimensional extension is [9], with an account in [10]. Here the functional depends only on the smooth terminal field $Q^{1/2}W_1$. We apply (4.36) only to bounded truncations; no unproved exponential integrability is required to invoke it.

Fix c as in (4.31). Choose p such that

$$1 < p < 2, \qquad \frac{pb}{2\gamma} < c. \tag{4.37}$$

Choose $1 < a < p$ and define ℓ and P by

$$\frac{1}{\ell} = \frac{1}{a} - \frac{1}{p}, \qquad P = \frac{a}{a-1}, \qquad 0 < s < \frac{2}{\ell} < 1. \tag{4.38}$$

Let

$$X = H^{-s,P}(\mathbb{T}^2), \qquad X^* = H^{s,a}(\mathbb{T}^2). \tag{4.39}$$

These are Bessel-potential spaces, with $\|f\|_{H^{s,a}} = \left\|(I-\Delta)^{s/2}f\right\|_{L^a}$ and the corresponding negative norm. The constant function 1 has norm one in X. Pairing X with X^* has norm at most the product of the two norms.

Lemma 4.6. *For every mean-zero $u \in H^1(\mathbb{T}^2)$,*

$$\|e^u\|_{H^{s,a}} \le C(1+\|\nabla u\|_2)\left(\int e^{pu}\,\mathrm{d}x\right)^{1/p}. \tag{4.40}$$

The constant depends on a, p, s, not on u.

Proof. Write $\delta_z f = f(\cdot+z) - f$. The inequality $|e^x - e^y| \le |x-y|(e^x+e^y)$ and Hölder give

$$\|\delta_z e^u\|_{L^a} \le 2\,\|e^u\|_{L^p}\,\|\delta_z u\|_{L^\ell}\,. \tag{4.41}$$

On the torus, Gagliardo–Nirenberg and the translation bounds imply, for $|z| \le 1$,

$$\begin{aligned}\|\delta_z u\|_{L^\ell} &\le C\left(\|\delta_z u\|_2^{2/\ell}\,\|\nabla\delta_z u\|_2^{1-2/\ell} + \|\delta_z u\|_2\right)\\ &\le C|z|^{2/\ell}\,\|\nabla u\|_2\,.\end{aligned}$$

For $0 < s < 1$, the heat-semigroup formula for $(I-\Delta)^{s/2}$ gives the bound

$$\|f\|_{H^{s,a}} \le C\left(\|f\|_{L^a} + \int_{|z|\le 1}\frac{\|\delta_z f\|_{L^a}}{|z|^{2+s}}\,dz\right). \tag{4.42}$$

For clarity, this is a sufficient difference-quotient bound (the $B^s_{a,1}$ bound), not an assertion that the Slobodeckij and Bessel norms are identical. It follows by writing the fractional power as an integral of $f - e^{t\Delta}f$ and using the Gaussian heat-kernel estimate. Substituting (4.41) makes the radial integral finite because $s < 2/\ell$. Also $\|e^u\|_{L^a} \le \|e^u\|_{L^p}$ on a probability space. Approximation by smooth functions, with (2.13), removes any smoothness restriction. □

Lemma 4.7. *Set $\rho_u = e^u / \int e^u \,\mathrm{d}x$ for mean-zero u. For every $L < \infty$ there is C_L such that, when $\|\nabla u\|_2 \le L$,*

$$\|\rho_u\|_{H^{s,a}} \le C_L, \tag{4.43}$$

$$\|\rho_u - 1\|_{H^{s,a}} \le C_L \|\nabla u\|_2, \tag{4.44}$$

$$\|\rho_u - 1 - u\|_{H^{s,a}} \le C_L \|\nabla u\|_2^2. \tag{4.45}$$

Proof. Put $r = \|\nabla u\|_2$. The Sobolev embeddings on the two-dimensional compact torus and (2.13) imply, for every fixed $j \in \mathbb{N}$ and finite exponents needed below,

$$\left\| |u|^j e^{\theta|u|} \right\|_{L^a} \le C_{L,j,\theta,a} r^j.$$

Taylor's formula therefore gives $\|e^u - 1 - u\|_{L^a} \le C_L r^2$. Since $a < 2$, choose finite q with $1/a = 1/2 + 1/q$. Then

$$\|\nabla(e^u - 1 - u)\|_{L^a} = \|(e^u - 1)\nabla u\|_{L^a} \le \|e^u - 1\|_{L^q} r \le C_L r^2.$$

The analogous first-order estimates and the embedding $W^{1,a} \hookrightarrow H^{s,a}$ give the unnormalized versions of the assertions.

Let $Z(u) = \int e^u \,\mathrm{d}x$. Jensen gives $Z(u) \ge 1$, and mean zero plus the Taylor estimate gives $|Z(u) - 1| \le C_L r^2$. Dividing by $Z(u)$ proves the result; for the second-order assertion use the exact identity

$$\rho_u - 1 - u = \frac{e^u - 1 - u}{Z(u)} + \left(\frac{1}{Z(u)} - 1 \right)(1 + u).$$

The $H^{s,a}$ norm of $1 + u$ is bounded on the chosen H^1 ball. □

Lemma 4.8. *For all sufficiently large N, (4.30) holds for every $0 < \varepsilon \le 1$.*

Proof. Let $\Phi_\varepsilon \sim \mathcal{N}(0, G_\varepsilon)$. For fixed positions, the Gaussian exponential identity yields

$$\mathbb{E} \exp\left\{ \sqrt{b/N} \sum_i \xi_i \Phi_\varepsilon(x_i) \right\} = \exp\left\{ b E_{N,\varepsilon} + \frac{b q_N}{2} G_\varepsilon(0) \right\}.$$

The diagonal is spatially constant and $G_\varepsilon(0) = \operatorname{Tr} G_\varepsilon$. Tonelli's theorem gives

$$Z_{N,\varepsilon}(b) = e^{-b q_N \operatorname{Tr} G_\varepsilon / 2} \mathbb{E} \exp\left\{ \sum_i \log \int e^{t_i \Phi_\varepsilon} \,\mathrm{d}x \right\}. \tag{4.46}$$

The quadratic factor $\exp(b q_N \|\Phi_\varepsilon\|_2^2 / 2)$ is integrable by (2.24). It changes the covariance to $Q_{N,b,\varepsilon}$ and contributes $\det(I - b q_N G_\varepsilon)^{-1/2}$. Combining this factor with the explicit diagonal subtraction gives precisely $\det_2(I - b q_N G_\varepsilon)^{-1/2}$, proving the identity. This calculation uses the signed ξ_i without taking their absolute values. □

Lemma 4.9. *For fixed $0 < b < \gamma$ and $Y = Y_{N,b,\varepsilon}$, the stationary covariance kernel satisfies*

$$|Q_{N,b,\varepsilon}(x, y)| \le C \left(1 + \log_+ \frac{1}{\operatorname{dist}(x, y)} \right), \qquad x \ne y, \tag{4.47}$$

uniformly for all sufficiently large N and all $0 < \varepsilon \le 1$.

Proof. The heat-kernel representation on the mean-zero space is

$$G_\varepsilon(x) = \int_\varepsilon^\infty (p_t(x) - 1)\, dt.$$

For $t \le 1$, the torus heat kernel is bounded by $C t^{-1} \exp(-\operatorname{dist}(x, 0)^2 / (Ct)) + C$; for $t \ge 1$, $p_t - 1$ decays exponentially. Splitting the integral at $t = \operatorname{dist}(x, 0)^2$ proves the claimed logarithmic bound for G_ε.

The remaining covariance is

$$Q_{N,b,\varepsilon} - G_\varepsilon = b q_N G_\varepsilon^2 (I - b q_N G_\varepsilon)^{-1}.$$

Its Fourier coefficients are absolutely summable, uniformly in N, ε, because $q_N \le M^2$, (2.24) bounds the denominator away from zero, and $\sum_{k \ne 0} \lambda_k^{-2} < \infty$. Its kernel is therefore uniformly bounded. This proves (4.47). □

Lemma 4.10. *Fix the spaces in (4.39), and let $m \geq \max\{P, 4\}$ be finite. Wick ordering is with respect to $Q = Q_{N,b,\varepsilon}$. There is $C_m < \infty$, independent of k, N, ε, such that for every integer $k \geq 1$,*

$$\left\|Y^{:k:}\right\|_{L^m(\Omega;X)} \leq k!\, C_m^k. \tag{4.48}$$

Proof. Let j_s be the positive, normalized kernel of $(I-\Delta)^{-s/2}$ on the full torus, including constants. Stationarity and Wick's formula give

$$\mathbb{E}\left|(j_s * Y^{:k:})(x)\right|^2 = k! \int j_{2s}(z) Q(z)^k \, dz. \tag{4.49}$$

The integral is nonnegative as written; we bound it by inserting absolute values. Since $0 < s < 1$, $j_{2s}(z) \leq C_s \operatorname{dist}(z,0)^{2s-2} + C_s$. By Lemma 4.9,

$$\mathbb{E}\left|(j_s * Y^{:k:})(x)\right|^2 \leq k!\, C^k \left(1 + \int_0^1 r^{2s-1}(1 + |\log r|)^k \, dr\right) \leq (k!)^2 C_s^{2k}.$$

For the last inequality one may expand the polynomial and use $\int_0^1 r^{2s-1} |\log r|^j \, dr = j!/(2s)^{j+1}$.

The smoothed Wick power is still in the kth homogeneous chaos. Apply (4.35) pointwise in x, and then use Minkowski, since $m \geq P$:

$$\left\|Y^{:k:}\right\|_{L^m(\Omega;X)} \leq \left(\int \left\|(j_s * Y^{:k:})(x)\right\|_{L^m(\Omega)}^P \, \mathrm{d}x\right)^{1/P} \leq k!\, C_m^k.$$

All constants are independent of the cutoff. □

Lemma 4.11. *There exist $t_0 \in (0,1]$ and random variables $K = K_{N,b,\varepsilon} \geq 1$ such that $\sup_{N,\varepsilon} \mathbb{E}K^m < \infty$ and, simultaneously for $|t| \leq t_0$,*

$$\|Y\|_X + \left\|Y^{:2:}\right\|_X \leq K, \tag{4.50}$$

$$\|M_t - 1\|_X \leq |t| K, \tag{4.51}$$

$$\left\|M_t - 1 - tY - \tfrac{1}{2}t^2 Y^{:2:}\right\|_X \leq |t|^3 K, \tag{4.52}$$

where

$$\sigma = Q(x,x), \qquad M_t = e^{tY - t^2\sigma/2}. \tag{4.53}$$

Proof. For each positive cutoff, Y is smooth almost surely, so the Wick generating series is

$$M_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} Y^{:k:}.$$

Choose t_0 so that $t_0 C_m < 1/2$. By Lemma 4.10, after multiplying by a harmless deterministic constant if needed,

$$K = 1 + \|Y\|_X + \left\|Y^{:2:}\right\|_X + \sum_{k=3}^{\infty} \frac{t_0^{k-3}}{k!} \left\|Y^{:k:}\right\|_X$$

has uniformly bounded $L^m(\Omega)$ norm. Absolute convergence in $L^m(\Omega;X)$ and the same random majorant for every $|t| \leq t_0$ give all the asserted inequalities. The identity agrees with the pointwise smooth exponential, and the bounds extend simultaneously in t by continuity. Importantly, t_0 may be small; it is made compatible with all t_i by increasing N_*, not by restricting the cutoff. □

4.4. **Stochastic coercivity and cutoff removal.**

Proof of Theorem 4.5. Keep $0 < b < \gamma$ and the exponents from Subsection 4.3. In this proof abbreviate

$$Y = Y_{N,b,\varepsilon}, \quad Q = Q_{N,b,\varepsilon}, \quad a_N = bq_N, \quad R_N = R_{N,b}, \quad r = \|\nabla h\|_2.$$

All Wick products use the tilted covariance Q, not G_ε. Increase N_* so that

$$\max_i |t_i| \leq \sqrt{b/N}\, M \leq t_0. \tag{4.54}$$

The bound $\sup \mathbb{E}K^2 < \infty$ now follows from Lemma 4.11.

The exact cancellation of the divergent variance. Since $\sum_i t_i^2 = a_N$ and $\|Y\|_2^2 = \langle Y^{:2:}, 1\rangle + \sigma$,

$$R_N(Y+h) = \sum_i \log \int M_{t_i} e^{t_i h} \, \mathrm{d}x - \frac{a_N}{2}\|h\|_2^2 - a_N\langle Y, h\rangle - \frac{a_N}{2}\langle Y^{:2:}, 1\rangle. \tag{4.55}$$

The terms proportional to σ cancel exactly. No bound involving σ is used subsequently. This is the source of true cutoff uniformity.

Shifts with $r \le R\sqrt{N}$. Fix $R \ge 1$, to be chosen after the small loss parameter. Put

$$u_i = t_i h, \qquad Z_i = \int e^{u_i}\,\mathrm{d}x, \qquad \rho_i = e^{u_i}/Z_i.$$

Then $\|\nabla u_i\|_2 \le \sqrt{b} M R$, so Lemma 4.7 applies with a constant denoted C_R. Since $M_{t_i} > 0$ and ρ_i is a probability density,

$$\begin{aligned} \log \int M_{t_i} e^{u_i}\,\mathrm{d}x &= \log Z_i + \log\langle M_{t_i}, \rho_i\rangle \\ &\le \log Z_i + \langle M_{t_i} - 1, \rho_i\rangle. \end{aligned} \tag{4.56}$$

Write

$$M_{t_i} - 1 = t_i Y + \frac{t_i^2}{2} Y^{:2:} + t_i^3 T_i, \qquad \|T_i\|_X \le K. \tag{4.57}$$

If $t_i = 0$, choose T_i arbitrarily, since its coefficient is zero.

Substituting (4.57) in (4.55), using $\langle Y, 1\rangle = 0$ and $\sum_i t_i^2 = a_N$, gives

$$\begin{aligned} R_N(Y+h) \le R_N(h) &+ \sum_i t_i \langle Y, \rho_i - 1 - t_i h\rangle \\ &+ \frac{1}{2}\sum_i t_i^2 \langle Y^{:2:}, \rho_i - 1\rangle + \sum_i t_i^3 \langle T_i, \rho_i\rangle. \end{aligned} \tag{4.58}$$

This displays both cancellations: $\sum_i t_i^2 \langle Y, h\rangle$ removes the mixed quadratic term, while the constants in ρ_i remove the Wick-square term. No independence is involved.

By duality and (4.43)–(4.45),

$$\begin{aligned} R_N(Y+h) &\le R_N(h) + C_R K \left(\sum_i |t_i|^3\right)(r^2 + r + 1) \\ &\le R_N(h) + \frac{C_R K}{\sqrt{N}}(r^2 + r + 1), \end{aligned} \tag{4.59}$$

since $\sum_i |t_i|^3 \le b^{3/2} M^3/\sqrt{N}$. In the present regime, $r^2/\sqrt{N} \le Rr$. Consequently, for every $\delta > 0$, Young's inequality yields

$$R_N(Y+h) \le R_N(h) + \delta r^2 + C_{R,\delta}(1 + K^2). \tag{4.60}$$

Applying (H) to $\sqrt{b/N}\,h$ gives

$$R_N(h) \le \frac{b}{2\gamma} r^2 - \frac{b q_N}{2}\,\|h\|_2^2\,. \tag{4.61}$$

Thus, for $r \le R\sqrt{N}$,

$$R_N(Y+h) \le \left(\frac{b}{2\gamma} + \delta\right) r^2 - \frac{b q_N}{2}\,\|h\|_2^2 + C_{R,\delta}(1 + K^2). \tag{4.62}$$

Large constants depending on R are harmless: R is fixed before N and ε vary.

Shifts with $r \ge R\sqrt{N}$. Here we use the global bound rather than a Taylor expansion of ρ_i. By Lemma 4.11, $\|M_{t_i}\|_X \le 1 + |t_i| K$. Duality and Lemma 4.6 imply

$$\int M_{t_i} e^{t_i h}\,\mathrm{d}x \le C(1 + |t_i|K)(1 + |t_i| r)\left(\int e^{p t_i h}\,\mathrm{d}x\right)^{1/p}. \tag{4.63}$$

The left-hand side is positive. Taking logarithms and using $\log(1 + x) \le x$ for the factor involving K,

$$\begin{aligned} \sum_i \log \int M_{t_i} e^{t_i h}\,\mathrm{d}x &\le \frac{1}{p}\sum_i \log \int e^{p t_i h}\,\mathrm{d}x + N \log C + K \sum_i |t_i| + \sum_i \log(1 + |t_i| r) \\ &\le \frac{pb}{2\gamma} r^2 + N \log C + B\sqrt{N} K + N \log\left(1 + \frac{Br}{\sqrt{N}}\right), \end{aligned} \tag{4.64}$$

where $B = \sqrt{b} M$ and (H) was applied to $p\sqrt{b/N}\,h$.

