Smooth isometric bending of a closed sphere

Boyang Hu
Tsinghua University

17 November 2026

Abstract

We construct a smooth embedded sphere $\Sigma \subset \mathbb{R}^3$ together with a smooth one-parameter family of isometric embeddings of Σ whose images are not all congruent. The family is periodic; an inner disk and a complementary cap stay fixed pointwise; and the area, the enclosed volume and the total mean curvature do not change. This gives a flexible closed surface in the smooth category.

The mechanism is elementary. A disk whose metric is rotationally symmetric, but whose embedding is not, can be turned inside a fixed shell provided it is exactly planar near its rim. The whole difficulty is to produce such a disk. This reduces to a scalar equation of Monge–Ampère type on an annulus, the Darboux equation, for a height function which must be nonradial and yet constant to infinite order at both boundary circles.

We solve this equation by a Nash–Moser argument around an infinitesimal bending of a radial profile. The profile is forced to change the sign of its curvature infinitely often near each boundary circle, so the linearized equation changes type on infinitely many circles accumulating at both ends. We build the profile from explicitly solvable one-dimensional models, describe its accumulating ends by an exact three-term recurrence, and invert the linearized operator on the whole annulus. Two modifications of the linearization, each proportional to an unsolved equation, keep the type-change circles and the characteristic return maps fixed during the iteration, so that all linearized operators act on one domain.

2020 Mathematics Subject Classification. 53A05, 53C24 (primary); 35M10, 35J96, 58C15 (secondary).
Keywords. Isometric deformations, flexible surfaces, bendings, Darboux equation, Monge–Ampère equations of mixed type, Nash–Moser iteration.

hexagon:2611.00534v1

Contents

1 Introduction **3**
1.1 The result . . . 3
1.2 The mechanism . . . 4
1.3 The analytic core . . . 4
1.4 Three difficulties, and how they are met . . . 6
1.5 Why the problem was open, and what is new . . . 7
1.6 Organization of the proof . . . 8
1.7 Context . . . 9
1.8 Conventions . . . 11

I The mechanism and the equation **11**

2 From an annular height to a flexible sphere **11**
2.1 Setting . . . 12
2.2 Removing one coordinate from the metric . . . 12
2.3 Development onto a round disk . . . 13
2.4 A stationary cap which distinguishes shapes . . . 13
2.5 The annulus implies the theorem . . . 15

3 The height equation **16**
3.1 The curvature residual and its derivative . . . 17
3.2 Why the profile must oscillate at a flat end . . . 19
3.3 Two resonance conditions in a finite model . . . 20
3.4 Normalization and the conservation law . . . 22
3.5 The first residual correction . . . 23

II The reference profile **23**

4 The reference profile: requirements and solvable models **23**
4.1 Angular modes and the design problem . . . 24
4.2 What the reference profile must provide . . . 24
4.3 Layout of the construction and vocabulary . . . 26
4.4 From a Schrödinger potential to a radial profile . . . 27
4.5 The central oscillator . . . 28
4.6 The Morse model . . . 31

5 Reaching a flat endpoint with an infinite chain of cells **32**
5.1 The budget of lengths, amplitudes and errors . . . 33
5.2 One cell . . . 33
5.3 Joining the cells . . . 35
5.4 Why a mass–resistance chain appears . . . 36
5.5 The exact reduction to a recurrence . . . 38
5.6 Localized bounds for the deep inverse . . . 41
5.7 Frequency bands: one transition cell at every frequency . . . 42
5.8 Choosing the amplitudes so that the lines do not meet . . . 42
5.9 Summary . . . 45

6 The inverse on the whole annulus **45**
6.1 Overview . . . 45
6.2 Completing the profile while keeping the seed . . . 46
6.3 A tool: cutting a model inside a positive buffer . . . 47
6.4 The inner region . . . 49
6.5 The outer region: an inverse without division by a boundary value . . . 51
6.6 Matching the two remaining boundary values . . . 53
6.7 The matched inverse and its weighted bound . . . 55
6.8 The zero mode, the domain, and the final border . . . 58
6.9 What Part III uses . . . 59

III The nonlinear problem **60**

7 A geometry that stays fixed during the iteration 60
7.1 Overview of Part III . . . 60
7.2 Weights and function spaces . . . 61
7.3 The geometry at the roots, and the travel-angle controls . . . 67
7.4 The state and one neighbourhood for all components . . . 69
7.5 Fixing the return maps by their own residual . . . 71
7.6 Coordinates on a negative component . . . 74
7.7 Coordinates on a positive component . . . 77
7.8 The objects of a state and their bounds . . . 78
7.9 The fixed transmission domain . . . 88

8 Inverting the corrected linearization near the reference 88
8.1 The linear problem in canonical form . . . 89
8.2 Two endpoint exponents give decay off the diagonal . . . 91
8.3 Positive cylinders: an estimate without boundary conditions . . . 92
8.4 The relative operator . . . 94
8.5 Solving, and recovering all derivatives . . . 95

9 From the linear seed to a nonlinear solution 98
9.1 The residual and its derivative . . . 99
9.2 The control column gains all coefficient weights . . . 100
9.3 Solving the corrected system . . . 100
9.4 The quadratic error and the Nash–Moser theorem . . . 101

10 The endpoint gain and the closed surface 102
10.1 Two identities for means . . . 103
10.2 Two weak factors give one strong remainder . . . 103
10.3 Recovering the mean and the controls . . . 105
10.4 The annular theorem . . . 106
10.5 Proof of Theorems B and A . . . 107

A Why a finite angular ansatz cannot work 108

B External results and the order of choices 110

C Index of notation 111

Statement on the use of AI 115

1 Introduction

1.1 The result

A *bending* of a closed surface is a continuous family of isometric embeddings of one fixed Riemannian metric into $\mathbb{R}^3$. It is nontrivial when the shape of the surface changes, that is, when the images are not all congruent. Connelly's flexible sphere [Con77] shows that nontrivial bendings exist for embedded polyhedra. Problem 1.1 of Ghomi's list [Gho19], which is attributed there to Euler and Maxwell, asks whether a closed C^2 surface in $\mathbb{R}^3$ can be flexible; for the history of the question see also [Ale47, Sab23]. We construct a C^∞ example.

Theorem A. *There are a smooth surface Σ diffeomorphic to S^2, a smooth Riemannian metric g_Σ on it, and a jointly smooth map $\mathcal{F} : \Sigma \times \mathbb{R} \to \mathbb{R}^3$ such that the maps $F_\tau = \mathcal{F}(\,\cdot\,, \tau)$ satisfy*

$$F_\tau \text{ is an embedding}, \qquad F_\tau^* g_{\mathrm{E}} = g_\Sigma, \qquad F_{\tau+1} = F_\tau \qquad \text{for all } \tau \in \mathbb{R},$$

where g_{E} is the Euclidean metric. At least two of the images $F_\tau(\Sigma)$ are not congruent under the Euclidean isometry group. An inner disk and a complementary cap of Σ are fixed pointwise throughout the family. The area, the enclosed volume and the total oriented mean curvature of $F_\tau(\Sigma)$ do not depend on τ.

Three conventions are implicit in this statement. A *closed* surface is compact and without boundary. *Nontrivial* always refers to noncongruence of the images: relabelling the points of a fixed surface by a family of intrinsic isometries moves material points but does not change the shape, and is not a bending in our sense. Families of this kind are the *slide bendings* studied by Sabitov [Sab95, Sab21, Sab23]; they need not be induced by rigid motions, although the surface does not change as a set. The mean curvature is $(k_1 + k_2)/2$ for the outward normal.

1.2 The mechanism

The family in Theorem A is completely explicit once one particular disk is available. Let D be a disk with a rotationally symmetric metric g, and write ϱ_ϑ for its intrinsic rotation by the angle ϑ. Suppose $F : D \to \mathbb{R}^3$ is an isometric embedding. If R_ϑ denotes the rotation of $\mathbb{R}^3$ about the vertical axis, then

$$F_\tau = R_{-2\pi\tau} \circ F \circ \varrho_{2\pi\tau} \tag{1.1}$$

is again an isometric embedding of (D, g) for every τ, and its image is the rotated copy $R_{-2\pi\tau}F(D)$ of the original image. Two cases are uninteresting. If $F(D)$ is a surface of revolution, the image does not move at all. If $F(D)$ is not a surface of revolution, the image does move, but in general so does its boundary curve, and nothing can be attached to it.

There is a third possibility. Suppose that $F(D)$ is not a surface of revolution, but that near the boundary of D the embedding is the inclusion of a flat horizontal annulus, $F(x) = (x, c)$. There the two rotations in (1.1) cancel, so $F_\tau = F$ near the boundary for every τ. All the maps F_τ can then be glued to one and the same complementary cap. The result is a closed surface in which a wavy annulus turns relative to a stationary shell (Figures 1 and 2). If the shell carries a marker which breaks its rotational symmetry, no rigid motion can account for the relative rotation, and the shape of the closed surface changes. In the terminology above, $F \circ \varrho_{2\pi\tau}$ is a slide bending of the disk, and the rotation $R_{-2\pi\tau}$ in (1.1) is chosen so that the collar stays fixed pointwise instead of the image.

The motion itself therefore requires no analysis; in particular, no infinitesimal bending has to be integrated in the variable τ. Everything rests on the existence of the disk, and the difficulty lies in the requirement that it be *exactly* planar on a collar of its boundary while being nonradial inside.

1.3 The analytic core

Such a disk is produced as follows. Fill the middle of the disk by a flat piece, so that only an annulus $(r_-, r_+) \times \mathbb{T}$ remains, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. On the annulus take the metric of a radial graph with slope w,

$$g = (1 + w(r)^2)\, dr^2 + r^2\, d\theta^2, \tag{1.2}$$

and look for an isometric embedding of the form (P, z), where z is a height function and P takes values in the horizontal plane. Such an embedding exists locally exactly when the

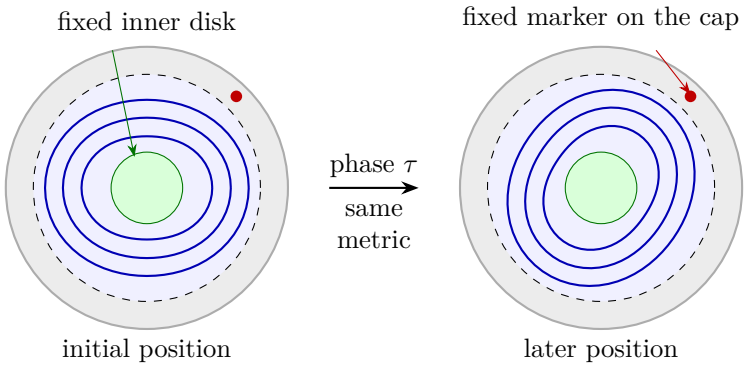


Figure 1: The relative rotation, seen from above. The blue curves are schematic height contours of the moving annulus; they are not numerical data for the constructed surface. The dashed circle bounds the planar collar, which stays fixed pointwise, as does the green inner disk. The red marker on the cap also stays fixed while the nonradial part turns, so no ambient rotation accounts for the whole motion.

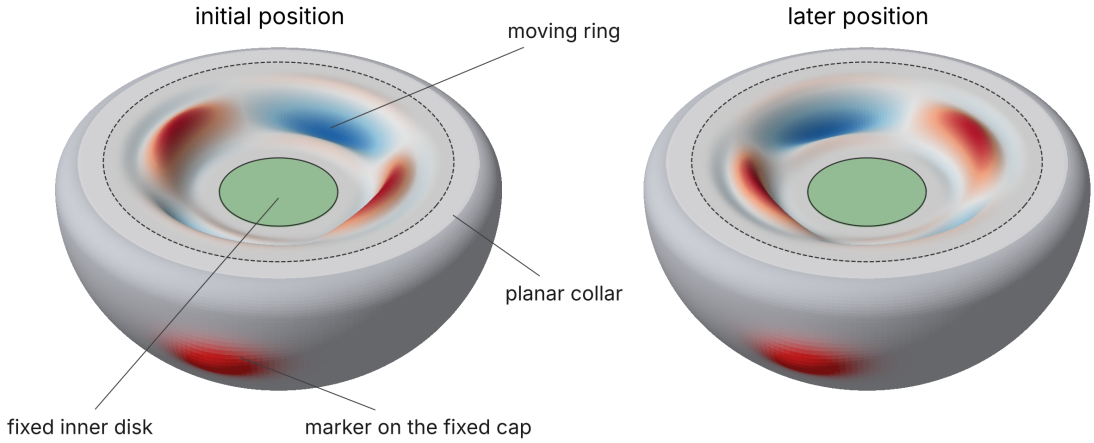


Figure 2: A schematic view of the closed surface in $\mathbb{R}^3$ at two values of τ. *This is an illustration and not a computation of the surface of Theorem A:* the nonradial part is magnified by many orders of magnitude, and the infinitely many small oscillations near the two edges of the moving ring are not drawn. The colours show the sign of the nonradial part of the height, which has angular period π. Between the two pictures this pattern has turned about the axis; the inner disk, the planar collar outside the dashed circle and the cap with its marker have not moved.

remainder metric $g - dz \otimes dz$ is positive definite and flat, and flatness is a single scalar equation for z, the Darboux equation. The disk of Section 1.2 is then obtained from the following statement, which is the main analytic result of the paper.

Theorem B. *There exist radii* $0 < r_- < r_+$*, a smooth function* $w > 0$ *on* (r_-, r_+) *and a smooth function* z *on* $(r_-, r_+) \times \mathbb{T}$ *with the following properties, where* g *is the metric* (1.2)*.*

(i) $|dz|_g < 1$*, and the metric* $g - dz \otimes dz$ *is flat.*

(ii) $z(r, \theta + \pi) = z(r, \theta)$*, and* z *is not independent of* θ*.*

(iii) w *extends smoothly by zero across* r_- *and* r_+*, and* z *extends smoothly by a constant across each of them.*

Condition (iii) says that w, and z minus its boundary value, vanish to infinite order at both boundary circles. After a slight enlargement of the annulus this gives open collars on which $w = 0$ and z is constant, which is the exact planarity needed for the mechanism. A small error at the boundary would be useless; this is why flatness to infinite order is the right target. The precise form of Theorem B, with quantitative information on w and z, is Theorem 10.3.

The passage from Theorem B to Theorem A is carried out in Section 2 and uses only classical differential geometry. The rest of the paper proves Theorem B.

1.4 Three difficulties, and how they are met

We look for the height in the form $z = h + qu$, where $h' = w$ is the radial height whose graph has the metric g, and q is a small parameter. To first order in q the Darboux equation is

$$\mathscr{L}_w u = (w u_r)_r + \frac{w'}{r}\, u_{\theta\theta} = 0. \tag{1.3}$$

A nonzero solution of this equation which is constant to infinite order at both ends is an infinitesimal version of what we want, and the plan is to choose w so that one exists and then to correct it to a solution of the nonlinear equation. Three features of the problem shape everything that follows.

Flat ends force infinitely many changes of type. For $w > 0$ the operator (1.3) is elliptic where $w' > 0$ and hyperbolic where $w' < 0$. At a flat end w tends to zero. If w were monotone near that end, no nonzero angular mode of (1.3) could vanish there together with its flux (Proposition 3.2). So w must oscillate: near each end there are infinitely many circles on which the equation changes type, and they accumulate at the boundary. We build such a profile from infinitely many copies of one explicitly solvable cell, with lengths and amplitudes decaying fast enough for smoothness (Sections 4 and 5).

Two unrelated resonance conditions. On each hyperbolic annulus, characteristics turn by a fixed *travel angle* T, and solving along them involves dividing by $\sin(jT)$ at the angular frequency j. Independently, the solutions on neighbouring annuli must be matched, and the matching fails at eigenvalue-like resonances. A finite model in which both sets of resonances can be computed exactly (Section 3.3) shows that neither condition implies the other. In the actual profile the travel angles are quadratic irrationals fixed by the choice of the cells, while matching resonances are avoided by adjusting the relative amplitudes of consecutive cells. The outcome is a two-sided inverse of $\mathscr{L}_w$ on the whole annulus, in spaces weighted towards the flat ends (Section 6).

The inverse has to survive perturbation. The Nash–Moser method needs an inverse of the linearization at every state near the reference one, not only at the reference. For a nearby state both the type-change circles and the characteristic return maps move, so its linearization does not even act on the same space. We remove this problem by subtracting from the linearization two terms, each proportional to an equation that is not yet solved: one fixes the type-change circles, the other fixes the return maps. Both terms vanish at a solution, and they produce exactly the quadratic error allowed by Hamilton's form of the implicit function theorem [Ham82] (Sections 7–9).

1.5 Why the problem was open, and what is new

What the existing theory provides. A closed surface which can be bent is not convex: two isometric closed convex surfaces are congruent [CV27, Her43, Pog73]. Since a closed surface of nonnegative curvature is convex, the curvature of a flexible closed surface takes both signs, and the equation for its isometric embeddings is of mixed type. For such equations the existence theorems known to us are local or semi-global. The local ones construct an embedding of a neighbourhood of a point at which the curvature changes sign, under a hypothesis on the way it vanishes: cleanly across a curve [Lin86, Han05], to finite order across a curve [Khu07], or to finite order on two curves which cross [HK10]. The semi-global ones construct an embedding of a narrow band along one closed curve on which the curvature vanishes: on both sides of the curve when the curvature changes sign cleanly [Don93] or to finite order [Cao19], and on one side of it in [Li12]. The angular variable is periodic there, as it is here, but a single curve of type change is involved, and in [Li12, Cao19] the embedding has finite regularity. A theorem of either kind produces one surface, and a bending is a path of surfaces. In the other direction, an infinitesimal bending of a surface with a planar patch can be written down at once (Section 1.7), but nothing guarantees that it is the derivative of a bending.

What was missing. The mechanism of Section 1.2 is elementary, and it turns one surface into a path. What was not available is a surface to which it applies: a nonradial isometric embedding of a rotationally symmetric disk which is exactly planar near the boundary. The existence of such a disk is a problem of a different kind from the local and the semi-global ones. The equation has to be solved in C^∞ on a whole annulus, with conditions at both boundary circles. It changes type on infinitely many circles which accumulate at the boundary (Proposition 3.2), so no hypothesis of finite order holds there. And its linearization has the two families of resonances of Section 1.4.

One circumstance works in our favour. In an embedding problem the metric is given. Here it is not: Ghomi's question asks for some surface, so the profile w is ours to choose, and we choose it so that the linearized equation can be solved explicitly.

The new steps.

1. *The profile.* A profile w_* with infinitely many changes of type at each end, for which the linearized equation has a solution $\phi_*(r)\cos 2\theta$ flat at both ends, has no such solution at the higher even frequencies, and has, after bordering by one scalar, an inverse bounded in weighted spaces at all frequencies (Proposition 4.2). Its ingredients are classical: two solvable models, a chain recurrence, Schur complements; so is the design of a meridian on which only finitely many modes are admissible (Section 1.7). What is new is their assembly into one inverse on the whole annulus, and in particular the choice of the amplitudes of the cells which avoids the matching resonances at all frequencies simultaneously (Lemma 5.9).

2. *The second residual correction.* Subtracting from the linearization a multiple of the residual of the equation is a device of Han and Khuri [HK10, pp. 650–653]; here it fixes the circles on which the type changes (Section 3.5). On an annulus of hyperbolic type which is traversed completely this is not enough, because the return map of the characteristics also depends on the state. We add an equation which says that the return maps are rotations in suitable charts, with the charts as new unknowns, and subtract from the linearization a second term, proportional to the residual of this equation (Section 7.5).

3. *The inverse at nearby states.* After the two corrections all linearizations act on one domain, and they are inverted by a Neumann series on a single Hilbert space, although the reference inverse loses weight (Section 8).

The Nash–Moser theorem itself is used as a black box, in Hamilton's form.

1.6 Organization of the proof

Figure 3 shows the logical structure, and Table 1 lists what each section contributes. The paper has three parts.

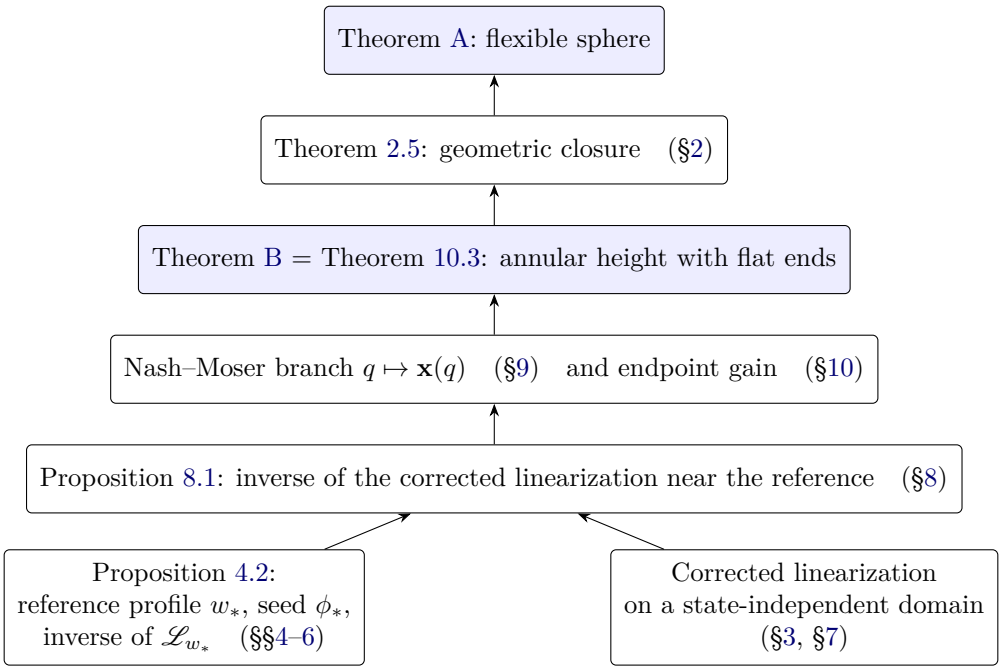


Figure 3: The structure of the proof. An arrow means "is used to prove".

Part I (Sections 2 and 3) contains the geometry and the derivation of the scalar equation. It is self-contained and elementary.

Part II (Sections 4–6) is one-dimensional. After separation of angular modes, the linear equation (1.3) becomes a family of ordinary differential equations, and the whole part is devoted to the construction of one profile w_* for which these equations have the right kernel and a controlled inverse. Its only output is Proposition 4.2; a reader willing to accept that proposition can go directly from Section 4.2 to Part III.

Part III (Sections 7–10) solves the nonlinear equation near the reference and proves the two theorems.

The estimates are arranged so that precision is spent where the decisions above are implemented: the matching of infinitely many intervals, the common domain of the perturbed

Section	What it provides
§2	Theorem B implies Theorem A: planar development, the marked cap, noncongruence, invariants.
§3	The Darboux equation as the curvature of a connection form; its linearization and conservation law; why the ends must oscillate; the two resonances in a finite model; the first residual correction.
§4	Statement of what the reference profile must provide (Proposition 4.2); the two solvable models from which it is built.
§5	The infinite chain of cells at each end; the mass–resistance recurrence and the decaying solution; choice of the cell amplitudes.
§6	The inverse of $\mathscr{L}_{w_*}$ on the whole annulus; proof of Proposition 4.2.
§7	Weighted spaces; the unknowns of the nonlinear problem; the second residual correction; coordinates in which every nearby linearization has the same transmission conditions.
§8	The inverse of the corrected linearization at nearby states, with tame estimates (Proposition 8.1).
§9	The coupled linear system and Hamilton's theorem: a smooth branch of solutions.
§10	Improved decay of the metric corrections; the precise annular theorem (Theorem 10.3); proof of Theorem A.
Appendix A	A side remark: no finite angular ansatz can solve the nonlinear equation.
Appendix B	The external results in the form used, and the order in which the constants are chosen.
Appendix C	Index of notation.

Table 1: Contents by section.

operators, and the decay at the endpoints are proved in detail, while the spectra of the models and the elementary operator theory are quoted. The specific numbers that appear in the construction, such as the exponents $2n^3/3$ and $n^{5/2}$, the power 12 and the weight exponent $3/4$, are each introduced together with the inequalities they have to satisfy; none of them is optimal, and Table 2 collects them.

1.7 Context

Continuous families versus pairs. Ghomi's problem [Gho19, Problem 1.1] asks for a continuous family. In his survey of the work of Cohn-Vossen, Alexandrov puts the question in the words "Do there exist closed surfaces that admit a continuous finite bending?" [Ale47]. Pairs are easier to obtain than paths. Replacing a dimple by its reflection across a flat rim gives two different isometric realizations of one metric, and an infinitesimal bending z of a closed surface x gives the isometric pair $x \pm tz$ [CV30, Ale47]; neither gives a path between the two surfaces. Sa Earp and Toubiana [SET02] construct smooth closed surfaces, some of genus zero, which have isometric and noncongruent smooth realizations arbitrarily close to them in every C^k topology, as well as families of isometric surfaces parametrized by a nowhere dense compact subset of $[0, 1]$. Ghomi, Qiu, Raffaelli and Zhang [GQRZ26] recently used the second observation to construct, for every genus $g \geq 1$, a metric on the closed surface of genus g with uncountably many smooth tight isometric embeddings no two of which are congruent. They state that it is not known whether these surfaces are flexible, and that no smooth counterpart of Connelly's flexible polyhedra has been found. Theorem A provides a path in the smooth category.

Infinitesimal versus finite bendings. A normal vector field supported inside a planar patch has zero first variation of the metric. This observation already appears in Cohn-Vossen's discussion of closed infinitesimally nonrigid surfaces [CV30, footnote 2]; it does not solve the nonlinear isometry equations. Efimov's survey [Efi48a] and his work on deformations at points of zero curvature [Efi48b] belong to the historical setting of this distinction, and the later survey [IKS91, Sections 1.1–1.4] treats finite bendings, their infinitesimal equations and higher-order extension. Our starting point is also an infinitesimal bending, the solution of (1.3); the analysis of Parts II and III is what turns it into an actual height.

Surfaces of revolution and their harmonics. The main subject of Cohn-Vossen's paper is close to Part II. He expands an infinitesimal bending of a closed surface of revolution in Fourier modes in the angle, obtains for each mode an ordinary differential equation along the meridian, and constructs meridians for which a mode of order at least two has a solution regular at both poles; he also shows that his surfaces admit infinitesimal bendings of second order [CV30, Ale47]. In §6 of that paper he considers the extension of an infinitesimal bending to higher order for a fixed metric. With a single admissible mode m, the second order involves the modes 0 and $2m$; at the third order the mode m reappears, with a compatibility condition which he leaves open.

Later work sharpens the control of the admissible modes. Reshetnyak [Res62, Theorem 1*] and Trotsenko [Tro80, Theorem 1*] show that for every $m \geq 1$ there are distinct integers $k_1, \dots, k_m \geq 2$ and a closed surface of revolution, smooth in the first paper and analytic in the second, for which the equation of the mode k has a nonzero solution regular at both poles exactly when $k \in \{k_1, \dots, k_m\}$. The integers are not prescribed in advance. Milka [Mil65, §2] exchanges the roles of the unknowns: he prescribes a component of the bending field in a given mode and recovers the meridian by integration. For surfaces of revolution which are flattened at the poles, Sabitov shows that in the non-analytic class there are nonrigid ones with arbitrary prescribed orders of flattening [Sab13], and gives conditions under which a given harmonic of an infinitesimal bending extends to second order [Sab14].

The design problem of Section 4 is of the same kind. What the nonlinear problem adds are three requirements: the ends are not poles on the axis but circles of positive radius at which the profile is flat to infinite order; the single admissible mode is prescribed; and the inverse on all other modes is bounded uniformly in the frequency.

Rigidity. On the rigid side, hypotheses on curvature and on boundary data are decisive. Besides the theorem on closed convex surfaces quoted in Section 1.5, Ghomi and Spruck [GS20, Theorem 1.1] prove that two properly oriented $C^{2,1}$ positively curved isometric immersions which agree on a curve segment agree everywhere on their connected domain. The annulus of Theorem B has curvature of both signs, with infinitely many transitions at each flat end, so neither strict positivity nor a finite collection of nondegenerate interfaces describes its geometry.

Other categories. Connelly's flexible embedded polyhedral sphere [Con77] establishes flexibility for polyhedra. Rounding its edges would change the metric, so it does not furnish a smooth bending. The C^1 theorems of Nash and Kuiper [Nas54, Kui55] concern another regularity regime; in the smooth problem the second fundamental form and the type of the Darboux equation must be controlled at every scale.

Smooth isometric embedding. Our analytic method is closest to the theory of smooth isometric embedding. Smoothing already compensates for the loss of derivatives in Nash's argument [Nas56]. Han, Hong and Lin [HHL03, Theorem 1.1] use a Nash–Moser–Hörmander

scheme for nonpositive curvature under a specified degeneracy condition, and Han and Khuri treat local problems of mixed type with finite-order changes of type [HK10, HK13]; the monograph [HH06] is a general reference. The modification of the linearization by a multiple of the residual in [HK10, pp. 650–653] is the precedent for our residual corrections. What these works do not have to face is the simultaneous control of infinitely many interfaces and of both flat ends on a single closed domain.

The invariants. The three invariants in Theorem A follow directly from the form (1.1) of the motion. They should be compared with general results of different scope: Sabitov's volume theorem [Sab96] is polyhedral, and Almgren and Rivin [AR98] prove the invariance of the mean-curvature integral under smooth bendings. Motions in which curvature precesses around a shell also arise in the mechanical model of Hamouche, Maurini, Vidoli and Vincenti [HMVV17]; that comparison supplies neither the flat collars nor the closed embedded surface.

1.8 Conventions

Throughout, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, the angular average of a function is

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f \, d\theta,$$

and R_ϑ denotes a rotation by the angle ϑ, of the plane, of $\mathbb{R}^3$ about the vertical axis, or of the circle ($R_\vartheta \theta = \theta + \vartheta$), as the context requires. A prime denotes a derivative with respect to the radius r unless another variable is indicated by a subscript.

The letters C and c denote positive constants, large and small respectively, whose values may change from line to line; a subscript indicates dependence, as in C_s. We write $a \asymp b$ when $cb \le a \le Cb$. Constants that are fixed once and referred to later carry a distinguishing mark, as in C_*, c_0 or N.

The paper uses a fairly large number of symbols. Each of Parts II and III begins by fixing its vocabulary, and Appendix C is an index of notation. Some letters have a local meaning inside a single proof; this is always said where it happens.

Part I
The mechanism and the equation

2 From an annular height to a flexible sphere

This section proves that Theorem B implies Theorem A. It uses no analysis beyond the classical theory of surfaces, and it fixes exactly what the remaining sections have to construct. The steps are as follows.

- An isometric embedding with prescribed height z exists locally if and only if the remainder metric $g - dz \otimes dz$ is positive and flat (Lemma 2.2).
- On a disk whose rim is planar, a positive flat remainder metric develops onto a round disk, and the development is the identity near the rim (Proposition 2.3).
- The resulting graph is closed by a cap which carries a marker (Lemma 2.4).
- Turning the graph inside the cap gives the family of Theorem A (Theorem 2.5).

2.1 Setting

Throughout the section $0 < r_- < r_+$, and on the annulus $(r_-, r_+) \times \mathbb{T}$ with coordinates (r, θ) we consider a metric

$$g = E(r)\, dr^2 + r^2\, d\theta^2, \qquad E = 1 + w^2, \tag{2.1}$$

where w is a smooth function of r. This is the metric induced on the graph $(r\cos\theta, r\sin\theta, h(r))$ of a radial function with $h' = w$. We identify (r, θ) with the point $x = (r\cos\theta, r\sin\theta)$ of the plane, and write D_ρ for the closed disk of radius ρ centred at the origin.

Definition 2.1. A smooth function f on (r_-, r_+), or on $(r_-, r_+) \times \mathbb{T}$, is *flat* at an endpoint if it extends smoothly by zero across that endpoint; equivalently, every derivative of f tends to zero there. A function z has a *constant extension* at an endpoint if z has a limit c there, independent of θ, and $z - c$ is flat.

If w is flat and z has constant extensions at both endpoints, we may extend w by zero and z by its limits, and regard them as defined on a slightly larger annulus. On the added collars $w = 0$ and z is constant. All statements of this section are therefore formulated for functions which are constant on open collars of the two ends.

2.2 Removing one coordinate from the metric

A map $F = (P, z)$ into $\mathbb{R}^2 \times \mathbb{R}$ is isometric for g exactly when

$$P^* g_{\mathbb{R}^2} = \bar{g}, \qquad \bar{g} := g - dz \otimes dz.$$

Thus $\bar{g}$ must be a positive definite flat metric, and for an embedding its planar development P must be injective. Flatness is a scalar equation for z. For a symmetric 2-tensor B we write $\det_g B = \det(B_{ij})/\det(g_{ij})$, and $\nabla_g^2 z$ is the Hessian of z for the metric g.

Lemma 2.2 (Darboux equation)**.** *Let z be a smooth function on a surface with metric g, and assume $\sigma := |dz|_g^2 < 1$. Then $\bar{g} = g - dz \otimes dz$ is positive definite, and its curvature is*

$$K_{\bar{g}} = \frac{K_g(1-\sigma) - \det_g \nabla_g^2 z}{(1-\sigma)^2}. \tag{2.2}$$

In particular $\bar{g}$ is flat if and only if

$$\det_g \nabla_g^2 z = K_g\,(1 - |dz|_g^2). \tag{2.3}$$

Proof. In a g-orthonormal coframe (e^1, e^2) with $dz = \sqrt{\sigma}\, e^1$ we have $\bar{g} = (1-\sigma)\, e^1 \otimes e^1 + e^2 \otimes e^2$. This is positive definite, and $|dz|_{\bar{g}}^2 = \sigma/(1-\sigma)$.

Consider the product $(\Sigma, \bar{g}) \times \mathbb{R}$ and in it the graph $x \mapsto (x, z(x))$. The metric induced on the graph is $\bar{g} + dz \otimes dz = g$. Its upward unit normal has vertical component $(1 + |dz|_{\bar{g}}^2)^{-1/2} = \sqrt{1-\sigma}$. The vertical coordinate is a function on the product with vanishing Hessian, and its restriction to the graph is z; hence the intrinsic Hessian of z is the second fundamental form II of the graph times the vertical component of the normal,

$$\nabla_g^2 z = \sqrt{1-\sigma}\ \mathrm{II}.$$

The sectional curvature of the product on the tangent plane of the graph is $K_{\bar{g}}$ times the squared area distortion of the horizontal projection, which is $1 - \sigma$. The Gauss equation for the graph therefore reads

$$K_g = (1-\sigma) K_{\bar{g}} + \det_g \mathrm{II} = (1-\sigma) K_{\bar{g}} + (1-\sigma)^{-1} \det_g \nabla_g^2 z,$$

and solving for $K_{\bar{g}}$ gives (2.2). □

Equation (2.3) is thus exactly the compatibility condition for the two horizontal coordinates once the vertical coordinate has been chosen. It guarantees local developments. The global reconstruction, including injectivity, is the subject of the next proposition.

2.3 Development onto a round disk

A flat metric on an annulus may have nontrivial holonomy around the hole. Filling the inner collar by a flat disk removes the holonomy, and the half-turn symmetry of z forces the inner disk to stay centred.

Proposition 2.3 (Planar reconstruction)**.** *Suppose that w vanishes on open collars of both ends of (r_-, r_+), that z equals a constant c_L on the inner collar and a constant c_R on the outer one, and that $z(r, \theta + \pi) = z(r, \theta)$. Suppose that $\bar{g} = g - dz \otimes dz$ is positive definite and flat. Extend w by 0 and z by c_L to $r \leq r_-$. Then there is an orientation-preserving diffeomorphism $P : D_{r_+} \to D_{r_+}$ such that*

$$P^* g_{\mathbb{R}^2} = \bar{g}, \qquad P = \mathrm{id} \text{ on the outer collar}, \qquad P = R_\gamma \text{ on the inner disk } D_{r_-}$$

for some angle γ. Consequently $F = (P, z) : D_{r_+} \to \mathbb{R}^3$ is an isometric embedding of (D_{r_+}, g), and its image is a graph over D_{r_+}.

Proof. Step 1: a normalized development. On the inner disk and on the outer collar we have $w = 0$ and $dz = 0$, so there $\bar{g} = g = dr^2 + r^2 d\theta^2$ is the Euclidean metric in the coordinate x. In particular $\bar{g}$ is a smooth flat metric on the whole disk D_{r_+}, including its centre. Choose a global oriented $\bar{g}$-orthonormal coframe η_1, η_2 which equals dx_1, dx_2 on the outer collar, and let α be its connection form: $d\eta_1 = \alpha \wedge \eta_2$, $d\eta_2 = -\alpha \wedge \eta_1$. Flatness means $d\alpha = 0$, and since the disk is simply connected, $\alpha = d\psi$ with $\psi = 0$ on the outer collar. The complex form $e^{i\psi}(\eta_1 + i\eta_2)$ is then closed. Its primitive P, normalized to be the identity on the outer collar, is an orientation-preserving local isometry from $(D_{r_+}, \bar{g})$ to the plane.

Step 2: the inner disk is rotated about its centre. Two orientation-preserving local isometries from a connected flat surface to the plane differ by a fixed Euclidean motion; this follows by differentiating their relative rotation. Let $I(x) = -x$ be the half-turn. It preserves $\bar{g}$ because g is rotationally symmetric and $z \circ I = z$. Both P and $-P \circ I$ are orientation-preserving local isometries equal to the identity on the outer collar, so they coincide: P is odd. On the inner disk $\bar{g}$ is Euclidean, hence P is a Euclidean motion there, and an odd Euclidean motion is a rotation R_γ about the origin.

Step 3: P is a diffeomorphism of the disk. Extend P by the identity outside D_{r_+}. The extension is a local diffeomorphism of $\mathbb{R}^2$ which is the identity outside a compact set; it is proper, hence a covering map, and it has one sheet because it is injective near infinity. So it is a diffeomorphism of $\mathbb{R}^2$, and since it fixes the exterior of the disk it maps D_{r_+} onto itself.

Step 4: conclusion. The map $F = (P, z)$ is injective because P is, and its induced metric is $P^* g_{\mathbb{R}^2} + dz \otimes dz = \bar{g} + dz \otimes dz = g$. Its image is the graph of $z \circ P^{-1}$ over D_{r_+}. □

2.4 A stationary cap which distinguishes shapes

The graph of Proposition 2.3 is horizontal near its boundary, so it can be closed by a cap. A rotationally symmetric cap would hide the relative rotation: the closed surfaces obtained for different τ would be congruent. We therefore put on the cap a marker which is recognized by every congruence.

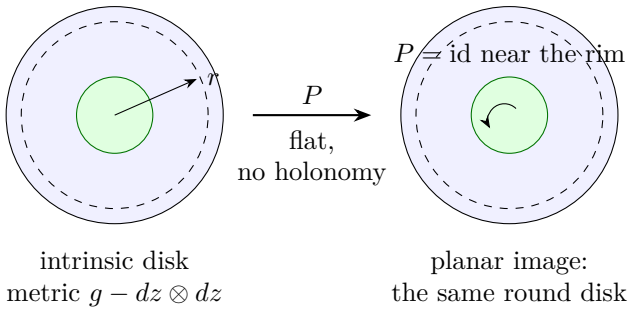


Figure 4: The development P. Filling the inner collar removes the holonomy of the annulus; the normalization fixes the outer collar; the half-turn symmetry keeps the inner disk centred, so that P only rotates it. Properness upgrades the local development to a diffeomorphism. The sketch records these constraints, not the pointwise distortion of P.

Lemma 2.4 (The marked cap)**.** *Let S_0 be a smooth graph over D_{r_+} which is horizontal near its boundary. Then S_0 extends to a smooth embedded sphere $S_0 \cup S_1$. The complementary cap S_1 can be chosen so that, for any given constant $K_{\max}$, it contains a point p with*

$$K(p) > K_{\max}, \qquad K(p) > K(p') \ \text{ for every other point } p' \text{ of } S_1,$$

and with distinct principal curvatures at p.

We shall take $K_{\max}$ to be the maximum of the Gaussian curvature of S_0. Then p is the unique point of the closed surface at which the Gaussian curvature is largest.

Proof. Step 1: the unmarked cap. Translate so that the boundary height is zero, and extend the horizontal collar to a radius $R > r_+$. Choose a smooth nondecreasing function $\chi : [0,1] \to [0,1]$ which is 0 near 0, is 1 near 1, and satisfies $\chi(1-t) = 1 - \chi(t)$. For $L > 0$ consider the unit-speed meridian curve $s \mapsto (r(s), z(s))$, $0 \le s \le L$, which starts at $(R, 0)$ and satisfies

$$r'(s) = \cos(\pi\chi(s/L)), \qquad z'(s) = -\sin(\pi\chi(s/L)).$$

Its tangent turns from the outward horizontal direction to the inward one. By the symmetry of χ the curve stays at radius at least R and returns to radius R, and its height decreases by a positive constant times L. Take L so large that the curve ends below the minimum of the graph S_0, and complete it by the horizontal segment from its endpoint to the axis. Rotating this meridian gives a smooth embedded surface which meets S_0 only along the attaching collar.

Step 2: the marker. Place a bump in the horizontal annulus $r_+ < r < R$. In Cartesian coordinates (x, y) centred at a point of that annulus, choose constants $A, B > 0$ with $A \neq B$, and $k, \rho > 0$; put $\xi = Ax^2 + By^2$ and

$$u(x,y) = \frac{k}{2} \int_{\xi}^{\rho^2} e^{-s/(\rho^2 - s)}\, ds \quad (\xi < \rho^2), \qquad u = 0 \quad (\xi \ge \rho^2).$$

This function is smooth. With $\Xi = \xi/(\rho^2 - \xi)$ one computes $u_x = -kAx\, e^{-\Xi}$ and $u_y = -kBy\, e^{-\Xi}$, and the Gaussian curvature of the graph of u is

$$K = \frac{ABk^2 e^{-2\Xi}\,(1 - 2\xi\,\Xi')}{\left[1 + k^2 e^{-2\Xi}(A^2x^2 + B^2y^2)\right]^2}, \qquad \Xi' = \frac{d\Xi}{d\xi} > 0. \tag{2.4}$$

At the centre $K = ABk^2$. At every other point of the support the numerator is either nonpositive or strictly smaller than ABk^2, and the denominator is at least one. So the centre p is the unique maximum of K on the bump, with value ABk^2. First choose k so that ABk^2 exceeds both $K_{\max}$ and the curvature of the unmarked cap. Then choose ρ so small that the support $\{\xi < \rho^2\}$ fits inside the horizontal annulus; the bump has height at most $k\rho^2/2$ and does not meet the rest of the surface. At p the gradient of u vanishes and its Hessian is $\operatorname{diag}(-kA, -kB)$, so the principal curvatures are $-kA$ and $-kB$ for the upward normal. They are distinct. □

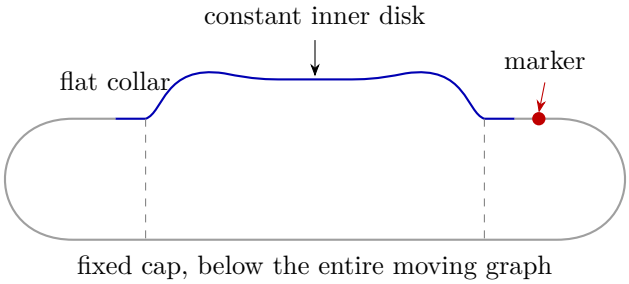


Figure 5: A meridional sketch of the closed surface. The blue graph depends on the angle; all its rotated copies stay inside the central cylinder and above the bottom of the cap. The marker lies in the stationary horizontal extension of the collar.

2.5 The annulus implies the theorem

Theorem 2.5 (Geometric closure)**.** *Let g be a metric of the form (2.1) and let z be a smooth function on the annulus. Suppose that*

(i) *$w = 0$ on open collars of both ends, and z is constant on each of these collars;*

(ii) *$z(r, \theta + \pi) = z(r, \theta)$, and z is not independent of θ;*

(iii) *$g - dz \otimes dz$ is positive definite and flat.*

Then all the conclusions of Theorem A hold.

Proof. Step 1: the family. Let P and $F = (P, z)$ be given by Proposition 2.3, and attach to the graph $F(D_{r_+})$ the marked cap of Lemma 2.4, with $K_{\max}$ the maximum of the Gaussian curvature of $F(D_{r_+})$. Let Σ be the resulting sphere, with its induced metric g_Σ. On the disk put

$$F_\tau = R_{-2\pi\tau} \circ F \circ \varrho_{2\pi\tau}, \qquad \varrho_\vartheta(r, \theta) = (r, \theta + \vartheta),$$

and let F_τ be the inclusion on the cap. The intrinsic rotation ϱ_ϑ preserves g, so each F_τ is isometric on the disk. On the outer collar $F(x) = (x, c_R)$, and on the inner disk $F(x) = (R_\gamma x, c_L)$; in both cases the two rotations cancel and $F_\tau = F$. Hence F_τ agrees with the inclusion near the rim and is smooth on Σ, jointly in the point and in τ, and $F_{\tau+1} = F_\tau$. The image $F_\tau(D_{r_+}) = R_{-2\pi\tau} F(D_{r_+})$ is a graph over the same round disk with the same range of heights, so it meets the cap only along the collar: F_τ is an embedding. The inner disk and the cap are fixed pointwise.

Step 2: the disk images are not all equal. Suppose $R_\vartheta F(D_{r_+}) = F(D_{r_+})$ for every ϑ. Then $F^{-1} \circ R_\vartheta \circ F$ is an isometry of (D_{r_+}, g), and on the outer collar it equals ϱ_ϑ. An isometry of a connected Riemannian manifold is determined by its value and differential at one point, so

$F^{-1}R_\vartheta F = \varrho_\vartheta$ on the whole disk. Comparing the vertical components of $R_\vartheta F = F\varrho_\vartheta$ gives $z(r, \theta + \vartheta) = z(r, \theta)$ for all ϑ, contrary to assumption (ii).

Step 3: the closed images are not all congruent. The Gaussian curvature of $F_\tau(\Sigma)$ is determined by g_Σ, so by the choice of the marker the point p is, for every τ, the unique point of $F_\tau(\Sigma)$ of largest Gaussian curvature; it lies on the cap and is the same point of $\mathbb{R}^3$ for all τ. Let Φ be a Euclidean isometry mapping $F_0(\Sigma)$ onto $F_\tau(\Sigma)$. It preserves the Gaussian curvature, so $\Phi(p) = p$. It maps the bounded component of the complement to the bounded component, so it preserves the outward normal at p, and then it preserves each of the two principal directions at p, which are well defined because the principal curvatures are distinct and which are the same for all τ because the cap is fixed. There are at most four such isometries. Thus, if all the images were congruent, the family $\tau \mapsto F_\tau(\Sigma)$ would be a continuous family of compact sets taking values in a finite set, hence constant. The part of $F_\tau(\Sigma)$ inside the closed cylinder over D_{r_+} and above the bottom of the cap is $F_\tau(D_{r_+})$, so the disk images would all be equal, which Step 2 excludes.

Step 4: the invariants. The area is constant because the metric is. The moving part $F_\tau(D_{r_+})$ is a rotated copy of $F(D_{r_+})$ and the rest of the surface is stationary, so the integral of the oriented mean curvature is constant. Finally let Z_τ be the height of $F_\tau(D_{r_+})$ as a graph over D_{r_+}, and let c_B be the height of the bottom of the cap. The part of the enclosed volume inside the cylinder over D_{r_+} is

$$\int_{D_{r_+}} (Z_\tau(y) - c_B)\, dy = \int_{D_{r_+}} (Z_0(R_{2\pi\tau} y) - c_B)\, dy = \int_{D_{r_+}} (Z_0(y) - c_B)\, dy,$$

and the remaining volume is bounded by the stationary part of the surface. □

Proof that Theorem B implies Theorem A. Let w and z be as in Theorem B. Extend w by zero and z by its two limiting constants to a slightly larger annulus. The extended functions are smooth, and on the added collars $g - dz \otimes dz = g$ is Euclidean, hence positive and flat. All the hypotheses of Theorem 2.5 hold. □

What remains is therefore precise: construct one nonradial height on an annulus with a positive flat remainder metric and with constant extensions at both ends. The family of embeddings then follows without any further existence argument.

3 The height equation

The geometric reduction leaves two things to choose: the rotational metric, that is the slope w, and the nonradial height z. This section derives the equation that relates them and extracts from it the information that guides both choices.

- Section 3.1 writes the curvature of the remainder metric as the exterior derivative of a connection form. This gives the equation, its linearization in divergence form, and a conservation law.
- Section 3.2 shows that a flat end at which w is monotone carries no infinitesimal bending. This is why the profile must oscillate.
- Section 3.3 computes, in a finite model, the two different resonance sets mentioned in the introduction.
- Section 3.4 normalizes the equation in the small parameter, and Section 3.5 introduces the first of the two residual corrections.

3.1 The curvature residual and its derivative

Let h be a radial function with $h' = w$, so that g in (2.1) is the metric of its graph and $E = 1 + w^2$. For a height $z(r, \theta)$ we use the abbreviations

$$k = z_r, \qquad p = \frac{z_\theta}{r}, \qquad D = E(1 - p^2) - k^2, \qquad Q = k + Ep_\theta. \tag{3.1}$$

In the coordinates (r, θ) the remainder metric $\bar{g} = g - dz \otimes dz$ has determinant $r^2 D$, and $1 - |dz|_g^2 = D/E$. Hence

$$\bar{g} \text{ is positive definite } \iff D > 0 \iff |dz|_g < 1.$$

We call D the *discriminant*. When $D > 0$, an oriented $\bar{g}$-orthonormal coframe is

$$\eta_1 = \frac{\sqrt{D}}{\sqrt{1 - p^2}}\, dr, \qquad \eta_2 = r\sqrt{1 - p^2}\, d\theta - \frac{kp}{\sqrt{1 - p^2}}\, dr.$$

For the radial height $z = h$ it reduces to the polar coframe dr, $r\, d\theta$ of the plane, whose connection form is $d\theta$.

All adjoints of scalar operators in this paper are taken with respect to $dr\, d\theta$, and source terms are densities for this measure. The measure will be written out whenever coordinates are changed.

Proposition 3.1 (Connection form, residual, linearization)**.** *Assume $D > 0$.*

(i) *The connection form α of the coframe (η_1, η_2), defined by $d\eta_1 = \alpha \wedge \eta_2$ and $d\eta_2 = -\alpha \wedge \eta_1$, is*

$$\alpha = D^{-1/2} \left(d\theta + \frac{k}{1 - p^2}\, dp \right). \tag{3.2}$$

(ii) *Define the* curvature residual $\mathcal{N}_h(z) = -\sqrt{D}\, K_{\bar{g}}$. *Then $d\alpha = r\, \mathcal{N}_h(z)\, dr \wedge d\theta$ and*

$$r\, \mathcal{N}_h(z) = D^{-3/2} \left[Qk_r - Er\, p_r^2 - \frac{E'}{2} (1 - p^2 + kp_\theta) \right]. \tag{3.3}$$

In particular $\bar{g}$ is flat if and only if

$$Qk_r - Er\, p_r^2 - \frac{E'}{2} (1 - p^2 + kp_\theta) = 0. \tag{3.4}$$

(iii) *For fixed w, the derivative of the residual in the direction η is in divergence form,*

$$r\, D_z \mathcal{N}_h(z)\, \eta = \partial_r(a\eta_r + b\eta_\theta) + \partial_\theta(b\eta_r + c\eta_\theta), \tag{3.5}$$

with coefficients

$$a = \frac{Q}{D^{3/2}}, \qquad b = -\frac{Ep_r}{D^{3/2}}, \qquad c = \frac{Ek_r - E'k/2}{r\, D^{3/2}}. \tag{3.6}$$

(iv) *The determinant of the coefficient matrix is*

$$ac - b^2 = \frac{K_0}{D^2} + \frac{E}{D^{3/2}}\, \mathcal{N}_h(z), \qquad K_0 = \frac{ww'}{r}. \tag{3.7}$$

Proof. Equality of mixed derivatives of z gives the compatibility relation

$$r\,p_r = k_\theta - p. \tag{3.8}$$

Step 1: the connection form. Write $\alpha = \alpha_r\,dr + \alpha_\theta\,d\theta$. The second structure equation determines α_θ:

$$\alpha_\theta\,\frac{\sqrt{D}}{\sqrt{1-p^2}} = \partial_r(r\sqrt{1-p^2}) + \partial_\theta\left(\frac{kp}{\sqrt{1-p^2}}\right) = \frac{1}{\sqrt{1-p^2}} + \frac{kp_\theta}{(1-p^2)^{3/2}},$$

where (3.8) was used. This is the $d\theta$-component of (3.2). The first structure equation then reads $\alpha_r\,r\sqrt{1-p^2} + \alpha_\theta\,kp/\sqrt{1-p^2} = -\partial_\theta(\sqrt{D}/\sqrt{1-p^2})$, and with $D_\theta = -2Epp_\theta - 2kk_\theta$ and (3.8) it is satisfied by $\alpha_r = D^{-1/2}kp_r/(1-p^2)$.

Step 2: the residual. For an orthonormal coframe $d\alpha = -K_{\bar g}\,\eta_1\wedge\eta_2$, and here $\eta_1\wedge\eta_2 = r\sqrt{D}\,dr\wedge d\theta$; this is the relation between $d\alpha$ and $\mathcal{N}_h$. To compute $d\alpha$, note that

$$\alpha = D^{-1/2}\,d\theta + B_1\,dp, \qquad B_1 = \frac{k}{(1-p^2)\sqrt{D}},$$

and that $D_r = E'(1-p^2) - 2Epp_r - 2kk_r$. The coefficient of $dr\wedge d\theta$ in $D^{3/2}\,d\alpha$ is

$$-\tfrac12 D_r + \frac{D}{1-p^2}(k_rp_\theta - k_\theta p_r) - \frac{k}{2(1-p^2)}(D_rp_\theta - D_\theta p_r).$$

Since $D_rp_\theta - D_\theta p_r = E'(1-p^2)p_\theta - 2k(k_rp_\theta - k_\theta p_r)$ and $D + k^2 = E(1-p^2)$, this equals

$$-\frac{E'}{2}(1 - p^2 + kp_\theta) + k_r(k + Ep_\theta) - Ep_r(k_\theta - p),$$

which is the bracket in (3.3) by (3.8). As a check, the Christoffel symbols $\Gamma^r_{rr} = E'/2E$, $\Gamma^r_{\theta\theta} = -r/E$, $\Gamma^\theta_{r\theta} = 1/r$ and the curvature

$$K_g = \frac{E'}{2rE^2} \tag{3.9}$$

of g give the same expression from the invariant form (2.3) of the Darboux equation.

Step 3: the variation. Vary z by η, so that $\delta k = \eta_r$ and $\delta p = \eta_\theta/r$. Then

$$\delta\alpha = \delta(D^{-1/2})\,d\theta + \delta B_1\,dp - \delta p\,dB_1 + d(B_1\,\delta p), \qquad \delta(D^{-1/2}) = D^{-3/2}\,(Ep\,\delta p + k\,\delta k).$$

Regard B_1 as a function of (k,p,E). Using $D + k^2 = E(1-p^2)$ one finds $\partial_k B_1 = ED^{-3/2}$ and $\partial_E B_1 = -k/(2D^{3/2})$. In $\delta B_1\,dp - \delta p\,dB_1$ the terms with $\partial_p B_1$ cancel, and since E depends on r only,

$$\delta B_1\,dp - \delta p\,dB_1 = ED^{-3/2}\,(\delta k\,dp - \delta p\,dk) + \frac{kE'}{2D^{3/2}}\,\delta p\,dr.$$

Collect the coefficients of $d\theta$ and of dr. By (3.8) and $Q = k + Ep_\theta$,

$$\begin{aligned} d\theta:&\quad D^{-3/2}\big[(k+Ep_\theta)\,\delta k + E\,(p-k_\theta)\,\delta p\big] = D^{-3/2}\big[Q\,\eta_r - Ep_r\,\eta_\theta\big],\\ dr:&\quad D^{-3/2}\Big[Ep_r\,\delta k - (Ek_r - \tfrac12 E'k)\,\delta p\Big] = D^{-3/2}\Big[Ep_r\,\eta_r - \frac{Ek_r - \frac12 E'k}{r}\,\eta_\theta\Big]. \end{aligned}$$

With the coefficients (3.6) this is

$$\delta\alpha = d(B_1\,\delta p) + (a\eta_r + b\eta_\theta)\,d\theta - (b\eta_r + c\eta_\theta)\,dr. \tag{3.10}$$

Taking the exterior derivative removes the exact term and proves (3.5). The formula also identifies $a\eta_r + b\eta_\theta$ as the flux of the linearized equation through a circle $r = \text{const}$; we call it the *normal flux*.

Step 4: the determinant. Let N be the bracket in (3.3). From $kQ + D = E(1 - p^2 + kp_\theta)$ one finds

$$ac - b^2 = \frac{E}{rD^3}\left(N + \frac{E'D}{2E}\right),$$

and $N = rD^{3/2}\mathcal{N}_h(z)$, $E' = 2ww'$ give (3.7). □

The linearization at the radial height. At $z = h$ we have $k = w$, $p = 0$, $D = 1$, and the coefficients are $a = w$, $b = 0$, $c = w'/r$. The linearized operator is therefore

$$\mathscr{L}_w u = (wu_r)_r + \frac{w'}{r}\,u_{\theta\theta}. \tag{3.11}$$

For $w > 0$ it is elliptic where $w' > 0$ and hyperbolic where $w' < 0$; by (3.9) these are the regions of positive and negative Gaussian curvature of g. A simple zero of w' is called a *turning circle*. Note that it is the derivative of the slope that changes sign there, not the slope itself: w stays positive.

