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Even Asymptotic Expansions and Uniform Hard-Edge Approximations for Wigner $3j$ and $6j$ Symbols at Fixed Rank

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Abstract

We study Wigner $3j$ symbols with two large and one small angular momentum, and the $6j$ symbols with one small entry (Edmonds' limit), both at fixed rank: the small labels stay fixed while the large ones grow. Their classical limits are Wigner rotation matrix elements. Combining a Regge symmetry with the Weyl shift $j \mapsto j + \frac{1}{2}$ singles out midpoint variables: for $3j$, the large scale $N = j_- + j_+ + 1$ ($j_\mp$ the two large angular momenta) and $x = 2\mu/N$ (2μ the difference of the two large magnetic indices), and, for $6j$, large lengths A, B, C shifted by offsets that depend on the small labels. Among constant shifts of the large labels and magnetic coordinate, and up to rank-independent gauge factors, these are the only centerings that remove the generic first-order term of the exact recurrences. In these variables the normalized symbols are values of analytic functions of the inverse square of the large scale, for integer and half-integer small labels alike (including spin–orbit coefficients): the expansions contain only even powers and converge. The first correction is a closed formula in one Wigner d function and its derivative, improving the error from second to fourth order in the inverse large scale; all higher corrections are constructed as local differential operators. The exact finite problems are Hahn and Racah spectral systems with undeformed Wigner spectra. Near the ends of the physical lattice (the hard edges), an explicit product of Pochhammer factors carries the difference between the discrete and continuum weights; extracting it gives expansions that are uniform on the whole lattice, including the $6j$ corner where two large lengths nearly coincide ($A \approx B$). We also show that centered Hahn and Racah polynomials are even deformations of Jacobi polynomials.

1 Introduction

Wigner $3j$ symbols with two large and one small angular momentum are the coupling coefficients of squeezed configurations: they appear in Gaunt integrals and the spin-weighted couplings of full-sky cosmic microwave background (CMB) bispectrum and lensing estimators [1, 2], in the Hönl–London line strengths of rotational spectroscopy [3, 4], and, with half-integer small labels, in spin–orbit coupling. The $6j$ symbols with one small entry are the recoupling coefficients of a small-rank tensor operator acting on one of two coupled large angular momenta [5]. In both cases the classical leading approximation is a Wigner rotation matrix element [5, 6, 7]. This paper shows why the averaged (midpoint) variables are the right ones, gives the corrections to all orders, and makes the approximation uniform up to the edges of the physical lattice, all at fixed rank: the small labels are fixed while the large ones grow.

For the $3j$ symbol the leading form goes back to Edmonds, who stated without proof that it tends to a Wigner d function whose angle is set by the magnetic index γ of one large spin c, $\cos\theta = \gamma/\sqrt{c(c+1)}$ [5]. Brussaard and Tolhoek gave a heuristic derivation [6], Ponzano and Regge's triangle geometry replaces $\sqrt{c(c+1)}$ by $c + \frac{1}{2}$ [33], and Flude proved the limit with this one-sided scale [7, Thm. 1]. McFarlane replaced the one-sided angle by an average over the two large spins, $\cos\theta = (m+m')/(l+l'+1)$ for $\langle L\, M\, l'\, m' | l\, m\rangle$, in which the difference $l - l'$ becomes the second magnetic index of the d function [27]. He chose this average by matching the exact rank recurrence to that of the d functions: the diagonal coefficients then agree exactly, and the

off-diagonal ones differ only through $k^2/(l+l'+1)^2$. Morehead derived the same form from the Weyl correspondence, which evaluates matrix elements on the average quantized torus, proved that its relative error is $O(l^{-2})$, and noted that one-sided choices such as $\cos\theta = m/l$ leave a first-order error [29, Eq. (3.31) and Appendix]. Carron and Lewis rederived the symmetric form from the Regge symmetry, with normalization $\sqrt{2\ell+1}$ [1, App. B, Eq. (B19)]. For a $3j$ coefficient with a small angular momentum L of magnetic index M, two large sides $j_\mp = \ell \mp \sigma/2$ and magnetic indices $m_\mp = \pm\mu - M/2$, it reads

$$(-1)^{2L+j_+ +m_+}\sqrt{2\ell+1}\begin{pmatrix} L & j_- & j_+ \\ M & m_- & m_+ \end{pmatrix} \longrightarrow d^L_{M\sigma}(\theta), \qquad \cos\theta = \frac{2\mu}{2\ell+1}. \tag{1}$$

Here $2\ell+1 = j_- + j_+ + 1$ is the mean of $2j_-+1$ and $2j_++1$, twice the midpoint of the Weyl lengths $j_\mp + \frac{1}{2}$. The normalization $[(2j_-+1)(2j_++1)]^{1/4}$ of McFarlane and Morehead differs from $\sqrt{2\ell+1}$ only by a factor $1+O(\ell^{-2})$, and the sign selects one analytic branch across both large-spin parity classes (Section 4). We prove that the physical branch is analytic in $(2\ell+1)^{-2}$, and that the midpoint variables

$$\boxed{N = 2\ell+1 = j_- + j_+ + 1, \qquad x = \frac{2\mu}{N}} \tag{2}$$

are the only constant shifts of N and 2μ for which the generic first-order term of the exact magnetic difference equation (30) vanishes, up to rank-independent gauge factors (Proposition 4.3). The one-sided scale $c+\frac{1}{2}$ of Ponzano–Regge and Flude is such a shift ($2c+1 = N \mp \sigma$ and $2\gamma = \pm 2\mu - M$ for $c = j_\mp$, the sign of μ being a reflection $x \mapsto -x$), so unless $\sigma = M = 0$ it carries a first-order error. Proposition 4.3 thus makes McFarlane's recurrence matching and Morehead's remark on one-sided expansion points precise, and Theorem 5.6 strengthens Morehead's $O(N^{-2})$ error to a convergent expansion in N^{-2} with explicit coefficients.

The directly usable consequence is the next term. Writing $\Phi_L^{M,\sigma}$ for the normalized coefficient (17) and putting

$$f(x) = d^L_{M\sigma}(\arccos x), \qquad J = L(L+1), \tag{3}$$

one has, uniformly on compact subsets of $-1 < x < 1$,

$$\boxed{\Phi_L^{M,\sigma} = f + \frac{A_2(x)f'(x) + B_2(x)f(x)}{12N^2(x^2-1)} + O(N^{-4}),} \tag{4}$$

where

$$\begin{aligned} A_2(x) &= 2J(x^3 - x) - 4x(M^2+\sigma^2) + 2M\sigma(x^2+3) + 4x, \\ B_2(x) &= J(x^2-3) + 2M\sigma x. \end{aligned} \tag{5}$$

The Regge-symmetric d approximation therefore has an intrinsic $O(N^{-2})$ error in the bulk (away from the ends $x = \pm 1$ of the grid), and one d matrix and one derivative improve it to $O(N^{-4})$ (Theorem 6.1).

For the one-small $6j$ symbol the classical limit of Edmonds and Flude is

$$\begin{Bmatrix} c & b & a \\ f & a+n & b+m \end{Bmatrix} \sim \text{phase} \times (\text{large normalization})^{-1} d^f_{nm}(\phi) \tag{6}$$

with ϕ a law-of-cosines angle of the large labels, for fixed f, m, n and large a, b, c [5, 7] (the symbol is that of Section 7 with its columns permuted). McFarlane gave an averaged version, again by matching rank recurrences [28, Eqs. (16), (A10)]. In terms of the Weyl-shifted lengths

$$A = a + \frac{n+1}{2}, \qquad B = b + \frac{m+1}{2}, \qquad C = c + \frac{1}{2}, \tag{7}$$

which are the midpoints of the Weyl lengths of the pairs $(a, a+n)$ and $(b, b+m)$ and the Weyl length of c, his angle and normalization agree through first order with $x = \cos\theta = (A^2 + B^2 - C^2)/(2AB)$ and $1/(2\sqrt{AB})$; his cosine is $x + (m^2+n^2)/(8AB)$. Within the same class of f-independent gauges, (7) is the unique affine centering of the exact recurrence (Proposition 7.2), and the centering must depend on the offsets: replacing every large spin by $j + \frac{1}{2}$ (the case $u = v = w = \frac{1}{2}$ of Proposition 7.2) leaves a generic first-order error unless $m = n = 0$, and Flude's variables leave one even then when $f > 0$, through his normalization (Section 9.3). The first correction is again one d function and one derivative, with coefficients homogeneous of degree -2 in (A, B, C) (Theorem 9.1).

Results. The small labels remain fixed: L, M, σ as $N \to \infty$ for $3j$, and f, m, n as $A, B \to \infty$ for $6j$. In both families they may be integers or half-integers; for $3j$ the half-integer case is that of large spins in opposite integer/half-integer classes, such as spin–orbit coefficients. The bulk $6j$ results require C to grow within a strict-triangle cone; the uniform results allow every physical C, including bounded C near the corner $A = B$, provided A/B stays bounded away from 0 and ∞ ($\lambda \le A/B \le \lambda^{-1}$ for a fixed $\lambda \in (0,1)$).

Family	Region	Result
$3j$ bulk	x in a compact subset of $(-1,1)$	Convergent even expansion in N^{-2}; explicit terms through N^{-4} (Theorems 5.6, 6.1, 6.3).
$3j$ uniform	whole physical grid	$\Phi = E_N[d + O(N^{-2})W]$ with an explicit edge factor E_N, and an all-order even expansion with polynomial coefficients after $E_N W$ is extracted (Theorem 10.1).
$6j$ bulk	compact strict-triangle cone	Homogeneous all-even expansion; explicit degree -2, recursive degree -4 (Theorems 8.7, 9.1, 9.2 and (245)).
$6j$ uniform	whole c-lattice, $\lambda \le A/B \le \lambda^{-1}$	$\Phi_f = E_{A,B}[d + O((A+B)^{-2})W]$, including the corner $A \approx B$; all orders (Theorem 10.1).

Here Φ and Φ_f are the normalized symbols, E_N and $E_{A,B}$ explicit edge factors, and W is the endpoint envelope of the Wigner function, so the uniform estimates are additive and remain valid at zeros of the leading mode. At the $3j$ and separated $6j$ hard edges (the ends of the lattice, where the link coefficients of the exact recurrences vanish) they give relative Laguerre laws, and the lower $6j$ corner carries a second explicit factor (Corollaries 10.4 and 10.5, Remark 10.7). Behind these formulas are exact structures that we also prove: the finite problems are Hahn and Racah spectral systems with the undeformed spectra $L(L+1)$ and $f(f+1)$, unitarily equivalent to truncated continuum Wigner problems (Theorem 5.1 and Corollary 5.3; Theorem 8.2 and Corollary 8.3), and every correction is a local differential operator constructed recursively (Theorems 6.5 and 9.2).

The even expansions rest on exact square-root-free closed forms of the physical kernels (Lemmas 5.4 and 8.5) in which the large scale enters only through its inverse square, so no odd power can occur. Midpoint centering alone does not force this: it makes the centered recurrences invariant under reversal of the large scale, and it is the physical initial data that select the even sector (Remark 3.3).

Using the results. In either family: (i) compute the centered variables, N and x, or A, B, C and the law-of-cosines x; (ii) in the bulk, use (4), respectively add the correction $G_{f,2}$ of (238) to the Wigner function, for a fourth-order error; (iii) near the edges, use the uniform form of Corollary 10.3, which multiplies by the explicit edge factor and is valid on the whole lattice

(for $6j$ when A/B is bounded away from 0 and ∞). Table 1 and Figure 1 show the accuracy at moderate sizes.

A companion paper [8] treats the regime in which the rank grows with the large scale, for a class of even centered Wigner–Jacobi chains that contains the $3j$ and $6j$ rank recurrences of this paper. The exact N^{-2} coefficients proved here are inputs of that analysis; none of its results is needed here.

Relation to earlier work. Recurrence methods have long been central to exact evaluation and semiclassical analysis of Wigner symbols [26, 9, 10]; classical-limit treatments and higher-order asymptotics from integral representations give complementary background [6, 11]. The Hahn, Jacobi and Racah ingredients, the $3j$–Hahn/dual-Hahn, Regge and recurrence-eigenvalue viewpoints, the Jacobi–Hahn Bernstein identity and the beta/binomial conditional-expectation model are classical [12, 13, 14, 15, 16]. A parity-separated all-order $1/N$ expansion is known for the equal-large-spin fuzzy-sphere family [17], and higher-order $6j$ expansions exist in the distinct all-large Ponzano–Regge regime [18, 19]. Semiclassical approximations with small and large angular momenta include a uniform rotation-matrix approximation of the $6j$ symbol valid near caustics [20], the $9j$ symbol with one small and eight large angular momenta [21], and an elementary Ponzano–Regge-based method for $3nj$ symbols with small and large angular momenta [22]. New here are the uniqueness of the $3j$ and $6j$ midpoints within their affine classes, including the offset dependence of the $6j$ centering; the physical branch normalization in the fixed-one-small regime; exact evenness and convergence at fixed rank, including the unequal-edge $6j$ case; the closed first corrections and the homogeneous all-order local calculus; and the exact edge factorization, which packages the fixed-degree Hahn–Jacobi and Racah–Jacobi limits into expansions uniform on the whole lattice.

Organization. Sections 2 and 3 fix conventions and two elementary analytic lemmas. Sections 4–6 treat the $3j$ symbol: variables, exact recurrences and the uniqueness of the midpoint; the finite Hahn system, the closed form and the even theorem; and the local corrections. Sections 7–9 treat the one-small $6j$ symbol in parallel, with $(N, x; L, M, \sigma)$ replaced by $(A, B, C; f, m, n)$ and the Hahn system by the Racah system on the c-lattice, dual to the rank recurrence. Section 10 proves the uniform expansions up to the hard edges, and Section 11 compares all approximations with exact symbols. Appendix A records classical algebraic structures of the finite $3j$ problem that the main results do not use; the other appendices contain the all-order constructions and the longer algebra.

2 Conventions

We use the Condon–Shortley Wigner symbols and the Wigner small-d matrices in the convention used by Flude [7]; see also the standard angular-momentum conventions in [23]. For fixed j, $d^j_{m'm}(\theta) = \langle j\, m'|e^{-i\theta J_y}|j\, m\rangle$. The $3j$ and $6j$ recurrences used below are DLMF 34.3.14–34.3.15 and 34.5.11–34.5.13 respectively [24].

The phrase *analytic in* N^{-2} always means the following. Physical Wigner symbols exist only on an admissible integer/half-integer subsequence. We construct, from the exact recurrence and exact physical initial data, an explicit closed form (Lemmas 5.4 and 8.5) in $t = N^{-2}$ near $t = 0$ whose values on that subsequence equal the normalized Wigner coefficient. Whenever a radius $|t| < \rho$ is quoted, t is complexified: the closed form is holomorphic in that disk after choosing the square-root branch equal to the positive physical branch at $t = 0$. The coefficient functions in the real shape variables are therefore real analytic, while the dilation variable is holomorphic. Taylor expansion in t then gives the asymptotic series. In the $3j$ sections the small labels L, M, σ are either all integers or all half-integers; the two large representations may lie in the same or in opposite integer/half-integer classes.

Notation guide. The two recoupling problems use deliberately different alphabets. In the $3j$ part, N, x are the centered large scale and midpoint coordinate and L, M, σ are fixed small labels. In the $6j$ part, A, B, C are the centered large lengths and f, m, n are fixed small labels. We reserve $\mathfrak{C}_r$ and $\mathfrak{D}_m$ for the central-factorial seed products, C_D for the beta–binomial conditional-expectation channel, and $\mathcal{C}$ for the physical c-lattice. Blackboard/sans-serif letters denote operators; calligraphic letters mostly denote spaces, lattices, recurrence operators, and the differential operators of the local calculus ($\mathcal{G}_{2r}$, $\widehat{\mathcal{D}}_{2r}$, $\mathcal{N}_s$, $\mathcal{J}$).

3 Centered shifts and the analytic principle

The difficult input is symbol-specific: the exact Regge/Weyl centers, the physical seeds, the finite spectral systems and the local correction operators. Once those are known, the exact recurrences and starts give the physical kernel as an explicit closed form (Lemma 5.4), and the two elementary lemmas below, together with those closed forms, are all the analysis the even expansions need.

Lemma 3.1 (Centered-shift product)**.** *Let $Q \subset \frac{1}{2}\mathbb{Z}$ be a finite multiset invariant under $q \mapsto -q$, let $r = |Q|$, and define*

$$\Pi_Q(X) := \prod_{q \in Q} (X + q). \tag{8}$$

For $S > 0$,

$$N^{-r}\Pi_Q(NS) = \prod_{q \in Q} (S + qN^{-1}) \tag{9}$$

is an even polynomial in N^{-1}, equivalently a polynomial in $t = N^{-2}$. Moreover,

$$\log[N^{-r}\Pi_Q(NS)] = r \log S - \frac{\sum_{q \in Q} q^2}{2S^2} N^{-2} + O(N^{-4}). \tag{10}$$

For the consecutive symmetric set $Q = \{-(r-1)/2, \dots, (r-1)/2\}$,

$$\Pi_Q(X) = \frac{\Gamma(X + (r+1)/2)}{\Gamma(X - (r-1)/2)}. \tag{11}$$

Here the ratio of Gamma functions is the polynomial $\prod_{q \in Q}(X + q)$; its apparent singularities, where both Gamma functions have poles, are removable.

Proof. Pair the factors with shifts q and $-q$. Every pair contributes $S^2 - q^2N^{-2}$, while a possible central factor $q = 0$ contributes S. This proves exact evenness. Expanding $\sum_q \log(S + qN^{-1})$ gives (10), since all odd power sums vanish. Equation (11) is the elementary finite Gamma-ratio identity. □

Thus, after the Weyl shift, Regge symmetry exposes finite large-label shift multisets centered at zero, and the resulting factorial products are even before any asymptotic expansion is taken.