The mean-zero Sobolev and Poincaré embeddings give $\|h\|_{H^{s,a}} \le Cr$. Hence

$$a_N|\langle Y,h\rangle| + \frac{a_N}{2}|\langle Y^{:2:},1\rangle| \le CK(r+1). \tag{4.65}$$

Inserting (4.64) and (4.65) into (4.55),

$$\begin{aligned} R_N(Y+h) &\le \frac{pb}{2\gamma}r^2 - \frac{bq_N}{2}\|h\|_2^2 \\ &\quad + N\left[\log C + \log\left(1+\frac{Br}{\sqrt{N}}\right)\right] + B\sqrt{N}K + CK(r+1). \end{aligned} \tag{4.66}$$

Given $\delta > 0$, choose $R \ge 1$ so large that

$$\log C + \log(1+Bz) \le \delta z^2 \qquad (z \ge R).$$

The deterministic logarithmic term in (4.66) is then at most δr^2. Also $\sqrt{N} \le r/R$, so Young's inequality bounds all remaining random terms by $\delta r^2 + C_\delta(1+K^2)$. We obtain

$$R_N(Y+h) \le \left(\frac{pb}{2\gamma} + 2\delta\right) r^2 - \frac{bq_N}{2}\|h\|_2^2 + C_\delta(1+K^2). \tag{4.67}$$

The near-one estimate $\|M_t\|_X \le 1 + |t|K$ is important: replacing it by a crude CK bound would introduce $N \log K$, which this argument would not absorb uniformly.

Choice of constants and adapted controls. First choose p by (4.37), then choose $\delta > 0$ so that $pb/(2\gamma) + 2\delta < c$, and then choose R as above. Combining (4.62) and (4.67) yields

$$R_N(Y+h) \le cD(h) - \frac{bq_N}{2}\|h\|_2^2 + C(1+K^2).$$

Because $c < 1/2$ and $q_N \ge 0$,

$$cD(h) - \frac{bq_N}{2}\|h\|_2^2 \le c\big(D(h) - bq_N\|h\|_2^2\big).$$

This proves (4.33), simultaneously for every mean-zero $h \in H^1$ on the full-probability event where the Wick bounds hold. The approximation argument used in Subsection 4.3 also extends the estimates from smooth shifts to all such shifts.

For a Cameron–Martin shift,

$$D(h) - bq_N\|h\|_2^2 \le \langle h, (G_\varepsilon^{-1} - bq_N I)h\rangle = \|h\|_{\mathrm{CM}(Q)}^2, \tag{4.68}$$

since $G_\varepsilon^{-1} = Ae^{\varepsilon A} \ge A$. For $h = h(v)$,

$$\|h(v)\|_{\mathrm{CM}(Q)}^2 = \left\|\int_0^1 v_t\,dt\right\|_2^2 \le \int_0^1 \|v_t\|_2^2\,dt. \tag{4.69}$$

Taking expectations proves (BD), by the uniform second moment of K. The derivation never conditions on or decouples h from Y. This completes the proof of Theorem 4.5. □

Corollary 4.12. *For some $\alpha > 1$,*

$$\sup_{N \ge N_*,\, 0<\varepsilon\le 1} \mathbb{E}\exp\{\alpha R_{N,b}(Y_{N,b,\varepsilon})^+\} < \infty. \tag{4.70}$$

Furthermore,

$$\sup_{0<\varepsilon\le 1} \left|\mathbb{E}e^{R_{N,b}(Y_{N,b,\varepsilon})} - 1\right| \longrightarrow 0. \tag{4.71}$$

Proof of Corollary 4.12: exponential integrability. Choose $1 < \alpha < 1/(2c)$ and set

$$F_L(f) = \alpha \min\{R_{N,b}(f)^+, L\}.$$

This is a bounded measurable functional of the smooth terminal Gaussian field. From (4.33), (4.68), and (4.69), its value on a controlled field satisfies

$$F_L(Y+h(v)) \le \alpha c\int_0^1 \|v_t\|_2^2\,dt + \alpha C(1+K^2).$$

The right side is nonnegative; thus it also bounds the positive part before truncation. Apply the bounded formula (4.36):

$$\log \mathbb{E} e^{F_L(Y)} \le \alpha C \mathbb{E}(1+K^2) + \sup_v \left(\alpha c - \frac{1}{2}\right) \mathbb{E} \int_0^1 \|v_t\|_2^2 \, dt \le C_\alpha.$$

The constants are uniform in L, N, ε. Monotone convergence as $L \to \infty$ proves (4.70). This argument is not an invocation of an unbounded-functional formula with unverified hypotheses. □

Proof of Corollary 4.12: convergence of the remainder. At $h = 0$, (4.55) becomes

$$R_{N,b}(Y) = \sum_i \log \int M_{t_i} \, \mathrm{d}x - \frac{bq_N}{2} \langle Y^{:2:}, 1 \rangle.$$

Since $\langle Y, 1 \rangle = 0$, write

$$\int M_{t_i} \, \mathrm{d}x = 1 + z_i, \qquad z_i = \frac{t_i^2}{2} \langle Y^{:2:}, 1 \rangle + t_i^3 \langle T_i, 1 \rangle.$$

Uniformly in i, $|z_i| \le CK/N$. On the event $CK/N \le 1/2$, the elementary bound $|\log(1+z) - z| \le Cz^2$ gives

$$|R_{N,b}(Y)| \le C \left(\frac{K}{\sqrt{N}} + \frac{K^2}{N} \right). \tag{4.72}$$

The complement has probability tending to zero uniformly in ε, by the uniform second moment of K. Therefore $R_{N,b}(Y) \to 0$ in probability uniformly in ε. Equation (4.70) gives uniform integrability of $e^{R_{N,b}(Y)}$, with an exponent $\alpha > 1$. Splitting according to $|R_{N,b}(Y)| \le \delta$ and then using Hölder on the complement proves, in fact,

$$\sup_{0<\varepsilon\le 1} \mathbb{E} |e^{R_{N,b}(Y_{N,b,\varepsilon})} - 1| \longrightarrow 0.$$

This implies (4.71). □

Lemma 4.13. *For every fixed $b < \gamma$,*

$$\sup_{N \ge N_b,\, 0<\varepsilon\le 1} Z_{N,\varepsilon}(b) < \infty. \tag{4.73}$$

Moreover, the determinant in (4.30) converges as $\varepsilon \downarrow 0$ for each N, and then as $N \to \infty$ after replacing q_N by q_2.

Proof. Write $g_{k,\varepsilon} = e^{-\varepsilon\lambda_k}/\lambda_k$. The spectral gap gives $0 \le bq_N g_{k,\varepsilon} \le 1 - \kappa$, with $\kappa = 1 - b/\gamma > 0$. Since

$$0 \le -\log(1-x) - x \le C_\kappa x^2 \qquad (0 \le x \le 1-\kappa),$$

we have

$$0 \le \log \det{}_2(I - bq_N G_\varepsilon)^{-1/2} \le C_{b,M,\kappa} \sum_{k\ne 0} \lambda_k^{-2} < \infty.$$

The identity (4.30) and (4.70) now prove (4.73). Dominated convergence of the logarithmic products gives the determinant limits. □

Completion of the proof of Theorem 4.4. For each fixed sufficiently large N, the kernels $G_\varepsilon(x_i - x_j)$ converge to $G(x_i - x_j)$ away from collisions, a set of full P_N measure. Hence $E_{N,\varepsilon} \to E_N$ almost surely. Fatou's lemma and Lemma 4.13 give $Z_N(b) \le C_b$ for every fixed $b < \gamma$ and sufficiently large N. This proves finiteness and (4.25) without assuming monotonicity of the signed interaction.

For later use, if $0 < \beta < b < \gamma$, the same bound at b makes $e^{\beta E_{N,\varepsilon}}$ uniformly integrable as $\varepsilon \downarrow 0$ for every such N. Consequently

$$Z_{N,\varepsilon}(\beta) \longrightarrow Z_N(\beta). \tag{4.74}$$

The determinant asymptotics and the Gibbs fluctuation law will be identified together in section 6.1, using one common change of measure from the independent reference ensemble. □

4.5. **Exponential-scale bounds below collapse.**

Proposition 4.14. *Assume (2.3) and (B). For every fixed $\beta < b < \gamma$,*

$$\sup_{N\ge N_b,\,0<\varepsilon\le 1} \frac{1}{N}\log Z_{N,\varepsilon}(b) < \infty, \qquad \sup_{N\ge N_b} \frac{1}{N}\log Z_N(b) < \infty. \tag{4.75}$$

In particular, (P) holds. Neither uniqueness nor homogeneous spectral stability is assumed.

Proof. We prove (B) ⇒ (P) using the *untilted* Gaussian field. This proof is not an invocation of the strict-(H) estimate outside its range.

Fix $\beta < b < \gamma$ as in (B). Let $Y_\varepsilon \sim \mathcal{N}(0, G_\varepsilon)$, $t_i = \sqrt{b/N}\,\xi_i$, and

$$M_{t,\varepsilon} = \exp\{tY_\varepsilon - t^2 G_\varepsilon(0)/2\}.$$

By the same Gaussian calculation as (4.46),

$$Z_{N,\varepsilon}(b) = \mathbb{E}e^{\mathcal{V}_{N,b,\varepsilon}(Y_\varepsilon)}, \qquad \mathcal{V}_{N,b,\varepsilon}(Y_\varepsilon + h) = \sum_i \log \int M_{t_i,\varepsilon} e^{t_i h}\,\mathrm{d}x. \tag{4.76}$$

Wick ordering in this subsection is with respect to G_ε, not the tilted covariance. Its kernel satisfies the same uniform logarithmic estimate. The proofs of Lemmata 4.10 and 4.11 therefore apply verbatim after deleting the bounded covariance correction. In particular, for sufficiently large N,

$$\|M_{t_i,\varepsilon}\|_{H^{-s,P}} \le 1 + |t_i| K_\varepsilon, \qquad \sup_\varepsilon \mathbb{E}K_\varepsilon^2 < \infty. \tag{4.77}$$

Choose $1 < p < 2$ with $pb < \gamma$, and choose $1 < a < p$, $s > 0$ as in (4.38). The deterministic exponential estimate (4.40) and duality give, with $r = \|\nabla h\|_2$,

$$\int M_{t_i,\varepsilon} e^{t_i h}\,\mathrm{d}x \le C(1 + |t_i|K_\varepsilon)(1 + |t_i| r)\left(\int e^{p t_i h}\,\mathrm{d}x\right)^{1/p}.$$

The estimate is pathwise for every shift. Summing logarithms, using $|\xi_i| \le M$ and (B), gives

$$\begin{aligned}\mathcal{V}_{N,b,\varepsilon}(Y_\varepsilon + h) &\le \frac{N}{p} L_{\nu_N}(p\sqrt{b/N}\,h) + N\log C + B\sqrt{N}K_\varepsilon + N\log(1 + Br/\sqrt{N}) \\ &\le \frac{pb}{2\gamma} r^2 + \frac{NA_\gamma}{p} + N\log C + B\sqrt{N}K_\varepsilon + N\log(1 + Br/\sqrt{N}),\end{aligned} \tag{4.78}$$

where $B = \sqrt{b}M$. For any $\delta > 0$,

$$N\log(1 + Br/\sqrt{N}) \le \delta r^2 + C_\delta N, \qquad B\sqrt{N}K_\varepsilon \le CN + CK_\varepsilon^2.$$

Choose δ small enough to obtain

$$\mathcal{V}_{N,b,\varepsilon}(Y_\varepsilon + h) \le cD(h) + C(N + K_\varepsilon^2), \qquad c < \tfrac{1}{2}. \tag{4.79}$$

All constants are independent of N, ε once N is sufficiently large.

For $h(v) = G_\varepsilon^{1/2}\int_0^1 v_t\,dt$, spectral calculus and Jensen give

$$D(h(v)) \le \left\|\int_0^1 v_t\,dt\right\|_2^2 \le \int_0^1 \|v_t\|_2^2\,dt.$$

Choose $1 < \alpha < 1/(2c)$ and apply the bounded BD formula to $\alpha\min\{\mathcal{V}^+_{N,b,\varepsilon}, L\}$. The bound (4.79) makes the controlled contribution at most

$$\alpha c\int_0^1 \|v_t\|_2^2\,dt + \alpha C(N + K_\varepsilon^2).$$

After subtracting the control cost and taking the supremum, its logarithmic exponential moment is at most $C_\alpha N + C_\alpha$. Monotone convergence in L proves the same bound for $e^{\alpha\mathcal{V}^+}$. In particular

$$Z_{N,\varepsilon}(b) \le e^{C_b N}.$$

For each fixed such N, the regularized off-diagonal energy converges almost surely to the singular one. Fatou gives $Z_N(b) \le e^{C_b N}$, proving (P). No condition on $I - bq_N G$ was used. □

5. Marked large deviations and macroscopic laws

5.1. **The free energy and the large deviation principle.** The partition bounds now yield a large-deviation principle for the whole marked empirical measure. Only an exponential-scale bound with a strict temperature margin is needed; homogeneous uniqueness and positivity of the homogeneous Hessian play no role.

Set $K = [-M, M]$, $E = \mathbb{T}^2 \times K$, and $\mathcal{X} = \mathcal{P}(E)$, with weak convergence. The law of the empirical measure under the Gibbs ensemble is

$$\Pi_{N,\beta} = (\widehat{\rho}_N)_{\#}\mu_{N,\beta} \in \mathcal{P}(\mathcal{X}). \tag{5.1}$$

For $\rho \in \mathcal{X}$, define

$$\mathsf{H}_\nu(\rho) = \begin{cases} \mathrm{Ent}(\rho \mid dx\,\nu(d\xi)), & \rho(\mathbb{T}^2, d\xi) = \nu(d\xi), \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.2}$$

For finite entropy, $\rho(dx, d\xi) = r_\xi(x)\,dx\,\nu(d\xi)$, and $\mathsf{H}_\nu(\rho) = H_\nu(r)$. Extend $F_{\beta,\nu}$ from (2.9) to $\mathcal{X}$ by $+\infty$ when $\mathsf{H}_\nu = +\infty$. The extension is not a convention for subtracting two infinities: the energy is finite on the finite-entropy domain, and the subcritical estimates below force infinite cost at its complement.

Energy normalization. For the large-deviation scale we use the additional notation

$$e_N = \frac{E_N}{N} = \frac{1}{N^2}\sum_{i<j}\xi_i\xi_j G(x_i - x_j), \qquad \mathfrak{e}(\rho) = \frac{1}{2}\mathcal{E}_\nu(r), \qquad F_{\beta,\nu}(\rho) = \mathsf{H}_\nu(\rho) - \beta\mathfrak{e}(\rho). \tag{5.3}$$

Thus the Gibbs exponent is $N\beta e_N = \beta E_N$. The quadratic quantity $\mathcal{E}_\nu$ in (2.8) has *no* factor $1/2$, whereas $\mathfrak{e}$ does. The singular continuum energy must never be evaluated on an atomic empirical measure without the off-diagonal convention.

Theorem 5.1. *Assume (2.3) and (P). Let*

$$m_{\beta,\nu} = \min_{\rho\in\mathcal{X}} F_{\beta,\nu}(\rho), \qquad I_{\beta,\nu} = F_{\beta,\nu} - m_{\beta,\nu}. \tag{5.4}$$

Then $I_{\beta,\nu}$ is a good rate function and

$$p_N(\beta) := \frac{1}{N}\log Z_N(\beta) \longrightarrow -m_{\beta,\nu} =: p(\beta). \tag{5.5}$$

On the compact space $\mathcal{P}(\mathcal{X})$, with weak convergence,

$$\Theta \longmapsto \frac{1}{N}\,\mathrm{Ent}(\Theta \mid \Pi_{N,\beta}) \quad \xrightarrow{\Gamma} \quad \Theta \longmapsto \int_{\mathcal{X}} I_{\beta,\nu}(\rho)\,\Theta(d\rho). \tag{5.6}$$

Equivalently, $\Pi_{N,\beta}$ satisfies the speed-N large deviation principle: for every Borel $A \subset \mathcal{X}$,

$$-\inf_{A^\circ} I_{\beta,\nu} \le \liminf_N \frac{1}{N}\log\Pi_{N,\beta}(A), \tag{5.7}$$

$$\limsup_N \frac{1}{N}\log\Pi_{N,\beta}(A) \le -\inf_{\overline{A}} I_{\beta,\nu}. \tag{5.8}$$

The equivalent Laplace principle is, for every $\Psi \in C(\mathcal{X})$,

$$-\frac{1}{N}\log\int e^{-N\Psi(\rho)}\,\Pi_{N,\beta}(d\rho) \longrightarrow \inf_{\rho\in\mathcal{X}}\{\Psi(\rho) + I_{\beta,\nu}(\rho)\}. \tag{5.9}$$

In particular all these conclusions hold throughout $0 < \beta < \beta_{\mathrm{coll}}$ by Corollary 5.7.