3.2 Why the profile must oscillate at a flat end

It is tempting to take for w a simple bump, to solve $\mathscr{L}_w u = 0$ in the middle of the annulus, and to flatten the ends afterwards. The following observation shows that a nonzero infinitesimal height cannot survive this. Separating variables, $u = f(r)\cos(j\theta)$, the equation $\mathscr{L}_w u = 0$ becomes $(wf')' - j^2 w' f/r = 0$. Rigidity statements for a surface of revolution one of whose parallels is fixed go back to Minagawa and Radó [MR53]; for generalizations see [Sab76]. The proposition below is a statement of this type at a flat end, where the equation degenerates, and we prove it directly.

Proposition 3.2 (The monotone-end obstruction)**.** *Let $w > 0$ be C^1 on (r_-, r_+) with $w(r_-) = 0$ and $w' \geq 0$ near r_-. Let $j \geq 1$ and let f be a solution of*

$$(wf')' - j^2\,\frac{w'}{r}\,f = 0$$

which extends continuously to r_- with

$$f(r_-) = 0, \qquad \lim_{r\downarrow r_-} w(r)f'(r) = 0.$$

Then $f = 0$. The same holds at the right endpoint if $w(r_+) = 0$ and $w' \leq 0$ near r_+.

Proof. Integrate the equation from r_- and put $f^*(r) = \max_{r_-\leq t\leq r}|f(t)|$. Since $w' \geq 0$ and $\int_{r_-}^{r} w' = w(r)$,

$$|f'(r)| = \frac{j^2}{w(r)}\left|\int_{r_-}^{r}\frac{w'(t)}{t}\,f(t)\,dt\right| \leq \frac{j^2}{r_-}\,f^*(r).$$

Integrating once more gives $f^*(r) \leq (j^2/r_-)\int_{r_-}^{r} f^*$, so $f^* = 0$ near r_- by Gronwall's inequality, and $f = 0$ everywhere by uniqueness for the ordinary differential equation in the interior. At the right endpoint integrate backwards and use $\int_{r}^{r_+}|w'| = w(r)$. The argument applies to complex-valued f as well. □

A nonconstant angular mode of a height with constant extensions satisfies both boundary conditions of the proposition. Consequently a positive profile with only finitely many changes of monotonicity cannot carry a nonzero angular mode which is flat at the endpoints. Within our construction the infinitely many turning circles are therefore forced. Their spacings and the amplitude of w between them will decrease so fast that w nevertheless extends smoothly by zero (Figure 6).

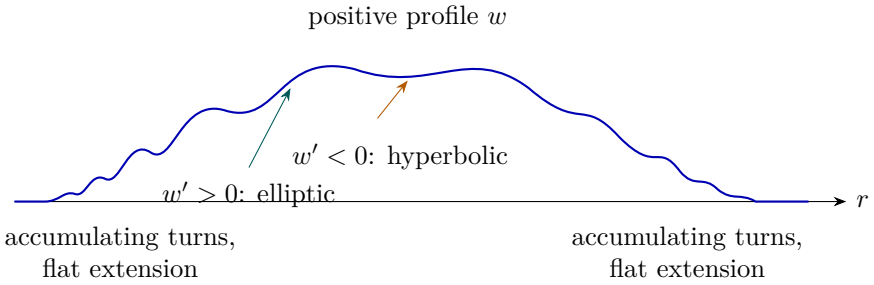


Figure 6: The shape of an admissible profile. Only a few representative turns are drawn, and not to scale. Where w rises the linearized equation is elliptic, where it falls it is hyperbolic. Every derivative of w tends to zero at the two accumulation points.

3.3 Two resonance conditions in a finite model

Before constructing the profile we examine a model with exactly two turning circles, in which everything can be computed. It is a guide to the construction of Part II and is not used in any proof. Its purpose is to show that an inverse of $\mathscr{L}_w$ involves two different sets of small denominators.

The model. Put $\epsilon = \sqrt{2}/100$ and $\rho(X) = \epsilon^2 X(1-X)$, and let $r(X)$ solve, on $-1 \le X \le 2$,

$$r_{XX} = -\rho\, r, \qquad r(0) = 1, \qquad r_X(0) = \frac{1}{10}. \tag{3.12}$$

On this interval $|\rho| \le 2\epsilon^2$, so the integral form of (3.12) bounds $R = \max|r|$ by $R \le 1.2 + 4\epsilon^2 R$. Hence $R < 1.21$, $r > 0.89$ and $|r_X - \frac{1}{10}| \le 4\epsilon^2 R < 0.001$. In particular r is increasing, and we may define a positive profile by

$$w(r(X)) = r_X(X).$$

Then $w' = dw/dr = -\rho\, r/r_X$, so w has exactly two turning circles, at $X = 0$ and $X = 1$, and the equation is hyperbolic between them. For an angular mode $u = U(X)\cos(j\theta)$ the operator (3.11) becomes, with $J = j\epsilon$,

$$L_j U = \frac{1}{r_X}\bigl(U_{XX} + J^2 X(1-X)\, U\bigr). \tag{3.13}$$

First denominator: characteristics. On $0 < X < 1$ the characteristics of $\mathscr{L}_w$ have slopes $d\theta/dX = \pm\epsilon\sqrt{X(1-X)}$. Between the two turning circles each of them turns by the *travel angle*

$$T = \epsilon \int_0^1 \sqrt{X(1-X)}\, dX = \frac{\epsilon\pi}{8}. \tag{3.14}$$

The two families of characteristics translate a function on the circle by $+T$ and by $-T$, and reconciling them requires inverting the difference of the two translations,

$$\mathcal{D}_T f(\theta) = f(\theta + T) - f(\theta - T), \qquad \mathcal{D}_T \sin(j\theta) = 2\sin(jT)\cos(j\theta). \tag{3.15}$$

So the characteristic problem costs the divisor $\sin(jT)$. Here $jT/\pi = j\sqrt{2}/800$ is a quadratic irrational multiple of j. If m is the integer nearest to $j\sqrt{2}/800$, the integer $2j^2 - 640000\,m^2$ is nonzero; factoring it, and using $j\sqrt{2}/800 + |m| \le j$ and $|\sin(\pi x)| \ge 2\,\mathrm{dist}(x, \mathbb{Z})$, we obtain

$$|\sin(jT)| \ge \frac{1}{320000\,j} \qquad (j \ge 1). \tag{3.16}$$

The inverse of $\mathcal{D}_T$ exists on functions of mean zero and loses one angular derivative. No choice of T avoids a loss altogether. If T/π is rational, $\sin(jT) = 0$ for infinitely many even j. If it is irrational, apply the pigeonhole principle to the fractional parts of $2kT/\pi$, $0 \le k \le N$: two of them differ by at most $1/N$, which gives an even $j \le 2N$ with $|\sin(jT)| \le \pi/N$. So the divisors are never bounded below, and a bound like (3.16) is the best one can ask for.

Second denominator: matching. Now consider the boundary value problem for (3.13). The substitution $x = \sqrt{J}\,(X - \frac{1}{2})$ turns the homogeneous equation into the harmonic oscillator equation

$$(-\partial_x^2 + x^2)\,U = \frac{J}{4}\,U. \tag{3.17}$$

As boundary condition at each end of the interval we require that (U, U_X) be proportional to the data of the solution of (3.17) on the whole line which decays in that direction. These are legitimate boundary conditions for (3.13); the radius need not be extended beyond $[-1, 2]$. The homogeneous problem then has a nonzero solution exactly when one solution decays in both directions, that is, when $J/4$ is an eigenvalue of the oscillator. The eigenvalues are the odd integers $2m + 1$ [Tes09, Section 8.3], so the resonances are

$$\frac{J}{4} = 2m + 1, \qquad \text{equivalently} \qquad jT = (m + \tfrac{1}{2})\pi. \tag{3.18}$$

Away from them the inhomogeneous problem is solved by the resolvent of the oscillator: multiply the source by $-r_X/J$, extend it by zero outside $|x| \le 3\sqrt{J}/2$, apply the resolvent on the whole line, and restrict. Thus the boundary value problem has precisely the denominators (3.18).

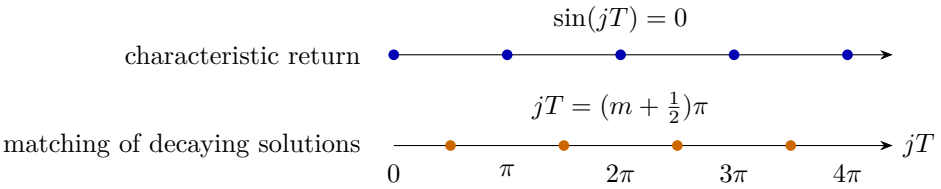


Figure 7: The two resonance sets of the finite model, as functions of the phase jT. Avoiding one row does not avoid the other. In the infinite construction both must be kept at a controlled distance.

Remark on the half-integer shift. The shift by $\pi/2$ between the two rows of Figure 7 has a local explanation. Near $X = 0$ put $y = J^{2/3}X$; then

$$U_{yy} + yU = J^{-2/3}y^2\,U.$$

On a fixed y-interval the solutions are, up to an error $O(J^{-2/3})$ in value and first derivative, combinations of the Airy functions $\mathrm{Ai}(-y)$ and $\mathrm{Bi}(-y)$; this follows by variation of constants and Gronwall's inequality. The asymptotics of the Airy functions carry a phase shift $\pi/4$ at each turning point [Nat, 9.7.5–9.7.12], and two turning points give $\pi/2$. This is only a heuristic; (3.18) was proved by the exact oscillator computation.

What the model teaches. In the actual construction the travel angle of every hyperbolic annulus will be a quadratic irrational multiple of π, which gives a bound of the type (3.16); and the matching problem, now between infinitely many pieces, will be controlled by spectral gaps of the individual pieces together with a choice of their relative heights.

3.4 Normalization and the conservation law

The nonlinear unknown is $z = h + qu$ with a small parameter q. Since $\mathcal{N}_h(h) = 0$, the residual of z is divisible by q, and we define the *normalized residual*

$$\mathcal{F}_q(w,u) = \begin{cases} r\,\mathcal{N}_h(h+qu)/q, & q \neq 0, \\ \mathscr{L}_w u, & q = 0. \end{cases} \tag{3.19}$$

It is smooth in q through $q = 0$, because

$$\mathcal{F}_q(w,u) = \int_0^1 r\, D_z\mathcal{N}_h(h+tqu)\, u\, dt. \tag{3.20}$$

It depends on h only through w, and it does not change when a constant is added to u.

Symmetry class. From now on all heights are even in θ and invariant under $\theta \mapsto \theta + \pi$. This class is preserved by products and by $\mathcal{F}_q$. Its Fourier modes are $\cos(j\theta)$ with $j \geq 0$ even.

Lemma 3.3 (Conservation law)**.** *Let w be flat at both endpoints and let u have constant extensions at both endpoints, with $D > 0$ for $z = h + qu$. Then*

$$\int_{r_-}^{r_+} \langle \mathcal{F}_q(w,u) \rangle\, dr = 0. \tag{3.21}$$

The same holds for every derivative of $\mathcal{F}_q(w,u)$ with respect to u, q or w in directions preserving these conditions.

Proof. By Proposition 3.1(ii) and Stokes' theorem, $\iint r\mathcal{N}_h(z)\, dr\, d\theta$ is the difference of the periods $\oint \alpha$ over the two boundary circles. At a boundary circle $w = 0$ and $dz = 0$ to infinite order, so $D = 1$ and $\alpha = d\theta$ there: both periods equal 2π. Differentiating the identity gives the last statement. □

The connection period. The angular mean of the equation is most transparent at the level of the connection form. For a height $z = h + U$ define

$$\mathfrak{F}_w(U) = \langle \alpha_\theta \rangle - 1 = \left\langle \frac{1 + kp_\theta/(1-p^2)}{\sqrt{D}} \right\rangle - 1, \tag{3.22}$$

the normalized period of α over the circle of radius r. Since $\partial_r \alpha_\theta - \partial_\theta \alpha_r = r\mathcal{N}_h$, its radial derivative is $r\langle \mathcal{N}_h(h+U) \rangle$, and it vanishes at the endpoints. So the mean of the equation $\mathcal{N}_h(h+U) = 0$ is equivalent to

$$\mathfrak{F}_w(U) = 0 \qquad \text{for all } r.$$

Formula (3.10) gives the derivative of $\mathfrak{F}_w$ with respect to a radial variation $\bar{U}(r)$ of the height, and its full derivative at $U = 0$:

$$\partial_{\bar{U}'} \mathfrak{F}_w(U) = \langle a \rangle, \qquad D_U \mathfrak{F}_w(0)[U] = w\langle U_r \rangle. \tag{3.23}$$

The second formula says that, to first order, the period sees only the mean of the height. This cancellation is the reason why the mean of the solution will decay twice as fast as its oscillating part (Section 10).

3.5 The first residual correction

By (3.7), the type of the true linearization (3.5) is governed by $K_0/D^2 + ED^{-3/2}\mathcal{N}_h(z)$. At a solution the second term vanishes and the type changes exactly where w' does. Away from a solution the second term moves the turning circles, which is inconvenient: the circles would depend on the unknown. We remove that term by changing the tangential coefficient c only.

Lemma 3.4 (Fixing the type-change set)**.** *On the set of heights with $D \geq \frac{1}{2}$ and $Q \geq \frac{7}{8}w > 0$ put*

$$B = \frac{E}{rQ}, \qquad \bar{c} = c - B\,r\,\mathcal{N}_h(z), \qquad \bar{L} = \partial_r(a\partial_r + b\partial_\theta) + \partial_\theta(b\partial_r + \bar{c}\,\partial_\theta).$$

Then

$$a\bar{c} - b^2 = \frac{K_0}{D^2}, \qquad D_u\mathcal{F}_q(w,u)\,\eta = \bar{L}\eta + q\,\partial_\theta(B\,\mathcal{A}\,\eta_\theta), \qquad \mathcal{A} = \mathcal{F}_q(w,u), \tag{3.24}$$

where $\bar{L}$ is formed with the coefficients of $z = h + qu$. The operator $\bar{L}$ is symmetric for $dr\,d\theta$, annihilates constants, and has the same normal flux $a\eta_r + b\eta_\theta$ as the true derivative.

Proof. The determinant changes by $-aB\,r\mathcal{N}_h = -ED^{-3/2}\mathcal{N}_h$, which cancels the unwanted term of (3.7). The operator changes by $-\partial_\theta(B\,r\mathcal{N}_h\,\partial_\theta)$, and $r\mathcal{N}_h(h+qu) = q\mathcal{A}$. Only the tangential coefficient was modified, which gives the remaining assertions. □

Thus $\bar{L}$ is elliptic exactly where $w' > 0$ and hyperbolic exactly where $w' < 0$, whatever the height. The price is the term $q\,\partial_\theta(B\mathcal{A}\eta_\theta)$, which is the product of the residual $\mathcal{A}$ and the increment η. An error of this bilinear form is exactly what Hamilton's version of the implicit function theorem tolerates (Section 9). This is the first residual correction. The second, which fixes the characteristic return maps, is made in Section 7.

Characteristics. For later reference we record the characteristic directions of $\bar{L}$ on a component of $\{w' < 0\}$:

$$\frac{d\theta}{dr} = d \pm v, \qquad d = \frac{b}{a} = \frac{E(p - k_\theta)}{rQ}, \qquad v = \frac{\sqrt{b^2 - a\bar{c}}}{a} = \frac{\sqrt{-ww'/r}\,\sqrt{D}}{Q}. \tag{3.25}$$

We call v the *characteristic speed*. At the radial height $d = 0$ and $v = \sqrt{-w'/(rw)}$. At a turning circle v is a smooth positive factor times the square root of the distance into the hyperbolic side. The coordinates of Section 7 are built to respect this degeneracy; treating all intervals as uniformly elliptic or uniformly hyperbolic would lose the geometry of the problem.

Part II
The reference profile

4 The reference profile: requirements and solvable models

Part II constructs one radial profile w_* at which the linearized equation has a single flat infinitesimal bending and a controlled inverse. This section states precisely what is needed (Proposition 4.2), explains the layout of the construction and its vocabulary, and introduces the two explicitly solvable models from which w_* is assembled.

4.1 Angular modes and the design problem

Functions in our symmetry class have the Fourier expansion

$$u(r,\theta) = u_0(r) + \sum_{\substack{j\geq 2\\ j \text{ even}}} u_j(r)\, e_j(\theta), \qquad e_j(\theta) = \sqrt{2}\cos(j\theta),$$

and on the jth mode the operator (3.11) acts as the ordinary differential operator

$$L_j u = (wu')' - j^2\,\frac{w'}{r}\,u, \qquad \text{that is,} \qquad \mathscr{L}_w(u\,e_j) = (L_j u)\,e_j. \tag{4.1}$$

When the dependence on the profile matters we write $L_{j,w}$. We want a profile for which

- at $j = 2$ there is exactly one solution of $L_2\phi = 0$, up to multiples, which is flat at both endpoints. This is the seed of the nonradial height. The frequency two is the lowest one compatible with the half-turn symmetry required in Theorem 2.5;
- at every even $j \geq 4$ there is no such solution, and L_j has a controlled inverse. The nonlinear equation creates all higher even frequencies, so all of them must be invertible on the same profile.

Two freedoms are available, and separating them is the organizing idea of the construction. The *shape* of a piece of the profile determines the homogeneous solutions on that piece, hence its spectrum and its travel angles. Multiplying a piece by a positive constant, its *amplitude*, changes none of these, but it changes the scale of the flux wu' by which solutions on neighbouring pieces are compared. We use the first freedom to make each piece explicitly solvable, and the second to prevent resonances when the pieces are joined.

4.2 What the reference profile must provide

We state the result of Part II before starting the construction. It involves three scales attached to the nth piece of the profile near an endpoint.

Definition 4.1 (The endpoint scales)**.** The pieces of the profile approaching either endpoint are called *cells* and are indexed by integers $n \geq N$. We set

$$A_n \asymp e^{-n^2}, \qquad S_n = \tfrac{2}{3}\,n^3, \quad H_n = e^{-S_n}, \qquad q_n = n^{5/2}\log(e+n), \quad \mathcal{E}_n = e^{q_n}. \tag{4.2}$$

- A_n is the *length scale*: cell n has radial length comparable to A_n.
- H_n is the *decay scale*: the distinguished solution of $L_j u = 0$ at the endpoint decays like H_n across the cells.
- $\mathcal{E}_n$ is the *coefficient allowance*: reciprocals and derivatives of the coefficients on cell n are bounded by powers of $\mathcal{E}_n$.

Two expressions built on the third scale are used throughout the paper:

$$\begin{aligned} &\textit{coefficient power} \text{ on cell } n\text{:} && \text{a quantity bounded by } C\,\mathcal{E}_n^{C};\\ &\textit{coefficient bounds} \text{ on cell } n\text{:} && \text{all derivatives of order } s \text{ bounded by } C_s\,\mathcal{E}_n^{C_s}, \end{aligned} \tag{4.3}$$

with constants independent of n. On the finitely many pieces in the middle of the annulus the same words refer to fixed positive constants. For the purpose of estimates a cell includes fixed portions of its neighbours; such an enlargement changes the index by a bounded amount, which is immaterial.

The three scales are ordered: for all fixed $a, b, \varepsilon > 0$,

$$\mathcal{E}_n^a \, e^{bn^2} \le C_{a,b,\varepsilon} \, e^{\varepsilon S_n}, \qquad \text{because} \qquad n^2 = o(q_n), \quad q_n = o(S_n). \tag{4.4}$$

So lengths are much larger than coefficient allowances, which are much larger than the decay scale. Section 5 derives all three scales from the construction.

Weighted norms. Choose a smooth partition of unity ϑ_m on (r_-, r_+) with bounded overlap, subordinate to the cells of both tails and to the finite middle part; the index m records both the endpoint and the cell. Fix a smooth function χ_L of r equal to one near the left endpoint and to zero near the right one. For exponents $0 < \gamma < 1$ and $\mathsf{a} \ge 0$, the *source norm* of a pair (f, c), consisting of a function f and a scalar c, is

$$\|(f,c)\|_{Y_{\gamma,s}}^2 = |c|^2 + \sum_m \mathcal{E}_m^{2\mathsf{a}} H_m^{-2\gamma} \|\vartheta_m f\|_{L^2(dr) H^s(\mathbb{T})}^2. \tag{4.5}$$

The *output norm* is defined for pairs (u, ν) of a function and a scalar. Write $u = a_L \chi_L + v$ with a constant a_L; then the squared output norm is

$$|a_L|^2 + |\nu|^2 + \sum_m \mathcal{E}_m^{-2\mathsf{b}} H_m^{-2\gamma} \|\vartheta_m v\|_{L^2(dr\,d\theta)}^2.$$

A finite source norm means decay like H_m^γ towards the endpoints, with a powers of the coefficient allowance to spare. A finite output norm means the same decay for v, after the loss of $\mathsf{a} + \mathsf{b}$ such powers, while u itself tends to a_L at the left endpoint and to 0 at the right one.

Proposition 4.2 (The reference profile)**.** *There are radii $0 < r_- < r_+$ and a smooth function $w_* > 0$ on (r_-, r_+), flat at both endpoints, such that all zeros of w_*' in (r_-, r_+) are simple, with the following properties.*

(i) (Kernel.) *There is a nonzero function ϕ_*, flat at both endpoints, with $L_{2,w_*}\phi_* = 0$. Every solution of $L_{2,w_*}\phi = 0$ which is bounded near both endpoints is a multiple of ϕ_*, and for every even $j \ge 4$ the equation $L_{j,w_*}\phi = 0$ has no nonzero solution which is bounded near both endpoints.*

(ii) (Travel angles.) *For every component m of $\{w_*' < 0\}$ the travel angle $T_m = \int \sqrt{-w_*'/(r w_*)}\, dr$ over that component satisfies*

$$|\sin(kT_m)| \ge \frac{\gamma_m}{k} \quad (k \ge 1), \qquad \gamma_m^{-1} \le \mathcal{E}_m^C, \qquad T_m/\pi \text{ is a quadratic irrational.} \tag{4.6}$$

(iii) (Border.) *There is a smooth function χ_*, compactly supported in a component of $\{w_*' > 0\}$, such that*

$$g_* = L_{2,w_*}'[w_*\chi_*]\,\phi_*, \qquad \int g_*\phi_*\, dr = 1, \qquad \ell_* u = \int g_* u_2\, dr. \tag{4.7}$$

Here $L_{2,w}'[\omega]\phi = (\omega\phi')' - 4\omega'\phi/r$ is the derivative of $L_{2,w}\phi$ with respect to the profile w in the direction ω.

(iv) (Inverse.) *The bordered operator*

$$\mathbb{L}_{w_*}(u, \nu) = (\mathscr{L}_{w_*} u + \nu\, g_* e_2,\ \ell_* u) \tag{4.8}$$

has a two-sided inverse on the sources (f, c) satisfying the compatibility condition $\int \langle f \rangle\, dr = 0$, where u is required to tend to zero at the right endpoint and to an arbitrary constant at the left one. For every compact interval $[\gamma_-, \gamma_+] \subset (0, 1)$ there are exponents a, b such that this inverse is bounded from $Y_{\gamma,0}$ to the output norm, uniformly for $\gamma \in [\gamma_-, \gamma_+]$. With the same exponents, the part v of the output satisfies

$$\sum_m \mathcal{E}_m^{-2\mathsf{b}} H_m^{-2\gamma} \, \|\vartheta_m\, \partial_r v\|_{L^2(dr\,d\theta)}^2 \le C\, \|(f,c)\|_{Y_{\gamma,1}}^2. \tag{4.9}$$

(v) (Independence of the exponent.) *The profile, the function ϕ_* and the border are the same for all exponents γ.*

Some comments on the statement.

The border. Since L_2 has a one-dimensional kernel, and is formally symmetric, it has a one-dimensional cokernel as well. The standard remedy is to *border* the operator: one adds a scalar unknown ν multiplying a fixed function g_* (the *column*) and a scalar equation $\ell_* u = c$ (the *row*). The column removes the cokernel and the row removes the kernel; the inverse of the bordered operator is then described by the Schur complement formula (for the general method, also called the Grushin problem, see [SZ07]). Our column has a meaning: if the profile is deformed to $w_* e^{\nu \chi_*}$, then $g_* e_2$ is the derivative of $\mathscr{L}_w(\phi_* e_2)$ with respect to ν. So ν will be an actual parameter of the metric in the nonlinear problem.

The compatibility condition and the free constant. These concern the mode $j = 0$, where $L_0 u = (wu')'$. By Lemma 3.3 the sources of interest have $\int \langle f \rangle \, dr = 0$; and the difference of the two endpoint values of the height is not prescribed.

Endpoint-flat kernel. The precise meaning of this term, and of the two-sided inverse, is given by the weighted graph domain of Definition 6.11.

What is proved here, and what later. Proposition 4.2 is an estimate at angular order zero, together with its first radial derivative. Estimates of all orders are recovered from the differential equation in Section 8, after the weighted spaces of the nonlinear problem have been introduced. Keeping the two tasks apart means that the geometry of the profile does not depend on the order of regularity.

The proof of Proposition 4.2 occupies the rest of this section and the next two; it is completed in Section 6.8.

4.3 Layout of the construction and vocabulary

Figure 8 shows how w_* is assembled. The following terms are used throughout Parts II and III.

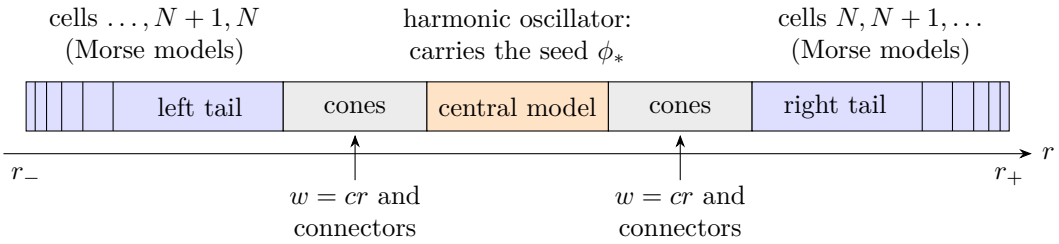


Figure 8: The layout of the reference profile w_* (schematic). The cells shrink and accumulate at both endpoints. Consecutive pieces are separated by intervals on which $w_*' > 0$.

Positive and negative components. A *positive component* is a component of $\{w' > 0\}$; the equation is elliptic there. A *negative component* is a component of $\{w' < 0\}$; the equation is hyperbolic there. They are separated by the turning circles, also called *roots*.

Travel angle. The travel angle of a negative component is $T = \int \sqrt{-w'/(rw)}\, dr$ over that component: the angle by which a characteristic of $\mathscr{L}_w$ turns while crossing it.

Model. A piece of the profile on which, in a suitable *model coordinate* X, every L_j is conjugate to an explicitly solvable Schrödinger operator (Section 4.4). There are two kinds: the *central model* and the Morse *cells*.

Cell. The nth cell consists, in the direction of increasing r, of a *short positive ramp*, one negative component, and a *long positive ramp* (Figure 10).

Join. A very short positive interval between two consecutive cells, on which w rises steeply from the amplitude of one cell to that of the next.

Cone. An interval on which $w = cr$. There $L_j u = c\,[(ru')' - j^2 u/r]$ has the solutions $r^{\pm j}$.

Connector. A positive interval joining the central model or a tail to a cone, built so that the seed solution is preserved exactly (Section 6.2).

Tail. The union of the cells and joins near one endpoint.

Buffer. A positive interval, viewed as a region across which the solutions of $L_j u = 0$ grow or decay exponentially; it decouples the pieces on its two sides.

Natural line, incoming line. At a point of a tail, and for a fixed frequency j, the *natural line* is the one-dimensional space of solutions of $L_j u = 0$ that decay towards the endpoint (Lemma 5.6). The *incoming line* is the one-dimensional space of solutions satisfying the condition imposed on the interior side. A homogeneous solution of the whole problem exists exactly when the two lines coincide.

Outward. Towards the nearer endpoint.

The plan of Part II is as follows.

- Section 4.4 explains how a Schrödinger potential determines a radial profile. Sections 4.5 and 4.6 treat the central model and the Morse model on the whole line: spectrum, spectral gap at energy zero, travel angle.
- Section 5 builds the tails. It proves that at every frequency the natural line exists, is unique, and consists of flat functions, and it chooses the amplitudes of the cells so that natural and incoming lines stay apart.
- Section 6 joins all pieces and proves that the local information combines to an inverse on the whole annulus.

4.4 From a Schrödinger potential to a radial profile

It is easier to design the spectrum first and the radius afterwards.

Lemma 4.3 (Liouville transformation)**.** *Let X range over an interval, let $r(X)$ be a smooth function with $r > 0$ and $r_X > 0$, let $\mathcal{P}(X) > 0$ be smooth, and let $\lambda > 0$. Define a profile w and two functions ρ, V_0 of X by*

$$w(r(X)) = \lambda\,\mathcal{P}(X)\,r_X(X), \qquad \rho = -\frac{r_{XX} + (\mathcal{P}_X/\mathcal{P})\,r_X}{r}, \qquad V_0 = \frac{(\sqrt{\mathcal{P}})_{XX}}{\sqrt{\mathcal{P}}}. \tag{4.10}$$

Then $w' = -\lambda\mathcal{P}\rho\, r/r_X$, and for every j and every function $\psi(X)$,

$$L_j(\mathcal{P}^{-1/2}\psi) = -\frac{\lambda\sqrt{\mathcal{P}}}{r_X}\,(-\psi_{XX} + (V_0 - j^2\rho)\,\psi). \tag{4.11}$$

Proof. For a function u of r we have $wu_r = \lambda\mathcal{P}u_X$, hence $(wu_r)_r = \lambda(\mathcal{P}u_X)_X/r_X$. Differentiating $w = \lambda\mathcal{P}r_X$ and using the definition of ρ gives the formula for w'. Therefore $L_j u = (\lambda/r_X)\,[(\mathcal{P}u_X)_X + j^2\mathcal{P}\rho\,u]$, and with $u = \mathcal{P}^{-1/2}\psi$ one has $(\mathcal{P}u_X)_X = \sqrt{\mathcal{P}}\,(\psi_{XX} - V_0\psi)$. □

We use the lemma in the opposite direction. Given a density $\mathcal{P}$ and a function ρ, the homogeneous equation $L_j u = 0$ is equivalent to the Schrödinger equation

$$H_j\psi := -\psi_{XX} + (V_0 - j^2\rho)\,\psi = 0,$$

provided the linear equation $r_{XX} + (\mathcal{P}_X/\mathcal{P})\,r_X + \rho\, r = 0$ for the radius has a solution which is positive and increasing. Three features of the dictionary are used constantly.

- The amplitude λ does not enter the homogeneous equation.
- The profile is falling where $\rho > 0$ and rising where $\rho < 0$. The negative components are the components of $\{\rho > 0\}$, the turning circles are the zeros of ρ, and the travel angle of a negative component is

$$T = \int \sqrt{-w'/(rw)}\,dr = \int_{\{\rho>0\}} \sqrt{\rho}\,dX. \tag{4.12}$$

- Frequency enters only through the coupling constant j^2 in front of ρ.

We use two potentials with complementary roles (Figure 9). The harmonic oscillator, in the middle, has a zero mode at $j = 2$ and is used to create the seed. The Morse potential, near the ends, has a spectral gap at energy zero for all j, and its cells are short enough in radius to accumulate at a finite point. It is the explicit spectra of the two models, not merely the shapes of their potentials, that make them useful.

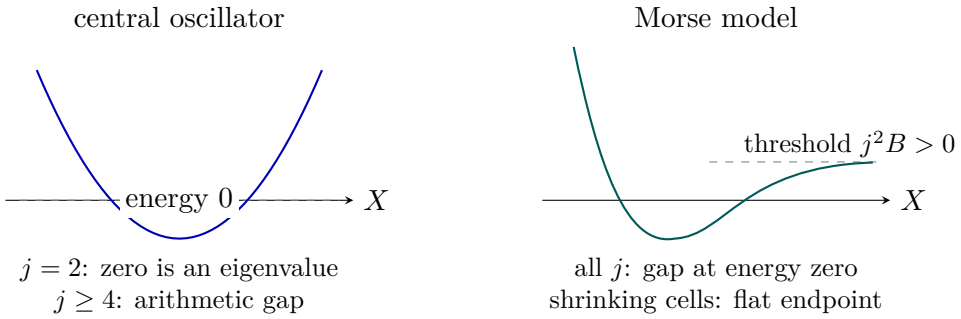


Figure 9: The potentials $V_0 - j^2\rho$ of the two models on the whole line, drawn schematically with an arbitrary vertical scale. Only finite pieces of them are used in the profile, and the pieces are separated by regions where the potential is positive.

4.5 The central oscillator

Definition and spectrum. Fix the constants

$$A_c = \frac{1}{\sqrt{2}}, \qquad a_c = \sqrt{4A_c - 2}, \qquad c_c = 3A_c - 2 > 0, \qquad \alpha_c = \frac{c_c + 1}{2}, \tag{4.13}$$

and take

$$\mathcal{P} = e^{-2a_c X}, \qquad \rho = A_c - X^2.$$

Then $V_0 = a_c^2$, and the Schrödinger operator of Lemma 4.3 is a shifted harmonic oscillator,

$$H_{j,c} = -\partial_X^2 + j^2X^2 + a_c^2 - j^2A_c,$$

with eigenvalues [Tes09, §8.3, equations (8.33)–(8.42)]

$$E_{m,j} = j(2m+1) + a_c^2 - j^2A_c, \qquad m = 0, 1, 2, \ldots \tag{4.14}$$

The value of a_c is chosen so that $E_{0,2} = 0$: at $j = 2$ the eigenvalues are $E_{m,2} = 4m$, and the kernel is spanned by the ground state e^{-X^2}. The irrationality of A_c keeps all other even frequencies away from zero:

Lemma 4.4 (Spectral gap of the central model)**.** *Let $\delta_c = \frac{3}{4}(3 - 2\sqrt{2}) > 0$. Then*

$$\|H_{j,c}^{-1}\| \le \frac{2}{\delta_c} \quad (j \ge 4 \text{ even}), \qquad \ker H_{2,c} = \operatorname{span}\{e^{-X^2}\}, \tag{4.15}$$

and the inverse of $H_{2,c}$ on the orthogonal complement of its kernel has norm $1/4$. Here $H_{j,c}$ is the self-adjoint operator on $L^2(\mathbb{R})$ defined by its closed quadratic form.

Proof. Only the first bound needs proof; it says $|E_{m,j}| \ge \delta_c/2$ for even $j \ge 4$. Put $\Phi_j = jA_c - a_c^2/j$, so that $E_{m,j} = j\,(2m + 1 - \Phi_j)$. Since $A_c^2 = \frac{1}{2}$,

$$N_j := \frac{j^2}{2} - \Phi_j^2 = 2A_c a_c^2 - \frac{a_c^4}{j^2} \in \Big[\tfrac{13}{4} - \tfrac{3\sqrt{2}}{2},\ 4 - 2\sqrt{2}\Big) \qquad (j \ge 4),$$

and this interval lies in $(1, 2)$ at distance δ_c from the integers. Suppose $|E_{m,j}| < 1$. Then $2m + 1 + \Phi_j < 2j$, and because $j^2/2 - (2m + 1)^2$ is an integer,

$$\delta_c \le \big|(2m + 1)^2 - \Phi_j^2\big| = \frac{|E_{m,j}|}{j}\,(2m + 1 + \Phi_j) < 2\,|E_{m,j}|.$$

If $|E_{m,j}| \ge 1$ there is nothing to prove. □

The model is a radial profile. We need a positive increasing solution of the radius equation, which for the present $\mathcal{P}$ and ρ reads $r_{XX} - 2a_c r_X + (A_c - X^2)\,r = 0$. An arbitrary solution would not do; here is an explicit one.

Lemma 4.5 (The central radius)**.** *Let*

$$\psi_1(X) = e^{-X^2/2} \int_0^\infty t^{\alpha_c - 1} e^{-t^2/2 + \sqrt{2}\,Xt}\,dt, \qquad r = e^{a_c X}\psi_1, \qquad \upsilon = \frac{r_X}{r}. \tag{4.16}$$

Then r is a positive solution of the radius equation, increasing from 0 to ∞, with $\upsilon > a_c$. Moreover

$$\upsilon = \begin{cases} -X + a_c - \alpha_c/X + O(X^{-3}), & X \to -\infty, \\ X + a_c + (\alpha_c - 1)/X + O(X^{-3}), & X \to +\infty, \end{cases} \tag{4.17}$$

and these expansions may be differentiated any fixed number of times. In particular $\upsilon \asymp 1 + |X|$.

Proof. An integration by parts in t shows $\psi_1'' = (X^2 + c_c)\,\psi_1$. Since $c_c - a_c^2 + A_c = 0$, the function $r = e^{a_c X}\psi_1$ solves the radius equation. Clearly $\psi_1 > 0$, and $\psi_1, \psi_1' \to 0$ as $X \to -\infty$; then $\psi_1'' > 0$ implies $\psi_1' > 0$, so $\upsilon = a_c + \psi_1'/\psi_1 > a_c$.

The leading terms of (4.17) can be read off from the equation: as $|X| \to \infty$ the solutions of $\psi'' = (X^2 + c_c)\psi$ behave like $|X|^{\alpha_c - 1}e^{X^2/2}$ or like $|X|^{-\alpha_c}e^{-X^2/2}$, and ψ_1 is of the second kind at $-\infty$ and of the first kind at $+\infty$. To prove the expansions with their remainders, use the integral. For the expansion at $X = -L$, substitute $t = z/(\sqrt{2}L)$: the remaining integral is $\Gamma(\alpha_c)(1 + O(L^{-2}))$ and may be differentiated under an exponentially integrable majorant. For $X > 0$, complete the square and expand $t^{\alpha_c - 1}$ about $t = \sqrt{2}X$ against the Gaussian; the region $|t - \sqrt{2}X| > X/2$ contributes an exponentially small amount. Logarithmic differentiation gives (4.17), and with it the range of r. □

The seed and the two ends. By Lemma 4.3 the kernel of $H_{2,c}$ corresponds to the solution

$$\phi_0 = \mathcal{P}^{-1/2} e^{-X^2} = e^{a_c X - X^2}, \qquad \frac{\partial_y \phi_0}{\phi_0} = \frac{a_c - 2X}{\upsilon} \longrightarrow \begin{cases} 2 & \text{as } X \to -\infty, \\ -2 & \text{as } X \to +\infty, \end{cases} \tag{4.18}$$

of $L_2\phi_0 = 0$, where $y = \log r$. So at its two ends ϕ_0 behaves like the solutions r^2 and r^{-2} on a cone. The profile itself also becomes conical: writing $w = re^f$,

$$1 + f_y = \frac{rw'}{w} = \frac{X^2 - A_c}{\upsilon^2} \longrightarrow 1 \qquad (X \to \pm\infty). \tag{4.19}$$

Both limits hold with any fixed number of y-derivatives, by (4.17). We may therefore cut the model so far out on both sides that the two end regions are positive, with uniformly bounded jets in the logarithmic coordinate y. The retained piece contains exactly two turning circles, at $X = \pm\sqrt{A_c}$. By (4.12) its travel angle is

$$T = \int_{-\sqrt{A_c}}^{\sqrt{A_c}} \sqrt{A_c - X^2}\, dX = \frac{\pi A_c}{2} = \frac{\pi}{2\sqrt{2}},$$

and $|\sin(kT)| \geq c/k$ for all $k \geq 1$, because $k^2 - 8m^2$ is a nonzero integer for all integers $k \geq 1$ and m.

The inverse in radial units. Estimates in the radial variable are made in a weighted L^2 space. Define

$$\varpi = \frac{12\, w}{r^2}, \qquad B_j = -\varpi^{-1} L_j, \qquad \|u\|^2_{L^2(\varpi\, dr)} = 12\lambda\, \|\upsilon\psi\|^2_{L^2(dX)} \quad \text{for } u = \mathcal{P}^{-1/2}\psi. \tag{4.20}$$

The operator B_j is symmetric in $L^2(\varpi\, dr)$, and on a cone it is $\frac{1}{12}(-\partial_y^2 + j^2)$. By (4.11), B_j corresponds to $\frac{1}{12}\upsilon^{-2} H_{j,c}$, so in the variable $\upsilon\psi$ its inverse is $12\, \upsilon H_{j,c}^{-1} \upsilon$, not merely $H_{j,c}^{-1}$.

Lemma 4.6. *The operators $\upsilon H_{j,c}^{-1}\upsilon$, $j \geq 4$ even, are bounded on $L^2(dX)$ uniformly in j. The same holds at $j = 2$ for the inverse on the complement of the kernel.*

Proof. The potential $V_j = j^2X^2 + a_c^2 - j^2A_c$ satisfies $V_j \geq cj^2\upsilon^2 - Cj^2$. For $\psi = H_{j,c}^{-1}G$ this and Lemma 4.4 give $cj^2\|\upsilon\psi\|^2 \leq \langle G, \psi\rangle + Cj^2\|\psi\|^2 \leq Cj^2\|G\|^2$, so $\|\upsilon H_{j,c}^{-1}\| \leq C$, and by taking adjoints $\|H_{j,c}^{-1}\upsilon\| \leq C$. Now let $\psi = H_{j,c}^{-1}(\upsilon F)$. In the same way

$$cj^2\|\upsilon\psi\|_2^2 \leq \|\upsilon\psi\|_2 \|F\|_2 + Cj^2\|F\|_2^2,$$

and absorbing the first term on the right proves the claim. At $j = 2$, add to $H_{2,c}$ the orthogonal projection onto e^{-X^2}, apply the same argument, and subtract the projection, whose weighted norm is finite. □

A temporary border. During the construction, mode two of the central model is bordered by a compactly supported function g_0 with $\int g_0\phi_0\, dr = 1$ and by the row $\ell_0 u = \int g_0 u\, dr$. A source is made orthogonal to ϕ_0 by subtracting a multiple of g_0, and the row is fulfilled by adding a multiple of ϕ_0; so the bordered inverse obeys the bounds of Lemma 4.6. The scalar is normalized so that under a rescaling $w \mapsto \lambda w$ of the amplitude one has $g_0 \mapsto \sqrt{\lambda}\, g_0$ and $\phi_0 \mapsto \phi_0/\sqrt{\lambda}$; then row and column have bounded norms in the Hilbert space $L^2(\varpi\, dr)$. This auxiliary border is replaced by the final one (4.7) in Section 6.8, after all geometric choices have been made.

4.6 The Morse model

Definition and spectrum. For the tails take $\mathcal{P} = 1$, so $V_0 = 0$, and for parameters $A > 4B > 0$

$$\rho(X) = A\,(e^{-X} - e^{-2X}) - B, \qquad H_{j,M} = -\partial_X^2 - j^2\rho.$$

The potential $-j^2\rho$ is confining as $X \to -\infty$ and tends to $j^2 B$ exponentially as $X \to +\infty$. Hence the essential spectrum of $H_{j,M}$ is $[j^2B, \infty)$ [Tes09, Theorem 6.19]. Below it there are finitely many simple eigenvalues, given by Morse's formula [Mor29]

$$E_{m,j} = j^2 B - (\alpha_j - m)^2, \qquad \alpha_j = \frac{j\sqrt{A} - 1}{2}, \qquad 0 \le m < \alpha_j. \tag{4.21}$$

For completeness we recall the factorization, in the sense of Infeld and Hull [IH51], which proves (4.21) and fixes its normalization. Let $\omega = j\sqrt{A}$, and for real α put

$$D_\alpha = \partial_X + \alpha - \omega e^{-X}, \qquad H_\alpha = -\partial_X^2 + \omega^2 e^{-2X} - (2\alpha + 1)\,\omega e^{-X} + j^2 B,$$

so that $H_{j,M} = H_{\alpha_j}$. A direct computation gives

$$H_\alpha = D_\alpha^* D_\alpha + j^2 B - \alpha^2, \qquad H_{\alpha-1} = D_\alpha D_\alpha^* + j^2 B - \alpha^2.$$

The null function of D_α is $e^{-\alpha X - \omega e^{-X}}$, which is square integrable exactly when $\alpha > 0$. Passing from H_α to $H_{\alpha-1}$ thus removes the ground state $j^2B - \alpha^2$ and keeps the rest of the spectrum. For $-1 < \alpha \le 0$ one has $H_\alpha - j^2 B = D_0^* D_0 - 2\omega\alpha\, e^{-X} \ge 0$, so there are no eigenvalues below $j^2 B$. So every eigenvalue below $j^2 B$ of $H_{j,M}$ is met in the descent and is one of the numbers (4.21). Conversely none of them is lost on the way back: if $H_{\alpha-1}\phi = (j^2B - \alpha_0^2)\phi$ with $0 < \alpha_0 < \alpha$, then $\|D_\alpha^*\phi\|^2 = (\alpha^2 - \alpha_0^2)\|\phi\|^2 > 0$, so $D_\alpha^*\phi$ is an eigenfunction of H_α with the same eigenvalue. Each eigenvalue is simple, because the Wronskian of two square integrable eigenfunctions is constant and vanishes at the confining end. The identities are justified on the form domain by integration against cutoffs: an eigenfunction below the threshold decays exponentially as $X \to +\infty$, and the form bound controls it as $X \to -\infty$.

An arithmetic choice of the parameters. We only need the resolvent at energy zero, and we need the travel angle. One irrationality controls both.

Lemma 4.7 (Gap and travel angle of a Morse model)**.** *Let $d \ge 2$ be an integer which is not a square, and let*

$$A = \left(d^{-1/2} + 2d^{-9/2}\right)^2, \qquad B = d^{-9}. \tag{4.22}$$

Then, for every integer $j \ge 2$,

$$\|H_{j,M}^{-1}\| \le 8d^9. \tag{4.23}$$

The travel angle of the negative component $\{\rho > 0\}$ is

$$T = \int_{\{\rho>0\}} \sqrt{\rho}\, dX = \pi\Big(\frac{\sqrt{A}}{2} - \sqrt{B}\Big) = \frac{\pi}{2\sqrt{d}}, \tag{4.24}$$

and

$$|\sin(kT)| \ge \frac{1}{3k\sqrt{d}} \qquad (k \ge 1). \tag{4.25}$$

Proof. Note that $\sqrt{A} - 2\sqrt{B} = d^{-1/2}$.

Step 1: the gap. For integers $j \ge 1$ and $m \ge 0$ we claim

$$\Big|\frac{j}{\sqrt{d}} - (2m+1)\Big| \ge \frac{1}{4j\sqrt{d}}.$$

If the left side is at least $\frac{1}{2}$ this is clear. Otherwise $(2m+1)\sqrt{d} < 2j$, and the claim follows by factoring the nonzero integer $j^2 - d(2m+1)^2$. Now consider an eigenvalue (4.21) and write $\kappa = \alpha_j - m = (j\sqrt{A} - (2m+1))/2 > 0$. Then $j\sqrt{B} - \kappa = \frac{1}{2}\left((2m+1) - j/\sqrt{d}\right)$, hence

$$|E_{m,j}| = |j\sqrt{B} - \kappa|\,(j\sqrt{B} + \kappa) \geq \frac{j\sqrt{B}}{2}\left|\frac{j}{\sqrt{d}} - (2m+1)\right| \geq \frac{1}{8d^5}.$$

The essential spectrum starts at $j^2 B \geq 4d^{-9}$. Both lower bounds are at least $1/(8d^9)$, which proves (4.23).

Step 2: the travel angle. With $t = e^{-X}$ we have $\rho = A(t-t_-)(t_+-t)$, where $t_- + t_+ = 1$ and $t_- t_+ = B/A$, and $dX = -dt/t$. The substitution $t = c + h\cos\xi$ with $c = \frac{1}{2}$ and $c^2 - h^2 = B/A$ gives the elementary integral

$$\int_{t_-}^{t_+} \frac{\sqrt{(t-t_-)(t_+-t)}}{t}\,dt = \pi(c - \sqrt{c^2-h^2}),$$

and (4.24) follows.

Step 3: the small divisors. We have $kT/\pi = k/(2\sqrt{d})$. If $k < \sqrt{d}$, its distance to $\mathbb{Z}$ is $k/(2\sqrt{d})$ itself. Otherwise let m be the nearest integer; factoring the nonzero integer $k^2 - 4dm^2$ and using $2m\sqrt{d} \leq 2k$ gives a distance of at least $1/(6k\sqrt{d})$. In both cases $|\sin(\pi x)| \geq 2\,\mathrm{dist}(x,\mathbb{Z})$ yields (4.25). □

Thus a single choice of parameters gives, for one Morse model on the whole line, both of the separations of Section 3.3: a spectral gap and a Diophantine travel angle. They remain different conditions when models are joined. A junction does not change any travel angle, since it adds only positive intervals; but it can create a new matching resonance. That remaining issue is resolved by the choice of amplitudes in the next section.

5 Reaching a flat endpoint with an infinite chain of cells

By Proposition 3.2 the profile must have infinitely many turning circles near each endpoint. We obtain them by repeating one Morse cell infinitely often, with rapidly shrinking lengths and amplitudes. This section constructs such a *tail* and analyses the equation $L_j u = 0$ on it. Its results are the following.

- The tail is a smooth profile which extends by zero across the endpoint, with explicit bounds on all coefficients (Lemmas 5.1 and 5.2).
- On the part of the tail which is deep enough for a given frequency j, the equation reduces *exactly* to a three-term recurrence with a dominant diagonal. Hence there is a unique solution decaying towards the endpoint, it loses the factor $e^{-2n^2+o(n^2)}$ across cell n, and the operator has a bounded inverse (Lemma 5.6).
- The accumulated loss $e^{-2n^3/3}$ beats every fixed number of derivatives, so the decaying solution is flat at the endpoint.
- The amplitudes of the cells can be chosen so that, at every frequency, the decaying solution is quantitatively transverse to the solution arriving from the interior (Lemma 5.9).

5.1 The budget of lengths, amplitudes and errors

For $n \geq N$ let d_n be the least integer which is not a square and is at least e^{n^2}; thus $d_n = e^{n^2} + O(1)$. Let A_n and B_n be the Morse parameters (4.22) for $d = d_n$, so that

$$A_n \asymp e^{-n^2}, \qquad B_n = d_n^{-9} \asymp A_n^9.$$

The amplitude of cell n is

$$\lambda_n = e^{-n^{5/2}} \prod_{k=N}^{n} \sigma_k, \qquad 1 \leq \sigma_k \leq 2. \tag{5.1}$$

The *amplitude ratios* σ_k are free parameters until Section 5.8; they change $\log \lambda_n$ by at most $O(n)$, which does not affect any of the comparisons of scales below. The initial index N will be taken large.

Table 2 lists the scales and the reason for each. The choices can be understood before any computation. A derivative of fixed order on a cell of length e^{-n^2} costs $e^{C_s n^2}$. For the profile to be flat its amplitude must decay faster than that, and the exponent $n^{5/2}$ achieves it. The solution must decay faster still, in order to pay for reciprocals of the coefficients; its cubic exponent is not chosen but comes out of the recurrence. These requirements leave the whole interval $2 < 5/2 < 3$ for the middle exponent; no optimal value is singled out.

Quantity	Size	Purpose
Length of cell n	$A_n \asymp e^{-n^2}$	The lengths are summable, so the cells accumulate at a finite endpoint.
Amplitude of cell n	$\lambda_n = e^{-n^{5/2}+O(n)}$	Beats the cost $e^{C_s n^2}$ of s derivatives: the profile is flat.
Length of a join	$\asymp A_n^{12}$	A large change of amplitude is crossed with negligible transfer error.
Matching angle	$\delta_n = c_0 e^{-n^2}$	The excluded sets of amplitude ratios are summable over all frequency bands.
Decay of the solution	$H_n = e^{-S_n}$, $S_n = \frac{2}{3}n^3$	Produced by the recurrence; gives integrability and flatness of the selected solution.
Coefficient allowance	$\mathcal{E}_n = e^{n^{5/2}\log(e+n)}$	Absorbs fixed powers of coefficients and of changes of coordinates, while staying below $e^{\varepsilon S_n}$.

Table 2: The scales of the tail. Their compatibility is expressed by (4.4) and verified in this section.

All comparisons of scales concern a fixed number of derivatives; the constants may grow with that number.

5.2 One cell

Let $A = A_n$, $B = B_n$ and $\rho = A(e^{-X} - e^{-2X}) - B$. Let $X_L < X_R$ be the two zeros of ρ; then $X_L = O(B/A)$ and $X_R = \log(A/B) + O(B/A)$. With $\mathcal{P} = 1$, Lemma 4.3 says that a positive increasing solution r of

$$r_{XX} = -\rho\, r, \qquad \text{together with} \qquad w = \lambda\, r_X, \tag{5.2}$$

defines a profile on which $L_j = -(\lambda/r_X)\, H_{j,M}$. We take the solution with

$$r(X_R) = 1, \qquad r_X(X_R) = \epsilon_R A^5,$$

for a fixed small $\epsilon_R > 0$, and we retain the interval

$$X_c \le X \le X_R + c_R A^{-4}, \qquad X_c = -\log \tfrac{3}{2},$$

for a fixed sufficiently small $c_R > 0$. It consists of three parts (Figure 10):

- the *short positive ramp* $[X_c, X_L]$, where $\rho < 0$ and w rises;
- the *negative component* (X_L, X_R), where $\rho > 0$ and w falls; its length in X is about $\log(A/B) \asymp n^2$;
- the *long positive ramp* $[X_R, X_R + c_R A^{-4}]$, where $-B \le \rho < 0$ and w rises slowly.

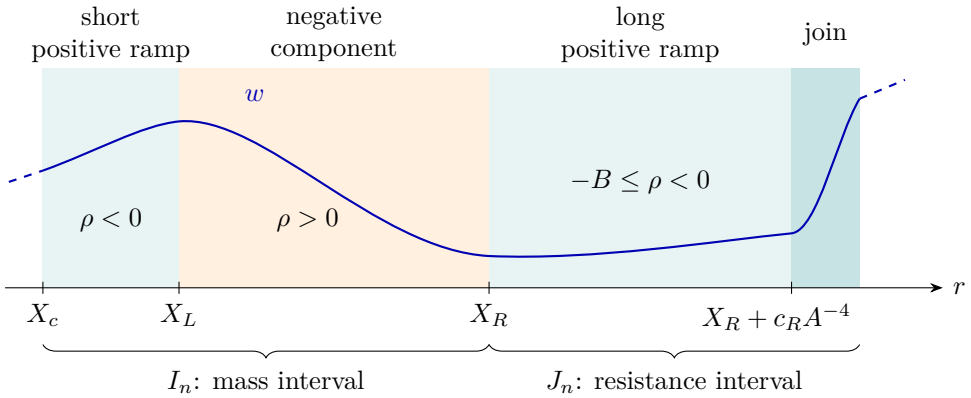


Figure 10: One cell and the join following it, in the direction of increasing r (not to scale). The tick marks give the values of the model coordinate X. Each of the three parts of the cell has radial length comparable to A_n; the join has length comparable to A_n^{12} or $A_{n\pm1}^{12}$. Along the join the profile rises by a factor $e^{4n^2+o(n^2)}$. The intervals I_n and J_n are used in Section 5.4.

Lemma 5.1 (The cell is a radial profile)**.** *On the retained interval $r_X > 0$. More precisely, with constants independent of n,*

$$r_X(X_c) = \tfrac{3}{8}A + O(A^2), \qquad r(X_c) = 1 - \tfrac{15}{16}A + O(A^2), \tag{5.3}$$

and

$$\begin{aligned} &r = 1 + O(A), \qquad cA^5 \le r_X \le CA, \qquad r_X \asymp A^5 \text{ on the long positive ramp,} \\ &\Delta r \asymp A, \qquad \int_{X_c}^{X_R} r_X^2\, dX \asymp A^2, \qquad \int_{\text{each positive ramp}} \sqrt{-\rho}\, dX \asymp \sqrt{A}. \end{aligned} \tag{5.4}$$

Here Δr is the radial length of each of the three parts.

Proof. On $[X_c, X_R]$ the integral $\int (Y - X_c)|\rho(Y)|\, dY$ is $O(A)$, so the Volterra equation

$$r(X) = 1 - r_X(X_R)\,(X_R - X) - \int_X^{X_R} (Y - X)\,\rho(Y)\,r(Y)\, dY$$

gives $r = 1 + O(A)$ there. For (5.3) evaluate the leading terms with $t = e^{-Y}$:

$$\int_0^{3/2} (1 - t)\, dt = \frac{3}{8}, \qquad \int_0^{3/2} \log\frac{3/2}{t}\,(1 - t)\, dt = \frac{15}{16};$$

the constant B contributes only $O(B\log^2(A/B)) = o(A^2)$.

On the negative component $r_X(X) = r_X(X_R) + \int_X^{X_R} \rho\, r > 0$. On the short positive ramp $r_{XX} = -\rho r > 0$, so the minimum of r_X is its value at X_c, which is positive by (5.3). On the long positive ramp $0 < r_{XX} \le Br$; since $B \cdot c_R A^{-4} \asymp c_R A^5$, comparison with $r_{XX} = Br$ shows $r_X \asymp A^5$ there, and r changes by $O(A)$. Taking c_R small keeps r in a fixed positive range throughout. The remaining statements follow by integration: $r_X \asymp A$ on a fixed part of $[X_c, X_R]$ and $r_X \le CAe^{-X} + CA^5$ on all of it; on the long ramp $-\rho \asymp B$ away from X_R and $\sqrt{B}\,A^{-4} \asymp \sqrt{A}$; on the short ramp $-\rho \asymp A$ on a fixed subinterval. □

The last quantity in (5.4) measures the strength of a positive ramp as a buffer. On a ramp the equation $L_j u = 0$ reads $u_{XX} = j^2|\rho|\,u$, whose solutions grow or decay like $\exp(\pm j\int\sqrt{|\rho|}\,dX)$. We call

$$\tau_{n,j} = j\sqrt{A_n} \tag{5.5}$$

the *action* of the positive ramps of cell n at frequency j. The ramps of cell n decouple their two sides when $\tau_{n,j}$ is large, and are transparent when it is small.