Lemma 3.2 (Uniform Taylor remainder)**.** *Let $K \subset \mathbb{R}^p$ be compact and let $F(z, t)$ be real analytic on a neighborhood of $K \times \{0\}$ in $\mathbb{R}^p \times \mathbb{R}$. Then there is $\rho > 0$ such that for every $R \geq 1$, with $F_r(z) := \partial_t^r F(z, 0)/r!$,*

$$F(z, t) = \sum_{r=0}^{R-1} F_r(z) t^r + O_{K,R}(t^R) \qquad \text{uniformly on } K \times [0, \rho]. \tag{12}$$

Proof. By compactness the neighborhood contains $K \times [-\rho, \rho]$ for some $\rho > 0$. There F is C^∞ and every $\partial_t^R F$ is bounded. Taylor's theorem with Lagrange remainder in t gives the bound $\sup_{K \times [0,\rho]} |\partial_t^R F|\, t^R/R!$. □

Remark 3.3 (Role of the seed and of the width)**.** *Midpoint centering and evenness are distinct. The uniqueness results (Propositions 4.3 and 7.2) identify the centering at which the generic first-order term of the recurrence vanishes, and in the centered variables the exact recurrences are invariant under reversal $h \mapsto -h$ of the inverse large scale. This alone does not make the physical solution even: a reversal-invariant recurrence can have solutions that change sign or acquire an odd normalization under $h \mapsto -h$. In general, if a fixed-width recurrence is equivariant under $h \mapsto -h$ with an involution R acting on a finite-dimensional space of values, and its complete analytic boundary data determine the solution uniquely and obey $Y_j(z,-h) = \mathsf{R}Y_j(z,h)$ at every supplied start j, then $Y_k(z,-h) = \mathsf{R}Y_k(z,h)$ for all k, since both sides solve the same recurrence with the same data; the $\mathsf{R} = +1$ part is even in h and the $\mathsf{R} = -1$ part is odd. For the scalar recurrences of this paper $\mathsf{R} = I$, and the physical initial data (the extremal $3j$ seed and the positive stretched $6j$ seed, with two starts when $M = \sigma = 0$, respectively $m = n = 0$) are even, so the physical kernels are even. The closed forms of Lemmas 5.4 and 8.5 make this explicit: the large scale enters them only through h^2. The case $\mathsf{R} = I$ of this principle, with coefficients and seed analytic in h^2, is what defines the even centered chains of the companion paper [8]. The mechanism does not control a recurrence whose width, matrix dimension or propagation index grows with the large scale; that is the boundary with growing-rank semiclassics.*

4 The $3j$ symbol: variables, exact recurrences and the midpoint

Fix small labels L, M, σ, either all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$, with

$$L \geq L_0 := \max(|M|, |\sigma|). \tag{13}$$

Let

$$N = 2\ell + 1, \qquad j_- := \frac{N - 1 - \sigma}{2}, \qquad j_+ := \frac{N - 1 + \sigma}{2}, \tag{14}$$

and write

$$m_- := \mu - \frac{M}{2}, \qquad m_+ := -\mu - \frac{M}{2}, \qquad x := \frac{2\mu}{N}. \tag{15}$$

Then $M + m_- + m_+ = 0$, $\sigma = j_+ - j_-$, and x is the midpoint magnetic coordinate of the symmetric McFarlane–Morehead form, normalized as by Carron and Lewis. In the spherical-bispectrum specialization $j_\pm$ are integer multipoles, so admissibility implies $\sigma \equiv N - 1 \pmod 2$. Since $\sigma = j_+ - j_-$, the small labels are half-integers exactly when j_- and j_+ lie in opposite integer/half-integer classes, as for the spin–orbit coefficients $\langle l\, m_l; \frac{1}{2}\, m_s | j\, m_j \rangle$ ($L = \frac{1}{2}$, $M = m_s$, and $\sigma = \pm\frac{1}{2}$ distinguishes $j = l \pm \frac{1}{2}$) or couplings to a spin-$\frac{3}{2}$ representation; then $N \in \mathbb{Z} + \frac{1}{2}$, and $\ell = (N-1)/2$ and μ are quarter-integers, not spins, while $m_\pm$ and $j_\pm$ are spins. In both cases $2L$, $L \pm M$, $L \pm \sigma$, $M \pm \sigma$, $N - L$ and $j_+ + m_+$ are integers, which is all that the constructions below use.

The centering is tied to an exact Regge exchange [25]. Writing $j_\pm = \ell \pm \sigma/2$,

$$\begin{pmatrix} L & \ell - \sigma/2 & \ell + \sigma/2 \\ M & \mu - M/2 & -\mu - M/2 \end{pmatrix} = \begin{pmatrix} L & \ell + M/2 & \ell - M/2 \\ -\sigma & \mu + \sigma/2 & -\mu + \sigma/2 \end{pmatrix}. \tag{16}$$

Thus the ordinary magnetic index and the displacement of the two large representation labels are exchanged.

Define

$$\Phi_L^{M,\sigma}(x; h) := (-1)^{2L + j_+ + m_+} \sqrt{N} \begin{pmatrix} L & j_- & j_+ \\ M & m_- & m_+ \end{pmatrix}, \qquad h := N^{-1}. \tag{17}$$

The choice $(-1)^{j_+ + m_+}$ is important when both integer and half-integer large-spin classes occur along the same fixed-x subsequence: it removes the otherwise alternating factor $(-1)^{2j_+}$ and

selects one analytic Wigner-d branch across both classes. The factor $(-1)^{2L}$ equals 1 for integer labels; for half-integer labels it is the N-independent sign without which the same branch would be $-d^L_{M\sigma}$ (the exponent $j_+ + m_+$ is an integer in every case).

Exact rank recurrence. Substitution of (14)–(15) into the standard rank recurrence gives

$$\begin{aligned}&(2L+1)[L(L+1)x - M\sigma]\Phi_L \\ &\quad = (L+1)R_L\sqrt{1-L^2h^2}\,\Phi_{L-1} + LR_{L+1}\sqrt{1-(L+1)^2h^2}\,\Phi_{L+1},\end{aligned} \tag{18}$$

where

$$R_k := \sqrt{(k^2-M^2)(k^2-\sigma^2)}. \tag{19}$$

Every large-parameter factor is an analytic function of h^2.

Observation 4.1 (Exact rank three-term deformation factor)**.** *For $L \geq L_0$ with $L > 0$, divide* (18) *by $L(L+1)$ and define*

$$a_k^{(N)} := a_k^{(0)}\sqrt{1-\frac{k^2}{N^2}}, \qquad a_k^{(0)} := \frac{\sqrt{(k^2-M^2)(k^2-\sigma^2)}}{k}. \tag{20}$$

Then the exact recurrence has the three-term form

$$a_{L+1}^{(N)}\Phi_{L+1} + a_L^{(N)}\Phi_{L-1} = b_L^{(0)}(x)\Phi_L, \qquad b_L^{(0)}(x) := (2L+1)\left(x - \frac{M\sigma}{L(L+1)}\right). \tag{21}$$

After division by $2L+1$ and the standard diagonal reweighting, (21) *is the symmetric Jacobi eigenvalue problem with spectral parameter x. The $N = \infty$ coefficients are exactly those of the Wigner-d rank recurrence; this comparison of the two rank recurrences is McFarlane's, who used it to choose the midpoint angle [27, Sec. 4]. At finite N the entire off-diagonal deformation is therefore carried by the single factor*

$$\sqrt{1-k^2/N^2}. \tag{22}$$

It is even in N^{-1}, has the local expansion

$$\sqrt{1-k^2/N^2} = 1 - \frac{k^2}{2N^2} + O(k^4/N^4), \tag{23}$$

and vanishes at $k = N$, closing the finite rank chain exactly. The exceptional $L = 0$ sector is supplied by the separate two-start data below.

Remark 4.2 (Local versus accumulated deformation)**.** *Equation* (23) *is a* local *defect of size $O(k^2/N^2)$, not a uniform $L = o(N)$ error estimate: coherent transport through ranks $k \leq L$ heuristically, and generically, accumulates its off-diagonal part to the scale*

$$\sum_{k\leq L}\frac{k^2}{N^2} \sim \frac{L^3}{3N^2}. \tag{24}$$

Growing rank, where this accumulation matters, is the subject of the companion paper [8]; the present paper uses only fixed-rank consequences of (21)*.*

Initial data. For the extremal case $M = L$, put

$$u_\pm = \frac{1 \pm x}{2}, \qquad \mathfrak{C}_r(u,h) := \prod_{q=-(r-1)/2}^{(r-1)/2} (u+qh), \quad \mathfrak{C}_0 = 1. \tag{25}$$

The set of q values is symmetric, so Lemma 3.1 makes $\mathfrak{C}_r(u,h)$ exactly even in h. Put

$$\Pi_L(h^2) := \mathfrak{C}_{2L+1}(1,h) = \prod_{\substack{0<q\le L \\ q\in L+\mathbb{Z}}} (1-q^2h^2), \tag{26}$$

which is $\prod_{k=1}^{L}(1-k^2h^2)$ for integer L and $\prod_{q=1/2}^{L}(1-q^2h^2)$ for half-integer L. The one-term Racah formula gives exactly

$$\begin{aligned}\Phi_L^{L,\sigma}(x;h) = (-1)^{L-\sigma}&\sqrt{\frac{(2L)!}{(L+\sigma)!(L-\sigma)!}} \\ &\times \frac{\sqrt{\mathfrak{C}_{L+\sigma}(u_+,h)\mathfrak{C}_{L-\sigma}(u_-,h)}}{\sqrt{\Pi_L(h^2)}}.\end{aligned} \tag{27}$$

The sign $(-1)^{L-\sigma}$ is that of $d^L_{L\sigma}$; for integer labels it equals $(-1)^{L+\sigma}$. For general M, σ, standard $3j$ symmetries and Regge exchange reduce the minimal-rank seed to this case up to an N-independent sign, which is made explicit in the proof of Lemma 5.4. When $L_0 > 0$, the factor $R_{L_0} = 0$ kills the backward term in (18), so this single minimal-rank value is already a complete start of the recursion. This is always the case for half-integer labels, where $L_0 \ge \frac{1}{2}$. The exceptional integer start $M = \sigma = 0$ is

$$\Phi_0^{0,0} = 1, \qquad \Phi_1^{0,0} = \frac{x}{\sqrt{1-h^2}}, \tag{28}$$

because the recurrence at $L = 0$ is then the identity $0 = 0$.

4.1 Exact magnetic difference equation and uniqueness of the midpoint

The three-term recurrence of the $3j$ symbol in its magnetic indices [9, 24], written in the variables (14)–(15) with the normalization (17) (a unit step in μ moves x by $2h$), gives an exact equation directly in x. Define

$$a_h(y) = \sqrt{[(1-y)^2 - h^2(\sigma-M)^2][(1+y)^2 - h^2(\sigma+M)^2]}. \tag{29}$$

Then

$$\begin{aligned}&a_h(x+h)\Phi(x+2h) + a_h(x-h)\Phi(x-2h) - 2(1-x^2)\Phi(x) \\ &\quad + 2h^2[2L(L+1) - M^2 - \sigma^2 + 1]\Phi(x) = 0.\end{aligned} \tag{30}$$

The operator is exactly invariant under $h \mapsto -h$: the two shifted terms exchange.

Proposition 4.3 (Uniqueness of the affine $3j$ midpoint)**.** *Consider the natural affine recenterings*

$$\widetilde{N} = N + \delta, \qquad \widetilde{x} = \frac{2\mu + \eta}{\widetilde{N}}, \tag{31}$$

with fixed $\delta, \eta = O(1)$. Admit a scalar gauge $\Phi = (1 + \widetilde{N}^{-1} g(\widetilde{x}))\widetilde{\Phi}$ and multiplication of the whole equation by $1 + \widetilde{N}^{-1} c(\widetilde{x})$, with g and c independent of the rank L. If the continuum operator obtained from the exact recurrence is required to have no generic $O(\widetilde{N}^{-1})$ term at fixed $\widetilde{x}$, then uniquely

$$\delta = 0, \qquad \eta = 0. \tag{32}$$

Thus $N = j_- + j_+ + 1$ and $x = 2\mu/N$ are the unique variables in this affine and gauge class with vanishing generic subprincipal term.

Proof. Let $X = \widetilde{x}$ and $\varepsilon = \widetilde{N}^{-1}$. The canonical coordinate is

$$x = \frac{\widetilde{N}X - \eta}{\widetilde{N} - \delta} = X + (\delta X - \eta)\varepsilon + O(\varepsilon^2). \tag{33}$$

Transform the leading Wigner operator

$$(1-x^2)\partial_x^2 - 2x\partial_x + J - \frac{\Sigma - 2\tau x}{1-x^2}, \qquad \Sigma = M^2 + \sigma^2, \quad \tau = M\sigma, \tag{34}$$

to the variable X. The $O(\varepsilon)$ coefficients of ∂_X^2 and ∂_X are respectively

$$2X\eta - 2\delta, \qquad 2\eta. \tag{35}$$

The zeroth-order terms change at $O(\varepsilon)$ by the rank-independent amount $-\varepsilon(\delta X - \eta)\,\partial_X\big[(\Sigma - 2\tau X)/(1-X^2)\big]$. Since $\partial_X(1+\varepsilon g)\widetilde{\Phi} = (1+\varepsilon g)\partial_X\widetilde{\Phi} + \varepsilon g'\widetilde{\Phi}$, the gauge and the multiplier act, to first order in ε, by multiplying the whole operator by $1 + \varepsilon(c(X) + g(X))$ and adding

$$2\varepsilon(1-X^2)g'(X)\,\partial_X + \varepsilon\big[(1-X^2)g''(X) - 2Xg'(X)\big]; \tag{36}$$

the coefficient of J becomes $1 + \varepsilon(c+g)$, the coefficient of ∂_X^2 becomes $(1-X^2)(1+\varepsilon(c+g)) + \varepsilon(2X\eta - 2\delta)$, and that of ∂_X becomes $-2X(1+\varepsilon(c+g)) + \varepsilon(2\eta + 2(1-X^2)g')$. Since c, g are independent of the rank, and J varies with the rank, there is no $O(\varepsilon)$ term in the coefficient of J only if $c + g \equiv 0$. The principal coefficient is then $(1-X^2) + \varepsilon(2X\eta - 2\delta)$, so $2X\eta - 2\delta$ must vanish identically in X, which forces $\eta = \delta = 0$; the ∂_X coefficient then forces $g' \equiv 0$, so the added zeroth-order term vanishes, as does the potential shift since $\delta = \eta = 0$. (Thus the admitted gauge and multiplier are at most opposite constants, which leave the operator unchanged.) Finally, the exact centered difference operator contains only even powers of the step h, since a_h is even in h and the two shifts occur symmetrically, and $h = \varepsilon + O(\varepsilon^2)$; so the displayed first-order term comes entirely from the affine change of variables. □

Remark 4.4 (Discrete Weyl ordering and the subprincipal term)**.** *The midpoint cancellation has a literal Weyl-symbol form. Define the lattice Weyl quantization by*

$$\mathrm{Op}_h^W[b(x)e^{ikp}]\psi(x) := b\left(x + \frac{kh}{2}\right)\psi(x+kh). \tag{37}$$

Then the hopping part of (30) *is exactly*

$$\mathrm{Op}_h^W[2a_h(x)\cos(2p)]\psi. \tag{38}$$

Because $a_h(x)$ is even in h, the exact Weyl symbol contains no $O(h)$ subprincipal term. By Proposition 4.3, within the affine class (31) *and its admitted gauge class the Regge midpoint is precisely the centering that puts the finite operator in this form, the discrete counterpart of the Weyl-shifted variables of improved semiclassical angular-momentum formulas such as Morehead's [29].*

Dividing by $4h^2$ and expanding gives at leading order

$$(1-x^2)f'' - 2xf' + \left[L(L+1) - \frac{M^2 + \sigma^2 - 2M\sigma x}{1-x^2}\right] f = 0, \tag{39}$$

the Wigner-d equation.

5 The finite Hahn system and the even $3j$ theorem

5.1 Exact finite spectral system

The physical midpoint grid is

$$\mathcal{X}_N = \{-1 + (1 + |M + \sigma|)h + 2nh : n = 0, \dots, N - L_0 - 1\}. \tag{40}$$

Define

$$(\mathsf{H}_h\psi)(x) := -\frac{a_h(x+h)\psi(x+2h) + a_h(x-h)\psi(x-2h) - 2(1-x^2)\psi(x)}{4h^2} + \frac{M^2+\sigma^2-1}{2}\psi(x). \tag{41}$$

Theorem 5.1 (Exact finite Wigner spectral system)**.** *For every admissible $N \geq L_0 + 1$, H_h is real symmetric on $\ell^2(\mathcal{X}_N, 2h)$ and*

$$\operatorname{spec}(\mathsf{H}_h) = \{L(L+1) : L = L_0, \dots, N-1\}. \tag{42}$$

Its eigenfunctions are $\Phi_L^{M,\sigma}$ with the normalization

$$2h \sum_{x \in \mathcal{X}_N} \Phi_L(x)\Phi_{L'}(x) = \frac{2}{2L+1}\delta_{LL'}, \tag{43}$$

and

$$h \sum_{L=L_0}^{N-1} (2L+1)\Phi_L(x)\Phi_L(y) = \delta_{xy}. \tag{44}$$

Proof. Equation (30) is equivalent to $\mathsf{H}_h\Phi_L = L(L+1)\Phi_L$. The link coefficient from x to $x+2h$ is $-a_h(x+h)/(4h^2)$ and equals the reverse link coefficient, so the matrix is symmetric. The boundary rows close exactly: using $2L_0 = |M+\sigma| + |M-\sigma|$, one has $x_{\max} = 1 - (1 + |M-\sigma|)h$ and $x_{\min} = -1 + (1 + |M+\sigma|)h$, hence $a_h(x_{\max} + h) = a_h(x_{\min} - h) = 0$. No value outside the physical lattice is therefore required. The admissible ranks $L_0, \dots, N-1$ give exactly $N - L_0$ distinct eigenvalues, matching the grid dimension. The normalization and completeness relations are the Clebsch–Gordan unitarity relations rewritten in the midpoint coordinate. □

Remark 5.2 (Exact discrete–continuum mass matching)**.** *Taking $L = L'$ in* (43) *gives $2h\sum_{x\in\mathcal{X}_N} |\Phi_L^{M,\sigma}|^2 = 2/(2L+1)$. Independently, Wigner-d orthogonality gives*

$$\int_{-1}^{1} |d^L_{M\sigma}(\arccos x)|^2 \, \mathrm{d}x = \frac{2}{2L+1}. \tag{45}$$

Thus the finite kernel and its continuum limit carry the same mass; each identity has its own proof. Expanding the discrete identity with the midpoint quadrature gives an independent scalar normalization condition at each order; the central-factorial seed is the more direct normalization proof used in Theorems 6.1 and 6.3.