Remark 5.2. The limit in (5.6) is $\int I_{\beta,\nu}\,d\Theta$, not $I_{\beta,\nu}(\int \rho\,d\Theta)$. It is a statement on *laws of marked empirical measures*. Ordinary Γ-convergence of regularized continuum free energies on $\mathcal{X}$ does not by itself imply it. The correspondence with the LDP is Mariani's entropy characterization [17, Theorems 3.3–3.5]; related distinctions between levels of free-energy convergence are discussed in [3]. Our proof first establishes an unnormalized lifted free-energy limit and then derives the pressure, avoiding circular normalization.

5.2. **Entropy compactness and exact-mark recovery.** We first control the singular continuum energy on entropy sublevels. This product-energy estimate will be used in recovery and lower semicontinuity. It does not concern arbitrary correlated configuration laws.

Let $\ell = 1/m$ with integer $m \geq 4$, and partition the torus into half-open squares Q of side ℓ. Put

$$k_\ell(x,y) = \left(\log \frac{\ell}{d_{\mathbb{T}^2}(x,y)}\right)_+. \tag{5.10}$$

Only equal or neighboring squares can contribute. Each square has at most nine such neighbors, including itself. If U_Q, U_R are independent uniform points in prescribed squares, then

$$\mathbb{E}e^{ak_\ell(U_Q,U_R)} \leq e^{Ca} \qquad (0 \leq a \leq 1), \tag{5.11}$$

uniformly in ℓ, Q, R. Indeed, for fixed x,

$$\ell^{-2}\int_R e^{k_\ell(x,y)}\,dy \leq 1 + \ell^{-2}\int_{d(x,y)<\ell} \frac{\ell}{d(x,y)}\,dy \leq C_0.$$

The last integral is bounded after rescaling. Convexity of $a \mapsto \log \mathbb{E}e^{ak_\ell}$ between 0 and 1 gives (5.11) with $C = \log C_0$.

Lemma 5.3. *For every probability density f on $\mathbb{T}^2$ with $H(f) = \int f \log f \,\mathrm{d}x < \infty$, put $p_Q = \int_Q f\,\mathrm{d}x$. Then*

$$\iint k_\ell(x,y)f(x)f(y)\,\mathrm{d}x\,dy \leq C\left(\max_Q p_Q\right)\big(H(f)+1\big). \tag{5.12}$$

In particular, for every $L < \infty$ the left side tends to zero as $\ell \downarrow 0$, uniformly over $H(f) \leq L$.

Proof. If $p_Q > 0$, write $\sigma_Q = p_Q^{-1} f\mathbf{1}_Q\,dx$ and let u_Q denote normalized Lebesgue measure on Q. Set $h_Q = \mathrm{Ent}(\sigma_Q \mid u_Q) \geq 0$. The entropy chain rule gives

$$\sum_Q p_Q h_Q = H(f) - \sum_Q p_Q \log\frac{p_Q}{\ell^2} \leq H(f). \tag{5.13}$$

The finite sum on the right is nonnegative, being a discrete relative entropy. For neighboring squares, the entropy variational inequality and (5.11) with $a = 1$ imply

$$\iint k_\ell\,d\sigma_Q d\sigma_R \leq h_Q + h_R + C.$$

This is also valid when $Q = R$, because the two samples in the product measure are independent. Multiplying by $p_Q p_R$ and summing over neighbors bounds the result by

$$18\left(\max_Q p_Q\right)\sum_Q p_Q h_Q + 9C\max_Q p_Q,$$

which proves (5.12).

By contraction to the event Q,

$$H(f) \geq p_Q \log\frac{1}{|Q|} - \log 2.$$

Indeed this is the binary relative-entropy bound, with binary entropy at most $\log 2$ and the complementary contribution $-(1-p_Q)\log(1-|Q|) \geq 0$. Consequently

$$\max_Q p_Q \leq \frac{H(f)+\log 2}{\log(1/\ell^2)}.$$

Insert this in (5.12) to obtain uniform vanishing on every entropy sublevel. Terms with zero mass contribute zero throughout. □

Corollary 5.4. *Suppose $\rho_n \Rightarrow \rho$ and $\sup_n \mathsf{H}_\nu(\rho_n) < \infty$. Then $\mathsf{H}_\nu(\rho) < \infty$ and*

$$\mathfrak{e}(\rho_n) \longrightarrow \mathfrak{e}(\rho). \tag{5.14}$$

For every finite-entropy marked measure the logarithmic interaction is absolutely integrable.

Proof. The spatial marginal has density f_n with $H(f_n) \le \mathsf{H}_\nu(\rho_n)$ by entropy contraction. Set $v_n(dx) = \int \xi\, \rho_n(dx, d\xi)$, so $|v_n| \le M f_n\, dx$. These are the uncentered vorticity measures; their centered versions are $w_n = v_n - q_1 dx$. Since G annihilates constants, $\mathfrak{e}(\rho_n) = \frac{1}{2}\iint G\, dv_n\, dv_n$. The domination is used for v_n, not for w_n. Define

$$G_\ell(x,y) = G(x-y) - \frac{1}{2\pi} k_\ell(x,y). \tag{5.15}$$

This kernel extends continuously across the diagonal: the logarithmic singularities cancel, leaving $-(2\pi)^{-1}\log\ell$ plus the continuous regular part. Away from the diagonal it is continuous as well. For fixed ℓ, the marked product integral with kernel $\xi\zeta G_\ell(x,y)$ is weakly continuous. The absolute tail is at most $M^2 \iint k_\ell f_n f_n/(4\pi)$, uniformly small by Lemma 5.3. Relative-entropy lower semicontinuity gives the same entropy bound for the limit. First let $n \to \infty$ with ℓ fixed and then let $\ell \downarrow 0$ to prove (5.14). The same estimates at one finite-entropy density prove absolute integrability. □

We give one explicit construction, avoiding a separate unproved approximation assumption. It will be used at both β and the larger parameter b in (P).

Lemma 5.5. *Let $\mathsf{H}_\nu(\rho) < \infty$. There are configuration laws $R_N \ll P_N$ such that*

$$(\widehat{\rho}_N)_{\#} R_N \Rightarrow \delta_\rho, \tag{5.16}$$

$$\frac{1}{N}\operatorname{Ent}(R_N \mid P_N) \longrightarrow \mathsf{H}_\nu(\rho), \tag{5.17}$$

$$\mathbb{E}_{R_N} e_N \longrightarrow \mathfrak{e}(\rho). \tag{5.18}$$

The laws can be chosen as products of smooth strictly positive spatial densities, one for each prescribed mark.

Proof. Disintegrate $\rho(dx, d\xi) = r_\xi(x)\, dx\, \nu(d\xi)$. Let $P_t = e^{t\Delta}$ and set $r_\xi^t = P_t r_\xi$. For each fixed $t > 0$, the positive heat kernel gives deterministic bounds

$$0 < c_t \le r_\xi^t(x) \le C_t < \infty,$$

uniformly in ξ, x, and bounds for all spatial derivatives. Write $\rho^t = r_\xi^t\, dx\, \nu$. The common heat-approximation argument in Subsection 2.1 gives $\mathsf{H}_\nu(\rho^t) \to \mathsf{H}_\nu(\rho)$ and $\mathfrak{e}(\rho^t) \to \mathfrak{e}(\rho)$ as $t \downarrow 0$.

Choose couplings $\pi_N(d\xi, d\zeta)$ of ν and ν_N with $\int |\xi - \zeta|\, d\pi_N = W_1(\nu, \nu_N) \to 0$. By disintegration, define

$$r^t_{N,\zeta}(x) = \int r^t_\xi(x)\, \pi_N(d\xi \mid \zeta), \qquad \rho_{N,t} = r^t_{N,\zeta}(x)\, dx\, \nu_N(d\zeta). \tag{5.19}$$

These are normalized densities with the same bounds c_t, C_t. Uniform continuity of test functions in the mark variable and the coupling estimate show $\rho_{N,t} \Rightarrow \rho^t$ for fixed t. Jensen gives

$$\operatorname{Ent}(\rho_{N,t} \mid dx\, \nu_N) \le \mathsf{H}_\nu(\rho^t).$$

Relative entropy is jointly lower semicontinuous under weak convergence of both arguments (by its continuous-test variational formula). Since $dx\,\nu_N \Rightarrow dx\,\nu$, the reverse limit inequality holds. Thus these entropies converge.

The vorticity densities satisfy the uniform estimate

$$\left\| \int \zeta r^t_{N,\zeta}\, \nu_N(d\zeta) - \int \xi r^t_\xi\, \nu(d\xi) \right\|_\infty \le C_t W_1(\nu_N, \nu).$$

The same holds for the centered densities with $C_t + 1$ in place of C_t. Hence their quadratic energies converge.

Now set

$$R_{N,t} = \bigotimes_{i=1}^N r^t_{N,\xi_i}(x_i)\, dx_i. \tag{5.20}$$

For each bounded test function, the empirical average has variance $O(N^{-1})$, uniformly in the marks. Its expectation is the integral against $\rho_{N,t}$. A countable dense family of tests proves $(\widehat{\rho}_N)_{\#} R_{N,t} \Rightarrow \delta_{\rho^t}$. Entropy additivity gives

$$\frac{1}{N}\operatorname{Ent}(R_{N,t} \mid P_N) = \operatorname{Ent}(\rho_{N,t} \mid dx\, \nu_N).$$

Independence gives, writing $\mathfrak{e}(\rho_{N,t})$ for the same centered quadratic formula with marginal ν_N,

$$\mathbb{E}_{R_{N,t}} e_N = \mathfrak{e}(\rho_{N,t}) - \frac{1}{2N^2} \sum_i \xi_i^2 \iint G(x-y) r^t_{N,\xi_i}(x) r^t_{N,\xi_i}(y)\, \mathrm{d}x\, dy.$$

The second term is $O_t(N^{-1})$, since $G \in L^1$ and the densities are bounded. This proves all three limits at fixed t with target ρ^t. Choose $t_j \downarrow 0$ and then increasing N_j making all fixed-t_j errors at most $1/j$ for $N \geq N_j$. Taking $t = t_{j(N)}$ proves (5.16)–(5.18). The mark marginal remains exactly ν_N at every step. □

Lemma 5.6. *Assume (P). Then $F_{b,\nu}(\rho) \geq -C_b$ on the finite-entropy domain, and*

$$F_{\beta,\nu}(\rho) \geq (1-\beta/b)\mathsf{H}_\nu(\rho) - \beta C_b/b. \tag{5.21}$$

The extension $F_{\beta,\nu} = +\infty$ off that domain is lower semicontinuous, bounded below, and has compact sublevels.

Proof. The Gibbs entropy inequality at b, applied to the smooth product recovery laws of Lemma 5.5, gives

$$\frac{1}{N} \log Z_N(b) \geq b\mathbb{E}_{R_N} e_N - \frac{1}{N} \operatorname{Ent}(R_N \mid P_N).$$

The energies are integrable under these smooth product laws, so this variational inequality can also be obtained by truncating the energy before passage to the limit. By (P) and recovery, $b\mathfrak{e}(\rho) - \mathsf{H}_\nu(\rho) \leq C_b$. Use

$$F_{\beta,\nu} = (1-\beta/b)\mathsf{H}_\nu + (\beta/b)F_{b,\nu}$$

to obtain (5.21).

For a sequence with bounded $F_{\beta,\nu}$, this estimate bounds the entropies. The mark marginal condition is closed under weak convergence, entropy is lower semicontinuous, and energy is continuous on the resulting entropy sublevel by Corollary 5.4. This proves lower semicontinuity, including at infinite-entropy limits. Compactness of $\mathcal{X}$ gives compact sublevels and attainment of the minimum. □

Corollary 5.7. *Assume (2.3). For every $0 < \beta < \beta_{\mathrm{coll}}$, the extended free energy $F_{\beta,\nu}$ is bounded below and lower semicontinuous on $\mathcal{X}$, with compact sublevel sets. Its minimum is attained, and every minimizer has the conditional exponential form (2.18), with a smooth spatial potential. Moreover,*

$$\beta < \beta_{\mathrm{coll}} \implies (B) \implies (P). \tag{5.22}$$

More precisely, if $\beta < b < \gamma$ in (B), then

$$\sup_{N \geq N_b,\, 0<\varepsilon\leq 1} \frac{1}{N} \log Z_{N,\varepsilon}(b) < \infty, \qquad \sup_{N \geq N_b} \frac{1}{N} \log Z_N(b) < \infty. \tag{5.23}$$

No homogeneous uniqueness or spectral stability is required.

Proof. Lemma 2.15 transfers the strict-subcritical Ricciardi–Suzuki bound to the empirical intensity laws, giving (B). The untilted Gaussian argument in Subsection 4.5 proves (5.23) and hence (P). Lemma 5.6 then gives lower semicontinuity, coercivity, and attainment on $\mathcal{X}$. The minimizing-pair correspondence and Lemma 2.11 identify every minimizing conditional density and give its smooth potential. These are the same variational and partition arguments used above; no second proof is needed here. □

5.3. **Microscopic collision control.** The argument here is a direct torus proof of the logarithmic quasi-continuity mechanism used in [16, equation (20), Lemma 3.3]. In particular, the topology and the off-diagonal convention are explicit, and no bounded-domain adaptation is left implicit.

Put

$$T_{N,\ell} = \frac{1}{N^2} \sum_{i<j} k_\ell(x_i, x_j). \tag{5.24}$$

Use the same grid as in Subsection 5.2 and write $n_Q = \#\{i : x_i \in Q\}$. Let $\mathcal{O}_{N,\ell,\alpha} = \{\max_Q n_Q \leq \alpha N\}$.

Lemma 5.8. *There is a universal C such that for every $t > 0$, $\alpha > 0$ with $20t\alpha < 1$, all sufficiently large N, and every $\ell = 1/m$ with $m \geq 4$,*

$$\mathbb{E}_{P_N}\left[e^{tNT_{N,\ell}} \mathbf{1}_{\mathcal{O}_{N,\ell,\alpha}}\right] \leq e^{Ct\alpha N}. \tag{5.25}$$

Consequently,

$$P_N(T_{N,\ell} > \delta, \mathcal{O}_{N,\ell,\alpha}) \leq \exp\{-N(t\delta - Ct\alpha)\}. \tag{5.26}$$

Proof. Condition on the square containing each labeled point. Under this conditioning the positions remain independent, each uniform in its prescribed square. Form a simple graph on the labels, joining two labels whenever their assigned squares are equal or neighbors. All nonzero k_ℓ terms are edges of this graph. On $\mathcal{O}_{N,\ell,\alpha}$, its maximum degree is at most $9\alpha N$ and its number of edges is at most $(9/2)\alpha N^2$.

A greedy edge coloring uses at most $2\Delta - 1$ colors for a graph of maximum degree $\Delta \geq 1$: when an edge is colored, at most $2\Delta - 2$ colors are forbidden by its incident edges. Hence the edge set is the union of $q \leq 18\alpha N + 1$ matchings. If there are no edges, the assertion is immediate. Let $\mathcal{M}_1, \dots, \mathcal{M}_q$ be those matchings. Hölder's inequality gives

$$\mathbb{E}\exp\left\{\frac{t}{N}\sum_e k_\ell(e)\right\} \leq \prod_{j=1}^{q}\left(\mathbb{E}\exp\left\{\frac{qt}{N}\sum_{e\in\mathcal{M}_j} k_\ell(e)\right\}\right)^{1/q},$$

where expectations are conditional on the square assignments. Edges within one matching use disjoint labels and are therefore independent. For large N, $qt/N \leq 1$ by the assumed margin, so (5.11) bounds the right side by

$$\exp\{Ct|\mathcal{E}_{\mathrm{graph}}|/N\} \leq e^{Ct\alpha N}.$$

This bound is uniform over all admissible assignments. Summing with their conditional probabilities proves (5.25), without a combinatorial factor for the number of assignments. Markov's inequality proves (5.26). □

Lemma 5.9. *Let d be any metric inducing weak convergence on $\mathcal{X}$. For each ρ with $\mathsf{H}_\nu(\rho) < \infty$ and each $\delta > 0$,*

$$\lim_{r\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log P_N\big(d(\widehat{\rho}_N,\rho) < r, |e_N - \mathfrak{e}(\rho)| > \delta\big) = -\infty. \tag{QC}$$

Proof. The spatial marginal $f\,dx$ of ρ is absolutely continuous. Given $\alpha > 0$, choose the grid fine enough that $\max_Q \int_Q f\,\mathrm{d}x < \alpha/2$. Every boundary has zero limiting mass. Therefore there is a weak neighborhood of ρ such that empirical measures in that neighborhood have $n_Q/N \leq \alpha$ for every square.