5.3 Joining the cells

Radii. Since (5.2) is linear in r, we may multiply r by a *radius factor* $\mathfrak{r}_n > 0$ without changing ρ or the operators $H_{j,M}$. Between a cell and its neighbour in the direction of increasing r we insert a *join*: an interval of length

$$\ell_{n,m} = c_C\, r_o\, \min(A_n, A_m)^{12}, \qquad m = n \pm 1, \tag{5.6}$$

where n, m are the indices of the two cells, r_o is the radius at which the join starts, and $c_C > 0$ is a fixed constant. The radius factor of the neighbour is chosen so that it starts at $r_o + \ell_{n,m}$. By Lemma 5.1 the ratio of consecutive radius factors is $1 + O(A_n + A_m)$, so the recursion converges at either end; on the left the limiting radius may be prescribed. Hence all radii stay in a compact subset of $(0, \infty)$, the $\mathfrak{r}_n$ are bounded above and below, and the cells accumulate at finite endpoints.

The profile on a join. Let W_L and W_R be the values of w at the two ends of a join. At the right endpoint the join leads from the end of the long ramp of cell n, where $w \asymp \lambda_n A_n^5$, to the beginning of cell $n+1$, where $w \asymp \lambda_{n+1} A_{n+1}$; at the left endpoint it leads from cell $n+1$ to cell n. In both cases the profile rises, and

$$\log \frac{W_R}{W_L} = 4n^2 + O(n^{3/2} + n). \tag{5.7}$$

To match the two neighbouring profiles to infinite order, let f_L, f_R be the logarithms of the two neighbouring model profiles, choose cutoffs χ_L, χ_R with disjoint supports, equal to one near the left and right end of the join respectively, and define $f = \log w$ on the join by

$$f' = \chi_L f_L' + \chi_R f_R' + C_1\,(1 - \chi_L - \chi_R), \qquad C_1 = \frac{\log(W_R/W_L) - \int \chi_L f_L' - \int \chi_R f_R'}{\int (1 - \chi_L - \chi_R)}, \tag{5.8}$$

with $f = \log W_L$ at the left end. Then $f = \log W_R$ at the right end, and $w = e^f$ agrees with the two models to infinite order. The two integrals in the numerator are $O(\ell_{n,m}/\min(A_n, A_m))$, while the required change of amplitude is of order n^2. Hence $C_1 > 0$, and after N is increased

$$\frac{w'}{w} \ge \frac{2}{r} \qquad \text{on every join.} \tag{5.9}$$

Why the exponent 12. The exponent in (5.6) looks extravagant; it is chosen so that a join is invisible to every frequency that has to cross it carefully. The contrast of amplitudes across the join costs a factor $e^{4n^2+o(n^2)}$. The largest frequency assigned to cell n (Section 5.7) satisfies $j^2 \le e^{n^2+o(n^2)}$. The length of the join contributes $e^{-12n^2+o(n^2)}$. The product is $e^{-7n^2+o(n^2)}$, far below the matching angle e^{-n^2} that we shall allow. The same choice makes the mass and the resistance of a join negligible in Section 5.4. No minimal exponent is needed.

Lemma 5.2 (Flatness and coefficient bounds)**.** *Let d_∂ be the distance to the endpoint. On cell n and on the joins adjacent to it,*

$$cA_{n+1} \le d_\partial \le CA_n, \qquad \log \frac{d_0}{d_\partial} = n^2 + O(n) \tag{5.10}$$

for a fixed $d_0 > 0$, and for every $s \ge 0$

$$\begin{aligned} e^{-n^{5/2}-Cn^2} \le w \le e^{-n^{5/2}+Cn^2}, \\ |\partial_r^s w| \le C_s\, e^{-n^{5/2}+C_s n^2}, \qquad |\partial_r^s (w^{-1})| \le C_s\, e^{n^{5/2}+C_s n^2}. \end{aligned} \tag{5.11}$$

The constants do not depend on the amplitude ratios $\sigma_k \in [1,2]$. Consequently w extends smoothly by zero across the endpoint, and all fixed-order derivatives of w and of w^{-1} on cell n are bounded by coefficient powers $C_s \mathcal{E}_n^{C_s}$.

Proof. The length of cell n is comparable to A_n by Lemma 5.1, and $A_{n+1}/A_n = e^{-2n-1}(1+o(1))$; summing the lengths of the cells beyond n gives (5.10). All lengths of intervals and their reciprocals are bounded by e^{Cn^2}. On a model $\partial_r = r_X^{-1}\partial_X$ and $r_X \ge c\,\mathfrak{r}_n A_n^5$, and the differential equation (5.2) bounds every fixed derivative of $\log r_X$ by $e^{C_s n^2}$. On a join (5.8) gives the same bound for $\log w$. A derivative of $e^{\pm \log w}$ is $e^{\pm \log w}$ times a polynomial in the derivatives of $\log w$, which proves (5.11). Since $e^{-n^{5/2}+C_s n^2} d_\partial^{-k} \to 0$ for all s, k by (5.10), the extension by zero is smooth. The last statement follows from $n^2 = o(q_n)$. □

5.4 Why a mass–resistance chain appears

We now study $L_j u = 0$ on a tail. The equation has the unpleasant feature that its zeroth-order coefficient $j^2 w'/r$ is enormous on the joins. A change of the unknown removes it.

Lemma 5.3 (Gauge)**.** *Let $y = \log r$ and $u = r^{j^2} V$. Then*

$$r^2 L_j u = r^{j^2} \frac{w}{\mathsf{p}_j} \left[(\mathsf{p}_j V_y)_y + \Lambda\, \mathsf{p}_j V \right], \qquad \mathsf{p}_j = w\, r^{2j^2-1}, \qquad \Lambda = j^2(j^2-1). \tag{5.12}$$

Proof. In the variable y one has $r^2 L_j u = w u_{yy} + (w_y - w) u_y - j^2 w_y u$. Substituting $u = e^{j^2 y} V$, the terms containing $w_y V$ cancel, and what remains is $e^{j^2 y}\,[w V_{yy} + ((2j^2-1)w + w_y) V_y + (j^4 - j^2) w V]$. □

So $L_j u = 0$ is equivalent to the Sturm–Liouville equation $-(\mathsf{p}_j V_y)_y = \Lambda\, \mathsf{p}_j V$, in which the profile enters only through the density p_j. For $j = 2$ this density is $\mathsf{p} = w r^7$ and $\Lambda = 12$.

Mass and resistance. Divide each cell into two intervals (Figure 10): the *mass interval* I_n, consisting of the short positive ramp and the negative component, and the *resistance interval* J_n, consisting of the long positive ramp and the join that follows it in the direction of increasing r. For a density p define

$$\mu_n = \int_{I_n} \mathsf{p}\, dy, \quad R_n^I = \int_{I_n} \frac{dy}{\mathsf{p}}, \quad \nu_n = \int_{J_n} \mathsf{p}\, dy, \quad R_n = \int_{J_n} \frac{dy}{\mathsf{p}}. \tag{5.13}$$

We call μ_n, ν_n *masses* and R_n^I, R_n *resistances*. The names come from the following elementary observation about an equation $-(\mathsf{p} V_y)_y = \Lambda \mathsf{p} V$.

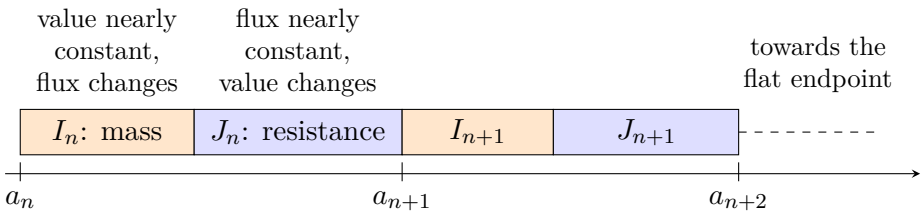
- On an interval of small resistance the value V is nearly constant, and the flux $\mathsf{p}V_y$ changes by $-\Lambda$ times the mass times that value.
- On an interval of small mass the flux is nearly constant, and the value changes by the flux times the resistance.

In our tails I_n carries almost all the mass and almost no resistance, and J_n the opposite. Let a_n be the value of V on I_n and φ_n the flux on J_n. The two rules give

$$\varphi_n = \varphi_{n-1} - \Lambda\mu_n a_n, \qquad a_{n+1} = a_n + R_n\varphi_n.$$

Eliminating the fluxes yields a three-term recurrence for the a_n whose diagonal coefficient $\Lambda\mu_n$ is very large compared with the off-diagonal coefficients $1/R_n$. For a solution that decays, the dominant balance is $a_{n+1} \approx -a_n/(\Lambda\mu_{n+1}R_n)$: the values alternate in sign and decrease rapidly (Figure 11). Section 5.5 turns this picture into an exact statement.

The picture is that of a vibrating string. In the variable $x = \int dy/\mathsf{p}$ the equation reads $-V_{xx} = \Lambda\,\mathsf{p}^2\,V$: the resistance of an interval is its length in x, and its mass is $\int \mathsf{p}^2\,dx = \int \mathsf{p}\,dy$. A string whose mass sits at isolated points satisfies a three-term recurrence, as in Stieltjes's work on continued fractions; for the theory of strings with a general distribution of mass see [KK74]. We use only the picture, not that theory.



$$b_n y_n = c_{n-1}y_{n-1} + c_n y_{n+1}, \qquad a_n = (-1)^n y_n$$
$$b_n \approx \Lambda\mu_n, \qquad c_n \approx R_n^{-1}$$

Figure 11: The intervals of a tail and the node equation satisfied by the values a_n of a solution at the entrances of the mass intervals. Large products $\Lambda\mu_n R_n$ make successive node values of the decaying solution small.

Lemma 5.4 (Masses and resistances of the tail)**.** *For the density $\mathsf{p} = wr^7$ on the tail constructed above,*

$$\mu_n \asymp \lambda_n A_n^2, \qquad R_n^I \le \frac{C}{\lambda_n}\log\frac{A_n}{B_n}, \qquad \nu_n \le C\lambda_n A_n^6, \qquad R_n \asymp \frac{1}{\lambda_n A_n^4}, \tag{5.14}$$

hence

$$\mu_n R_n \asymp A_n^{-2}, \qquad \mu_n R_n^I \le C A_n^2 \log\frac{A_n}{B_n}, \qquad \nu_n R_n \le C A_n^2, \tag{5.15}$$

and

$$\log(\mu_{n+1}R_n) = 2n^2 + O(n^{3/2} + n). \tag{5.16}$$

Proof. Since $dy = dr/r$ and $dr = r_X\,dX$, the measures in (5.13) are $wr^6\,dr$ and $dr/(wr^8)$, and on a model $w\,dr = \lambda r_X^2\,dX$ and $dr/w = dX/\lambda$. The radius is bounded above and below, so (5.4) gives (5.14) for the model parts. A join of length ℓ adds at most $W_R\ell$ to the mass and ℓ/W_L to the resistance of J_n; relative to ν_n and R_n these are $e^{-7n^2+O(n^{3/2}+n)}$ and $e^{-11n^2+O(n)}$. The products (5.15) follow, and so does (5.16) from $\log(\lambda_{n+1}/\lambda_n) = O(n^{3/2})$ and $\log(A_{n+1}^2/A_n^4) = 2n^2 + O(n)$. □

Other frequencies. For a general frequency the density is p_j. After division by the constant $r_\partial^{2(j^2-4)}$, where r_∂ is the radius of the endpoint,

$$\frac{\mathsf{p}_j}{\mathsf{p}} = \Big(\frac{r}{r_\partial}\Big)^{2(j^2-4)},$$

and on the part of the tail beyond cell N' the logarithm of this ratio is bounded by $Cj^2A_{N'}$. So Lemma 5.4 holds for p_j, with uniform constants, on every part of the tail where $j^2A_{N'}$ is bounded. We shall use the slightly stronger condition

$$j^2 A_{N'}\sqrt{\log(A_{N'}/B_{N'})} \le c \tag{5.17}$$

for a small constant c; by (5.15) it makes $\Lambda\mu_n R_n^I$ and $\Lambda\nu_n R_n$ small for all $n \ge N'$.

5.5 The exact reduction to a recurrence

Use the coordinate $x = \log r$ at the right endpoint and $x = -\log r$ at the left endpoint, so that x always increases outward. Definition 5.5 below is formulated for an abstract sequence of intervals I_n, J_n in outward order, in which a resistance interval J_n follows each mass interval I_n. At the right endpoint these are the intervals of the cells. At the left endpoint the outward order of the intervals of the cells is $J_N, I_N, J_{N+1}, I_{N+1}, \ldots$; there the definition is applied with the interval J_{n+1} of the cells in the place of its J_n, and with J_N as a separate *initial resistance edge* J_0, whose resistance and mass are denoted R_0 and ν_0. The shift of index affects only the lower-order terms of (5.16).

The following definition isolates what the reduction needs.

Definition 5.5 (Deep tail)**.** Let $\mathsf{p} > 0$ be a density on a half-line in x which is divided into consecutive intervals I_n, J_n ($n \ge N'$), possibly preceded by an initial resistance edge J_0, and let $\Lambda > 0$. With the masses and resistances (5.13) formed with dx, the data form a *deep tail* with constants $\eta \ll 1 \ll K$ if for all n

$$\begin{gathered}\Lambda\mu_n R_n^I + \Lambda\nu_n R_n \le \eta, \qquad \Lambda\mu_n R_n \ge K, \quad \Lambda\mu_{n+1}R_n \ge K,\\ \nu_n \le \eta\,\min(\mu_n, \mu_{n+1}),\end{gathered} \tag{5.18}$$

and if, when there is an initial edge, $\Lambda\mu_{N'}R_0 \ge K$, $\Lambda\nu_0 R_0 \le \eta$ and $\nu_0 \le \eta\mu_{N'}$.

By Lemma 5.4 and the remark on other frequencies, the part of a constructed tail beyond cell N' is a deep tail for p_j and $\Lambda = j^2(j^2-1)$ whenever (5.17) holds and N' is large enough; η and K^{-1} can be made as small as we wish.

Lemma 5.6 (Deep inverse and the natural line)**.** *On a deep tail, with η and K^{-1} sufficiently small, let*

$$A_{\mathsf{p}} = -\mathsf{p}^{-1}\partial_x\,\mathsf{p}\,\partial_x$$

act in $L^2(\mathsf{p}\,dx)$ with the Dirichlet condition at the entrance and the Friedrichs condition at the outer endpoint.

(i) $\|(A_{\mathsf{p}} - \Lambda)^{-1}\| \le C/\Lambda$.

(ii) *Up to multiples, the equation $(A_{\mathsf{p}} - \Lambda)V = 0$ has exactly one solution which is square integrable for $\mathsf{p}\,dx$ near the outer endpoint. Its values a_n at the entrances of the mass intervals I_n are nonzero and alternate in sign: $a_n = (-1)^n y_n$ with $y_n > 0$ and*

$$\frac{y_{n+1}}{y_n} = \frac{1 + O(\eta) + O(K^{-1})}{\Lambda\mu_{n+1}R_n}, \qquad \frac{\sqrt{\mu_{n+1}}\,y_{n+1}}{\sqrt{\mu_n}\,y_n} \le \frac{C}{K} < \frac{1}{2}. \tag{5.19}$$

(iii) *For the constructed tails and* $\mathsf{p} = \mathsf{p}_j$*, the corresponding solution* $u = r^{j^2}V$ *of* $L_j u = 0$ *is smooth and flat at the endpoint.*

The line spanned by the solution in (ii) is the *natural line* of Section 4.3.

Proof. The reduction is carried out on the differential equation itself, so that the operator norm and the endpoint condition come out of one argument. An *edge* is $\mathsf{E}_n = I_n \cup J_n$; its two endpoints are *nodes*, and node n is the entrance of I_n.

Step 1: one edge in the resistance coordinate. On E_n introduce $ds = dx/\mathsf{p}$. The edge has length $L_n = R_n^I + R_n$ in s; with $\omega = \mathsf{p}^2$ we have $\mathsf{p}\,dx = \omega\,ds$ and

$$(A_{\mathsf{p}} - \Lambda)u = f \quad \Longleftrightarrow \quad -u_{ss} - \Lambda\,\omega\,u = \omega f.$$

The Dirichlet Green function of $-\partial_s^2$ on the edge is $G_0(s,t) = \min(s,t)\,[L_n - \max(s,t)]/L_n$. The small quantities are

$$\delta_n^{\mathrm{edge}} = \Lambda \int \omega(s)\,G_0(s,s)\,ds \le 3\eta, \qquad \frac{R_n^I}{L_n} \le \frac{\eta}{K},$$

by the first line of (5.18). The inequality $G_0(s,z)G_0(z,t) \le G_0(s,t)G_0(z,z)$ bounds the Neumann series of the Dirichlet Green function G_Λ of $-\partial_s^2 - \Lambda\omega$ term by term. With $l_0 = 1 - s/L_n$, $l_1 = s/L_n$, and v_0, v_1 the solutions of the homogeneous equation on the edge with boundary values $(1,0)$ and $(0,1)$, the same argument applies with one factor replaced by l_i, and we obtain

$$G_0 \le G_\Lambda \le (1 - \delta_n^{\mathrm{edge}})^{-1} G_0, \qquad l_i \le v_i \le (1 - \delta_n^{\mathrm{edge}})^{-1}\, l_i. \tag{5.20}$$

For u vanishing at both ends of the edge, $\Lambda \int \omega |u|^2 \le \delta_n^{\mathrm{edge}} \int |u_s|^2$. Hence the direct sum G_D of the Dirichlet inverses of all edges has norm at most $C\eta/\Lambda$ on $L^2(\mathsf{p}\,dx)$.

Step 2: the boundary matrix of an edge. For boundary values (a, a') at the two ends of E_n the homogeneous solution is $av_0 + a'v_1$, and its outward derivatives at the two ends, in the variable s, are given by the matrix

$$\begin{pmatrix} d_{n,0} & -c_n \\ -c_n & d_{n,1} \end{pmatrix}, \qquad c_n = \frac{1}{L_n} + \Lambda \int \omega\, l_0 v_1, \qquad d_{n,i} = \frac{1}{L_n} - \Lambda \int \omega\, l_i v_i;$$

the formulas follow from Green's identity with the functions l_i. Since $l_0 l_1 = G_0(s,s)/L_n$, (5.20) gives

$$c_n = \frac{1 + O(\eta)}{R_n}, \qquad b_n := -d_{n-1,1} - d_{n,0} = \Lambda\mu_n\,(1 + O(\eta) + O(K^{-1})). \tag{5.21}$$

For the second formula: on I_n we have $l_0 = 1 + O(R_n^I/L_n)$, so that interval contributes $\Lambda\mu_n(1 + O(\eta))$ to $\Lambda \int \omega\, l_0 v_0$, and J_n contributes at most $C\Lambda\nu_n \le C\eta\Lambda\mu_n$. The preceding edge contributes to $\Lambda \int \omega\, l_1 v_1$ at most $C\Lambda$ times $\nu_{n-1} + \mu_{n-1}(R_{n-1}^I/L_{n-1})^2$, and the second term is at most $\eta^2 K^{-2}\mu_n$ by (5.18). The two terms $1/L_n$ and $1/L_{n-1}$ are at most $\Lambda\mu_n/K$ each. The same computation covers an initial resistance edge.

Step 3: interpolation of node values. For a sequence $a = (a_n)$ of node values let Ta be the function which on each edge is the homogeneous solution with these boundary values, $(Ta)|_{\mathsf{E}_n} = a_n v_0 + a_{n+1} v_1$. Let $\mathfrak{h} = \ell^2(\mu_n)$. Then

$$c\,\|a\|_{\mathfrak{h}}^2 \le \|Ta\|_{L^2(\mathsf{p}\,dx)}^2 \le C\,\|a\|_{\mathfrak{h}}^2. \tag{5.22}$$

Indeed, by (5.20) the function Ta is close to a_n on I_n, which carries the mass μ_n. On I_n the neighbouring value enters with the factor $l_1 \le R_n^I/L_n$, and the sum of these contributions is $O(\eta^2/K^2)\,\|a\|_{\mathfrak{h}}^2$ after a shift of index. The contribution of J_n is controlled by the last inequality of (5.18).

Step 4: the node operator. The quadratic form of $A_{\mathsf{p}} - \Lambda$ on interpolants is, by Step 2,

$$(\mathcal{D}a, a)_{\mathfrak{h}} = -\sum_n b_n |a_n|^2 - 2\,\mathrm{Re}\sum_n c_n\, a_n \overline{a_{n+1}},$$

which defines a symmetric operator $\mathcal{D}$ on $\mathfrak{h}$. In the coordinates $\sqrt{\mu_n}\,a_n$ its diagonal entries have size Λ by (5.21), while each off-diagonal entry is bounded by $C/(R_n\sqrt{\mu_n\mu_{n+1}}) \le C\Lambda/K$. By diagonal dominance

$$\|\mathcal{D}^{-1}\| \le \frac{C}{\Lambda},$$

independently of the number of edges.

Step 5: the inverse. These algebraic facts give an inverse of the operator because they respect its form domain. On the one hand, the energy of an interpolant is bounded by $C\Lambda\|a\|_{\mathfrak{h}}^2$, by the formula for $\mathcal{D}$ and (5.22). On the other hand, averaging $u(x_n) - u(x)$ over I_n, where x_n is node n, gives

$$\mu_n\,|u(x_n)|^2 \le 2\,\|u\|^2_{L^2(I_n,\mathsf{p}\,dx)} + 2\mu_n R_n^I \int_{I_n} |u_s|^2\,ds,$$

so every function in the form domain has node values in $\mathfrak{h}$ and splits as $u = Ta + u^\circ$ with u° vanishing at all nodes. The two parts are orthogonal for the form of $A_{\mathsf{p}} - \Lambda$, because Ta solves the homogeneous equation on each edge. Therefore

$$(A_{\mathsf{p}} - \Lambda)^{-1} = G_D + T\,\mathcal{D}^{-1}T^*, \qquad (T^*f)_n = \frac{1}{\mu_n}\int \mathsf{p}\,(T\mathbf{e}_n)\,f\,dx, \tag{5.23}$$

where $\mathbf{e}_n$ is the nth unit sequence. Truncations of the tail by a Dirichlet wall at a far node obey the same formula and converge strongly, since the diagonally dominant series for $\mathcal{D}^{-1}$ converges uniformly on finite compressions; their limit is the Friedrichs realization identified by the decomposition of the form domain. Steps 1, 3 and 4 now give (i).

Step 6: the decaying solution. A homogeneous solution is the interpolant of its node values, and continuity of the flux at node n reads $-b_n a_n = c_{n-1}a_{n-1} + c_n a_{n+1}$. With $a_n = (-1)^n y_n$ this is

$$b_n\,y_n = c_{n-1}\,y_{n-1} + c_n\,y_{n+1}.$$

Put $\epsilon_n = c_n^2/(b_n b_{n+1}) \le C/K^2$ and $t_n = b_{n+1}y_{n+1}/(c_n y_n)$. Eliminating backwards from the outer end,

$$t_n = \frac{1}{1 - \epsilon_{n+1}t_{n+1}}.$$

For large K these maps preserve the interval $[1,2]$ and contract it by a factor at most $\frac12$. Hence the solutions with a Dirichlet wall at a far node converge, as the wall recedes, to a unique sequence with $t_n \in [1,2]$ for all n. By (5.21) it satisfies (5.19), and by (5.22) its interpolant is square integrable.

Any solution independent of this one has node values of the form

$$z_n = y_n\Big(C + \mathcal{W}\sum_{k<n}\frac{1}{c_k y_k y_{k+1}}\Big), \qquad \mathcal{W} \neq 0.$$

Consecutive terms of the sum grow by a factor at least cK^2, so eventually $|z_{n+1}/z_n| \asymp \Lambda\mu_n R_n$ and $\sqrt{\mu_{n+1}}\,|z_{n+1}| \ge cK\sqrt{\mu_n}\,|z_n|$. On I_n the contribution of the neighbouring node is at most

$C\eta$ times that of node n itself, so the $L^2(\mathsf{p}\,dx)$ norm of this solution diverges. This proves the uniqueness in (ii); in the language of Sturm–Liouville theory, the outer endpoint is of limit-point type.

Step 7: flatness for the constructed tails. Summing (5.16) in (5.19) gives, for $k \ge m$,

$$\left|\frac{a_k}{a_m}\right| \le \mathcal{E}_k^C\,\mathcal{E}_m^C\,e^{-(S_k-S_m)}, \qquad \text{since} \quad 2\sum_{n=m}^{k-1} n^2 = \tfrac{2}{3}(k^3-m^3) - (k^2-m^2) + \tfrac{1}{3}(k-m), \tag{5.24}$$

where the accumulated errors $O(n^{3/2}+n)$ and the quadratic terms are absorbed by $q_k + q_m$. By Lemma 5.2 every radial derivative of fixed order costs at most $e^{C_s n^2}$ on cell n, and the normalized gauge factor $(r/r_\partial)^{j^2}$ is bounded with bounded inverse in the range (5.17). Thus, after a normalization at the entrance, the solution and all its derivatives of fixed order are bounded on cell n by $\mathcal{E}_n^{C_s} e^{-S_n}$, which tends to zero faster than every power of d_∂. This is (iii). □

Remark 5.7. The contraction in Step 6 is uniform in the amplitude ratios, so the natural line depends continuously on $(\sigma_k)_{k\ge N}$ in the product topology. This is used in Lemma 5.9.

5.6 Localized bounds for the deep inverse

Later we need more than the operator norm: the kernel of the deep inverse decays between distant cells.

The node Green matrix. Let N be the symmetric tridiagonal matrix with diagonal entries b_n and off-diagonal entries $-c_n$. Up to the signs $(-1)^n$ and the weights μ_n it represents $-\mathcal{D}$, and by Step 4 it is positive with a bounded inverse in the coordinates $\sqrt{\mu_n}\,a_n$. Let $\Gamma = \mathsf{N}^{-1}$. Its diagonal satisfies $b_m^{-1} \le \Gamma_{mm} \le C b_m^{-1}$, by the Neumann series with positive terms. For fixed m the sequence $n \mapsto \Gamma_{nm}$, $n \ge m$, is a decaying solution of the recurrence, so by uniqueness

$$\Gamma_{nm} = \Gamma_{mm}\,\frac{y_n}{y_m} \le \frac{C}{\Lambda\mu_m}\prod_{k=m}^{n-1}\frac{1+C(\eta+K^{-1})}{\Lambda\mu_{k+1}R_k} \qquad (n \ge m), \tag{5.25}$$

and symmetry gives the bound for $n < m$. The kernel of $T\mathcal{D}^{-1}T^*$ in (5.23) is built from Γ with additional signs, which do not matter for absolute bounds, and with at most two interpolation functions at each of its two arguments; the kernel of G_D lives on single edges. With (5.24) we conclude: for the constructed tails, the part of the kernel of $(A_\mathsf{p} - \Lambda)^{-1}$ leading from cell m to cell n has operator norm at most

$$C\,\mathcal{E}_n^C\,\mathcal{E}_m^C\,e^{-|S_n-S_m|}. \tag{5.26}$$

Return to radial units. Let $g = (r/r_\partial)^{j^2}$ and $\mathsf{p}^\circ = g^2 w/r$, a constant multiple of p_j. By Lemma 5.3,

$$L_j(gV) = -\frac{gw}{r^2}\,(A_{\mathsf{p}^\circ} - \Lambda)V, \qquad \mathsf{p}^\circ\,dx = \frac{g^2 w}{r^2}\,dr, \qquad G_{L_j}(r,r') = -g(r)\,g(r')\,G_A(x,x'),$$

where G_{L_j} and G_A are the kernels of the inverses of L_j, with respect to dr, and of $A_{\mathsf{p}^\circ} - \Lambda$, with respect to $\mathsf{p}^\circ dx$. The gauge factors are bounded above and below, so (5.26) holds for the inverse of L_j in the dr pairing. In terms of the operator $B_j = -\varpi^{-1}L_j$ of (4.20),

$$B_j = \frac{1}{12}\,g\,(A_{\mathsf{p}_j} - j^2(j^2-1))\,g^{-1}, \qquad \|B_{j,D}^{-1}\| \le \frac{C}{j^4} \tag{5.27}$$

on a deep tail, where $B_{j,D}$ has the Dirichlet condition at the entrance.

The entrance impedance. For the natural solution, the boundary matrix of Step 2 gives at a mass entrance $\mathsf{p}V_x/(\Lambda\mu_n V) = 1+O(\eta)+O(K^{-1})$; if an initial resistance edge comes first, the ratio of flux to value is instead $-(1+O(\eta)+O(K^{-1}))/R_0$. In both cases the boundary term of Green's identity at the outer endpoint vanishes for the natural solution: at node n it is bounded by $C\Lambda\mu_n|a_n|^2 \to 0$, and finiteness of mass and energy identifies this with the limit at the endpoint.

5.7 Frequency bands: one transition cell at every frequency

How a cell acts on a solution depends on the frequency. If the action $\tau_{n,j} = j\sqrt{A_n}$ of its ramps is large, the cell is isolated from its neighbours and is governed by its own spectral gap (Lemma 4.7). If the condition (5.17) holds, the cell is part of a deep tail and is governed by the recurrence. The following definition arranges that, at each frequency, at most one cell at each end falls between the two regimes. Fix a large constant C_* and put

$$\mathsf{j}_n = C_*\, q_n\, A_n^{-1/2}, \qquad \mathrm{Band}_n = \{j \text{ even}:\ \mathsf{j}_{n-1} \le j < \mathsf{j}_n\} \quad (n > N), \tag{5.28}$$

and let the first band Band_N consist of all even j with $2 \le j < \mathsf{j}_N$. The *bands* Band_n, $n \ge N$, partition the even frequencies. For $j \in \mathrm{Band}_n$:

- every earlier cell $m < n$ has large action, $\tau_{m,j} = j\sqrt{A_m} \ge C_* q_m$. These cells, together with the central model, form the *inner region*;
- cell n is the *transition cell*;
- the cells $m > n$ form a deep tail, because

$$j^2 A_{n+1}\sqrt{\log(A_{n+1}/B_{n+1})} \le C\, C_*^2\, q_n^2\, \frac{A_{n+1}}{A_n}\,(n+1) \longrightarrow 0. \tag{5.29}$$

So there is no range of frequencies left unexamined between the local spectral inverses and the deep inverse. The decomposition depends on the frequency; the profile does not. This structure is the key to the assembly of the global inverse in Section 6.

An auxiliary cone at the entrance of a tail. To define an incoming line for the first band, attach at the entrance of each tail an auxiliary cone $w = cr$. For the right tail the cone is continued inward to $r = 0$, for the left tail outward to $r = \infty$. The transition between cone and first cell is a short positive interval with $rw'/w \ge 1$, constructed by applying (5.8) to $\log(w/r)$. On the cone, in $y = \log r$,

$$B_j = \tfrac{1}{12}\,(-\partial_y^2 + j^2), \qquad \varpi\, dr = 12c\, dy;$$

the solutions regular at the far end of the cone are r^j for the right tail and r^{-j} for the left tail, and the Dirichlet inverse of B_j has norm at most $12/j^2$. The cone serves only to specify an incoming line. The final profile retains a finite piece of it, the *cone head*.

5.8 Choosing the amplitudes so that the lines do not meet

Fix an endpoint, a band index n and a frequency $j \in \mathrm{Band}_n$. Consider the join which precedes cell n in the outward order (for $n = N$: the transition from the auxiliary cone). Let W_{in} be the value of the profile at its inner end and $\sigma_n W^0_{\mathrm{out}}$ its value at the outer end. By (5.1), W_{in} does not depend on σ_n, and W^0_{out} is the outer value for $\sigma_n = 1$. Put

$$\Xi_n = \frac{W^0_{\mathrm{out}}}{W_{\mathrm{in}}}, \qquad \Upsilon_n = \max(\Xi_n, \Xi_n^{-1}), \qquad \log\Upsilon_n = 4n^2 + O(n^{3/2}+n). \tag{5.30}$$

At the first join the amplitude c of the auxiliary cone is chosen, independently of σ_N, so that the contrast is the same. The radii and all cut points are independent of the amplitude ratios.

The role of σ_n is the following. Everything beyond the join is multiplied by σ_n; this does not change the homogeneous equation there, hence not the natural line. The incoming line is determined by what lies before the join, and does not depend on σ_n either. What σ_n changes is the scale of the flux wu' by which the two lines are compared across the join. We show that this is enough to move them apart.

A state variable adapted to the join. On a join w'/w is huge, and the pair (u, wu') is not a convenient state. Define instead

$$\zeta_j = \frac{ru_r}{j^2} - u. \tag{5.31}$$

If $L_j u = 0$, then $(w\zeta_j)' = -(j^2-1)\, wu_r/j^2$, in which w' no longer appears. For the state $Y = (u,\ w\zeta_j/W_{\mathrm{in}})$ this gives

$$Y_r = \frac{1}{r}\begin{pmatrix} j^2 & j^2\, W_{\mathrm{in}}/w \\ -(j^2-1)\, w/W_{\mathrm{in}} & -(j^2-1) \end{pmatrix} Y. \tag{5.32}$$

The coefficients involve only the sizes of w and w^{-1} on the join, which lie between those at its ends.

In this subsection the *incoming line* at the join means the line of the solution which vanishes at a cut inside the positive ramp preceding the join. The true incoming line differs from it by an exponentially small amount, which is accounted for in Section 6.6. Let

$$\delta_n = c_0\, e^{-n^2}.$$

Lemma 5.8 (The excluded amplitude ratios for one frequency)**.** *Let C_* be large. For every $n \geq N$, every $j \in \mathrm{Band}_n$ and every choice of the ratios σ_k, $k \neq n$, the set of $\sigma_n \in [1,2]$ for which the incoming and the natural line at the join preceding cell n make an angle of sine less than δ_n, in the state (5.32) at the outer end of the join, is contained in an interval of length at most $C\delta_n$. The constant C does not depend on n, on j, on the other ratios or on the contrast Ξ_n.*

Proof. Step 1: the join is almost transparent. By the Volterra series, the transfer matrix of (5.32) across the join differs from the identity by at most $e^{\eta_{n,j}} - 1$, where

$$\eta_{n,j} = Cj^2\, \frac{\ell_n}{r_{\min}}\, \Upsilon_n \leq C\, C_*^2\, q_n^2\, e^{-7n^2 + O(n^{3/2}+n)} = o(e^{-n^2}), \tag{5.33}$$

ℓ_n being the length of the join. This is the computation behind the exponent 12 in (5.6): the steepness of the profile on the join causes no difficulty.

Step 2: the incoming line has a definite slope. Normalize the solution which defines the incoming line to $u = 1$ at the inner end of the join. Then

$$a := \frac{\zeta_j}{u} \in \left[-\tfrac{3}{2}, -\tfrac{1}{2}\right]. \tag{5.34}$$

Indeed, on the ramp $u_{XX} = j^2|\rho|\, u$, and comparison with the Riccati equation for a constant potential gives $|u_X/u| \leq Cj\sqrt{b}\, \coth(cj\sqrt{A})$, where b is the size of $|\rho|$ on the ramp and $A = A_{n-1}$. The ramp is either the short one, with $|\rho| \asymp A$, length $\asymp 1$ in X and $r_X \asymp A$, or the long one, with $|\rho| \asymp A^9$, length $\asymp A^{-4}$ and $r_X \asymp A^5$. In both cases $r_X \asymp \sqrt{Ab}$, so

$$\left|\frac{ru_r}{j^2 u}\right| \leq \frac{C}{j\sqrt{A_{n-1}}} < \frac{1}{2}$$

once C_* is large, because $j\sqrt{A_{n-1}} \geq C_* q_{n-1}$. At the left endpoint only the sign of ru_r changes. In the first band the cone gives $ru_r/(j^2 u) = \pm 1/j$, which is at most $\frac{1}{2}$ in absolute value even for $j = 2$.

Step 3: the excluded set. Let (β_1, β_2) span the natural line in the frame (u, ζ_j) at the outer end of the join, computed for $\sigma_n = 1$. In the common frame Y the natural line is then spanned by

$$\mathbf{b}(\sigma) = (\beta_1,\ \sigma\, \Xi_n\, \beta_2), \qquad \sigma = \sigma_n,$$

while the incoming line at the inner end is spanned by $(1, a)$; see Figure 12. If $\beta_1 = 0$ the angle between the two lines is bounded below. Otherwise put $b_\sigma = \sigma \Xi_n \beta_2/\beta_1$; the sine of the angle is

$$\frac{|a - b_\sigma|}{\sqrt{1 + a^2}\,\sqrt{1 + b_\sigma^2}}.$$

If this is small, b_σ lies in a fixed compact interval of negative numbers, on which $|db_\sigma/d\sigma| = |b_\sigma|/\sigma$ is bounded below. Therefore the set of $\sigma \in [1, 2]$ for which the sine is less than $2\delta_n$ is contained in an interval of length at most $C\delta_n$. By Step 1 the state of the incoming solution at the outer end of the join is $(1, a) + O(\eta_{n,j})$, and normalization of vectors is Lipschitz near $(1, a)$. Since $\eta_{n,j} = o(\delta_n)$ by (5.33), an angle of sine less than δ_n at the outer end forces a sine less than $2\delta_n$ for the line $(1, a)$, and the claim follows. None of the bounds used depends on j, on the other ratios or on the size of Ξ_n. □

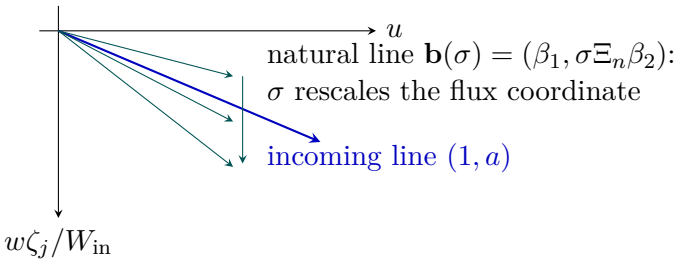


Figure 12: Selection of an amplitude ratio in the plane of boundary states. The slope of the incoming line lies in a fixed negative interval. As σ varies, the natural line sweeps past it with speed bounded below, so the two lines are close only for a short interval of σ.

Lemma 5.9 (Simultaneous selection of the amplitudes)**.** *If N is large enough, then at each endpoint the amplitude ratios $\sigma_n \in [1, 2]$, $n \geq N$, can be chosen so that for every n and every $j \in \mathrm{Band}_n$ the incoming and natural lines at the join preceding cell n make an angle of sine at least δ_n, in the sense of Lemma 5.8.*

Proof. The band Band_n contains at most $\mathsf{j}_n = C_* q_n e^{n^2/2}(1 + o(1))$ frequencies, so the total length of the excluded intervals satisfies

$$\sum_{n \geq N} C\, \#\,\mathrm{Band}_n\ \delta_n \leq C\, C_* c_0 \sum_{n \geq N} q_n\, e^{-n^2/2} < \frac{1}{2} \tag{5.35}$$

for large N. The excluded interval of Lemma 5.8 for a pair (n, j) depends on the ratios σ_k with $k \neq n$, but the bound for its length does not, and this is all we need.

Consider first finitely many bands $N \leq n \leq N_1$, with $\sigma_k = 1$ for $k > N_1$, and Lebesgue measure on the cube $[1, 2]^{N_1 - N + 1}$. By Fubini's theorem and the uniform one-dimensional bound, the set of ratios violating the condition for a given pair (n, j) has measure at most $C\delta_n$, and by (5.35) the union of these sets has measure less than one. So a good choice exists

for every N_1. Extend each of these choices by ones and pass to a subsequence converging in every coordinate. For a fixed frequency, the transfer across finitely many cells is continuous in the ratios, and the natural line is continuous in the product topology by Remark 5.7. Each condition on the angle is closed, hence holds for the limit. The two endpoints are treated independently. □

5.9 Summary

We have constructed, at each endpoint, a smooth flat tail with the following properties, for every even frequency j with band index n: beyond cell n the equation $L_j u = 0$ has a unique decaying solution, which is flat at the endpoint, and L_j has a Dirichlet inverse with the localized bound (5.26); at the entrance of cell n the natural line makes an angle at least δ_n with the line arriving through the preceding positive ramp.

We have *not* yet shown that gluing the tails to the central model preserves the spectral gaps of the pieces. That global question is the subject of the next section. Its answer uses both halves of the present construction: the selected angle at the transition cell, and the strong attenuation through the positive ramp that precedes it.

6 The inverse on the whole annulus

Spectral gaps of the individual pieces do not automatically survive gluing. A homogeneous solution can leave one piece, return through its neighbour, and satisfy a new matching condition. We must therefore solve the matching problem for the whole annulus, not just list the inverses of the pieces. This section does so and completes the proof of Proposition 4.2.

6.1 Overview

Everything is done one frequency at a time. Fix an even frequency j and let n be its band index, $j \in \mathrm{Band}_n$ (Section 5.7). At each end the annulus is divided into three regions (Figure 13).

- The *inner region*: the central model, the cells $m < n$ of both tails, and the positive intervals between them. All these pieces have large action at the frequency j, so each is governed by its own spectral inverse, and neighbouring pieces are coupled only through exponentially small terms.
- The *outer core* at each end: the join preceding cell n and cell n itself. Nothing is small here, but the core is a single cell: crossing it costs a known finite factor, and the amplitude selection of Lemma 5.9 keeps the matching angle at its entrance above δ_n.
- The *deep tail* at each end: the cells $m > n$, governed by the recurrence of Lemma 5.6.

The inverse is assembled in four steps: an inner inverse G_h with zero boundary values at the two entrances of the outer cores (Section 6.4); an outer inverse R_o at each end, with the natural condition at the endpoint (Section 6.5); a 2×2 linear system for the two boundary values that remain, one for each end (Section 6.6); and weighted estimates for the resulting formula (Section 6.7). Solving on the pieces first and then solving a finite system for the values on the cuts is again an instance of the Schur complement formula [SZ07]; the matrices S and D_b below are Schur complements in this sense.

One principle organizes the estimates. A correction created at the transition cell is large: it costs δ_n^{-1}, the transfer factor $e^{C_o \tau_{n,j}}$ of the core, and coefficient powers. If its output stays in cell n or beyond, this cost is absorbed by the coefficient weight allowed in the output norm there. If its output returns to the interior, it must first cross the positive ramp preceding

the transition join, which attenuates it by $e^{-c\tau_{n-1,j}}$; and $\tau_{n-1,j}/\tau_{n,j} \asymp e^n$, so the attenuation wins.

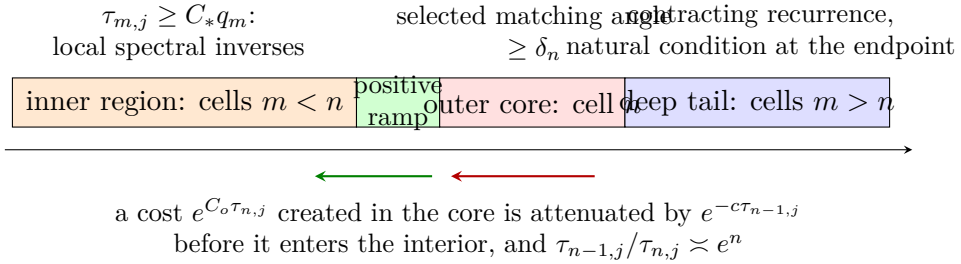


Figure 13: The decomposition of one end of the annulus at a frequency $j \in \mathrm{Band}_n$. The other end is treated at the same time. The position of the division changes with the frequency; the profile is chosen once.

Notation of this section. We write w for the profile. Estimates are made in the Hilbert space $L^2(d\mu)$, $d\mu = \varpi\, dr$, with the conventions

$$B_j = -\varpi^{-1} L_j, \qquad F = -\varpi^{-1} f, \qquad \langle u, F\rangle_{L^2(d\mu)} = -\int \overline{u}\, f\, dr, \tag{6.1}$$

so that $L_j u = f$ is the same as $B_j u = F$; we also write $\mathscr{H}_j = -L_j$. For a border column $k = -\varpi^{-1} g_0$ the Hilbert adjoint is $k^* u = -\ell_0 u$; so after a change of sign of the scalar equation the bordered operator is symmetric. This convention is used in all energy identities below. Finite matrices that act on boundary values are written in sans-serif type when their letter is also used for a scale ($\mathsf{A}, \mathsf{B}, \mathsf{K}, \mathsf{S}$). The letters λ, Λ, κ, T, D, E have the local meanings stated in each subsection.

6.2 Completing the profile while keeping the seed

We first finish the construction of the profile, in such a way that the solution of $L_2\phi = 0$ is preserved exactly. The device is to prescribe the solution and to solve for the profile.

Lemma 6.1 (Connectors)**.** *On an interval put $y = \log r$ and $w = re^f$, and let ϕ be a nonvanishing function with logarithmic slope $\mathsf{k} = \phi_y/\phi$. Then $L_2\phi = 0$ is equivalent to*

$$f_y = \frac{\mathsf{k}^2 + \mathsf{k}_y - 4}{4 - \mathsf{k}}, \qquad \textit{that is,} \qquad 1 + f_y = \frac{rw'}{w} = \frac{\mathsf{k}^2 - \mathsf{k} + \mathsf{k}_y}{4 - \mathsf{k}}. \tag{6.2}$$

Proof. In the variable y the equation $r^2 L_2\phi = 0$ reads $w\phi_{yy} + (w_y - w)\phi_y - 4w_y\phi = 0$. Divide by $w\phi$ and use $\phi_{yy}/\phi = \mathsf{k}_y + \mathsf{k}^2$ and $w_y/w = 1 + f_y$. □

Thus any function $\mathsf{k}(y)$ with $\mathsf{k} < 4$ and $\mathsf{k}^2 - \mathsf{k} + \mathsf{k}_y > 0$ produces, by integrating first f_y and then $\phi_y = \mathsf{k}\phi$, a positive interval together with a solution of $L_2\phi = 0$ on it. For constant $\mathsf{k} = \pm 2$ the right side of (6.2) is 1: the interval is a cone and $\phi = r^{\pm 2}$. This is more reliable than interpolating w and hoping to keep the kernel.

The central side. By (4.18) the logarithmic slope of the central solution ϕ_0 tends to 2 on the left and to -2 on the right, with all derivatives. Cut the central model far out on both sides and interpolate k to the constant ± 2 over a unit interval of y, with cutoffs that match all derivatives. Since the right side of (6.2) is 1 at the constant, closeness in C^1 gives $1 + f_y \geq \frac{3}{4}$. This *connector* ends in an exact cone on which $\phi = r^2$ (left) or $\phi = r^{-2}$ (right).

The tail side. On the auxiliary cone of a tail with selected amplitudes, the natural solution at frequency two is $C_+r^2 + C_-r^{-2}$. At the right tail the incoming line is spanned by r^2, so the nonzero angle of Lemma 5.9 gives $C_- \neq 0$; far along the cone, towards $r = 0$, the slope of the natural solution tends to -2. At the left tail the incoming line is r^{-2}, hence $C_+ \neq 0$ and the slope tends to 2 towards $r = \infty$. In both cases the natural solution does not vanish far along the cone, and the same interpolation of k connects it to an exact cone on which the solution is r^{-2} (right tail) or r^2 (left tail), keeping $1 + f_y \geq \frac{3}{4}$.

Assembly. The exact cones on the central side and on the tail side carry the same solution, r^{-2} on the right and r^2 on the left. A change of radius

$$\widetilde{r} = Rr, \qquad \widetilde{w}(Rr) = R\,w(r) \tag{6.3}$$

preserves w/r, the operators B_j and the measure $\varpi\,dr$; it places each tail at the required radius. A rescaling of the amplitude of the tail then matches the additive constant of f, without changing homogeneous solutions or travel angles, and a constant multiple of the tail solution matches its value. Cone segments of arbitrary length may be left between the connectors.

The result is one smooth profile w_*, flat at both endpoints by Lemma 5.2, and one function ϕ_* with $L_{2,w_*}\phi_* = 0$ which is flat at both endpoints by Lemma 5.6(iii). Only positive intervals were added, so the negative components and their travel angles are those of the models; part (ii) of Proposition 4.2 follows from Section 4.5 and Lemma 4.7, with $\gamma_m^{-1} = 3\sqrt{d_m}$. From now on $w = w_*$. It remains to prove that there is no kernel besides ϕ_* and that the inverse satisfies the stated estimates.

6.3 A tool: cutting a model inside a positive buffer

The spectral information of Section 4 concerns models on the whole line, whereas the profile contains only finite pieces of them. The following lemma says that cutting a model inside a region where its potential is large and positive changes its inverse, and its boundary data, by an exponentially small amount. It is formulated for the Schrödinger operator in the model coordinate.

Lemma 6.2 (Cutting inside a positive buffer)**.** *Let $H_0 = -\partial_X^2 + V$ be self-adjoint on a full model, possibly with a compact symmetric scalar border supported away from the cuts, and let $\mathcal{H}$ denote the resulting operator. Assume $\mathcal{H}$ is invertible with $\|\mathcal{H}^{-1}\| \leq B_0$. Let there be one or two cuts c_i, and at each cut numbers $m_i > 0$ and $\Sigma_i > 0$ such that*

- *$V \geq m_i^2$ on the exterior of the cut and on the inward interval of length $(\Sigma_i + 6)/m_i$ adjacent to it (the* buffer*);*
- *$V \leq Cm_i^2$ on two layers of width $4/m_i$: one at the cut and one at the inner end of the buffer;*
- *an exterior wall, if there is one, is at distance at least $4/m_i$ from the cut.*

Let α_i^{in} be the least energy $\int(|v'|^2 + V|v|^2)\,dX$ of a function on the buffer with $v = 1$ at the cut and $v = 0$ at the inner end of the buffer, and α_i^{out} the least energy of a function on the exterior with $v = 1$ at the cut. Then $\alpha_i^{\mathrm{in}}, \alpha_i^{\mathrm{out}} \asymp m_i$. Put $D = \mathrm{diag}(\alpha_i^{\mathrm{in}} + \alpha_i^{\mathrm{out}})$.

Let $\mathcal{H}_D$ be the operator on the retained interval with the Dirichlet condition at the cuts; let T be its Poisson map, which sends boundary values at the cuts to the homogeneous solution with zero scalar equation; and let Λ be its matrix of outward derivatives at the cuts. If

$B_0 m_i^2 e^{-\Sigma_i}$ is sufficiently small at every cut, then

$$\begin{aligned} \|\mathcal{H}_D^{-1}\| \le B_0 + C\sum_i m_i^{-2}, \qquad \|TD^{-1/2}\| \le C\Big(\sum_i m_i^{-2}\Big)^{1/2}, \\ D^{-1/2}(\Lambda - \operatorname{diag}\alpha_i^{\mathrm{in}})D^{-1/2} = O(E), \qquad |E_{ik}| \le CB_0\, m_i m_k\, e^{-\Sigma_i - \Sigma_k}. \end{aligned} \tag{6.4}$$

In particular Λ is positive definite.

In words: the cut model has essentially the inverse of the full model; a boundary value at a cut penetrates only a distance $1/m_i$; and the response of the cut model to a boundary value is, up to a doubly exponentially attenuated error, that of the buffer alone.

Proof. Step 1: the energies. Expanding $\int |v' \pm m_i v|^2 \ge 0$ shows that the energy of a function with $v = 1$ at the cut is at least m_i. A test function supported within distance $2/m_i$ of the cut gives the matching upper bound.

Step 2: an exact Green column at each cut. Join at the cut the minimizers of the two energies. After division by $\alpha_i^{\mathrm{in}} + \alpha_i^{\mathrm{out}}$ this is the Green function, with pole at the cut, of the operator with a Dirichlet wall at the inner end of the buffer. Multiply it by a cutoff which vanishes near that wall, and call the result K_i. Comparison with the solutions of $-v'' + m_i^2 v = 0$ gives

$$\mathcal{H}(K_i, 0) = (\delta_{c_i} + f_i,\, 0), \qquad \|K_i\|_2 \le C m_i^{-3/2}, \qquad \|f_i\|_2 \le C m_i^{1/2} e^{-\Sigma_i}.$$

The commutator term f_i is an ordinary L^2 function supported in the inner layer, and the support of the cutoff does not meet the border. Therefore

$$\boldsymbol{g}_i = (K_i, 0) - \mathcal{H}^{-1}(f_i, 0)$$

satisfies $\mathcal{H}\boldsymbol{g}_i = (\delta_{c_i}, 0)$ exactly.

Step 3: the matrix of cut values. Let Γ_{ik} be the value at c_i of the field component of $\boldsymbol{g}_k$. The correction $\mathcal{H}^{-1}(f_k, 0)$ solves the homogeneous equation in the buffer at c_i and satisfies the exterior condition, so on its way from the inner layer to the cut it decays by a further factor $e^{-\Sigma_i}$. Passing from an L^2 bound to a pointwise bound costs $C m_i^{1/2}$: where $V \ge m_i^2$ the modulus of this solution is monotone, so its value at a point is at most $m_i^{1/2}$ times its L^2 norm over the preceding interval of length $1/m_i$. With $\|\mathcal{H}^{-1}(f_k, 0)\| \le B_0\, C m_k^{1/2} e^{-\Sigma_k}$ and $D_{ii} \asymp m_i$ this gives

$$D^{1/2}\,\Gamma\, D^{1/2} = I + E, \qquad |E_{ik}| \le CB_0\, m_i m_k\, e^{-\Sigma_i - \Sigma_k},$$

which also follows from Green's identity applied to two of the commutator terms. Under the smallness assumption this matrix is invertible.

Step 4: the cut operator. Let ι extend a source on the retained interval by zero, let $\mathcal{R}$ restrict a field to the retained interval, both keeping the scalar, and let $\boldsymbol{g}_I$ be the restricted Green columns. Subtracting the cut values gives

$$\mathcal{H}_D^{-1} = \mathcal{R}\,\mathcal{H}^{-1}\iota - \boldsymbol{g}_I\,\Gamma^{-1}\boldsymbol{g}_I^*, \qquad T = \boldsymbol{g}_I\,\Gamma^{-1}.$$

The kernel is trivial: a homogeneous solution with zero values at the cuts, extended by zero, solves the full equation with delta sources at the cuts, and the invertibility of Γ forces their coefficients to vanish. Moreover

$$\|\boldsymbol{g}_i\|\, D_{ii}^{1/2} \le C(m_i^{-1} + B_0\, m_i\, e^{-\Sigma_i}) \le C m_i^{-1},$$

which gives the first two bounds in (6.4). Finally, the interior and exterior derivatives of a Green column add up to the coefficient of its delta function, so $\Lambda = \Gamma^{-1} - \operatorname{diag}\alpha_i^{\mathrm{out}}$; this proves the third bound and the positivity. The symmetric border cancels in all the Green identities used. □

Translation to radial quantities. Lemma 6.2 is applied to the models through the dictionary of Lemma 4.3. For $u = \mathcal{P}^{-1/2}\psi$,

$$\|u\|^2_{L^2(d\mu)} = 12\lambda \int \upsilon^2 |\psi|^2 \, dX, \qquad w u_r = \lambda\sqrt{\mathcal{P}}\Big(\psi_X - \frac{\mathcal{P}_X}{2\mathcal{P}}\,\psi\Big), \tag{6.5}$$

where $\upsilon = r_X/r$; working with $\sqrt{\lambda}\,\psi$ removes the amplitude λ from all formulas. The quantity m_i of the lemma is a lower bound for $\sqrt{V} = j\sqrt{|\rho|}$ on the buffer. The buffer is a fixed fraction of the ramp at a distance from the turning point, where ρ vanishes. For the two ramps of a Morse cell m, with ΔX the length of the buffer,

$$(m_i,\ \Delta X,\ \upsilon) \asymp (j\sqrt{A_m},\ 1,\ A_m) \text{ (short ramp)}, \qquad (jA_m^{9/2},\ A_m^{-4},\ A_m^5) \text{ (long ramp)}.$$

In both cases the buffer has action $\Sigma_i = c\,\tau_{m,j}$, with $\tau_{m,j} = j\sqrt{A_m}$ as in (5.5), and $\upsilon/m_i \le C/j$. The inverse of the full Morse model costs a coefficient power, by (4.23). For the central model, a cut at $|X| = X_0$ has $m_i \asymp jX_0$ and action $c\,jX_0^2$, and the weighted inverse of the full model is bounded by Lemma 4.6.

By the second formula in (6.5), the Liouville factor contributes to the radial flux. At the cuts of the central model, with $\epsilon_i = \pm 1$ the outward sign and $\mathcal{P}_i$ the value of $\mathcal{P}$ at the cut, the matrix of outward radial fluxes is

$$\Lambda_r = \lambda\ \mathrm{diag}(\sqrt{\mathcal{P}_i})\,\Lambda\ \mathrm{diag}(\sqrt{\mathcal{P}_i}) + \lambda\, a_c\ \mathrm{diag}(\epsilon_i \mathcal{P}_i). \tag{6.6}$$

Its leading diagonal part is therefore

$$\kappa_i = \lambda\,\mathcal{P}_i\,(\alpha_i^{\mathrm{in}} + \epsilon_i a_c), \tag{6.7}$$

which is positive and comparable to $\lambda\mathcal{P}_i m_i$ at far cuts. For a Morse cell $\mathcal{P} = 1$ and $\kappa_i = \lambda\alpha_i^{\mathrm{in}}$. The term with a_c must be kept for the fluxes to match exactly.

6.4 The inner region

Fix $j \in \mathrm{Band}_n$. The inner domain consists of the central model, the cells $m < n$ of both tails, and the positive intervals between them; its two outer boundary points are the entrances of the joins that precede cell n at the two ends. We impose zero boundary values there. At $j = 2$ the central model carries its temporary border.