5.2 Explicit Hahn realization of the finite kernel

The normalized $3j$ wavefunction is an ordinary Hahn polynomial with explicit parameters. It is enough to display one sign/order chamber, since the other sign/order chambers follow from the exact Regge and simultaneous-sign symmetries. Assume

$$M \geq |\sigma|, \tag{46}$$

and set

$$\alpha_H := M - \sigma, \qquad \beta_H := M + \sigma, \qquad D := N - M - 1, \qquad n := L - M. \tag{47}$$

Thus α_H, β_H, D, n are nonnegative integers, for integer and half-integer labels alike, and $0 \le n \le D$ for every admissible $N \ge L + 1$. Measure the physical lattice from the upper endpoint by

$$u := \frac{N - 1 - M + \sigma - 2\mu}{2} = j_+ + m_+, \qquad x_u = 1 - \frac{\alpha_H + 1 + 2u}{N}, \qquad u = 0, \dots, D. \tag{48}$$

Use the DLMF Hahn convention

$$Q_n(u; \alpha_H, \beta_H, D) = {}_3F_2\left(\begin{matrix} -n,\ n + \alpha_H + \beta_H + 1,\ -u \\ \alpha_H + 1,\ -D \end{matrix}; 1\right), \tag{49}$$

with weight and squared norm

$$w_H(u) := \frac{(\alpha_H + 1)_u}{u!} \frac{(\beta_H + 1)_{D-u}}{(D-u)!}, \tag{50}$$

$$h_{H,n} := \sum_{u=0}^{D} w_H(u) Q_n(u; \alpha_H, \beta_H, D)^2. \tag{51}$$

Then the physical phase convention (17) gives the exact identity

$$\Phi^{M,\sigma}_{M+n}(x_u; N^{-1}) = (-1)^{\alpha_H} \sqrt{\frac{N}{2(M+n)+1}} \sqrt{\frac{w_H(u)}{h_{H,n}}}\, Q_n(u; \alpha_H, \beta_H, D). \tag{52}$$

This is the classical Clebsch–Gordan/Hahn correspondence [13, 14] in the exact Regge-midpoint variables of the present paper. In other chambers retain the upper-edge orientation and set

$$\alpha_H = |M - \sigma|, \qquad \beta_H = |M + \sigma|, \qquad D = N - L_0 - 1, \qquad n = L - L_0, \tag{53}$$

with $u = (N - 1 - |M - \sigma| - 2\mu)/2$. The same normalized Hahn expression then holds with the N-independent phase

$$\epsilon(M, \sigma) = \begin{cases} (-1)^{M-\sigma}, & \sigma \le M, \\ 1, & \sigma > M. \end{cases} \tag{54}$$

in place of $(-1)^{\alpha_H}$. This phase follows directly from the exact $3j$ symmetries (simultaneous magnetic reversal and exchange of the last two columns) and the Regge exchange (16), applied to the normalized physical kernel (17). No Taylor expansion is used here. In particular, transposition $M \leftrightarrow \sigma$ multiplies the physical kernel by $(-1)^{M-\sigma}$ while leaving $|M \pm \sigma|$ unchanged, so no phase depending only on these absolute values can work in every chamber.

The degree and spectral variables have the chamber-independent form

$$L_0 = \max(|M|, |\sigma|), \qquad D = N - L_0 - 1, \qquad n = L - L_0, \qquad \alpha_H + \beta_H = 2L_0, \tag{55}$$

so the dual-Hahn spectral coordinate is exactly

$$n(n + \alpha_H + \beta_H + 1) = L(L+1) - L_0(L_0 + 1). \tag{56}$$

On the continuum side, in the variable $(1 - x)/2$, the same α_H, β_H occur in the Jacobi-polynomial representation of $d^L_{M\sigma}$. Thus the Hahn-to-Jacobi contraction is explicit at the level of eigenfunctions (Lemma 5.5 below); its algebraic form, an exact finite contraction of the Hahn quadratic algebra to the Jacobi algebra, is Proposition A.1 in Appendix A.

There is also an exact spectral transmutation. Put $\Sigma = M^2 + \sigma^2$, $\tau = M\sigma$ and write

$$\mathsf{H}_0 = -\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right] + \frac{\Sigma - 2\tau x}{1-x^2}, \tag{57}$$

and set

$$u_L(x) = \sqrt{\frac{2L+1}{2}}\, d^L_{M\sigma}(\arccos x), \qquad v_L(x_j) = \sqrt{\frac{2L+1}{2}}\, \Phi^{M,\sigma}_L(x_j;h). \tag{58}$$

Let $\mathcal{V}_N = \mathrm{span}\{u_L : L_0 \le L \le N-1\}$.

Corollary 5.3 (Exact finite-to-continuum spectral map)**.** *The map*

$$\mathsf{S}_N u_L = v_L \tag{59}$$

is unitary from $\mathcal{V}_N$ to $\ell^2(\mathcal{X}_N, 2h)$, has kernel

$$K_N(x_j, y) = \sum_{L=L_0}^{N-1} \frac{2L+1}{2}\, \Phi^{M,\sigma}_L(x_j;h) d^L_{M\sigma}(\arccos y), \tag{60}$$

and satisfies

$$\mathsf{H}_h \mathsf{S}_N = \mathsf{S}_N(\mathsf{H}_0|_{\mathcal{V}_N}). \tag{61}$$

Proof. Unitarity needs only that $\{u_L\}$ is an orthonormal basis of $\mathcal{V}_N$ (Wigner orthogonality) and that $\{v_L\}_{L=L_0}^{N-1}$ is an orthonormal basis of $\ell^2(\mathcal{X}_N, 2h)$ (Theorem 5.1: (43), and $N - L_0 = |\mathcal{X}_N|$ ranks); the kernel (60) is the expansion of S_N in these bases. The intertwining (61) needs only the eigen-equations $\mathsf{H}_0 u_L = L(L+1)u_L$ and $\mathsf{H}_h v_L = L(L+1)v_L$, the latter being (30); in it H_h acts on lattice functions, which is legitimate because the boundary link coefficients vanish ($a_h(x_{\max} + h) = a_h(x_{\min} - h) = 0$, proof of Theorem 5.1), so no value outside $\mathcal{X}_N$ enters. The simplicity of the spectrum is used only for the uniqueness of such an intertwining unitary (up to the phases of the eigenvectors), not for its existence. □

The same map is also the polar factor of a natural finite-to-continuum contraction, a Bernstein–Durrmeyer conditional expectation with a beta prior (Proposition A.4 in Appendix A); this realization is classical and is not used below.

5.3 The edge factor and the square-root-free closed form

The even theorem rests on an exact closed form of the physical kernel in which the large scale enters only through its square. It factors out an explicit ratio of discrete and continuum weights, which is also what controls the hard edges in Section 10. In this subsection and in Section 10 we write $\varsigma := (1-x)/2$.

For an integer $p \ge 0$ and $X \ge (p+1)/2$ define, with the central product $\mathfrak{C}_p(X) := \mathfrak{C}_p(X, 1)$ of the extremal seed,

$$\begin{aligned} \mathcal{R}_p(X) &:= \frac{\sqrt{\mathfrak{C}_p(X)}}{X^{p/2}} = \left[\prod_{q=-(p-1)/2}^{(p-1)/2} \left(1 + \frac{q}{X}\right)\right]^{1/2}, \\ \mathfrak{R}_{p,r} &:= \mathcal{R}_p\Big(r + \frac{p+1}{2}\Big) = \frac{\sqrt{(r+1)_p}}{\left(r + \frac{p+1}{2}\right)^{p/2}}. \end{aligned} \tag{62}$$

Thus $\mathcal{R}_0 = \mathcal{R}_1 = 1$, and $\mathcal{R}_p(X)^2$ is the ratio of the discrete Pochhammer weight $\mathfrak{C}_p(X)$ to its continuum power X^p. By Lemma 3.1, $\mathcal{R}_p(X)^2$ is even in X^{-1} and

$$\mathcal{R}_p(X) = 1 - \frac{p(p^2-1)}{48X^2} + O(X^{-4}), \qquad \text{in particular} \quad \mathfrak{R}_{p,r} = 1 + O(r^{-2}). \tag{63}$$

For the $3j$ problem put

$$p_\pm := |M \mp \sigma|, \qquad E_N(x) := \mathcal{R}_{p_+}\Big(\frac{N(1-x)}{2}\Big)\,\mathcal{R}_{p_-}\Big(\frac{N(1+x)}{2}\Big),$$
$$W_{3j}(x) := \Big(\frac{1-x}{2}\Big)^{p_+/2}\Big(\frac{1+x}{2}\Big)^{p_-/2}. \tag{64}$$

The magnetic inequalities give the exact extreme grid points

$$x_{\max} = 1 - \frac{1+p_+}{N}, \qquad x_{\min} = -1 + \frac{1+p_-}{N}, \tag{65}$$

and the nth points inward are $x_n^\pm = \pm[1-(p_\pm+1+2n)/N]$. At x_n^+ one has $N(1-x)/2 = n + (p_+ + 1)/2$, so the first factor of E_N is exactly $\mathfrak{R}_{p_+,n}$; similarly at the lower edge.

For integers $d \geq 0$ and parameters $\alpha, \beta \geq 0$ write

$$\widehat{P}_d^{(\alpha,\beta)}(\varsigma) := \frac{(\alpha+1)_d\, d!}{(-d)_d\,(d+\alpha+\beta+1)_d}\,{}_2F_1\!\left(\begin{matrix}-d,\ d+\alpha+\beta+1\\ \alpha+1\end{matrix};\varsigma\right) \tag{66}$$

for the monic Jacobi polynomial in $\varsigma = (1-x)/2$. It is the monic degree-d eigenpolynomial of the hypergeometric operator

$$\mathcal{L}_0^{(\alpha,\beta)} = \varsigma(\varsigma-1)\partial_\varsigma^2 + [(\alpha+\beta+2)\varsigma - (\alpha+1)]\partial_\varsigma, \qquad \mathcal{L}_0^{(\alpha,\beta)}\widehat{P}_d = d(d+\alpha+\beta+1)\widehat{P}_d, \tag{67}$$

and the standard Jacobi form of the Wigner functions [5, 23], written with this monic normalization, reads

$$d^L_{M\sigma}(\arccos x) = k_{3j}\, W_{3j}(x)\, \widehat{P}^{(p_+,p_-)}_{L-L_0}(\varsigma)\,. \tag{68}$$

The constant is explicit: for integers $k, p, q \geq 0$ put

$$\kappa_k(p,q) := (-1)^k \sqrt{\frac{k!\,(k+p+q)!}{(k+p)!\,(k+q)!}}\;\frac{(k+p+q+1)_k}{k!}, \tag{69}$$

the factor $(-1)^k(k+p+q+1)_k/k!$ being the leading coefficient of $P_k^{(p,q)}(1-2\varsigma)$ in ς, so that the monic normalization of $\widehat{P}$ is in ς (the monic normalization in x would differ by $(-2)^k$). Then

$$k_{3j} = \epsilon(M,\sigma)\,\kappa_{L-L_0}(p_+,p_-), \tag{70}$$

with ϵ the phase (54); the sign follows from the chamber $M \geq |\sigma|$ by the symmetries $d^L_{M\sigma} = (-1)^{M-\sigma} d^L_{\sigma M} = d^L_{-\sigma,-M}$.

Recall $\Pi_L(h^2) = \mathfrak{C}_{2L+1}(1,h)$ from (26).

Lemma 5.4 (Square-root-free closed form of the $3j$ kernel)**.** *Fix L, M, σ in any chamber. On the physical grid*

$$\Phi_L^{M,\sigma} = E_N\, W_{3j}\, \Pi_L^{-1/2}\, \widetilde{P}_L(\varsigma; h^2), \tag{71}$$

with positive square roots, where $\widetilde{P}_L$ is a polynomial of degree $L - L_0$ in $\varsigma = (1-x)/2$ whose coefficients are polynomials in h^2. It satisfies the square-root-free recurrence

$$(2L+1)[L(L+1)(1-2\varsigma) - M\sigma]\widetilde{P}_L = (L+1)R_L(1-L^2h^2)\widetilde{P}_{L-1} + LR_{L+1}\widetilde{P}_{L+1}, \tag{72}$$

has the minimal-rank value, a constant,

$$\widetilde{P}_{L_0} = \epsilon(M,\sigma)\sqrt{\frac{(2L_0)!}{p_+!\,p_-!}}, \tag{73}$$

with the phase ϵ of (54)*, has the h-independent leading coefficient k_{3j} of* (70)*, and at zero scale is the monic Jacobi polynomial times this constant:*

$$\widetilde{P}_L(\varsigma; 0) = k_{3j}\,\widehat{P}^{(p_+,p_-)}_{L-L_0}(\varsigma). \tag{74}$$

Proof. Since $\Pi_{L+1} = \Pi_L(1-(L+1)^2h^2)$ and $\Pi_{L-1} = \Pi_L(1-L^2h^2)^{-1}$, substituting (71) into (18) cancels the common factor $E_N W_{3j}\Pi_L^{-1/2}$ and turns the square roots into $1-L^2h^2$ on the backward term and 1 on the forward term, which is (72). It therefore suffices to check the starts and to define $\widetilde{P}_L$ by the recurrence.

Starts. In the extremal chamber $M = L_0$, $\mathfrak{C}_r(u,h) = h^r\mathfrak{C}_r(u/h)$ turns the seed (27) into $\Phi_{L_0} = E_N W_{3j}\Pi_{L_0}^{-1/2}\widetilde{P}_{L_0}$ with the constant $\widetilde{P}_{L_0} = (-1)^{L_0-\sigma}\sqrt{(2L_0)!/((L_0+\sigma)!(L_0-\sigma)!)}$. The other chambers follow from the exact symmetries, which leave $p_\pm$, E_N, W_{3j}, Π_L, R_k and $M\sigma$ unchanged and multiply Φ (hence $\widetilde{P}_{L_0}$) by a sign; this gives (73), the displayed sign $(-1)^{L_0-\sigma}$ being the value of ϵ in the chamber $M = L_0$ only. If $L_0 = 0$ the scalar starts are $\widetilde{P}_0 = 1$ and $\widetilde{P}_1 = 1-2\varsigma$.

Polynomial structure. For $L \geq L_0$ with $L > 0$ one has $R_{L+1} > 0$ and $R_{L_0} = 0$, so (72) reads

$$\widetilde{P}_{L+1} = \frac{(2L+1)\big[L(L+1)(1-2\varsigma) - M\sigma\big]\widetilde{P}_L - (L+1)R_L(1-L^2h^2)\widetilde{P}_{L-1}}{LR_{L+1}}, \tag{75}$$

with $\widetilde{P}_{L_0-1}$ not occurring. The diagonal coefficient is affine in ς and does not involve h, and the backward coefficient is a polynomial in h^2. Induction on L therefore shows that $\widetilde{P}_L$ is a polynomial in ς and h^2, of degree $L-L_0$ in ς, and that its ς-leading coefficient obeys $\operatorname{lc}(\widetilde{P}_{L+1}) = -2(2L+1)(L+1)R_{L+1}^{-1}\operatorname{lc}(\widetilde{P}_L)$: the backward term has lower degree, so this coefficient does not depend on h.

Zero scale. At $h = 0$ the recurrence (72), with $x = 1-2\varsigma$, is the standard rank recurrence (219) for $d^L_{M\sigma}(\arccos x)$ (with $(f;n,m)$ replaced by $(L;M,\sigma)$), the common factor W_{3j} having been removed. By (68) the functions $k_{3j}W_{3j}\widehat{P}_{L-L_0}$ are the solution of that recurrence, and for $L = L_0$ they equal $W_{3j}\widetilde{P}_{L_0}$ because $\kappa_0(p_+,p_-) = \sqrt{(2L_0)!/(p_+!p_-!)}$ (for $L_0 = 0$ the same holds for the two starts 1 and x). Since the forward coefficients LR_{L+1} are non-zero, the recurrence determines the solution from its starts, which gives (74). In particular the leading coefficient is k_{3j} at every h. □

Lemma 5.5 (Centered Hahn polynomials are even deformations of Jacobi polynomials)**.** *In the Hahn realization of Section 5.2, in any chamber, put $\alpha = \alpha_H = p_+$, $\beta = \beta_H = p_-$, $d = L - L_0$, and $\varsigma = (1-x_u)/2 = (u+\frac{\alpha+1}{2})/N$. Define*

$$q_d(\varsigma;h^2) := k_{3j}^{-1}\,\widetilde{P}_{L_0+d}(\varsigma;h^2), \qquad \textit{that is} \qquad \widetilde{P}_L = k_{3j}\,q_{L-L_0}(\varsigma;h^2), \tag{76}$$

with $\widetilde{P}$ from Lemma 5.4; it depends on (M,σ,L) only through (α,β,d). Then

- *(a) $q_d(\varsigma;h^2)$ is monic of degree d in ς with coefficients that are polynomials in h^2, and $q_d(\varsigma;0) = \widehat{P}_d^{(\alpha,\beta)}(\varsigma)$;*
- *(b) at every admissible $N > L$, $Q_d(u;\alpha,\beta,D) = \kappa_N\,q_d\big(\frac{u+(\alpha+1)/2}{N};N^{-2}\big)$ as polynomials in u, with $\kappa_N \neq 0$ independent of u;*
- *(c) there is C, depending on α,β,d only, with*

$$\sup_{0\leq\varsigma\leq1}\big|q_d(\varsigma;t) - \widehat{P}_d^{(\alpha,\beta)}(\varsigma)\big| \leq Ct \qquad (0\leq t\leq 1); \tag{77}$$

in particular $\sup_{0\leq\varsigma\leq1}|q_d(\varsigma;N^{-2}) - \widehat{P}_d^{(\alpha,\beta)}(\varsigma)| \leq CN^{-2}$.