Let G_ℓ be the continuous truncated kernel (5.15). The corresponding discrete energy satisfies

$$e_{N,\ell} = \mathfrak{e}_\ell(\widehat{\rho}_N) - \frac{1}{2N^2}\sum_i \xi_i^2 G_\ell(0), \tag{5.27}$$

where $\mathfrak{e}_\ell(\sigma) = \frac{1}{2}\iint \xi\zeta G_\ell(x,y)\,d\sigma\,d\sigma$ is weakly continuous. The diagonal correction is $O_\ell(N^{-1})$. Moreover

$$|e_N - e_{N,\ell}| \leq \frac{M^2}{2\pi}T_{N,\ell}, \qquad \mathfrak{e}_\ell(\rho) \longrightarrow \mathfrak{e}(\rho)$$

by Lemma 5.3.

Fix a requested exponential rate $L < \infty$. Choose t large enough that $t\delta'$ exceeds $2L$, where $\delta' > 0$ is $\pi\delta/(2M^2)$. Then choose α small enough that $20t\alpha < 1$ and $Ct\alpha < L$. Take ℓ sufficiently small for the occupancy and continuum-tail requirements, and finally shrink the weak neighborhood so that the continuous truncated energies differ by less than $\delta/4$. Choose the continuum tail smaller than $\delta/4$ as well. For large N, (5.27) also makes the diagonal error less than $\delta/4$. The event in (QC) is then contained in $\{T_{N,\ell} > \delta', \mathcal{O}_{N,\ell,\alpha}\}$. Equation (5.26) bounds its limsup logarithmic probability by $-L$. Since L is arbitrary, (QC) follows. The signs of the marks were used only through $|\xi_i\xi_j| \leq M^2$. □

5.4. **Lifted free energies and pressure.** Let X be compact metric, $a_N \to \infty$, and $Q_N \in \mathcal{P}(X)$. For a lower semicontinuous $I : X \to [0,\infty]$, Mariani [17, Theorems 3.3–3.5] identifies the LDP with the entropy variational limits. In this compact case,

$$Q_N \text{ satisfies the speed-}a_N \text{ LDP with rate } I \iff a_N^{-1}\,\mathrm{Ent}(\,\cdot\,|\,Q_N) \xrightarrow{\Gamma} \left[\Theta \mapsto \int_X I\,d\Theta\right] \tag{5.28}$$

on $\mathcal{P}(X)$. The lower probability bound corresponds to entropy recovery (it is enough to recover Dirac laws, then mix); the weak upper probability bound corresponds to the entropy Γ-liminf. Compactness supplies exponential tightness and hence the full upper bound. We apply this with $X = \mathcal{X} = \mathcal{P}(\mathbb{T}^2 \times K)$, not with $X = \mathbb{T}^2 \times K$. The Γ-convergence is established below; (5.28) is used only to identify it with the LDP.

We prove Theorem 5.1 under (P), deriving the normalization rather than assuming it. Set

$$W_N(A) = \mathbb{E}_{P_N}\left[e^{N\beta e_N}\mathbf{1}_{\{\widehat{\rho}_N\in A\}}\right], \qquad W_N(\mathcal{X}) = Z_N(\beta), \tag{5.29}$$

and define

$$\mathscr{F}_N(\Theta) = \frac{1}{N}\operatorname{Ent}(\Theta \mid \Pi_{N,\beta}) - p_N(\beta). \tag{5.30}$$

This uses ordinary relative entropy with a probability reference and subtracts a finite constant. There is no ambiguous convention for entropy relative to an unnormalized measure. Jensen gives $Z_N(\beta)\geq 1$ since $\mathbb{E}_{P_N}e_N = 0$, while Hölder and (P) give $p_N(\beta)\leq \beta C_b/b$. Thus $\mathscr{F}_N$ is bounded below uniformly in N.

For $V\in C(E)$, independence gives

$$\Lambda_N(V) := \frac{1}{N}\log \mathbb{E}_{P_N} e^{N\langle V,\widehat{\rho}_N\rangle} = \int \log \int e^{V(x,\xi)}\,dx\,\nu_N(d\xi) \longrightarrow \Lambda_\nu(V). \tag{5.31}$$

The integrand is continuous in ξ. Its convex dual is

$$\sup_{V\in C(E)}\{\langle V,\rho\rangle - \Lambda_\nu(V)\} = \mathsf{H}_\nu(\rho). \tag{5.32}$$

For the correct mark marginal, the upper bound follows by applying the entropy inequality separately for each mark. The reverse follows from the ordinary joint entropy variational formula and $\Lambda_\nu(V)\leq \log\int e^V\,dx\,\nu$ by Jensen. For a wrong mark marginal, tests depending only on ξ and multiplication by arbitrary constants make the supremum infinite.

Exponential Chebyshev, followed by continuity of $\langle V,\cdot\rangle$ and the supremum over V, gives

$$\lim_{r\downarrow 0}\limsup_N \frac{1}{N}\log P_N(d(\widehat{\rho}_N,\rho)<r) \leq -\mathsf{H}_\nu(\rho). \tag{5.33}$$

No reference LDP is assumed here. Its lower bound could also be obtained from the product recovery construction, but only this local upper bound is needed for the following Γ-liminf.

If $\mathsf{H}_\nu(\rho)<\infty$, split a small ball into the good event $|e_N-\mathfrak{e}(\rho)|\leq\delta$ and its complement A_N. On the good part the weight is at most $e^{N\beta(\mathfrak{e}(\rho)+\delta)}$. On the bad part, Hölder gives

$$\mathbb{E}_{P_N}[e^{N\beta e_N}\mathbf{1}_{A_N}] \leq Z_N(b)^{\beta/b}P_N(A_N)^{1-\beta/b} \leq e^{C_b\beta N/b}P_N(A_N)^{1-\beta/b}. \tag{5.34}$$

By Lemma 5.9, its local rate is $-\infty$. Equations (5.33) and (5.34), followed by $\delta\downarrow 0$, give

$$\lim_{r\downarrow 0}\limsup_N \frac{1}{N}\log W_N(B_r(\rho)) \leq -F_{\beta,\nu}(\rho). \tag{5.35}$$

If $\mathsf{H}_\nu(\rho)=\infty$, apply Hölder to the whole ball and use (5.33); the resulting bound is $-\infty$. This also handles wrong mark marginals.

The Γ-liminf. Let $f\in C(\mathcal{X})$ satisfy $f\leq F_{\beta,\nu}$. For any $\delta>0$ and each center ρ of finite $F_{\beta,\nu}$, (5.35) and continuity of f provide a small neighborhood on which $f\leq f(\rho)+\delta$ and whose eventual logarithmic W_N mass is at most $-F_{\beta,\nu}(\rho)+\delta$. At a center of infinite F, use any finite rate exceeding $\|f\|_\infty+1$. Compactness of $\mathcal{X}$ gives a finite subcover. Summing the integrals over it yields

$$\limsup_N \frac{1}{N}\log\int e^{Nf}\,dW_N \leq 0, \tag{5.36}$$

after $\delta\downarrow 0$.

For $\Theta_N\Rightarrow\Theta$, the entropy variational inequality gives

$$\mathscr{F}_N(\Theta_N) \geq \int f\,d\Theta_N - \frac{1}{N}\log\int e^{Nf}\,dW_N.$$

Consequently $\liminf_N \mathscr{F}_N(\Theta_N)\geq\int f\,d\Theta$. By Lemma 5.6, $F_{\beta,\nu}$ is lower semicontinuous and bounded below, so on the compact metric space it is the increasing pointwise limit of continuous minorants. Monotone convergence after shifting by a constant proves

$$\liminf_N \mathscr{F}_N(\Theta_N) \geq \int F_{\beta,\nu}(\rho)\,\Theta(d\rho). \tag{5.37}$$

The Γ-limsup at Dirac laws and mixtures. Take ρ with finite $F_{\beta,\nu}$ and product recovery laws R_N from Lemma 5.5. At each N, their smooth densities make the microscopic logarithmic energy integrable. The Gibbs identity and contraction of relative entropy under $\widehat{\rho}_N$ imply

$$\begin{aligned}\mathscr{F}_N((\widehat{\rho}_N)_{\#}R_N) &\le \frac{1}{N}\operatorname{Ent}(R_N \mid \mu_{N,\beta}) - p_N(\beta)\\ &= \frac{1}{N}\operatorname{Ent}(R_N \mid P_N) - \beta\mathbb{E}_{R_N}e_N.\end{aligned} \tag{5.38}$$

The right side converges to $F_{\beta,\nu}(\rho)$, while the empirical law tends to δ_ρ. This proves recovery at every Dirac law of finite cost. At infinite cost, the upper-bound requirement is vacuous and any convergent sequence suffices.

For a finitely supported law $\Theta = \sum_j a_j\delta_{\rho_j}$ of finite cost, mix the corresponding empirical recovery laws with the same weights. Convexity of relative entropy gives the required upper bound $\sum_j a_j F_{\beta,\nu}(\rho_j)$. For arbitrary Θ with finite $\int F\,d\Theta$, partition the compact space into finitely many sets of diameter tending to zero, and in each positive-mass cell choose a finite-cost point with F no larger than the conditional average plus a vanishing error. The resulting finitely supported laws converge weakly to Θ and have limsup costs at most $\int F\,d\Theta$. A second diagonal choice over N yields

$$(\Gamma\text{-}\limsup_N \mathscr{F}_N)(\Theta) \le \int F_{\beta,\nu}\,d\Theta. \tag{5.39}$$

For infinite integral the assertion is again vacuous. Equations (5.37)–(5.39) prove

$$\mathscr{F}_N \xrightarrow{\Gamma} \mathscr{F}(\Theta) := \int F_{\beta,\nu}\,d\Theta. \tag{5.40}$$

Pressure and normalization. The minimum of $\mathscr{F}_N$ is $-p_N(\beta)$, attained at $\Pi_{N,\beta}$. Compactness of $\mathcal{P}(\mathcal{X})$ makes this family equicoercive. The lower bound along minimizers and a recovery sequence at a minimizer of $\mathscr{F}$ therefore give convergence of minima:

$$-p_N(\beta) \longrightarrow \min_\Theta \int F_{\beta,\nu}\,d\Theta = \min_\rho F_{\beta,\nu}(\rho) = m_{\beta,\nu}.$$

Adding the convergent constants $p_N(\beta)$ to (5.40) proves (5.6). Mariani's entropy characterization (5.28) gives the full LDP, because $\mathcal{X}$ is compact. The Laplace principle follows by the compact-space Laplace–Varadhan equivalence. This completes the proof of Theorem 5.1 without assuming the pressure limit in advance.

Why microscopic entropy alone is insufficient. Take all marks equal to one. Let C_N be the product of uniform laws on a fixed ball of radius e^{-N}, and set

$$Q_N = (1 - N^{-1})P_N + N^{-1}C_N.$$

Then $(\widehat{\rho}_N)_{\#}Q_N \Rightarrow \delta_{dx\otimes\delta_1}$, because the exceptional component has probability N^{-1}. Convexity gives

$$\operatorname{Ent}(Q_N \mid P_N) \le N^{-1}\operatorname{Ent}(C_N \mid P_N) = 2N + O(1).$$

But scaling the logarithmic kernel gives $\mathbb{E}_{C_N}G(x_i - x_j) = N/(2\pi) + O(1)$, and therefore

$$\mathbb{E}_{Q_N}e_N = \frac{1}{N}\frac{N(N-1)}{2N^2}\left(\frac{N}{2\pi} + O(1)\right) \longrightarrow \frac{1}{4\pi},$$

whereas the continuum energy of the weak limiting profile is zero. Thus a direct passage of the mean energy under arbitrary entropy-bounded configuration laws would be invalid. The local singular estimate and the strict margin $b > \beta$ in (5.34) avoid precisely this defect.

5.5. **Equilibrium mixtures and the balanced empirical law.** The LDP identifies the equilibrium set. Its distribution within that set requires more information whenever several phases coexist.

Corollary 5.10. *Let $\mathcal{M}_\beta = \operatorname{argmin} F_{\beta,\nu}$. For every open neighborhood U of $\mathcal{M}_\beta$,*

$$\limsup_N \frac{1}{N}\log\Pi_{N,\beta}(U^c) \le -\inf_{U^c} I_{\beta,\nu} < 0 \tag{5.41}$$

when $U^c \neq \varnothing$ and the infimum is finite, with the evident superexponential interpretation if it is infinite. Every subsequential weak limit Π of $\Pi_{N,\beta}$ is supported on $\mathcal{M}_\beta$.

For k uniformly sampled distinct labels, the corresponding marked marginal converges along that subsequence to

$$\int_{\mathcal{M}_\beta} \rho^{\otimes k}\,\Pi(d\rho). \tag{5.42}$$

If the minimizer is unique, this is ordinary marked propagation of chaos. In general it is a mixture of product laws, and the LDP does not determine the mixing weights. More precisely, fix $\rho_* \in \mathcal{M}_\beta$, let d metrize weak convergence, and let $\Phi \in C(E^k)$. If $Y_{N,1}, \dots, Y_{N,k}$ are the uniformly sampled distinct marked labels, then

$$\lim_{r\downarrow 0} \limsup_{N\to\infty} \left| \mathbb{E}_{\mu_{N,\beta}} \left[\Phi(Y_{N,1}, \dots, Y_{N,k}) \mid d(\widehat{\rho}_N, \rho_*) < r\right] - \int \Phi\, d\rho_*^{\otimes k} \right| = 0. \tag{5.43}$$

The conditioning events have positive probability for every sufficiently large N. No convergence for a fixed radius, and no unconditioned phase weight, is asserted.

Proof of Corollary 5.10. The good rate has zero set $\mathcal{M}_\beta$. On the compact closed complement of an open neighborhood of that set, lower semicontinuity gives a strictly positive minimum, unless the cost is infinite. The LDP upper bound proves (5.41). It also implies that every subsequential empirical-law limit is supported on $\mathcal{M}_\beta$.

Given a marked configuration, sampling k labels with replacement has conditional law $\widehat{\rho}_N^{\otimes k}$. The probability of a repeated label is at most $k(k-1)/(2N)$. The map $\rho \mapsto \rho^{\otimes k}$ is weakly continuous on the compact space, so convergence of the empirical laws gives (5.42). With a unique minimizer the mixing law is its Dirac mass. For a fixed ball around ρ_*, the LDP lower bound and $I(\rho_*) = 0$ show that its probability has zero exponential rate and is eventually positive. Conditional sampling differs from $\widehat{\rho}_N^{\otimes k}$ by at most $k(k-1)/(2N)$ in total variation. Consequently the absolute difference in (5.43) is bounded by

$$\sup_{d(\rho,\rho_*)<r} \left| \int \Phi\, d\rho^{\otimes k} - \int \Phi\, d\rho_*^{\otimes k} \right| + \frac{k(k-1)}{N} \|\Phi\|_\infty.$$

Continuity on the compact empirical-measure space proves the stated double-limit conclusion without a fixed-radius convergence assumption. □

Contraction to signed vorticity. The continuous map $T\rho = \int_K \xi\, \rho(\cdot, d\xi)$ gives a contracted LDP for the empirical signed vorticity, in the weak topology on signed measures with total variation at most M, with rate

$$I^{\mathrm{vort}}_{\beta,\nu}(w) = \inf\{I_{\beta,\nu}(\rho) : T\rho = w\}. \tag{5.44}$$

The marked LDP contains more information: cancellation in $T\rho$ can hide different distributions of positive and negative species. The LDP is an equilibrium result, not a theorem about dynamical transition times, and does not imply a CLT at a degenerate minimum.

The contraction statement (5.44) follows because $(x, \xi) \mapsto \xi\varphi(x)$ is continuous for every continuous spatial test φ. The image lies in a weakly compact set of signed measures with uniformly bounded total variation. Applying the LDP to preimages of closed and open sets, or the compact-space contraction principle, yields the displayed infimum rate.

The LDP adds a probabilistic interpretation of the threshold comparison but does not calculate the minimizing set. Throughout the subcollapse interval,

$$p(\beta) = \begin{cases} 0, & \beta \le \beta_H, \\ > 0, & \beta_H < \beta < \beta_{\mathrm{coll}}. \end{cases} \tag{5.45}$$

For $\rho_0 = dx\,\nu(d\xi)$,

$$I_{\beta,\nu}(\rho_0) = p(\beta). \tag{5.46}$$

Thus, between a genuine phase transition and homogeneous spectral instability, the homogeneous profile can have a strictly positive Hessian and still have strictly positive equilibrium large-deviation cost. At coexistence all minimizers have zero rate, but their relative probabilities can differ on subexponential scales. Linear-source logarithmic moment-generating functions alone may recover only a convex envelope; (5.9) retains the full, possibly nonconvex rate.

To prove (5.45), use the entropy–energy characterization of β_H: $F_{\beta,\nu} \ge 0$ for $\beta \le \beta_H$, and the homogeneous state has value zero. For $\beta > \beta_H$, the defining quotient supplies a state of negative free energy. Within the subcollapse interval pressure convergence therefore gives exactly (5.45). Since

$F_{\beta,\nu}(\rho_0) = 0$, its rate is $-m_{\beta,\nu} = p(\beta)$, proving (5.46). Convexity of p also follows directly from $p(\beta) = \sup_\rho\{\beta\mathfrak{e}(\rho) - \mathsf{H}_\nu(\rho)\}$.