Decomposition. Cut each model inside its positive ramps, at points to which Lemma 6.2 applies. The inner domain is then the union of the cut models and of the *gaps* between them; a gap consists of the outer parts of two ramps and of the join, connector or cone between them, and is a positive interval. On every gap the quadratic form of $\mathscr{H}_j$ is coercive:

$$Q_j(u) = \int \Big(w|u_r|^2 + j^2\,\frac{w'}{r}\,|u|^2 \Big)\, dr \ge c\, j^2\, \|u\|^2_{L^2(d\mu)}, \tag{6.8}$$

with $c = \frac{1}{6}$ on the joins, where $rw'/w \ge 2$ by (5.9), and a smaller fixed c on cone heads and on the connectors of the central model. Consequently the Poisson map P_C of a gap and its matrix Λ_C of outward fluxes satisfy $\|P_C a\| \le (C/j)\,(a^*\Lambda_C a)^{1/2}$.

Let D_h be the direct sum of the Dirichlet inverses of the cut models and of the gaps. Let T_h map values at the cuts to the function which is homogeneous on every cut model and every gap with these values (and zero scalar equation). At the cuts define

$$\mathsf{S}_0 = \mathrm{diag}(\kappa_i) + \bigoplus_C \Lambda_C, \qquad \mathsf{S} = \mathsf{S}_0 + E_h, \tag{6.9}$$

where S is the matrix of total outward fluxes of T_h, and E_h collects the errors of Lemma 6.2. A short gap can have a large off-diagonal entry in Λ_C; this entry is kept in S_0 and is not treated as an error. The matrix S_0 is positive, consists of blocks of size at most two, and dominates $D_\kappa = \operatorname{diag}(\kappa_i)$. A zero-boundary solution with source F has outgoing flux $-T_h^* F$ at the cuts, and matching values and fluxes gives the exact inner inverse

$$G_h = D_h + T_h\, \mathsf{S}^{-1}\, T_h^*, \tag{6.10}$$

provided S is invertible.

Weights. To obtain estimates which are uniform in the number of cells, and which carry the weights of Proposition 4.2, let W be a positive function equal to one on the finite middle part and of size

$$W_m = \Big(\frac{\mathcal{E}_m}{\mathcal{E}_N}\Big)^{\mathsf{a}} \Big(\frac{H_m}{H_N}\Big)^{-\gamma} \tag{6.11}$$

on cell m. We take W constant on each gap, equal to the geometric mean of the two neighbouring values, and let it vary inside the cells. For $|\gamma| \le 1$,

$$\Big|\log \frac{W_{m+1}}{W_m}\Big| \le C m^2 + C\,\mathsf{a}\, m^{3/2} \log(e+m), \tag{6.12}$$

so this choice changes the norms on each cell by coefficient powers only. Let $\Omega = \operatorname{diag}(W(c_i))$ be the weights at the cuts. Since the two ends of a gap have the same weight, Ω commutes with S_0.

Smallness of the coupling. By Lemma 6.2 and (6.5), the errors E_h, normalized by $\kappa_i^{1/2}$ on both sides and conjugated by the weights, are bounded by

$$\mathcal{E}_m^{C(\mathsf{a})} \langle j\rangle^C e^{-c\tau_{m,j}}, \qquad C\langle j, X_0\rangle^C e^{-cjX_0^2}, \qquad C_N \langle j\rangle^C e^{-cjL} \tag{6.13}$$

for a cell $m < n$, for the central cuts at $|X| = X_0$, and for a cone head of length L in $\log r$, respectively. The Poisson function of a model cut has a leading part of norm at most $(C/j)\sqrt{\kappa_i}\,|a_i|$ in $L^2(d\mu)$, because $\upsilon/m_i \le C/j$, and a remainder bounded by the first quantity in (6.13).

These bounds have to be small uniformly over all frequencies of all bands beyond m, that is, for all j with $t := \tau_{m,j} \ge C_* q_m$, however large. Since $\log\langle j\rangle \le \log(1+t) + Cm^2$, the logarithm of the first quantity in (6.13) is at most

$$c_1\, q_m + c_2 \log(1+t) - c_3 t \le c_1\, q_m - \tfrac{3}{4} c_3 t + C \le -\tfrac{1}{2} c_3 t + C \qquad \text{if } C_* \ge 4c_1/c_3, \tag{6.14}$$

with fixed $c_1, c_2, c_3 > 0$; here $m^2 = o(q_m)$ was absorbed in c_1, and $c_2 \log(1+t) \le c_3 t/4 + C$. So after C_* is fixed, increasing N makes these errors smaller than any prescribed tolerance, simultaneously for all $m < n$ and all admissible j. The other two quantities in (6.13) are made small by the choice of the central cuts and of the lengths of the cone heads. The constants are thus chosen in the order

$$\mathsf{a} \;\longrightarrow\; C_* \;\longrightarrow\; N \;\longrightarrow\; X_0,\ L. \tag{6.15}$$

Lemma 6.3 (The inner inverse)**.** *With the choices* (6.15)*, for both signs,*

$$\|\mathsf{S}_0^{-1/2}\, \Omega^{\pm 1} E_h\, \Omega^{\mp 1} \mathsf{S}_0^{-1/2}\| < \tfrac{1}{2}, \qquad \|W^{\pm 1}\, T_h\, \Omega^{\mp 1} \mathsf{S}_0^{-1/2}\| \le \frac{C}{j}. \tag{6.16}$$

Hence S *is invertible,* (6.10) *is a two-sided inverse of the inner problem, and there is a fixed exponent* I *such that, with* $W_1 = W\,(\mathcal{E}/\mathcal{E}_N)^{-\mathsf{I}}$*,*

$$\|W_1\, G_h\, W^{-1}\| \le C, \qquad \|W^{\pm 1}\, G_h\, W^{\mp 1}\| \le \mathcal{E}_n^C. \tag{6.17}$$

Proof. For the first bound in (6.16), normalize the error of each model by $\kappa_i^{1/2}$ at its two cuts. Every cut belongs to at most two such blocks, so their sum has norm at most twice the largest of their norms, and $\|\mathsf{S}_0^{-1/2} D_\kappa^{1/2}\| \le 1$ on both sides. For the second, consider the Poisson function of one model cut, with boundary value u_i. In the model coordinate (6.4) gives $\|\psi\|_{L^2(dX)} \le C m_i^{-1} D_{ii}^{1/2} |\psi_i|$, and by (6.5) and (6.7) this is $\|u\|_{L^2(d\mu)} \le C(\upsilon/m_i)\sqrt{\kappa_i}\,|u_i| \le (C/j)\sqrt{\kappa_i}\,|u_i|$ for the part of the function in the buffer; the rest is bounded by (6.13). Now sum the squares of the norms of the local Poisson functions over the cut models, which are disjoint, and over the gaps, which have bounded overlap with them. Both estimates are independent of the number of cells.

They give the inverse as follows. Since Ω commutes with S_0,

$$\Omega^{\pm1}\mathsf{S}^{-1}\Omega^{\mp1} = \mathsf{S}_0^{-1/2}\big(I + \mathsf{S}_0^{-1/2}\Omega^{\pm1}E_h\Omega^{\mp1}\mathsf{S}_0^{-1/2}\big)^{-1}\mathsf{S}_0^{-1/2},$$

and the middle factor has norm at most 2. Hence, using (6.16) with both signs,

$$\|W^{\pm1}\,T_h\mathsf{S}^{-1}T_h^*\,W^{\mp1}\| \le 2\,\|W^{\pm1}T_h\Omega^{\mp1}\mathsf{S}_0^{-1/2}\|\,\|\mathsf{S}_0^{-1/2}\Omega^{\pm1}T_h^*W^{\mp1}\| \le \frac{C}{j^2}.$$

So the coupling term of (6.10) loses nothing. The Dirichlet inverses in D_h cost coefficient powers of their own cells, which is the loss l in the first bound of (6.17); in the second bound all cells are estimated by the largest one. The invertibility of the Dirichlet blocks and of S gives both inverse identities, and the symmetry of G_h is the corresponding Green identity. □

The first estimate in (6.17) is localized: an inner source in cell m produces an output which is controlled with the weight of cell m, up to the loss l. It will handle ordinary inner sources. The second, coarser estimate will be applied only to boundary terms which already carry an exponentially small factor.

6.5 The outer region: an inverse without division by a boundary value

Fix an end. The outer region consists of the *outer core*, namely the join preceding cell n and cell n, and of the deep tail beyond it. The large positive ramp before the join belongs to the inner region. At the right end the deep tail begins with a mass interval, at the left end with an initial resistance edge (Section 5.5). Let r_o be a radial coordinate increasing outward, and choose a scale $\lambda > 0$ with $e^{-Cn^2} \le \lambda/\lambda_n \le e^{Cn^2}$. We describe solutions on the core by the state (u, z) with $z = w u_{r_o}/\lambda$.

Lemma 6.4 (Inputs from the core and the deep tail)**.** *Let $\mathsf{T}_{\mathrm{core}}$ be the transfer matrix of $L_j u = 0$ across the core in the state (u, z), and let P_d be the natural solution on the deep tail with value one at its entrance. Then*

$$\begin{gathered}\|\mathsf{T}_{\mathrm{core}}\| + \|\mathsf{T}_{\mathrm{core}}^{-1}\| \le \Pi_n := e^{Cn^2 + C_o\tau_{n,j}}, \qquad \int_{\mathrm{core}} \varpi\, dr_o \le \lambda\, e^{Cn^2}, \\ \|P_d\|_{L^2(d\mu)} \le \sqrt{\lambda}\, e^{Cn^2}, \qquad \|B_{d,D}^{-1}\| \le C j^{-4},\end{gathered} \tag{6.18}$$

where $B_{d,D}$ is B_j on the deep tail with the Dirichlet condition at its entrance.

Proof. The last bound is (5.27). The mass of the core is $12\lambda_n \int (r_X/r)^2\, dX$ plus the negligible mass of the join, and is estimated by (5.4). The contraction (5.19) bounds $\|P_d\|^2$ by a constant times the first mass of the deep tail; if an initial resistance edge comes first, the value at the next mass entrance is at most $C/(\Lambda\mu R_0)$ and the mass of the edge is small, with the same conclusion.

For the transfer matrix, consider first a positive piece. In the state (u, z) the equation is $u_{r_o} = (\lambda/w)\, z$, $z_{r_o} = (j^2 w_r/(\lambda r))\, u$. Replace z by ςz with a constant $\varsigma > 0$ and apply

Gronwall's inequality; the best choice of ς balances the integrals of the two coefficients, and $\varsigma^{\pm 1} \le e^{Cn^2}$ on the pieces of the core. This bounds the transfer in either direction by

$$e^{Cn^2} \exp\Big(2j \sqrt{\int \frac{dr_o}{w} \int \frac{w_r}{r}\, dr_o} \Big).$$

The product under the square root is $O(A_n)$ on each ramp, by Lemma 5.1, and smaller on the join. On the negative component use $u_{XX} + j^2 \rho\, u = 0$ and the energy $\mathsf{e} = |u_X|^2 + (1 + j^2\rho)|u|^2$, which satisfies

$$|\mathsf{e}_X| \le \Big[1 + \frac{j^2 |\rho_X|}{1 + j^2 \rho} \Big] \mathsf{e}.$$

The component has length $O(n^2)$ in X and ρ has a single maximum, so the integral of the second term is at most $2 \log(1 + j^2 \max \rho)$. Converting between (u, u_X) and (u, z) costs e^{Cn^2}. □

A Robin inverse. Let V be the natural solution on the outer region. Its value at the entrance $r_{o,-}$ of the core may vanish, so we do not divide by it. Instead normalize its full state there, and let U be the solution with the perpendicular state:

$$v = \Big(V,\ \frac{f_V}{\lambda} \Big)(r_{o,-}), \quad |v| = 1, \qquad f_u := w u_{r_o}, \qquad \Big(U,\ \frac{f_U}{\lambda} \Big)(r_{o,-}) = v^\perp := (-v_2, v_1). \tag{6.19}$$

Then $w\,(U V_{r_o} - U_{r_o} V) = -\lambda$, and the kernel

$$K_o(r_o, t_o) = \lambda^{-1}\, U(\min(r_o, t_o))\, V(\max(r_o, t_o))$$

defines operators

$$R_o F = \int K_o(r_o, t_o)\, F(t_o)\, d\mu(t_o), \qquad G_o f = -\int K_o(r_o, t_o)\, f(t_o)\, dt_o, \tag{6.20}$$

which invert B_j in $L^2(d\mu)$ and L_j for the pairing dr_o, respectively, with the natural condition at the endpoint and the Robin condition $v \cdot (u, f_u/\lambda) = 0$ at the entrance. For a solution $u = R_o F$, Green's identity with V gives the complementary boundary quantity,

$$v^\perp \cdot \Big(u,\ \frac{f_u}{\lambda} \Big)(r_{o,-}) = \frac{1}{\lambda} \int V\, \varpi\, F\, dr_o, \qquad \|V\|_{L^2(d\mu)} \le \sqrt{\lambda}\, \mathcal{E}_n^C\, \Pi_n; \tag{6.21}$$

the boundary term at the endpoint vanishes by Section 5.6.

Lemma 6.5 (The outer inverse)**.** *The operator R_o satisfies*

$$\|R_o\| \le \mathcal{E}_n^C\, \Pi_n^2, \tag{6.22}$$

and, for the localizations ϑ_k, ϑ_m to cells $k, m \ge n - O(1)$,

$$\|\vartheta_k\, R_o\, \vartheta_m\| \le \mathcal{E}_k^C\, \mathcal{E}_m^C\, \Pi_n^2\, e^{-|S_k - S_m|}. \tag{6.23}$$

The localized bounds (6.23) hold also for G_o between the spaces $L^2(dr_o)$.

The global bound (6.22) has no counterpart for G_o: no inverse of L_j is bounded on $L^2(dr_o)$ over the whole outer region. Indeed, let u_ϵ be a fixed smooth bump rescaled to the points at distance between ϵ and 2ϵ from the flat endpoint. Since w is flat there and $L_j u = (w u')' - j^2 w' u / r$,

$$\|L_j u_\epsilon\|_{L^2(dr_o)} \le C_N\, \epsilon^N\, \|u_\epsilon\|_{L^2(dr_o)} \qquad \text{for every } N.$$

Only the weighted bound and the localized bounds are used below.

Proof. The factor λ^{-1} in K_o causes no loss. On the core it is cancelled by the mass bound in (6.18). On the deep tail, restricting the kernel gives exactly

$$(R_o)_{d,d} = B_{d,D}^{-1} + P_d\, K_o(r_{o,+}, r_{o,+})\, P_d^*, \qquad |K_o(r_{o,+}, r_{o,+})| \le \lambda^{-1}\, \mathcal{E}_n^C\, \Pi_n^2,$$

where $r_{o,+}$ is the entrance of the deep tail, and the two factors P_d cancel λ^{-1}. The blocks between core and deep tail contain one factor P_d each. This proves (6.22). The localized bound follows from (5.26) for the first term and from (5.24) for the factors P_d: a product of two such decay factors, both measured from the entrance, is bounded by $e^{-|S_k - S_m|}$. Passing to $L^2(dr_o)$ in a localized bound multiplies source and output by $\varpi^{\mp 1/2}$ on their cells m and k, which are powers of $\mathcal{E}_m$ and $\mathcal{E}_k$. These powers are not bounded uniformly in the cell, which is why the global bound is not transferred. □

6.6 Matching the two remaining boundary values

Let i label the two truncation points, where the inner domain meets the outer core at each end. Put

$$\lambda_i = \frac{j^2\, W_{\mathrm{in},i}}{r_{\mathrm{in},i}},$$

with $W_{\mathrm{in},i}$ and $r_{\mathrm{in},i}$ the profile and the radius at that point; its ratio to λ_n lies between e^{-Cn^2} and e^{Cn^2}, so Section 6.5 applies with $\lambda = \lambda_i$. We use the normalized boundary coordinates

$$x_i = \sqrt{\lambda_i}\, u_i, \qquad z_i = \frac{f_{u,i}}{\sqrt{\lambda_i}}.$$

In these coordinates the stand-in solution of Section 5.8, which vanishes at a cut in the preceding positive ramp, has the ratio of outward flux to value

$$d_i = \frac{f_u}{\lambda_i u} = \frac{r\, u_{r_o}}{j^2 u} \in (0, \tfrac{1}{2}], \qquad D_0 := \mathrm{diag}(d_i),$$

by the estimate proving (5.34); in the first band $d_i = 1/j$.

The true inner boundary matrix. Let B_h be the inner operator in $L^2(d\mu)$, including the symmetric border, and q_h its quadratic form. We compare the true matrix of normalized outward fluxes of the inner problem with D_0. Take the stand-in solution with boundary value $\lambda_i^{-1/2}$, multiply it by a cutoff near the inner end of its ramp, and extend it by zero to a field L_i on the inner domain; let $\mathsf{F}_i = B_h \mathsf{L}_i$, with zero scalar component. Separate fixed fractions of the ramp are reserved for this cutoff and for the buffer of the neighbouring model. Let L and F have the columns L_i and F_i, and put $P_h = \mathsf{L} - G_h \mathsf{F}$.

Lemma 6.6 (The inner boundary matrix)**.** *The field $P_h x$ is the solution of the homogeneous inner problem with normalized boundary values x. For both signs, and with ω_i the constant value of W on the gap which carries the cutoff,*

$$\|W^{\pm 1}\mathsf{L}_i\|\, \omega_i^{\mp 1} \le \frac{C\sqrt{d_i}}{j}, \qquad \|W^{\pm 1}\mathsf{F}_i\|\, \omega_i^{\mp 1} \le \sqrt{d_i}\, \mathcal{E}_n^C\, e^{-\Sigma_{b,i}},$$
$$\Sigma_{b,i} = c\,\tau_{n-1,j} \quad \textit{(in the first band: } \Sigma_{b,i} = c\,jL\textit{)}. \tag{6.24}$$

The matrix D_b of normalized outward fluxes of P_h, defined by $x^* D_b x = \mathsf{q}_h(P_h x, P_h x)$, is

$$D_b = \mathsf{q}_h(\mathsf{L}, \mathsf{L}) - \mathsf{F}^* G_h \mathsf{F} = D_0 + E_b, \qquad |(E_b)_{ik}| \le \mathcal{E}_n^C\, \sqrt{d_i d_k}\, e^{-\Sigma_{b,i} - \Sigma_{b,k}}. \tag{6.25}$$

Proof. The field $\mathsf{L}x$ has the boundary values x and $B_h\mathsf{L}x = \mathsf{F}x$, while $G_h\mathsf{F}x$ vanishes at the boundary and $B_hG_h\mathsf{F}x = \mathsf{F}x$; so P_hx is the homogeneous solution with boundary values x. The bounds (6.24) follow from positivity and the commutator formula

$$\mathscr{H}_j(\chi u) - \chi\,\mathscr{H}_j u = -2w\chi' u' - (w\chi')' u.$$

The first bound uses the energy d_i of the stand-in solution and (6.8); the second uses the attenuation up to the cutoff layer. The two fields L_i have disjoint supports, and their energy matrix is diagonal with entries $d_i\,(1 + O(\mathcal{E}_n^C e^{-2\Sigma_{b,i}}))$.

For (6.25), complete the square in the quadratic form. Since $B_hG_h\mathsf{F} = \mathsf{F}$ and $G_h\mathsf{F}x$ vanishes at the boundary, integration by parts gives

$$\mathsf{q}_h(\mathsf{L}x, G_h\mathsf{F}x) = \langle \mathsf{F}x, G_h\mathsf{F}x\rangle = \mathsf{q}_h(G_h\mathsf{F}x, G_h\mathsf{F}x),$$

so in the expansion of $\mathsf{q}_h(\mathsf{L}x - G_h\mathsf{F}x, \mathsf{L}x - G_h\mathsf{F}x)$ the two cross terms cancel one of the two correction terms. The error E_b consists of $-\mathsf{F}^*G_h\mathsf{F}$ and of the deviation of the energy matrix of the fields L_i from D_0; both are bounded by (6.24) and the second estimate in (6.17). This is why the error contains *two* attenuation factors. The border is symmetric and the fields L_i have zero scalar component, so no scalar term is missing. □

Any dependence of the interior on the selected tails enters only through E_b.

The matching system. Let v_i be the normalized entrance state (6.19) of the natural solution V_i at the end i, and put

$$\mathsf{A} = \operatorname{diag}(v_{1,i}), \qquad \mathsf{B} = \operatorname{diag}(v_{2,i}), \qquad \mathcal{V} = \big(V_i/\sqrt{\lambda_i}\big)_i,$$

so that $\mathsf{A}^2 + \mathsf{B}^2 = I$.

Lemma 6.7 (The matching system)**.** *Let F_h be a source in the inner domain and F_o a source in the outer regions. Consider the field which is $G_hF_h + P_hx$ in the inner domain and, in the outer region at the end i, is R_oF_o plus a multiple of V_i. Its values and fluxes are continuous at the truncation points if and only if the normalized boundary values x and fluxes z satisfy*

$$\begin{aligned} z &= D_bx - P_h^*F_h, \qquad -\mathsf{B}x + \mathsf{A}z = \mathcal{V}^*F_o, \\ \textit{that is,}\qquad \mathsf{K}x &= \mathcal{V}^*F_o + \mathsf{A}P_h^*F_h, \qquad \mathsf{K} = \mathsf{A}(D_0 + E_b) - \mathsf{B}. \end{aligned} \tag{6.26}$$

The coefficients of the natural solutions $V_i/\sqrt{\lambda_i}$ in the outer regions are then the entries of $\mathsf{A}x + \mathsf{B}z$.

Proof. The outward flux of P_hx is D_bx, and by Green's identity that of G_hF_h, which vanishes at the boundary, is $-P_h^*F_h$. The outer solution at the end i is R_oF_o plus a multiple of V_i; by (6.19) and (6.21) its normalized state (x_i, z_i) at the truncation point is $(\mathcal{V}^*F_o)_i\, v_i^\perp$ plus a multiple of v_i, the multiple being the coefficient of $V_i/\sqrt{\lambda_i}$ in the outer solution. Continuity of values and fluxes at the truncation points means that the two normalized states agree. Taking the component along $v_i^\perp$ gives the second equation of (6.26), and the component along v_i shows that the coefficients of the natural solutions are $\mathsf{A}x + \mathsf{B}z$. Eliminating z gives the last line. □

Lemma 6.8 (The matching matrix)**.** *After C_*, N and the lengths of the cone heads have been chosen suitably, the matrix K is invertible and $\|\mathsf{K}^{-1}\| \le C\delta_n^{-1}$, uniformly in n and in $j \in \mathrm{Band}_n$.*

Proof. The diagonal part $\mathsf{A}D_0 - \mathsf{B}$ of K is controlled by the selected angle. The state used in Lemma 5.9 is $(u, -u \pm f_u/\lambda_i)$ at the truncation point, which differs from $(u, f_u/\lambda_i)$ by a fixed linear map, and transport across the join is close to the identity by (5.33). Hence

$$|v_{1,i} d_i - v_{2,i}| = \sqrt{1 + d_i^2}\, |\sin \angle(v_i, (1, d_i))| \geq c\, \delta_n. \tag{6.27}$$

This holds also when $v_{1,i} = 0$. The direction $(1, d_i)$ was fixed before the amplitude ratio was varied. It therefore suffices to have

$$\delta_n^{-1} \|E_b\| < \frac{c}{4}, \qquad \text{and we also arrange} \qquad \mathcal{E}_n^{C_\mathrm{b}}\, \delta_n^{-1}\, e^{C_o \tau_{n,j} - c\tau_{n-1,j}} \leq e^{-c\tau_{n-1,j}/2} \tag{6.28}$$

for use in Section 6.7. Here C_b is the largest coefficient exponent which occurs in (6.24), (6.25) and in the bounds (6.32) below. Like every coefficient exponent it depends only on the exponent a of the weights, and it is fixed before C_* and N. Both inequalities hold because

$$\frac{\tau_{n-1,j}}{\tau_{n,j}} \asymp e^n, \qquad \tau_{n-1,j} \geq C_* q_{n-1}, \qquad \frac{q_n}{q_{n-1}} \to 1, \qquad \frac{n^2}{q_{n-1}} \to 0.$$

The first relation absorbs $C_o \tau_{n,j}$ into a quarter of $c\tau_{n-1,j}$, and the others absorb $Cq_n + n^2 + |\log c_0|$ into another quarter, provided C_* is chosen after the coefficient exponent C_b and N is large; what remains is the factor $e^{-c\tau_{n-1,j}/2}$. The first inequality in (6.28) even has two attenuation factors, by (6.25). In the first band the role of $\tau_{n-1,j}$ is played by jL, and the length L of the cone heads is taken large. □

6.7 The matched inverse and its weighted bound

Put $\mathsf{K}_1 = \mathsf{A} + \mathsf{B}D_b$. Solving (6.26) and reconstructing the coefficient of the natural solution in the outer regions, which is $\mathsf{K}_1 x - \mathsf{B}P_h^* F_h$, gives the inverse for the frequency j on the whole annulus,

$$G = \begin{pmatrix} G_h & 0 \\ 0 & R_o \end{pmatrix} + \begin{pmatrix} P_h & 0 \\ 0 & \mathcal{V} \end{pmatrix} \begin{pmatrix} \mathsf{K}^{-1}\mathsf{A} & \mathsf{K}^{-1} \\ \mathsf{K}^{-T} & \mathsf{K}_1\mathsf{K}^{-1} \end{pmatrix} \begin{pmatrix} P_h^* & 0 \\ 0 & \mathcal{V}^* \end{pmatrix}, \tag{6.29}$$

in the decomposition into inner and outer sources and fields. Since D_b is symmetric and A, B are commuting diagonal matrices with $\mathsf{A}^2 + \mathsf{B}^2 = I$, direct multiplication gives

$$\mathsf{K}\mathsf{A} = \mathsf{A}\mathsf{K}^T, \qquad \mathsf{K}^T\mathsf{K}_1 = \mathsf{K}_1^T\mathsf{K}, \qquad \mathsf{K}_1\mathsf{K}^{-1}\mathsf{A} - \mathsf{B} = \mathsf{K}^{-T}.$$

These identities prove the formula and show that the middle matrix is symmetric; its norm is $O(\delta_n^{-1})$ by Lemma 6.8. The boundary weights at the two ends are taken equal, which is possible because the band index is the same at both ends.

We now prove that G is bounded between the weighted spaces. The direct terms G_h and R_o are covered by Lemmas 6.3 and 6.5. The correction terms are of two kinds, according to where their output lies, and each kind is paid for in its own way.

Proposition 6.9 (Weighted bound at one frequency)**.** *For every compact interval $[\gamma_-, \gamma_+] \subset (0,1)$ there are exponents a, $\mathsf{l} > \mathsf{a}$ and choices of C_*, N, X_0, L such that, with the weights W of (6.11) and $W_1 = W\,(\mathcal{E}/\mathcal{E}_N)^{-\mathsf{l}}$,*

$$\|W_1\, G\, W^{-1}\| \leq C \tag{6.30}$$

for all even $j \geq 2$ and all $\gamma \in [\gamma_-, \gamma_+]$, where at $j = 2$ the temporary border is included. The same bound holds between the spaces of source densities for dr and fields in $L^2(dr)$, after an adjustment of the exponents.

Proof. Step 1: terms whose source and output are both outer. These are the terms of (6.29) built from R_o, $\mathcal{V}$ and the leading parts L of P_h; their sources and outputs lie in cells of index at least $n - O(1)$. By (6.23) and $\|\mathsf{K}^{-1}\| \le C\delta_n^{-1}$, with fixed constants α_1, α_2,

$$\|\vartheta_k\, G_{\mathrm{out}}\, \vartheta_m\| \le C \exp\{\alpha_1\,(q_k + q_m) + \alpha_2\, q_n - |S_k - S_m|\};$$

here α_2 accounts for the cost of the core, since $\delta_n^{-1} e^{C_o \tau_{n,j}} \le c_0^{-1} \exp(n^2 + C_o C_* q_n)$. The output is outer, so $q_n \le c_1 q_k + C_1$; put $\alpha_3 = \alpha_2 c_1$. Conjugating by the weights gives, with $\Delta S = S_k - S_m$,

$$\begin{aligned} \log \|\vartheta_k\, W_1 G_{\mathrm{out}} W^{-1} \vartheta_m\| \le C &- (\mathsf{l} - 2\alpha_1 - \alpha_3)\, q_k \\ &+ (\alpha_1 - \mathsf{a})\,(q_m - q_k) - |\Delta S| + \gamma\, \Delta S. \end{aligned} \tag{6.31}$$

Choose $\mathsf{l} > \max(\mathsf{a}, 2\alpha_1 + \alpha_3)$; this is done after C_*. Since q_n varies slowly compared with S_n,

$$|q_m - q_k| \le \varepsilon\, |S_m - S_k| + C_\varepsilon, \qquad \varepsilon = \frac{1 - \gamma_+}{2\,(1 + |\alpha_1 - \mathsf{a}|)},$$

and (6.31) is at most $C_\gamma - \frac{1}{2}(1 - \gamma_+)\,|S_k - S_m|$. With $s_0 = \inf_m (S_{m+1} - S_m) > 0$ and $\eta = \frac{1}{2}(1 - \gamma_+)$,

$$\sup_k \sum_m e^{-\eta |S_k - S_m|} \le 1 + \frac{2}{e^{\eta s_0} - 1},$$

and the same holds with k and m exchanged. Schur's test for block kernels, with the bounded overlap of the localizers and the two ends, bounds this part of $W_1 G W^{-1}$. Note that raising the output exponent l helps precisely because the output index k is at least about n; it would not help for an output at a fixed place in the interior.

Step 2: terms which contain a column $G_h\mathsf{F}$. Insert $P_h = \mathsf{L} - G_h\mathsf{F}$ in (6.29). The terms not covered by Step 1 are those in which the column $G_h\mathsf{F}$ occurs at least once. With the symmetric matrix $\mathsf{X} = \mathsf{K}^{-1}\mathsf{A} = \mathsf{A}\mathsf{K}^{-T}$ there are five of them:

	term	from source to output
(a)	$-(G_h\mathsf{F})\,\mathsf{K}^{-1}\,\mathcal{V}^*$	outer to inner
(a′)	$-\mathcal{V}\,\mathsf{K}^{-T}\,(G_h\mathsf{F})^*$	inner to outer
(b)	$-(G_h\mathsf{F})\,\mathsf{X}\,\mathsf{L}^*$	inner to inner
(b′)	$-\mathsf{L}\,\mathsf{X}\,(G_h\mathsf{F})^*$	inner to inner
(c)	$(G_h\mathsf{F})\,\mathsf{X}\,(G_h\mathsf{F})^*$	inner to inner

Each is a product of a column, a matrix indexed by the truncation points, and a row. Let Ω be the common weight at the two truncation points, and W_h, W_o the restrictions of W to the inner and outer regions. Since Ω is a multiple of the identity, it commutes with K^{-1} and with X. For both signs,

$$\begin{gathered} \|W_h^{\pm 1}\, G_h\mathsf{F}\, \Omega^{\mp 1}\| \le \mathcal{E}_n^{C_1}\, e^{-c\tau_{n-1,j}}, \qquad \|W_h^{\pm 1}\, \mathsf{L}\, \Omega^{\mp 1}\| \le \mathcal{E}_n^{C_1}, \\ \|\mathsf{K}^{-1}\| + \|\mathsf{X}\| \le C\delta_n^{-1}, \\ \|W_o^{\pm 1}\, \mathcal{V}\, \Omega^{\mp 1}\| \le C_\gamma\, \mathcal{E}_n^{C_2}\, e^{C_o \tau_{n,j}}. \end{gathered} \tag{6.32}$$

The first bound is the second bound of (6.17) applied to (6.24), with $d_i \le \frac{1}{2}$; the second is the first bound of (6.24). In both, the weight ω_i of (6.24) has been replaced by Ω: the two belong to the same cell or to neighbouring cells, so by (6.12) their ratio is a coefficient power. The third bound is Lemma 6.8. For the fourth, the part of the column $\mathcal{V}$ in cell k is bounded

by $\mathcal{E}_n^C \mathcal{E}_k^C\, e^{C_o \tau_{n,j}}\, e^{-(S_k - S_n)}$; multiplying by either power of the weight changes the exponent by $(C \pm \mathsf{a})(q_k - q_n) - (1 \mp \gamma)(S_k - S_n)$, and the slow variation of q used in Step 1 makes the sum of the squares finite, leaving a coefficient power at n and a constant C_γ.

The norm of a row is the norm of the adjoint column, so $\|\Omega^{\pm 1}\, \mathcal{V}^*\, W_o^{\mp 1}\| = \|W_o^{\mp 1}\, \mathcal{V}\, \Omega^{\pm 1}\|$, and likewise for $(G_h \mathsf{F})^*$ and L^*. Insert $\Omega^{-1}\Omega$ on both sides of the middle matrix and multiply the bounds (6.32), taking the column with the upper signs and the row, through its adjoint, with the lower signs:

$$\begin{aligned} \text{(a), (a}'\text{):} &\quad C_\gamma\, \mathcal{E}_n^{C_1 + C_2}\, \delta_n^{-1}\, e^{C_o \tau_{n,j} - c\tau_{n-1,j}}, \\ \text{(b), (b}'\text{):} &\quad C\, \mathcal{E}_n^{2C_1}\, \delta_n^{-1}\, e^{-c\tau_{n-1,j}}, \\ \text{(c):} &\quad C\, \mathcal{E}_n^{2C_1}\, \delta_n^{-1}\, e^{-2c\tau_{n-1,j}}. \end{aligned} \tag{6.33}$$

These are bounds for the norms of $W\,(\cdot)\,W^{-1}$ of the five terms. By the second inequality in (6.28), in which $C_\mathrm{b} \geq \max(C_1 + C_2,\, 2C_1)$, each of them is at most $C_\gamma\, e^{-c\tau_{n-1,j}/2} \leq C_\gamma$; in the first band $c\,\tau_{n-1,j}$ is replaced by $c\,jL$. Finally $\mathcal{E}/\mathcal{E}_N \geq 1$ on the tails and is bounded below on the middle part, so $W_1 \leq C\,W$, and the bounds hold with the output weight W_1.

Step 3: conclusion. All terms of (6.29) have been covered. The passage between $L^2(d\mu)$ and the spaces for dr costs coefficient powers on each cell, which are absorbed by enlarging a and l. □

Lemma 6.10 (One geometry for all exponents)**.** *The profile, the amplitude ratios, the central cuts and the lengths of the cone heads can be chosen once, so that* (6.30) *holds for every compact interval* $[\gamma_-, \gamma_+] \subset (0, 1)$*. Only the constants and the exponents* a, l *depend on the interval.*

Proof. All the finite matching problems, that is the inversion of S and of K, involve the weights only through the ratios of neighbouring weights, and (6.12) is uniform for $|\gamma| \leq 1$. Fix one exponent a large enough for the conversion of sources, and choose C_*, N, the central cuts and the cone heads using that uniform bound. The invertibility of S and K then does not depend on γ.

With the geometry fixed, the only place where γ matters is the summation over the cells of the deep tail. Conjugation by $H^{-\gamma}$ changes the exponential in (6.23) to

$$e^{-|S_k - S_m| + \gamma(S_k - S_m)} \leq e^{-(1-|\gamma|)\,|S_k - S_m|},$$

whose row and column sums are finite for $|\gamma| \leq \gamma_+ < 1$. The coefficient factors are absorbed by $|q_k - q_m| \leq \varepsilon |S_k - S_m| + C_\varepsilon$, which holds because $q'(x)/S'(x) \to 0$; here ε is chosen after γ_+, which changes the summation constant and the output exponent but not the geometry. The terms which enter the interior already carry the attenuation (6.28). So the dependence on the interval is confined to constants which multiply an existing bound. □

One radial derivative. Derivatives of the solution are estimated from the equation, not by differentiating the construction. On a slightly enlarged cell m,

$$\begin{aligned} \|u_r\|_2 &\leq \mathcal{E}_m^C (\|f\|_2 + (1 + j)\,\|u\|_2), \\ \|u_{rr}\|_2 &\leq \mathcal{E}_m^C (\|f\|_2 + (1 + j^2)\,\|u\|_2). \end{aligned} \tag{6.34}$$

For the first, use the interpolation inequality $\|u_r\|_2^2 \leq C \|u\|_2 (\|u_{rr}\|_2 + \|u\|_2)$ on an interval, substitute $w u_{rr} = f - w' u_r + j^2 w' u / r$, and absorb the resulting term $C\|u\|_2 \|u_r\|_2$ by Young's inequality; then substitute back for the second. Rescaling the interval and dividing by w cost coefficient powers, and a cutoff on the enlarged cell gives the estimate without any boundary condition. At $j = 2$ the source f is replaced by $f - \nu g_0$, whose second term is compactly

supported and controlled by $|\nu|$. Summing the first inequality over the modes and the cells, with the weights of the output norm, gives (4.9): the factor $1+j$ is one angular derivative of u, and the inverse commutes with ∂_θ. In an outer core, any fixed power of j is itself a coefficient power, because $j < \mathsf{j}_n$.

6.8 The zero mode, the domain, and the final border

It remains to treat the mode $j = 0$, to say on which domain the operator is inverted, and to replace the temporary border.

The zero mode. For a source f_0 with $\int f_0\, dr = 0$ put

$$p_0(r) = \int_{r_-}^{r} f_0(t)\, dt = -\int_{r}^{r_+} f_0(t)\, dt, \qquad u_0(r) = -\int_{r}^{r_+} \frac{p_0(t)}{w(t)}\, dt. \tag{6.35}$$

Then $(wu_0')' = f_0$, and u_0 vanishes at the right endpoint. Summing over the cells beyond n gives $|p_0| \le \mathcal{E}_n^C H_n^\gamma \|f_0\|_{Y_{\gamma,0}}$ on cell n of either tail. Division by w and one more integration cost coefficient powers, so u_0 converges to a constant a_L at the left endpoint, a_L is a bounded functional of the source, and $u_0 - a_L\chi_L$ has the required decay at both ends. The solution is unique: a homogeneous solution with nonzero flux would be a multiple of $\int dr/w$, which is excluded by its growth, and a constant is excluded by the condition at the right endpoint.

Definition 6.11 (The weighted graph domain)**.** Let $\mathcal{Y}_\gamma^{\mathrm{rad}}$ be the space of sources (f, c) with finite norm (4.5) at $s = 0$ and with $\int \langle f\rangle\, dr = 0$. Let $\mathcal{X}_\gamma^{\mathrm{rad}}$ be the space of pairs (u, ν), $u = a_L\chi_L + v$, with finite output norm

$$|a_L|^2 + |\nu|^2 + \sum_m \mathcal{E}_m^{-2\mathsf{b}} H_m^{-2\gamma} \|\vartheta_m v\|^2_{L^2(dr\, d\theta)},$$

the angular measure being normalized. The *graph domain* is

$$\mathcal{D}_\gamma^{\mathrm{rad}} = \{U = (u,\nu) \in \mathcal{X}_\gamma^{\mathrm{rad}} :\ \mathbb{L}_w U \in \mathcal{Y}_\gamma^{\mathrm{rad}}\}, \qquad \|U\|_{\mathcal{D}_\gamma^{\mathrm{rad}}} = \|U\|_{\mathcal{X}_\gamma^{\mathrm{rad}}} + \|\mathbb{L}_w U\|_{\mathcal{Y}_\gamma^{\mathrm{rad}}},$$

where the differential expression is understood in the sense of distributions. The definition is used first with the temporary border and then with the final one.

The kernel statement of Proposition 4.2(i) will be proved for the kernel on this domain. This suffices, because the domain contains every solution of $L_j u = 0$, $j \ge 2$, which is bounded near both endpoints: the total mass $\int \mathsf{p}_j\, dx$ of a tail is finite, so a bounded solution is square integrable for $\mathsf{p}_j\, dx$, lies on the natural line by Lemma 5.6(ii), and therefore decays like H_m by (5.24).

Lemma 6.12 (Two-sided inverse and closure)**.** *Let $\mathbb{G}_w$ be defined on $\mathcal{Y}_\gamma^{\mathrm{rad}}$ by the matched inverses (6.29) on the modes $j \ge 2$ and by (6.35) on the mode $j = 0$. Then $\mathbb{G}_w : \mathcal{Y}_\gamma^{\mathrm{rad}} \to \mathcal{D}_\gamma^{\mathrm{rad}}$ is bounded, with $\mathsf{b} = \mathsf{l} - \mathsf{a}$ adjusted as in Proposition 6.9, and*

$$\mathbb{L}_w \mathbb{G}_w = I, \qquad \mathbb{G}_w \mathbb{L}_w = I \quad \text{on } \mathcal{D}_\gamma^{\mathrm{rad}}. \tag{6.36}$$

The graph domain is the closure, in the graph norm, of the smooth fields obtained from smooth compactly supported sources with finitely many modes. Green's identity holds on it with the pairing dr, including the border.

Proof. Step 1: right inverse. On smooth compactly supported sources with finitely many modes, every block of the construction is an inverse of the corresponding piece, and values and fluxes match exactly at all interfaces. So $\mathbb{L}_w\mathbb{G}_w = I$ on these sources, and by the bound (6.30) and density it holds on $\mathcal{Y}^{\mathrm{rad}}_\gamma$.

Step 2: uniqueness. Let $U \in \mathcal{D}^{\mathrm{rad}}_\gamma$ with $\mathbb{L}_w U = 0$; we show $U = 0$. At each frequency $j \geq 2$ the component of U is a solution of the homogeneous equation which is locally H^2, because $w > 0$ in the interior; so its values and fluxes agree at every interface. The weights place it in $L^2(\mathsf{p}_j\, dx)$ near each endpoint, since $\gamma S_m - Cq_m \to +\infty$ for every fixed C. By Lemma 5.6(ii) it lies on the natural line at both ends. Its two boundary values at the truncation points then satisfy the homogeneous system (6.26) and vanish because K is invertible, and the inner and outer inverses show that it is zero. The zero mode is unique by the computation above.

Step 3: closure and reciprocity. If a sequence converges in the graph norm, its limit and the limit of its images are related by $\mathbb{L}_w$ in the sense of distributions, so the graph domain is complete. Smooth compactly supported sources with finitely many modes are dense in $\mathcal{Y}^{\mathrm{rad}}_\gamma$: truncate and mollify, and restore the integral condition with a fixed bump of integral one. Their images are smooth, and flat at the endpoints in all modes $j \geq 2$. By Steps 1 and 2 every element of the graph domain is a limit of such images. Green's identity follows from the local identities and the symmetry of the middle matrix in (6.29), with the conversion (6.1). □

The final border. The function ϕ_* of Section 6.2 spans the kernel of the unbordered operator at $j = 2$ in the graph domain; a larger kernel would contradict the injectivity of the bordered operator. For the same reason $\ell_0\phi_* \neq 0$. Now choose a function $\chi \geq 0$, not identically zero, supported in a positive connector on the right of the central model where $\phi_*'/\phi_* < 0$. Integrating by parts,

$$a_* := \int \phi_*\, L'_{2,w}[w\chi]\, \phi_*\, dr = \int w\chi\Big(-\phi_*'^2 + \frac{8\phi_*\phi_*'}{r} - \frac{4\phi_*^2}{r^2}\Big) dr < 0. \tag{6.37}$$

Put

$$\chi_* = \frac{\chi}{a_*}, \qquad g_* = L'_{2,w}[w\chi_*]\, \phi_*, \qquad \ell_* u = \int g_* u_2\, dr, \tag{6.38}$$

so that $\int g_*\phi_*\, dr = 1$, that is, $\ell_*(\phi_* e_2) = 1$. The bordered equation with the final border is solved as follows: for a source f_2 in the second mode, the scalar unknown is $\nu = \int \phi_* f_2\, dr$; the source $f_2 - \nu g_*$ is orthogonal to ϕ_* and is solved with the temporary border; and a multiple of ϕ_* is added to satisfy the row ℓ_*. This change of finite rank is bounded in all the weighted spaces and preserves the inverse identities and the closure of the graph. From now on $\mathbb{L}_{w_*}$ and $\mathbb{G}_{w_*}$ refer to the final border.

Proof of Proposition 4.2. The profile and part (ii) were obtained in Section 6.2; the simple zeros of w_*' are the zeros of ρ in the models. Parts (i) and (iv) are Lemma 6.12 with the final border, together with Proposition 6.9 and (6.34); part (iii) is (6.38); and part (v) is Lemma 6.10. □

6.9 What Part III uses

The nonlinear argument uses the reference profile only through Proposition 4.2, the coefficient bounds of Lemma 5.2, and the localized form of the inverse estimate: the block of $\mathbb{G}_{w_*}$ from cell m to cell k is bounded by a coefficient power times $e^{-\gamma(S_k - S_m)}$, for every γ in a compact subinterval of $(0, 1)$, whatever the sign of $S_k - S_m$; this is (6.30) read block by block.

What has been proved is an estimate at angular order zero, without loss of angular derivatives, with its first radial derivative, on the maximal graph domain. Higher radial derivatives will be obtained from the equation, which replaces two radial derivatives by at most two angular ones and lower-order terms. At each fixed total order s the *complete* product of coefficients produced by this recursion is bounded by $C_s\mathcal{E}_m^{C_s}$. To absorb such products for all s at once we shall use the slightly larger weight

$$M_m = \exp\{1 + m^{5/2}\log^2(e+m)\}, \tag{6.39}$$

which satisfies, for every fixed s and every $\varepsilon > 0$,

$$\mathcal{E}_m^{C_s} \le C_s M_m, \qquad |\log M_k - \log M_m| \le \varepsilon\,|S_k - S_m| + C_\varepsilon.$$

The first inequality holds because $\log(e+m) \to \infty$, the second because the ratio of the derivatives of $\log M$ and of S tends to zero. So complete products of coefficients cost one power of M, whatever the order.

Part III
The nonlinear problem

7 A geometry that stays fixed during the iteration

7.1 Overview of Part III

We now solve the nonlinear equation $\mathcal{F}_q(w,u) = 0$ near the reference pair (w_*, Φ_*), where

$$\Phi_* = \phi_*\, e_2 = \phi_*\sqrt{2}\cos 2\theta, \qquad \mathscr{L}_{w_*}\Phi_* = 0, \qquad \ell_*\Phi_* = 1.$$

The inverse of $\mathscr{L}_{w_*}$ constructed in Part II is useful only if the linearization at a nearby state can be compared with it. Two features of the linearization would normally move with the state: the circles on which it changes type, and the angular positions at which characteristics arrive after crossing a negative component. The first was fixed in Lemma 3.4 by a correction proportional to the residual. The second is fixed in the same spirit, at the price of enlarging the problem.

Unknowns. The profile is allowed to vary in a controlled family,

$$w = w_*\exp\Big(\nu\chi_* + \sum_m \kappa_m\chi_m\Big), \tag{7.1}$$

where χ_* is the function of Proposition 4.2(iii) and, for every negative component m, χ_m is a bump supported inside that component (Lemma 7.6). In addition every negative component carries a circle diffeomorphism $h_{L,m}$, a *chart* of its left boundary circle. The *state* is $\mathbf{x} = (u, \nu, \kappa, h_L)$ with $\kappa = (\kappa_m)_m$ and $h_L = (h_{L,m})_m$; see Table 3.

Equations. There are two. The first is the height equation $\mathcal{A} := \mathcal{F}_q(w,u) = 0$. The second, $\mathfrak{e} = 0$, is imposed by us: it says that on every negative component the two characteristic return maps of the linearization, read in the charts, are the rotations by $\pm T_m$, where T_m is the travel angle of the reference profile. The *return residual* $\mathfrak{e}_m$ is an even function on the circle. Its mean is controlled by the number κ_m, and its oscillating part by the odd function $h_{L,m} - \mathrm{id}$; solving for the latter involves the small divisors $\sin(jT_m)$ of Proposition 4.2(ii).

Unknown	Lives in	Role
u	$\Phi_* + \ker \ell_* \subset \mathcal{X}_\beta$	The height is $z = h + qu$ with $h' = w$.
ν	$\mathbb{R}$	Amplitude of the metric control χ_*; removes the obstruction in the second angular mode.
$\kappa = (\kappa_m)$	$\mathcal{K}_\beta$	Amplitudes of the metric controls χ_m; adjust the mean travel angle of each negative component.
$h_L = (h_{L,m})$	$\mathrm{id} + \mathcal{Z}_{\beta,\mathrm{odd}}$	Charts of the left boundary circles; straighten the return maps.

Table 3: The unknowns of the nonlinear problem. The spaces are defined in Definition 7.1; q is a parameter.

The corrected linearization. The derivative of $\mathcal{A}$ with respect to u will be written as

$$D_u\mathcal{A} = \widehat{L}_{\mathbf{x}} + (\text{a term proportional to } \mathcal{A}) + (\text{a term proportional to } \mathfrak{e}),$$

where $\widehat{L}_{\mathbf{x}}$ changes type exactly at the turning circles of w_* and has exactly the return maps $R_{\pm T_m}$ in the charts. After a change of coordinates all operators $\widehat{L}_{\mathbf{x}}$ act on one set of cylinders with one set of transmission conditions, independent of $\mathbf{x}$. This makes the comparison with the reference operator possible.

Plan. This section introduces the weighted spaces (Section 7.2), the controls and the neighbourhood of the reference state on which everything is defined (Sections 7.3–7.4), the return residual and the operator $\widehat{L}$ (Section 7.5), and the coordinates (Sections 7.6–7.7). Section 8 inverts $\widehat{L}_{\mathbf{x}}$, Section 9 solves the coupled system by the Nash–Moser method, and Section 10 concludes.

The construction involves infinitely many components, and one smallness condition has to serve for all of them. The room for this comes from a simple fact, used again and again: the deformation decays like H^β, which is far smaller than any power of the coefficient allowance, while the reference profile decays only like $e^{-n^{5/2}}$.

7.2 Weights and function spaces

Continuous weights. Let d_∂ be the distance to the nearer endpoint. Near that endpoint put, with the constant d_0 of Lemma 5.2,

$$x_\partial = \sqrt{\log \frac{d_0}{d_\partial}}, \quad H = e^{-2x_\partial^3/3}, \quad E_0 = e^{1+x_\partial^{5/2}\log(e+x_\partial)}, \quad M = e^{1+x_\partial^{5/2}\log^2(e+x_\partial)}, \tag{7.2}$$

and extend these functions smoothly and positively through the middle of the annulus. By (5.10), $x_\partial = n + O(1)$ on cell n. Hence H agrees with H_n, and E_0 with $\mathcal{E}_n$, up to coefficient powers, and M is the weight (6.39). For every fixed A and $\epsilon > 0$,

$$E_0^A \le C_A\, M, \qquad H^\epsilon M^A \le C_{A,\epsilon}. \tag{7.3}$$

The extra logarithm in M is deliberate. However many coefficients appear after differentiating a fixed expression, their *whole product* is bounded by a power of E_0 and thus costs one power of M, a cost that does not grow with the number of derivatives. And the cubic exponent of H still dominates every power of M.

A unit-scale derivative. Near the ends put

$$\ell = \frac{d_\partial}{1 + x_\partial}, \qquad d\varrho = \frac{dr}{\ell}, \qquad \mathsf{D} = \partial_\varrho = \ell\, \partial_r,$$

with a positive extension in between. The variable ϱ ranges over $\mathbb{R}$, and

$$|\mathsf{D} x_\partial| = \frac{1}{2x_\partial(1 + x_\partial)}, \qquad |\mathsf{D} \log H| = \frac{x_\partial}{1 + x_\partial}.$$

All D-derivatives of positive order of $\log H$, $\log M$ and $\log \ell$ are bounded: in the variable ϱ the weights vary on the unit scale. An ordinary radial derivative ∂_r costs a factor bounded by a power of E_0.

The exponent β. We fix $\beta = \frac{3}{4}$. What matters is

$$\tfrac{1}{2} < \beta < 1. \tag{7.4}$$

The upper inequality leaves room for two neighbouring exponents $\beta \pm \epsilon$ in $(0, 1)$, at which the reference inverse is available. The lower inequality makes a product of two quantities of size H^β smaller than H; this gives the improved estimate for the metric in Theorem 10.3.

Definition 7.1 (Spaces of heights, sources and circle data)**.** All functions are real, even in θ and π-periodic. For $\gamma > 0$ and integers $s \geq 0$ put

$$\|u\|^2_{\gamma,s} = \sum_{a+b+c \leq s} \big\| M^c\, \mathsf{D}^a \partial_\theta^b (H^{-\gamma} u) \big\|^2_{L^2(d\varrho\, d\theta)}. \tag{7.5}$$

- $\mathcal{X}^0_\gamma$ is the space of functions with all these norms finite.
- $\mathcal{X}_\gamma = \mathcal{X}^0_\gamma \oplus \mathbb{R}\chi_L$, where χ_L is a fixed function of r, equal to one near the left endpoint and to zero near the right one, whose derivative is supported in a compact subset of a positive component. The coefficient of χ_L is the difference of the two endpoint values of a height.
- $\mathcal{Y}_\beta = \{ f \in \mathcal{X}^0_\beta : \mathsf{Q} f = 0 \}$ is the space of compatible sources, where $\mathsf{Q} f = \int_{r_-}^{r_+} \langle f \rangle\, dr$.
- For a sequence $a = (a_m)$ of functions on the circle, one for each negative component,

$$\|a\|^2_{\mathcal{Z}_{\gamma,s}} = \sum_m \sum_{b+c \leq s} M_m^{2c} H_m^{-2\gamma} \|\partial_\theta^b a_m\|^2_{L^2(\mathbb{T})}, \tag{7.6}$$

  and $\mathcal{Z}_\gamma$ is the space with all these norms finite. Its subspace of sequences of constants is $\mathcal{K}_\gamma$, and $\mathcal{Z}_{\gamma,\mathrm{odd}}$, $\mathcal{Z}_{\gamma,\mathrm{even}}$ are its subspaces of odd and of even functions.

When no exponent is displayed, a norm of a height or of a source is the one with $\gamma = \beta$.

Note that the index s counts derivatives and powers of M together. All these spaces consist of smooth functions which are flat at the endpoints; the exponent γ only measures how fast they decay.

Tame spaces and tame maps. We use the terminology of [Ham82, Part II]. A *graded* Fréchet space is one whose topology is defined by an increasing sequence of norms $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \cdots$. A linear map L between graded spaces is *tame* if there is a number d, its *loss*, with $\|Lf\|_s \leq C_s \|f\|_{s+d}$ for all s. For a Banach space B, let $\Sigma(B)$ be the space of sequences $(f_k)_{k \geq 0}$ in B for which all the norms $\|f\|_s = \sum_k e^{sk} \|f_k\|_B$ are finite. A graded space F is *tame* if there are tame linear maps $J : F \to \Sigma(B)$ and $P : \Sigma(B) \to F$ with $PJ = I$. A map

$\mathcal{P}$ on an open subset of a graded space is *tame* if it is continuous and every point has a neighbourhood on which $\|\mathcal{P}(f)\|_s \le C_s(1+\|f\|_{s+d})$ for all s; it is *smooth tame* if it is smooth and all its derivatives are tame. Since a derivative of order k is linear in its k increments, this amounts to estimates of the form

$$\big\|D^k\mathcal{P}(f)[h_1,\dots,h_k]\big\|_s \le C_{s,k}\Big[\sum_i \|h_i\|_{s+d_k}\prod_{j\neq i}\|h_j\|_b + (1+\|f\|_{s+d_k})\prod_j\|h_j\|_b\Big] \tag{7.7}$$

with a fixed low order b. *In every term only one factor carries the high order.* All estimates of this part have this shape. Two features will be checked each time: the loss does not depend on s, and the neighbourhood is defined by norms of one fixed order.

The following elementary lemma produces estimates of this shape. It replaces the usual interpolation inequalities for sup norms by inequalities for the Fourier transform, which can be verified in a few lines.

Lemma 7.2 (Estimates with one high factor)**.** *Let* C *be the circle* $\mathbb{T}$ *or the cylinder* $\mathbb{R}\times\mathbb{T}$*, let* $\|\cdot\|_{H^s}$ *denote the Sobolev norms on* C*, and let* $|v|_{\mathcal{F}} = \|\widehat{v}\|_{L^1}$ *be the* L^1 *norm of the Fourier transform. Then* $\sup|v| \le |v|_{\mathcal{F}} \le C\,\|v\|_{H^b}$*, where* $b=1$ *on the circle and* $b=2$ *on the cylinder.*

(i) (Interpolation.) *If* $k\le i\le m$ *and the point* (i,j) *lies on the segment from* (k,l) *to* (m,n)*, then*

$$\|f\|_{H^i}\,\|g\|_{H^j} \le \|f\|_{H^k}\,\|g\|_{H^l} + \|f\|_{H^m}\,\|g\|_{H^n}.$$

(ii) (Products.) *If* $k\ge 1$ *and* $\alpha_1,\dots,\alpha_k$ *are multi-indices with* $|\alpha_1|+\cdots+|\alpha_k|\le s$*, then*

$$\Big\|\prod_{i=1}^k \partial^{\alpha_i} v_i\Big\|_{L^2} \le C_k\sum_{i=1}^k \|v_i\|_{H^s}\prod_{j\neq i}|v_j|_{\mathcal{F}}.$$

In particular $\|v_1v_2\|_{H^s} \le C_s\big(\|v_1\|_{H^s}|v_2|_{\mathcal{F}} + |v_1|_{\mathcal{F}}\|v_2\|_{H^s}\big)$.