Proof. The recurrence (72) involves (M,σ) only through $M\sigma = (\beta^2-\alpha^2)/4$, $M^2+\sigma^2 = (\alpha^2+\beta^2)/2$ and $M^2\sigma^2$, and the start (73) only through α,β and a sign that cancels in q_d; hence the stated dependence. Part (a) is Lemma 5.4: q_d is monic because k_{3j} is the leading coefficient of $\widetilde{P}_L$, its coefficients are polynomials in h^2, and (74) is $q_d(\cdot\,;0) = \widehat{P}_d^{(\alpha,\beta)}$.

(b) By (52) and $Q_0 = 1$,

$$\frac{\Phi_L^{M,\sigma}(x_u)}{\Phi_{L_0}^{M,\sigma}(x_u)} = \sqrt{\frac{2L_0+1}{2L+1}}\sqrt{\frac{h_{H,0}}{h_{H,d}}}\,Q_d(u;\alpha,\beta,D), \tag{78}$$

and by (71) the same ratio equals $(\Pi_{L_0}/\Pi_L)^{1/2}\widetilde{P}_L(\varsigma_u;N^{-2})/\widetilde{P}_{L_0}$, the edge factor and W_{3j} cancelling (they are positive on the grid, and $\Pi_L>0$ for $N>L$). So $\widetilde{P}_L(\varsigma_u;N^{-2})=c\,Q_d(u;\alpha,\beta,D)$ at $u=0,\dots,D$, with $c\neq 0$ independent of u. Both sides are polynomials in u of degree d, and the lattice has $D+1=N-L_0\geq d+1$ points, so they coincide as polynomials; this is (b), with $\kappa_N=k_{3j}/c$.

(c) Write $q_d(\varsigma;t)=\sum_{i\leq d}(a_i(0)+t\,b_i(t))\varsigma^i$ with polynomials b_i; then $a_i(0)$ are the coefficients of $\widehat{P}_d$ by (a), and for $0\leq\varsigma,t\leq 1$ the difference is at most $t\sum_i\max_{[0,1]}|b_i|$. □

For $d=1$ the zero of the Hahn polynomial already sits exactly at the image of the Jacobi zero, $q_1(\varsigma;h^2)=\widehat{P}_1(\varsigma)$: by (75) (or the scalar start $1-2\varsigma$) $\widetilde{P}_{L_0+1}$ is the constant $\widetilde{P}_{L_0}$ times an affine function of ς that does not involve h. In general the lemma is the polynomial form of the Hahn–Jacobi contraction (Proposition A.1), and evenness is visible because the recurrence (72) and its start involve the large scale only through h^2 (Remark 3.3).

5.4 The even theorem

Theorem 5.6 (Regge-midpoint even expansion)**.** *Fix L,M,σ and a compact $K\Subset(-1,1)$. There exist an open interval U with $K\subset U\Subset(-1,1)$, a number $\rho>0$, and a function $G_L^{M,\sigma}(x,t)$ real analytic on $U\times(-\rho,\rho)$, such that, for every sufficiently large admissible N and every grid point $x\in\mathcal{X}_N\cap K$,*

$$\Phi_L^{M,\sigma}(x;N^{-1})=G_L^{M,\sigma}(x,N^{-2}). \tag{79}$$

Moreover

$$G_L^{M,\sigma}(x,0)=d_{M\sigma}^{L}(\arccos x). \tag{80}$$

Consequently, for every fixed R,

$$\Phi_L^{M,\sigma}(x;N^{-1})=\sum_{r=0}^{R-1}\frac{f_{2r}^{L,M,\sigma}(x)}{N^{2r}}+O_K(N^{-2R}), \tag{81}$$

uniformly on physical grid points in K. No odd power of N^{-1} occurs.

Proof of Theorem 5.6. The proof uses the exact closed form of Lemma 5.4, which is derived from (18) and the seeds alone. At every physical grid point, with the edge factor E_N and the weight W_{3j} of (64),

$$\Phi_L^{M,\sigma}(x;h)=E_N(x)\,W_{3j}(x)\,\Pi_L(h^2)^{-1/2}\,\widetilde{P}_L(\tfrac{1-x}{2};h^2), \tag{82}$$

where $\widetilde{P}_L$ is a polynomial in $(1-x)/2$ and h^2. Let $Q_\pm$ be the symmetric shift sets of sizes $p_\pm$ in E_N and $Q_\pm^{>}$ their positive elements. Pairing the shifts $\pm q$ (Lemma 3.1) gives $E_N(x)=\mathcal{E}(x,N^{-2})$ with

$$\mathcal{E}(x,t)^2:=\prod_{q\in Q_+^{>}}\Big(1-\frac{4q^2t}{(1-x)^2}\Big)\prod_{q\in Q_-^{>}}\Big(1-\frac{4q^2t}{(1+x)^2}\Big). \tag{83}$$

Define

$$G_L^{M,\sigma}(x,t):=\mathcal{E}(x,t)\,W_{3j}(x)\,\Pi_L(t)^{-1/2}\,\widetilde{P}_L(\tfrac{1-x}{2};t). \tag{84}$$

Then $\mathcal{E}^2$ and Π_L are polynomials equal to 1 at $t=0$, W_{3j} is real analytic and positive on $(-1,1)$, and $\widetilde{P}_L$ is a polynomial. Hence for compact $K\Subset(-1,1)$ there are an open interval U with $K\subset U\Subset(-1,1)$ and $\rho>0$ such that both square roots are the positive analytic branches on $U\times(-\rho,\rho)$, and $G_L^{M,\sigma}$ is real analytic there. By construction $\Phi_L^{M,\sigma}(x;N^{-1})=G_L^{M,\sigma}(x,N^{-2})$ at every admissible grid point $x\in K$ with $N>\rho^{-1/2}$ (and $N>L$). The function G depends on the scale only through $t=h^2$, so no odd power of N^{-1} can occur; this is the positive reflection sector of Remark 3.3, with $\mathsf{R}=I$.

At $t = 0$ one has $\mathcal{E} = \Pi_L = 1$ and $G_L^{M,\sigma}(x,0) = W_{3j}\widetilde{P}_L(\cdot\,;0) = k_{3j}W_{3j}\widehat{P} = d^L_{M\sigma}(\arccos x)$ by (74) and (68). Equivalently, at $t = 0$ the recurrence (72) and the start $\widetilde{P}_{L_0}$ are the rank recurrence and the extremal value of $d^L_{M\sigma}$.

Finally Lemma 3.2 with $F = G_L^{M,\sigma}$, $z = x$ and $f_{2r}^{L,M,\sigma}(x) := \partial_t^r G_L^{M,\sigma}(x,0)/r!$ gives (81), uniformly on K with remainder $O_K(t^R) = O_K(N^{-2R})$. □

Remark 5.7 (Spin–orbit coefficients)**.** *For $L = \frac{1}{2}$ one has $L_0 = \frac{1}{2}$, so $\widetilde{P}_{1/2}$ is a constant, $p_\pm \le 1$ gives $E_N = 1$, and $\Pi_{1/2} = 1 - h^2/4$. Lemma 5.4 then gives, for all $M, \sigma \in \{\pm\frac{1}{2}\}$ and every physical grid point, the exact closed form $G = d^{1/2}_{M\sigma}(\arccos x)/\sqrt{1 - t/4}$, that is*

$$\Phi^{M,\sigma}_{1/2}(x;h) = \frac{d^{1/2}_{M\sigma}(\arccos x)}{\sqrt{1-h^2/4}} = d^{1/2}_{M\sigma}(\arccos x)\Big[1+\frac{1}{8N^2}+\frac{3}{128N^4}+r_N\Big], \quad 0 \le r_N \le \frac{1}{64N^6}. \tag{85}$$

Indeed $(1 - t/4)^{-1/2} = \sum_{k\ge0}\binom{2k}{k}(t/16)^k$ has positive terms whose ratios are $\frac{2k+1}{8(k+1)}t < t/4$, so for $t = N^{-2} \le 4/9$ (that is $N \ge \frac{3}{2}$, every physical N) the tail after $k = 2$ is at most $\frac{5}{1024}t^3(1-t/4)^{-1} \le \frac{45}{8192}t^3 < \frac{1}{64}t^3$. Since $|d^{1/2}_{M\sigma}| \le 1$, this gives $|\Phi^{M,\sigma}_{1/2} - d^{1/2}_{M\sigma}(1 + \frac{1}{8N^2} + \frac{3}{128N^4})| \le \frac{1}{64N^6}$. Up to the standard phase and $\sqrt{2j+1}$ normalization these are the spin–orbit Clebsch–Gordan coefficients $\langle l\, m_l; \frac{1}{2}\, m_s | j\, m_j\rangle$ with $m_s = M$ and $j = l \pm \frac{1}{2}$ according to $\sigma = \pm\frac{1}{2}$: in the centered variables they are a Wigner-d value times one x-independent even factor.

Remark 5.8 (Fixed midpoint coordinate versus moving sampling)**.** *The even expansion is a statement at fixed midpoint coordinate x. If one follows a lattice point x_N that moves with N, odd powers appear kinematically by composition: for fixed μ, $x_N = 2\mu/N$ and*

$$d^L_{M\sigma}(\arccos x_N) = d^L_{M\sigma}(\pi/2) + \frac{2\mu}{N}\,\frac{\mathrm{d}}{\mathrm{d}x}d^L_{M\sigma}(\arccos x)\Big|_{x=0} + O(N^{-2}). \tag{86}$$

This N^{-1} term comes from moving the evaluation point, not from an odd deformation of the finite problem, so numerical tests of the intrinsic expansion should hold x fixed along an admissible subsequence. For the kernel itself this is a theorem: for fixed μ the points $x_N = 2\mu/N$ lie in a fixed compact $K \Subset (-1,1)$, so Theorem 5.6 (case $R = 1$) and the Taylor expansion of $d^L_{M\sigma}(\arccos x)$ at $x = 0$ give

$$\Big|\Phi^{M,\sigma}_L(x_N; N^{-1}) - d^L_{M\sigma}(\pi/2) - \frac{2\mu}{N}\,\frac{\mathrm{d}}{\mathrm{d}x}d^L_{M\sigma}(\arccos x)\Big|_{x=0}\Big| \le \frac{C}{N^2}. \tag{87}$$

Corollary 5.9 (Coefficientwise Regge covariance)**.** *The exact N-independent symmetries of the normalized kernel hold as identities of the closed form $G_L^{M,\sigma}$ and are therefore inherited by each Taylor coefficient. The Regge identity (16) acts on the finite labels as $(M,\sigma) \mapsto (-\sigma,-M)$, so*

$$f_{2r}^{L,-\sigma,-M}(x) = f_{2r}^{L,M,\sigma}(x), \tag{88}$$

$$f_{2r}^{L,-M,-\sigma}(x) = (-1)^{M-\sigma} f_{2r}^{L,M,\sigma}(x), \tag{89}$$

$$f_{2r}^{L,\sigma,M}(x) = (-1)^{M-\sigma} f_{2r}^{L,M,\sigma}(x). \tag{90}$$

The differential correction operators themselves are invariant under the bare transposition $M \leftrightarrow \sigma$; the phase in (90) belongs to the Wigner-d function.

Proof. If $L_0 = 0$, then $M = \sigma = 0$ and all three transformations fix the labels, so the identities are immediate. Otherwise, by (84) and Lemma 5.4, $G_L^{M,\sigma}(x,t)$ depends on (M,σ) only through $p_\pm = |M \mp \sigma|$ (hence $\mathcal{E}$, W_{3j}), L_0 (hence the degree), Π_L (which depends only on L), the recursion coefficients R_k and $M\sigma$ of (72), and the constant start $\widetilde{P}_{L_0} = \epsilon(M,\sigma)\sqrt{(2L_0)!/(p_+!\,p_-!)}$, with ϵ the phase (54). Under $(M,\sigma) \mapsto (-\sigma,-M)$, $(-M,-\sigma)$ and (σ,M) the quantities $p_\pm$, L_0, R_k and $M\sigma$ are unchanged, and ϵ is multiplied by 1, $(-1)^{M-\sigma}$ and $(-1)^{M-\sigma}$ respectively (a direct check of the two cases $\sigma \le M$, $\sigma > M$). Since (72) is linear, $\widetilde{P}_L$ is multiplied by the same sign for every L, so the three symmetries hold as exact identities of $G_L^{M,\sigma}(x,t)$ for all (x,t) in its domain, with these signs. Differentiating r times in t at $t = 0$ gives (88)–(90). □

6 Local corrections for the $3j$ symbol

In this section $\Sigma = M^2 + \sigma^2$ and $\tau = M\sigma$, as in (57).

The remaining question is one of *locality*: when does the exact nonlocal unitary S_N of Corollary 5.3 admit a local differential expansion after sampling? For compact $K \Subset (-1, 1)$ define

$$(\mathsf{E}_{N,K} g)(x_j) := g(x_j), \qquad x_j \in \mathcal{X}_N \cap K. \tag{91}$$

The correction operators below are the terms of this sampled local expansion.

Write

$$\Phi = f_0 + h^2 f_2 + h^4 f_4 + \cdots, \qquad f_0 = d^L_{M\sigma}(\arccos x). \tag{92}$$

Expanding (30) and reducing higher derivatives modulo (39) gives the first canonical Taylor coefficient.

Theorem 6.1 (Next-to-leading squeezed $3j$ formula)**.** *Let $f(x) = d^L_{M\sigma}(\arccos x)$ and put $J = L(L+1)$.*

$$A_2(x) = 2J(x^3 - x) - 4x(M^2 + \sigma^2) + 2M\sigma(x^2 + 3) + 4x, \tag{93}$$

$$B_2(x) = J(x^2 - 3) + 2M\sigma x. \tag{94}$$

Then, uniformly on compact $K \Subset (-1, 1)$,

$$\boxed{\Phi^{M,\sigma}_L(x; N^{-1}) = f(x) + \frac{1}{N^2} \frac{A_2(x) f'(x) + B_2(x) f(x)}{12(x^2 - 1)} + O_K(N^{-4}).} \tag{95}$$

Thus

$$f_2(x) = \frac{A_2(x) f'(x) + B_2(x) f(x)}{12(x^2 - 1)}. \tag{96}$$

Proof. By Theorem 5.6, $\Phi_L = d_L + t f_{2,L} + O(t^2)$ with $d_L = d^L_{M\sigma}(\arccos x)$ and $t = N^{-2}$, where the closed form (84) satisfies (18) identically in (x, t) by (72), so its Taylor coefficients satisfy the recurrence order by order. Insert the expansion into (18) and expand $\sqrt{1 - k^2 t} = 1 - \frac{1}{2} k^2 t + O(t^2)$. In the notation (297) of Appendix B, the coefficient of t is

$$\mathcal{L}_{\mathrm{W}}(f_2)_L = \tfrac{1}{2}\big[L R_{L+1}(L+1)^2 d_{L+1} + (L+1) R_L L^2 d_{L-1}\big] = -(2L+1)\lambda_L\big[-\tfrac{1}{2}\mathcal{N}_1[I]\big] d_L, \tag{97}$$

by the neighboring-moment identity (303) with $s = 1$. By (298), $\mathsf{D}d$ is a particular solution for every rank-independent operator D with $[x, \mathsf{D}] = -\frac{1}{2}\mathcal{N}_1[I]$. With $V = (\Sigma - 2\tau x)/(1 - x^2)$, $\Sigma = M^2 + \sigma^2$ and $\tau = M\sigma$, one finds $-\frac{1}{2}\mathcal{N}_1[I] = \frac{1}{2}x(1 - x^2)\partial_x^2 + \frac{1}{2}(1 - 3x^2)\partial_x + \frac{1}{2}(\tau - x - xV)$, and (304) gives

$$\mathsf{D} = -\frac{x(1 - x^2)}{6}\partial_x^3 + \frac{3x^2 - 1}{4}\partial_x^2 - \frac{\tau - x - xV}{2}\partial_x. \tag{98}$$

Reducing the higher derivatives with (39),

$$\mathsf{D} d_L = \frac{A_2 d_L' + B_2 d_L}{12(x^2 - 1)} - \psi_2(x) d_L, \qquad \psi_2(x) = \frac{p_+^2}{8(1 - x)^2} + \frac{p_-^2}{8(1 + x)^2}, \quad p_\pm = |M \mp \sigma|. \tag{99}$$

Since ψ_2 does not depend on L, $\psi_2 d$ solves the homogeneous recurrence, and so does the difference F_L between $f_{2,L}$ and the right side of (96). For $L_0 > 0$ the recurrence is one-sided at L_0 (because $R_{L_0} = 0$), so $F = 0$ once $F_{L_0} = 0$. Since A_2, B_2 depend on (M, σ) only through Σ and τ, and f_2 and f acquire the same sign under the Regge and sign symmetries (Corollary 5.9), it suffices to treat the chamber $M \geq |\sigma|$, in which the minimal rank $L = M$ is the extremal seed. There, differentiating the exact central-factorial formula (27) with respect to t gives

$$\frac{f_2}{f} = \frac{L(L+1)(2L+1)}{12} - \frac{r_+(r_+^2 - 1)}{48 u_+^2} - \frac{r_-(r_-^2 - 1)}{48 u_-^2}, \tag{100}$$

with $L = M$, $r_\pm = L \pm \sigma$ and $u_\pm = (1 \pm x)/2$, while substituting the extremal logarithmic derivative $f'/f = (\sigma - Lx)/(1 - x^2)$ at $L = M$ into (96) gives the same expression; hence $F_{L_0} = 0$. For $M = \sigma = 0$ the homogeneous solutions are fixed by the values at $L = 0, 1$: the exact starts $\Phi_0 = 1$ and $\Phi_1 = x/\sqrt{1-t}$ give $f_2 = 0$ and $f_2 = x/2$, which (96) reproduces ($J = 0$ and $J = 2$). Theorem 5.6 supplies the uniform $O_K(N^{-4})$ remainder. □

Remark 6.2 (Scalar Gaunt specialization)**.** *For integer large multipoles with $M = 0$ and $\mu = 0$, one has $x = 0$, $\sigma = j_+ - j_-$ and*

$$f(0) = d^L_{0\sigma}\left(\frac{\pi}{2}\right). \tag{101}$$

The first two corrections reduce to the particularly simple multiplicative expansion

$$(-1)^{j_+}\sqrt{N}\begin{pmatrix} L & j_- & j_+ \\ 0 & 0 & 0 \end{pmatrix} = d^L_{0\sigma}\left(\frac{\pi}{2}\right)\left[1 + \frac{J}{4N^2} + \frac{J(5J-4)}{32N^4} + O(N^{-6})\right], \qquad J = L(L+1). \tag{102}$$

For this 000 *case the expansion is already implicit in the exact separable Carron–Lewis formula;* (102) *is included as a useful implementation check. The practical extension established here is the general-magnetic correction* (4)*.*

Put $\Sigma = M^2 + \sigma^2$, $\tau = M\sigma$ and $s = 1 - x^2$. The next coefficient has the same first-order form.