Neither the LDP nor the entropy Γ-limit determines coexistence weights. For example, the two-atom laws $(1 - e^{-\sqrt{N}})\delta_{\rho_1} + e^{-\sqrt{N}}\delta_{\rho_2}$ have zero speed-N rate at both points, but converge weakly to δ_{ρ_1}. This simple example explains why zero exponential cost is weaker than a positive limiting phase weight.

For $N = 2n$, prescribe n signs $+1$ and n signs -1, write their positions as $y_1, \dots, y_n$ and $z_1, \dots, z_n$, and put

$$L_n^+ = \frac{1}{n}\sum_i \delta_{y_i}, \qquad L_n^- = \frac{1}{n}\sum_i \delta_{z_i}, \qquad \mathfrak{q}_N = \tfrac{1}{2}(L_n^+ - L_n^-), \qquad \omega_N^\pm = \sqrt{N}\mathfrak{q}_N.$$

Write $H_N^\pm = \sum_{i<j} \sigma_i\sigma_j G(X_i - X_j)$, $Z_N^\pm(\beta) = \int e^{\beta H_N^\pm/N}\, dX^N$, and $\nu_{N,\beta}^\pm$ for the corresponding Gibbs law.

Corollary 5.11. *For $0 < \beta < 16\pi$,*

$$\lim_{N=2n\to\infty} \frac{1}{N}\log Z_N^\pm(\beta) = p(\beta). \tag{5.47}$$

The empirical pair has a speed-N LDP with good rate

$$I_\beta(\rho_+, \rho_-) = \mathcal{A}_\beta(\rho_+, \rho_-) + p(\beta). \tag{5.48}$$

For every weak neighborhood $\mathcal{V}$ of $\mathcal{M}_\beta$, some $c_\mathcal{V} > 0$ satisfies

$$\nu_{N,\beta}^\pm((L_n^+, L_n^-) \notin \mathcal{V}) \le e^{-c_\mathcal{V} N} \tag{5.49}$$

for all sufficiently large even N.

Proof. The map $(\rho_+, \rho_-) \mapsto \frac{1}{2}\rho_+ \otimes \delta_1 + \frac{1}{2}\rho_- \otimes \delta_{-1}$ is a homeomorphism onto the marked measures with the prescribed balanced marginal. Under this map $F_{\beta,\nu} = \mathcal{A}_\beta$ and $\beta_{\mathrm{coll}} = 16\pi$. The assertions are exactly the specialization of Theorem 5.1 and its equilibrium-set concentration consequence. □

The elementary finiteness estimate is worth recording separately. If $G \ge -B$, then

$$Z_{2n}^\pm(\beta) \le e^{\beta B n/2} Z_n(\beta/2)^2 < \infty \quad (\beta < 16\pi).$$

For $n \ge 2$, collapse the positive particles at one scale while keeping the negative particles separated. The relative radial integral is proportional, up to nonsingular factors, to

$$\int_0^\varepsilon r^{2(n-1)-1-\beta(n-1)/(8\pi)}\, dr.$$

It diverges for $\beta \ge 16\pi$. This is the collapse threshold, not the symmetry-breaking transition.

Corollary 5.12. *For $0 < \beta \le 4\pi^2$, (L_n^+, L_n^-) converges in probability to (dx, dx). For $4\pi^2 < \beta < 16\pi$, it converges in distribution to*

$$(\rho_{+,\beta}(\cdot - a)dx, \rho_{-,\beta}(\cdot - a)dx), \qquad a \sim \mathrm{Haar}(\mathbb{T}^2),$$

where $\rho_{\pm,\beta} = e^{\pm U_{m_\beta}}/A_{m_\beta}$.

Proof. The large-deviation principle concentrates every subsequential law on the minimizing set. Below and at $4\pi^2$ that set is a singleton. Above $4\pi^2$ it is one free translation orbit by Theorem 3.10. Every subsequential law is invariant under translations, and the unique invariant probability on a compact transitive orbit is the pushforward of Haar probability. Compactness of the empirical-measure space therefore gives full-sequence convergence. □

For fixed numbers k, ℓ of observed positive and negative labels, the corresponding marginals converge to the mixture of $\rho_+^{\otimes k} \otimes \rho_-^{\otimes \ell}$ under this orbit law. Indeed, sampling distinct labels differs from independent sampling from the empirical measures by at most $O((k^2 + \ell^2)/n)$. In the supercritical interval the original field $\omega_N^\pm$ is not tight: the compact family of nonzero charges $\frac{1}{2}(\rho_{+,\beta}(\cdot - a) - \rho_{-,\beta}(\cdot - a))$ is uniformly separated from zero by finitely many smooth tests, and (5.49) makes the squared norm of the corresponding finite-dimensional projection of $\omega_N^\pm$ at least cN with probability tending to one exponentially.

6. Homogeneous statistics and the common Gaussian limit

6.1. **Partition asymptotics and Gaussian limits.** The partition bound is the only model-specific input in the following change of measure. The cluster argument provides it for one species, and Theorem 4.4 provides it for arbitrary bounded strengths under (H). We first state the consequences under (H), then give the common independent-reference argument. All finite-particle assertions concern sufficiently large N; an exceptional finite initial segment is allowed for a general triangular array of strengths.

Theorem 6.1. *Assume (2.3) and (H). At the chosen $0 < \beta < \gamma$,*

$$Z_N(\beta) \longrightarrow Z_\infty(\beta) := \det{}_2(I - \beta q_2 G)^{-1/2}, \qquad \frac{1}{N} \log Z_N(\beta) \longrightarrow 0. \tag{6.1}$$

If $\beta < b < \gamma$ and $p = b/\beta$, then

$$\sup_{N \geq N_b} \mathbb{E}_{P_N} \left(\frac{d\mu_{N,\beta}}{dP_N} \right)^p < \infty. \tag{6.2}$$

In particular $\mathrm{Ent}(\mu_{N,\beta} \mid P_N)$ is uniformly bounded, and its value divided by N tends to zero.

Theorem 6.2. *Under the same hypotheses, for every $s > 1$, under $\mu_{N,\beta}$,*

$$\eta_N \Rightarrow \eta_\beta \quad \text{in } H_0^{-s}(\mathbb{T}^2), \qquad \eta_\beta \sim \mathcal{N}(0, C_\beta), \qquad C_\beta = q_2(I - \beta q_2 G)^{-1}. \tag{6.3}$$

The covariance acts on mean-zero spatial test functions. Explicitly,

$$\mathbb{E}\langle \eta_\beta, \varphi \rangle \langle \eta_\beta, \psi \rangle = q_2 \langle \varphi, (I - \beta q_2 G)^{-1} \psi \rangle. \tag{6.4}$$

No convergence rate for $\nu_N \Rightarrow \nu$ is required with the centering in (2.6). If $q_2 = 1$, the covariance is $(I - \beta G)^{-1}$.

Interpretation. The mean vorticity may be nonzero, but it is spatially constant in the homogeneous phase. Signed cancellations alone would not establish species-wise chaos; this is why Theorem 6.5 treats the full marked empirical measure. The auxiliary Gaussian field used in the partition representation has covariance asymptotic to G, whereas the limiting vorticity has covariance asymptotic to the identity. These two Gaussian fields must not be confused.

Lemma 6.3. *The following two independent-reference limits hold, jointly with every fixed finite collection of empirical test-function coordinates.*

Weighted homogeneous reference. *Under $P_N = dx^{\otimes N}$ and (2.3), for every $s > 1$,*

$$(\eta_N, E_N) \Rightarrow \left(\sqrt{q_2} W_0, \frac{q_2}{2} \sum_{j=1}^{\infty} g_j (Z_j^2 - 1) \right) \quad \text{in } H_0^{-s}(\mathbb{T}^2) \times \mathbb{R}. \tag{6.5}$$

Here (e_j) is a real orthonormal Laplacian eigenbasis of L_0^2, $g_j = \lambda_j^{-1}$, the Z_j are independent standard real normal variables, and $W_0 = \sum_j Z_j e_j$. The energy series converges in L^2 and is denoted by $\frac{q_2}{2} :\langle W_0, G W_0 \rangle:$.

Fixed-species reference. *Let Σ be a fixed finite set of types, let $n_{N,\sigma}/N \to p_\sigma > 0$ with $\sum_\sigma p_\sigma = 1$, and prescribe exactly $n_{N,\sigma}$ independent positions with law $\mu_\sigma(dx) = \rho_\sigma(x)dx$, where ρ_σ is smooth and strictly positive. Set*

$$\mathcal{H} = \bigoplus_{\sigma \in \Sigma} L_0^2(\mu_\sigma), \qquad \langle f, g \rangle_{\mathcal{H}} = \sum_\sigma p_\sigma \int f_\sigma g_\sigma \, d\mu_\sigma.$$

Let $K_{\sigma\tau}(x, y) = K_{\tau\sigma}(y, x)$ be square-integrable and separately canonical, that is, $\int K_{\sigma\tau}(x, y) \mu_\tau(dy) = 0$ for every type pair and almost every x. Its self-adjoint Hilbert–Schmidt operator is

$$(T_K f)_\sigma(x) = \sum_\tau p_\tau \int K_{\sigma\tau}(x, y) f_\tau(y) \, \mu_\tau(dy).$$

Define the species fluctuation measures by their exact finite populations,

$$\zeta_N^\sigma = \frac{1}{\sqrt{N}} \left(\sum_{\sigma_i = \sigma} \delta_{X_i} - n_{N,\sigma} \mu_\sigma \right).$$

Thus each component has zero mass. Centering by $Np_\sigma\mu_\sigma$ would require an additional rate for $n_{N,\sigma}/N \to p_\sigma$ and is not used. For separately centered test functions put

$$W_N(f) = \sum_\sigma \zeta_N^\sigma(f_\sigma) = \frac{1}{\sqrt{N}} \sum_i f_{\sigma_i}(X_i), \qquad U_N(K) = \frac{1}{N} \sum_{i<j} K_{\sigma_i\sigma_j}(X_i, X_j). \tag{6.6}$$

Then

$$(W_N, U_N(K)) \Longrightarrow \left(W, \frac{1}{2} I_2(T_K)\right), \tag{6.7}$$

where W is isonormal over $\mathcal{H}$. The first component is interpreted as the species empirical fluctuation distributions in $H^{-s}(\mathbb{T}^2)^\Sigma$. For a self-adjoint Hilbert–Schmidt operator B on the reference Hilbert space, $I_2(B) = \sum_j b_j(W(v_j)^2 - 1)$ in a real orthonormal eigenbasis; this series converges in L^2.

Proof. Both assertions follow from the same finite-rank identity and canonical-kernel approximation. First consider a fixed finite set of smooth centered test functions. In the homogeneous case their coordinates are weighted sums

$$S_{N,j} = \langle \eta_N, e_j \rangle = \frac{1}{\sqrt{N}} \sum_i \xi_i e_j(x_i),$$

whose independent summands are bounded by a constant times $N^{-1/2}$, with zero means and covariance $q_N I \to q_2 I$. In the fixed-species case the same triangular-array central limit theorem applies to $N^{-1/2} f_{\sigma_i}(X_i)$, with limiting covariance $\sum_\sigma p_\sigma \int f_\sigma g_\sigma \, d\mu_\sigma$. Smooth separately centered functions are dense in $\mathcal{H}$; approximation in L^2 therefore includes any fixed collection of $\mathcal{H}$ coordinates.

The distribution-valued empirical fields are tight. For the weighted homogeneous field the exact bound is

$$\mathbb{E}_{P_N} \sum_{j>m} (1+\lambda_j)^{-s} |S_{N,j}|^2 = q_N \sum_{j>m} (1+\lambda_j)^{-s} \longrightarrow 0$$

uniformly in N, since $s > 1$. For a species fluctuation $N^{-1/2} \sum_{\sigma_i=\sigma} (\delta_{X_i} - \mu_\sigma)$, its Fourier-coordinate variance is at most $C n_{N,\sigma}/N$, since the reference density is bounded. The corresponding H^{-s} tail is therefore bounded by the same summable Fourier tail, uniformly in N. The finite collection of species causes no further tightness issue. These estimates combine with finite-dimensional convergence to give the asserted Hilbert-space convergence.

For the finite-rank energy step, choose symmetric approximants whose ranges are spanned by finitely many smooth separately centered functions. To do so, take increasing finite-dimensional subspaces of this dense class and compress the Hilbert–Schmidt operator to them. These compressions preserve separate centering and converge in Hilbert–Schmidt norm. Each approximant can then be diagonalized within its smooth finite-dimensional range. For any such representation $K_{ij}^{(m)}(x,y) = \sum_{r \le m} \theta_r a_r(i,x) a_r(j,y)$, the off-diagonal convention gives the exact identity

$$\frac{1}{N} \sum_{i<j} K_{ij}^{(m)}(X_i, X_j) = \frac{1}{2} \sum_{r \le m} \theta_r \left[\left(\frac{1}{\sqrt{N}} \sum_i a_r(i, X_i) \right)^2 - \frac{1}{N} \sum_i a_r(i, X_i)^2 \right]. \tag{6.8}$$

The preceding central limit theorem treats the first term; the law of large numbers treats the diagonal correction, jointly with any fixed collection of other empirical coordinates. In particular, for the spatial spectral truncation $G^{(m)}(x,y) = \sum_{j \le m} g_j e_j(x) e_j(y)$,

$$E_N^{(m)} = \frac{1}{2} \sum_{j \le m} g_j \left[S_{N,j}^2 - \frac{1}{N} \sum_i \xi_i^2 e_j(x_i)^2 \right]. \tag{6.9}$$

The subtracted average has mean q_N and variance $O(N^{-1})$, so its limit is q_2. In the fixed-species case an orthonormal finite-rank representation in $\mathcal{H}$ has limiting diagonal averages one. This yields the corresponding finite-rank limits in both assertions.

Finally, if L_{ij} is any separately canonical residual kernel, independence shows that distinct unordered edges have zero covariance, even when they share one vertex. Thus

$$\mathbb{E} \left| \frac{1}{N} \sum_{i<j} L_{ij}(X_i, X_j) \right|^2 = \frac{1}{N^2} \sum_{i<j} \int |L_{ij}|^2 \, d\mu_i \, d\mu_j. \tag{6.10}$$

For weighted homogeneous kernels this specializes to

$$\mathbb{E}_{P_N}\left|\frac{1}{N}\sum_{i<j}\xi_i\xi_j L(x_i,x_j)\right|^2 = \frac{1}{N^2}\sum_{i<j}\xi_i^2\xi_j^2\,\|L\|_2^2 \le \frac{M^4}{2}\,\|L\|_2^2\,. \tag{6.11}$$

For the fixed-species reference it is bounded by a constant times $\sum_{\sigma,\tau} p_\sigma p_\tau \,\|L_{\sigma\tau}\|^2_{L^2(\mu_\sigma\otimes\mu_\tau)}$, since the finite set of ratios $n_{N,\sigma}/(Np_\sigma)$ is bounded for large N. The corresponding limiting chaos tail has variance $\frac{1}{2}\,\|T_L\|^2_{\mathrm{HS}}$; in the homogeneous case it is $q_2^2\sum_{j>m} g_j^2/2$. Square-integrable kernels can be approximated by symmetric finite-rank kernels made from smooth separately centered functions. In the homogeneous case take $L = G - G^{(m)}$. All these residual bounds vanish with the approximation, proving both joint limits by the usual bounded-Lipschitz approximation argument. □

Lemma 6.4. *Let W be isonormal over a real separable Hilbert space $\mathcal{K}$, and let B be a positive self-adjoint Hilbert–Schmidt operator with $\|B\| < 1$. Then*

$$\mathbb{E}e^{I_2(B)/2} = \det{}_2(I-B)^{-1/2}. \tag{6.12}$$

Under this normalized weight the Gaussian coordinates have covariance $(I-B)^{-1}$: for $f,g\in\mathcal{K}$ their covariance is $\langle f,(I-B)^{-1}g\rangle_{\mathcal{K}}$. In particular, if $0<\beta<\lambda_1/q_2$, the weight

$$\exp\left\{\frac{\beta q_2}{2}\sum_j g_j(Z_j^2-1)\right\}$$

has mean $\det_2(I-\beta q_2 G)^{-1/2}$, and the covariance of $\sqrt{q_2}W_0$ under it is $q_2(I-\beta q_2 G)^{-1}$.