(iii) (Functions.) *Let* v *be a real function and let* Φ *be smooth on a neighbourhood of an interval* I *which contains* 0 *and the closure of the range of* v*. Then* $\|\Phi(v)-\Phi(0)\|_{H^s} \le C_s\,\|v\|_{H^s}$ *for every* $s\ge 0$*, where* C_s *depends only on an upper bound for* $|v|_{\mathcal{F}}$ *and on the suprema over* I *of* $|\Phi^{(m)}|$ *for* $1\le m\le \max\{1,s\}$*. The same holds for a smooth function of several real functions, with* I *replaced by a convex set which contains* 0 *and the closure of the joint range.*

(iv) (Composition and inversion on the circle.) *Let* h *be the lift of a diffeomorphism of the circle with* $h'\ge c_0>0$*. Then, for* $s\ge 1$*,*

$$\|u\circ h\|_{H^s} \le C_s\big(\|u\|_{H^s} + \|u\|_{H^3}\,\|h'-1\|_{H^{s-1}}\big), \qquad \|(h^{-1})'-1\|_{H^{s-1}} \le C_s\,\|h'-1\|_{H^{s-1}},$$

where C_s *depends only on* c_0 *and* $\|h'-1\|_{H^3}$.

Proof. The first assertion is the Cauchy–Schwarz inequality for $\widehat{v} = \langle\zeta\rangle^{-b}\cdot\langle\zeta\rangle^b\widehat{v}$, since $\langle\zeta\rangle^{-2b}$ is summable over the frequencies of C exactly when $2b$ exceeds its dimension.

(i) Let $i=\vartheta k+(1-\vartheta)m$; then $j=\vartheta l+(1-\vartheta)n$. By Hölder's inequality for the Fourier transform, $\|f\|_{H^i}\le\|f\|_{H^k}^{\vartheta}\|f\|_{H^m}^{1-\vartheta}$, and likewise for g. The product of the two right sides is $X^{\vartheta}Y^{1-\vartheta}\le X+Y$ with $X=\|f\|_{H^k}\|g\|_{H^l}$ and $Y=\|f\|_{H^m}\|g\|_{H^n}$.

(ii) The Fourier transform of the product at the frequency ζ is bounded by the integral of $\prod_i|\zeta_i|^{|\alpha_i|}|\widehat{v}_i(\zeta_i)|$ over $\zeta_1+\cdots+\zeta_k=\zeta$. Since $\langle\zeta_i\rangle\ge 1$ and the exponents add up to at most s,

$$\prod_i|\zeta_i|^{|\alpha_i|} \le \max_i\langle\zeta_i\rangle^s \le \sum_i\langle\zeta_i\rangle^s.$$

So the Fourier transform of the product is bounded by a sum of k convolutions, in each of which one factor is $\langle\zeta\rangle^s|\widehat{v}_i| \in L^2$ and the others are $|\widehat{v}_j| \in L^1$. Young's inequality and Plancherel's theorem give the claim.

(iii) A derivative ∂^α of order between 1 and s of $\Phi(v)$ is a sum of terms $\Phi^{(m)}(v)\prod_{i=1}^m \partial^{\alpha_i} v$ with $\alpha_1+\cdots+\alpha_m=\alpha$; bound $\Phi^{(m)}(v)$ by its supremum and apply (ii). In L^2, $|\Phi(v)-\Phi(0)| \le \sup_I|\Phi'|\,|v|$, by the mean value theorem on the segment from 0 to the value of v; this segment lies in I.

The interval I cannot be replaced by the range of v. For the constant function $v=1$ on the circle and $\Phi_M(x)=M(2x-x^2)$ we have $\Phi_M(0)=0$ and $\Phi_M'=0$ on the range $\{1\}$, while $\|\Phi_M(v)\|_{H^1} = M\,\|v\|_{H^1}$. The first derivative is needed also for $s=0$, as $\Phi(x)=Mx$ shows.

(iv) Change of variables gives $\|f\circ h\|_{L^2} \le c_0^{-1/2}\|f\|_{L^2}$. For $s\le 3$ the chain rule expresses the derivatives of $u\circ h$ through $u^{(m)}\circ h$, $m\le s$, and bounded functions, which proves the first estimate without its second term. Let $s\ge 4$ and assume the estimate for $s-1$. Since $(u\circ h)' = (u'\circ h)\,h'$, part (ii) on the circle gives

$$\|u\circ h\|_{H^s} \le C\big(\|u\|_{L^2} + \|u'\circ h\|_{H^{s-1}}\,\|h'\|_{H^1} + \|u'\circ h\|_{H^1}\,(1+\|h'-1\|_{H^{s-1}})\big).$$

By the induction hypothesis $\|u'\circ h\|_{H^{s-1}} \le C(\|u\|_{H^s}+\|u\|_{H^4}\|h'-1\|_{H^{s-2}})$, and by (i), applied to the points $(3,s-1)$, $(4,s-2)$ and $(s,2)$,

$$\|u\|_{H^4}\,\|h'-1\|_{H^{s-2}} \le \|u\|_{H^3}\,\|h'-1\|_{H^{s-1}} + \|u\|_{H^s}\,\|h'-1\|_{H^2}.$$

This proves the first estimate. For the inverse $\iota = h^{-1}$ we have $\iota'-1 = G\circ\iota$ with $G=-(h'-1)/h'$, a smooth function of $h'-1$. By the chain rule ι' is bounded below by $1/\sup h'$ and $\|\iota'-1\|_{H^3}$ is bounded in terms of c_0 and $\|h'-1\|_{H^3}$; and $\|G\|_{H^k}\le C\|h'-1\|_{H^k}$ by (iii). The first estimate, with ι in the place of h, gives

$$\|\iota'-1\|_{H^{s-1}} \le C\big(\|G\|_{H^{s-1}} + \|G\|_{H^3}\,\|\iota'-1\|_{H^{s-2}}\big),$$

and induction on s proves the second estimate. □

Functions on a cylinder with boundary, such as the pieces of Section 8, are first extended to $\mathbb{R}\times\mathbb{T}$ by a fixed extension operator which is bounded in all Sobolev norms; parts (i)–(iii) then apply to them as well.

Lemma 7.3 (Flows on the circle)**.** *Let $G(t,\theta)$ be smooth and periodic in θ, and let Θ solve $\partial_t\Theta = G(t,\Theta)$ for $0\le t\le 1$ with $\Theta(0,\theta)=\theta$. Put $\Lambda_s = \int_0^1 \|G_\theta(t,\cdot)\|_{H^s}\,dt$. Then*

$$\Theta_\theta = \exp\int_0^t G_\theta(\tau,\Theta(\tau,\theta))\,d\tau \in \big[e^{-C\Lambda_1}, e^{C\Lambda_1}\big], \qquad \sup_t \|\Theta_\theta(t,\cdot)-1\|_{H^s} \le C_s\,\Lambda_s,$$

where C_s depends only on Λ_4. If G_* does not depend on θ and $\Theta_* = \theta + \int_0^t G_*$ is its flow, then $\sup_t\|\Theta-\Theta_*\|_{H^s} \le C_s\int_0^1\|(G-G_*)(t,\cdot)\|_{H^s}\,dt$, with C_s depending only on Λ_4 and on $\int_0^1\|(G-G_*)(t,\cdot)\|_{H^3}\,dt$. If G depends smoothly on parameters, every derivative of Θ with respect to the parameters solves a linear equation $\partial_t\Theta' = (G_\theta\circ\Theta)\,\Theta' + R$ with the same integrating factor, where R involves derivatives of G and derivatives of Θ of lower order. A derivative of order k with respect to the parameters satisfies estimates of the same form in which the norms of G and of its parameter derivatives are taken at the order $s+k$ instead of s.

The loss of k angular derivatives in the last statement cannot be avoided. Let $G = a\,\alpha(t) + \beta(t)\,f(\theta)$, where α and β have integral one and are supported in $(0,\frac13)$ and $(\frac23,1)$. The flow at time one is the time-one flow of f applied to $\theta+a$. At $f=0$ its derivative in the direction h of f is $h(\theta+a)$, and k further derivatives with respect to a give $h^{(k)}(\theta+a)$. This is why the loss d_k in (7.7) is allowed to depend on k.

Proof. The formula for Θ_θ follows from $\partial_t \Theta_\theta = (G_\theta \circ \Theta)\,\Theta_\theta$, and the bounds from $|G_\theta| \le C\|G_\theta\|_{H^1}$. The derivatives $\partial_\theta^k \Theta$, $2 \le k \le 4$, solve linear equations with the same integrating factor, whose sources are polynomials in the lower derivatives and in $(\partial_\theta^m G) \circ \Theta$, $2 \le m \le 4$; hence they are bounded in terms of Λ_4. Now let $y = \Theta_\theta - 1$, so that $\partial_t y = (G_\theta \circ \Theta)(1 + y)$. By Lemma 7.2(ii) and (iv), with the low norms just bounded,

$$\|\partial_t y\|_{H^s} \le C_s\big(\|G_\theta(t,\cdot)\|_{H^s} + \|G_\theta(t,\cdot)\|_{H^3}\,\|y\|_{H^s}\big),$$

and Gronwall's inequality gives the second assertion. For the last one, $\partial_t(\Theta - \Theta_*) = (G - G_*)(t, \Theta)$; apply Lemma 7.2(iv) and integrate. For the parameters, differentiate the equation k times. By the chain rule the source R is a sum of products of derivatives of Θ of order less than k with functions $(\partial_\theta^a \partial^b G) \circ \Theta$, where ∂^b denotes b derivatives with respect to the parameters and $a + b \le k$; so at most k additional angular derivatives fall on G or on one of its parameter derivatives. Induction on k, with Lemma 7.2(ii) and (iv) for the products and compositions and Gronwall's inequality as before, gives the last statement. □

The point of the last statement is that a flow with a large speed which does not depend on the angle, as for the reference profile, does not enter the constants: only the angular derivative of the speed does.

Lemma 7.4 (Calculus in the weighted spaces)**.** *Let $\gamma, \gamma_1, \gamma_2 > 0$ and let $s \ge 0$ be an integer.*

(i) (Tameness.) *The spaces of Definition 7.1, graded by s, are tame Fréchet spaces.*

(ii) (Products.) *For $u_1 \in \mathcal{X}^0_{\gamma_1}$ and $u_2 \in \mathcal{X}^0_{\gamma_2}$,*

$$\|u_1 u_2\|_{\gamma_1+\gamma_2,s} \le C_s\big(\|u_1\|_{\gamma_1,s}\,\|u_2\|_{\gamma_2,2} + \|u_1\|_{\gamma_1,2}\,\|u_2\|_{\gamma_2,s}\big). \tag{7.8}$$

(iii) (Reference functions.) *If c is a function of r with $|\mathsf{D}^a c| \le E_0^A$ for $0 \le a \le s$, then*

$$\|c\,u\|_{\gamma,s} \le C_{s,A}\,\|u\|_{\gamma,s+1}, \qquad \text{and} \qquad \|\partial_r u\|_{\gamma,s} \le C_s\,\|u\|_{\gamma,s+2}. \tag{7.9}$$

(iv) (Functions of small quantities.) *Let $\Phi(t,\rho)$ be smooth for t in the cube $[-\delta,\delta]^p$ and ρ in a compact set $K \subset \mathbb{R}^{p'}$. Let $\varepsilon_1, \dots, \varepsilon_p \in \mathcal{X}^0_\gamma$ with $\sup|\varepsilon_k| \le \delta$, and let $\rho_1, \dots, \rho_{p'}$ be functions of r with values in K which are bounded together with all their D-derivatives. Then*

$$\|\Phi(\varepsilon,\rho) - \Phi(0,\rho)\|_{\gamma,s} \le C_s \sum_k \|\varepsilon_k\|_{\gamma,s}, \tag{7.10}$$

where C_s depends only on s, on Φ, on the bounds for the ρ_l and on $\max_k \|\varepsilon_k\|_{\gamma,2}$.

(v) (Flat extension.) *For $u \in \mathcal{X}^0_\gamma$ and all $a, b, k \ge 0$,*

$$d_\partial^{-k}\,|\partial_r^a \partial_\theta^b u| \le C_{a,b,k}\,H^\gamma\,E_0^{C_{a,b,k}}\,\|u\|_{\gamma,a+b+2}. \tag{7.11}$$

Consequently u extends smoothly by zero across both endpoints.

Parts (i), (ii) *and* (iv) *hold also in the spaces $\mathcal{Z}_\gamma$ of circle data, with the low order* 1 *in place of* 2.

The meaning of (7.8) is that weights add under multiplication and only one factor carries the high order. The meaning of (7.9) is that a factor bounded by a coefficient power costs *one* order, whatever the power: this is why the weight M was introduced.

Proof. Step 1: tameness. Write $v = H^{-\gamma}u$. Choose functions $\phi_i(\varrho)$, $i \in \mathbb{Z}$, with bounded derivatives, finite overlap and $\sum_i \phi_i^2 = 1$, supported on intervals of unit length. Since $\log M$ has bounded derivatives, $M \asymp M_i := M(i)$ on the support of ϕ_i, and the norm of v is equivalent to

$$\sum_{i,j} \int (M_i^2 + \xi^2 + j^2)^s \, |\widehat{\phi_i v}(\xi, j)|^2 \, d\xi.$$

Let $\omega_i = (M_i^2 + \xi^2 + j^2)^{1/2}$. A smooth partition ζ_k in the variable $\log \omega_i$ gives maps J and P with $PJ = I$ to and from the space of exponentially decreasing sequences in $\ell_i^2 L_\xi^2 \ell_j^2$: namely $(Jv)_{k,i} = \zeta_k(\omega_i)\, \widehat{\phi_i v}$ and $Pb = \sum_i \phi_i \, \mathcal{F}^{-1} \sum_k \widetilde{\zeta}_k(\omega_i)\, b_{k,i}$, where $\widetilde{\zeta}_k$ equals one on the support of ζ_k. By the Cauchy–Schwarz inequality in k, $\|Jv\|_s \le C_s \|v\|_{s+1}$, and by finite overlap $\|Pb\|_s \le C_s \|b\|_s$. This exhibits the space as a tame direct summand of a model space, and provides smoothing operators. Parity projections preserve the construction, and the scalar multiple of χ_L is a finite-dimensional summand. If ψ is compactly supported with $\mathsf{Q}\psi = 1$, then $f \mapsto f - \psi\, \mathsf{Q} f$ is a projection onto $\mathcal{Y}_\beta$; the functional Q is continuous because it is the pairing of $H^{-\beta} f$ with the square integrable function $H^\beta \ell$. The same Fourier construction applies to the spaces of circle data.

Step 2: products. Let $v_k = H^{-\gamma_k} u_k$, so that $H^{-\gamma_1-\gamma_2} u_1 u_2 = v_1 v_2$. On the support of ϕ_i the norm of order s of v is equivalent to $\sum_{c \le s} M_i^c \, \|\phi_i v\|_{H^{s-c}}$. Write $\phi_i v_1 v_2 = (\phi_i v_1)(\widetilde{\phi}_i v_2)$ with a cutoff $\widetilde{\phi}_i$ equal to one on the support of ϕ_i, apply Lemma 7.2(ii) on the cylinder, and attach the power M_i^c to the factor of high order. Summing the squares over i, with the low factor estimated uniformly in i, gives (7.8). Products of several factors, and products of derivatives of total order at most s, are estimated in the same way.

Step 3: reference functions. By Leibniz' rule, $M^c\, \mathsf{D}^a \partial_\theta^b (H^{-\gamma} c\, u)$ is a sum of terms $(\mathsf{D}^{a_1} c) \cdot M^c\, \mathsf{D}^{a-a_1} \partial_\theta^b (H^{-\gamma} u)$, and $|\mathsf{D}^{a_1} c| \le E_0^A \le C_A M$ by (7.3). Each term is therefore bounded by a term of the norm of order $s+1$ of u. For the second estimate, $\partial_r = \ell^{-1} \mathsf{D}$; the operator D costs one order, because $\mathsf{D} \log H$ is bounded with all its derivatives, and ℓ^{-1} is a reference function of the kind just treated.

Step 4: functions of small quantities. Write $\Phi(\varepsilon, \rho) - \Phi(0, \rho) = \sum_k \varepsilon_k \, \Phi_k(\varepsilon, \rho)$ with the smooth functions $\Phi_k(t, \rho) = \int_0^1 \partial_{t_k} \Phi(\tau t, \rho)\, d\tau$. Then $H^{-\gamma}(\Phi(\varepsilon, \rho) - \Phi(0, \rho)) = \sum_k (H^{-\gamma} \varepsilon_k)\, \Phi_k(\varepsilon, \rho)$. A derivative of $\Phi_k(\varepsilon, \rho)$ is a sum of products of derivatives of the ε_l and of the ρ_l with bounded coefficients; the derivatives of the ρ_l are bounded, and the products of derivatives of the ε_l are estimated on the support of ϕ_i by Lemma 7.2(ii), as in the proof of part (iii) of that lemma. Since $H \le C$, Sobolev norms of ε_l on the support of ϕ_i are bounded by its weighted norms. The product estimate of Step 2, with the power M_i^c attached to the factor of high order, gives (7.10).

Step 5: flat extension. The operator ∂_r^a is a combination of D-derivatives of order at most a with coefficients bounded by $E_0^{C_a}$, and $d_\partial^{-1} \le E_0^C$. By the Sobolev embedding on unit intervals of ϱ, the supremum of $|\mathsf{D}^{a'} \partial_\theta^b (H^{-\gamma} u)|$, $a' \le a$, is bounded by $\|u\|_{\gamma, a+b+2}$. This gives (7.11). Its right side tends to zero at the endpoints by (7.3), so all derivatives of the extension by zero exist and vanish at the endpoints. □

The small quantities. In the height equation and its linearization, part (iv) of the lemma is applied to the following six functions only. Let $Z = qu$ and $f = \nu \chi_* + \sum_m \kappa_m \chi_m$, so that $w = w_* e^f$, and put

$$\varepsilon_1 = f, \quad \varepsilon_2 = \frac{Z_r}{w_*}, \quad \varepsilon_3 = \frac{Z_{\theta\theta}}{r\, w_*}, \quad \varepsilon_4 = \frac{Z_\theta}{r}, \quad \varepsilon_5 = \frac{Z_{r\theta} - Z_\theta / r}{w_*}, \quad \varepsilon_6 = \frac{f'}{r\, v_0^2}, \tag{7.12}$$

the last one on the negative components, where $v_0 = \sqrt{-w_*'/(r w_*)}$; it is supported in the supports of the controls χ_m. Each is a reference function times a derivative of positive order of the height, or a control term. The bounded parameters are $\rho = (r, w_*, w_*')$. With $E = 1 + w_*^2 e^{2\varepsilon_1}$,

$$\frac{w}{w_*} = e^{\varepsilon_1}, \qquad \frac{Q}{w} = 1 + e^{-\varepsilon_1}(\varepsilon_2 + E\,\varepsilon_3), \qquad D = 1 - w_*^2\,(2e^{\varepsilon_1}\varepsilon_2 + \varepsilon_2^2) - E\,\varepsilon_4^2,$$
$$d = -\frac{E}{r}\,\frac{e^{-\varepsilon_1}\,\varepsilon_5}{Q/w}, \qquad v = v_0\,\sqrt{1-\varepsilon_6}\,\frac{\sqrt{D}}{Q/w}, \tag{7.13}$$

where $d \pm v$ are the characteristic speeds (3.25). The coefficients (3.6) are $a = w\,(Q/w)\,D^{-3/2}$, $b = -(E/r)\,w_*\varepsilon_5\,D^{-3/2}$ and $c = r^{-1}(w' + EZ_{rr} - ww'Z_r)\,D^{-3/2}$; the residual $\mathcal{N}_h$ and $\bar{c}$ are polynomials in w, w' and the derivatives of Z of order at most two, times powers of D and Q. On the neighbourhood of Lemma 7.7 below, $\varepsilon_1, \dots, \varepsilon_6$ are small in the supremum norm, so the reciprocals and roots in these formulas are evaluated where they are smooth.

7.3 The geometry at the roots, and the travel-angle controls

Scales in a cell. Recall from Section 5 that on cell n

$$A_n \asymp e^{-n^2}, \quad B_n \asymp A_n^9, \quad c\,\mathfrak{r}_n A_n^5 \le r_X \le C\,\mathfrak{r}_n A_n, \quad w_* = \lambda_n\, r_X, \quad \log\lambda_n = -n^{5/2} + O(n),$$

with $\mathfrak{r}_n$ bounded above and below, that $r_X = \partial r/\partial X$ satisfies $r_{XX} = -\rho\, r$ with $\rho = A_n(e^{-X} - e^{-2X}) - B_n$, that $cA_{n+1} \le d_\partial \le CA_n$, and that every fixed derivative of $w_*^{\pm 1}$ costs a coefficient power (Lemma 5.2). These comparisons persist on a bounded enlargement of a cell.

The characteristic speed near a root. Let x be the distance from a root of a negative component, counted positively into the component and negatively on the positive side. On a cell, $w_* = \lambda_n r_X$ and $r_{XX} = -\rho\, r$ give $w_*' = -\lambda_n \rho\, r / r_X$, so the reference characteristic speed (3.25) is

$$v_0 = \sqrt{-\frac{w_*'}{r w_*}} = \frac{\sqrt{\rho}}{r_X} = \sqrt{x}\; c_i(x), \qquad i \in \{L, R\},$$

where c_i is smooth and positive on a neighbourhood of the root, on both sides.

Lemma 7.5 (Root zones)**.** *At the left and right roots of the negative component of cell n,*

$$c_L(0) \asymp \mathfrak{r}_n^{-3/2} A_n^{-1}, \qquad c_R(0) \asymp \mathfrak{r}_n^{-3/2} A_n^{-3}, \tag{7.14}$$

and for every k, on the interval $|x| \le c\,\mathfrak{r}_n A_n^5$,

$$\Big|\partial_r^k \Big(\frac{c_i}{c_i(0)}\Big)^{\pm 1}\Big| \le C_k\,(\mathfrak{r}_n A_n^5)^{-k}. \tag{7.15}$$

Fix a sufficiently small $\epsilon_{\mathrm{root}} > 0$ and put

$$\delta_n^{\mathrm{root}} = \epsilon_{\mathrm{root}}\,\mathfrak{r}_n\,A_n^{20}. \tag{7.16}$$

The interval $|x| < 8\delta_n^{\mathrm{root}}$ is called the zone *of the root. The zones of the two roots are disjoint and lie inside the cell, and on them $c_i(x)/c_i(0) \in [\frac{1}{2}, 2]$. The finitely many roots of the central model have zones with the same properties and fixed constants.*

Proof. Step 1: the value at the root. Let X_i be the value of the model coordinate at the root. Since $v_0^2 = \rho/r_X^2$ and $x = |r - r(X_i)|$, division by x and passage to the limit give

$$c_i(0)^2 = \frac{|\rho_X(X_i)|}{r_X(X_i)^3}.$$

With $t = e^{-X}$ the roots solve $A_n(t - t^2) = B_n$; hence $e^{-X_L} = 1 + O(A_n^8)$ and $e^{-X_R} \asymp A_n^8$, and $|\rho_X(X_L)| \asymp A_n$, $|\rho_X(X_R)| \asymp A_n^9$. By Lemma 5.1 and the choice of the solution in Section 5.2, $r_X(X_L) \asymp \mathfrak{r}_n A_n$ and $r_X(X_R) = \epsilon_R\, \mathfrak{r}_n A_n^5$. This gives (7.14).

Step 2: derivatives. Division of a function with a simple zero by the distance to the zero is the operation $f(s)/s = \int_0^1 f'(ts)\, dt$; it costs one derivative and preserves bounds. On $|X - X_i| \le c$ the quotient $\rho/(X_i - X)$, with the sign which makes it positive, is comparable to $|\rho_X(X_i)|$, and its derivatives in X are bounded by multiples of it; this is read off from the explicit function ρ. At a root $r_{XX} = 0$ and $r_{XXX} = -\rho_X r$, and the differential equation shows that r_X is comparable to $r_X(X_i)$ on $|X - X_i| \le c$, with derivatives in X bounded by multiples of it. The same then holds for the quotient $(r - r(X_i))/(X - X_i)$. Since

$$c_i^2 = \frac{\rho}{X_i - X} \cdot \frac{X_i - X}{r(X_i) - r} \cdot \frac{1}{r_X^2},$$

with the signs chosen as above, and $\partial_r = r_X^{-1}\partial_X$ with $r_X \ge c\, \mathfrak{r}_n A_n^5$, we obtain (7.15) on the image of $|X - X_i| \le c$, an interval of radial length at least $c\, \mathfrak{r}_n A_n^5$ on either side of the root. At the left root the same argument gives the better scale $\mathfrak{r}_n A_n$.

Step 3: zones. The width $8\delta_n^{\mathrm{root}}$ divided by $\mathfrak{r}_n A_n^5$ is $8\epsilon_{\mathrm{root}} A_n^{15}$. So for small ϵ_{root} the case $k = 1$ of (7.15) keeps $c_i(x)/c_i(0)$ in $[\frac12, 2]$ on the zone. The zones lie in the cell and are disjoint because each part of the cell has radial length comparable to A_n. The central model has finitely many roots, at which $c_i(0) > 0$, and fixed zones are chosen there. The width of a zone, its reciprocal, and the derivatives of cutoffs adapted to it are coefficient powers. □

Lemma 7.6 (Travel-angle controls)**.** *For a profile w let $T_n(w) = \int \sqrt{-w'/(rw)}\, dr$ be the travel angle of its nth negative component. There are functions χ_n, supported in $\delta_n^{\mathrm{root}} < x < 2\delta_n^{\mathrm{root}}$ near the left root of that component, such that*

$$\frac{d}{dt}\Big|_{t=0} T_n(w_* e^{t\chi_n}) = 1, \qquad \|\partial_r^k \chi_n\|_\infty \le C_k\, \mathfrak{r}_n^{-k} A_n^{9-20k}.$$

There is also a cutoff function of r which rises from 0 to 1 on $3\delta_n^{\mathrm{root}} < x < 4\delta_n^{\mathrm{root}}$, with the corresponding bounds. On the supports of χ_n and of the derivative of this cutoff,

$$v_0 \asymp \mathfrak{r}_n^{-1} A_n^9, \qquad \Big|\frac{w_*'}{w_*}\Big| \asymp \mathfrak{r}_n^{-1} A_n^{18}.$$

Proof. Let $\phi \ge 0$ be a nonzero smooth function supported in $(1, 2)$ and put $f_n = \phi(x/\delta_n^{\mathrm{root}})$. For the variation $w_* e^{t f_n}$ the speed changes by $\delta v_0 = -f_n'/(2rv_0)$, so

$$\frac{d}{dt}\Big|_{t=0} T_n = \frac12 \int \Big(\frac{1}{rv_0}\Big)' f_n\, dr = -\frac{(\delta_n^{\mathrm{root}})^{-1/2}}{4\, r_{n,L}\, c_L(0)} \int_1^2 t^{-3/2}\phi(t)\, dt\ (1 + O(A_n^{19})),$$

where $r_{n,L}$ is the radius of the root. This is nonzero and comparable to $-A_n^{-9}$. Let χ_n be f_n divided by this number; the bounds follow by scaling. The bounds for v_0 follow from (7.14), and $|w_*'/w_*| = rv_0^2$. The cutoff is the integral of a rescaled bump. □

7.4 The state and one neighbourhood for all components

The state space is

$$\mathbf{x} = (u, \nu, \kappa, h_L), \qquad u \in \Phi_* + \ker \ell_* \subset \mathcal{X}_\beta, \quad \nu \in \mathbb{R}, \quad \kappa \in \mathcal{K}_\beta, \quad h_L - \mathrm{id} \in \mathcal{Z}_{\beta,\mathrm{odd}}. \tag{7.17}$$

Each $h_{L,m}$ is an increasing map of $\mathbb{R}$ which commutes with the translation by π, that is, the lift of a diffeomorphism of the circle $\mathbb{R}/\pi\mathbb{Z}$. The normalization $\ell_* u = 1$ removes the freedom of adding a multiple of the kernel Φ_*. The size of a state, together with the parameter, is measured by the seminorms

$$\mathfrak{n}_s = |q| + \|u - \Phi_*\|_s + |\nu| + \|\kappa\|_{\mathcal{K}_{\beta,s}} + \|h_L - \mathrm{id}\|_{\mathcal{Z}_{\beta,s}}. \tag{7.18}$$

The unscaled u appears here, and not only qu, because the derivative of $\mathcal{F}_0(w,u) = \mathscr{L}_w u$ with respect to the metric depends on u even at $q = 0$.

We shall also need localized versions. For a component m, positive or negative, consider the component together with the neighbouring components that enter the construction of its coordinates; for a positive component these are the two adjacent negative ones. These neighbourhoods have bounded overlap. Let $\mathfrak{g}_{s,m}$ be the norm of order s of $(qu, \nu, \kappa, h_L - \mathrm{id})$ restricted to this neighbourhood, where only derivatives of positive order of the height are used, and compact scalars are counted only where their supports meet the neighbourhood. Then $(\sum_m \mathfrak{g}_{s,m}^2)^{1/2} \le C_s\, \mathfrak{n}_s$.

For the joint problem we also use the localized seminorm $\mathfrak{n}_{s,m}$ of $(q\,\Phi_*,\ u-\Phi_*,\ \nu,\ \kappa,\ h_L-\mathrm{id})$, defined in the same way. For $|q| \le 1$ the identity $qu = q\Phi_* + q(u - \Phi_*)$ gives $\mathfrak{g}_{s,m} \le C_s\, \mathfrak{n}_{s,m}$, and $(\sum_m \mathfrak{n}_{s,m}^2)^{1/2} \le C_s\, \mathfrak{n}_s$.

The neighbourhood $\mathfrak{N}$. All statements of Part III about states are made on one set,

$$\mathfrak{N} = \{(q, \mathbf{x}) :\ \mathfrak{n}_{b_0} < \epsilon_0\}, \tag{7.19}$$

in which the order b_0 and the number $\epsilon_0 > 0$ are fixed once. They are determined by six requirements, each of the form $b_0 \ge b^{(i)}$, $\epsilon_0 \le \epsilon^{(i)}$:

	imposed in	purpose
(N1)	Lemma 7.7	type and denominators
(N2)	Lemma 7.8	flows of $\bar{L}$ close to the reference flows
(N3)	Proposition 7.9	corrected speeds of the same sign
(N4)	Proposition 7.11	coordinate maps invertible
(N5)	Lemma 8.4	relative operator of norm less than $\frac{1}{2}$
(N6)	Proposition 9.2	control equation solvable

The numbers $b^{(i)}$ and $\epsilon^{(i)}$ of the ith requirement depend on the reference profile, on β, and on the constants of the statements with smaller index; they do not depend on the order s of any estimate. Since $\mathfrak{n}_b \le \mathfrak{n}_{b'}$ for $b \le b'$, a statement which holds on $\{\mathfrak{n}_b < \epsilon\}$ holds with the same constants on every smaller set $\{\mathfrak{n}_{b'} < \epsilon'\}$, $b' \ge b$, $\epsilon' \le \epsilon$. So the six requirements can be imposed one after the other, and we let b_0 be the largest of the $b^{(i)}$ and ϵ_0 the smallest of the $\epsilon^{(i)}$. In the proof of the ith statement only the requirements up to i are used.

Lemma 7.7 (One neighbourhood for the type and the denominators)**.** *On $\mathfrak{N}$ the profile w has the same positive and negative components and the same germs at the roots as w_*, and, with $Z = qu$ and $E = 1 + w^2$,*

$$\frac{w}{w_*} \in [\tfrac{1}{2}, 2], \qquad \frac{Q}{w} \in [\tfrac{7}{8}, \tfrac{9}{8}], \qquad D \in [\tfrac{1}{2}, \tfrac{3}{2}], \qquad 1 - \Big(\frac{Z_\theta}{r}\Big)^2 \ge \tfrac{3}{4}$$

on the whole annulus. Here

$$Q = w + Z_r + \frac{E\, Z_{\theta\theta}}{r}, \qquad D = 1 - 2wZ_r - Z_r^2 - E\Big(\frac{Z_\theta}{r}\Big)^2 \tag{7.20}$$

are the quantities (3.1) *for* $z = h + Z$.

Proof. Only finitely many derivatives of low order enter these conditions.

Step 1: low-order bounds on all components. On the neighbourhood, $\|u\|_{b_0} \le \|\Phi_*\|_{b_0} + \epsilon_0$ and $|q| < \epsilon_0$, so the geometric displacement $(Z, \nu, \kappa, h_L - \mathrm{id})$ is of size $C\epsilon_0$ in the norms of order b_0. Let K be the largest order of a derivative that occurs in the conditions, and take $b_0 \ge K + 2$. The Sobolev embedding on unit intervals of the variable ϱ, and the conversion of D into ∂_r at the cost of coefficient powers, give on component m

$$\max_{1\le a+b\le K} |\partial_r^a \partial_\theta^b Z| + |\kappa_m|\, \|\chi_m\|_{C^K} + \|h_{L,m} - \mathrm{id}\|_{C^K} \le C\, \epsilon_0\, H_m^\beta\, \mathcal{E}_m^C. \tag{7.21}$$

Only derivatives of positive order of Z occur: the endpoint constant of a height does not decay, but it does not enter any coefficient. The compact control $\nu\chi_*$ obeys the same bound on the finitely many components it meets. Now

$$\sup_m H_m^\beta\, \mathcal{E}_m^A < \infty \qquad \text{and} \qquad H_m^\beta\, \mathcal{E}_m^A \longrightarrow 0 \qquad \text{for every fixed } A, \tag{7.22}$$

because $\log \mathcal{E}_m = o(S_m)$. Hence, for any fixed coefficient power, a condition $\epsilon_0 \le \epsilon$ makes the right side of (7.21) times that power as small as we wish on all components at once. Requirement (N1) is the condition of this kind needed in Steps 2 and 3; the later requirements are of the same form.

Step 2: the type. Write $w = w_* e^f$ with $f = \nu\chi_* + \sum_m \kappa_m \chi_m$. The controls vanish near every root, so the germs of w at the roots are those of w_*. On the support of χ_n, Lemma 7.6 gives

$$\frac{|\kappa_n \chi_n'|}{|w_*'/w_*|} \le C\, |\kappa_n|\, A_n^{-29},$$

which is small by Step 1. On the support of χ_* the function w_*'/w_* has a positive minimum. So $w'/w = w_*'/w_* + f'$ has the sign of w_*'/w_* away from the roots, and $|f| \le \log 2$.

Step 3: the denominators. By (5.11), $w^{-1} = w_*^{-1} e^{-f}$ is bounded by a coefficient power. So (7.20), (7.21) and (7.22) make $(Q - w)/w$, $D - 1$ and Z_θ/r uniformly small. □

The second lemma of this subsection controls the characteristic flows of $\bar{L}$. Everything constructed afterwards is built from them.

Lemma 7.8 (Speeds and flows of $\bar{L}$)**.** *On* $\mathfrak{N}$ *the following holds on every negative component* m. *The characteristic speeds* $c_\pm = d \pm v$ *of* $\bar{L}$ *satisfy*

$$\int \|\partial_\theta c_\pm(r, \cdot)\|_{H_\theta^s}\, dr \le C_s\, \mathcal{E}_m^{C_s}\, H_m^\beta\, \mathfrak{g}_{s+3,m}, \tag{7.23}$$

the integral being taken over the component. The flows of $c_\pm$ *across the component and their inverses have angular derivatives in* $[\frac{1}{2}, 2]$, *and the constants of Lemma 7.3 for them are the same for all* m. *In the zone of a root,* $v = \sqrt{x}\, c_0(x, \theta)$ *with*

$$c_0 = c_i(x)\, \sqrt{1 - \varepsilon_6}\, \frac{\sqrt{D}}{Q/w}, \qquad \tfrac{1}{4}\, c_i(0) \le c_0 \le 4\, c_i(0). \tag{7.24}$$

Proof. Step 1: the speeds. By (7.13), d and $v/v_0 - 1$ are smooth functions of the small quantities (7.12) which vanish when these are zero, and only the small quantities depend on θ. Apply Lemma 7.2(ii) and (iii) on the circle for each r, and the Cauchy–Schwarz inequality in r. The reference functions $1/w_*$, v_0 and $1/(rv_0^2)$ on the supports of the controls are bounded by coefficient powers (Lemmas 5.2 and 7.6), and the length of the component is bounded. The small quantities contain at most two derivatives of Z, and ∂_θ adds one. This gives (7.23); the factor H_m^β is the weight of the deformation on the component.

Step 2: the flows. Rescale the radial variable on the component to the unit interval; this multiplies the speeds by its length. By (7.22) the right side of (7.23) for $s \le 4$ is as small as we wish on the neighbourhood, uniformly in m. Lemma 7.3 gives the statement about the flows; the derivative of an inverse flow is the reciprocal of that of the flow. The speed v_0 of the reference profile may be large, but it does not depend on θ and so does not enter.

Step 3: the root factor. In the zone of a root $v_0 = \sqrt{x}\, c_i(x)$, so the formula for v in (7.13) gives (7.24). The bounds follow from $c_i(x)/c_i(0) \in [\frac{1}{2}, 2]$ (Lemma 7.5) and the smallness of the quantities (7.12). □

The objects constructed in the next three subsections, and the estimates they satisfy, are collected in Table 4 and Proposition 7.11 of Section 7.8. The constructions themselves use only the two lemmas of this subsection.

7.5 Fixing the return maps by their own residual

Characteristics of $\bar{L}$. Start from the operator $\bar{L}$ of Lemma 3.4, with coefficients $(a, b, \bar{c})$, on a negative component. By (3.25) its characteristic slopes are

$$c_\pm = d \pm v, \qquad d = \frac{b}{a}, \qquad v = \frac{\sqrt{b^2 - a\bar{c}}}{a}.$$

Because the height is even in θ, the reflection $\mathsf{s}\theta = -\theta$ exchanges the two families: $c_+ = -c_- \circ \mathsf{s}$. Let f be the *minus return map*: the map from the left boundary circle of the component to the right one along the characteristics of slope c_-. Let T be the travel angle of the reference profile on this component. At the reference state $f = R_{-T}$.

Charts and the return residual. Given the chart h_L of the left circle, define

$$\widetilde{h}_R = R_{-T} \circ h_L \circ f^{-1}, \qquad h_R = \mathsf{O}\,\widetilde{h}_R, \qquad \mathfrak{e} = (\mathsf{E}\,\widetilde{h}_R) \circ h_R^{-1}, \tag{7.25}$$

where O and E denote the odd and even parts of a function of θ. The derivative of h_R is an average of two positive derivatives, so h_R is increasing; it is the chart of the right circle. By construction $\widetilde{h}_R = (\mathrm{id} + \mathfrak{e}) \circ h_R$, hence

$$(\mathrm{id} + \mathfrak{e}) \circ h_R \circ f \circ h_L^{-1} = R_{-T}. \tag{7.26}$$

So $\mathfrak{e} = 0$ means that the minus return map is the rotation R_{-T} when read in the charts h_L and h_R. Since the charts are odd and the plus return map is $\mathsf{s} \circ f \circ \mathsf{s}$, the plus return map is then R_T. Thus the single even equation $\mathfrak{e} = 0$ imposes both return conditions. Its mean concerns the travel angle, its oscillating part the parametrization of the boundary circles.

The derivative with respect to the chart. Vary the chart by $\delta h_L = \psi \circ h_L$ with ψ odd, keeping f fixed. Then

$$\begin{aligned} D_h\mathfrak{e}[\psi] &= \mathcal{M}_T\psi + \mathcal{Q}_h(\mathfrak{e},\psi), \qquad \mathcal{M}_T\psi = \mathsf{E}\big[\psi(\theta+T)\big], \\ \mathcal{Q}_h(\mathfrak{e},\psi) &= \mathsf{E}\Big[\mathfrak{e}\int_0^1 \psi'(\theta+T+t\mathfrak{e})\,dt\Big] - \mathfrak{e}'\,\mathsf{O}\big[\psi(\theta+T+\mathfrak{e})\big]. \end{aligned} \tag{7.27}$$

Indeed, the variation of $\widetilde{h}_R$, composed with h_R^{-1}, is $\psi(\theta+T+\mathfrak{e})$; its odd part is the variation of h_R in the same coordinates, and the formula follows by differentiating $\mathfrak{e}\circ h_R = \mathsf{E}\,\widetilde{h}_R$. The term $\mathcal{Q}_h$ is bilinear in $(\mathfrak{e},\psi)$ in the sense needed later and vanishes when $\mathfrak{e}=0$. The leading term is explicit:

$$\mathcal{M}_T(\sin j\theta) = \sin(jT)\cos j\theta, \qquad \text{so} \qquad \mathcal{M}_T^{-1}(g_j\cos j\theta) = \frac{g_j}{\sin(jT)}\,\sin j\theta \quad (j>0 \text{ even}). \tag{7.28}$$

By Proposition 4.2(ii) the inverse loses one angular derivative and a coefficient power. It does not exist on constants: the mean of $\mathfrak{e}$ cannot be corrected by a chart. This is the reason for the radial controls κ_m.

Proposition 7.9 (The return correction)**.** *On $\mathfrak{N}$ the operator $\bar{L}$ can be replaced by a symmetric operator in divergence form*

$$\widehat{L} = \partial_r(a\,\partial_r + \widehat{b}\,\partial_\theta) + \partial_\theta(\widehat{b}\,\partial_r + \widehat{c}\,\partial_\theta)$$

which has the same coefficient a, coincides with $\bar{L}$ on the positive components and near the roots, and whose two return maps $\widehat{f}_\pm$ on every negative component satisfy

$$h_R \circ \widehat{f}_\pm \circ h_L^{-1} = R_{\pm T}$$

exactly. The difference has the form

$$\bar{L} - \widehat{L} = \mathscr{E}_{\mathbf{x}}(\mathfrak{e}), \tag{7.29}$$

where $(e,\eta)\mapsto \mathscr{E}_{\mathbf{x}}(e)\eta$ is bilinear with values in $\mathcal{Y}_\beta$, and

$$\|\mathscr{E}_{\mathbf{x}}(e)\eta\|_{\beta,s} \le C_s\big[\|e\|_{\mathcal{Z}_{\beta,s+5}}\,\|\eta\|_{\beta,b_0} + \|e\|_{\mathcal{Z}_{\beta,b_0}}\,\|\eta\|_{\beta,s+5} + \mathfrak{n}_{s+5}\,\|e\|_{\mathcal{Z}_{\beta,b_0}}\,\|\eta\|_{\beta,b_0}\big]. \tag{7.30}$$

Its derivatives with respect to the state satisfy estimates of the shape (7.7).

Proof. Step 1: deforming the flow. Let Φ_r be the flow of the minus characteristics from the left boundary to the radius r, so that f is its value at the right boundary. The return map we want is $f_* = h_R^{-1}\circ R_{-T}\circ h_L$. With $\mathsf{g} = h_R\circ f$, (7.26) gives

$$K := f^{-1}\circ f_* = \mathsf{g}^{-1}\circ(\mathrm{id}+\mathfrak{e})\circ\mathsf{g}, \qquad \Delta := K - \mathrm{id}.$$

Let $\chi(r)$ be the cutoff of Lemma 7.6, rising from 0 to 1 inside the component, and put

$$J_r = \mathrm{id} + \chi(r)\,\Delta, \qquad \widehat{\Phi}_r = \Phi_r\circ J_r.$$

Each J_r is increasing, being a convex combination of increasing maps, and $\widehat{\Phi}_r$ runs from the identity to $f\circ K = f_*$. The curves $r\mapsto\widehat{\Phi}_r(\theta_0)$ have the slope

$$\widehat{c}_-(r,\theta) = c_-(r,\theta) + \Phi_r'(\Phi_r^{-1}\theta)\,\chi'(r)\,\Delta(J_r^{-1}\Phi_r^{-1}\theta). \tag{7.31}$$

Step 2: the new coefficients. Define $\widehat{c}_+ = -\widehat{c}_- \circ \mathsf{s}$ and

$$\widehat{d} = \frac{\widehat{c}_+ + \widehat{c}_-}{2}, \quad \widehat{v} = \frac{\widehat{c}_+ - \widehat{c}_-}{2}, \qquad \widehat{b} = a\,\widehat{d}, \quad \widehat{c} = a\,(\widehat{d}^{\,2} - \widehat{v}^{\,2}).$$

Then $\widehat{L}$ has the characteristic slopes $\widehat{c}_\pm$, its minus return map is f_*, and its plus return map is the reflection of f_*; these are the asserted returns. On the support of χ' the speed v is bounded below by a coefficient power (Lemma 7.6). The correction in (7.31) is at most $C\epsilon_0 H_m^\beta \mathcal{E}_m^C$: indeed $\Phi_r' \le 2$ by Lemma 7.8, χ' is a coefficient power, and Δ is bounded by $\mathfrak{e}$, which by its definition and (7.21) is bounded by the low-order norms of the deformation on the component. By (7.22) the correction is less than $v/2$ on $\mathfrak{N}$; this is requirement (N3). So $\widehat{v} > 0$. Near the roots and on positive components nothing is changed.

Step 3: the error is bilinear. From the definition of K,

$$\Delta(\theta) = \mathfrak{e}(\mathsf{g}\theta) \int_0^1 (\mathsf{g}^{-1})'(\mathsf{g}\theta + t\,\mathfrak{e}(\mathsf{g}\theta))\,dt.$$

Keep the state and the points at which the functions in this formula are evaluated, and replace only the leading factor $\mathfrak{e}$ by an independent function e. This defines increments $d_1[e]$ and $v_1[e]$ of d and v which are linear in e and equal the actual increments $d_1 = \widehat{d} - d$, $v_1 = \widehat{v} - v$ when $e = \mathfrak{e}$. The coefficients of $\bar{L} - \widehat{L}$ are then $\delta b[\mathfrak{e}]$ and $\delta c[\mathfrak{e}]$, where

$$\delta b[e] = -a\,d_1[e], \qquad \delta c[e] = -a\,(2d + d_1)\,d_1[e] + a\,(2v + v_1)\,v_1[e],$$

and $\mathscr{E}_{\mathbf{x}}(e)\eta$ is the corresponding expression in divergence form applied to η. It is bilinear in (e, η). It is supported in the interior of the component, so it is flat at the endpoints and has integral zero.

Step 4: the estimate. We first list the factors of $\delta b[e]$ and $\delta c[e]$. By (7.31) and the formula for Δ in Step 3, the increment of the minus speed is

$$\widehat{c}_-[e] - c_- = (\Phi_r' \circ \Phi_r^{-1})\;\chi'(r)\;(e\,\mathsf{m}) \circ \mathsf{h}_r, \qquad \mathsf{m} = \int_0^1 (\mathsf{g}^{-1})'(\,\cdot + t\,\mathfrak{e})\,dt, \quad \mathsf{h}_r = \mathsf{g} \circ J_r^{-1} \circ \Phi_r^{-1},$$

and $d_1[e]$, $v_1[e]$ are the half sum and the half difference of this function and its reflection. So $\delta b[e]$ and $\delta c[e]$ are sums of products of the following factors:

(1) the coefficients a, d, v of row 1 and the increments d_1, v_1, which are of the form considered here with $e = \mathfrak{e}$;

(2) the reference function χ';

(3) $\Phi_r' \circ \Phi_r^{-1}$ and $\mathsf{m} \circ \mathsf{h}_r$;

(4) $e \circ \mathsf{h}_r$, which occurs exactly once in each product.

Fix r in the support of χ' and estimate each factor in H^s_θ. The maps Φ_r, f, h_L, h_R, J_r and their inverses have derivatives bounded above and below (Lemma 7.8 and Step 2). By Lemma 7.3 and (7.23),

$$\|\Phi_r' - 1\|_{H^s} + \|f' - 1\|_{H^s} \le C_s\,\mathcal{E}_m^{C_s}\,\mathfrak{g}_{s+3,m},$$

and $h_L' - 1$ is part of the state. By Lemma 7.2(iv), applied to each composition and each inverse in the definitions of h_R, $\mathfrak{e}$, g, J_r, h_r and m, the same bound holds for $\mathsf{h}_r' - 1$ in H^{s-1} and for the factors (3) minus one in H^s, and

$$\|e \circ \mathsf{h}_r\|_{H^s} \le C_s(\|e\|_{H^s} + \|e\|_{H^3}\,\|\mathsf{h}_r' - 1\|_{H^{s-1}}).$$

The factors (1) differ from functions of r by at most $C_s\mathcal{E}_m^{C_s}\mathfrak{g}_{s+3,m}$ in H^s_θ, by Lemma 7.2(iii) and (7.13), and the functions of r and the factor (2) are bounded by coefficient powers. The product estimate of Lemma 7.2(ii), with the high order on one factor at a time, gives on the component m and for each r

$$\|\delta b[e]\|_{H^s_\theta} + \|\delta c[e]\|_{H^s_\theta} \le C_s\,\mathcal{E}_m^{C_s}\,(\|e\|_{H^s} + (1+\mathfrak{g}_{s+3,m})\,\|e\|_{H^3}).$$

A radial derivative of a factor is expressed through the equations of the flows:

$$\partial_r\Phi_r = c_-(r,\Phi_r), \qquad \partial_r\Phi_r^{-1} = -c_-\,(\Phi_r^{-1})', \qquad \partial_r J_r = \chi'\,\Delta.$$

So it is again a sum of products of factors of the four kinds and of speeds, in which one factor carries one more angular derivative, and the bound just proved holds for $\partial_r\delta b[e]$ and $\partial_r\delta c[e]$ with $s+1$ in place of s and H^4 in place of H^3. The same is true of further radial derivatives.

We now count. The expression $\mathscr{E}_{\mathbf{x}}(e)\eta = \partial_r(\delta b[e]\,\eta_\theta) + \partial_\theta(\delta b[e]\,\eta_r + \delta c[e]\,\eta_\theta)$ contains two derivatives of η and one derivative of the coefficients. Its norm of order s therefore involves η to the order $s+2$, e to the order $s+1$, and the state to the order $s+4$, with one of the three at the high order in each term. The reference functions cost one further order by (7.9). So the loss is at most 5, as stated in (7.30). Each term contains the factor e and a derivative of positive order of η, both of weight H^β_m; after division by the weight of the source space a factor H^β_m is left, which by (7.3) absorbs the coefficient powers and the powers of M. The product estimate (7.8) and summation over the components give (7.30).

The state enters $\delta b[e]$ and $\delta c[e]$ through the factors (1) and (3) and through the map h_r. A derivative of order j with respect to the state of a flow, of a composition or of an inverse is a sum of products of the same kinds in which at most j additional angular derivatives fall on one factor (Lemma 7.3 and the chain rule). The count above, with $5+j$ in place of 5, gives the estimates (7.7) for the derivatives with respect to the state. □

Combining Lemma 3.4 with Proposition 7.9 we obtain the exact identity

$$D_u\mathcal{F}_q(w,u)\,\eta = \widehat{L}\eta + q\,\partial_\theta(B\,\mathcal{A}\,\eta_\theta) + \mathscr{E}_{\mathbf{x}}(\mathfrak{e})\,\eta, \qquad \mathcal{A} = \mathcal{F}_q(w,u), \quad B = \frac{E}{rQ}. \tag{7.32}$$

Both additional terms vanish at a solution of $(\mathcal{A},\mathfrak{e}) = 0$. We shall invert $\widehat{L}$ and carry these terms as errors in the nonlinear theorem.

7.6 Coordinates on a negative component

Travel coordinates. Let $S_\pm(r,\theta)$ be the angle at which the plus or minus characteristic of $\widehat{L}$ through (r,θ) meets the left boundary circle. Put

$$U = h_L\circ S_+, \qquad V = h_L\circ S_-, \qquad \sigma = \frac{V-U}{2}, \qquad \varphi = \frac{U+V}{2}. \tag{7.33}$$

On the left boundary $(\sigma,\varphi) = (0,h_L)$, and by the exact returns of $\widehat{L}$, on the right boundary $(\sigma,\varphi) = (T,h_R)$. Moreover $U_\theta, V_\theta > 0$ and

$$\det\frac{\partial(\sigma,\varphi)}{\partial(r,\theta)} = \widehat{v}\,U_\theta V_\theta > 0.$$

For fixed φ the function σ increases strictly from 0 to T, so (σ,φ) is a diffeomorphism of the interior of the component onto the fixed cylinder $(0,T)\times\mathbb{T}$. In these coordinates the principal part of $\widehat{L}$, as a divergence expression, has the density $\delta\,\mathrm{diag}(1,-1)$ with $\delta = a\widehat{v}$: it is a wave operator, degenerate at the two ends, where δ vanishes.

The root coordinate. At a root the speed is $\widehat{v} = \sqrt{x}\, c_0(x,\theta)$ with c_0 smooth and positive, so σ is of order $x^{3/2}$. The coordinates which are smooth at the two roots are

$$\xi_L = \big(\tfrac{3}{2}\sigma\big)^{2/3}, \qquad \xi_R = \big(\tfrac{3}{2}(T-\sigma)\big)^{2/3},$$

used on the halves $\sigma \le T/2$ and $\sigma \ge T/2$ of the component (Figure 14). For an ordinary differential equation with a simple turning point this is Langer's change of variable [Lan49], [Olv74, Chapter 11]. When the coefficients depend on the angle, smoothness of ξ_L at the root is not obvious. To prove it, with estimates, we solve the characteristic equation in a form in which $x^{3/2}$ is replaced by an independent variable ζ:

$$\Theta_t = 2xt\, d(xt^2,\Theta) + 2\zeta t^2\, c_0(xt^2,\Theta), \qquad 0 \le t \le 1. \tag{7.34}$$

For $\zeta = \pm x^{3/2}$ this is the equation of the plus and minus characteristics, parametrized by $x' = xt^2$. We use it for $0 \le x \le 3\delta_n^{\mathrm{root}}$, where $\widehat{L} = \bar{L}$, so that d and c_0 are the functions of Lemma 7.8; on the rest of the component the characteristics are smooth curves in (r,θ) and no such device is needed. The inverse of the time-one map of (7.34), followed by the boundary chart, is one smooth function $G(x,\zeta,\theta)$, and

$$U = G(x, x^{3/2}, \theta), \qquad V = G(x, -x^{3/2}, \theta).$$

That both families come from one function, with one boundary chart, is essential for the cancellation in the following lemma.

Lemma 7.10 (Substitution at a root)**.** *Let $G(x,\zeta,\theta)$ be smooth on the rectangle $0 \le x \le 1$, $|\zeta| \le 1$, and define*

$$\mathsf{Ev}\, G = \frac{G(x,x^{3/2},\theta) + G(x,-x^{3/2},\theta)}{2}, \qquad \mathsf{Od}\, G = \frac{G(x,x^{3/2},\theta) - G(x,-x^{3/2},\theta)}{2x^{3/2}}.$$

Both functions extend smoothly to $x = 0$, with $\mathsf{Od}\, G = \partial_\zeta G(0,0,\theta)$ there, and for every integer $s \ge 0$

$$\|\mathsf{Ev}\, G\|_{C^s_{x,\theta}} \le C_s \|G\|_{C^s_{x,\zeta,\theta}}, \qquad \|\mathsf{Od}\, G\|_{C^s_{x,\theta}} \le C_s \|G\|_{C^{s+1}_{x,\zeta,\theta}}. \tag{7.35}$$

The estimates hold also after any fixed number of differentiations with respect to parameters.

The point of the lemma is that the half-integer power $x^{3/2}$ does not cost half-integer many derivatives: an even function of ζ is a function of $\zeta^2 = x^3$.

Proof. Step 1: G even in ζ. Fix $0 < x_0 \le 1$ and let G_s be the Taylor polynomial of degree s of G at $(x_0,0)$ in the variables (x,ζ). Its monomials $(x-x_0)^i\zeta^{2k}$ become $(x-x_0)^i x^{3k}$ after the substitution. Their derivative of order $a \le s$ in x at x_0 either vanishes or carries a factor x_0^{3k-a+i} with nonnegative exponent, and their coefficients involve at most s derivatives of G. For the remainder $G - G_s$, a term of the chain rule with i derivatives falling on the first argument and k on the second is bounded by

$$C_s\|G\|_{C^s}\; x_0^{3(s-i-k)/2}\; x_0^{3k/2-(a-i)} = C_s\|G\|_{C^s}\; x_0^{3s/2-a-i/2},$$

again with a nonnegative exponent. When the exponent is zero the remainder still tends to zero as $x_0 \to 0$, by the continuity of the derivatives of order s. The polynomial terms converge to the appropriate Taylor coefficients at $x = 0$. So the derivatives of the substituted function up to order s are continuous up to $x = 0$ and bounded as claimed. Derivatives in θ commute with the construction and reduce the remaining order.

Step 2: the general case. For a general G apply Step 1 to its even part to obtain the first bound. For the second, the function

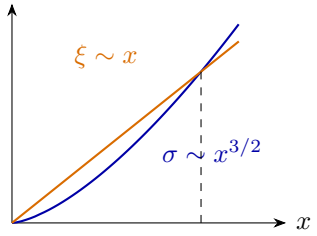
$$K(x,\zeta,\theta)=\frac{1}{2}\int_{-1}^{1}\partial_\zeta G(x,t\zeta,\theta)\,dt$$

is smooth and even in ζ, satisfies $\|K\|_{C^s}\le C_s\|G\|_{C^{s+1}}$, and $\mathsf{Od}\,G=K(x,x^{3/2},\theta)$; apply Step 1 to K. Both operations are linear in G, so they commute with differentiation with respect to parameters. □

Applied to the function G of the flow (7.34), the lemma gives

$$\varphi=\mathsf{Ev}\,G,\qquad \sigma=x^{3/2}\chi_0(x,\theta),\quad \chi_0(0,\theta)=\tfrac{2}{3}\,c_0(0,\theta)\,h_L'(\theta)>0,\qquad \xi_L=x\,(\tfrac{3}{2}\chi_0)^{2/3}.$$

So ξ_L is a smooth function with nonvanishing radial derivative up to the root, and (ξ_L,φ) are smooth coordinates there. The right root is treated in the same way with the chart h_R. Rescaling from the unit rectangle of the lemma to a zone of a root introduces powers of the zone width and its reciprocal, which are coefficient powers; their complete product costs one power of M. The substitution Ev uses s derivatives of its argument and Od uses $s+1$, and these counts do not grow with the output order.