Theorem 6.3 (Fourth-order Regge-symmetric squeezed $3j$ formula)**.** *There are rational functions $A_4, B_4 \in \mathbb{Q}(x, J, \Sigma, \tau)$, displayed in Appendix C, such that, with f and J as in Theorem 6.1,*

$$\mathcal{D}_4 = A_4(x)\partial_x + B_4(x) \tag{103}$$

and, uniformly on compact $K \Subset (-1, 1)$,

$$\boxed{\Phi^{M,\sigma}_L(x; N^{-1}) = f(x) + \frac{f_2(x)}{N^2} + \frac{\mathcal{D}_4 f(x)}{N^4} + O_K(N^{-6}).} \tag{104}$$

The displayed N^{-4} coefficient is the canonical t^2 Taylor coefficient of the physical interpolation.

Proof. The case $r = 2$ of the recursion of Appendix B, which is proved there without using the present theorem, gives the t^2 coefficient as $\widehat{\mathcal{D}}_4 d_L$ with $\widehat{\mathcal{D}}_4 = \mathcal{J}(\mathsf{B}_4^{(3j)}) + \psi_4$, exactly as the case $r = 1$ gave f_2 in the previous proof. Reducing the derivatives of order two and higher with (39) gives a first-order representative $A_4\partial_x + B_4$; this reduction is an exact symbolic computation with rational functions of (x, J, Σ, τ), and its result is displayed in Appendix C. The scalar ψ_4 is fixed in the chamber $M \geq |\sigma|$ by the t^2 coefficient (330) of the exact central-factorial seed (the other chambers follow from Corollary 5.9 as in the previous proof), and for $M = \sigma = 0$ by the two exact starts. Since Theorem 5.6 gives analyticity in $t = N^{-2}$, the next remainder is $O_K(N^{-6})$. □

The polynomial dependence on $J = L(L+1)$ in Theorems 6.1 and 6.3 can now be promoted to a universal differential operator on a fixed continuum spectral window. Define $\widehat{\mathcal{D}}_2$ by putting every occurrence of J to the right as H_0:

$$\widehat{\mathcal{D}}_2 = \frac{1}{12(x^2-1)}\Big[2(x^3 - x)\partial_x\mathsf{H}_0 + (x^2 - 3)\mathsf{H}_0 + \{-4x\Sigma + 2\tau(x^2 + 3) + 4x\}\partial_x + 2\tau x\Big]. \tag{105}$$

Define $\widehat{\mathcal{D}}_4$ from Appendix C by the same right-ordering prescription: replace each term $a(x)J^p\partial_x$ of $A_4\partial_x$ by $a(x)\partial_x\mathsf{H}_0^p$ and each term $b(x)J^p$ of B_4 by $b(x)\mathsf{H}_0^p$.

Corollary 6.4 (Sampled local form of the exact transmutation through fourth order)**.** *Fix $L_* \geq L_0$ and $K \Subset (-1,1)$, independently of N, and put $\mathcal{V}_{L_*} = \mathrm{span}\{u_L : L_0 \leq L \leq L_*\}$. Then, for all sufficiently large admissible N,*

$$(\mathsf{S}_N g)|_{\mathcal{X}_N \cap K} = \mathsf{E}_{N,K}\left[I + N^{-2}\widehat{\mathcal{D}}_2 + N^{-4}\widehat{\mathcal{D}}_4\right] g + O_{K,L_*}(N^{-6}) \tag{106}$$

uniformly for g in the unit sphere of the finite-dimensional space $\mathcal{V}_{L_*}$ and pointwise on $\mathcal{X}_N \cap K$ (hence also in $\ell^2(\mathcal{X}_N \cap K, 2h)$).

Proof. For $g = u_L$, right ordering makes $\mathsf{H}_0^p u_L = J^p u_L$, so (106) is exactly Theorems 6.1 and 6.3 evaluated on the physical grid. There are only finitely many basis vectors with $L \leq L_*$, so the individual uniform remainders combine into a uniform operator estimate on $\mathcal{V}_{L_*}$. □

Theorem 6.5 (All-order local differential form of the exact $3j$ transmutation)**.** *Fix M, σ with $M \pm \sigma \in \mathbb{Z}$ (so that L, M, σ are all integers or all half-integers), put $L_0 = \max(|M|, |\sigma|)$, choose a finite spectral cutoff $L_* \geq L_0$, and let $K \Subset (-1,1)$. For every $r \geq 0$ there is a rank-independent finite-order differential operator $\widehat{\mathcal{D}}_{2r}$, generated by rational functions of x, ∂_x, and the continuum Wigner operator H_0, such that $\widehat{\mathcal{D}}_0 = I$ and, for every fixed R,*

$$(\mathsf{S}_N g)|_{\mathcal{X}_N \cap K} = \mathsf{E}_{N,K}\left[\sum_{r=0}^{R-1} N^{-2r}\widehat{\mathcal{D}}_{2r}\right] g + O_{K,L_*,R}(N^{-2R}) \tag{107}$$

uniformly for g in the unit sphere of $\mathcal{V}_{L_*}$ and pointwise on $\mathcal{X}_N \cap K$. The operators are constructive and their only possible real singularities are at $x = \pm 1$. The first two nontrivial members are exactly $\widehat{\mathcal{D}}_2$ and $\widehat{\mathcal{D}}_4$ above.

Proof. This is the local calculus of Proposition B.2 (Appendix B) for the rank recurrence (18). For a rank-independent A, commuting it through the undeformed Wigner rank recurrence expresses the defect through $[x, \mathsf{A}]$, and the weighted neighboring moments L^{2s}, $(L+1)^{2s}$ reduce to a finite local operator on the central mode. With $\sqrt{1-k^2 t} = \sum_{s \geq 0} c_s k^{2s} t^s$ and $\Phi = \sum_r t^r \widehat{\mathcal{D}}_{2r} d$, the coefficient of t^r in (18) becomes

$$[x, \widehat{\mathcal{D}}_{2r}]\, d_L = \sum_{s=1}^{r} c_s\, \mathcal{N}_s[\widehat{\mathcal{D}}_{2(r-s)}]\, d_L, \tag{108}$$

which is solved by $\widehat{\mathcal{D}}_{2r} = \mathcal{J}(\mathsf{B}_{2r}^{(3j)}) + \psi_{2r}(x)$ with $\mathsf{B}_{2r}^{(3j)} = \sum_{s=1}^{r} c_s \mathcal{N}_s[\widehat{\mathcal{D}}_{2(r-s)}]$ and $[x, \mathcal{J}(\mathsf{B})] = \mathsf{B}$. For $L_0 > 0$ (in particular for all half-integer labels), ψ_{2r} is fixed by the exact central-factorial seed at the minimal rank, and Regge/transposition covariance carries the construction through all nondegenerate chambers. For $M = \sigma = 0$ one takes $\psi_{2r} = 0$ ($r \geq 1$), which reproduces $\Phi_0 = 1$, and Proposition B.2(c) gives

$$\widehat{\mathcal{D}}_{2r} x = \frac{\binom{2r}{r}}{4^r}\, x, \tag{109}$$

the Taylor coefficient of $x/\sqrt{1 - N^{-2}}$, so the second start is also reproduced at every order. The recurrence then propagates the same local operator family to all fixed ranks in every physical sector, and these series are the Taylor coefficients of the physical kernel (Appendix B.1, paragraph 'Link to the physical kernels'). Since the scalar Taylor series converges on the fixed window and only finitely many modes $L \leq L_*$ occur, the remainder in (107) is uniform. □

The exact edge coefficient

$$a_h(x)^2 = [(1-x)^2 - h^2(\Sigma - 2\tau)][(1+x)^2 - h^2(\Sigma + 2\tau)] \tag{110}$$

shows simultaneously why the local coefficients have rational Regge-invariant form and why their only real singularities can occur at $x = \pm 1$.

7 The one-small $6j$ symbol: centering and recurrence

Consider

$$W_f(a,b,c;m,n) := \begin{Bmatrix} a & b & c \\ b+m & a+n & f \end{Bmatrix}, \tag{111}$$

where f, m, n are fixed admissible integer or half-integer quantities and a, b, c are large. The triangle $(f, b+m, b)$ requires $f - m \in \mathbb{Z}$, and similarly $f - n \in \mathbb{Z}$, so m and n have the same parity class as f.

Define

$$A = a + \frac{n+1}{2}, \qquad B = b + \frac{m+1}{2}, \qquad C = c + \frac{1}{2}, \tag{112}$$

and the centered angle

$$x := \cos\theta = \frac{A^2 + B^2 - C^2}{2AB}. \tag{113}$$

For occasional dilation arguments we write

$$A = \Lambda\alpha, \qquad B = \Lambda\beta, \qquad C = \Lambda\gamma, \tag{114}$$

with a fixed strict-triangle shape (α, β, γ). The symbol Λ is only a derivational bookkeeping scale and has no intrinsic meaning; all final $6j$ correction operators below are stated directly as homogeneous functions of (A, B, C). Geometrically, A, B, C are the Weyl-shifted large edge lengths of the Edmonds triangle and (113) is exactly its Euclidean law-of-cosines angle.

Set

$$\xi := a + b + c + f + n \in \mathbb{Z}, \qquad \Phi_f(A,B,C) := (-1)^{\xi} 2\sqrt{AB}\, W_f(a,b,c;m,n). \tag{115}$$

This phase is chosen so that the limiting branch is $d^f_{nm}(\theta)$ in the convention of Flude. Replacing n by m in ξ multiplies the entire fixed-(m,n) family by $(-1)^{m-n}$ and instead selects the transposed d^f_{mn} convention; the distinction is invisible when $m - n$ is even.

7.1 Regge symmetry and the Weyl shift explain the midpoint

The midpoint (112) has a direct Regge interpretation. Permute the columns of (111) to write

$$\begin{Bmatrix} c & a & b \\ f & b+m & a+n \end{Bmatrix}. \tag{116}$$

Applying the Regge symmetry [30] (DLMF 34.5.9) gives exactly

$$\begin{Bmatrix} a & b & c \\ b+m & a+n & f \end{Bmatrix} = \begin{Bmatrix} c & b+(m-n)/2 & a+(n-m)/2 \\ f & a+(m+n)/2 & b+(m+n)/2 \end{Bmatrix}. \tag{117}$$

Now add the $SU(2)$ Weyl shift $j \mapsto j + \frac{1}{2}$ to each large entry. The five large Weyl lengths in (117) become

$$C, \qquad B - \frac{n}{2}, \qquad A - \frac{m}{2}, \qquad A + \frac{m}{2}, \qquad B + \frac{n}{2}. \tag{118}$$

Thus Regge symmetry reorganizes the large representation labels into two symmetric Weyl pairs centered at A and B. The finite differences m and n become precisely the magnetic indices of the limiting Wigner rotation. Equivalently, A and B are the midpoints of the Weyl lengths of the two pairs $(a, a+n)$ and $(b, b+m)$ that form triangles with f, and C is the Weyl length of c.

7.2 Exact centered Racah recurrence

Write the $6j$ in the form

$$\begin{Bmatrix} f & b+m & b \\ c & a & a+n \end{Bmatrix}. \tag{119}$$

Apply the standard DLMF recurrence 34.5.11 in its first entry f. With

$$R_k := \sqrt{(k^2-m^2)(k^2-n^2)}, \tag{120}$$

we have, from DLMF 34.5.13,

$$\begin{aligned} E(k) &= \sqrt{(k^2-m^2)[(2B)^2-k^2](k^2-n^2)[(2A)^2-k^2]} \\ &= 4AB\, R_k\, \eta_k, \end{aligned} \tag{121}$$

where

$$\eta_k := \sqrt{\left(1-\frac{k^2}{4A^2}\right)\left(1-\frac{k^2}{4B^2}\right)}. \tag{122}$$

In the notation of DLMF 34.5.11–34.5.13 we take $(j_1, j_2, j_3) = (f, b+m, b)$ and $(l_1, l_2, l_3) = (c, a, a+n)$, and write $J_r = j_r(j_r+1)$, $L_r = l_r(l_r+1)$. A direct simplification of the diagonal coefficient (Appendix D) gives

$$\begin{aligned} &(J_3+J_2-J_1)(L_3+L_2-J_1) - 2(J_3L_3+J_2L_2-J_1L_1) \\ &\quad = -4AB\left\{f(f+1)[x-q_f]-mn\right\}, \end{aligned} \tag{123}$$

with

$$q_f := \frac{2f(f+1)+1-m^2-n^2}{8AB}. \tag{124}$$

The factor $(-1)^f$ in (115) reverses the sign of the neighboring recurrence terms. Dividing the exact DLMF recurrence by $-4AB$ therefore yields:

Lemma 7.1 (Exact midpoint Racah recurrence)**.** *The normalized coefficient* (115) *satisfies exactly*

$$\begin{aligned} &(2f+1)\{f(f+1)[x-q_f]-mn\}\Phi_f \\ &\quad = fR_{f+1}\eta_{f+1}\Phi_{f+1} + (f+1)R_f\eta_f\Phi_{f-1}. \end{aligned} \tag{125}$$

Under the formal dilation (114), x is independent of Λ and

$$q_f = \Lambda^{-2}\frac{2f(f+1)+1-m^2-n^2}{8\alpha\beta}, \tag{126}$$

$$\eta_k = \sqrt{\left(1-\frac{k^2}{4\alpha^2\Lambda^2}\right)\left(1-\frac{k^2}{4\beta^2\Lambda^2}\right)}. \tag{127}$$

Thus every coefficient of the exact recurrence is holomorphic in Λ^{-2} near the origin after choosing the branch equal to 1 there.

7.3 The midpoint is selected within the admitted gauge class

The absence of a first-order term is not obtainable from an arbitrary shift.

Proposition 7.2 (Uniqueness of the affine centering in the canonical gauge)**.** *Work with the normalized kernel* (115)*. Admit only (i) an additional scalar renormalization of the kernel independent of the recurrence index f and (ii) multiplication of the whole three-term equation by a scalar independent of f. Consider*

$$A' = a + u, \qquad B' = b + v, \qquad C' = c + w, \tag{128}$$

with fixed $u, v, w = O(1)$, and hold the corresponding angle

$$x' = \frac{A'^2 + B'^2 - C'^2}{2A'B'} \tag{129}$$

fixed under a proportional dilation $A' = \Lambda\alpha$, $B' = \Lambda\beta$, $C' = \Lambda\gamma$. If the generic Λ^{-1} subprincipal operator vanishes on an open set of strict-triangle shapes, then uniquely

$$u = \frac{n+1}{2}, \qquad v = \frac{m+1}{2}, \qquad w = \frac{1}{2}. \tag{130}$$

The uniqueness is relative to this canonical, f-independent gauge class and the declared recurrence; the affine coordinates are not claimed to be invariant under arbitrary f-dependent recoupling gauges.

Proof. Write the canonical centered lengths in terms of the trial ones as

$$A = A' + \frac{\Delta_A}{2}, \qquad B = B' + \frac{\Delta_B}{2}, \qquad C = C' + \frac{\Delta_C}{2}, \tag{131}$$

where

$$\Delta_A = n + 1 - 2u, \qquad \Delta_B = m + 1 - 2v, \qquad \Delta_C = 1 - 2w. \tag{132}$$

After division of the exact DLMF recurrence by $4A'B'$, the two off-diagonal square roots have the common first-order expansion

$$\frac{E(k)}{4A'B'R_k} = 1 + \frac{s}{\Lambda} + O(\Lambda^{-2}), \qquad s := \frac{\Delta_A}{2\alpha} + \frac{\Delta_B}{2\beta}, \tag{133}$$

independently of the neighboring rank k; the k-dependence begins only at order Λ^{-2}. Thus an allowed scalar normalization of the whole equation can remove this common factor, but nothing more.