Proof. For the first m eigen-coordinates of B, with eigenvalues b_j, the expectation factors as

$$\prod_{j\le m} e^{-b_j/2}(1-b_j)^{-1/2}.$$

Its logarithm converges absolutely because $\sum_j b_j^2<\infty$ and $\sup_j b_j<1$. Choose $a>1$ with $a\,\|B\|<1$; the same product estimate for aB supplies a uniform L^a bound for the finite-rank weights. Their exponents converge in L^2, so uniform integrability proves (6.12). For any fixed number of eigen-coordinates, independence integrates out the tail factor, and completing the square changes the variance of each coordinate from one to $(1-b_j)^{-1}$. The finite-dimensional marginals determine the Gaussian law and, by L^2 approximation, the covariance identity holds for all $f,g\in\mathcal{K}$. The homogeneous specialization follows by taking $B=\beta q_2 G$; the variances of $\sqrt{q_2}Z_j$ become $q_2/(1-\beta q_2 g_j)$. They are uniformly bounded, so this field belongs to H_0^{-s} for every $s>1$. □

Common proof of Theorems 6.1 and 6.2. Choose $\beta<b<\gamma$. By Theorem 4.4,

$$\sup_{N\ge N_b}\mathbb{E}_{P_N}(e^{\beta E_N})^{b/\beta} = \sup_{N\ge N_b} Z_N(b) < \infty.$$

Thus the Gibbs weights at β are uniformly integrable. The same argument applies whenever the bound is supplied by the cluster expansion instead of Theorem 4.4. For bounded continuous F on $H_0^{-s}(\mathbb{T}^2)$, truncate the exponential as a function of the energy and use Lemma 6.3; the displayed moment bound then removes the truncation. This yields

$$\mathbb{E}_{P_N}[F(\eta_N)e^{\beta E_N}] \longrightarrow \mathbb{E}\left[F(\sqrt{q_2}W_0)e^{\frac{\beta q_2}{2}:\langle W_0,GW_0\rangle:}\right].$$

Taking $F=1$ and applying Lemma 6.4 proves (6.1). Dividing the same identity by the converging normalizations proves (6.3) and (6.4). This proves convergence as probability measures on H_0^{-s}, and not only finite-dimensional convergence. No source-dependent Boué–Dupuis estimate is required. The determinant limit is positive and finite, so the logarithm of the partition function is $O(1)$ and its value divided by N tends to zero.

Since $\mathbb{E}_{P_N}E_N=0$, Jensen gives $Z_N(\beta)\ge 1$. For $p=b/\beta>1$,

$$\mathbb{E}_{P_N}\left(\frac{d\mu_{N,\beta}}{dP_N}\right)^p = \frac{Z_N(b)}{Z_N(\beta)^p} \le C_b, \tag{6.13}$$

which proves (6.2). Finally, for a probability density $f=d\mu/dP$, Jensen's inequality under μ gives

$$\mathrm{Ent}(\mu\mid P) = \mathbb{E}_\mu \log f \le \frac{1}{p-1}\log \mathbb{E}_\mu f^{p-1} = \frac{1}{p-1}\log \mathbb{E}_P f^p.$$

This proves the uniform relative-entropy bound and the vanishing specific entropy. □

6.2. **Propagation of chaos and species specializations.**

Theorem 6.5. *Assume (2.3) and (H), as in Theorem 6.1. Under $\mu_{N,\beta}$,*

$$\widehat{\rho}_N \longrightarrow dx\,\nu(d\xi) \quad \text{in probability for the weak topology.} \tag{6.14}$$

If k distinct labels are sampled uniformly, independently of the configuration, their marked positions converge weakly to $(dx\,\nu(d\xi))^{\otimes k}$. In addition, for any deterministic sequence of k distinct labels $i_1(N),\dots,i_k(N)$,

$$\big\|\mathrm{Law}_{\mu_{N,\beta}}(x_{i_1(N)},\dots,x_{i_k(N)}) - dx^{\otimes k}\big\|_{\mathrm{TV}} \longrightarrow 0. \tag{6.15}$$

Fixed-label intensities remain deterministic; they are not independent samples from ν. No total-variation convergence of empirical intensity laws is assumed.

Proof of Theorem 6.5: empirical measures and random labels. Let $p = b/\beta > 1$ as in (6.13). For any event A_N, Hölder gives

$$\mu_{N,\beta}(A_N) \le C_b^{1/p} P_N(A_N)^{1-1/p}. \tag{6.16}$$

For a continuous test function Φ on the compact marked space $\mathbb{T}^2 \times [-M, M]$,

$$\mathbb{E}_{P_N}\langle \widehat{\rho}_N, \Phi\rangle = \int \Phi(x,\xi)\,\mathrm{d}x\,\nu_N(d\xi) \longrightarrow \int \Phi(x,\xi)\,\mathrm{d}x\,\nu(d\xi),$$

and the variance is at most $\|\Phi\|_\infty^2/N$. A countable uniformly dense family of tests proves convergence in probability of the empirical measure under P_N; (6.16) transfers it to $\mu_{N,\beta}$.

Conditionally on the marked configuration, independent uniform sampling of k labels with replacement has law $\widehat{\rho}_N^{\otimes k}$. The collision probability for the sampled labels is at most $k(k-1)/(2N)$. Thus sampling without replacement has the same weak limit, namely $(dx\,\nu(d\xi))^{\otimes k}$. This is a statement about random labels, and does not use exchangeability of the full quenched ensemble. □

Proof of Theorem 6.5: fixed tagged positions. Let I_N be the set of k tagged labels, and define the bulk energy by deleting all pairs meeting I_N while retaining the denominator N:

$$E_N^{\mathrm{bulk}} = \frac{1}{N} \sum_{\substack{i<j \\ i,j\notin I_N}} \xi_i\xi_j G(x_i - x_j).$$

The degeneracy calculation in (6.11) gives

$$\mathbb{E}_{P_N}|E_N - E_N^{\mathrm{bulk}}|^2 \le C\frac{k}{N}. \tag{6.17}$$

Both energies are also uniformly tight, since their second moments are uniformly bounded.

Conditionally on the untagged positions, integration in the tagged uniform positions gives $\mathbb{E}[E_N \mid (x_i)_{i\notin I_N}] = E_N^{\mathrm{bulk}}$. Conditional Jensen therefore yields

$$\mathbb{E}_{P_N} e^{bE_N^{\mathrm{bulk}}} \le \mathbb{E}_{P_N} e^{bE_N} \le C_b. \tag{6.18}$$

Equations (6.17), tightness, and continuity of the exponential show that $e^{\beta E_N} - e^{\beta E_N^{\mathrm{bulk}}} \to 0$ in probability. The bounds at $b > \beta$ imply uniform integrability of both weights, so

$$\mathbb{E}_{P_N}|e^{\beta E_N} - e^{\beta E_N^{\mathrm{bulk}}}| \longrightarrow 0. \tag{6.19}$$

Let $Z_N^{\mathrm{bulk}} = \mathbb{E}e^{\beta E_N^{\mathrm{bulk}}}$. Both normalizations are at least one by Jensen. Elementary algebra gives

$$\mathbb{E}_{P_N}\left|\frac{e^{\beta E_N}}{Z_N(\beta)} - \frac{e^{\beta E_N^{\mathrm{bulk}}}}{Z_N^{\mathrm{bulk}}}\right| \le 2\mathbb{E}_{P_N}|e^{\beta E_N} - e^{\beta E_N^{\mathrm{bulk}}}|.$$

The normalized bulk measure has tagged marginal exactly $dx^{\otimes k}$. Total variation decreases under taking marginals, so (6.15) follows. The estimates depend on k, M, b but not on the chosen labels. □

The one-species fluctuation corollary.

Theorem 6.6. *For every $0 \le \beta < 8\pi$, $\sup_{N\ge 2} Z_N(\beta) < \infty$, $\quad Z_N(\beta) \longrightarrow \det_2(I - \beta G)^{-1/2}$. Moreover the centered empirical vorticity under the Gibbs measure converges weakly in $H^{-s}(\mathbb{T}^2)$, $s > 1$, to the energy–enstrophy Gaussian measure with Radon–Nikodym density $\det_2(I-\beta G)^{1/2} e^{\beta Y}$ with respect to zero-average white noise.*

Proof. The scalar inequality gives (H) for $\nu = \delta_1$ at every $\beta < 8\pi$. Either the splitting-and-cluster proof or the Boué–Dupuis proof supplies the uniform partition bound. The finitely many initial partition functions are finite by (EI); at $\beta = 0$ the assertions are the independent product limits. The shared Gaussian change of measure gives the stated determinant and covariance $(I - \beta G)^{-1}$, without a second limit argument. □

The sharp balanced inequality makes the general homogeneous theorem explicit up to the transition. At $\beta = 0$ the assertions are the independent product limits.

Theorem 6.7. *For every $0 \le \beta < 4\pi^2$,*

$$Z_N^{\pm}(\beta) \longrightarrow \det{}_2(I - \beta G)^{-1/2}. \tag{6.20}$$

Under the neutral Gibbs law, the signed field $\omega_N^{\pm}$ converges in $H^{-s}(\mathbb{T}^2)$, $s > 1$, to μ_β. The total-density field

$$\zeta_N = \frac{1}{\sqrt{N}} \left(\sum_{i=1}^{N} \delta_{X_i} - N dx \right)$$

converges jointly to independent zero-mean white noise.

Proof. For the balanced law $q_2 = 1$ and the sharp inequality gives (H) with any $\beta < \gamma < 4\pi^2$. Apply the common partition and fluctuation theorem. For the joint statement use the finite-species reference limit of Lemma 6.3: the charge and total-density sectors are orthogonal. The interaction operator is G on the charge sector and zero on the total-density sector. The Gaussian change of measure of Lemma 6.4 therefore tilts only the charge field, leaving independent white noise in the total-density sector. □

7. Balanced phase-centered fluctuations and critical information

7.1. **The phase-centered limit and its Gaussian covariance.** Fix $4\pi^2 < \beta < 16\pi$. In this section write

$$u = U_{m_\beta}, \qquad A = \int e^u = \int e^{-u}, \qquad \rho_\pm = \frac{e^{\pm u}}{A}, \qquad q = \frac{1}{2}(\rho_+ - \rho_-).$$

Then $u = \beta G * q$ and

$$p(\beta) = \log A - \frac{1}{2\beta} \int |\nabla u|^2 = \log A - \frac{\beta}{2} Q(q) > 0. \tag{7.1}$$

The neutral Gibbs law is denoted by $\nu_{N,\beta}^{\pm}$.

Let

$$\mathcal{H} = L_0^2(\rho_+ dx) \oplus L_0^2(\rho_- dx), \qquad \langle f, g \rangle_{\mathcal{H}} = \tfrac{1}{2} \sum_{\sigma = \pm 1} \int f_\sigma g_\sigma \rho_\sigma.$$

Thus each component has zero mean under its own species density. Put

$$r_\sigma = G * \rho_\sigma, \qquad b_{\sigma\tau} = \langle \rho_\sigma, G * \rho_\tau \rangle, \qquad B = \frac{\beta}{4}(b_{++} + b_{--}).$$

The separately canonical kernel and its operator on $\mathcal{H}$ are

$$k_{\sigma\tau}(x, y) = \sigma\tau \, [G(x - y) - r_\tau(x) - r_\sigma(y) + b_{\sigma\tau}], \tag{7.2}$$

$$(Tf)_\sigma(x) = \frac{1}{2} \sum_\tau \int k_{\sigma\tau}(x, y) f_\tau(y) \rho_\tau(y) dy. \tag{7.3}$$

The operator T is self-adjoint, positive, and Hilbert–Schmidt. Indeed, if $g_f = \frac{1}{2}(\rho_+ f_+ - \rho_- f_-)$, then

$$\langle f, Tf \rangle_{\mathcal{H}} = Q(g_f) \ge 0,$$

and its kernel is square integrable because $G \in L^2$ and the densities are bounded.

Define the two translation scores, their Fisher matrix, and the orthogonal projection onto their span:

$$(t_j)_\sigma(x) = -\sigma \partial_j u(x), \qquad j = 1, 2, \tag{7.4}$$

$$J_{ij} = \langle t_i, t_j \rangle_{\mathcal{H}} = \tfrac{1}{2} \int (\rho_+ + \rho_-) \partial_i u \, \partial_j u, \tag{7.5}$$

$$Pf = \sum_{i,j=1}^{2} t_i (J^{-1})_{ij} \langle t_j, f \rangle_{\mathcal{H}}, \qquad \Pi = I - P. \tag{7.6}$$

The scores belong to $\mathcal{H}$, J is positive definite, and $Tt_j = \beta^{-1} t_j$. On the square J is a scalar multiple of the identity, but no such simplification is needed below. Proposition 3.12 and its density-space counterpart below give

$$\ker(I - \beta T) = \operatorname{Ran} P, \qquad \langle f, (I - \beta T) f \rangle_{\mathcal{H}} \geq \gamma_\beta \, \|f\|_{\mathcal{H}}^2 \quad (f \in \Pi \mathcal{H}) \tag{7.7}$$

for a positive γ_β. Let $S = \beta T - P$. Then S is positive and Hilbert–Schmidt, $S = 0$ on $P\mathcal{H}$, and $\|S\| < 1$. Define

$$D_\beta = \det{}_2(I - S)^{-1/2} = \prod_{\lambda_j(T) \neq \beta^{-1}} [(1 - \beta \lambda_j(T)) e^{\beta \lambda_j(T)}]^{-1/2}.$$

The two translation eigenvalues, and only those two, are omitted from the product.

For an empirical pair, define its phase, on a fixed small neighborhood of the minimizing orbit, by the unique maximum-likelihood estimator

$$a_N = \arg\max_{a \in \mathbb{T}^2} \left\{ \tfrac{1}{2} \int \log \rho_+(x - a) \, dL_n^+(x) + \tfrac{1}{2} \int \log \rho_-(x - a) \, dL_n^-(x) \right\}.$$

Existence and uniqueness on that neighborhood are proved in Subsection 7.3. Define a_N arbitrarily outside it, an event of exponentially small Gibbs probability. The phase-centered fluctuation measures are

$$\eta_N^\sigma = \frac{1}{\sqrt{N}} \left(\sum_{\sigma_i = \sigma} \delta_{X_i - a_N} - \frac{N}{2} \rho_\sigma \, dx \right).$$

For a test pair f, let $f_\sigma^0 = f_\sigma - \int f_\sigma \rho_\sigma$.

Theorem 7.1. *For every fixed $4\pi^2 < \beta < 16\pi$ and every $s > 1$,*

$$(a_N, \eta_N^+, \eta_N^-) \implies (a, \eta_\beta^+, \eta_\beta^-) \quad \text{in } \mathbb{T}^2 \times H^{-s}(\mathbb{T}^2)^2, \tag{7.8}$$

where a is Haar-uniform, independent of the centered Gaussian pair η_β. Its covariance is

$$\operatorname{Cov}(\eta_\beta(f), \eta_\beta(g)) = \langle \Pi f^0, (I - \beta T)^{-1}_{\Pi \mathcal{H}} \Pi g^0 \rangle_{\mathcal{H}}. \tag{7.9}$$

The signed fluctuation $\eta_\beta^+ - \eta_\beta^-$ is supported on $\langle \eta_\beta^+ - \eta_\beta^-, \partial_j u \rangle = 0$, $j = 1, 2$. The partition function satisfies

$$Z_N^\pm(\beta) = \frac{\sqrt{\det J}}{2\pi} \, e^{-B-1} D_\beta \, N e^{N p(\beta)} (1 + o(1)). \tag{7.10}$$

No uniformity as $\beta \downarrow 4\pi^2$ or $\beta \uparrow 16\pi$ is claimed.

Proposition 7.2. *For the operator and score projection in (7.3) and (7.6), the kernel and gap assertions (7.7) hold.*

Proof. For the density Hessian, perturb ρ_σ by $\rho_\sigma(1 + \varepsilon f_\sigma)$ with bounded $f \in \mathcal{H}$. Its second variation is $\|f\|_{\mathcal{H}}^2 - \beta Q(g_f) = \langle f, (I - \beta T) f \rangle_{\mathcal{H}}$. Global minimality makes this form nonnegative, first on bounded functions and then on $\mathcal{H}$ by density. If $f = \beta T f$, let $h_0 = \beta G * g_f$, which has mean zero. Then $f_\sigma = \sigma(h_0 - \int \rho_\sigma h_0)$, and the equation for h_0 is the stream Hessian equation. Elliptic regularity makes it smooth. Thus f is a linear combination of the scores. Conversely, differentiating the mean-field equation gives $Tt_j = \beta^{-1} t_j$. Finally compactness of T makes the rest of its spectrum strictly smaller than β^{-1}, proving the positive gap on $\Pi \mathcal{H}$.

□

7.2. **Localized exponential integrability.** Let $P_N = \rho_+^{\otimes n} \otimes \rho_-^{\otimes n}$. For any convolution kernel R, use superscripts to denote its separately canonical version k^R as in (7.2), and put

$$Y_N(R) = \frac{1}{N} \sum_{i<j} k^R_{\sigma_i\sigma_j}(X_i, X_j).$$

The unrestricted absolute multigraph expansion is not sufficient at the present temperatures: it removes the opposite-sign cancellation and has a smaller global integrability range. We instead localize spatial occupancies before taking absolute values.