The travel coordinate σ has zero derivative at the root; its 2/3 power has a nonzero derivative.

Figure 14: Regularization at a turning circle; x is the distance to the root. The graph shows the leading powers with unit coefficients. The positive coefficient of x in ξ is controlled uniformly at all turning circles.

The canonical form. Let $i\in\{L,R\}$, let $\Psi_i=(\xi_i,\varphi)$ be the coordinate map on the corresponding half of the component, and put

$$\alpha_i=\frac{\delta\circ\Psi_i^{-1}}{\sqrt{\xi_i}}=\frac{a\,|\det D\Psi_i|}{U_\theta V_\theta}\circ\Psi_i^{-1},\qquad h_i=\alpha_i^{-1/2}.$$

The second expression shows that α_i is smooth and positive up to the root. In the coordinates Ψ_i the principal density of $\widehat{L}$ is $\alpha_i\,\mathrm{diag}(1,-\xi_i)$, and conjugation by h_i removes its first-order terms: for $u=(h_iW_i)\circ\Psi_i$, the equation $\widehat{L}u=f$ becomes

$$\begin{aligned}(\partial_{\xi_i}^2-\xi_i\,\partial_\varphi^2+V_i)W_i&=F_i,\\ V_i&=-\alpha_i^{-1/2}\,(\partial_{\xi_i}^2-\xi_i\,\partial_\varphi^2)\,\alpha_i^{1/2},\\ F_i&=h_i\Big(\frac{f}{|\det D\Psi_i|}\circ\Psi_i^{-1}\Big).\end{aligned}\tag{7.36}$$

The principal part is the Tricomi operator $\partial_\xi^2-\xi\,\partial_\varphi^2$ [Tri23], the same for all states; only the potential V_i depends on the state. The transformation of the source preserves the pairing: $\int uf\,dr\,d\theta=\int W_iF_i\,d\xi_i\,d\varphi$. We call W_i the *canonical unknown*, h_i the *conjugating factor* and F_i the *canonical source*.

Transmission in the middle. The two halves meet at $\sigma = T/2$, where $\xi_L = \xi_R = \xi_c := (3T/4)^{2/3}$. There $\alpha_L = \alpha_R$ and $\partial_\sigma \log \alpha_R - \partial_\sigma \log \alpha_L = \xi_c^{-3/2}$. Continuity of u and of its derivative therefore reads

$$\begin{pmatrix} W_R \\ \partial_{\xi_R} W_R \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/(2\xi_c) & -1 \end{pmatrix} \begin{pmatrix} W_L \\ \partial_{\xi_L} W_L \end{pmatrix} \qquad \text{at } \xi_L = \xi_R = \xi_c. \tag{7.37}$$

This matrix does not depend on the state.

Boundary data at a root. At a root, where the boundary chart is h_{bd} ($= h_L$ or h_R), the value of u and its inward normal flux p_{in} are

$$u = (h_i W_i)(0, h_{\mathrm{bd}}), \qquad p_{\mathrm{in}} = \frac{h_{\mathrm{bd}}'}{h_i(0, h_{\mathrm{bd}})} \left(\partial_{\xi_i} W_i + (\partial_{\xi_i} \log h_i)\, W_i\right)(0, h_{\mathrm{bd}}). \tag{7.38}$$

No tangential derivative of W_i appears, because $a\varphi_r + \widehat{b}\varphi_\theta = 0$ at the root. The identity is exact, and it includes the zero mode.

Bounds. At the reference state all these objects are explicit functions of r; their bounds by coefficient powers are proved in Lemma 7.12. For a general state they are estimated in Proposition 7.11. These objects are rows 5–11 of Table 4.

7.7 Coordinates on a positive component

The reference coordinate. On a positive component, continue the root coordinates of the two adjacent roots across them and put $y_i = -\xi_i$, which is positive on the positive side; here $i = L$ at the left end of the positive component and $i = R$ at its right end. Choose a coordinate $y \in [0, 1]$, a function of r, equal to y_L near the left end and to $1 - y_R$ near the right end, by prescribing its derivative: the given derivatives on disjoint zones of the two ends and a positive constant in between, interpolated with cutoffs. Similarly interpolate the logarithms of the two conjugating factors h_i, taking the conjugating factor equal to one in the middle. The explicit formulas are (7.47) and (7.48); that all these reference quantities have coefficient bounds is Lemma 7.13.

The coordinate of a general state. For a general state the coordinates and conjugating factors of the adjacent negative components differ from the reference ones. Extend these *differences* into the positive component: the jets of the coordinates up to order four and the jets of the logarithms of the conjugating factors up to order three, by Taylor polynomials multiplied with cutoffs. The cutoffs are chosen so that the reference coordinate is unchanged on a fixed compact interval J_* of radii which contains the support of g_* in the normalization ℓ_* and the support of the metric control χ_*. Since the differential is close to the reference one, the resulting map $\Psi_{\mathbf{x}}$ is a diffeomorphism onto $[0, 1] \times \mathbb{T}$: it is proper, locally invertible, and of degree one on the boundary (rows 12–14 of Table 4).

The canonical operator. Let $h_{\mathbf{x}}$ be the conjugating factor in the original variables, $\Psi_{\mathbf{x}}$ the coordinate map and $J_{\mathbf{x}}$ its Jacobian determinant. For the equation $-\widehat{L}u = F_{\mathrm{ph}}$ on a positive component, the canonical operator and source are

$$P_{\mathbf{x}} U = -\Big(\frac{h_{\mathbf{x}}}{|J_{\mathbf{x}}|}\, \widehat{L}\big[h_{\mathbf{x}}\, (U \circ \Psi_{\mathbf{x}})\big]\Big) \circ \Psi_{\mathbf{x}}^{-1}, \qquad F = \Big(\frac{h_{\mathbf{x}} F_{\mathrm{ph}}}{|J_{\mathbf{x}}|}\Big) \circ \Psi_{\mathbf{x}}^{-1}. \tag{7.39}$$

These formulas again preserve the pairing. At the reference state

$$P_* = -\partial_y(a\,\partial_y) - c\,\partial_\varphi^2 + V, \qquad a = h_*^2\, w_*\, y_r, \quad c = \frac{h_*^2\, w_*'}{r\, y_r},$$

where in this formula a and c denote the coefficients of P_*. Near a root $a = 1$ and $c = y_i$ exactly: P_* is the continuation $-\partial_{y_i}^2 - y_i\,\partial_\varphi^2$ of the Tricomi operator. On the whole component, with

$$\mathfrak{y} = y\,(1-y),$$

we have

$$a \geq \mathcal{E}_m^{-C}, \qquad c \geq \mathcal{E}_m^{-C}\,\mathfrak{y}, \tag{7.40}$$

and all derivatives of the coefficients of fixed order are bounded by coefficient powers (Lemma 7.13); away from the roots this comes from the lower bounds for w_*'/w_* on the positive parts of the models and on the joins.

The difference vanishes to third order at the roots. Since the jets specified above match the principal densities of $P_{\mathbf{x}}$ and P_* to second order at both roots,

$$P_{\mathbf{x}} - P_* = -\partial_a\big(\mathfrak{y}^3\, e_{\mathbf{x}}^{ab}\,\partial_b\big) + \delta V_{\mathbf{x}}, \tag{7.41}$$

with smooth coefficients $e_{\mathbf{x}}^{ab}$, where a, b are indices running over y, φ, and a potential $\delta V_{\mathbf{x}}$ on which no boundary condition is imposed. Division by $\mathfrak{y}^3$ has a fixed cost: if F, F' and F'' vanish at 0, then

$$\frac{F(y)}{y^3} = \frac{1}{2}\int_0^1 (1-t)^2\, F'''(ty)\,dt,$$

each further derivative differentiates the integrand, and the factor $t^{-1/2}$ produced by rescaling in L^2 is integrable; this is row 16 of Table 4. The cubic vanishing is what will allow a second-order perturbation to be controlled without boundary conditions (Section 8.3).

7.8 The objects of a state and their bounds

Table 4 lists every object that Sections 7.4–7.7 attach to a state: its defining formula, the earlier rows from which it is built, and a number k. The number k counts derivatives: *the object, measured at order s, involves derivatives of the state of order at most $s + k$.* Each entry of the last column can be checked against the defining formula. An object built from earlier rows inherits their largest k, and the following operations increase it:

- a derivative of an earlier object: by 1;
- the divided difference Od at a root (Lemma 7.10): by 1;
- a Taylor polynomial of degree p at a boundary circle: by p;
- division by $\mathfrak{y}^3$: by 3.

Solving an ordinary differential equation, composing maps, inverting a map, and applying a smooth function do not increase k. These rules are stated with their hypotheses in Lemma 7.15.

Table 4: The objects attached to a state. A circle marks positive quantities whose reciprocals are bounded as well.

	Object	Defined by	Built from	k
On a negative component				
1	coefficients $a, b, \bar{c}$ of $\bar{L}$; speeds $c_\pm = d \pm v$; root factor c_0°	(3.6), Lemma 3.4, (7.24)	the state	2
2	flows of $c_\pm$; return map f	Section 7.5	row 1	2
3	chart h_R; return residual $\mathfrak{e}$	(7.25)	row 2, h_L	2
4	Δ; corrected speeds $\widehat{c}_\pm$; coefficients $\widehat{b}, \widehat{c}$; $\delta b[e]$, $\delta c[e]$	(7.31), Proposition 7.9	rows 2, 3; one derivative of a flow	3
5	travel coordinates U, V, σ, φ	(7.33)	row 4, h_L	3
6	root function $G(x, \zeta, \theta)$	(7.34)	rows 1, 3, h_L	2
7	root coordinates ξ_L, ξ_R	$\xi = x\,(\frac{3}{2}\,\mathsf{Od}\,G)^{2/3}$	row 6; Od	3
8	coordinate map $\Psi_i = (\xi_i, \varphi)$ and its inverse	Section 7.6	rows 5, 7	3
9	Jacobian $\lvert\det D\Psi_i\rvert^\circ$; density α_i°; conjugating factor h_i°; source factor $(h_i/\lvert\det D\Psi_i\rvert)^\circ$	(7.36) and the lines before it	rows 1, 5, 8; one derivative	4
10	boundary term $\partial_{\xi_i} \log h_i$	(7.38)	row 9; one derivative	5
11	potential V_i	(7.36)	row 9; two derivatives	6
On a positive component				
12	coordinate map $\Psi_{\mathbf{x}}$ and its inverse	Section 7.7	row 8; Taylor polynomials of degree 4	7
13	conjugating factor $h_{\mathbf{x}}^\circ$	Section 7.7	row 9; Taylor polynomials of degree 3	7
14	Jacobian $J_{\mathbf{x}}^\circ$; source factor $(h_{\mathbf{x}}/\lvert J_{\mathbf{x}}\rvert)^\circ$	(7.39)	rows 12, 13; one derivative	8
15	principal coefficients of $P_{\mathbf{x}}$	(7.39)	rows 1, 12–14	8
16	divided coefficients $e_{\mathbf{x}}^{ab}$	(7.41)	row 15; division by $\mathfrak{y}^3$	11
17	potential of $P_{\mathbf{x}}$	(7.39)	rows 1, 13, 14; two derivatives of $h_{\mathbf{x}}$	9

Proposition 7.11 (Uniform bounds for the objects of a state)**.** *On $\mathfrak{N}$ the following holds for every object $\mathfrak{o}$ of Table 4 on every component m, with k the number in its row. Let $\mathfrak{o}_*$ be the value of $\mathfrak{o}$ at the reference state.*

The norms C^s are taken in the variables of the object, after rescaling to unit size: on the component for the functions of (r, θ) in rows 1–5 and 12–14; on the part $\sigma_* \le \frac{3}{4}T$, respectively $\sigma_* \ge \frac{1}{4}T$, of the component for ξ_i and Ψ_i; on the fixed cylinders for the functions of (ξ_i, φ)

and of (y,φ); on the set $0 \le x \le 3\delta^{\mathrm{root}}$, $|\zeta| \le (3\delta^{\mathrm{root}})^{3/2}$ for row 6; and on the circle for f, h_R, $\mathfrak{e}$ and Δ. The speeds of rows 1 and 4, the flows of row 2 and the travel coordinates of row 5 are not smooth in r at the roots, where they are smooth functions of x, $\sqrt{x}$ and θ. For these objects $\|\cdot\|_{C^s}$ denotes the norm C^s in (r,θ) on the part of the component at distance at least $2\delta^{\mathrm{root}}$ from both roots, plus the supremum over the remaining radii of the norm C^s in θ. Near the roots they are used only through $d = b/a$, the root factor c_0 and rows 6–9, which are smooth up to the roots. Here x is the distance from a root and δ^{root} is the number (7.16) of the cell.

(i) (Reference.) *The object $\mathfrak{o}_*$ depends on r only, apart from the terms θ in U_*, V_*, φ_* and in the flows, and $\|\mathfrak{o}_*\|_{C^s} \le C_s\,\mathcal{E}_m^{C_s}$. For the quantities marked with a circle the reciprocal satisfies the same bound.*

(ii) (Differences.)

$$\|\mathfrak{o} - \mathfrak{o}_*\|_{C^s} \le C_s\,\mathcal{E}_m^{C_s}\,H_m^{\beta}\,\mathfrak{g}_{s+k+3,m}. \tag{7.42}$$

In particular the differentials of the coordinate maps, the Jacobians and the conjugating factors remain comparable to their reference values.

(iii) (Derivatives with respect to the state.) *For increments $\eta_1,\dots,\eta_j$ of the state,*

$$\begin{aligned}\big\|D^j\mathfrak{o}\,[\eta_1,\dots,\eta_j]\big\|_{C^s} \le C_{s,j}\,\mathcal{E}_m^{C_{s,j}}\,H_m^{\beta}\Big[\sum_i |\eta_i|_{s+k+3+j}\prod_{l\ne i}|\eta_l|_{b_0} \\ + (1+\mathfrak{n}_{s+k+3+j,m})\prod_l |\eta_l|_{b_0}\Big],\end{aligned} \tag{7.43}$$

where $|\eta|_s$ denotes the localized seminorm of order s of an increment on the neighbourhood of m. The loss grows with the order j of the derivative, as (7.7) allows; the neighbourhood and the low order b_0 do not depend on j.

In particular (7.42) holds for all objects with $\mathfrak{g}_{s+d_{\mathrm{geo}},m}$ on the right, where

$$d_{\mathrm{geo}} = 14.$$

The Sobolev embedding on a cell costs two derivatives; the constant 3 in $s+k+3$ leaves one to spare. The proof occupies the rest of this subsection. The reference values are treated first (Lemmas 7.12–7.14); they involve functions of r only. Lemma 7.15 then states the rules of the count, and the rows of the table are taken in order, each with its formula, the rules applied to it, and the hypotheses of these rules.

Lemma 7.12 (Root charts of the reference profile)**.** *On each half of the negative component of cell n, the root coordinate of the reference profile is the function of r*

$$\xi = x\Big[\tfrac{3}{2}\int_0^1 \sqrt{t}\;c_i(xt)\,dt\Big]^{2/3}, \tag{7.44}$$

where x is the distance from the root of that half, and

$$\mathcal{E}_n^{-C} \le |\xi_r| \le \mathcal{E}_n^{C}, \qquad \|\xi\|_{C^s} + \|r(\xi)\|_{C^s} \le C_s\,\mathcal{E}_n^{C_s}. \tag{7.45}$$

Moreover

$$\alpha_i = w_*\,|\xi_r|, \qquad h_i = \alpha_i^{-1/2}, \tag{7.46}$$

and the functions $\alpha_i^{\pm1}$, $h_i^{\pm1}$, $|\xi_r|^{\pm1}$, the potential $V_i = -\alpha_i^{-1/2}\partial_\xi^2\alpha_i^{1/2}$, the source factors $h_i/|\xi_r|$ and $|\xi_r|/h_i$, and $\partial_\xi \log h_i$ have derivatives of order s bounded by $C_s\mathcal{E}_n^{C_s}$. These bounds persist when the half is rescaled to unit length.

Proof. At the reference state the charts are the identity, so $\varphi = \theta$ and, on each half, $\xi = (\frac{3}{2}\sigma_i)^{2/3}$, where $\sigma_i = \int v_0\,dr$ is the travel angle from the root of that half.

Step 1: near the root. Since $v_0 = \sqrt{x}\,c_i(x)$, the substitution $x' = xt$ gives $\sigma_i = x^{3/2}\int_0^1 \sqrt{t}\,c_i(xt)\,dt$, which is (7.44). The bracket is smooth and positive, and equals $c_i(0)$ at $x = 0$; so $|\xi_r| = c_i(0)^{2/3}$ at the root. On the zone $|x| < 8\delta_n^{\mathrm{root}}$ the bracket lies between $\frac{1}{2}c_i(0)$ and $2c_i(0)$, and by (7.15) its logarithmic derivative times $|x|$ is at most $C\epsilon_{\mathrm{root}}A_n^{15}$. Hence $|\xi_r|$ is comparable to $c_i(0)^{2/3}$ on the zone, and the derivatives of ξ have coefficient bounds there.

Step 2: away from the zones. Differentiating $\xi^{3/2} = \frac{3}{2}\sigma_i$ gives $|\xi_r| = v_0/\sqrt{\xi}$, so we need upper and lower bounds for v_0 and ξ. In the model coordinate the zone of the left root ends at a distance comparable to A_n^{19} from X_L, and that of the right root at a distance comparable to A_n^{15} from X_R, because r_X is comparable to $\mathfrak{r}_n A_n$ and $\mathfrak{r}_n A_n^5$ there. The function ρ has one maximum on the component and vanishes only at its two ends, where $|\rho_X|$ is comparable to A_n and A_n^9. Hence $\rho \geq cA_n^{24}$ outside the zones, and

$$c\,\mathfrak{r}_n^{-1}A_n^{11} \leq v_0 = \frac{\sqrt{\rho}}{r_X} \leq C\,\mathfrak{r}_n^{-1}A_n^{-9/2}.$$

By Step 1, $\xi \geq cA_n^{20}$ at the end of its zone, hence beyond it, and $\xi \leq \xi_c = (3T_n/4)^{2/3} \asymp A_n^{1/3}$. So $|\xi_r|$ is bounded above and below by powers of A_n. The higher derivatives follow by differentiating $\xi_r = \pm v_0\,\xi^{-1/2}$, using the lower bounds for r_X, ρ and ξ on this region.

Step 3: the inverse. From $r(\xi(r)) = r$ we get $r_\xi = 1/\xi_r$, and every higher derivative of $r(\xi)$ is a polynomial in lower ones and in derivatives of ξ, divided by ξ_r. This proves (7.45). The length ξ_c of a half and its reciprocal are powers of A_n, so rescaling to unit length costs a fixed power of A_n for each derivative.

Step 4: the factors. For the reference operator $a = w_*$, $\widehat{b} = 0$, $\delta = w_* v_0$, $U_\theta = V_\theta = 1$ and $|\det D\Psi_i| = |\xi_r|$. Hence $\alpha_i = \delta/\sqrt{\xi} = w_*|\xi_r|$, which is (7.46). By (5.11) and (7.45) the functions $\alpha_i^{\pm 1}$ have coefficient bounds, and the remaining quantities are obtained from α_i and ξ_r by products, powers and differentiation in ξ. All of them are functions of r, so no angular frequency is involved. □

Lemma 7.13 (The reference coordinate on a positive component)**.** *Let (r_1, r_2) be a positive component with index n, let y_L, y_R be the continued root coordinates at its two ends, and let η_L, η_R be cutoffs with disjoint supports in the zones of the two ends, equal to one on smaller zones. Define*

$$\begin{aligned} y_r &= \eta_L\,(y_L)_r - \eta_R\,(y_R)_r + \mu_0\,(1 - \eta_L - \eta_R), \qquad y(r) = \int_{r_1}^{r} y_r\,dr, \\ \mu_0 &= \frac{1 - \int \eta_L\,(y_L)_r\,dr + \int \eta_R\,(y_R)_r\,dr}{\int (1 - \eta_L - \eta_R)\,dr}, \end{aligned} \tag{7.47}$$

and

$$\log h_* = \eta_L \log h_L + \eta_R \log h_R, \qquad h_i = \left(w_*\,|(y_i)_r|\right)^{-1/2}. \tag{7.48}$$

Then y is a diffeomorphism of $[r_1, r_2]$ onto $[0, 1]$ with $y = y_L$ near r_1 and $y = 1 - y_R$ near r_2, and $y_r^{\pm 1}$, $h_*^{\pm 1}$, the inverse function $r(y)$ and the source factors h_*/y_r, y_r/h_* have derivatives of order s bounded by $C_s\mathcal{E}_n^{C_s}$. The coefficients of the reference operator $P_* = -\partial_y(a\,\partial_y) - c\,\partial_\varphi^2 + V$ are

$$a = h_*^2\,w_*\,y_r, \qquad c = \frac{h_*^2\,w_*'}{r\,y_r}, \qquad V = -\frac{h_*}{y_r}\left(w_*\,h_{*,r}\right)_r. \tag{7.49}$$

Near the ends $a = 1$ and $c = y_i$; on the whole component (7.40) holds, and the derivatives of order s of a, c, V and a^{-1} are bounded by $C_s \mathcal{E}_n^{C_s}$. On the component which contains the interval J_*, the zones can be chosen disjoint from J_*.

Proof. Step 1: the zones. The continued coordinates $y_i = -\xi_i$ have positive inward derivatives with the bounds of Lemma 7.12, since (7.15) holds on both sides of the root. The increment of y_i over its zone is at most $8\delta^{\text{root}} \sup|\xi_r| = O(A_n^{18})$. So the two zones together account for less than $\frac{1}{4}$ of the total increment 1 on all but finitely many components, and on those as well if ϵ_{root} is small. This is the last of the finitely many smallness conditions on the constant ϵ_{root} of Lemma 7.5; like the others it involves the reference profile only. The interval J_* has positive distance from the roots.

Step 2: the coordinate. Note that $(y_R)_r < 0$, since y_R increases inwards from the right end. The numerator of μ_0 lies between $\frac{3}{4}$ and 1, and its denominator is comparable to $r_2 - r_1$; so μ_0 and μ_0^{-1} are coefficient powers. The function y_r is a convex combination of positive functions with coefficient bounds above and below, and its integral over the component is 1 by the choice of μ_0. Hence $y = y_L$ near r_1 and $y = 1 - y_R$ near r_2, and the inverse function is estimated as in Step 3 of the preceding proof.

Step 3: the conjugating factor. The logarithms $\log h_i$ are bounded by $C(n^{5/2} + n^2)$ by (5.11), and their derivatives of positive order have coefficient bounds. Differentiating (7.48) and exponentiating gives the bounds for $h_*^{\pm 1}$.

Step 4: the coefficients. The formulas (7.49) follow by inserting $u = h_* U(y(r)) \cos(j\theta)$ into $-(h_*/y_r)\, \mathscr{L}_{w_*} u$; the terms with U_y which do not belong to $(aU_y)_y$ cancel. In a zone $h_* = (w_* |(y_i)_r|)^{-1/2}$ gives $a = 1$. Differentiating (7.44) gives $\xi\, \xi_r^2 = x\, c_i(x)^2 = -w_*'/(r w_*)$ on both sides of the root, so $w_*'/(r w_*) = y_i\, (y_i)_r^2$ and $c = y_i$. The upper bounds and the lower bound for a follow from Steps 2 and 3.

Step 5: the lower bound for c. It suffices to show

$$w_*'(r) \geq \mathcal{E}_n^{-C} \min(r - r_1,\, r_2 - r), \tag{7.50}$$

because $y(1-y) \leq \mathcal{E}_n^C \min(r - r_1, r_2 - r)$ by the bounds for y_r. On a short positive ramp $-\rho \geq c A_n \min(|X - X_L|, 1)$, and on a long positive ramp $-\rho \geq c A_n^9 \min(|X - X_R|, 1)$; since $w_*' = -\lambda_n \rho\, r / r_X$, the bounds of Lemma 5.1 give (7.50) on the ramps. On a join $w_*'/w_* \geq 2/r$ by (5.9), and on the connectors of Lemma 6.1 $w_*'/w_* \geq 3/(4r)$; with the lower bound for w_* this gives (7.50) there, since the distance to the ends is bounded. □

Lemma 7.14 (Conversion of weights and derivatives)**.** *On cell n and its neighbours, the continuous weights H, E_0, M and the discrete weights H_n, $\mathcal{E}_n$, M_n differ by factors bounded by $\mathcal{E}_n^C$. Changes between the variables r, ϱ, the root coordinates and the coordinates y transfer derivatives of total order s to derivatives of total order at most s, with coefficients bounded by $E_0^{C_s}$; so do the source factors and cutoffs. A restriction to a circle costs one radial derivative.*

Proof. By (5.10), $x_\partial = n + O(1)$ on cell n, so

$$\log \frac{H}{H_n} = O(n^2), \qquad \log \frac{E_0}{\mathcal{E}_n} = O(n^{3/2} \log(e+n)), \qquad \log \frac{M}{M_n} = O(n^{3/2} \log^2(e+n)),$$

and the same bounds hold for the ratios of the weights of neighbouring cells; all of them are $O(\log \mathcal{E}_n)$. In the variable ϱ,

$$|\mathsf{D} \log M| \leq C\, (1 + x_\partial)^{-1/2} \log^2(e + x_\partial),$$

which is bounded together with its derivatives, and the same is true of $\log H$, $\log E_0$ and $\log \ell$; so the weights are uniformly comparable on unit intervals of ϱ, and $\ell^{\pm 1}$ is bounded by a power of E_0. The chain rule expresses derivatives of total order s in one set of variables through derivatives of order at most s in another, with coefficients which are polynomials in the derivatives of the coordinate functions and in reciprocals of their first derivatives; these are bounded by Lemmas 7.12 and 7.13. The inequality $E_0^{C_s} \le C_s M$ is applied to the complete product. The trace inequality on an interval of unit length costs one derivative, and the rescaling to unit length costs coefficient powers. □

Lemma 7.15 (The rules of the count)**.** *Let Ω and Ω' be products of finitely many unit intervals and circles, with the norms C^s of functions which are smooth up to the boundary; for a circle factor, maps and segments are understood in the lift. In each statement C_s denotes a polynomial in the quantities named there, whose coefficients depend only on s and on the domains.*

(R1) (Products and functions.) *If $|\alpha_1| + \cdots + |\alpha_p| \le s$, then*

$$\sup \Big| \prod_{i=1}^{p} \partial^{\alpha_i} v_i \Big| \le C_s \sum_{i=1}^{p} \|v_i\|_{C^s} \prod_{l \neq i} \|v_l\|_{C^0}.$$

Let v and δ be functions on Ω with values in $\mathbb{R}^p$, and let Φ be smooth on a neighbourhood of the set I of the values of $v + t\delta$, $0 \le t \le 1$. Then

$$\|\Phi(v+\delta) - \Phi(v)\|_{C^s} \le C_s \, \|\delta\|_{C^s},$$

where C_s is a polynomial in $\|v\|_{C^s}$, $\|\delta\|_{C^0}$ and the suprema over I of the derivatives of Φ of orders $1, \dots, s+1$.

(R2) (Derivatives.) *$\|\partial v\|_{C^s} \le \|v\|_{C^{s+1}}$ for a derivative ∂ of the first order.*

(R3) (Composition and inversion.) *Let $h, h_* \colon \Omega' \to \Omega$ be smooth maps and u, u_* functions on Ω. For $s \ge 1$,*

$$\begin{aligned} \|u \circ h\|_{C^s} &\le C_s(\|u\|_{C^s} + \|u\|_{C^1} \, \|h\|_{C^s}), \\ \|u \circ h - u_* \circ h_*\|_{C^s} &\le C'_s(\|u - u_*\|_{C^s} + \|h - h_*\|_{C^s}), \end{aligned}$$

where C_s is a polynomial in $\|h\|_{C^1}$ and C'_s is a polynomial in $\|u_*\|_{C^{s+1}}$, $\|h_*\|_{C^s}$, $\|u - u_*\|_{C^1}$ and $\|h - h_*\|_{C^1}$. Let now h and h_* be diffeomorphisms of Ω' onto sets which contain Ω, and let h_*^{-1} be defined and smooth on a product Ω_* of intervals and circles which contains both images. Then, on Ω,

$$\|h^{-1} - h_*^{-1}\|_{C^s} \le C''_s \, \|h - h_*\|_{C^s},$$

where C''_s is a polynomial in $\|h_*\|_{C^{s+1}}$, in the norm C^{s+1} of h_*^{-1} on Ω_*, in $\|h - h_*\|_{C^2}$ and in $\sup |(Dh)^{-1}|$.

(R4) (Flows.) *Let P be a product of unit intervals, or a point. Let $G(t, p, \theta)$ and $G_*(t, p)$ be continuous for $0 \le t \le 1$, $p \in P$, $\theta \in \mathbb{T}$, and smooth in (p, θ), and let Θ and Θ_* be the solutions of $\partial_t \Theta = G(t, p, \Theta)$ and $\partial_t \Theta_* = G_*(t, p)$ with the initial value θ. Put $|F|_s = \sup_t \|F(t, \cdot)\|_{C^s(P \times \mathbb{T})}$. Then*

$$\sup_t \|\Theta(t, \cdot) - \Theta_*(t, \cdot)\|_{C^s} \le C_s \, |G - G_*|_s,$$

where C_s is a polynomial in $|G_*|_s$, $|G - G_*|_1$ and $\exp |\partial_\theta G|_0$.

(R5) (Substitution at a root.) $\|\mathsf{Ev}\,G\|_{C^s} \le C_s\|G\|_{C^s}$ *and* $\|\mathsf{Od}\,G\|_{C^s} \le C_s\|G\|_{C^{s+1}}$ (Lemma 7.10).

(R6) (Taylor polynomials.) *For a function* $F(x,\theta)$ *on* $[0,1]\times\mathbb{T}$ *let* $T_pF(x,\theta) = \sum_{i\le p}\partial_x^i F(0,\theta)\,x^i/i!$, *considered for* $|x|\le 1$. *Then* $\|T_pF\|_{C^s} \le C_s\|F\|_{C^{s+p}}$.

(R7) (Division.) *If* F, F_y *and* F_{yy} *vanish at* $y=0$, *then* $\|F/y^3\|_{C^s} \le C_s\|F\|_{C^{s+3}}$.

Proof. We use the interpolation inequalities $\|v\|_{C^i} \le C_s\|v\|_{C^0}^{1-i/s}\|v\|_{C^s}^{i/s}$, $0\le i\le s$, which hold on such domains [Ham82, Part II, Section 2.2], [Hör76, Appendix A].

(R1) Let $a_i = \|v_i\|_{C^0} \le b_i = \|v_i\|_{C^s}$ and $\vartheta_i = |\alpha_i|/s$, so that $\sum\vartheta_i \le 1$. By interpolation the product is at most

$$C\prod_i a_i^{1-\vartheta_i}b_i^{\vartheta_i} = C\prod_l a_l\prod_i\Big(\frac{b_i}{a_i}\Big)^{\vartheta_i} \le C\prod_l a_l\,\max_i\frac{b_i}{a_i} \le C\sum_i b_i\prod_{l\ne i}a_l.$$

For the second statement write $\Phi(v+\delta)-\Phi(v) = \sum_i \delta_i\,\Phi_i$ with $\Phi_i = \int_0^1 \partial_i\Phi(v+t\delta)\,dt$. A derivative of order between 1 and s of $\partial_i\Phi(w)$, $w = v+t\delta$, is a sum of terms $(\partial^\gamma\partial_i\Phi)(w)\prod_l\partial^{\alpha_l}w$ with $|\gamma|$ factors, whose orders add up to at most s. By the first statement $\|\Phi_i\|_{C^s} \le C(1+\|v\|_{C^s}+\|\delta\|_{C^s})$, with C a polynomial in the sup norms. The first statement, applied to $\delta_i\Phi_i$, gives the claim; note that $\|\delta\|_{C^0}\|\delta\|_{C^s}$ is of the required form.

(R3) A derivative of order j, $1\le j\le s$, of $u\circ h$ is a sum of terms $(\partial^\beta u)\circ h\,\prod_{i=1}^{|\beta|}\partial^{\gamma_i}h$ with $|\gamma_i|\ge 1$ and $\sum|\gamma_i| = j$. Such a term is bounded by $\|Du\|_{C^{|\beta|-1}}\prod\|Dh\|_{C^{|\gamma_i|-1}}$, and $(|\beta|-1)+\sum(|\gamma_i|-1) = j-1\le s-1$. Interpolating Du and Dh between C^0 and C^{s-1}, as in (R1), gives the first estimate. For the second,

$$u\circ h - u_*\circ h_* = (u-u_*)\circ h + \Big[\int_0^1 Du_*(h_*+t(h-h_*))\,dt\Big](h-h_*).$$

Apply the first estimate to the first term and to the integrand, and (R1) to the product. No derivative of $u-u_*$ of order higher than s occurs, because the additional derivative falls on u_*.

For the inverses let $\iota = h^{-1}$ and $\iota_* = h_*^{-1}$. Since $D\iota = (Dh)^{-1}\circ\iota$, the first estimate and induction on s give $\|\iota\|_{C^s} \le C(1+\|h\|_{C^s})$, with C a polynomial in $\|h\|_{C^2}$ and $\sup|(Dh)^{-1}|$. For $y\in\Omega$ and $x = \iota(y)$ the segment from $h_*(x)$ to $h(x) = y$ lies in Ω_*, which is convex in the lift. Therefore $\iota_*(y)-\iota(y) = \iota_*(h(x))-\iota_*(h_*(x))$, that is,

$$\iota-\iota_* = -\Big(\Big[\int_0^1 D\iota_*(h_*+t(h-h_*))\,dt\Big](h-h_*)\Big)\circ\iota,$$

where the bracket is a function on Ω'. The first estimate, (R1) and the bound for ι give the claim.

(R4) Let $\delta G = G-G_*$ and $\delta\Theta = \Theta-\Theta_*$, and let $\Xi_t(p,\theta) = (p,\Theta(t,p,\theta))$. Since G_* does not depend on θ, a derivative ∂^α in (p,θ) of order $s\ge 1$ satisfies

$$\partial_t\,\partial^\alpha\delta\Theta = (\partial_\theta\delta G)\circ\Xi_t\;\partial^\alpha\delta\Theta + (\partial_\theta\delta G)\circ\Xi_t\;\partial^\alpha\Theta_* + \sum(\partial_p^a\partial_\theta^b\delta G)\circ\Xi_t\prod_{i=1}^b\partial^{\gamma_i}\Theta,$$

where in the sum $|a|+\sum|\gamma_i| = s$ and $1\le|\gamma_i|<s$. The first term gives the integrating factor $\exp\int(\partial_\theta\delta G)\circ\Xi_t\,dt$, which is bounded by $\exp|\partial_\theta G|_0$. In the remaining terms write $\Theta = \Theta_*+\delta\Theta$. The derivatives of Θ_* are bounded by $1+|G_*|_s$, and by induction $\|\delta\Theta\|_{C^i} \le C|\delta G|_i$ for $i<s$. A product of $|\delta G|_{|a|+b}$ with factors $|\delta G|_{|\gamma_i|}$, i in a set S, has orders $n_0,\dots,n_{|S|}\ge 1$ with $\sum(n_l-1)\le s-1$; interpolation between $|\delta G|_1$ and $|\delta G|_s$ bounds it by $C\,|\delta G|_1^{|S|}\,|\delta G|_s$. Integration in t gives the claim; the case $s=0$ follows from $\partial_t\delta\Theta = \delta G\circ\Xi_t$.

(R6) is evident, and (R7) follows from $F(y)/y^3 = \frac12\int_0^1(1-t)^2F'''(ty)\,dt$. □

Proof of Proposition 7.11. Part (i). At the reference state $a = w_*$, $b = 0$, $\bar c = w_*'/r$, $c_\pm = \pm v_0$ and $c_0 = c_i$; the flows are translations of the angle by $\pm \int v_0\, dr$, the return map is R_{-T}, the charts are the identity and $\mathfrak{e} = 0$, $\Delta = 0$. So rows 1–6 are explicit, and their bounds are those of $w_*^{\pm 1}$ (Lemma 5.2), of v_0 and of c_i (Lemma 7.5); for $x \geq 2\delta^{\mathrm{root}}$ the function $v_0 = \sqrt{x}\, c_i(x)$ and its reciprocal have coefficient bounds. Rows 7–11 are Lemma 7.12, and rows 12–17 are Lemma 7.13.

Part (ii): the scheme. Fix the component m and rescale as in the statement. For an object $\mathfrak{o}$ write $\delta\mathfrak{o} = \mathfrak{o} - \mathfrak{o}_*$. We prove, for the rows in their order and for all $s \geq 1$,

$$\|\delta\mathfrak{o}\|_{C^s} \leq C_s\, \mathcal{E}_m^{C_s}\, H_m^\beta\, \mathfrak{g}_{s+k+2,m}. \tag{7.51}$$

This gives (7.42) for $s \geq 1$, and for $s = 0$ because $\|\cdot\|_{C^0} \leq \|\cdot\|_{C^1}$. Three remarks apply to every row.

(a) Entry. The state enters the formulas through f, f', the derivatives of $Z = qu$ of orders one and two, and the chart h_L. By the Sobolev embedding on unit intervals of ϱ, as in (7.21), and by Lemma 7.14,

$$\sum_{1\leq a+b\leq 2} \|\partial_r^a \partial_\theta^b Z\|_{C^s} + \|f\|_{C^{s+1}} \leq C_s\, \mathcal{E}_m^{C_s}\, H_m^\beta\, \mathfrak{g}_{s+4,m}, \qquad \|h_L - \mathrm{id}\|_{C^s} \leq C_s\, H_m^\beta\, \mathfrak{g}_{s+1,m}. \tag{7.52}$$

The embedding costs two derivatives on the component and one on a circle.

(b) Low orders. When a row is treated, (7.51) is known for the earlier rows. Let $b_0 \geq 15$; then for $s \leq 2$ the right side of (7.51) is at most $C\epsilon_0 H_m^\beta \mathcal{E}_m^C$ for every row. By (7.22) this quantity, multiplied by any fixed coefficient power, is as small as we wish on all components at once if ϵ_0 is small. Requirement (N4) consists of $b_0 \geq 15$ and of finitely many conditions of this kind:

- the quantities marked with a circle differ from their reference values by at most half of these;
- the differentials of the coordinate maps satisfy (7.54) below;
- the low norms of differences on which the constants of Lemma 7.15 depend are at most 1, and $|\sigma - \sigma_*| \leq T/8$.

(c) Constants. In every application of Lemma 7.15 below, the reference objects have norms bounded by coefficient powers, by part (i); the functions Φ are polynomials, exponentials of small quantities, the functions of (7.13) on the ranges of Lemma 7.7, and powers of quantities marked with a circle, on intervals where their derivatives are bounded by coefficient powers; the low norms of the differences are bounded by (b); and $\exp|\partial_\theta G|_0$ is bounded, because the speeds of the reference state do not depend on θ and the angular derivatives of the differences are small by (b). So each constant of the lemma is at most $C_s\mathcal{E}_m^{C_s}$, and each rule takes the form

$$\|\delta(\text{output})\|_{C^s} \leq C_s\, \mathcal{E}_m^{C_s} \sum \|\delta(\text{input})\|_{C^{s+l}}, \tag{7.53}$$

with $l = 0$ for (R1), (R3), (R4) and for Ev in (R5), $l = 1$ for (R2) and for Od in (R5), $l = p$ for (R6) and $l = 3$ for (R7). Hence the number k of an object is the largest of the numbers $k + l$ of its inputs; this is the rule stated before Table 4. For each row we now give the formula, the rules, and the hypotheses that have to be checked.

Row 1 ($k = 2$). By (7.13) and the lines after it, each of a, b, $\bar c$, d, v/v_0 and c_0/c_i is a reference function times a smooth function of reference functions and of f, f' and the derivatives of Z of orders one and two. Rule (R1) and (7.52) give (7.51) with $k = 2$. For the speeds $c_\pm = d \pm v$ the factor v_0 has coefficient bounds on $x \geq 2\delta^{\mathrm{root}}$, and is bounded on the rest of the component, where only derivatives in θ are taken.

Row 2 ($k = 2$). Rescale r to the unit interval. The flow Φ_r of c_- from the left boundary solves $\partial_r \Phi_r = c_-(r, \Phi_r)$, and $c_{-,*} = -v_0(r)$ does not depend on θ. Rule (R4), with r as the time and without parameters, bounds $\sup_r \|\Phi_r - \Phi_{r,*}\|_{C^s_\theta}$ by $\sup_r \|\delta c_-(r,\cdot)\|_{C^s_\theta}$, which is the norm of row 1. On $x \geq 2\delta^{\mathrm{root}}$ the radial derivatives follow from $\partial_r(\Phi_r - \Phi_{r,*}) = (\delta c_-)(r, \Phi_r)$ by induction on their number, with (R3) for the composition. The flow of c_+ is the reflection of that of c_-, and f is the value of Φ_r at the right boundary.

Row 3 ($k = 2$). By (7.25), $\widetilde{h}_R = R_{-T} \circ h_L \circ f^{-1}$, $h_R = \mathsf{O}\,\widetilde{h}_R$ and $\mathfrak{e} = (\mathsf{E}\,\widetilde{h}_R) \circ h_R^{-1}$, with the reference values $f_* = R_{-T}$, $\widetilde{h}_{R,*} = h_{R,*} = \mathrm{id}$. Rule (R3) on the circle applies to the two inverses and the two compositions; the parity projections are linear and bounded. The inputs are f ($k = 2$) and h_L, for which (7.52) is used.

Row 4 ($k = 3$). First $\Delta = \mathsf{g}^{-1} \circ (\mathrm{id} + \mathfrak{e}) \circ \mathsf{g} - \mathrm{id}$ with $\mathsf{g} = h_R \circ f$; (R3) gives $k = 2$. By (7.31),

$$\widehat{c}_- - c_- = (\Phi_r' \circ \Phi_r^{-1})\; \chi'\, \Delta \circ J_r^{-1} \circ \Phi_r^{-1}, \qquad J_r = \mathrm{id} + \chi\, \Delta,$$

which is supported in $3\delta^{\mathrm{root}} < x < 4\delta^{\mathrm{root}}$ at the left root, where the flows are smooth in (r, θ). The factor Φ_r' is a derivative of row 2, so (R2) gives $k = 3$ for it; the maps $(r, \theta) \mapsto (r, \Phi_r\theta)$ and $(r, \theta) \mapsto (r, J_r\theta)$ are inverted and composed by (R3), and the product is formed by (R1). The coefficients $\widehat{b} = a\,\widehat{d}$ and $\widehat{c} = a\,(\widehat{d}^{\,2} - \widehat{v}^{\,2})$ are polynomials in these and in row 1; near the roots they are those of row 1. For $\delta b[e]$ and $\delta c[e]$, which are linear in e, the statement refers to the factors (1)–(3) and to the map h_r in Step 4 of the proof of Proposition 7.9.

Row 5 ($k = 3$). By Step 1 of the same proof the flow of $\widehat{c}_-$ is $\Phi_r \circ J_r$, so $S_-(r,\cdot) = J_r^{-1} \circ \Phi_r^{-1}$, and S_+ is its reflection. Rule (R3) gives the bounds for $S_\pm$, and then for $U = h_L \circ S_+$ and $V = h_L \circ S_-$; the functions σ and φ are linear in U and V. The inputs are rows 2 and 4 and h_L, so $k \leq 3$. The norm is that of row 2.

Row 6 ($k = 2$). On $0 \leq x \leq 3\delta^{\mathrm{root}}$ the operators $\widehat{L}$ and $\bar{L}$ coincide, so the functions d and c_0 in (7.34) are those of row 1. After the rescaling of x and ζ to unit size, which multiplies d and c_0 by coefficient powers, (7.34) is a flow of the kind considered in (R4), with the parameters $p = (x, \zeta)$ and the speed $2xt\, d(xt^2, \theta) + 2\zeta t^2\, c_0(xt^2, \theta)$. The speed of the reference state is $2\zeta t^2 c_i(xt^2)$, which does not depend on θ. Since $0 \leq t \leq 1$, the chain rule bounds the difference of the two speeds, in the norm $|\cdot|_s$ of (R4), by $C_s(\|\delta d\|_{C^s} + \|\delta c_0\|_{C^s})$. Rule (R4) bounds the time-one map, (R3) its inverse with respect to θ, with (x, ζ) carried along, and (R3) again the composition with the boundary chart h_L, or h_R of row 3.

Row 7 ($k = 3$). On $x \leq 3\delta^{\mathrm{root}}$, $\xi_i = x\,(\frac{3}{2}\chi_0)^{2/3}$ with $\chi_0 = \mathsf{Od}\,G$. The operation Od is linear, so (R5) gives $\|\delta\chi_0\|_{C^s} \leq C_s\|\delta G\|_{C^{s+1}}$, which is $k = 3$. The reference value $\chi_{0,*} = \int_0^1 \sqrt{t}\, c_i(xt)\, dt$ lies between $\frac{1}{3}c_i(0)$ and $\frac{4}{3}c_i(0)$, and by (b) $|\delta\chi_0| \leq \frac{1}{2}\chi_{0,*}$; so (R1) applies to the power $\frac{2}{3}$. On the rest of the half, $\xi_L = (\frac{3}{2}\sigma)^{2/3}$ and $\xi_R = (\frac{3}{2}(T - \sigma))^{2/3}$ with σ of row 5, and σ_*, respectively $T - \sigma_*$, is bounded below by a coefficient power there; (R1) applies again.

Row 8 ($k = 3$). The second component of Ψ_i is $\varphi = \mathsf{Ev}\,G$ on $x \leq 3\delta^{\mathrm{root}}$, where (R5) gives $k = 2$, and $\varphi = (U + V)/2$ elsewhere, by row 5. For the inverse we use (R3). Its hypotheses are checked as follows. Consider Ψ_L and $\Psi_{L,*}$ on the fixed set $\Omega' = \{\sigma_* \leq \frac{3}{4}T\}$. Since $|\sigma - \sigma_*| \leq T/8$, both images contain the cylinder $[0, \xi_c] \times \mathbb{T}$ of the half and are contained in the image Ω_* of $\{\sigma_* \leq \frac{7}{8}T\}$ under $\Psi_{L,*}$, a cylinder on which $\Psi_{L,*}^{-1}$ has coefficient bounds by (7.45). Both maps are diffeomorphisms onto their images: (σ, φ) is a diffeomorphism of the open component (Section 7.6), ξ_L is an increasing function of σ, and by row 7 the map is smooth up to the root. Finally, by the case $s = 1$ of rows 5 and 7 and by (b),

$$\big\|(D\Psi_i - D\Psi_{i,*})\,(D\Psi_{i,*})^{-1}\big\|_{C^0} \leq C\,\epsilon_0\, H_m^\beta\, \mathcal{E}_m^C < \tfrac{1}{4}, \tag{7.54}$$

so $(D\Psi_i)^{-1}$ is bounded by a coefficient power. The right half is treated in the same way on $\{\sigma_* \geq \frac{1}{4}T\}$.

Row 9 ($k = 4$). The Jacobian $|\det D\Psi_i|$ is a polynomial in first derivatives of row 8: (R2) and (R1). The density is

$$\alpha_i = \frac{a\,|\det D\Psi_i|}{U_\theta V_\theta} \circ \Psi_i^{-1}$$

(Section 7.6). On $x \geq 2\delta^{\mathrm{root}}$ the product $U_\theta V_\theta$ consists of derivatives of row 5. On $x \leq 3\delta^{\mathrm{root}}$ it is Ev of the function $G_\theta(x,\zeta,\theta)\,G_\theta(x,-\zeta,\theta)$, which is even in ζ; so it is smooth up to the root, and (R2), (R1) and (R5) give $k = 3$ for it. It is bounded below, because the angular derivatives of the flows and of the charts lie in fixed compact subintervals of $(0,\infty)$. Rule (R1) for the quotient and (R3) for the composition with Ψ_i^{-1} give $k = 4$ for α_i. The Jacobian of Ψ_i does not vanish at the root, unlike that of the coordinates (σ,φ); this is why this formula is used instead of $\alpha_i = \delta/\sqrt{\xi_i}$. The conjugating factor $h_i = \alpha_i^{-1/2}$ and the source factor are powers and quotients of quantities marked with a circle: (R1).

Rows 10 and 11 ($k = 5,\ 6$). These are $\partial_{\xi_i} \log h_i$ and $V_i = -\alpha_i^{-1/2}(\partial_{\xi_i}^2 - \xi_i\partial_\varphi^2)\alpha_i^{1/2}$: one and two applications of (R2) to row 9, and (R1).

Row 12 ($k = 7$). Let x be the distance from the root i at an end of the positive component, and for a function F on the adjacent negative component which is smooth up to the root let T_pF be its Taylor polynomial of degree p in the radial variable at the root, as in (R6), continued to the positive side. By the construction of Section 7.7,

$$\Psi_{\mathbf{x}} - \Psi_* = \sum_{i\in\{L,R\}} \eta_i\,(\mp T_4\,\delta\xi_i,\ T_4\,\delta\varphi),$$

with cutoffs η_i supported in the zones of the two ends and the signs of (7.47). Rule (R6) with $p = 4$, applied to row 8, gives $k = 3 + 4 = 7$; the cutoffs are reference functions. The differential of $\Psi_{\mathbf{x}}$ satisfies (7.54), by the case $s = 1$ of what has just been proved. Hence $\Psi_{\mathbf{x}}$ is a local diffeomorphism of the closed component into the cylinder $[0,1]\times\mathbb{T}$. It maps each boundary circle to a boundary circle by the chart of that circle, because $\xi_i = 0$ at the root for every state. A proper local diffeomorphism between cylinders which is a diffeomorphism of the boundaries is a diffeomorphism. Its inverse is estimated by (R3), with $\Omega_* = \Omega = [0,1]\times\mathbb{T}$.

Row 13 ($k = 7$). Similarly

$$\log h_{\mathbf{x}} - \log h_* = \sum_i \eta_i\ T_3\,\delta \log(h_i \circ \Psi_i),$$

where $h_i \circ \Psi_i$ is the conjugating factor of row 9 as a function of (r,θ). Rules (R3) and (R1) give $k = 4$ for $\log(h_i \circ \Psi_i)$, (R6) with $p = 3$ gives $k = 7$, and (R1) is used for the exponential.

Row 14 ($k = 8$). The Jacobian $J_{\mathbf{x}}$ is a polynomial in first derivatives of row 12: (R2), (R1). It is bounded below by (7.54). The source factor is a quotient.

Row 15 ($k = 8$). The principal coefficients are the entries of

$$\Big(\frac{h_{\mathbf{x}}^2}{|J_{\mathbf{x}}|}\,D\Psi_{\mathbf{x}}\,A\,D\Psi_{\mathbf{x}}^T\Big) \circ \Psi_{\mathbf{x}}^{-1},$$

where A is the matrix of the coefficients $a, b, \bar{c}$ of row 1. The inputs have $k = 7,\ 8,\ 8,\ 2$ and 7; (R1) and (R3) apply.

Row 16 ($k = 11$). Let F be the difference of a principal coefficient of row 15 from its reference value. We claim that F vanishes with its first two derivatives in y at both ends. On the negative side of a root the identity $h_i^2\,|\det D\Psi_i|^{-1}\,D\Psi_i\,A\,D\Psi_i^T = \mathrm{diag}(1,-\xi_i)$ holds, as an identity between functions of (r,θ) which are smooth up to the root, with $h_i \circ \Psi_i$ written h_i; this is the canonical form of Section 7.6. By construction the radial jets at the root of $\Psi_{\mathbf{x}}$ of order four and of $h_{\mathbf{x}}$ of order three are those of Ψ_i and h_i, and A is smooth across the root. The matrix of row 15 contains one derivative of $\Psi_{\mathbf{x}}$ and none of $h_{\mathbf{x}}$; so its radial jet of

order three at the root is that of $\operatorname{diag}(1, y_i)$, where $y_i = -\xi_i$ is y at the left end and $1 - y$ at the right end. The same is true at the reference state, where the coefficients are $a = 1$ and $c = y_i$ near the root. Composition with $\Psi_{\mathbf{x}}^{-1}$ preserves the vanishing at the boundary, which proves the claim. Now (R7) applies near the left end; near the right end it applies with $1 - y$ in the place of y; and in the middle $\mathfrak{y}^{-3}$ is a smooth function. This gives $k = 8 + 3 = 11$.

Row 17 ($k = 9$). The potential is $-(h_{\mathbf{x}}/|J_{\mathbf{x}}|)\,\bar{L}[h_{\mathbf{x}}]$ composed with $\Psi_{\mathbf{x}}^{-1}$. It contains two derivatives of $h_{\mathbf{x}}$, which gives $7 + 2 = 9$ by (R2), the Jacobian of row 14, and the coefficients of $\bar{L}$ with their first derivatives; (R1) and (R3) apply.

This completes the proof of (7.51). The last statement of part (ii) is contained in (b).

Part (iii). The derivatives with respect to the state are obtained by differentiating the same formulas, and each rule has a differentiated form. A variation of a composition $u \circ h$ is $\dot{u} \circ h + (Du \circ h)\,\dot{h}$; a variation of an inverse is $-(Dh^{-1})\,(\dot{h} \circ h^{-1})$, as in (10.5) below; a variation of a flow solves the linear equation $\partial_t \dot{\Theta} = (\partial_\theta G \circ \Xi_t)\,\dot{\Theta} + \dot{G} \circ \Xi_t$ with the integrating factor of (R4); and every other operation in the table is a product, a smooth function, a derivative, a substitution (R5), a Taylor polynomial or a division, which are differentiated by the Leibniz rule or are linear. In each case the variation is estimated by the same rules, and it contains one more spatial derivative of one earlier object: Du in the first case, Dh^{-1} in the second, $\partial_\theta G$ in the third (Lemma 7.3 and the example after it). After j differentiations no factor has more than j additional derivatives, which is the index $s + k + 3 + j$ in (7.43); the products are estimated by (R1), with the high order on one factor at a time, and this gives the two terms of (7.43). The height enters through $Z = qu$, whose first derivative is $\dot{q}\,u + q\,\dot{u}$, whose second derivative is $q_1 u_2 + q_2 u_1$, and whose higher derivatives vanish. This is why the seminorms (7.18) contain the unscaled u: they control all derivatives also at $q = 0$. □

7.9 The fixed transmission domain

We have represented a height on the annulus by canonical unknowns on a collection of cylinders: two half-cylinders $[0, \xi_c] \times \mathbb{T}$ for each negative component and one cylinder $[0, 1] \times \mathbb{T}$ for each positive component. The cylinders do not depend on the state. Neither do the conditions that link them:

- at the middle of a negative component, the condition (7.37);
- at a root, equality of the canonical values and of the inward derivatives with opposite signs. Indeed, the coordinates and the conjugating factors of the two adjacent components have the same value and the same first derivative at the root, so (7.38) takes this form.

We call the set of canonical unknowns satisfying these conditions the *transmission domain*. A smooth element of it which solves the equation corresponds to a function which is smooth in the original variables across the roots: equality of the values gives equality of all tangential derivatives; equality of the fluxes gives equality of the first normal derivative; and the equation can be solved for $a\,u_{rr}$, because $a > 0$, so that repeated differentiation matches all higher normal derivatives.

This is the result of the section: all the operators $\widehat{L}_{\mathbf{x}}$ act, in canonical form, on one domain. It is what makes it possible to regard them as perturbations of the reference operator in the next section.

8 Inverting the corrected linearization near the reference

The goal of this section is Proposition 8.1: for every state $\mathbf{x}$ in the neighbourhood of Proposition 7.11, the operator $\widehat{L}_{\mathbf{x}}$, bordered by the derivative of the equation with respect to

the metric parameter ν, has an inverse which satisfies tame estimates.

The argument has to respect one constraint. The reference inverse loses coefficient weight, and a tame estimate with a loss is not a bounded inverse on a single Banach space; it cannot by itself justify a Neumann series. We therefore proceed in three steps.

1. We put the problem in canonical form and identify a single Hilbert space of sources on which the *relative* operator $\mathscr{R}_{\mathbf{x}} = (\mathbb{P}_{\mathbf{x}} - \mathbb{P}_*)\,\mathbb{G}_*$ is bounded, with small norm. The loss of the reference inverse is compensated by the factor H^β carried by every difference of coefficients (Sections 8.1–8.4).
2. A Neumann series on that one space gives the solution.
3. All higher regularity of this same solution is then recovered from the equation: angular derivatives by commutators, radial derivatives by solving the equation for its highest radial derivative (Section 8.5).

In this order of argument, one neighbourhood of the reference state serves for all orders of differentiation.

The result of the section is the following proposition. Its proof occupies Sections 8.1–8.5; the last lines are at the end of Section 8.5.

Proposition 8.1 (The inverse at a nearby state)**.** *Let $k_{\mathbf{x}} = D_\nu \mathcal{F}_q(w,u)$ be the derivative of the equation with respect to the scalar ν. For every state in $\mathfrak{N}$ and every $f \in \mathcal{Y}_\beta$ there is a unique pair $G_{\mathbf{x}} f = (\eta, \mu)$ with*

$$\widehat{L}_{\mathbf{x}}\eta + \mu\, k_{\mathbf{x}} = f, \qquad \eta \in \mathcal{X}_\beta \cap \ker \ell_*, \qquad \mu \in \mathbb{R},$$

where η tends to zero at the right endpoint and to a constant at the left endpoint. For every s, with the base order b of Lemma 8.2 and the number d_* fixed below,

$$\|G_{\mathbf{x}} f\|_s \le C_s\big[\|f\|_{s+d_*} + (1+\mathfrak{n}_{s+d_*})\,\|f\|_{b+d_*}\big]. \tag{8.1}$$

The map $(\mathbf{x}, q, f) \mapsto G_{\mathbf{x}} f$ is smooth tame.