The canonical law-of-cosines angle is

$$x = \frac{A^2 + B^2 - C^2}{2AB} = x' + \frac{X_1}{\Lambda} + O(\Lambda^{-2}), \qquad X_1 = \frac{\Delta_A}{2}\partial_\alpha x' + \frac{\Delta_B}{2}\partial_\beta x' + \frac{\Delta_C}{2}\partial_\gamma x'. \tag{134}$$

The diagonal carries the same factor. By (123) the exact diagonal coefficient divided by $4A'B'$ is

$$\frac{AB}{A'B'}\,(2f+1)\{\lambda_f[x - q_f] - mn\}, \qquad \lambda_f = f(f+1), \tag{135}$$

and $AB/(A'B') = 1 + s/\Lambda + O(\Lambda^{-2})$ is exactly the first-order part of (133). So the whole equation carries one f-independent factor at order Λ^{-1}, which the admitted normalization removes. Since q_f and $\eta_k - 1$ start at Λ^{-2}, the complete first-order operator is then

$$(2f+1)\lambda_f X_1. \tag{136}$$

This must vanish for every admissible rank, and any single admissible rank $f > 0$ already gives $X_1 = 0$; no distinction between $mn \neq 0$ and $mn = 0$ is needed. Differentiating the cosine law explicitly,

$$X_1 = \frac{\Delta_A(\alpha^2 - \beta^2 + \gamma^2)}{4\alpha^2\beta} + \frac{\Delta_B(\beta^2 - \alpha^2 + \gamma^2)}{4\alpha\beta^2} - \frac{\Delta_C\gamma}{2\alpha\beta}. \tag{137}$$

Multiplying by $4\alpha^2\beta^2$ gives the polynomial identity

$$\Delta_A\beta(\alpha^2 - \beta^2 + \gamma^2) + \Delta_B\alpha(\beta^2 - \alpha^2 + \gamma^2) - 2\Delta_C\alpha\beta\gamma = 0 \tag{138}$$

on an open set of strict-triangle shapes. The coefficient of γ forces $\Delta_C = 0$, and the coefficient of γ^2 then forces $\Delta_A\beta + \Delta_B\alpha = 0$ for all independent α, β, hence $\Delta_A = \Delta_B = 0$. This is the stated u, v, w.

All three conditions come from the first-order drift X_1 of the law-of-cosines point. The first-order deformation of the neighboring Racah coefficients is the common factor $AB/(A'B')$ of the whole equation and is absorbed by the admitted normalization. Thus, within the admitted gauge class, the midpoint is the unique affine choice for which the trial angle agrees with the exact recurrence angle, $x' = x + O(\Lambda^{-2})$. This is the $6j$ analogue of the fixed-point versus moving-sampling distinction in Remark 5.8. McFarlane's recurrence-matched angle [28, Eq. (A10)] is $x + (m^2 + n^2)/(8AB)$ in these variables, so it satisfies this first-order condition; the proposition shows that this condition forces the affine shifts (112). □

8 Stretched seed, finite Racah system and the even $6j$ theorem

8.1 Exact stretched initial data, including half-integer parity

The recurrence alone does not determine the parity of the physical solution; its initial normalization must have the same property. We now prove this exactly.

By ordinary $6j$ symmetries, it suffices for analyticity to treat the sector

$$m \geq |n|, \qquad m \geq 0. \tag{139}$$

Here the reduction can be written explicitly. Interchanging the first two columns sends

$$(a, b; m, n; A, B) \longmapsto (b, a; n, m; B, A). \tag{140}$$

Interchanging upper and lower entries in the first two columns and then swapping those columns sends the same numerical $6j$ symbol to the family with

$$(a, b; m, n) \longmapsto (a + n, b + m; -m, -n), \tag{141}$$

and its centered variables are exactly

$$A' = (a + n) + \frac{-n + 1}{2} = A, \qquad B' = (b + m) + \frac{-m + 1}{2} = B. \tag{142}$$

These two operations allow the larger of $|m|, |n|$ to be called m and then made nonnegative. The phase exponent ξ changes only by a fixed integer (for example by $n - m$ in either basic operation), so the normalization changes at most by a dilation-independent sign. Thus even holomorphic dependence on the inverse dilation is unaffected by the reduction.

At the lowest rank $f = m$, the triangle $(m, b + m, b)$ is stretched, so the Racah sum contains one term. Recall the central product (Section 5.3), for nonnegative integer r,

$$\mathfrak{C}_r(X) := \prod_{q=-(r-1)/2}^{(r-1)/2} (X + q), \qquad \mathfrak{C}_0 = 1. \tag{143}$$

Also define the parity-dependent set

$$Q_m := \begin{cases} \{1, 2, \ldots, m\}, & m \in \mathbb{Z}, \\ \{\frac{1}{2}, \frac{3}{2}, \ldots, m\}, & m \in \mathbb{Z} + \frac{1}{2}, \end{cases} \tag{144}$$

$$\mathfrak{D}_m(X) := \prod_{q \in Q_m} [(2X)^2 - q^2], \tag{145}$$

and

$$\chi_m(A, B) := \begin{cases} 1, & m \in \mathbb{Z}, \\ 4AB, & m \in \mathbb{Z} + \frac{1}{2}. \end{cases} \tag{146}$$

Lemma 8.1 (Positive stretched seed)**.** *In the sector* (139)*, let* $\mathcal{S}_m(A, B, C)$ *denote the right-hand side of*

$$\begin{aligned} \mathcal{S}_m = \frac{\Gamma(2m+1)}{\Gamma(m+n+1)\Gamma(m-n+1)} \frac{\chi_m(A,B)}{\mathfrak{D}_m(A)\mathfrak{D}_m(B)} \\ \times \mathfrak{C}_{m-n}(A - B + C)\mathfrak{C}_{m-n}(-A + B + C) \\ \times \mathfrak{C}_{m+n}(A + B - C)\mathfrak{C}_{m+n}(A + B + C). \end{aligned} \tag{147}$$

Then on every physical admissible point

$$\Phi_m = \sqrt{\mathcal{S}_m(A, B, C)}, \tag{148}$$

where the square root is the positive real one. In particular $\Phi_m^2 = \mathcal{S}_m$*.*

Proof. Insert $f = m$ into the Racah factorial definition of the $6j$ symbol. In the standard Racah summation variable z, the triangle $(m, b + m, b)$ is stretched. Two of the factorial constraints are then

$$z \geq a + b + c + m + n, \qquad z \leq a + b + c + m + n, \tag{149}$$

so the only possible summand is

$$z_0 = a + b + c + m + n. \tag{150}$$

Thus the Racah sum collapses exactly to one term, for integer or half-integer spins alike (all factorials being understood as gamma functions). Its Racah sign is $(-1)^{z_0}$. At $f = m$ the normalization exponent is now exactly

$$\xi = a + b + c + m + n = z_0, \tag{151}$$

so the Racah sign and the normalization phase cancel identically: $(-1)^{\xi + z_0} = +1$. Thus the physical stretched seed is the positive branch, in both integer and half-integer parity sectors. Squaring removes the remaining square roots. Group the four triangle-factor ratios into symmetric gamma ratios. Each numerator ratio is of the form

$$\frac{\Gamma(X + (r+1)/2)}{\Gamma(X - (r-1)/2)} = \mathfrak{C}_r(X), \tag{152}$$

with $r = m \pm n \in \mathbb{Z}_{\geq 0}$. The two large-spin denominator ratios give $\mathfrak{D}_m(A)\mathfrak{D}_m(B)$. If m is half-integral, the half-integer product starts at $q = \frac{1}{2}$ rather than $q = 1$; the uncancelled central factors contribute the additional $4AB$. Straight cancellation gives (147), and taking the positive physical branch gives (148). (The formula reduces directly to DLMF's $f = \frac{1}{2}$ special cases when $m = \frac{1}{2}$.) □

Under the formal dilation $A = \Lambda\alpha$, $B = \Lambda\beta$, $C = \Lambda\gamma$, every central factor has the form

$$\mathfrak{C}_r(\Lambda S) = \Lambda^r \prod_q (S + q\Lambda^{-1}). \tag{153}$$

By Lemma 3.1, the product after extraction of Λ^r is an even polynomial in Λ^{-1}. Each $\mathfrak{D}_m(\Lambda\alpha)$ is likewise a monomial in Λ times a polynomial in Λ^{-2}. The extra $4AB$ in the half-integer sector is a pure Λ^2 monomial and creates no odd correction. The powers of Λ cancel exactly in (147). Since the strict triangle inequalities make all four combinations $\alpha \pm \beta \pm \gamma$ appearing in the seed nonzero, the squared seed is holomorphic in $t = \Lambda^{-2}$ and strictly positive at $t = 0$. The square-root branch equal to the positive physical branch at $t = 0$ is therefore holomorphic near the origin and, by (148), is exactly the physical stretched seed.

Exceptional scalar start. If the canonical reduction gives $m = 0$, then necessarily $n = 0$, and (125) at $f = 0$ is the identity $0 = 0$. The two physical starts are instead

$$\Phi_0 = 1, \qquad \Phi_1 = \frac{x - \dfrac{1}{8AB}}{\sqrt{\left(1 - \dfrac{1}{4A^2}\right)\left(1 - \dfrac{1}{4B^2}\right)}}. \tag{154}$$

The first value is the zero-entry $6j$ identity; the second follows from the elementary $f = 1$ formula. Under any strict-triangle dilation both are holomorphic in the inverse-square scale and $\Phi_1 \to x = d^1_{00}(\theta)$.

8.2 Exact finite Racah spectral system on the physical c-lattice

The rank recurrence above treats f as the discrete variable and the centered angle x as the spectral coordinate. Racah bispectrality supplies the exact dual picture: for fixed a, b, m, n (with $b + m, a + n \geq 0$ and $f_1 \geq f_0$ below, so that the lattices are non-empty), the same $6j$ symbol is a finite orthogonal eigenfunction on the physical c-lattice, with $f(f+1)$ as undeformed spectrum.

The two c-triangles give

$$c_{\min} = \max\{|a - b|, |b + m - a - n|\} = |A - B| + \frac{|m - n|}{2}, \tag{155}$$

$$c_{\max} = \min\{a + b, a + b + m + n\} = A + B - 1 - \frac{|m + n|}{2}. \tag{156}$$

The admissible rank range is

$$f_0 := \max(|m|, |n|), \qquad f_1 := \min(2a + n, 2b + m) = 2\min(A, B) - 1. \tag{157}$$

Since

$$\frac{|m - n| + |m + n|}{2} = \max(|m|, |n|), \tag{158}$$

one has

$$c_{\max} - c_{\min} = f_1 - f_0. \tag{159}$$

Thus the c- and f-lattices have the same finite cardinality.

Put

$$z_c := c(c + 1) = \left(c + \frac{1}{2}\right)^2 - \frac{1}{4}, \tag{160}$$

and define the exact edge factor by taking $\mathcal{A}(c)$ to be the nonnegative square root, on every physical link, of

$$\begin{aligned}\mathcal{A}(c)^2 := {} & [c^2-(a-b)^2][(a+b+1)^2-c^2] \\ & \times [c^2-(b+m-a-n)^2][(a+b+m+n+1)^2-c^2].\end{aligned} \tag{161}$$

Equivalently, in centered variables the four factors vanish at $|c|$ equal to the absolute values of

$$A-B+\frac{m-n}{2},\quad -A+B+\frac{m-n}{2},\quad A+B-\frac{m+n}{2},\quad A+B+\frac{m+n}{2}. \tag{162}$$

Also set

$$\kappa_{\mathrm{R}} := 1-2(A^2+B^2)-\frac{m^2+n^2}{2}, \tag{163}$$

$$\Omega_{\mathrm{R}} := \frac{1}{16}\big[(m+n)^2-4(A+B)^2\big]\big[(m-n)^2-4(A-B)^2\big]. \tag{164}$$

For $c>0$ define

$$a_c^+ := \frac{\mathcal{A}(c+1)}{2(2c+1)(c+1)}, \qquad a_c^- := \frac{\mathcal{A}(c)}{2(2c+1)c}, \tag{165}$$

$$b_c := -\frac{z_c^2+\kappa_{\mathrm{R}} z_c+\Omega_{\mathrm{R}}}{2z_c}. \tag{166}$$

If $c=0$ is physical, then necessarily $A=B$ and $m=n$, so $\Omega_{\mathrm{R}}=0$; in that case (166) has the removable continuation

$$b_0 := -\frac{\kappa_{\mathrm{R}}}{2}. \tag{167}$$

The a_0^- term is simply absent.

Finally define the positive lattice weight

$$\omega_c := \frac{2c+1}{2AB}. \tag{168}$$

This has a direct midpoint meaning. If

$$\widehat{x}(s) := \frac{A^2+B^2-s^2}{2AB}, \tag{169}$$

then

$$\omega_c = \widehat{x}(c)-\widehat{x}(c+1), \tag{170}$$

so ω_c is exactly the law-of-cosines x-width of the cell centered at the Weyl point $C=c+\frac{1}{2}$.

Theorem 8.2 (Exact finite self-adjoint Racah–Wigner system)**.** *On $\ell^2(\mathcal{C},\omega)$, where*

$$\mathcal{C}=\{c_{\min},c_{\min}+1,\dots,c_{\max}\}, \tag{171}$$

define

$$(\mathsf{H}_{\mathrm{R}}q)_c = -a_c^+ q_{c+1}+b_c q_c-a_c^- q_{c-1}, \tag{172}$$

with the absent boundary terms omitted and with (167) *at the physical $c=0$ endpoint. Then H_{R} is self-adjoint and has simple undeformed spectrum*

$$\operatorname{spec}\mathsf{H}_{\mathrm{R}} = \{f(f+1): f=f_0,f_0+1,\dots,f_1\}. \tag{173}$$

An orthonormal eigenbasis is

$$v_f(c) := \sqrt{\frac{2f+1}{2}}\,\Phi_f(c), \qquad \mathsf{H}_{\mathrm{R}}v_f = f(f+1)v_f. \tag{174}$$

Equivalently, the physical kernels satisfy the exact discrete orthogonality and completeness relations

$$\sum_{c\in\mathcal{C}} \omega_c \Phi_f(c)\Phi_g(c) = \frac{2}{2f+1}\delta_{fg}, \tag{175}$$

$$\sum_{f=f_0}^{f_1} \frac{2f+1}{2}\Phi_f(c)\Phi_f(c') = \frac{\delta_{cc'}}{\omega_c}. \tag{176}$$

Proof. Apply DLMF 34.5.11–34.5.13 to

$$\begin{Bmatrix} c & a & b \\ f & b+m & a+n \end{Bmatrix}. \tag{177}$$

Writing the DLMF diagonal as a part independent of $f(f+1)$ plus $2z_c f(f+1)$, direct centered simplification gives

$$D_0(c) = z_c^2 + \kappa_{\mathrm{R}} z_c + \Omega_{\mathrm{R}}. \tag{178}$$

The c-dependent phase in (115) changes sign under $c \mapsto c \pm 1$, so solving the recurrence for $f(f+1)\Phi_f(c)$ gives exactly (172). The closure is exact because

$$\mathcal{A}(c_{\min}) = 0, \qquad \mathcal{A}(c_{\max}+1) = 0. \tag{179}$$

At a physical $c = 0$ endpoint, $A = B$ and $m = n$ make $\Omega_{\mathrm{R}} = 0$; the continued row (167) follows directly from the $c = 0$ and, when $c_{\max} \geq 1$, the $c = 1$ special $6j$ formulas (DLMF 34.5.1 and 34.5.7).

Self-adjointness in the weight (168) follows from the detailed-balance identity

$$\omega_c a_c^+ = \omega_{c+1} a_{c+1}^- = \frac{\mathcal{A}(c+1)}{4AB(c+1)}. \tag{180}$$

Finally DLMF 34.5.14, used with $j_3 = c$ and $l_3 = f$, gives (175) after inserting the normalization (115). The dimension identity (159) makes these orthogonal eigenvectors a complete basis, giving (176) and the stated spectrum. □

8.3 Explicit physical Racah-polynomial realization

In the canonical sector (139) one has $f_0 = m$ and the stretched kernel $\Phi_m(c)$ is positive on the physical lattice by Lemma 8.1. Define

$$\pi_c := \omega_c \frac{2m+1}{2}\Phi_m(c)^2 = \frac{(2c+1)(2m+1)}{4AB}\,\mathcal{S}_m(A,B,c+\tfrac{1}{2}). \tag{181}$$

Then $\pi_c > 0$ and $\sum_c \pi_c = 1$. For $r = 0, 1, \ldots, D_{\mathrm{R}}$, where

$$D_{\mathrm{R}} := f_1 - m = c_{\max} - c_{\min}, \tag{182}$$

put

$$\mathcal{U}_r(c) := \sqrt{\frac{2(m+r)+1}{2m+1}}\,\frac{\Phi_{m+r}(c)}{\Phi_m(c)}. \tag{183}$$

Set $y := c - c_{\min}$ and keep $D_{\mathrm{R}} = c_{\max} - c_{\min}$. In the standard terminating Racah convention [31] define

$$Q_r^{\mathrm{R}}(y) := {}_4F_3\left(\begin{matrix} -r,\ r+2m+1,\ -y,\ y+2c_{\min}+1 \\ m-n+1,\ m+n+c_{\min}+c_{\max}+2,\ -D_{\mathrm{R}} \end{matrix}; 1\right). \tag{184}$$

Equivalently these are the standard Racah parameters

$$\alpha_{\mathrm{R}} = m - n, \qquad \beta_{\mathrm{R}} = m + n, \qquad \gamma_{\mathrm{R}} = -D_{\mathrm{R}} - 1, \qquad \delta_{\mathrm{R}} = c_{\min} + c_{\max} + 1. \tag{185}$$

Indeed $\alpha_{\mathrm{R}} + \beta_{\mathrm{R}} = 2m$ and $\gamma_{\mathrm{R}} + \delta_{\mathrm{R}} + 1 = 2c_{\min} + 1$, so the Racah quadratic lattice is exactly

$$\zeta_c := z_c - z_{c_{\min}} = y(y + 2c_{\min} + 1) = (c - c_{\min})(c + c_{\min} + 1). \tag{186}$$

The orthonormality (188) below is (175) rewritten. Moreover, the rank recurrence (125) at the nodes x_c of (202) has c-independent coefficients and is one-sided at $f = m$ (because $R_m = 0$), so $\Phi_{m+r}(c)/\Phi_m(c)$ is a polynomial of exact degree r in x_c (for $m = 0$ the two starts (154) play this role), hence in ζ_c, which is affine in x_c. Orthonormal polynomials for a positive weight on the $D_{\mathrm{R}} + 1$ points of $\mathcal{C}$ are unique up to sign, and the weight π_c is proportional to the standard Racah weight with the parameters (185) [16, Sec. 9.2], as a comparison of the ratios π_{c+1}/π_c obtained from (147) shows. This is the classical identification of $6j$ symbols with Racah polynomials [32, 15]; in the present normalization it reads

$$\mathcal{U}_r(c) = \mathcal{U}_r(c_{\min})\, Q_r^{\mathrm{R}}(c - c_{\min}). \tag{187}$$

Thus the physical ratio (183) is the standard terminating Racah polynomial with the parameters (185), times its orthonormal endpoint normalization, and the orthogonality is simply

$$\sum_{c \in \mathcal{C}} \pi_c \mathcal{U}_r(c) \mathcal{U}_s(c) = \delta_{rs}. \tag{188}$$

So the positive stretched seed, multiplied by the exact cell weight, is the Racah orthogonality weight of the dual finite spectral problem.