For $0 < a < 1$, first let $C_a^{\mathrm{heat}} = e^{a\Delta}G$. Choose a real finite Fourier truncation C_a of C_a^{heat} such that $0 \le \widehat{C}_a(k) \le \widehat{G}(k)$, $\|C_a - C_a^{\mathrm{heat}}\|_\infty \le a$, and put $R_a = G - C_a$. This is possible by absolute convergence of the heat-smoothed Fourier series. Uniformly as $a \downarrow 0$,

$$\|R_a\|_1 \le 3a, \qquad \|R_a\|_2 \le C\sqrt{a}, \qquad \|R_a\|_r \le C_r a^{1/r} \quad (r > 2), \tag{7.11}$$

$$|R_a(x)| \le G_+(x) + C, \qquad R_a(x) \ge -2a. \tag{7.12}$$

Indeed $G - e^{a\Delta}G = \int_0^a (p_t - 1)dt$, giving the L^1 bound and the lower bound. Since $e^{a\Delta}G \ge \min G$, its upper bound has coefficient one in front of G_+. The Fourier coefficients are bounded by $C\min(a, |k|^{-2})$; Parseval and Hausdorff–Young give the remaining estimates. The finite Fourier truncation error does not change their orders.

Lemma 7.3. *Fix $T_0 < \infty$. There is a partition into sufficiently small square cells Q_c of side h, and a constant M_0, such that the event*

$$E_{N,h} = \big\{\#\{i : X_i \in Q_c\} \le M_0 N h^2 \text{ for every physical cell } c\big\} \tag{7.13}$$

contains a fixed weak neighborhood of $(\rho_+ dx, \rho_- dx)$, and

$$\sup_{\substack{N\ge 2\\ N \text{ even}}} \sup_{|t|\le T_0} \mathbb{E}_{P_N}\left[\mathbf{1}_{E_{N,h}} e^{tY_N(R_a)}\right] \le 1 + Ca^{1/4} \tag{7.14}$$

for sufficiently small $a > 0$. Constants may depend on h, T_0, ρ_+, ρ_-, but not on N, a.

Proof. Take $h = 1/L$, with L a sufficiently large multiple of three. A class c in the proof records both a species and a physical cell. Its reference probability is $p_c = \frac{1}{2}\int_{Q_c} \rho_{\sigma(c)}$, and its conditional law is $\mu_c(dx) = \mathbf{1}_{Q_c}(x)\rho_{\sigma(c)}(x)dx / \int_{Q_c} \rho_{\sigma(c)}$. All these conditional densities are bounded by a finite M_h. Conditional on class counts n_c and an assignment of labels to cells, the variables are independent with these conditional laws. Symmetry within species makes the choice of assignment immaterial. Write $D_c = n_c - Np_c$.

Exponential stability with a local count cap. For any selected set of v labels whose class counts obey the cap in (7.13), and any pair kernels satisfying $|\ell_{ij}(x,y)| \le G_+(x-y) + C_h$, we claim

$$\int \exp\left(\frac{q_0|t|}{N}\sum_{i<j}|\ell_{ij}|\right)\prod_{i=1}^{v}\mu_{c_i}(dx_i) \le C_h^v, \qquad |t| \le 2T_0, \tag{7.15}$$

where $q_0 = 4/3$ and C_h can also depend on T_0. Pairs at distance at least h contribute at most $C_h v^2/N \le C_h v$. A closer pair belongs to a block consisting of a 3×3 group of cells. Color the blocks by their center coordinates modulo three. There are nine colors, and blocks of the same color are disjoint on the torus. Overcounting nonnegative short-range interactions by all such blocks and applying Hölder over the nine colors reduces the claim to blocks containing $v_B \le 9M_0Nh^2$ selected labels.

In one block the coefficient in front of the pair sum is $9q_0|t|/N$. Edge-color the complete graph on its v_B labels into at most v_B matchings, and apply Hölder once more. Each matching consists of disjoint pairs, whose two-variable integrals are bounded by $M_h^2 \iint e^{b_0 G_+(x-y)}dxdy < \infty$ provided $b_0 < 4\pi$. Choose h so small that $81q_0(2T_0)M_0h^2 < b_0 < 4\pi$; then every edge exponent after the matching Hölder bound is at most b_0. The resulting block bound is $C_h^{v_B}$, and the block-color bound is C_h^v. This proves (7.15). The only singularity used here is $G(x) = (2\pi)^{-1}\log(1/|x|) + O(1)$.

Recentering within cells. Let $K = k^{R_a}$ on the tagged space and set, for x in class c,

$$r_d(x) = \int K(x,y)\mu_d(dy), \quad k_{cd} = \iint K(x,y)\mu_c(dx)\mu_d(dy), \quad h_{cd}(x) = r_d(x) - k_{cd},$$

$\ell_{cd}(x,y) = K(x,y) - r_d(x) - r_c(y) + k_{cd}$. The kernel ℓ_{cd} is canonical under each of its conditional measures. The original kernel is canonical under the reference mixture, so

$$\sum_d p_d r_d(x) = 0, \qquad \sum_d p_d k_{cd} = 0, \qquad \sum_d p_d h_{cd}(x) = 0. \tag{7.16}$$

Because the reference and conditional densities are bounded, (7.11) gives

$$\sup_{c,d} \big(\|r_d\|_\infty + |k_{cd}| + \|h_{cd}\|_\infty \big) \le C_h a. \tag{7.17}$$

It also gives the uniform row and pair estimates

$$\sup_{c,d,x} \|\ell_{cd}(x,\cdot)\|_{L^4(\mu_d)} \le C_h a^{1/4}, \qquad \sup_{c,d} \|\ell_{cd}\|^2_{L^8(\mu_c \otimes \mu_d)} \le C_h a^{1/4}. \tag{7.18}$$

Moreover $|\ell_{cd}| \le G_+ + C_h$, uniformly in a, so (7.15) applies.

Stopped multigraph expansion. Expand the conditional exponential in powers of the pair kernels $\ell_{c_i c_j}$, with coupling t/N. Absolute convergence follows from (7.15). A total-degree-one vertex vanishes by conditional canonicality. Every nonzero connected component on v vertices therefore contains a spanning unicyclic multigraph with v edge copies; a doubled edge is allowed. Reserve the skeleton factors and sum all remaining multiplicities using $m! \ge u!(m-u)!$. Hölder with exponents $4, 4/3$ separates the skeleton from the residual exponential.

Peel trees from the skeleton. For a cycle with possibly different vertex measures, each kernel $|\ell_{cd}|^4$ defines an operator between the corresponding L^2 spaces. Schur bounds control its operator norm by the fourth power of the row bound in (7.18); two Hilbert–Schmidt bounds close the cycle. Thus the skeleton integral to the power $1/4$ is at most $C_h^v (a^{1/4})^{v-1}$. The length-two cycle is exactly the L^8 integral. There are at most $v^v/2$ labeled unicyclic skeletons. Choosing their labels and using Stirling's bound gives total connected absolute activity $B_{N,v} \le C v^{-1/2} (C_h|t|)^v (a^{1/4})^{v-1}$. For sufficiently small a, summing in $v \ge 2$ and factoring into connected components yields

$$1 \le \mathbb{E}\left[e^{tN^{-1} \sum_{i<j} \ell_{c_i c_j}(X_i, X_j)} \,\middle|\, (n_c) \right] \le e^{C_h a^{1/4}}, \qquad |t| \le 2T_0. \tag{7.19}$$

This proof does not require identical conditional vertex laws.

Restoring scalar and one-body terms. Exactly,

$$\begin{aligned} \sum_{i<j} K_{ij} = {} & \sum_{i<j} \ell_{c_i c_j}(X_i, X_j) + \sum_i \sum_d (n_d - \mathbf{1}_{d=c_i}) h_{c_i d}(X_i) \\ & + \tfrac{1}{2} \sum_{c,d} (n_c n_d - \mathbf{1}_{c=d} n_c) k_{cd}. \end{aligned} \tag{7.20}$$

By (7.16), the one-body coefficient can replace n_d by D_d, while the quadratic scalar term replaces $n_c n_d$ by $D_c D_d$. Equations (7.17)–(7.20) show that the scalar term after division by N has absolute value at most $C_h a(1 + |D|^2/N)$. Each one-body summand is conditionally centered, with absolute value at most $C_h a(1+|D|)/N$ after division by N. Hoeffding's elementary bounded-variable moment estimate bounds its product exponential by $\exp[C_h a^2 (1+|D|^2)/N]$. Applying Hölder to this factor and the canonical factor, using (7.19) at $2t$, gives

$$\mathbb{E}[e^{tY_N(R_a)} \mid (n_c)] \le \exp\left(C_h a^{1/4} + C_h a \frac{|D|^2}{N} \right) \tag{7.21}$$

whenever the counts obey the cap.

Under P_N the count vectors are multinomial within each species. Since the number of cells is fixed, coordinatewise Hoeffding bounds imply $\sup_N \mathbb{E} e^{\eta |D|^2/N} < \infty$ for some $\eta > 0$. In particular, $\mathbb{E} e^{C_h a |D|^2/N} \le 1 + C_h' a$ for small a, uniformly in N. Averaging (7.21) over the capped counts proves (7.14).

Finally take M_0 larger than a fixed multiple of $\|\rho_+ + \rho_-\|_\infty$. Continuous functions dominating the cell indicators and supported on slightly enlarged cells have reference integrals $O(h^2)$. Finitely many weak inequalities on their empirical integrals give a fixed weak neighborhood contained in (7.13). This also avoids any issue from the cell boundaries. □

Let

$$Q_N = Y_N(G), \qquad Z_{N,j} = \frac{1}{\sqrt{N}} \sum_i (t_j)_{\sigma_i}(X_i), \qquad |Z_N|^2_{J^{-1}} = Z_N^T J^{-1} Z_N.$$

Lemma 7.4. *If $0 < c \le 1$ and $\lambda_{\max}(\beta T - cP) < 1$, there are $r > 1$ and a weak neighborhood W of $(\rho_+ dx, \rho_- dx)$ such that*

$$\sup_{\substack{N\ge 2\\ N \text{ even}}} \mathbb{E}_{P_N}\left[\mathbf{1}_{\{(L_n^+,L_n^-)\in W\}} \exp\left(r\left[\beta Q_N - \tfrac{c}{2}|Z_N|^2_{J^{-1}}\right]\right)\right] < \infty.$$

The same assertion holds with $P = 0$, with the score term omitted, whenever $\lambda_{\max}(\beta T) < 1$; it requires only smooth strictly positive reference densities, not a nonflat profile.

Proof. Choose $r, r_1 > 1$ so close to one that $rr_1\lambda_{\max}(\beta T - cP) < 1$, and let $r_2 = r_1/(r_1 - 1)$. Choose the cell partition in Lemma 7.3 for $T_0 = rr_2\beta$, then fix a sufficiently small a in that lemma. The cutoff C_a is finite rank. Its canonical operator T_a satisfies $0 \le T_a \le T$, by positivity of $G - C_a$. Thus the finite-rank operator $M_a = \beta T_a - cP$ has largest eigenvalue below $(rr_1)^{-1}$.

Let $e_1, \dots, e_d$ be an orthonormal basis in $\mathcal{H}$ containing its finite-dimensional range. These functions are smooth and bounded. Under P_N the vector $S_N = N^{-1/2}\sum_i (e_j)_{\sigma_i}(X_i)$ has mean zero and covariance I_d. For every $\varepsilon > 0$, Taylor expansion of its independent summand moment generating functions gives

$$\log \mathbb{E} e^{t\cdot S_N} \le \tfrac{1}{2}(1+\varepsilon)|t|^2, \qquad |t| \le \delta\sqrt{N}, \tag{7.22}$$

for a sufficiently small fixed $\delta > 0$. A finite net of unit vectors and Chernoff's bound then show that for any fixed $b < 1$, after decreasing δ,

$$\sup_N \mathbb{E}\left[\mathbf{1}_{\{|S_N|\le\delta\sqrt{N}\}} e^{b|S_N|^2/2}\right] < \infty. \tag{7.23}$$

Indeed the net and (7.22) give a tail bound $Ce^{-b'R^2/2}$ for $R \le \delta\sqrt{N}$, with $b < b' < 1$; integrate this bound over the truncated radius.

The finite-rank off-diagonal identity gives

$$\beta Y_N(C_a) - \tfrac{c}{2}|Z_N|^2_{J^{-1}} = \tfrac{1}{2}S_N^T M_a S_N - \frac{\beta}{2N}\sum_i k^{C_a}_{\sigma_i\sigma_i}(X_i, X_i).$$

The last term is bounded for the fixed cutoff. Choose a weak neighborhood W so that its empirical pairs imply both $|S_N| \le \delta\sqrt{N}$ and the cell cap. Hölder yields

$$\begin{aligned}&\mathbb{E}\mathbf{1}_W e^{r[\beta Q_N - c|Z_N|^2_{J^{-1}}/2]}\\ &\le \left(\mathbb{E}\mathbf{1}_W e^{rr_1[\beta Y_N(C_a) - c|Z_N|^2_{J^{-1}}/2]}\right)^{1/r_1}\left(\mathbb{E}\mathbf{1}_{E_{N,h}} e^{rr_2\beta Y_N(R_a)}\right)^{1/r_2}.\end{aligned}$$

The first factor is bounded by (7.23) and the second by Lemma 7.3. All choices are fixed before $N \to \infty$. With $P = 0$ and no score variables the identical proof gives the last assertion. □

Remark 7.5. Spatial localization is essential. The moment exponent $rr_2\beta$ in the last factor can be much larger than 16π, yet the capped local moment is finite and controlled: no microscopic cluster contains a fixed large fraction of all N labels. This does not assert existence of an unrestricted Gibbs measure at that enlarged parameter.

7.3. **Phase alignment and the change of measure.** For $f \in \mathcal{H}$ let

$$W_N(f) = \frac{1}{\sqrt{N}}\sum_i f_{\sigma_i}(X_i).$$

The fixed-species case of Lemma 6.3, with $p_+ = p_- = 1/2$, gives the isonormal process W over $\mathcal{H}$ and joint empirical convergence in $H^{-s}(\mathbb{T}^2)^2$ for every $s > 1$. For general test pairs use their separate centerings f^0. Their reference covariances are

$$\mathbb{E}[W_\sigma(f)W_\tau(g)] = \frac{\mathbf{1}_{\sigma=\tau}}{2}\left[\int fg\rho_\sigma - \int f\rho_\sigma \int g\rho_\sigma\right].$$

Use the centered second-chaos notation $I_2(K)$ of Lemma 6.3. The separately canonical statistic satisfies the joint limit

$$(W_N, Q_N, Z_N) \Longrightarrow \left(W, \tfrac{1}{2}I_2(T), (W(t_1), W(t_2))\right). \tag{7.24}$$

Here W_N in the first component can be interpreted as the empirical distribution pair in H^{-s}.

This is the finite-species, separately canonical case of Lemma 6.3; its common finite-rank and degenerate-kernel proof applies with the present $\mathcal{H}$ and T.

Set

$$B_N = \frac{\beta}{N}\sum_i r_{\sigma_i}(X_i) - B, \qquad B = \frac{\beta}{4}(b_{++} + b_{--}). \tag{7.25}$$

The functions r_σ are smooth and bounded, so B_N is uniformly bounded and $B_N \to B$ in P_N probability.

Lemma 7.6. *The unnormalized Gibbs measure has the exact density*

$$e^{\beta H_N^{\pm}/N}\, dX^N = e^{Np(\beta)} e^{\beta Q_N - B_N}\, dP_N. \tag{7.26}$$

Proof. Pair counting and $u = \beta G * q$ give

$$\sum_{i<j} k_{\sigma_i\sigma_j}(X_i, X_j) = H_N^{\pm} - \frac{N}{\beta}\sum_i \sigma_i u(X_i) + \sum_i r_{\sigma_i}(X_i) + \frac{N^2}{2}Q(q) - \frac{N}{4}(b_{++} + b_{--}).$$

On the other hand, $\log(dP_N/dX^N) = \sum_i \sigma_i u(X_i) - N\log A$. Combining these identities with (7.1) proves (7.26). □

For a pair of probability measures $\nu = (\nu_+, \nu_-)$ define

$$\ell_\nu(a) = \tfrac{1}{2}\sum_{\sigma=\pm1}\int \log \rho_\sigma(x-a)\, d\nu_\sigma(x).$$

At the reference pair $\rho = (\rho_+ dx, \rho_- dx)$,

$$\ell_\rho(0) - \ell_\rho(a) = \tfrac{1}{2}\sum_\sigma \mathrm{Ent}(\rho_\sigma \,\|\, \rho_\sigma(\cdot - a)) \geq 0, \tag{7.27}$$

$$D_a^2 \ell_\rho(0) = -J. \tag{7.28}$$

Equality in (7.27) occurs only for $a = 0$, because the orbit is free. The functions $\log\rho_\sigma = \sigma u - \log A$ are smooth, and their translates form compact sets in every C^r. Thus weak convergence of ν controls ℓ_ν and its first finitely many a-derivatives uniformly in a. Compactness of the orbit, the strict gap away from the maximizing phase, and (7.28) imply that in a fixed translation-invariant weak neighborhood $\mathcal{V}$ of the orbit, ℓ_ν has a unique maximizer $a(\nu)$, depending continuously on ν. Its Hessian is uniformly negative definite in the relevant local chart. Apply the implicit-function theorem to the score equation with the likelihood as a parameter in $C^2(\mathbb{T}^2)$ and the phase in a local Euclidean chart. The score derivative is the invertible matrix $-J$, so the phase depends continuously differentiably on that likelihood near the reference pair. The empirical score and Hessian estimates below use only integrals of smooth translates; no differentiability with respect to the weak topology on measures is assumed. In particular,

$$a(\tau_b\nu) = a(\nu) + b \pmod{\mathbb{Z}^2}. \tag{7.29}$$

Set $a_N = a(L_n^+, L_n^-)$ on $\mathcal{V}$.