Throughout this section and the next, d_* denotes one fixed integer, the largest of the following losses of derivatives:

- $d_{\mathrm{geo}} + 2$, for the coefficients of the relative operator and their commutators with Λ (Lemma 8.4); here $d_{\mathrm{geo}} = 14$ is the number of Proposition 7.11;
- d_{tot}, the loss in (8.24);
- the sum of d_{tot}, $L_0 + 1$ and the losses of the transformations between canonical and original variables (Proposition 7.11), for (8.1).

These numbers are determined by the reference profile and by the base order b; they do not depend on the orders s and t, nor on d_*. Every estimate in which d_* occurs remains true when d_* is replaced by a larger number, because the seminorms $\mathfrak{n}_s$ and $\mathfrak{g}_{s,m}$ are nondecreasing in s. So the maximum can be taken at the outset.

8.1 The linear problem in canonical form

Transformations. Use the cylinders of Section 7. Let $\Psi_{\mathbf{x}}$ denote the coordinate map on each piece, $J_{\mathbf{x}}$ its Jacobian determinant and $h_{\mathbf{x}}$ the conjugating factor in the original variables. Define

$$\mathscr{C}_{\mathbf{x}} u = \frac{u}{h_{\mathbf{x}}} \circ \Psi_{\mathbf{x}}^{-1}, \qquad \mathscr{S}_{\mathbf{x}} f = \frac{h_{\mathbf{x}} f}{|J_{\mathbf{x}}|} \circ \Psi_{\mathbf{x}}^{-1}, \qquad \mathbb{L}_{\mathbf{x}} = \mathscr{S}_{\mathbf{x}}\, \widehat{L}_{\mathbf{x}}\, \mathscr{C}_{\mathbf{x}}^{-1}.$$

Thus $\mathscr{C}_{\mathbf{x}}$ maps a height to its canonical unknown, $\mathscr{S}_{\mathbf{x}}$ maps a source to its canonical source, and $\mathbb{L}_{\mathbf{x}}$ is the operator in canonical form; on a positive cylinder $\mathbb{L}_{\mathbf{x}} = -P_{\mathbf{x}}$. The pairing with $dr\,d\theta$ is preserved. A *piece* is one of the cylinders: a half of a negative component, or a positive component. A function on the collection of all pieces is written $F = (F_m)$.

Source norms. For real t and integers $s \geq 0$ let $\mathbb{V}_{t,s}$ be the space of $F = (F_m)$ with

$$\|F\|_{t,s}^2 = \sum_m M_m^{2t}\, H_m^{-2\beta}\, \|F_m\|^2_{L^2_{\mathrm{rad}} H^s_\varphi} < \infty, \tag{8.2}$$

where m runs over all pieces, the norm is L^2 in the radial variable of the cylinder with s angular derivatives, and M_m, H_m are the weights of the cell containing the piece. No matching condition and no integral condition is imposed on sources. The index t counts powers of the weight M, the index s angular derivatives.

Scalars and the augmented operator. The linear problem involves three scalars besides the canonical unknown W.

- *The endpoint constant a_L.* A height is represented as

  $$\eta = \mathscr{C}_{\mathbf{x}}^{-1} W + a_L \chi_L, \qquad \ell_*^c W = 0,$$

  so that η tends to a_L at the left endpoint and to 0 at the right one. The coordinates agree with the reference ones on the interval J_* of Section 7.7, which contains the support of g_*; hence the normalization $\ell_* \eta = 0$ takes a form $\ell_*^c W = 0$ which does not depend on the state.

- *The metric increment μ.* It multiplies the column

  $$k_{\mathbf{x}} = D_\nu \mathcal{F}_q(w, u), \qquad k_{\mathbf{x}}^c = \mathscr{S}_{\mathbf{x}} k_{\mathbf{x}},$$

  where D_ν is the derivative of the residual with respect to ν for fixed u and q; it acts on the metric and on the radial height h. At the reference state $k_* = g_* e_2$, the border column of Proposition 4.2. This scalar removes the obstruction in the second mode.

- *An auxiliary scalar λ.* Choose a function ψ of r, compactly supported in J_*, with $\mathsf{Q}\psi = 1$, and put $c_* = \mathscr{S}_* \psi = \mathscr{S}_{\mathbf{x}} \psi$. The scalar λ multiplies c_* and removes the integral obstruction, so that arbitrary sources can be treated. For compatible sources it will turn out to be zero.

With $s_{\mathbf{x}} = \mathscr{S}_{\mathbf{x}} \widehat{L}_{\mathbf{x}} \chi_L$, the *augmented operator* is

$$\mathbb{P}_{\mathbf{x}} \mathbf{W} = \mathbb{L}_{\mathbf{x}} W + a_L\, s_{\mathbf{x}} + \mu\, k_{\mathbf{x}}^c + \lambda\, c_*, \qquad \mathbf{W} = (W, a_L, \mu, \lambda). \tag{8.3}$$

The norm of an output $\mathbf{W}$ is the norm (8.2) of W plus the absolute values of the three scalars, which are assigned to the finite middle part; we write $\|\mathbf{W}\|_{t,s}^{\mathrm{vol}}$.

The reference inverse in this form. At the reference state let $\mathfrak{g}_* = \mathscr{C}_* 1$ and $\mathsf{Q}_* F = \langle \mathfrak{g}_*, F \rangle$; this functional is continuous by the coefficient bounds, it is the integral Q in canonical form, and $\mathsf{Q}_* c_* = 1$. Given F, put $\lambda = \mathsf{Q}_* F$ and apply the inverse of Proposition 4.2 to the compatible source $F - \lambda c_*$. This defines a two-sided inverse $\mathbb{G}_*$ of $\mathbb{P}_*$. In the zero mode it is explicit: with $f_* = \mathscr{S}_*^{-1} F$,

$$p_0(r) = \int_{r_-}^{r} (\langle f_* \rangle - \lambda \psi)(t)\, dt, \qquad u_0(r) = -\int_r^{r_+} \frac{p_0(t)}{w_*(t)}\, dt,$$

as in (6.35); near either endpoint p_0 is the primitive starting at that endpoint, division by w_* costs a coefficient power and not an endpoint exponent, and subtracting the left endpoint value times χ_L gives the field and the scalar a_L.

8.2 Two endpoint exponents give decay off the diagonal

By Proposition 4.2(iv),(v) the reference inverse is bounded with the endpoint weight $H^{-\gamma}$ for every γ in a compact subinterval of $(0,1)$. Fix

$$0 < \epsilon < \min(\beta, 1-\beta).$$

The inverses at the exponents $\beta-\epsilon$ and $\beta+\epsilon$ coincide on smooth compactly supported sources: their difference is a homogeneous solution in the graph domain, hence zero by Lemma 6.12. So both estimates apply to the same operator $\mathbb{G}_*$. Using each of them on one side of the diagonal converts the two bounds into decay in the distance between source and output (Figure 15). Obtaining decay of an inverse away from the diagonal from its boundedness in a family of exponentially weighted spaces is the idea of Combes and Thomas [CT73].

Lemma 8.2 (A fixed loss of coefficient weight)**.** *There is a fixed L_0 such that for every real t and every $s \geq 0$*

$$\|\mathbb{G}_* F\|^{\mathrm{vol}}_{t-L_0,s} \leq C_{t,s}\, \|F\|_{t,s}. \tag{8.4}$$

The first radial derivative of $\mathbb{G}_* F$ satisfies the same bound with at most one additional angular derivative of the source. For a fixed integer $b \geq 2$,

$$\mathbb{D}_* := \mathbb{G}_*(\mathbb{V}_{0,b}), \qquad \|\mathbf{W}\|_{\mathbb{D}_*} = \|\mathbb{P}_* \mathbf{W}\|_{0,b}, \tag{8.5}$$

is a Hilbert space on which $\mathbb{P}_*$ is closed. It contains every smooth $\mathbf{W}$ in the transmission domain of Section 7.9 which satisfies the conditions at the endpoints and the normalization, and for which $\mathbb{P}_* \mathbf{W} \in \mathbb{V}_{0,b}$.

Proof. Step 1: blocks. Let $\bar{S}_m = S_m - S_N$ on the tails and $\bar{S}_m = 0$ on the finite middle. Let π_k be the restriction to the piece k and ι_m the extension by zero of a source on the piece m. By Section 6.9, the estimate at the exponent γ gives

$$\|\pi_k\, \mathbb{G}_*\, \iota_m\|_{H^s_\varphi \to H^s_\varphi} \leq C_s\, \mathcal{E}_k^{C_s}\, \mathcal{E}_m^{C_s}\, e^{-\gamma(\bar{S}_k - \bar{S}_m)}.$$

There is no loss of angular derivatives, because the reference inverse acts on each angular mode with bounds uniform in the mode. Multiply by the weights $H_k^{-\beta} H_m^{\beta}$ of (8.2), and use $\gamma = \beta + \epsilon$ when $\bar{S}_k \geq \bar{S}_m$ and $\gamma = \beta - \epsilon$ otherwise. The resulting block K_{km} satisfies

$$\|K_{km}\|_{H^s_\varphi \to H^s_\varphi} \leq C_s\, \mathcal{E}_k^{C_s}\, \mathcal{E}_m^{C_s}\, e^{-\epsilon|\bar{S}_k - \bar{S}_m|}. \tag{8.6}$$

The estimate for the first radial derivative in Proposition 4.2(iv) gives the same block bound with one more angular derivative.

Step 2: summation. For every $\delta > 0$, $|\log M_k - \log M_m| \leq \delta\, |\bar{S}_k - \bar{S}_m| + C_\delta$, because the derivative of $n^{5/2} \log^2(e+n)$ with respect to $2n^3/3$ tends to zero. With $\mathcal{E}^{C_s} \leq C_s M$ we get, for $L_0 \geq 2$,

$$M_k^{t-L_0}\, \|K_{km}\|\, M_m^{-t} \leq C_s\, M_k^{2-L_0} \Big(\frac{M_k}{M_m}\Big)^{t-1} e^{-\epsilon|\bar{S}_k - \bar{S}_m|} \leq C_{t,s}\, e^{-(\epsilon/2)|\bar{S}_k - \bar{S}_m|},$$

where δ was chosen with $|t-1|\,\delta \leq \epsilon/2$. This matrix has bounded row and column sums, since the numbers S_n are uniformly separated and each index carries a bounded number of pieces. Schur's test proves (8.4).

Step 3: the domain. By definition $\mathbb{D}_*$ is isometric to $\mathbb{V}_{0,b}$, hence complete. If $\mathbf{W}_i \to \mathbf{W}$ in the space of outputs and $\mathbb{P}_* \mathbf{W}_i \to F$ in $\mathbb{V}_{0,b}$, then $\mathbf{W}_i = \mathbb{G}_* \mathbb{P}_* \mathbf{W}_i \to \mathbb{G}_* F$, so $\mathbf{W} = \mathbb{G}_* F$: the operator is closed. Approximating a source by smooth compactly supported ones, the estimate

gives convergence of the solutions on every piece, the equation passes to the limit in the sense of distributions, and the bound for the first radial derivative identifies the traces, so the transmission conditions hold in the weak sense. Finally, a smooth $\mathbf{W}$ as in the last statement differs from $\mathbb{G}_*\mathbb{P}_*\mathbf{W}$ by a normalized homogeneous solution with the natural behaviour at the endpoints, which vanishes by Lemma 6.12. □

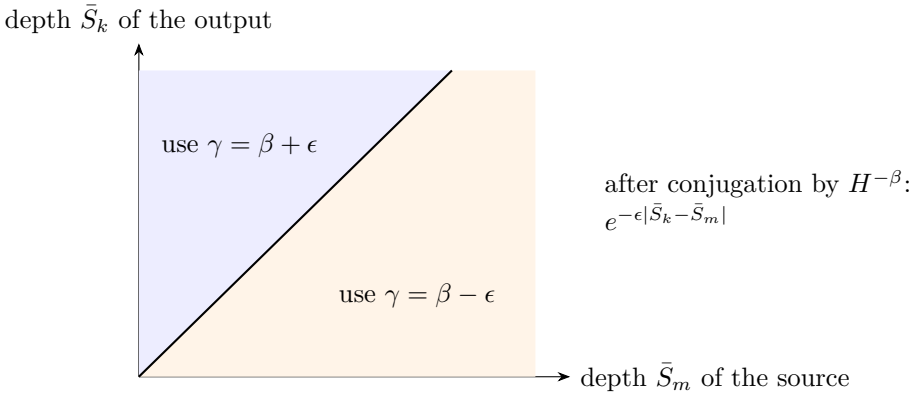


Figure 15: The same reference inverse is estimated with two endpoint weights, each used on one side of the diagonal. Together they give decay in the distance between the depths of the source and of the output.

8.3 Positive cylinders: an estimate without boundary conditions

On a positive cylinder the perturbation (7.41) is of second order, the same order as the operator. It can nevertheless be controlled by the operator, because its coefficients vanish to third order at the two ends. The following estimate is the reason for that requirement. In this subsection $\langle j\rangle = (1+j^2)^{1/2}$.

Lemma 8.3 (Weighted graph estimate)**.** *On $0<y<1$ consider, for an integer j,*

$$P_j u = -(a\,u')' + j^2 c\,u + V\,u = F, \qquad a \geq \mathcal{E}_m^{-C}, \quad c \geq \mathcal{E}_m^{-C}\,\mathfrak{y}, \quad \mathfrak{y} = y(1-y),$$

where the derivatives up to order two of a, c, V, and a^{-1}, are bounded by $\mathcal{E}_m^{C_1}$. For every solution $u \in L^2$ with $F \in L^2$, without any condition at the endpoints,

$$\begin{aligned} &\|\mathfrak{y}\,u'\|_2 + \langle j\rangle\,\|\mathfrak{y}^{3/2}u\|_2 + \|\mathfrak{y}^2 u''\|_2 + \langle j\rangle^2\,\|\mathfrak{y}^3 u\|_2 + \langle j\rangle\,\|\mathfrak{y}^{5/2}u'\|_2 \\ &\qquad \leq \mathcal{E}_m^{C}\,(\|F\|_2 + \|u\|_2). \end{aligned} \tag{8.7}$$

Proof. For fixed j a solution in L^2 with $F \in L^2$ is in H^2 up to the endpoints, since the equation is a regular ordinary differential equation; this justifies the integrations by parts below, but the uniform bounds come from the computation.

Step 1: first order. Multiply the equation by $\mathfrak{y}^2\bar{u}$ and integrate. Since $\mathfrak{y}$ vanishes at both ends,

$$\int a\,|(\mathfrak{y}u)'|^2 + j^2 \int c\,\mathfrak{y}^2|u|^2 = \mathrm{Re}\langle F, \mathfrak{y}^2 u\rangle + \int a\,|\mathfrak{y}'|^2|u|^2 - \int V\,\mathfrak{y}^2|u|^2.$$

This bounds $\mathfrak{y}\,u'$ and $j\,\mathfrak{y}^{3/2}u$.

Step 2: second order. The function $\widetilde{u} = \mathfrak{y}^2 u$ vanishes at both ends and satisfies

$$P_j\widetilde{u} = \mathfrak{y}^2 F - 2a\,(\mathfrak{y}^2)'u' - (a\,(\mathfrak{y}^2)')'u,$$

whose right side is controlled by Step 1. With $\mathsf{c} = c/a$ the same is true of $F_1 = -\widetilde{u}'' + j^2 \mathsf{c}\,\widetilde{u}$. Squaring and integrating the cross term by parts twice,

$$\|\widetilde{u}''\|_2^2 + j^4 \|\mathsf{c}\,\widetilde{u}\|_2^2 + 2j^2 \int \mathsf{c}\,|\widetilde{u}'|^2 = \|F_1\|_2^2 + j^2 \int \mathsf{c}''\,|\widetilde{u}|^2 .$$

The last integral is controlled by Step 1 because $\mathfrak{y}^4 \le C\mathfrak{y}^3$. Using $\mathsf{c} \ge \mathcal{E}_m^{-C}\mathfrak{y}$ and expressing $\widetilde{u}'$ and $\widetilde{u}''$ through u gives (8.7). □

Consequence for the perturbation. Every second-order term of (7.41) is controlled by the left side of (8.7): $\mathfrak{y}^3 u_{yy}$ by $\mathfrak{y}^2 u''$, $\mathfrak{y}^3 u_{y\varphi}$ by $\langle j\rangle \mathfrak{y}^{5/2} u'$, and $\mathfrak{y}^3 u_{\varphi\varphi}$ by $\langle j\rangle^2 \mathfrak{y}^3 u$. When a derivative falls on a coefficient, the remaining terms are controlled by the first two norms.

The coefficients depend on φ, and this dependence is handled by Fourier series. Write $u = \sum_j u_j(y)\, e^{ij\varphi}$ and, for one of the coefficients, $e = \sum_k e_k(y)\, e^{ik\varphi}$. The lth Fourier coefficient of $\mathfrak{y}^3 e\, u_{yy}$ is $\sum_k e_k\, \mathfrak{y}^3 u''_{l-k}$, so

$$\big\|(\mathfrak{y}^3 e\, u_{yy})_l\big\|_{L^2_y} \le \sum_k \epsilon_k\, g_{l-k}, \qquad \epsilon_k = \sup_y |e_k(y)|, \quad g_j = \|\mathfrak{y}^2 u_j''\|_{L^2_y} \le \mathcal{E}_m^C(\|F_j\|_2 + \|u_j\|_2),$$

the last inequality being Lemma 8.3 for the mode j. Since $\langle l\rangle^s \le 2^s(\langle k\rangle^s + \langle l-k\rangle^s)$, Young's inequality for sequences gives

$$\big\|\langle l\rangle^s (\epsilon * g)_l\big\|_{\ell^2} \le 2^s\big(\|\epsilon\|_{\ell^1}\, \|\langle j\rangle^s g\|_{\ell^2} + \|\langle k\rangle^s \epsilon\|_{\ell^2}\, \|g\|_{\ell^1}\big), \tag{8.8}$$

and $\|g\|_{\ell^1} \le C\|\langle j\rangle g\|_{\ell^2}$, $\|\epsilon\|_{\ell^1} \le C\|\langle k\rangle \epsilon\|_{\ell^2}$. The weighted norms of (ϵ_k) are bounded by norms of the coefficient e with one more derivative in y, hence by (7.42).

The same argument applies to every term of

$$\delta P_{\mathbf{x}} u = -\mathfrak{y}^3 e^{ab}_{\mathbf{x}}\, \partial_a \partial_b u - \partial_a(\mathfrak{y}^3 e^{ab}_{\mathbf{x}})\, \partial_b u + \delta V_{\mathbf{x}}\, u,$$

with the following coefficient e and sequence g_j; in each case $g_j \le \mathcal{E}_m^C(\|F_j\|_2 + \|u_j\|_2)$ by Lemma 8.3, because $\mathfrak{y} \le 1$.

term	coefficient e	g_j
$\mathfrak{y}^3 e^{yy}\, u_{yy}$	e^{yy}	$\|\mathfrak{y}^2 u_j''\|_2$
$2\,\mathfrak{y}^3 e^{y\varphi}\, u_{y\varphi}$	$e^{y\varphi}$	$\langle j\rangle\, \|\mathfrak{y}^{5/2} u_j'\|_2$
$\mathfrak{y}^3 e^{\varphi\varphi}\, u_{\varphi\varphi}$	$e^{\varphi\varphi}$	$\langle j\rangle^2\, \|\mathfrak{y}^3 u_j\|_2$
$\big[(\mathfrak{y}^3 e^{yy})_y + \mathfrak{y}^3 e^{y\varphi}_{\varphi}\big]\, u_y$	$3\mathfrak{y}' e^{yy} + \mathfrak{y}\, e^{yy}_y + \mathfrak{y}\, e^{y\varphi}_{\varphi}$	$\|\mathfrak{y}\, u_j'\|_2$
$\big[(\mathfrak{y}^3 e^{y\varphi})_y + \mathfrak{y}^3 e^{\varphi\varphi}_{\varphi}\big]\, u_\varphi$	$3\mathfrak{y}' e^{y\varphi} + \mathfrak{y}\, e^{y\varphi}_y + \mathfrak{y}\, e^{\varphi\varphi}_{\varphi}$	$\langle j\rangle\, \|\mathfrak{y}^{3/2} u_j\|_2$
$\delta V_{\mathbf{x}}\, u$	$\delta V_{\mathbf{x}}$	$\|u_j\|_2$

In the fourth and fifth lines a factor $\mathfrak{y}^2$ has been moved from the coefficient to g_j. The coefficients involve the functions $e^{ab}_{\mathbf{x}}$ of row 16 of Table 4 with their first derivatives, and the difference of the potentials of row 17. For the commutator with $\Lambda = |D_\varphi|$, the lth coefficient of $[\Lambda, e]\, v$ is $\sum_k (|l| - |l-k|)\, e_k v_{l-k}$, and $\big||l| - |l-k|\big| \le |k|$; so ϵ_k is replaced by $|k|\,\epsilon_k$, which costs one more angular derivative of the coefficient. Altogether, for $P_* u = F$ on the piece m and $s \ge b$,

$$\|\delta P_{\mathbf{x}} u\|_s \le C_s\, H^\beta_m\, \mathcal{E}_m^{C_s}\Big[\mathfrak{g}_{b_0,m}\,(\|F\|_s + \|u\|_s) + \mathfrak{g}_{s+d_*,m}\,(\|F\|_b + \|u\|_b)\Big], \tag{8.9}$$

and the same bound holds for $[\Lambda, \delta P_{\mathbf{x}}]u$. Here $\delta P_{\mathbf{x}} = P_{\mathbf{x}} - P_*$, the norms are those of $L^2_y H^s_\varphi$, and $\mathfrak{g}_{s,m}$ is the localized seminorm of Section 7.4.

An elementary estimate for the first radial derivative. On every piece, normalized to unit radial length, the equation at frequency j has the form $Au'' + Bu' + (j^2C + D_0)u = F$, with A bounded away from zero and all coefficients bounded by coefficient powers. From the equation and the interpolation inequality $\|u'\|_2^2 \le C\|u\|_2(\|u''\|_2 + \|u\|_2)$,

$$\|u'\|_2 \le \mathcal{E}_m^C(\|F\|_2 + \langle j\rangle\|u\|_2), \qquad \|u''\|_2 \le \mathcal{E}_m^C(\|F\|_2 + \langle j\rangle^2\|u\|_2), \tag{8.10}$$

with no assumption on the boundary values: substitute the equation for u'' into the interpolation inequality and absorb the term $\mathcal{E}_m^C\|u\|_2\|u'\|_2$ by Young's inequality. Smooth sources with finitely many frequencies are dense, and for limits of such sources these estimates identify the weak derivatives; so no regularity is assumed beforehand.

8.4 The relative operator

On smooth compactly supported sources define

$$\mathscr{R}_{\mathbf{x}} = (\mathbb{P}_{\mathbf{x}} - \mathbb{P}_*)\,\mathbb{G}_*.$$

If $\mathbb{G}_*F = (W, a_L, \mu, \lambda)$, then

$$\mathscr{R}_{\mathbf{x}}F = \delta\mathbb{L}_{\mathbf{x}}W + a_L\,(s_{\mathbf{x}} - s_*) + \mu\,(k^c_{\mathbf{x}} - k^c_*), \qquad \delta\mathbb{L}_{\mathbf{x}} = \mathbb{L}_{\mathbf{x}} - \mathbb{L}_*; \tag{8.11}$$

the column c_* does not depend on the state. On a negative piece $\delta\mathbb{L}_{\mathbf{x}}$ is the difference of the potentials V_i in (7.36), a multiplication operator. On a positive piece it is estimated by (8.9), the compactly supported scalar columns being counted as part of the source. In every case the difference of coefficients on the output piece k carries the factor H_k^β of (7.42).

Lemma 8.4 (The relative operator is small on one space)**.** *Let $b \ge 2$ be as in Lemma 8.2. On $\mathfrak{N}$,*

$$\|\mathscr{R}_{\mathbf{x}}\|_{\mathbb{V}_{0,b}\to\mathbb{V}_{0,b}} \le C\,\mathfrak{n}_{b_0} < \tfrac{1}{2}. \tag{8.12}$$

Moreover, for $t \ge 0$ and $s \ge b$, with $\Lambda = |D_\varphi|$,

$$\|\mathscr{R}_{\mathbf{x}}F\|_{t,s} + \|[\Lambda, \mathscr{R}_{\mathbf{x}}]F\|_{t,s} \le C_{t,s}\big[\epsilon_0\,\|F\|_{0,s} + \mathfrak{n}_{t+s+d_*}\,\|F\|_{0,b}\big]. \tag{8.13}$$

The operator $\mathscr{R}_{\mathbf{x}}$ extends by continuity to $\mathbb{V}_{0,b}$, and

$$\mathbb{P}_{\mathbf{x}} = (I + \mathscr{R}_{\mathbf{x}})\,\mathbb{P}_* \qquad \text{on } \mathbb{D}_*. \tag{8.14}$$

Note that (8.12) bounds an operator from one space to the same space, as a Neumann series requires.

Proof. Step 1: the base space. Combine the factor H_k^β with (8.6). With coefficients measured at low order, the block of $\mathscr{R}_{\mathbf{x}}$ from the piece m to the piece k has norm at most

$$C\,\mathfrak{n}_{b_0}\,M_k^C\,M_m^C\,e^{-\beta\bar S_k - \epsilon|\bar S_k - \bar S_m|}. \tag{8.15}$$

For every fixed C and $\delta > 0$ we have $M_i^C \le C_\delta\,e^{\delta\bar S_i}$, and $\bar S_m \le \bar S_k + |\bar S_k - \bar S_m|$. Choosing $\delta < \min(\beta/4, \epsilon/2)$ bounds (8.15) by

$$C'\,\mathfrak{n}_{b_0}\,e^{-(\beta/2)\bar S_k - (\epsilon/2)|\bar S_k - \bar S_m|},$$

which has bounded row and column sums. Increasing b_0 to cover the finitely many low derivatives used, and then decreasing ϵ_0, gives (8.12).

Step 2: higher orders. In the product estimates keep one coefficient factor at the high order. The same summation proves the bound for $\mathscr{R}_{\mathbf{x}}$ in (8.13); the output may carry any power M^t because of the factor H_k^{β}. For the commutator, assign the frequency zero to the scalars a_L, λ and the frequency two to μ. For multiplication by a coefficient, the matrix of the commutator in Fourier modes gains the factor $|k|-|j|$, which is bounded by $|k-j|$; so one more derivative of the coefficient gives the same bound. The commutators with the scalar columns are the terms $\Lambda(s_{\mathbf{x}}-s_*)\,a_L$ and $(\Lambda-2)(k_{\mathbf{x}}^c-k_*^c)\,\mu$, which are of the same type.

Step 3: the factorization. By Step 1 the operator extends to $\mathbb{V}_{0,b}$. Approximating by smooth compactly supported sources identifies the extension with the difference of the differential operators, in the sense of distributions, on $\mathbb{D}_*$; here it is used that the transmission conditions are the same for all states. This is (8.14). □

8.5 Solving, and recovering all derivatives

By (8.12) we may define, for $F \in \mathbb{V}_{0,b}$,

$$\widetilde{F} = (I + \mathscr{R}_{\mathbf{x}})^{-1} F, \qquad \mathbf{W} = \mathbb{G}_* \widetilde{F},$$

where the inverse is a Neumann series on $\mathbb{V}_{0,b}$. By (8.14), $\mathbb{P}_{\mathbf{x}}\mathbf{W} = F$. We now show that this solution is as regular as the source allows, with tame estimates. Three steps recover, in turn, the angular derivatives of $\widetilde{F}$, its coefficient weight, and the radial derivatives of W.

Step A: angular derivatives of $\widetilde{F}$. This step rests on a general lemma. Let $\mathcal{H}$ be a Hilbert space with a self-adjoint operator Λ whose spectrum lies in $\{0,1,2,\dots\}$, let $\|v\|_p = \|\langle\Lambda\rangle^p v\|$, and let $\mathcal{H}^p$ be the space of vectors for which this norm is finite.

Lemma 8.5 (Regularity of an inverse from a commutator)**.** *Let R be a bounded operator on $\mathcal{H}$ which maps every $\mathcal{H}^p$ into itself, and suppose that $I+R$ has a bounded inverse on $\mathcal{H}$. Suppose the commutator $\mathscr{K} = [\Lambda, R]$ satisfies, for all integers $p \geq 0$,*

$$\|\mathscr{K} v\|_p \leq a_p \|v\|_p + c_p \|v\|_0, \tag{8.16}$$

with nondecreasing constants a_p and c_p. Then $(I+R)^{-1}$ maps every $\mathcal{H}^p$ into itself, and for $p \geq 1$

$$\|(I+R)^{-1} f\|_p \leq C_p(\|f\|_p + (1+c_{p-1})\,\|f\|_0), \tag{8.17}$$

where C_p depends only on p, on $a_0,\dots,a_{p-1}$, on c_0 and on the norm of $(I+R)^{-1}$ on $\mathcal{H}$.

The constants c_p with $p \geq 1$ may be arbitrarily large: they enter linearly, and they do not enter the constant C_p.

Proof. Step 1: the solution is regular. Let $U_\delta = e^{i\delta\Lambda}$. For $v \in \mathcal{H}^1$ the function $t \mapsto U_{\delta-t} R U_t v$ is differentiable with derivative $-i\,U_{\delta-t}\mathscr{K} U_t v$, so

$$[U_\delta, R] = i \int_0^\delta U_{\delta-t}\,\mathscr{K}\,U_t\,dt,$$

first on $\mathcal{H}^1$ and then on $\mathcal{H}$, because $\mathscr{K}$ is bounded on $\mathcal{H}$ by (8.16) with $p=0$. Let $u = (I+R)^{-1} f$. Then

$$(I+R)\,\frac{U_\delta u - u}{\delta} = \frac{U_\delta f - f}{\delta} - \frac{[U_\delta, R]\,u}{\delta}.$$

If $f \in \mathcal{H}^1$ the right side is bounded in $\mathcal{H}$ uniformly in δ, hence so are the difference quotients of u; by Fatou's lemma for the spectral decomposition of Λ, $u \in \mathcal{H}^1$, and letting $\delta \to 0$ gives $(I+R)\Lambda u = \Lambda f - \mathscr{K} u$. Now let $p \geq 2$, $f \in \mathcal{H}^p$ and suppose $u \in \mathcal{H}^{p-1}$. Then

$$(I+R)\,\Lambda^{p-1}u = \Lambda^{p-1}f - \sum_{k=0}^{p-2} \Lambda^k\,\mathscr{K}\,\Lambda^{p-2-k}u,$$

and the right side lies in $\mathcal{H}^1$, because $\Lambda^{p-2-k}u \in \mathcal{H}^{k+1}$ and $\mathscr{K}$ maps $\mathcal{H}^{k+1}$ into itself. By the case just treated $\Lambda^{p-1}u \in \mathcal{H}^1$, that is, $u \in \mathcal{H}^p$. By induction, for $f \in \mathcal{H}^p$,

$$(I+R)\,\Lambda^{p}u = \Lambda^{p}f - \sum_{k=0}^{p-1} \Lambda^k\,\mathscr{K}\,\Lambda^{p-1-k}u. \tag{8.18}$$

Step 2: the estimate. For $p=1$, (8.18) and the bound a_0+c_0 for $\mathscr{K}$ on $\mathcal{H}$ give the claim. Let $p \geq 2$ and $A = 1 + c_{p-1}$. By (8.16), $\mathscr{K}$ is bounded on $\mathcal{H}$ with norm at most a_0+c_0, and from $\mathcal{H}^{p-1}$ with the norm $(\|v\|_{p-1}^2 + A^2\|v\|_0^2)^{1/2}$ to $\mathcal{H}^{p-1}$ with norm at most $a_{p-1}+1$. Both pairs of norms are weighted ℓ^2 norms over the spectrum of Λ. Interpolating between them, by the three-lines lemma applied to $z \mapsto \langle\Lambda\rangle^{(p-1)z}\,\mathscr{K}\,(\langle\Lambda\rangle^{2(p-1)} + A^2)^{-z/2}$ on vectors with finitely many nonzero spectral components, for which this family is analytic and bounded on the closed strip, and using $(\langle j\rangle^{2(p-1)} + A^2)^{\vartheta} \leq \langle j\rangle^{2(p-1)\vartheta} + A^{2\vartheta}$ for $0 \leq \vartheta \leq 1$, we get

$$\|\mathscr{K} v\|_k \leq C\big(\|v\|_k + A^{k/(p-1)}\,\|v\|_0\big), \qquad 0 \leq k \leq p-1.$$

Apply this to $v = \Lambda^{p-1-k}u$. Since $\|u\|_{p-1-k} \leq \|u\|_{p-1}^{1-k/(p-1)}\|u\|_0^{k/(p-1)}$, Young's inequality gives

$$A^{k/(p-1)}\,\|u\|_{p-1-k} \leq (A\|u\|_0)^{k/(p-1)}\,\|u\|_{p-1}^{1-k/(p-1)} \leq \|u\|_{p-1} + A\,\|u\|_0.$$

So every term of the sum in (8.18) is bounded in $\mathcal{H}$ by $C(\|u\|_{p-1}+A\|u\|_0)$. Applying $(I+R)^{-1}$ to (8.18),

$$\|u\|_p \leq C\big(\|f\|_p + \|u\|_{p-1} + A\,\|u\|_0\big).$$

Finally $\|u\|_{p-1} \leq \varepsilon\|u\|_p + C_\varepsilon\|u\|_0$, and $\|u\|_0 \leq C\|f\|_0$; absorbing the term $\varepsilon\|u\|_p$ proves (8.17). □

We apply the lemma with $\mathcal{H} = \mathbb{V}_{0,b}$, $\mathcal{H}^p = \mathbb{V}_{0,b+p}$, $\Lambda = |D_\varphi|$ and $R = \mathscr{R}_{\mathbf{x}}$. By (8.13) with $t=0$, the operator $\mathscr{R}_{\mathbf{x}}$ maps every $\mathbb{V}_{0,s}$ into itself, and its commutator with Λ satisfies (8.16) with fixed constants a_p and with $c_p = C_p\,\mathfrak{n}_{b+p+d_*}$; in particular c_0 is bounded on the neighbourhood. Hence $\widetilde{F} = (I+\mathscr{R}_{\mathbf{x}})^{-1}F$ has all the angular regularity of F, and

$$\|\widetilde{F}\|_{0,s} \leq C_s\big[\|F\|_{0,s} + (1+\mathfrak{n}_{s+d_*})\,\|F\|_{0,b}\big], \qquad s \geq b. \tag{8.19}$$

Step B: coefficient weight of $\widetilde{F}$. Use the identity $\widetilde{F} = F - \mathscr{R}_{\mathbf{x}}\widetilde{F}$ for the function already constructed. By (8.13) the last term has every coefficient weight once the angular regularity of $\widetilde{F}$ is known. Hence

$$\|\widetilde{F}\|_{t,s} \leq C_{t,s}\big[\|F\|_{t,s} + (1+\mathfrak{n}_{t+s+d_*})\,\|F\|_{0,b}\big]. \tag{8.20}$$

Step C: radial derivatives of W. The corrected source $\widetilde{F}$ is controlled only in its angular derivatives. For the radial derivatives of W we therefore use the equation $\mathbb{P}_{\mathbf{x}}\mathbf{W} = F$ with the *original* source. Subtract the three scalar columns,

$$F^{\circ} = F - a_L\, s_{\mathbf{x}} - \mu\, k^{c}_{\mathbf{x}} - \lambda\, c_{*},$$

and rescale every half of a negative component to unit radial length with a variable y; the positive pieces are already normalized, and the rescaling factors are coefficient powers. Solving the equation for its highest radial derivative gives on every piece

$$\partial_y^2 W = \sum_{(i,j)\in I} A_{ij}\, \partial_y^i \partial_\varphi^j W + B\, F^{\circ}, \qquad I = \{(1,1),(0,2),(1,0),(0,1),(0,0)\}. \tag{8.21}$$

On a negative half this is the equation $W_{\xi\xi} = \xi W_{\varphi\varphi} - VW + F^{\circ}$. On a positive piece the coefficient of ∂_y^2 is bounded below by $\mathcal{E}_m^{-C}$ on the whole neighbourhood. The coefficients A_{ij} and B obey the calculus of Proposition 7.11, with a fixed loss d_{geo} of derivatives of the state. Differentiating, for $a \geq 2$,

$$\begin{aligned}\partial_y^a \partial_\varphi^b W = {} & \sum_{(i,j)\in I} \sum_{p=0}^{a-2} \sum_{q=0}^{b} \binom{a-2}{p} \binom{b}{q} (\partial_y^p \partial_\varphi^q A_{ij})\, \partial_y^{a-2-p+i} \partial_\varphi^{b-q+j} W \\ & + \partial_y^{a-2} \partial_\varphi^b (B\, F^{\circ}).\end{aligned} \tag{8.22}$$

Every term on the right contains at most $a-1$ radial derivatives of W and has total order at most $a+b$, and a derivative that falls on a coefficient lowers the order of the remaining factor by the same amount. Iterating, the expansion terminates in angular derivatives of W, angular derivatives of $\partial_y W$, and derivatives of F°; at total order s their orders are at most s, $s-1$ and $s-2$. Starting from the angular regularity and the first radial derivative, which are known, this identity proves the existence of all mixed derivatives in turn.

To make this quantitative, fix a base order $b_1 \geq 4$ and for $s \geq b_1 + 2$ put

$$\mathfrak{a}_{s,m} = \|W_m\|_{L^2_y H^s_\varphi} + \|\partial_y W_m\|_{L^2_y H^{s-1}_\varphi} + \|F^{\circ}_m\|_{H^{s-2}_{y,\varphi}}.$$

The finite expansion of (8.22) gives

$$\|W_m\|_{H^s_{y,\varphi}} \leq C_s\, \mathcal{E}_m^{C_s} \left[\mathfrak{a}_{s,m} + (1 + \mathfrak{n}_{s+d_{\mathrm{geo}},m})\, \mathfrak{a}_{b_1,m}\right]. \tag{8.23}$$

Indeed, in each term of the expansion the orders of the derivatives of the coefficients and of the remaining factor add up to at most s. This is the situation of Lemma 7.2(ii), applied after extension to the full cylinder: one factor is taken in the norm of order s, or $s + d_{\mathrm{geo}}$ for a coefficient, and all the others in norms of the fixed order b_1. Where a factor is known only in the mixed norms of $\mathfrak{a}_{s,m}$, interpolate its intermediate norms between the corresponding entries of $\mathfrak{a}_{b_1,m}$ and $\mathfrak{a}_{s,m}$ by Lemma 7.2(i). The Sobolev embedding is used at the fixed order b_1 only. The reciprocal of the leading coefficient A is $A_*^{-1}(1 + (A - A_*)/A_*)^{-1}$, where $|A - A_*| \leq \frac{1}{2} A_*$; so Lemma 7.2(iii) applies to it with $\Phi(x) = (1+x)^{-1}$ on $|x| \leq \frac{1}{2}$.

The exponent C_s of the reference coefficients in (8.23) may grow with s. It is important to absorb the *complete* product only after the expansion: $\mathcal{E}_m^{C_s} \leq C_s M_m$ costs one power of the weight, whereas paying for each step of the recursion separately would not give a fixed loss. Define the mixed norms

$$\left(\|Z\|^{\mathrm{mix}}_{t,s}\right)^2 = \sum_m M_m^{2t}\, H_m^{-2\beta} \sum_{a+b\leq s} \|\partial_y^a \partial_\varphi^b Z_m\|_2^2.$$

The first entry of $\mathfrak{a}_{s,m}$ is controlled by (8.4). For the second, use the bound for the first radial derivative in Lemma 8.2 at the angular order $s-1$; its one additional angular derivative requires exactly the order s of $\widetilde{F}$. Summing (8.23) over the pieces yields

$$\|W\|^{\mathrm{mix}}_{t,s} \le C_{t,s}\big[\|\widetilde{F}\|_{t+L_0+1,s} + \|F\|^{\mathrm{mix}}_{t+1,s-2} + (1+\mathfrak{n}_{t+s+d_{\mathrm{tot}}})\,\|F\|^{\mathrm{mix}}_{0,b_1}\big], \tag{8.24}$$

where the fixed number d_{tot} covers L_0+1, d_{geo} and the base and embedding orders. The scalar columns are compactly supported and satisfy the same estimates, and the scalars themselves are bounded by the base inverse. In the term with the high norm of the state, the weight M^{t+L_0+1} is placed on the localized norm of the state, and the low norm of the field is taken in the output space of the reference inverse, with weight $-L_0$. The order b_0 controls the finitely many base norms that occur, which is part of requirement (N5); the finitely many orders below b_1+2 follow from the same computation.

Conclusion. Insert (8.20) into the first term of (8.24). The indices of weight and of derivatives in that term add up to $t+s+L_0+1$, and in the source term to $t+s-1$. So an estimate for all $t+s\le r$ loses a fixed number of orders of the grading, independent of r. The transformations between canonical and original variables have fixed tame losses by Proposition 7.11, and the jets match across the roots as explained in Section 7.9. All these losses are included in d_*, and we obtain (8.1).

Undoing the transformations, $\mathbf{W}$ corresponds to a height η and scalars μ,λ with

$$\widehat{L}_{\mathbf{x}}\eta + \mu\, k_{\mathbf{x}} + \lambda\,\psi = f, \qquad \ell_*\eta = 0.$$

Integrate this equation over the annulus. The fluxes of η vanish at the endpoints, so $\mathsf{Q}\widehat{L}_{\mathbf{x}}\eta = 0$; and $\mathsf{Q}k_{\mathbf{x}} = 0$, by differentiating the conservation law of Lemma 3.3 with respect to ν. Hence $\lambda = \mathsf{Q}f$, which is zero for $f\in\mathcal{Y}_\beta$.

End of the proof of Proposition 8.1. Existence and the estimate (8.1) have now been proved. The factorization (8.14) on $\mathbb{D}_*$ shows that $G_{\mathbf{x}}$ is a two-sided inverse, and uniqueness among smooth solutions follows because $\mathbb{D}_*$ contains all of them (Lemma 8.2). Continuity in the state follows from the resolvent identity, first in the base space and then, by the higher estimates, in every order. The coefficients form a smooth tame family. A smooth tame family of linear maps which has unique solutions and a continuous tame inverse has a smooth tame inverse (Theorem 9.3 below); its hypotheses, uniqueness and continuity, are exactly what has just been verified. □

9 From the linear seed to a nonlinear solution

We can now explain why the unknowns were chosen as in Table 3, and solve the system. The height solves the curvature equation. The compact metric parameter ν removes its obstruction in the second mode. One control κ_m per negative component adjusts the mean travel angle, and the odd charts adjust the oscillating part of the return maps; these two tasks are separated by taking the angular mean. The return residual depends on the height only through qu, so the feedback from the return equations to the height equation carries the small factor q, and the coupled linear system is solved by a perturbation of a triangular one.

Objects of the full system are written in bold type: the state $\mathbf{x}$, the residual $\mathbf{R}$, the approximate inverse $\mathbf{V}$ and the error $\mathbf{Q}$.

9.1 The residual and its derivative

For a state $\mathbf{x} = (u, \nu, \kappa, h_L)$ as in (7.17) and a parameter $q \in \mathbb{R}$ let

$$\mathbf{R}(\mathbf{x}, q) = (\mathcal{A}, \mathfrak{e}), \qquad \mathcal{A} = \mathcal{F}_q(w, u), \qquad \mathfrak{e} = \mathfrak{e}(qu, \kappa, h_L), \tag{9.1}$$

where w is the profile (7.1) and $\mathfrak{e} = (\mathfrak{e}_m)_m$ is the return residual (7.25) on every negative component. The return residual depends on the height through $Z = qu$, on κ and on h_L; it does not depend on ν or on the endpoint constant of u, because χ_* and the derivative of χ_L are supported in positive components. The spaces of increments and of residuals are

$$\mathfrak{X} = (\mathcal{X}_\beta \cap \ker \ell_*) \times \mathbb{R} \times \mathcal{K}_\beta \times \mathcal{Z}_{\beta,\mathrm{odd}}, \qquad \mathfrak{Y} = \mathcal{Y}_\beta \times \mathcal{Z}_{\beta,\mathrm{even}}.$$

The second factor of $\mathfrak{Y}$ includes the angular means; the first incorporates the conservation law of Lemma 3.3. By (3.20) and the calculus of Section 7, $\mathbf{R}$ is a smooth tame map into $\mathfrak{Y}$ on the open set $\mathfrak{N}$ of (7.19). It vanishes at

$$q = 0, \qquad \mathbf{x}_0 = (\Phi_*, 0, 0, \mathrm{id}),$$

since $\mathscr{L}_{w_*}\Phi_* = 0$ and the return maps of the reference profile are the rotations $R_{\pm T_m}$.

The derivative of the return residual. Write

$$C^\circ_{\mathbf{x}} = D_Z \mathfrak{e}, \qquad R^\circ_{\mathbf{x}} = \langle C^\circ_{\mathbf{x}} \rangle, \qquad \mathcal{D}_{\mathbf{x}} = D_\kappa \mathfrak{e}, \qquad \mathsf{J}_{\mathbf{x}} = \langle \mathcal{D}_{\mathbf{x}} \rangle. \tag{9.2}$$

The residual on the component m depends only on the height near that component, on its chart and on κ_m. So $\mathsf{J}_{\mathbf{x}}$ is a diagonal matrix. At a state with radial height and trivial charts, $\mathfrak{e}_m = T_m(w) - T_m(w_*)$, and by the normalization of the controls in Lemma 7.6, $\mathsf{J} = I$ at the reference. On a neighbourhood of low order the entries of $\mathsf{J}_{\mathbf{x}}$ stay uniformly away from zero, and $\mathsf{J}_{\mathbf{x}}^{-1}$ is smooth tame.

At the reference state, the mean R°_0 of the derivative with respect to the height annihilates all nonzero angular frequencies, by invariance under rotation, and on a radial increment Z it is

$$(R^\circ_0 Z)_m = -\int v_0 \, \frac{1 + w_*^2}{w_*} \, Z_r \, dr, \tag{9.3}$$

the integral being over the mth negative component. Indeed, for a radial height $h + Z$ the characteristic speed (3.25) is $v_0 \, w_* \sqrt{D}/(w_* + Z_r)$ with $D = 1 - 2w_* Z_r - Z_r^2$, and its logarithmic derivative at $Z = 0$ is $-(1 + w_*^2) Z_r / w_*$. In particular $R^\circ_0 \Phi_* = 0$.

The corrected linear system. Let $K_{\mathbf{x}} = D_\kappa \mathcal{A}$ and, as before, $k_{\mathbf{x}} = D_\nu \mathcal{A}$. For an increment $(\eta, \mu, \varkappa, \psi \circ h_L)$ of the state (u, ν, κ, h_L) and a right side $(f, g) \in \mathfrak{Y}$ we consider

$$\begin{aligned} \widehat{L}_{\mathbf{x}} \eta + k_{\mathbf{x}} \, \mu + K_{\mathbf{x}} \, \varkappa &= f, \qquad \ell_* \eta = 0, \\ q \, C^\circ_{\mathbf{x}} \eta + \mathcal{D}_{\mathbf{x}} \, \varkappa + \mathcal{M}_T \psi &= g. \end{aligned} \tag{9.4}$$

This is the derivative of $\mathbf{R}$ with respect to the state, except for three terms that have been dropped: the two residual corrections in (7.32) and the term $\mathcal{Q}_h$ in (7.27). Each of them is proportional to $\mathcal{A}$ or to $\mathfrak{e}$. The structure of the system is shown in Figure 16.

	η	μ	$\varkappa$	ψ	
curvature equation	$\widehat{L}$	k	K	0	
return equations	qC°	0	$\mathcal{D}$	$\mathcal{M}_T$	$\langle \mathcal{D} \rangle = \mathsf{J}, \quad \langle \mathcal{M}_T \psi \rangle = 0$

Figure 16: The corrected linear system and its four columns: the increments $\eta, \mu, \varkappa, \psi$ of the height, of ν, of the controls κ and of the charts. Taking the mean of the second row removes the chart column. The feedback from the second row to the height is multiplied by q.

9.2 The control column gains all coefficient weights

Let $G_{\mathbf{x}}$ be the inverse of Proposition 8.1 and put $U_{\mathbf{x}} = G_{\mathbf{x}} K_{\mathbf{x}}$: the response (η, μ) of the first equation to an increment of the controls. The natural space for increments of the controls is

$$B_\beta = \Big\{ \varkappa = (\varkappa_m) : \ \|\varkappa\|_{B_\beta}^2 = \sum_m H_m^{-2\beta} |\varkappa_m|^2 < \infty \Big\},$$

which has no weight M at all.

Lemma 9.1 (Smoothing by the control column)**.** *For every s, with a fixed loss d_*,*

$$\|K_{\mathbf{x}} \varkappa\|_{\mathcal{Y}_\beta, s} + \|U_{\mathbf{x}} \varkappa\|_{\mathcal{X}_\beta, s} + |(U_{\mathbf{x}} \varkappa)_\mu| \le C_s \, (1 + \mathfrak{n}_{s+d_*}) \, \|\varkappa\|_{B_\beta}, \tag{9.5}$$

where the middle term is the norm of the height component of $U_{\mathbf{x}} \varkappa$ and the last term is its scalar component.

Proof. Step 1: two decaying factors. Differentiating the formula (3.20) for $\mathcal{F}_q$ with respect to the controls gives

$$K_{\mathbf{x}} \varkappa = \int_0^1 D_\kappa \big[\, r \, D_z \mathcal{N}_h(h + tqu) \big] [\varkappa] \, u \, dt,$$

where $h' = w$ and the derivative D_κ acts on w and on h. Every term of the result contains the increment of the control and a derivative of positive order of the height u: the curvature residual does not change when a constant is added to the height. This is true also at $q = 0$, where the derivative is that of $\mathscr{L}_w u$. On the component m write $\varkappa_m = H_m^\beta b_m$. The derivative of the height contributes another factor H_m^β; the endpoint constant of u does not contribute, because the supports of the controls lie in negative components. After division by the weight H_m^β of the source space a factor H_m^β remains.

Step 2: summation. At the order s, all coordinate, cutoff and reciprocal factors and the derivatives of the control χ_m together cost a coefficient power $\mathcal{E}_m^{C_s}$ and a fixed power of M_m; the remaining factor H_m^β bounds their product, by (7.3). The product estimate (7.8) puts one factor of the state at the high order. The supports of the controls are disjoint, so summing over m gives the bound for $K_{\mathbf{x}}$ with $(\sum_m |b_m|^2)^{1/2} = \|\varkappa\|_{B_\beta}$. Composition with $G_{\mathbf{x}}$ adds its fixed loss. □

9.3 Solving the corrected system

Proposition 9.2 (The approximate right inverse)**.** *On $\mathfrak{N}$ the system (9.4) has for every $(f, g) \in \mathfrak{Y}$ the solution*

$$\begin{aligned} \varkappa &= (I - q \, \mathsf{J}^{-1} R^\circ U)^{-1} \, \mathsf{J}^{-1} \, (\langle g \rangle - q \, R^\circ G f), \\ (\eta, \mu) &= G f - U \varkappa, \\ \psi &= \mathcal{M}_T^{-1} (g - q \, C^\circ \eta - \mathcal{D} \varkappa), \end{aligned} \tag{9.6}$$

where the subscript $\mathbf{x}$ *is omitted and* R°, C° *act on the height component of* G *and* U. *The map*

$$\mathbf{V}(\mathbf{x}, q)\,(f, g) = (\eta, \mu, \varkappa, \psi \circ h_L)$$

is smooth tame, and linear in (f, g).

Proof. Step 1: elimination. The first row gives $(\eta, \mu) = Gf - U\varkappa$. Taking the angular mean of the second row removes $\mathcal{M}_T\psi$ and gives

$$(I - q\,\mathsf{J}^{-1}R^\circ U)\,\varkappa = \mathsf{J}^{-1}\,(\langle g\rangle - q\,R^\circ G f).$$

By Lemma 9.1, $R^\circ U$ is a bounded operator from B_β to the *same* space B_β, with norm controlled by finitely many norms of the state. Requirement (N6) is that b_0 controls these norms and that ϵ_0, which bounds $|q|$ because $|q| \le \mathfrak{n}_{b_0}$, is so small that $\|q\,\mathsf{J}^{-1}R^\circ U\|_{B_\beta \to B_\beta} < \frac{1}{2}$. This gives the first line of (9.6), and the second follows. By the equation for $\varkappa$, the argument of $\mathcal{M}_T^{-1}$ in the third line has mean zero on every component, so (7.28) applies.

Step 2: regularity. Higher norms of $\varkappa$ are read off from the identity

$$\varkappa = \mathsf{J}^{-1}(\langle g\rangle - q\,R^\circ G f) + q\,\mathsf{J}^{-1}R^\circ U\varkappa,$$

satisfied by the solution already found. By Lemma 9.1 the last term has all higher norms, bounded by $\|\varkappa\|_{B_\beta}$ and one high norm of the state; the first term has the norms of the sources with a fixed loss. So $\varkappa$ lies in all the spaces $\mathcal{K}_{\beta,s}$, with tame bounds, and substitution gives the estimates for η, μ and ψ.

Step 3: smoothness. The resolvent identity gives continuity in the base space, and the identity of Step 2 with its estimates gives continuity in every order. By Theorem 9.3 below, applied to the inverse in the first line, all three formulas are smooth tame. □

9.4 The quadratic error and the Nash–Moser theorem

We use two theorems of Hamilton. Tame spaces and smooth tame maps were defined in Section 7.2. Implicit function theorems which compensate a loss of derivatives by smoothing go back to Nash [Nas56] and Moser [Mos66]. Hörmander gave versions in [Hör76, Hör85]; the second is the one cited in the Nash–Moser–Hörmander iteration for local isometric embedding in [HHL03, Section 6], see also [HH06]. We use Hamilton's formulation [Ham82] because it allows an inverse of the linearization up to a quadratic error.

Theorem 9.3 (Hamilton: families of inverses)**.** *Let* U *be an open subset of a tame space, let* $\mathfrak{H}, \mathfrak{K}$ *be tame spaces, and let* $L : U \times \mathfrak{H} \to \mathfrak{K}$ *be a smooth tame map which is linear in its second argument. Suppose that* $L(x)$ *is bijective for every* $x \in U$, *and that the family of inverses* $(x, y) \mapsto L(x)^{-1}y$ *is continuous and tame. Then this family is smooth tame.*

This is [Ham82, Part II, Theorem 3.1.1]. Existence, uniqueness, continuity and tame bounds for the inverses are hypotheses; the theorem supplies their derivatives. It was used in this way in Propositions 8.1 and 9.2.

Theorem 9.4 (Hamilton: implicit function theorem with quadratic error)**.** *Let* $\mathfrak{X}, \mathfrak{P}, \mathfrak{Y}$ *be tame spaces and let* $\mathbf{R} : U \subset \mathfrak{X} \times \mathfrak{P} \to \mathfrak{Y}$ *be a smooth tame map on an open set. Suppose there are smooth tame maps* $\mathbf{V}(\mathbf{x}, q)\,\mathbf{y}$ *with values in* $\mathfrak{X}$, *linear in* $\mathbf{y} \in \mathfrak{Y}$, *and* $\mathbf{Q}(\mathbf{x}, q)\,[\mathbf{r}, \mathbf{y}]$ *with values in* $\mathfrak{Y}$, *bilinear in* $(\mathbf{r}, \mathbf{y}) \in \mathfrak{Y} \times \mathfrak{Y}$, *such that*

$$D_{\mathbf{x}}\mathbf{R}(\mathbf{x}, q)\,\mathbf{V}(\mathbf{x}, q)\,\mathbf{y} = \mathbf{y} + \mathbf{Q}(\mathbf{x}, q)\,[\mathbf{R}(\mathbf{x}, q), \mathbf{y}] \tag{9.7}$$

for all $(\mathbf{x}, q) \in U$ *and all* $\mathbf{y}$. *Then near every zero* $(\mathbf{x}_0, q_0)$ *of* $\mathbf{R}$ *there is a smooth tame map* $q \mapsto \mathbf{x}(q)$ *with* $\mathbf{x}(q_0) = \mathbf{x}_0$ *and* $\mathbf{R}(\mathbf{x}(q), q) = 0$.

This is [Ham82, Part III, Theorem 3.3.1]. The point of the hypothesis is that $\mathbf{V}$ need only be an inverse *up to an error proportional to the residual*; but the identity must hold exactly, with the two arguments of $\mathbf{Q}$ independent. The theorem gives a solution, not its uniqueness, which we do not need.

Proposition 9.5 (The quadratic error identity)**.** *For* $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in \mathfrak{Y}$ *and* $\mathbf{y} = (f, g) \in \mathfrak{Y}$*, let* $(\eta, \mu, \varkappa, \psi)$ *be given by* (9.6) *and define*

$$\mathbf{Q}(\mathbf{x}, q)\,[\mathbf{r}, \mathbf{y}] = \Big(q\,\partial_\theta(B\,\mathbf{r}_1\,\eta_\theta) + \mathscr{E}_{\mathbf{x}}(\mathbf{r}_2)\,\eta, \quad \mathcal{Q}_h(\mathbf{r}_2, \psi) \Big),$$

where in $\mathcal{Q}_h$ *the points of evaluation in* (7.27) *are those of the actual state and only the explicit factors* $\mathfrak{e}$*,* $\mathfrak{e}'$ *are replaced by* $\mathbf{r}_2$*,* $\mathbf{r}_2'$*, as was done for* $\mathscr{E}_{\mathbf{x}}$ *in Proposition 7.9. Then* $\mathbf{Q}$ *is a smooth tame family of bilinear maps* $\mathfrak{Y} \times \mathfrak{Y} \to \mathfrak{Y}$*, and* (9.7) *holds.*

Proof. For the increments (9.6), the identities (7.32) and (7.27) give the true derivative

$$D_{\mathbf{x}}\mathbf{R}\,(\eta, \mu, \varkappa, \psi \circ h_L) = \Big(f + q\,\partial_\theta(B\,\mathcal{A}\,\eta_\theta) + \mathscr{E}_{\mathbf{x}}(\mathfrak{e})\,\eta, \quad g + \mathcal{Q}_h(\mathfrak{e}, \psi) \Big), \tag{9.8}$$

which is (9.7) with $\mathbf{r} = \mathbf{R} = (\mathcal{A}, \mathfrak{e})$. The increments are linear in $\mathbf{y}$, so $\mathbf{Q}$ is bilinear.