Equivalently, the ground-state transform

$$(\mathcal{L}_{\mathrm{R}} p)(c) := \Phi_m(c)^{-1} \big(\mathsf{H}_{\mathrm{R}} - m(m+1)\big)\big(\Phi_m p\big)(c) \tag{189}$$

is therefore the standard finite Racah difference operator with parameters (185), and satisfies

$$\mathcal{L}_{\mathrm{R}} \mathcal{U}_r = r(r + 2m + 1)\mathcal{U}_r. \tag{190}$$

This is the exact polynomial form of the bispectral duality used implicitly in the hard-edge analysis in Section 10.

8.4 Exact Racah-to-Wigner transmutation

Let

$$\mathsf{H}_0^{(m,n)} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right] + \frac{m^2 + n^2 - 2mnx}{1 - x^2} \tag{191}$$

and define the normalized continuum modes

$$u_f(x) := \sqrt{\frac{2f+1}{2}}\, d_{nm}^f(\arccos x), \qquad f_0 \le f \le f_1. \tag{192}$$

They are orthonormal in $L^2(-1, 1)$ and satisfy

$$\mathsf{H}_0^{(m,n)} u_f = f(f+1) u_f. \tag{193}$$

Let

$$\mathcal{V}_{A,B}^{m,n} := \mathrm{span}\{u_f : f_0 \le f \le f_1\}. \tag{194}$$

Corollary 8.3 (Exact finite Racah-to-Wigner spectral map)**.** *The basis map*

$$\mathsf{S}_{\mathrm{R}} u_f = v_f \tag{195}$$

is unitary from $\mathcal{V}_{A,B}^{m,n}$ *onto* $\ell^2(\mathcal{C},\omega)$*, has kernel*

$$K_{\mathrm{R}}(c,x) = \sum_{f=f_0}^{f_1} \frac{2f+1}{2}\,\Phi_f(c) d_{nm}^f(\arccos x), \tag{196}$$

in the sense that

$$(\mathsf{S}_{\mathrm{R}} g)(c) = \int_{-1}^{1} K_{\mathrm{R}}(c,x) g(x)\,\mathrm{d}x. \tag{197}$$

Its inverse is the weighted adjoint

$$(\mathsf{S}_{\mathrm{R}}^{-1} q)(x) = \sum_{c\in\mathcal{C}} \omega_c K_{\mathrm{R}}(c,x) q(c), \qquad q \in \ell^2(\mathcal{C},\omega), \tag{198}$$

and it intertwines the exact finite and continuum Hamiltonians:

$$\mathsf{H}_{\mathrm{R}}\mathsf{S}_{\mathrm{R}} = \mathsf{S}_{\mathrm{R}}(\mathsf{H}_0^{(m,n)}|_{\mathcal{V}_{A,B}^{m,n}}). \tag{199}$$

Proof. Immediate from the continuum and finite orthonormal eigenbases and their common simple spectrum. □

Remark 8.4 (Exact Egorov form of the Racah transmutation)**.** *Let* X_c *be multiplication by* x_c *on* $\ell^2(\mathcal{C},\omega)$*, with* x_c *the node* (202) *below, let* X *be multiplication by* x *on* $L^2(-1,1)$*, and let* P *be the orthogonal projection onto* $\mathcal{V}_{A,B}^{m,n}$*. Then*

$$\mathsf{S}_{\mathrm{R}}^{*}\mathsf{X}_c\mathsf{S}_{\mathrm{R}} = \frac{2\mathsf{H}_0^{(m,n)} + 1 - m^2 - n^2}{8AB} + \sqrt{\Big(1 - \frac{\mathrm{ad}^2_{\mathsf{H}_0^{(m,n)}}}{16A^2}\Big)\Big(1 - \frac{\mathrm{ad}^2_{\mathsf{H}_0^{(m,n)}}}{16B^2}\Big)}\,(P\mathsf{X}P) \tag{200}$$

on $\mathcal{V}_{A,B}^{m,n}$*, where the square root is the positive multiplier on the matrix units of* $P\mathsf{X}P$*. Indeed,* (125) *and its zero-scale form* (219)*, multiplied by the orthonormalization factors* $\sqrt{(2f+1)/2}$*, give the action of* x_c *on* v_f *and of* x *on* u_f*. They differ only by the diagonal shift* q_f *and by the factor* η_f *on the link* $f-1 \leftrightarrow f$*, where* $\mathrm{ad}_{\mathsf{H}_0^{(m,n)}}$ *has eigenvalue* $\pm 2f$*; in the scalar sector the* $f=0$ *row is* (154)*. Since every entry of* $P\mathsf{X}P$ *is at most* 1 *in modulus and* η_f *decreases in* f*, Schur's test gives, on the span of* u_f *with* $f \le f_* < f_1$*,*

$$\|\mathsf{S}_{\mathrm{R}}^{*}\mathsf{X}_c\mathsf{S}_{\mathrm{R}} - P\mathsf{X}P\| \le \frac{2f_*(f_*+1)+1+m^2+n^2}{8AB} + 2(1-\eta_{f_*+1}) = O((f_*+1)^2(A^{-2}+B^{-2})), \tag{201}$$

so the coordinate is intertwined up to $o(1)$ *on every window* $f_* = o(\min(A,B))$*.*

For the physical midpoint nodes

$$x_c := \frac{A^2 + B^2 - (c+\frac{1}{2})^2}{2AB}, \tag{202}$$

define the sampling map $(\mathsf{E}_{A,B} g)(c) = g(x_c)$. On a strict-triangle cone $\mathcal{T}_\delta$ (218), in the canonical sector (139) (other chambers by the reductions of Section 8.1), on every fixed interior spectral window and compact $K \Subset (-1,1)$, Theorem 8.7 below and the local calculus of Section 9 give, in intrinsic homogeneous form,

$$\mathsf{S}_{\mathrm{R}} = \mathsf{E}_{A,B}\Big[I + \mathcal{G}_2(A,B,C) + \mathcal{G}_4(A,B,C) + O_K(H^{-6})\Big] \tag{203}$$

coefficientwise on every fixed finite set of Wigner ranks, pointwise at the nodes with $x_c \in K$, where $\mathcal{G}_{2r}$ is homogeneous of degree $-2r$ in (A,B,C). The centered $6j$ correction operators are thus the local differential normal form of an exact finite Racah–Wigner transmutation, not merely corrections to isolated symbols.

8.5 The edge factor and the square-root-free closed form

As for $3j$ (Section 5.3), the even theorem rests on a closed form of the physical kernel in which the large scale enters only through inverse squares. As there, $\varsigma := (1-x)/2$. Put $p_{\min} := |m-n|$, $p_{\max} := |m+n|$ and, with the edge factor $\mathcal{R}_p$ of (62),

$$\begin{aligned} E_{A,B}(C) &:= \mathcal{R}_{p_{\min}}(A-B+C)\,\mathcal{R}_{p_{\min}}(-A+B+C)\,\mathcal{R}_{p_{\max}}(A+B-C)\,\mathcal{R}_{p_{\max}}(A+B+C), \\ W_{6j}(x) &:= \left(\frac{1-x}{2}\right)^{p_{\min}/2}\left(\frac{1+x}{2}\right)^{p_{\max}/2}. \end{aligned} \tag{204}$$

The two c-triangles (155)–(156) give the exact centered endpoints

$$C_{\min} = |A-B| + \frac{1+p_{\min}}{2}, \qquad C_{\max} = A+B-\frac{1+p_{\max}}{2}, \tag{205}$$

and we write $C_r^{\min} = C_{\min}+r$, $C_r^{\max} = C_{\max}-r$ for the inward lattice points, with centered cosines $x_r^{\min}$, $x_r^{\max}$. All four arguments in (204) are at least $(p+1)/2$ on the physical lattice. The law of cosines gives

$$(A-B+C)(-A+B+C) = 4AB\,\frac{1-x}{2}, \qquad (A+B-C)(A+B+C) = 4AB\,\frac{1+x}{2}, \tag{206}$$

so at $C_r^{\min}$ one factor $-|A-B|+C$ equals $r+(p_{\min}+1)/2$, and at $C_r^{\max}$ the factor $A+B-C$ equals $r+(p_{\max}+1)/2$.

Since d^f_{nm} is the function $d^L_{M\sigma}$ of (68) with $(L,M,\sigma)=(f,n,m)$, its Jacobi representation is

$$d^f_{nm}(\arccos x) = k_{6j}\,W_{6j}(x)\,\widehat{P}^{(p_{\min},p_{\max})}_{f-f_0}(\varsigma)\,, \qquad k_{6j} = \epsilon(n,m)\,\kappa_{f-f_0}(p_{\min},p_{\max}), \tag{207}$$

and in the canonical chamber $m \ge |n|$ one has $\epsilon(n,m)=1$.

Put

$$\Pi_f(A,B) := \prod_{q\in Q_f}\Big(1-\frac{q^2}{4A^2}\Big)\Big(1-\frac{q^2}{4B^2}\Big) = \prod_{q\in Q_f}\eta_q^2, \tag{208}$$

with Q_f the parity-dependent set of Section 8.1 and the empty product equal to 1. In what follows $u=A^{-2}$, $v=B^{-2}$, $w=(AB)^{-1}$, $s_1=u+v$ and $s_2=uv$; note that $\eta_f^2 = 1-\frac{f^2}{4}s_1+\frac{f^4}{16}s_2$.

Lemma 8.5 (Square-root-free closed form of the $6j$ kernel)**.** *Fix admissible f,m,n in any chamber, with $f_0=\max(|m|,|n|)$, $p_{\min}=|m-n|$, $p_{\max}=|m+n|$ and $d=f-f_0$. On the physical lattice*

$$\Phi_f = \varepsilon_{mn}\,E_{A,B}\,W_{6j}\,\Pi_f^{-1/2}\,\widetilde{P}_f(\varsigma;A,B), \tag{209}$$

with positive square roots. The sign is $\varepsilon_{mn}=\epsilon(n,m)$ of (54), that is $(-1)^{m-n}$ for $m\le n$ and 1 for $m>n$. Here $\widetilde{P}_f(\varsigma;A,B) = \widetilde{P}_f(\varsigma;s_1,s_2,w)$ is the value, at $(s_1,s_2,w)=(A^{-2}+B^{-2},A^{-2}B^{-2},(AB)^{-1})$, of a polynomial in ς,s_1,s_2,w (so it is symmetric under $A\leftrightarrow B$), and it depends on (m,n) only through $(p_{\min},p_{\max})$. It satisfies the square-root-free recurrence

$$(2f+1)\{\lambda_f[1-2\varsigma-q_f]-mn\}\widetilde{P}_f = (f+1)R_f\eta_f^2\widetilde{P}_{f-1} + fR_{f+1}\widetilde{P}_{f+1}, \qquad \lambda_f=f(f+1), \tag{210}$$

with the start $\widetilde{P}_{f_0}=\sqrt{\binom{2f_0}{p_{\min}}}$ (for $f_0=0$ the two starts $\widetilde{P}_0=1$, $\widetilde{P}_1=1-2\varsigma-w/8$), and its monomials obey the weight bound

$$\varsigma^i s_1^a s_2^b w^c \text{ occurs in } \widetilde{P}_{f_0+d} \implies i+2a+2b+c \le d. \tag{211}$$

In particular $\widetilde{P}_f$ has degree d in ς with a constant leading coefficient, which is $\varepsilon_{mn}k_{6j} = \kappa_d(p_{\min},p_{\max})$, and at zero scale ($s_1=s_2=w=0$)

$$\widetilde{P}_f(\varsigma;0) = \kappa_d(p_{\min},p_{\max})\,\widehat{P}^{(p_{\min},p_{\max})}_d(\varsigma). \tag{212}$$

The sign ε_{mn} is exactly the transposition phase $d^f_{nm}=(-1)^{m-n}d^f_{mn}$: by (207), $\varepsilon_{mn}W_{6j}\widetilde{P}_f(\cdot\,;0) = d^f_{nm}(\arccos x)$ in every chamber.

Proof. Substituting (209) into (125), the factors $\eta_f \cdot \eta_f$ and η_{f+1}/η_{f+1} produced by $\Pi_{f\mp1}$ turn (125) into (210) (the constant sign ε_{mn} cancels). It therefore suffices to check the starts and to define $\widetilde{P}_f$ by the recurrence, in the ring $\mathbb{R}[\varsigma, s_1, s_2, w]$, in which A and B enter through η_f^2 symmetrically and through $q_f = w(2\lambda_f + 1 - m^2 - n^2)/8$.

Starts and chambers. In the canonical sector, Lemma 8.1 with $\mathfrak{C}_p(X) = X^p \mathcal{R}_p(X)^2$ and (206) gives $\Phi_m = E_{A,B} W_{6j} \Pi_m^{-1/2} \sqrt{\binom{2m}{m+n}}$, after cancelling the monomial factors in $\chi_m (4AB)^{2m} / (\mathfrak{D}_m(A)\mathfrak{D}_m(B))$ in either parity class of m; here $\binom{2m}{m+n} = \binom{2m}{p_{\min}}$. The other chambers follow from the two reductions of Section 8.1: each changes ξ by $m-n$, so multiplies Φ by $(-1)^{m-n}$, and leaves $E_{A,B}$, W_{6j}, Π_f and (210) unchanged (the first reduction exchanges A and B, and all three are symmetric) while replacing (m,n) by (n,m), respectively $(-m,-n)$. The chambers $|m| \le n$ and $|n| \le -m$ need one reduction, the chamber $n \le -|m|$ needs both, whose signs cancel; this is the sign ε_{mn}. In fact the coefficients of (210) involve (m,n) only through $mn = (p_{\max}^2 - p_{\min}^2)/4$ and $m^2+n^2 = (p_{\min}^2 + p_{\max}^2)/2$ (in q_f and in $R_f^2 = (f^2-m^2)(f^2-n^2)$), and $f_0 = (p_{\min}+p_{\max})/2$, so $\widetilde{P}_f$ is the same polynomial in every chamber, namely that of the canonical sector. The scalar starts for $f_0 = 0$ are $\widetilde{P}_0 = 1$ and, by (154), $\widetilde{P}_1 = 1 - 2\varsigma - w/8$.

Weight bound. Give ς and w weight 1 and s_1, s_2 weight 2, and say that a polynomial has weight $\le k$ if each of its monomials does. For $f \ge f_0$ with $f > 0$ one has $f R_{f+1} > 0$, so (210) reads

$$\widetilde{P}_{f+1} = \frac{D_f\, \widetilde{P}_f - (f+1) R_f\, \eta_f^2\, \widetilde{P}_{f-1}}{f R_{f+1}}, \qquad D_f = (2f+1)\{\lambda_f[1 - 2\varsigma - q_f] - mn\}, \tag{213}$$

where D_f has weight ≤ 1 (it is affine in ς and w), η_f^2 has weight ≤ 2, and R_f, $f R_{f+1}$ are constants. We claim that $\widetilde{P}_{f_0+k}$ has weight $\le k$. For $k = 0$ it is a constant. For $k = 1$: if $f_0 > 0$ then $R_{f_0} = 0$ and $\widetilde{P}_{f_0+1} = D_{f_0} \widetilde{P}_{f_0} / (f_0 R_{f_0+1})$ has weight ≤ 1, and if $f_0 = 0$ this is the explicit start $1 - 2\varsigma - w/8$. If the claim holds for $k \le d$, with $f = f_0 + d$ the first term of (213) has weight $\le 1 + d$ and the second weight $\le 2 + (d-1)$, so $\widetilde{P}_{f_0+d+1}$ has weight $\le d+1$. This is (211).