Under P_N, $a_N \to 0$, and the score equation at its maximum gives

$$\sqrt{N}\, a_N = J^{-1} Z_N + o_{P_N}(1). \tag{7.30}$$

More precisely, on a sufficiently small local neighborhood,

$$Z_N = \sqrt{N}\, J_N a_N, \qquad J_N = -\int_0^1 D_a^2 \ell_{(L_n^+, L_n^-)}(t a_N)\, dt, \tag{7.31}$$

and $J_N \to J$ under P_N. Both the phase and a_N in these formulas use its representative near zero.

For a test pair f, integration by parts yields

$$\tfrac{1}{2}\sum_\sigma \int \partial_j f_\sigma\, \rho_\sigma = \langle t_j, f^0\rangle_{\mathcal{H}}.$$

Combining the phase expansion with translation of the empirical measures gives

$$\eta_N(f) = W_N(f^0) - \sum_{i,j}\langle t_i, f^0\rangle_{\mathcal{H}}(J^{-1})_{ij} Z_{N,j} + o_{P_N}(1) \Longrightarrow W(\Pi f^0). \tag{7.32}$$

This holds jointly in $H^{-s}(\mathbb{T}^2)^2$: the reference empirical measures are tight there, small translations converge uniformly on compact subsets of that Hilbert space, and the translation expansion of the smooth deterministic densities has remainder $O(\sqrt{N}|a_N|^2) = o_{P_N}(1)$.

Exact group averaging. Let χ be any integrable function on $\mathbb{T}^2$, and let F_N be a bounded function of the phase-centered configuration. On $\mathcal{V}$, F_N and $\mathbf{1}_{\mathcal{V}}$ are invariant under simultaneous translations, whereas a_N transforms by (7.29). Translation invariance of the Gibbs law therefore gives the exact identity

$$\int \mathbf{1}_{\mathcal{V}} F_N \chi(a_N)\, d\nu^{\pm}_{N,\beta} = \left(\int_{\mathbb{T}^2} \chi(a) da\right) \int \mathbf{1}_{\mathcal{V}} F_N \, d\nu^{\pm}_{N,\beta}. \tag{7.33}$$

This also proves that, conditional on $\mathcal{V}$, the phase is exactly Haar-uniform and independent of the aligned configuration at every N.

Choose a small Euclidean ball D_δ in the phase chart about zero and put

$$\chi_N(a) = \mathbf{1}_{D_\delta}(a) e^{-N a^T J a/2}, \qquad c_N = \int \chi_N(a) da = \frac{2\pi}{N\sqrt{\det J}}(1 + o(1)). \tag{7.34}$$

Let $\Gamma_N = \{(L_n^+, L_n^-) \in \mathcal{V},\ a_N \in D_\delta\}$ and define the pinned product weight

$$\mathcal{W}_N = \mathbf{1}_{\Gamma_N} \exp\left(\beta Q_N - B_N - \frac{N}{2} a_N^T J a_N\right).$$

Equations (7.26) and (7.33) give, exactly,

$$\mathbb{E}_{P_N}[\mathcal{W}_N F_N] = e^{-Np(\beta)} Z_N^{\pm}(\beta) c_N \mathbb{E}_{\nu^{\pm}_{N,\beta}}[\mathbf{1}_{\mathcal{V}} F_N]. \tag{7.35}$$

In particular,

$$Z_N^{\pm}(\beta)\nu^{\pm}_{N,\beta}(\mathcal{V}) = \frac{e^{Np(\beta)}}{c_N} \mathbb{E}_{P_N} \mathcal{W}_N. \tag{7.36}$$

There is no coordinate Jacobian approximation in this identity.

First choose a small $\varepsilon > 0$ and put $c = (1+\varepsilon)^{-2} > 0$. By (7.7), $\lambda_{\max}(\beta T - cP) = \max\{1 - c, \lambda_{\max}(\beta T|_{\Pi\mathcal{H}})\} < 1$. Choose the neighborhood W and exponent $r > 1$ from Lemma 7.4. Shrink the orbit neighborhood $\mathcal{V}$ and the phase ball D_δ so that Γ_N implies $(L_n^+, L_n^-) \in W$, and so that throughout the segments in (7.31), $\left\|J^{-1/2} J_N J^{-1/2} - I\right\| \le \varepsilon$. This is possible by the continuity properties of the phase map and the smooth likelihood derivatives. Equation (7.31) then gives the deterministic bound on Γ_N

$$N a_N^T J a_N \ge (1+\varepsilon)^{-2} Z_N^T J^{-1} Z_N = c|Z_N|^2_{J^{-1}}.$$

Since B_N is uniformly bounded, Lemma 7.4 proves

$$\sup_N \mathbb{E}_{P_N} \mathcal{W}_N^r < \infty. \tag{7.37}$$

This is the required uniform integrability; it is a localized statement, not an unrestricted higher-temperature partition-function bound.

The product-law probability of Γ_N tends to one. From (7.24), (7.25), and (7.30),

$$\begin{aligned} \mathcal{W}_N &\Longrightarrow \exp\left[-B + \tfrac{1}{2} I_2(\beta T) - \tfrac{1}{2}\sum_{i,j} W(t_i)(J^{-1})_{ij} W(t_j)\right] \\ &= e^{-B-1} \exp\left[\tfrac{1}{2} I_2(\beta T - P)\right]. \end{aligned} \tag{7.38}$$

The final constant is e^{-1} because P has rank two: $I_2(P) = \sum_{i,j} W(t_i)(J^{-1})_{ij} W(t_j) - 2$. Since $S = \beta T - P$ is positive, self-adjoint and Hilbert–Schmidt with $\|S\| < 1$, Lemma 6.4 gives

$$\mathbb{E}\exp[\tfrac{1}{2} I_2(S)] = \det{}_2(I - S)^{-1/2} = D_\beta < \infty. \tag{7.39}$$

By (7.37)–(7.39),

$$\mathbb{E}_{P_N} \mathcal{W}_N \longrightarrow e^{-B-1} D_\beta. \tag{7.40}$$

The empirical-pair large-deviation principle gives $\nu^{\pm}_{N,\beta}(\mathcal{V}) \to 1$, with exponentially small complement. Equations (7.36), (7.34), and (7.40) prove (7.10).

For bounded continuous F on $H^{-s}(\mathbb{T}^2)^2$, use $F_N = F(\eta_N)$ in (7.35). Joint product convergence (7.32) and (7.38), together with (7.37), gives the phase-centered limit as the Gaussian reference law $f \mapsto W(\Pi f^0)$ tilted by

$$\frac{\exp[\tfrac{1}{2} I_2(S)]}{D_\beta}.$$

The covariance assertion of Lemma 6.4 identifies this centered Gaussian law. On $\Pi\mathcal{H}$ the operator is $S = \beta T$, which gives exactly (7.9). The component on $P\mathcal{H}$ has vanished from the aligned empirical field. Thus the limiting pair has the two stated signed constraints. They also hold exactly for η_N on $\mathcal{V}$, by the maximum-likelihood score equation. The exact independence assertion following (7.33) and the exponentially small complement of $\mathcal{V}$ now give the joint Haar-independent limit (7.8). This completes the proof of Theorem 7.1.

Laboratory-frame vorticity. Let $\eta_N^{\mathrm{ch}} = \eta_N^+ - \eta_N^-$, and write τ_a for translation by a. On the phase neighborhood,

$$\omega_N := \frac{1}{\sqrt{N}} \sum_i \sigma_i \delta_{X_i} = \sqrt{N}\, q(\cdot - a_N)dx + \tau_{a_N} \eta_N^{\mathrm{ch}}.$$

The theorem describes the order-one remainder after subtracting the random macroscopic dipole. Keeping the laboratory frame gives a Gaussian law conditional on the phase, with covariance translated by that phase. Moving into the phase frame gives a fixed Gaussian law independent of the uniform phase.

Why the reduced determinant has the stated constant. For a smooth finite-rank translation-invariant interaction, the canonical operator is trace class and $\operatorname{Tr} T = G(0) - \frac{1}{2}(b_{++} + b_{--})$, $\operatorname{Tr}(\beta T - P) = \beta \operatorname{Tr} T - 2$. Consequently the factor in (7.10) can be rewritten as $e^{-B-1} \det_2(I - \beta T + P)^{-1/2} = e^{-\beta G(0)/2} \det{}'(I - \beta T)^{-1/2}$. This is the ordinary finite-rank Laplace constant, with the two zero directions omitted. For the logarithmic kernel $G(0)$ is infinite and the ordinary determinant is not defined; the combined expression on the left is the finite, correctly renormalized form. In particular, neither the finite diagonal correction B nor the factor e^{-1} should be discarded.

Precision form and fixed populations. For a relative density fluctuation $f \in \mathcal{H}$, the precision is the density Hessian

$$\langle f, (I - \beta T) f \rangle_{\mathcal{H}} = \frac{1}{2} \sum_\sigma \int \rho_\sigma f_\sigma^2 - \frac{\beta}{4} Q(\rho_+ f_+ - \rho_- f_-),$$

restricted to $\Pi\mathcal{H}$. The two species mass constraints are built into $\mathcal{H}$; this is the quenched equal-population ensemble. Randomizing the populations would introduce additional finite-dimensional modes and is not covered by this covariance formula without modification.

7.4. **Critical behavior.** The preceding Gaussian theorem is for fixed $4\pi^2 < \beta < 16\pi$. Its proof is not uniform at either endpoint. At $4\pi^2$, the unique global minimizer is still flat, but the first Laplacian eigenspace is four-dimensional and the quadratic precision vanishes there. The following statement records what can be concluded at that parameter from the arguments included here.

Proposition 7.7. *At $\beta = 4\pi^2$,*

$$(L_n^+, L_n^-) \longrightarrow (dx, dx) \quad \textit{in probability}, \qquad \frac{1}{N} \log Z_N^{\pm}(4\pi^2) \longrightarrow 0, \qquad Z_N^{\pm}(4\pi^2) \geq cN$$

for all sufficiently large even N, with $c > 0$ independent of N.

The first two assertions follow from the sharp inequality and Corollary 5.11. We prove the last one using a finite mixture of product trial measures. The key observation is an exact finite-particle identity. Let $\rho^{(j)} = (\rho_+^{(j)}, \rho_-^{(j)})$ be any finite family of smooth positive pairs, let $P_{N,j}$ be the corresponding two-species product laws, and let J be a uniform label. Write $\overline{P}_N = K^{-1} \sum_j P_{N,j}$. The entropy chain rule gives

$$\operatorname{Ent}(\overline{P}_N \mid dx^N) = \frac{N}{K} \sum_j \mathcal{S}(\rho^{(j)}) - I(J; X),$$

where I is mutual information. Direct pair counting gives

$$\frac{\beta}{N} \mathbb{E}_{P_{N,j}} H_N^{\pm} - N\mathcal{S}(\rho^{(j)}) = -N\mathcal{A}_\beta(\rho^{(j)}) - B_j, \quad B_j = \frac{\beta}{4}[Q(\rho_+^{(j)}) + Q(\rho_-^{(j)})].$$

The Gibbs variational inequality with trial law $\overline{P}_N$ therefore yields

$$\log Z_N^{\pm}(\beta) \geq I(J; X) - \frac{1}{K} \sum_j [N\mathcal{A}_\beta(\rho^{(j)}) + B_j]. \tag{7.41}$$

We use the following elementary information estimate. Suppose the labels form an integer grid of dimension d, and an estimator $\widehat{J}(X)$ has integer displacement $\Delta = J - \widehat{J}$ with $\mathbb{E}|\Delta|^2 \leq C$. For a periodic grid use the coordinatewise wrapped displacement. Since J is recovered from $(\widehat{J}, \Delta)$,

$H(J \mid X) \le H(J \mid \widehat{J}) \le H(\Delta)$. Comparison with the probability mass $Z_d^{-1} e^{-|n|^2}$ on $\mathbb{Z}^d$ gives $H(\Delta) \le \mathbb{E}|\Delta|^2 + \log Z_d \le C'$. Consequently,

$$I(J;X) \ge \log K - C'. \tag{7.42}$$

The constants are independent of the size of the grid.

Proof of the lower bound in Proposition 7.7. Let $e_1, \dots, e_4$ be a real L^2-orthonormal basis of the first eigenspace, and write $h_a = \sum_{j=1}^4 a_j e_j$, $\rho_\pm^a = 1 \pm h_a$. For $|a|$ sufficiently small these are positive probability densities. Since $G h_a = \frac{1}{4\pi^2} h_a$, Taylor expansion of the entropy gives exact cancellation of the quadratic terms:

$$\mathcal{A}_{4\pi^2}(1+h_a, 1-h_a) = \frac{1}{12} \int h_a^4 + O(|a|^6) \le C|a|^4.$$

Also $B_a = \frac{1}{2}\|h_a\|_2^2$. Fix a small $r > 0$, put $L_N = \lfloor r N^{1/4} \rfloor$, and take the coefficient grid $a = N^{-1/2} k, \quad k \in \{-L_N, \dots, L_N\}^4$. It has $K_N = (2L_N+1)^4 \ge cN$ points. Uniformly on this grid, $N\mathcal{A}_{4\pi^2}(\rho^a) + B_a \le C$.

Under the product law of ρ^a, the empirical estimator

$$\widehat{a}_j = \frac{1}{2}\left(\int e_j \, dL_n^+ - \int e_j \, dL_n^- \right)$$

has mean a_j and mean-square error at most C/N. Rounding $\sqrt{N}\widehat{a}$ to the nearest grid point, with clipping at its boundary, produces an integer label estimator with uniformly bounded squared displacement. Equation (7.42) gives $I(J;X) \ge \log K_N - C$. Inserting this and the uniform action bound into (7.41) yields $\log Z_N^{\pm}(4\pi^2) \ge \log K_N - C \ge \log N - C$, as required. □

The local behavior of the action is described by its quartic expansion. Let

$$v = a_1 \cos 2\pi x + a_2 \sin 2\pi x + a_3 \cos 2\pi y + a_4 \sin 2\pi y, \qquad r_x^2 = a_1^2 + a_2^2, \quad r_y^2 = a_3^2 + a_4^2.$$

The Taylor expansion (3.42) and its first-eigenspace computation, with the β-dependent quadratic term restored, give

$$\mathcal{F}_\beta(v) = \frac{1}{4}\left(\frac{4\pi^2}{\beta} - 1 \right)(r_x^2 + r_y^2) + \frac{1}{64}(r_x^4 + r_y^4) + O(|a|^6). \tag{7.43}$$

For completeness, let P_1 be the first-eigenspace projection and work in mean-zero $C^{2,\alpha}$, $0 < \alpha < 1$. Set

$$\mathcal{R}_\beta(u) = -\beta^{-1}\Delta u - \frac{1}{2}\left(\frac{e^u}{A_+(u)} - \frac{e^{-u}}{A_-(u)} \right).$$

On the complement of E_1, the derivative of $(I - P_1)\mathcal{R}_{4\pi^2}$ at zero is $-\frac{1}{4\pi^2}\Delta - I$, an isomorphism onto the corresponding mean-zero $C^{0,\alpha}$ complement because the next eigenvalue is $8\pi^2$. The analytic implicit-function theorem gives a correction $w_\beta(v) \perp E_1$ for β near $4\pi^2$. The map $\mathcal{R}_\beta$ is odd, the linear cross term vanishes, and there is no quadratic nonlinearity. Hence $w_\beta(v) = O(|a|^3)$ uniformly for such β. Orthogonality eliminates the quadratic cross term in the action; its quartic mixed term is $O(|a|^6)$. Thus substituting $v + w_\beta(v)$ changes the action only at order six, so (7.43) is also the reduced quartic form. Writing $\delta = 1 - 4\pi^2/\beta$, the quartic polynomial in (7.43) is minimized at $r_x^2 = r_y^2 = 8\delta$ when $\delta > 0$, with value $-2\delta^2$; a stripe has the larger value $-\delta^2$. The exact classification already identifies the globally minimizing branch. This polynomial calculation only describes its leading local action.

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