It takes values in $\mathfrak{Y}$. Its first component has integral zero: one term is an angular derivative, the other is in divergence form with compact support inside the negative components. Its second component is even. Finally, every term is a product of two factors each of which decays like H^β, so by the calculus of Section 7 and the estimates for flows, $\mathbf{Q}$ is smooth tame with values in the spaces of weight β. □

Theorem 9.6 (The solution branch)**.** *There are* $q_1 > 0$ *and a smooth tame map*

$$q \mapsto \mathbf{x}(q) = (u_q, \nu_q, \kappa_q, h_{L,q}), \qquad |q| < q_1, \tag{9.9}$$

with $\mathbf{x}(0) = (\Phi_*, 0, 0, \mathrm{id})$ *and* $\mathbf{R}(\mathbf{x}(q), q) = 0$*. In particular* $u_q = \Phi_* + O(q)$ *in every norm of* $\mathcal{X}_\beta$*.*

Proof. We verify the hypotheses of Theorem 9.4 with $\mathfrak{P} = \mathbb{R}$; they are listed in Table 5. The order of the choices matters. First the reference profile, the spaces and the finite order b_0 are fixed; then ϵ_0; then the range of q, so that the control equation in Proposition 9.2 can be solved. At a zero of $\mathbf{R}$ the identity (9.7) shows that the derivative with respect to the state is surjective, with right inverse $\mathbf{V}$. □

This solves the nonlinear system in the scale of weight β. The next section extracts stronger information from the equations themselves, without a second iteration in another space.

10 The endpoint gain and the closed surface

The branch of Theorem 9.6 was constructed with the weight H^β, because the reference inverse is available only for exponents below one. This already suffices for the geometry: every element of $\mathcal{X}_\beta$ is flat at the endpoints. In this section we first show that the corrections to the metric are in fact of size $q^2 H^{2\beta}$, which quantifies how the final metric differs from the reference one, and then we state the annular theorem in its precise form and prove Theorems B and A.

The gain comes from two identities for angular means. The first recovers the mean of the height from an expression which is quadratic in the oscillating part. The second recovers the controls κ from that mean and from another quadratic expression. A product of two factors of size H^β has size $H^{2\beta}$, and $2\beta > 1$. In this section a quantity of size H^β is called *weak* and one of size $H^{2\beta}$ *strong*; both are flat at the endpoints.

Hypothesis	Where it is verified
Tame spaces	Lemma 7.4; the scalar factors and the affine slice $\ell_* u = 1$ are tame direct summands.
Open domain	The set $\mathfrak{N}$ of (7.19), defined by norms of one finite order (Lemmas 7.7 and 7.8, Proposition 7.11).
R smooth tame, with values in $\mathfrak{Y}$	Formula (3.20) and Lemma 3.3 for $\mathcal{A}$; the calculus of Lemmas 7.2, 7.3 and row 3 of Table 4 for $\mathfrak{e}$.
A zero	$(\mathbf{x}_0, 0)$: $\mathscr{L}_{w_*}\Phi_* = 0$, $\ell_*\Phi_* = 1$, and the reference return maps are rotations.
V smooth tame and linear	Proposition 9.2, which rests on Proposition 8.1.
Q smooth tame and bilinear	Proposition 9.5, with Proposition 7.9 for $\mathscr{E}_{\mathbf{x}}$.
The identity (9.7)	(9.8).

Table 5: The hypotheses of Theorem 9.4 and where they are verified.

10.1 Two identities for means

The mean of the height. Let $\mathfrak{F}_w(U)$ be the connection period (3.22). By (3.23) its derivative at $U = 0$ is $w\langle U_r\rangle$. Let

$$\mathcal{B}_{w,U}(v_1, v_2) = \int_0^1 (1-t)\, D_U^2 \mathfrak{F}_w(tU)[v_1, v_2]\, dt \tag{10.1}$$

be its Taylor remainder of second order. For a solution the period vanishes identically, because its radial derivative is the mean of the residual and it is zero at the endpoints. Hence, with $u^{\mathrm{av}} = \langle u\rangle$,

$$0 = \mathfrak{F}_w(qu) = q\, w\, (u^{\mathrm{av}})' + q^2\, \mathcal{B}_{w,qu}(u,u), \qquad \text{that is,} \qquad (u^{\mathrm{av}})' = -\frac{q}{w}\, \mathcal{B}_{w,qu}(u,u). \tag{10.2}$$

The mean of the return residual. Use the additive coordinates $\mathbf{v} = (Z, \kappa, a)$ with $Z = qu$ and $h_L = \mathrm{id} + a$, and put

$$\mathcal{C}(\mathbf{v}) = \langle \mathfrak{e}(\mathbf{v})\rangle, \qquad \mathcal{B}^\circ_{\mathbf{v}}(v_1, v_2) = \int_0^1 (1-t)\, D^2\mathcal{C}(t\mathbf{v})[v_1, v_2]\, dt.$$

By Section 9 the derivative of $\mathcal{C}$ at $\mathbf{v} = 0$ is the identity on κ, zero on the odd increments of the charts, and R_0° on the mean of the height. At a solution therefore

$$0 = \kappa + q\, R_0^\circ u^{\mathrm{av}} + \mathcal{B}^\circ_{\mathbf{v}}(\mathbf{v}, \mathbf{v}). \tag{10.3}$$

Formulas (10.2) and (10.3) already explain the exponent 2β. What remains is to verify the corresponding estimates in the stronger topology, with all derivatives in the parameter, so that the conclusion holds for the whole branch.

10.2 Two weak factors give one strong remainder

Fix a convex neighbourhood of $(U, \nu, \kappa, a) = (0, 0, 0, 0)$ in the spaces of weight β, small in one norm of finite order, on which D and Q/w are positive and the charts are uniformly increasing. Both Taylor remainders above are integrals over segments in this neighbourhood; the map $\mathcal{C}$ does not depend on ν.

Lemma 10.1 (Quadratic remainders in the doubled weight)**.** *The maps $(w, U, v_1, v_2) \mapsto w^{-1}\mathcal{B}_{w,U}(v_1, v_2)$ and $(\mathbf{v}, v_1, v_2) \mapsto \mathcal{B}^{\circ}_{\mathbf{v}}(v_1, v_2)$ are smooth tame with values in the space $\mathcal{X}^0_{2\beta}$ of radial functions and in $\mathcal{K}_{2\beta}$, respectively. For every number k of derivatives with respect to the state there is a loss d_k, independent of the order s, such that*

$$\begin{aligned} &\big\| D^k \mathcal{B}[\eta_1, \dots, \eta_k]\,(\eta_{k+1}, \eta_{k+2}) \big\|_{2\beta,s} \\ &\quad \le C_{s,k} \Big[\sum_{i=1}^{k+2} \|\eta_i\|_{s+d_k} \prod_{j \neq i} \|\eta_j\|_{b+d_k} + (1 + \|\mathbf{y}\|_{s+d_k}) \prod_{j} \|\eta_j\|_{b+d_k} \Big], \end{aligned} \tag{10.4}$$

where $\mathcal{B}$ is either of the two maps, $\mathbf{y}$ is the state, the norms of the arguments are those of weight β, and compact scalars are measured by their absolute values. The same bound holds for all derivatives of $\mathcal{C}$ of order at least two.

Proof. Step 1: counting weak factors. On the component m, every increment of a control or of a chart, and every derivative of positive order of a height, carries a factor H_m^{β}. The endpoint constants of heights are harmless: only derivatives of positive order enter, and those of χ_L are supported in a compact positive region and vanish on the supports of all controls on negative components. Differentiation does not destroy a weak factor: for $u = H^{\beta}\widetilde{u}$,

$$H^{-\beta}\, \mathsf{D}u = \mathsf{D}\widetilde{u} + \beta\, (\mathsf{D} \log H)\, \widetilde{u},$$

and $\mathsf{D} \log H$ is bounded. Products and restrictions to the pieces cost fixed numbers of derivatives and reference factors $\mathcal{E}_m^{C_{s,k}}$. So it suffices to show that every term of the expressions in question contains two weak factors, one from each of the two distinguished arguments: these give the weight $H_m^{2\beta}$ of the output; the highest derivative is placed on one argument or on one coefficient of the state; the complete product of coefficients is absorbed into one power of M; and finite overlap together with $\|ab\|_{\ell^2} \le \|a\|_{\ell^2} \|b\|_{\ell^\infty}$ sums over the components.

Step 2: the period. With $k = w + U_r$, $p = U_\theta / r$ and $E = 1 + w^2$, the integrand of $\mathfrak{F}_w(U)$ is

$$\frac{1 + k p_\theta / (1 - p^2)}{\sqrt{E(1 - p^2) - k^2}} - 1,$$

a smooth function of w and of the derivatives $(U_r, U_\theta / r, U_{\theta\theta} / r)$ on the neighbourhood. Each second derivative with respect to U contains one such derivative from each argument, and further differentiation with respect to the state does not remove them. The factor $w^{-1} = w_*^{-1} \exp(-\nu\chi_* - \sum_m \kappa_m \chi_m)$ is part of the product of coefficients; it costs a coefficient power, not an endpoint exponent.

Step 3: flows. For the map $\mathcal{C}$ consider a normalized characteristic flow $\dot{\Theta} = G(t, \Theta; \mathbf{y})$, in the form (7.34) near a root. Its first and second variations with respect to the state satisfy

$$\dot{\Theta}_i = G_\theta\, \Theta_i + G_i, \qquad \dot{\Theta}_{12} = G_\theta\, \Theta_{12} + G_{\theta\theta}\, \Theta_1 \Theta_2 + (G_1)_\theta\, \Theta_2 + (G_2)_\theta\, \Theta_1 + G_{12}.$$

The integrating factor of the homogeneous equation is bounded above and below, by Lemma 7.8. The source G_i contains one weak factor and G_{12} two, so by variation of constants Θ_i contains one and Θ_{12} two. Composition of flows does not change the weight of the component.

Step 4: inverses. If $\iota = h^{-1}$, then

$$\iota_i = -\frac{h_i \circ \iota}{h' \circ \iota}, \qquad \iota_{12} = -\frac{h_{12} \circ \iota + (h_1' \circ \iota)\, \iota_2 + (h_2' \circ \iota)\, \iota_1 + (h'' \circ \iota)\, \iota_1 \iota_2}{h' \circ \iota}. \tag{10.5}$$

The denominator is bounded below, and every term of the second numerator contains both variations. Apply this to f^{-1} and h_R^{-1} in (7.25); taking even or odd parts and means preserves the weights.

Step 5: higher variations. Label the directions of differentiation by a finite set I. In the chain rule for $G(t,\Theta;\mathbf{y})$, a subset $J\subset I$ acts directly on the state and the blocks B of a partition of $I\setminus J$ act on the flow, giving terms

$$(D_{\mathbf{y}}^{|J|}\partial_\theta^\ell G\,[\eta_j : j\in J])\circ\Theta\,\prod_B \Theta_B,$$

where ℓ is the number of blocks. Keep the term $G_\theta\,\Theta_I$ on the left side; all other terms involve only variations of lower order and contain every label exactly once. By induction both distinguished factors are retained, and the identity for inverses has the same structure. The additional weak factors are bounded in low norms. The tame estimates for products and compositions put only one factor at the high order, and the formulas for the coefficients, the regularization at the roots and the restrictions have fixed order; k variations add a cost depending on k but not on s. This proves (10.4).

Step 6: smoothness. These are estimates for the actual derivatives. Taylor's formula with integral remainder and one more variation gives continuity and differentiability in the norms of weight 2β, and by iteration smoothness in the stronger Fréchet topology. □

10.3 Recovering the mean and the controls

Lemma 10.2 (A primitive keeps the endpoint exponent)**.** *Let $\gamma>0$. For a radial function $f\in\mathcal{X}_\gamma^0$ let $\mathscr{P}f$ be its primitive which vanishes at the right endpoint. Then $\mathscr{P}f=v+a_L\chi_L$ with $v\in\mathcal{X}_\gamma^0$ and $a_L\in\mathbb{R}$, and $\mathscr{P}:\mathcal{X}_\gamma^0\to\mathcal{X}_\gamma$ is a tame linear map on radial functions.*

This is a Hardy inequality with weights [Muc72]. For the weights of this paper, which are exponential in the variable ϱ, the proof is short.

Proof. Near an endpoint, orient ϱ towards it. Then $(\log H)'\to -1$ in the variable ϱ, so far enough out

$$\frac{d}{d\varrho}\log H^\gamma \le -\frac{\gamma}{2}:$$

the ratio of H^γ at $\sigma\ge\varrho$ to its value at ϱ is at most $e^{-\gamma(\sigma-\varrho)/2}$. Near the right endpoint $\mathscr{P}f(\varrho)=-\int_\varrho^\infty f\,\ell\,d\sigma$, where the factor $\ell\le d_\partial$ comes from $dr=\ell\,d\varrho$ and is bounded. So the kernel of $\mathscr{P}$, conjugated by the weight $H^{-\gamma}$, is bounded by $Ce^{-\gamma(\sigma-\varrho)/2}$ for $\sigma\ge\varrho$ and vanishes otherwise. Its integrals in either variable are bounded, and Schur's test gives the bound in $L^2(d\varrho)$. One derivative of the primitive is ℓf; further derivatives with respect to ϱ fall on f, on the weight and on ℓ, whose logarithmic derivatives are bounded. A factor M^c with $c\ge0$ causes no loss, since $M(\varrho)\le M(\sigma)$ for $\sigma\ge\varrho$. Near the left endpoint the same argument applies to $\mathscr{P}f-a_L$, where $a_L=-\int_{r_-}^{r_+} f\,dr$ is a continuous functional of f, because this difference is the primitive which vanishes at the left endpoint. On the compact middle part integration is bounded in every Sobolev norm, and the derivative of χ_L is supported there. □

The mean. By (10.2) the mean of the height on the solution branch is given by the explicit map

$$\mathcal{U}(\mathbf{x},q)=-q\,\mathscr{P}[w^{-1}\mathcal{B}_{w,qu}(u,u)],\qquad u_q^{\mathrm{av}}=\mathcal{U}(\mathbf{x}(q),q).\tag{10.6}$$

By Lemma 10.1, $\mathcal{U}$ is smooth with values in $\mathcal{X}_{2\beta}$, and it is $O(q)$ in every norm; in particular the difference of the endpoint values of u_q is $O(q)$.

The controls. The operator R_0° of (9.3) involves only the derivative of its argument. It maps the part of $\mathcal{U}$ in $\mathcal{X}_{2\beta}^0$ to $\mathcal{K}_{2\beta}$ with a fixed loss, and annihilates the multiple of χ_L, whose derivative is supported in a positive component. By (10.3),

$$\kappa_q = -q\, R_0^\circ \mathcal{U}(\mathbf{x}(q), q) - \mathcal{B}^\circ_{\mathbf{v}(q)}(\mathbf{v}(q), \mathbf{v}(q)). \tag{10.7}$$

Since $\mathbf{v}(q) = O(q)$ on the branch, both terms on the right are $O(q^2)$ in every norm of $\mathcal{K}_{2\beta}$. The identity also shows that $q \mapsto \kappa_q$ is smooth with values in $\mathcal{K}_{2\beta}$, and that $\dot\kappa_0 = 0$, where the dot denotes the derivative with respect to q.

The compact parameter. Differentiating the equation $\mathcal{F}_q(w_q, u_q) = 0$ at $q = 0$,

$$\mathscr{L}_{w_*} \dot u_0 + \dot\nu_0\, k_* + K_*\, \dot\kappa_0 + \partial_q \mathcal{F}_q(w_*, \Phi_*)\big|_{q=0} = 0, \qquad \ell_* \dot u_0 = 0.$$

The last term is quadratic in the derivatives of Φ_*, which contain only the angular frequency two; products of $\cos 2\theta$ and $\sin 2\theta$ contain only the frequencies zero and four. So this term has no component in the second mode. Since also $\dot\kappa_0 = 0$, the second mode of the equation is the homogeneous bordered equation of Proposition 4.2, whose only solution is zero. Hence $\dot\nu_0 = 0$ and $\nu_q = O(q^2)$.

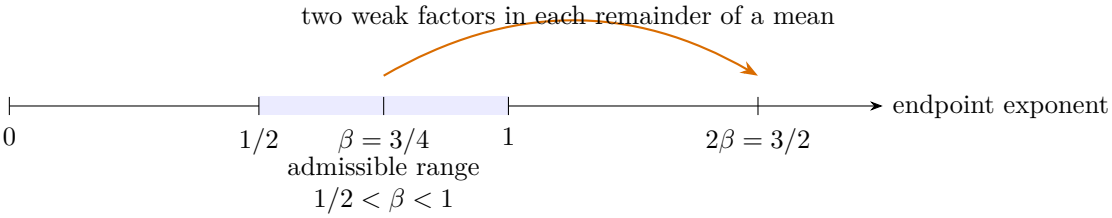


Figure 17: The two roles of the exponent $\beta = \frac{3}{4}$. Exponents near β and below one are needed for the reference inverse. The equations for the means multiply two factors of weight β and give an exponent above one, which yields the bound for the metric in Theorem 10.3. Flatness of the solution needs only a positive exponent.

10.4 The annular theorem

For a profile w let h_w denote the radial function with $h_w' = w$, with a fixed normalization of the additive constant.

Theorem 10.3 (A nonradial height with flat ends)**.** *Let w_*, ϕ_*, χ_* be as in Proposition 4.2 and let χ_m be the controls of Lemma 7.6. There are $q_0 > 0$ and a smooth family*

$$(u_q, \nu_q, \kappa_q, h_{L,q}), \qquad |q| < q_0,$$

through $(\Phi_*, 0, 0, \mathrm{id})$ at $q = 0$, such that the profile and the height

$$w_q = w_* \exp\Big(\nu_q \chi_* + \sum_m \kappa_{q,m} \chi_m\Big), \qquad z_q = h_{w_q} + q\, u_q$$

have the following properties, where $g_q = (1 + w_q^2)\, dr^2 + r^2\, d\theta^2$.

(i) (Equation.) *$|dz_q|_{g_q} < 1$ and $g_q - dz_q \otimes dz_q$ is flat; equivalently $\mathcal{N}_{h_{w_q}}(z_q) = 0$ with $D > 0$.*

(ii) (Profile.) *$w_q > 0$ on (r_-, r_+), and w_q extends smoothly by zero across both endpoints. The zeros of w_q' are those of w_*', and $w_q = w_*$ near each of them.*

(iii) (Height.) *z_q is even and π-periodic in θ, and has a constant extension at each endpoint; the two constants may differ. Moreover $\ell_* u_q = 1$, so z_q is not radial for $q \neq 0$.*

(iv) (Returns.) *On every negative component m, the two characteristic return maps $f_{q,m,\pm}$ of the linearized equation at z_q satisfy*

$$h_{R,q,m} \circ f_{q,m,\pm} \circ h_{L,q,m}^{-1} = R_{\pm T_m} \tag{10.8}$$

for odd increasing charts commuting with the translation by π, where T_m is the travel angle of w_*.

(v) (Estimates.) *For every s,*

$$\|u_q - \Phi_*\|_{\mathcal{X}_\beta,s} \le C_s\,|q|, \qquad |\nu_q| + \|\kappa_q\|_{\mathcal{K}_{2\beta},s} \le C_s\,q^2, \tag{10.9}$$

and the family is smooth in q for these topologies. In particular, for all $a \ge 0$ and $\epsilon > 0$,

$$|\partial_r^a (w_q - w_*)| \le C_{a,\epsilon}\,q^2\,H^{1-\epsilon}$$

near the endpoints, and $w_q - w_* = O(q^2)$ in every C^a norm.

Proof. The family is the branch of Theorem 9.6. Choose q_0 so that $[-q_0, q_0]$ lies in its domain; every seminorm of the branch is bounded on this compact interval, with constants depending on the order but with q_0 independent of the order.

Estimates. The first bound in (10.9) is contained in Theorem 9.6; the second and the smoothness in the stronger topology follow from (10.6), (10.7) and the bound for ν_q. On the tails χ_* vanishes, and differentiating the formula for w_q gives $|\partial_r^a(w_q - w_*)| \le C_a\,q^2\,H^{2\beta}E_0^{C_a}$; since $2\beta > 1$, (7.3) gives the stated bound. On the compact middle part use the smoothness of the family and $\nu_q = O(q^2)$.

Profile. The controls vanish near every turning circle, so the germ of the metric there is unchanged, and by Lemma 7.7 the sign of w_q' is that of w_*' elsewhere. Flatness of w_q follows from that of w_* and of the controls.

Equation and returns. At a zero of $\mathbf{R}$ both residual corrections vanish, so $\widehat{L}$ is the true linearization and its exact returns are (10.8). The inequality $D > 0$ of Lemma 7.7 is equivalent to $|dz_q|_{g_q} < 1$, and $\mathcal{A} = 0$ means that the curvature residual vanishes, which by Proposition 3.1 is the flatness of the remainder metric.

Height. The symmetry is that of the function spaces. Since $\ell_* u_q = 1$, the second angular mode of u_q is not zero. By Lemma 7.4 every element of $\mathcal{X}_\beta$ has constant extensions at the endpoints; the same holds for all derivatives in q. Finally h_{w_q} has constant extensions because w_q is flat. □

10.5 Proof of Theorems B and A

The construction involves two parameters with different roles, and it is worth separating them once more. The small parameter q was used to construct one nonradial isometric realization of one rotationally symmetric metric. The phase τ moves that realization.

Proof of Theorem B. Fix q with $0 < |q| < q_0$ and let $w = w_q$, $z = z_q$ be given by Theorem 10.3. Part (i) of that theorem is condition (i); parts (ii) and (iii) give conditions (ii) and (iii). □

Proof of Theorem A. By Section 2, Theorem B implies Theorem A. In detail: extend w by zero and z by its limiting constants to a slightly larger annulus. The remainder metric is Euclidean on the added collars, so the extended data satisfy the hypotheses of Theorem 2.5.

That theorem develops the graph, fills the inner disk, attaches the marked cap, and yields the family

$$F_\tau = R_{-2\pi\tau} \circ F \circ \varrho_{2\pi\tau}$$

on the disk, with the inclusion on the cap. On the collars the two rotations cancel pointwise; the marked cap makes two of the closed images noncongruent; and the same theorem gives embeddedness and the constancy of the area, of the enclosed volume and of the total oriented mean curvature. □

A Why a finite angular ansatz cannot work

This appendix is independent of the existence proof. It explains one feature of the construction: although the seed Φ_* has the single angular frequency two, the inverse of the linearized operator had to be constructed for *all* even frequencies. Proposition 3.2 ruled out a monotone flat end for the linearized equation. Another tempting simplification would be to look for the nonlinear height among functions with finitely many angular modes. The following proposition shows that this is impossible as soon as the height is required to be radial on a collar.

Proposition A.1. *Let $E > 0$ be a C^1 function on an interval $I \subset (0, \infty)$, and let z be a real C^2 solution of the Darboux equation (2.3) for the metric $E\,dr^2 + r^2\,d\theta^2$ on $I \times \mathbb{T}$. Suppose that*

$$z^2 = \sum_{|j| \le J} b_j(r)\, e^{ij\theta}, \qquad b_{-j} = \overline{b_j}, \qquad b_j \in C^2(I),$$

and that z is independent of θ for r in some nonempty open subinterval of I. Then z is independent of θ on all of I.

The hypothesis covers every height with finitely many angular modes, and every smooth square root of a trigonometric polynomial in θ.

The difficulty in the proof is the continuation through a zero of a Fourier coefficient. The highest coefficient satisfies an ordinary differential equation which is singular where the coefficient vanishes, and the coefficient is complex, so neither the standard uniqueness theorem nor an argument for real functions applies directly.

Proof. Step 1: a polynomial identity for $v = z^2$. Let $K = E'/(2rE^2)$ be the curvature of the metric. From $\nabla^2 v = 2z\,\nabla^2 z + 2\,dz \otimes dz$ and the expansion of a 2×2 determinant under a perturbation of rank one,

$$2v\,\det_g \nabla^2 v - \operatorname{cof}_g(\nabla^2 v)[dv, dv] = 8Kv^2 - 2Kv\,|dv|_g^2. \tag{A.1}$$

Indeed, on the left the terms containing $z^3 \operatorname{cof}_g(\nabla^2 z)[dz, dz]$ cancel, leaving $8z^4 \det_g \nabla^2 z$, and the Darboux equation gives the right side. No division by z was used, so the identity holds also where $z = 0$.

Step 2: the equation for the highest coefficient. Let $b = b_J$ be the coefficient of the highest frequency $J > 0$, and put $a = EJ^2/r$ and $d = E'J^2/(2r)$. In (A.1) the frequency $3J$ arises only from products of three copies of $b\,e^{iJ\theta}$. Write the entries of the Hessian of v as

$$A = v_{rr} - \frac{E'}{2E}\,v_r, \qquad B = v_{r\theta} - \frac{v_\theta}{r}, \qquad C = v_{\theta\theta} + \frac{r}{E}\,v_r,$$

so that, after multiplication by Er^2, the left side of (A.1) is $2v(AC-B^2)-(Cv_r^2-2Bv_rv_\theta+Av_\theta^2)$. Replacing $v, v_r, v_{rr}, v_\theta, v_{r\theta}, v_{\theta\theta}$ by $b, b', b'', iJb, iJb', -J^2b$ and including the cubic term of the right side, the coefficient of $e^{3iJ\theta}$ gives

$$\begin{aligned} 0 = (2bb' - ab^2)\, b'' - (b')^3 + ab\,(b')^2 - \frac{2a}{r}\, b^2 b' + \frac{2a}{r^2}\, b^3 \\ + d\, b^2 b' - \frac{2d}{r}\, b^3. \end{aligned} \tag{A.2}$$

In particular $b' = 0$ wherever $b = 0$.

Step 3: a first-order system where $b \neq 0$. On a component of $\{b \neq 0\}$ choose a continuous complex square root y of b. Substituting $b = y^2$ in (A.2) and dividing by $2y^4$,

$$(4y' - ay)\, y'' + a\Big(y' - \frac{y}{r}\Big)^2 + d\Big(yy' - \frac{y^2}{r}\Big) = 0. \tag{A.3}$$

Fix a compact subinterval of I and a constant $K_1 > 1 + \sup|a|$, and consider

$$U = \max(K_1|y|,\ |y'|) = \max\Big(K_1\sqrt{|b|},\ \frac{|b'|}{2\sqrt{|b|}}\Big).$$

We claim $|U'| \leq CU$ almost everywhere. Where $|y'| \geq K_1|y|$ we have $|4y' - ay| \geq 3|y'|$, and (A.3) gives $|y''| \leq C|y'|$. Where $K_1|y|$ is the larger term, its derivative is at most $K_1|y'| \leq K_1 U$ in absolute value. The maximum of two Lipschitz functions is Lipschitz, so the estimate holds across the points where the maximum switches. Consequently U is bounded above and bounded away from zero on the component, up to its endpoints in the compact subinterval.

Step 4: zeros are isolated and quadratic. Let r_0 be an endpoint, interior to I, of a component of $\{b \neq 0\}$. There $y \to 0$, so $U = |y'|$ nearby, and since $|y''| \leq C|y'|$ the derivative y' has a limit λ, which is not zero because U is bounded below. Hence

$$b'' = 2\,(y')^2 + 2\,y\,y'' \longrightarrow 2\lambda^2 \neq 0 \qquad (r \to r_0). \tag{A.4}$$

So r_0 is an isolated zero of b of order exactly two. Across such a zero U extends continuously, with the value $\sqrt{|b''|/2}$, and satisfies the same differential inequality.

Step 5: continuation. Consider the open set $\mathcal{O} = \{b \neq 0\} \cup \{b = 0,\ b'' \neq 0\}$. Suppose a component of $\mathcal{O}$ had a boundary point in the interior of I; there $b = b'' = 0$. If zeros of b accumulated at this point from inside the component, the lower bound for U at these quadratic zeros and the continuity of b'' would give a contradiction; if a last interval of $\{b \neq 0\}$ ended at this point, (A.4) would give a contradiction. Hence $\mathcal{O}$ is empty or equal to I. By assumption z^2 is radial on an open interval, so b vanishes there and $\mathcal{O} \neq I$. Therefore $b = 0$ on I.

Step 6: conclusion. Removing the highest coefficient and repeating the argument, all coefficients b_j with $j \neq 0$ vanish, so z^2 is radial. For each r the function $\theta \mapsto z(r,\theta)$ is continuous on the circle with constant square, hence constant. □

The proposition does not say that a single angular mode is useless: a single mode is exactly the right seed for the linearized problem. It says that the nonlinear correction of the seed cannot stay within any finite number of modes once it has to meet a radial collar.

B External results and the order of choices

The proof is self-contained except for the results listed in Table 6. The table records each of them in the form in which it is used, so that the reader can see exactly what is taken from the literature. The rigidity and flexibility theorems quoted in the introduction give context only; none of their conclusions is used.

Table 6: External results in the form used.

Result	Statement used	Where
Gauss equation and structure equations	For a surface with metric g and second fundamental form II in a three-manifold, $K_g = \overline{K} + \det_g \mathrm{II}$, with $\overline{K}$ the sectional curvature of the tangent plane. For an oriented orthonormal coframe with $d\eta_1 = \alpha \wedge \eta_2$, $d\eta_2 = -\alpha \wedge \eta_1$: $d\alpha = -K\,\eta_1 \wedge \eta_2$.	Lemma 2.2, Proposition 3.1
Primitives and coverings	A closed one-form on a simply connected domain is exact. A proper local diffeomorphism of the plane which is the identity outside a compact set is a diffeomorphism.	Proposition 2.3
Ordinary differential equations	Existence, uniqueness and smooth dependence on data and parameters; Gronwall's inequality.	radial models, flows, Proposition 3.2
Quadratic forms [Tes09, Theorems 2.12, 2.13]	A densely defined closed quadratic form which is bounded below is the form of a unique self-adjoint operator.	the models of Section 4; Lemma 5.6
Spectral theorem [Tes09, Chapter 3]	For a self-adjoint operator A and z outside its spectrum, $\|(A-z)^{-1}\| = \mathrm{dist}(z, \sigma(A))^{-1}$.	Lemmas 4.4, 4.7
Weyl's theorem [Tes09, Theorem 6.19]	If the difference of the resolvents of two self-adjoint operators is compact, their essential spectra coincide.	Section 4.6
Harmonic oscillator [Tes09, Section 8.3]	The operator $-\partial_x^2 + \omega^2 x^2$ on the line has the simple eigenvalues $(2n+1)\omega$, $n \geq 0$, and no other spectrum.	Section 3.3, Lemma 4.4
Countable products	A sequence in a countable product of compact intervals has a subsequence which converges in every coordinate.	Lemma 5.9
Three-lines lemma	Let T_z be a family of bounded operators such that, for every vector v, the function $z \mapsto T_z v$ is analytic for $0 < \mathrm{Re}\, z < 1$, and continuous and bounded for $0 \leq \mathrm{Re}\, z \leq 1$. If $\|T_z\| \leq M_0$ for $\mathrm{Re}\, z = 0$ and $\|T_z\| \leq M_1$ for $\mathrm{Re}\, z = 1$, then $\|T_z\| \leq M_0^{1-\vartheta} M_1^{\vartheta}$ for $\mathrm{Re}\, z = \vartheta$. Boundedness on the closed strip cannot be omitted.	Lemma 8.5
Extension operator [See64]	On a half-line or an interval there is a linear extension operator to the line which is bounded in every Sobolev norm H^s, $s \geq 0$.	after Lemma 7.2

Result	Statement used	Where
Hamilton's theorem on families of inverses	Theorem 9.3.	Propositions 8.1, 9.2
Hamilton's theorem with quadratic error	Theorem 9.4.	Theorem 9.6

The Airy asymptotics [Nat] quoted in Section 3.3 serve as a heuristic only.

Two elementary facts used repeatedly. *Schur's test.* If the blocks K_{nm} of an operator matrix satisfy $\sup_n \sum_m \|K_{nm}\| \le A$ and $\sup_m \sum_n \|K_{nm}\| \le B$, then the operator is bounded on the ℓ^2 sum with norm at most $\sqrt{AB}$: apply the Cauchy–Schwarz inequality with the weights $\|K_{nm}\|$ in each row, and then the bound for the columns. *Neumann series.* If $\|R\| < 1$ on a Banach space, then $\sum_{k\ge 0}(-R)^k$ converges in norm to a two-sided inverse of $I + R$.

The order of the choices. The constants of the construction are chosen in the following order; no later choice changes an earlier one.

1. *The models.* The parameters of the central oscillator and of the Morse models, the exponent 12 of the joins and the constant c_0 of the matching angle (Sections 4, 5).
2. *The profile.* The exponent a of the source weight, then C_*, then the initial index N, then the cuts of the central model and the lengths of the cone heads; then the amplitude ratios σ_n, by Lemma 5.9; then the connectors. These choices use only $|\gamma| \le 1$ and are therefore the same for all endpoint exponents (Lemma 6.10).
3. *The reference inverse.* A compact interval of endpoint exponents $\gamma \in (0, 1)$ changes the constants of the estimates, not the profile. The final border (g_*, ℓ_*) is fixed after the geometry (Section 6.8).
4. *The spaces.* The exponent $\beta = \frac{3}{4}$, the root zones of width δ_n^{root}, the controls χ_n and the cutoffs (Section 7.3).
5. *The neighbourhood.* The order b_0 and the number ϵ_0 of (7.19), by the requirements (N1)–(N6) of Section 7.4, in this order. The last of them also restricts the parameter q.
6. *The losses.* The number d_*, as the largest of the finitely many losses listed at the beginning of Section 8.

Estimates of higher order change constants and use a fixed shift of the order; they never require a new choice of the profile, of the neighbourhood or of the range of q. The two inequalities $\frac{1}{2} < \beta < 1$ serve different purposes: $\beta < 1$ leaves room for two neighbouring exponents at which the reference inverse is available, and $\beta > \frac{1}{2}$ makes the product of two decaying factors decay faster than H.

C Index of notation

The tables list the symbols that are used in more than one place, with the place where each is introduced. Letters not listed here have a meaning local to one proof or one subsection, stated there. A few letters are deliberately reused in different parts for unrelated objects, as indicated.

General, and Part I

Symbol	Meaning	Introduced
$\mathbb{T}$, $\langle f\rangle$, R_ϑ	circle $\mathbb{R}/2\pi\mathbb{Z}$; angular average; rotation by ϑ	§1.8
e_j	$\sqrt{2}\cos(j\theta)$	§4
w, h, E, g	profile (slope), radial height with $h'=w$, $E=1+w^2$, metric $E\,dr^2+r^2d\theta^2$	(2.1)
z, u, q	height $z=h+qu$; small parameter	§3.4
$\bar g$	remainder metric $g-dz\otimes dz$	§2
F_τ, τ, ϱ_ϑ	the bending, its phase, intrinsic rotation	(1.1)
k, p, D, Q	z_r, z_θ/r, discriminant, $k+Ep_\theta$	(3.1)
α	connection form of the remainder metric	(3.2)
$\mathcal{N}_h(z)$	curvature residual	(3.3)
$\mathcal{F}_q(w,u)$, $\mathcal{A}$	normalized residual	(3.19)
a, b, c	coefficients of the linearization	(3.6)
$\mathscr{L}_w$, L_j, $L'_{2,w}$	linearization at the radial height; its jth mode; derivative in w	(3.11), (4.1)
K_0, K_g	ww'/r; curvature of g	(3.7), (3.9)
B, $\bar c$, $\bar L$	first residual correction	Lemma 3.4
d, v, $c_\pm$	drift, speed and slopes of characteristics	(3.25)
$\mathfrak{F}_w$	connection period	(3.22)
T, T_m	travel angle of a negative component	§4.3

Part II

Symbol	Meaning	Introduced
X, $\mathcal{P}$, ρ, V_0, λ	model coordinate, density, potential data, amplitude of a model	Lemma 4.3
$H_{j,c}$, $H_{j,M}$	Schrödinger operators of the central and Morse models	§4.5, §4.6
υ	r_X/r	(4.16)
ϖ, $d\mu$, B_j	$12w/r^2$; $\varpi\,dr$; $-\varpi^{-1}L_j$	(4.20)
A_c, a_c, c_c, α_c, ϕ_0	constants and seed of the central model	(4.13), (4.18)
g_0, ℓ_0	temporary border	§4.5
A, B, d	Morse parameters (in §§4.6–5)	(4.22)
d_n, A_n, B_n	parameters of cell n; $A_n\asymp e^{-n^2}$	§5
λ_n, σ_n	amplitude of cell n; amplitude ratios	(5.1)
S_n, H_n, q_n, $\mathcal{E}_n$	$\frac{2}{3}n^3$, e^{-S_n}, $n^{5/2}\log(e+n)$, e^{q_n}	(4.2)
M_n	$\exp\{1+n^{5/2}\log^2(e+n)\}$	(6.39)
N, C_*, c_0	first cell index; band constant; angle constant	§5
X_c, X_L, X_R, $\mathfrak{r}_n$	cut and roots of a cell; radius factor	§5.2, §5.3
$\tau_{n,j}$	action $j\sqrt{A_n}$ of the ramps of cell n	(5.5)
p_j, Λ	gauge density wr^{2j^2-1}; $j^2(j^2-1)$ (in §5)	(5.12)

Symbol	Meaning	Introduced
I_n, J_n; μ_n, ν_n, R_n^I, R_n	mass and resistance intervals; masses and resistances	(5.13)
a_n, y_n, b_n, c_n	node values and coefficients of the recurrence	Lemma 5.6
j_n, Band_n	band edges; frequency bands	(5.28)
ζ_j, Ξ_n, Υ_n, δ_n	state variable on a join; amplitude contrast; matching angle	§5.8
w_*, ϕ_*, χ_*, g_*, ℓ_*	reference profile, seed, metric control, border	Prop. 4.2
$\mathbb{L}_{w_*}$, $\mathbb{G}_{w_*}$	bordered reference operator and its inverse	(4.8), Lemma 6.12
$Y_{\gamma,s}$, a, b, l	source norm; exponents of the coefficient allowance	(4.5), Lemma 6.3
ϑ_m, χ_L	partition of unity; endpoint step function	§4.2
G_h, R_o, G	inner, outer and matched inverses	(6.10), (6.20), (6.29)
S, K, A, B, D_0, D_b	matching matrices	§§6.4–6.6
Π_n	transfer bound $e^{Cn^2+C_o\tau_{n,j}}$	(6.18)
$\mathcal{D}_\gamma^{\mathrm{rad}}$	weighted graph domain	Def. 6.11

Part III

Symbol	Meaning	Introduced
x_∂, H, E_0, M	continuous versions of the endpoint scales	(7.2)
ℓ, ϱ, D	unit-scale radial variable and derivative	§7.2
β, ϵ	weight exponent $\frac{3}{4}$; gap between exponents	(7.4), §8.2
$\mathcal{X}_\gamma$, $\mathcal{X}_\gamma^0$, $\mathcal{Y}_\beta$, $\mathcal{Z}_\gamma$, $\mathcal{K}_\gamma$, Q	spaces of heights, sources, circle data; integral functional	Def. 7.1
$\mathbf{x} = (u, \nu, \kappa, h_L)$, Φ_*	state; reference height $\phi_* e_2$	(7.17)
χ_m	travel-angle controls	Lemma 7.6
$\mathfrak{n}_s$, $\mathfrak{g}_{s,m}$; b_0, ϵ_0, d_*	seminorms of the state; the neighbourhood; fixed loss	§7.4
f, h_R, $\mathfrak{e}$	minus return map, right chart, return residual	(7.25)
$\mathcal{M}_T$, $\mathcal{Q}_h$	derivative of $\mathfrak{e}$ in the chart	(7.27)
$\widehat{L}$, $\mathscr{E}_{\mathbf{x}}$	corrected linearization; its error	Prop. 7.9
σ, φ; ξ_L, ξ_R, ξ_c	travel coordinates; root coordinates	(7.33)
W_i, h_i, α_i, V_i	canonical unknown, conjugating factor, density, potential	(7.36)
y, $\mathfrak{y}$, $P_{\mathbf{x}}$, P_*	positive coordinate; $y(1-y)$; positive canonical operators	§7.7
$\mathscr{C}_{\mathbf{x}}$, $\mathscr{S}_{\mathbf{x}}$, $\mathbb{L}_{\mathbf{x}}$	transformations to canonical form	§8.1
$\mathbb{V}_{t,s}$	spaces of canonical sources	(8.2)
$\mathbb{P}_{\mathbf{x}}$, $\mathbf{W} = (W, a_L, \mu, \lambda)$	augmented operator and unknown	(8.3)
$k_{\mathbf{x}}$, $s_{\mathbf{x}}$, c_*	columns of the augmented operator	§8.1

Symbol	Meaning	Introduced
$\mathbb{G}_*$, $\mathbb{D}_*$, L_0	reference inverse, its domain, its loss	Lemma 8.2
$\lvert v\rvert_{\mathcal{F}}$	L^1 norm of the Fourier transform	Lemma 7.2
$\mathfrak{g}_{s,m}$, $\mathfrak{n}_{s,m}$	localized seminorms of the deformation and of the state	§7.4
δ_n^{root}	width of the root zones	(7.16)
$\mathscr{R}_{\mathbf{x}}$	relative operator	§8.4
$G_{\mathbf{x}}$	inverse at a nearby state	Prop. 8.1
$\mathbf{R}$, $\mathbf{V}$, $\mathbf{Q}$	residual, approximate inverse, quadratic error	§9
$\mathfrak{X}$, $\mathfrak{Y}$	spaces of increments and residuals	§9
C°, R°, $\mathcal{D}$, J, $K_{\mathbf{x}}$, $U_{\mathbf{x}}$	blocks of the linear system	(9.2)
$(\eta, \mu, \varkappa, \psi)$	increments of (u, ν, κ, h_L)	(9.4)
B_β	space of control increments	§9
$\mathcal{B}_{w,U}$, $\mathcal{B}^\circ_{\mathbf{v}}$, $\mathcal{C}$, $\mathscr{P}$, $\mathcal{U}$	remainders, mean return map, primitive, mean of the height	§10

Words with a fixed meaning.

- *Height* always refers to the height function $z = h + qu$ of the surface or to one of its parts (h, u, increments). The positive constant by which a piece of the profile is multiplied is its *amplitude* (λ_n, with *amplitude ratios* σ_n).
- *Cap* is the stationary part of the closed surface (Section 2). The two positive parts of a cell are its short and long *ramps* (Section 5.2).
- *Collar* is a neighbourhood of a boundary circle of the annulus on which the surface is planar. The neighbourhood $\lvert x\rvert < 8\delta_n^{\mathrm{root}}$ of a turning circle is the *zone* of that root (Lemma 7.5).
- A *coefficient power* is a bound $C\mathcal{E}_n^C$, and *coefficient bounds* are bounds $C_s\mathcal{E}_n^{C_s}$ for derivatives of order s; see (4.3). A *reference function* is a function of r determined by w_* alone, with coefficient bounds.
- The *conjugating factor* h_i or $h_{\mathbf{x}}$ is the positive function by which a height is divided to obtain its canonical unknown (Section 7.6).
- *Component* means a maximal interval on which w_*' has one sign; *cell*, *join*, *connector* and *tail* are the building blocks of the profile (Section 4.3); a *piece* is one of the cylinders of Section 8.1.

Letters with several meanings. The following letters have more than one meaning. Each meaning is confined to the sections named, and where two meanings meet in one section they differ by a subscript or an argument, not by the typeface alone.

- q is the small parameter of the height; q_n is the exponent of the coefficient allowance.
- λ is the amplitude of a model in Part II and the auxiliary scalar of the augmented operator in Section 8.
- μ is a mass in Section 5, the measure $d\mu = \varpi\, dr$ in Section 6, and the increment of ν in Part III; ν_n is a mass in Section 5 and ν the metric parameter elsewhere.

- κ_i is a boundary coefficient in Section 6; κ_m are the controls in Part III.
- τ is the phase of the bending; $\tau_{n,j}$ is an action.
- A, B are Morse parameters; the matrices A, B and the coefficient $B = E/(rQ)$ are distinguished by type or by context.
- Λ is $j^2(j^2-1)$ in Section 5, a matrix of outward derivatives in Section 6, and $|D_\varphi|$ in Section 8.
- G with a subscript or in blackboard type (G_h, G_o, $\mathbb{G}_*$, $G_{\mathbf{x}}$) is an inverse operator; $G(t,\theta)$ is the speed of a flow on the circle (Lemma 7.3, Section 10); and $G(x,\zeta,\theta)$ is the function of the regularized flow at a root (Section 7.6).
- D is the discriminant; D_u, D_ν, $D_{\mathbf{x}}$ are derivatives; D_0, D_b, D_h, D_κ are matrices or operators of Section 6.
- $E = 1 + w^2$; $\mathcal{E}_n$ is the coefficient allowance; E_b is an error matrix in Section 6; $\mathscr{E}_{\mathbf{x}}$ is the error of the return correction; E is the even projection.
- H_n is the decay scale and H^s a Sobolev space; $\mathscr{H}_j$ and $H_{j,c}$ are operators of Part II.
- T_m is a travel angle and $\mathbb{T}$ the circle; T and T_h are matrices in Section 6.
- a is the leading coefficient of the linearization; a_L is the endpoint constant of a height; a is a coefficient exponent; a_p are the constants of Lemma 8.5.
- $c_\pm$, $\widehat{c}_\pm$ are characteristic speeds and c_0, c_i root factors in Part III; $\bar{c}$, $\widehat{c}$ are coefficients; c is the scalar of a source pair and c_0 the constant of δ_n in Part II.
- d is the drift $(c_+ + c_-)/2$; d_i is a flux ratio in Section 6; d_k, d_*, d_{geo}, d_{tot} are losses of derivatives. The boundary-defining function $y(1-y)$ is written $\mathfrak{y}$.
- h is the radial height; h_L, h_R are charts of the circle; h_i, $h_{\mathbf{x}}$ are conjugating factors; the subscript h marks the inner problem of Section 6.
- R_τ is a rotation; R_n, R_n^I are resistances; R_o is the outer resolvent; R° and $\mathscr{R}_{\mathbf{x}}$ are operators of Part III.
- K is the matching matrix of Section 6; $\mathcal{K}_\gamma$ is the space of controls and $K_{\mathbf{x}}$ the derivative with respect to the controls in Part III.
- I_n, J_n are the mass and resistance intervals of a cell; the frequency bands are written Band_n.

Statement on the use of AI

The mathematical arguments of this paper were developed with GPT-6 Astra Pro. The paper was then organized and rewritten with Claude Fable 5.2, and in the course of this several arguments were replaced or expanded.

Theorem A has been formalized in Lean 4 with Mathlib:

`https://github.com/Rogerhu12/continuous-flexibility`

The statement is the file `src/Challenge/FlexibleSphereChallenge.lean`, written with notions of Mathlib only, and the theorem is `FlexibleSphere.exists_closedBending`. According to the repository, its proof contains no `sorry` and depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`. The formal statement is that of Theorem A:

a jointly smooth one-periodic family of smooth embeddings of the standard sphere which induce one metric, two of whose images are not congruent under any affine isometry, reflections included, with an inner disk and the complementary cap fixed pointwise, and with constant area, enclosed volume and total mean curvature. The formal proof follows the lemma structure of this paper, but its intermediate statements are not word for word those of the text, and it does not certify the text line by line.

The source of this paper is accompanied by four Python scripts (directory `anc`), which verify about one hundred of the explicit identities of the text, symbolically where possible. The proofs do not depend on them.

Boyang Hu organized the work and is responsible for the manuscript.

References

[Ale47] A. D. Alexandrov. On the works of S. E. Cohn-Vossen. *Uspekhi Matematicheskikh Nauk*, 2(3(19)):107–141, 1947. In Russian; English translation by A. Iacob, 2012, https://math.uni-koeln.de/sites/math_career/studium/Weitere_Dokumente/Stefan_Cohn_Vossen/Alexandrov_Iacob.pdf.

[AR98] F. J. Almgren, Jr. and I. Rivin. The mean curvature integral is invariant under bending. In *The Epstein Birthday Schrift*, volume 1 of *Geometry & Topology Monographs*, pages 1–21. 1998.

[Cao19] W. Cao. The semi-global isometric embedding of surfaces with curvature changing signs stably. *Proceedings of the American Mathematical Society*, 147(10):4343–4353, 2019.

[Con77] R. Connelly. A counterexample to the rigidity conjecture for polyhedra. *Publications Mathématiques de l'IHÉS*, 47:333–338, 1977.

[CT73] J. M. Combes and L. Thomas. Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators. *Communications in Mathematical Physics*, 34:251–270, 1973.

[CV27] S. Cohn-Vossen. Zwei Sätze über die Starrheit der Eiflächen. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 125–134, 1927.

[CV30] S. Cohn-Vossen. Unstarre geschlossene flächen. *Mathematische Annalen*, 102:10–29, 1930.

[Don93] G.-C. Dong. The semi-global isometric imbedding in $\mathbb{R}^3$ of two-dimensional Riemannian manifolds with Gaussian curvature changing sign cleanly. *Journal of Partial Differential Equations*, 6(1):62–79, 1993.

[Efi48a] N. V. Efimov. Qualitative problems of the theory of deformation of surfaces. *Uspekhi Matematicheskikh Nauk*, 3(2(24)):47–158, 1948. In Russian.

[Efi48b] N. V. Efimov. Research on the deformation of surfaces containing points with zero gaussian curvature. *Matematicheskii Sbornik (N.S.)*, 23(65)(1):89–125, 1948. In Russian.

[Gho19] M. Ghomi. Open problems in geometry of curves and surfaces, 2019.

[GQRZ26] M. Ghomi, H. Qiu, M. Raffaelli, and Y. Zhang. Nonrigidity of smooth tight surfaces, 2026. Preprint, arXiv:2611.17269v1, 12 November 2026.

[GS20] M. Ghomi and J. Spruck. Rigidity of nonnegatively curved surfaces relative to a curve. *International Mathematics Research Notices*, 2020(17):5387–5400, 2020.

[Ham82] R. S. Hamilton. The inverse function theorem of Nash and Moser. *Bulletin of the American Mathematical Society (New Series)*, 7(1):65–222, 1982.

[Han05] Q. Han. On the isometric embedding of surfaces with Gauss curvature changing sign cleanly. *Communications on Pure and Applied Mathematics*, 58(2):285–295, 2005.

[Her43] G. Herglotz. Über die Starrheit der Eiflächen. *Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität*, 15:127–129, 1943.

[HH06] Q. Han and J.-X. Hong. *Isometric Embedding of Riemannian Manifolds in Euclidean Spaces*, volume 130 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[HHL03] Q. Han, J.-X. Hong, and C.-S. Lin. Local isometric embedding of surfaces with nonpositive Gaussian curvature. *Journal of Differential Geometry*, 63(3):475–520, 2003.

[HK10] Q. Han and M. A. Khuri. On the local isometric embedding in $\mathbb{R}^3$ of surfaces with Gaussian curvature of mixed sign. *Communications in Analysis and Geometry*, 18(4):649–704, 2010.

[HK13] Q. Han and M. A. Khuri. Smooth solutions to a class of mixed type Monge–Ampère equations. *Calculus of Variations and Partial Differential Equations*, 47:825–867, 2013.

[HMVV17] W. Hamouche, C. Maurini, S. Vidoli, and A. Vincenti. Multi-parameter actuation of a neutrally stable shell: a flexible gear-less motor. *Proceedings of the Royal Society A*, 473(2204):20170364, 2017. arXiv:1707.09013.

[Hör76] L. Hörmander. The boundary problems of physical geodesy. *Archive for Rational Mechanics and Analysis*, 62:1–52, 1976.

[Hör85] L. Hörmander. On the Nash–Moser implicit function theorem. *Annales Academiae Scientiarum Fennicae, Series A I Mathematica*, 10:255–259, 1985.

[IH51] L. Infeld and T. E. Hull. The factorization method. *Reviews of Modern Physics*, 23:21–68, 1951.

[IKS91] I. Ivanova-Karatopraklieva and I. K. Sabitov. Deformation of surfaces. I. *Itogi Nauki i Tekhniki. Seriya Problemy Geometrii*, 23:131–184, 1991. In Russian; English translation in Journal of Mathematical Sciences 70 (1994), no. 2, 1685–1716.

[Khu07] M. A. Khuri. The local isometric embedding in $\mathbb{R}^3$ of two-dimensional Riemannian manifolds with Gaussian curvature changing sign to finite order on a curve. *Journal of Differential Geometry*, 76(2):249–291, 2007.

[KK74] I. S. Kac and M. G. Krein. On the spectral functions of the string. *American Mathematical Society Translations, Series 2*, 103:19–102, 1974.

[Kui55] N. H. Kuiper. On C^1-isometric imbeddings. I. *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series A*, 58:545–556, 1955.

[Lan49] R. E. Langer. The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point. *Transactions of the American Mathematical Society*, 67:461–490, 1949.

[Li12] C.-H. Li. The semi-global isometric embedding of surfaces with Gaussian curvature changing sign cleanly. *Science China Mathematics*, 55(12):2507–2515, 2012.

[Lin86] C.-S. Lin. The local isometric embedding in $\mathbb{R}^3$ of two-dimensional Riemannian manifolds with Gaussian curvature changing sign cleanly. *Communications on Pure and Applied Mathematics*, 39(6):867–887, 1986.

[Mil65] A. D. Milka. On points of relative non-rigidity of a convex surface of revolution. *Ukrainskii Geometricheskii Sbornik*, 1:65–74, 1965. In Russian.

[Mor29] P. M. Morse. Diatomic molecules according to the wave mechanics. II. vibrational levels. *Physical Review*, 34:57–64, 1929.

[Mos66] J. Moser. A rapidly convergent iteration method and non-linear partial differential equations. I. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (3)*, 20(2):265–315, 1966.

[MR53] T. Minagawa and T. Radó. On the infinitesimal rigidity of surfaces of revolution. *Mathematische Zeitschrift*, 59:151–163, 1953.

[Muc72] B. Muckenhoupt. Hardy's inequality with weights. *Studia Mathematica*, 44(1):31–38, 1972.

[Nas54] J. Nash. C^1 isometric imbeddings. *Annals of Mathematics*, 60(3):383–396, 1954.

[Nas56] J. Nash. The imbedding problem for Riemannian manifolds. *Annals of Mathematics*, 63(1):20–63, 1956.

[Nat] National Institute of Standards and Technology. Digital Library of Mathematical Functions: Chapter 9, Airy and related functions. https://dlmf.nist.gov/9.

[Olv74] F. W. J. Olver. *Asymptotics and Special Functions*. Academic Press, New York, 1974.

[Pog73] A. V. Pogorelov. *Extrinsic Geometry of Convex Surfaces*, volume 35 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1973.

[Res62] Y. G. Reshetnyak. On non-rigid surfaces of revolution. *Sibirskii Matematicheskii Zhurnal*, 3(4):591–604, 1962. In Russian.

[Sab76] I. K. Sabitov. Possible generalizations of the Minagawa–Rado lemma on the rigidity of a surface of revolution with a fixed parallel. *Mathematical Notes*, 19(1):74–79, 1976. Russian original: Mat. Zametki 19:1 (1976), 123–132.

[Sab95] I. K. Sabitov. Quasiconformal mappings of a surface generated by its isometric transformations, and bendings of the surface onto itself. *Fundamentalnaya i Prikladnaya Matematika*, 1(1):281–288, 1995. In Russian.

[Sab96] I. K. Sabitov. The volume of polyhedron as a function of its metric. *Fundamentalnaya i Prikladnaya Matematika*, 2(4):1235–1246, 1996. In Russian.

[Sab13] I. K. Sabitov. Infinitesimal and global rigidity and inflexibility of surfaces of revolution with flattening at the poles. *Sbornik: Mathematics*, 204(10):1516–1547, 2013.

[Sab14] I. K. Sabitov. Second-order infinitesimal bendings of surfaces of revolution with flattening at the poles. *Sbornik: Mathematics*, 205(12):1787–1814, 2014.

[Sab21] I. K. Sabitov. Bendable surfaces without visible deformations of their shape. *Mathematical Notes*, 110(1):126–134, 2021.

[Sab23] I. K. Sabitov. Infinitesimal sliding bendings of compact surfaces and Euler's conjecture. *Siberian Mathematical Journal*, 64(5):1213–1228, 2023.

[See64] R. T. Seeley. Extension of C^∞ functions defined in a half space. *Proceedings of the American Mathematical Society*, 15:625–626, 1964.

[SET02] R. Sa Earp and E. Toubiana. Discrete and nondiscrete isometric deformations of surfaces in $\mathbb{R}^3$. *Siberian Mathematical Journal*, 43(4):714–718, 2002.

[SZ07] J. Sjöstrand and M. Zworski. Elementary linear algebra for advanced spectral problems. *Annales de l'Institut Fourier*, 57(7):2095–2141, 2007.

[Tes09] G. Teschl. *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, volume 99 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[Tri23] F. Tricomi. Sulle equazioni lineari alle derivate parziali di secondo ordine, di tipo misto. *Atti della Accademia Nazionale dei Lincei, Serie 5*, 14:134–247, 1923.

[Tro80] D. A. Trotsenko. Nonrigid analytic surfaces of revolution. *Siberian Mathematical Journal*, 21(5):718–724, 1980. Russian original: Sibirsk. Mat. Zh. 21:5 (1980), 100–108.