Degree, leading coefficient, zero scale. By (211) the exponent i of ς is at most d, and the coefficient of ς^d has weight 0, so it is a constant. Set $s_1 = s_2 = w = 0$: then $q_f = 0$, $\eta_f = 1$, and (210) with $x = 1 - 2\varsigma$ is the standard rank recurrence (219) for the canonical Wigner functions d^f_{nm}, after division by the common factor W_{6j}. In the canonical chamber $\epsilon(n,m) = 1$, so by (207) the functions $\kappa_d W_{6j} \widehat{P}_d$ solve this recurrence, and for $d = 0$ they equal $W_{6j} \sqrt{\binom{2f_0}{p_{\min}}}$ since $\kappa_0(p,p') = \sqrt{(p+p')!/(p!\,p'!)}$ (for $f_0 = 0$, they agree with the two starts 1 and x). As the forward coefficients are non-zero, the solution is determined by its starts, which is (212); comparing leading coefficients gives κ_d for the constant leading coefficient. Finally $\varepsilon_{mn} k_{6j} = \epsilon(n,m)^2 \kappa_d = \kappa_d$, and (212) with (207) gives the last assertion. □

Lemma 8.6 (Centered Racah polynomials are even deformations of Jacobi polynomials)**.** *In any chamber put $p = p_{\min}$, $p' = p_{\max}$, $d = f - f_0$, $S = A + B$, $\varepsilon = S^{-1}$, $\delta = (A-B)/S$ and $\varsigma = (1-x)/2$. With $\widetilde{P}_f$ of Lemma 8.5 define $q_d(\varsigma; \varepsilon, \delta) := \kappa_d(p,p')^{-1} \widetilde{P}_{f_0+d}$ evaluated at*

$$(s_1, s_2, w) = \Big(\frac{8\varepsilon^2(1+\delta^2)}{(1-\delta^2)^2}, \frac{16\varepsilon^4}{(1-\delta^2)^2}, \frac{4\varepsilon^2}{1-\delta^2} \Big),$$

which are the values of $A^{-2} + B^{-2}$, $A^{-2}B^{-2}$, $(AB)^{-1}$ at $A = \frac{1+\delta}{2\varepsilon}$, $B = \frac{1-\delta}{2\varepsilon}$; since $\varepsilon_{mn} k_{6j} = \kappa_d$,

$$\widetilde{P}_f = \varepsilon_{mn} k_{6j}\, q_{f-f_0}(\varsigma; \varepsilon, \delta). \tag{214}$$

It depends on δ only through δ^2 and on the labels only through (p, p', d), and in every chamber $\Phi_f = E_{A,B} W_{6j} \Pi_f^{-1/2}\, k_{6j}\, q_d$. Then

(a) q_d *is monic of degree* d *in* ς*, the coefficient of* ς^{d-j} *is a polynomial of degree at most* j *in* ε^2 *whose coefficients are rational in* δ^2 *with denominator* $(1-\delta^2)^j$*, and* $q_d(\varsigma;0,\delta)=\widehat{P}_d^{(p,p')}(\varsigma)$*;*
(b) in the canonical chamber $m\ge|n|$*,* $Q_d^{\mathrm{R}}(c-c_{\min})=\kappa_{A,B}\,q_d(\varsigma_c;S^{-1},\delta)$ *as polynomials, with* $\kappa_{A,B}\neq 0$ *independent of* c*;*
(c) for fixed labels and $0\le\delta_0<1$ *there is* C *with*

$$\sup_{0\le\varsigma\le1,\ |\delta|\le\delta_0}\bigl|q_d(\varsigma;\varepsilon,\delta)-\widehat{P}_d^{(p,p')}(\varsigma)\bigr|\le C\varepsilon^2\qquad(0<\varepsilon\le1).\tag{215}$$

The restriction $\varepsilon\le1$ *is needed: the coefficients are polynomials in* ε^2 *of degree up to* d*, and for instance* q_2 *with* $(m,n)=(1,0)$ *has a non-zero* ε^4 *term. It excludes no physical case:* A *and* B *in* (112) *are at least* $\frac12$*, so* $S\ge1$*.*

Proof. That q_d depends only on δ^2 and on (p,p',d) is Lemma 8.5 (symmetry under $A\leftrightarrow B$ and dependence through $(p_{\min},p_{\max})$), and the last assertion in the display is (209) with (214).

(a) By (212), $q_d(\cdot\,;0,\delta)=\widehat{P}_d^{(p,p')}$, and q_d is monic since κ_d is the leading coefficient of $\widetilde{P}_f$. Consider a monomial $\varsigma^{d-j}s_1^as_2^bw^c$ of $\widetilde{P}_{f_0+d}$; by (211), $\mu:=2a+2b+c\le j$. Under the substitution it becomes

$$8^a16^b4^c\,\varepsilon^{2(a+2b+c)}\,\frac{(1+\delta^2)^a(1-\delta^2)^{j-\mu}}{(1-\delta^2)^j},\tag{216}$$

a polynomial in ε^2 of degree $a+2b+c\le\mu\le j$, with coefficient a polynomial in δ^2 over $(1-\delta^2)^j$. Summing over the monomials proves (a); the ε^0 term is the zero-scale value.

(b) In the canonical chamber, (183) and (187) show that Φ_f/Φ_m is a c-independent multiple of $Q_d^{\mathrm{R}}(c-c_{\min})$, and by (209) it equals $(\Pi_m/\Pi_f)^{1/2}\widetilde{P}_f(\varsigma_c)/\widetilde{P}_m$, the edge factor and W_{6j} cancelling. Hence $\widetilde{P}_f(\varsigma_c)=c'\,Q_d^{\mathrm{R}}(c-c_{\min})$ with $c'\neq0$ independent of c. Now $\varsigma_c=\bigl((c+\frac12)^2-(A-B)^2\bigr)/(4AB)$ is affine in $\lambda_c=(c-c_{\min})(c+c_{\min}+1)=(c+\frac12)^2-(c_{\min}+\frac12)^2$, and Q_d^{R} is a polynomial of degree d in λ_c, hence in ς. The two polynomials in ς of degree d agree on the $D_{\mathrm{R}}+1\ge d+1$ distinct points ς_c, so they are equal, which is (b) with $\kappa_{A,B}=\kappa_d/c'$. (The other chambers reduce to this one by the reductions of Section 8.1, which preserve $c_{\min}$, $c_{\max}$, D_{R} and the integrality conventions and replace (A,B) by (A,B) or (B,A).)

(c) Write the coefficient of ς^i in q_d as $a_i+\sum_{s=1}^{d-i}\varepsilon^{2s}M_{i,s}(\delta^2)/(1-\delta^2)^{d-i}$, where a_i is the corresponding coefficient of $\widehat{P}_d$ by (a). For $0<\varepsilon\le1$, $|\delta|\le\delta_0$ and $0\le\varsigma\le1$ the difference is at most $\varepsilon^2\sum_i\sum_{s\ge1}\max_{|\delta|\le\delta_0}|M_{i,s}(\delta^2)|(1-\delta_0^2)^{-(d-i)}$. □

For $d=1$ in the canonical chamber ($p=m-n$, $p'=m+n$) the Racah zero moves by the diagonal shift q_m of (124): q_1 is $D_m=(2m+1)m(m+1)\{1-2\varsigma-q_m-\frac{n}{m+1}\}$ in (213) normalized to be monic (for $m=0$ use the exceptional start), so

$$q_1(\varsigma;\varepsilon,\delta)=\varsigma-\frac{p+1}{2(m+1)}+\frac{(p+1)(p'+1)}{16AB},\qquad\frac{1}{16AB}=\frac{\varepsilon^2}{4(1-\delta^2)},\tag{217}$$

whereas the Hahn zero does not move (Lemma 5.5).

8.6 The even theorem

For $\delta>0$, define the compact-shape strict-triangle cone

$$\begin{aligned}&H:=\max(A,B,C),\\&\mathcal{T}_\delta:=\left\{(A,B,C)>0:\ \frac{\min(A,B,C,A+B-C,A+C-B,B+C-A)}{H}\ge\delta\right\}.\end{aligned}\tag{218}$$

On this cone the centered angle stays in a compact subset of $(-1,1)$ and all large linear factors in the stretched seed are uniformly separated from zero. The zero-scale recurrence is

$$(2f+1)[f(f+1)x-mn]y_f=fR_{f+1}y_{f+1}+(f+1)R_fy_{f-1},\tag{219}$$

the standard rank recurrence for $d^f_{nm}(\arccos x)$.

Theorem 8.7 (Uniform homogeneous Weyl-midpoint expansion for the one-small $6j$)**.** *Fix admissible f, m, n and $\delta > 0$. There are real-analytic coefficient functions*

$$G_{f,2r}(A, B, C), \qquad r = 0, 1, 2, \ldots, \tag{220}$$

on the interior of the strict-triangle cone, homogeneous of degree $-2r$, such that

$$G_{f,0}(A, B, C) = d^f_{nm}(\arccos x), \qquad x = \frac{A^2 + B^2 - C^2}{2AB}, \tag{221}$$

and, for every fixed R,

$$\Phi_f(A, B, C) = \sum_{r=0}^{R-1} G_{f,2r}(A, B, C) + O_{f,m,n,R,\delta}(H^{-2R}) \tag{222}$$

uniformly over all sufficiently large physical admissible triples in $\mathcal{T}_\delta$. Along each fixed normalized shape the series is the Taylor series of a holomorphic germ in Λ^{-2} with $\Lambda = A + B + C$; on the compact set of shapes determined by (218)*, its dilation radius has a positive uniform lower bound.*

Proof. The scale $H = \max(A, B, C)$ is used only to state uniformity; for the analytic construction use the positive homogeneous analytic scale

$$\Lambda := A + B + C, \qquad A = \Lambda\alpha, \quad B = \Lambda\beta, \quad C = \Lambda\gamma, \qquad t := \Lambda^{-2}. \tag{223}$$

The shape map $(A, B, C) \mapsto (\alpha, \beta, \gamma)$ is real analytic on the whole positive cone, including the hypersurfaces $A = B$, $B = C$, $C = A$. On $\mathcal{T}_\delta$ one has $H \le \Lambda \le 3H$ and every positive edge and strict-triangle linear form divided by Λ is at least $\delta/3$, so the normalized shapes lie in one compact subset of the open strict-triangle simplex.

By Lemma 8.5, at every physical triple

$$\Phi_f(A, B, C) = \varepsilon_{mn}\, E_{A,B}(C)\, W_{6j}(x)\, \Pi_f(A, B)^{-1/2}\, \widetilde{P}_f(\varsigma; A, B), \qquad \varsigma = \tfrac{1-x}{2}, \tag{224}$$

where $\varepsilon_{mn} = \pm 1$ is the chamber sign of the lemma; the lemma, which covers the exceptional two-start sector, is derived from the exact recurrence (125) and the stretched and exceptional starts above, and does not use the present theorem. Express every factor through the normalized shape (α, β, γ) and $t = \Lambda^{-2}$. The angle x, hence ς and $W_{6j}(x)$, depends on the shape only, and is real analytic on the open strict-triangle simplex, where $x \in (-1, 1)$. Put $\xi_1 = \alpha - \beta + \gamma$, $\xi_2 = -\alpha + \beta + \gamma$, $\xi_3 = \alpha + \beta - \gamma$, $\xi_4 = \alpha + \beta + \gamma$, all positive on the strict-triangle simplex. Pairing the shifts $\pm q$ in each factor $\mathcal{R}_p(\Lambda\xi)^2 = \prod_{q>0}(1 - q^2 t/\xi^2)$ of (204), and writing $A^{-2} = t/\alpha^2$, $B^{-2} = t/\beta^2$, $(AB)^{-1} = t/(\alpha\beta)$, one finds that $E^2_{A,B}$ and Π_f are polynomials in t with coefficients rational in the shape and equal to 1 at $t = 0$, and that $\widetilde{P}_f$ is a polynomial in (ς, t) with coefficients rational in the shape. Hence, on a neighborhood of the compact set $\mathcal{K}_\delta$ of normalized shapes of $\mathcal{T}_\delta$ times $\{t = 0\}$ (complexified in t if desired), both square roots are the positive analytic branches and

$$G_f(\alpha, \beta, \gamma; t) := \varepsilon_{mn}\, E\, W_{6j}\, \Pi_f^{-1/2}\, \widetilde{P}_f \tag{225}$$

is real analytic in the shape and holomorphic in t, with a positive lower bound on the dilation radius uniform over $\mathcal{K}_\delta$, and $\Phi_f(A, B, C) = G_f(A/\Lambda, B/\Lambda, C/\Lambda; \Lambda^{-2})$ at every physical triple. It depends on the scale only through $t = \Lambda^{-2}$, so no odd power occurs (the positive reflection sector of Remark 3.3). At $t = 0$, $G_f = \varepsilon_{mn} W_{6j} \widetilde{P}_f(\cdot\,; 0) = d^f_{nm}(\arccos x)$ by (212) and (207), which is (221); equivalently, at $t = 0$ the recurrence (210) is (219) and the start is the extremal Wigner-d value, the sign ε_{mn} being the transposition phase of Lemma 8.5.

If $a_{f,r}(\alpha,\beta,\gamma) := \partial_t^r G_f(\alpha,\beta,\gamma;0)/r!$ is the coefficient of t^r, define

$$G_{f,2r}(A,B,C) := \Lambda^{-2r} a_{f,r}(A/\Lambda, B/\Lambda, C/\Lambda). \tag{226}$$

Because $\Lambda = A+B+C$ is itself positive, homogeneous and real analytic, this formula proves that $G_{f,2r}$ is globally real analytic on the entire interior strict-triangle cone, not merely chamberwise, and is homogeneous of degree $-2r$. Lemma 3.2, applied on $\mathcal{K}_\delta$, gives (222) with remainder $O(\Lambda^{-2R})$, hence also $O(H^{-2R})$ because $H \le \Lambda$. □

Remark 8.8. *Flude proved the leading Edmonds limit under proportional large-spin scaling [7, Thm. 3]. Theorem 8.7 uses the same interior geometry but states the centered result directly on physical triples; the correction functions are homogeneous, so no fixed admissible ray is needed.*

9 Local corrections for the one-small $6j$ symbol

9.1 The first homogeneous correction

Write the homogeneous expansion through degree -2 as

$$\Phi_f = d_f + G_{f,2} + O(H^{-4}), \qquad d_f := d^f_{nm}(\theta), \tag{227}$$

and put

$$\begin{aligned} \varrho_{AB} &:= \frac{1}{8}\left(A^{-2}+B^{-2}\right), \qquad s := 1-x^2, \qquad \Sigma := m^2+n^2, \qquad \tau := mn, \\ V(x) &:= \frac{\Sigma - 2\tau x}{s}, \end{aligned} \tag{228}$$

so that the Wigner operator is $\mathsf{H}_0 = -\partial_x s \partial_x + V$ and the exact diagonal shift $q_f = (2f(f+1) + 1 - \Sigma)/(8AB)$ is already intrinsic. Define

$$c_0 := \frac{1-\Sigma}{8AB} + \varrho_{AB}\tau, \qquad c_1(x) := \frac{1}{4AB} - \varrho_{AB}x, \qquad \chi_{mn}(x) := \frac{(nx-m)(mx-n)}{s^2}, \tag{229}$$

and, with $\lambda_f = f(f+1)$,

$$P_f(x) := -c_0 + c_1(x)\frac{2-2\Sigma+4\tau x - \lambda_f s}{3s}, \tag{230}$$

$$Q_f(x) := \frac{\varrho_{AB}}{2}\lambda_f + c_1(x)\frac{\tau - \lambda_f x}{3s}. \tag{231}$$

All of these are homogeneous of degree -2 in (A,B,C).

For the stretched seed set

$$\Delta_1 = A-B+C, \quad \Delta_2 = -A+B+C, \quad \Delta_3 = A+B-C, \quad \Delta_4 = A+B+C. \tag{232}$$

The degree -2 seed coefficient is (Appendix E)

$$G_{m,2} = \kappa_m^{[2]} d^m_{nm}(\theta), \tag{233}$$

where

$$\begin{aligned} 48\kappa_m^{[2]} = {} & m(m+1)(2m+1)(A^{-2}+B^{-2}) \\ & - (m-n)[(m-n)^2-1](\Delta_1^{-2}+\Delta_2^{-2}) \\ & - (m+n)[(m+n)^2-1](\Delta_3^{-2}+\Delta_4^{-2}). \end{aligned} \tag{234}$$

At $f = m$ the extremal Wigner mode satisfies

$$\frac{d'_m}{d_m} = \frac{n - mx}{1 - x^2}. \tag{235}$$

The law of cosines gives the intrinsic identities

$$\begin{aligned} \Delta_1^{-2} + \Delta_2^{-2} &= \frac{A^{-2} + B^{-2} - (1+x)(AB)^{-1}}{(1-x)^2}, \\ \Delta_3^{-2} + \Delta_4^{-2} &= \frac{A^{-2} + B^{-2} + (1-x)(AB)^{-1}}{(1+x)^2}. \end{aligned} \tag{236}$$

Substitution of (235)–(236) into (234) gives exactly

$$\kappa_m^{[2]} = P_m(x)\frac{d'_m}{d_m} + Q_m(x). \tag{237}$$

Theorem 9.1 (Closed local degree -2 correction for the centered one-small $6j$)**.** *For every fixed admissible f, m, n in Theorem 8.7, in every sign/order chamber and on $-1 < x < 1$,*

$$\boxed{G_{f,2}(A,B,C) = P_f(x)\,\partial_x d^f_{nm}(\arccos x) + Q_f(x)\,d^f_{nm}(\arccos x),} \tag{238}$$

with P_f, Q_f from (230)–(231)*. Equivalently,*

$$\mathcal{G}_2(A,B,C) = -c_0\partial_x + \frac{\varrho_{AB}}{2}\mathsf{H}_0 + c_1(x)\left[\frac{2 - 2\Sigma + 4\tau x}{3s}\partial_x - \frac{1}{3}\partial_x\mathsf{H}_0 + \frac{\tau}{3s} - \frac{x}{3s}\mathsf{H}_0\right], \tag{239}$$

so that $G_{f,2} = \mathcal{G}_2 d_f$. Every coefficient of $\mathcal{G}_2$ is homogeneous of degree -2 in the physical centered lengths.

Proof. The exact recurrence (125) is the instance of (299) with $(a,b) = (n,m)$, $k = f$ and $q(\lambda) = (2\lambda + 1 - \Sigma)/(8AB)$, in which $e_1 = -\varrho_{AB}$ and the order is minus the homogeneous degree (Appendix B.3). By Proposition B.2 and the link to the physical kernel that follows it, in the canonical sector $G_{f,2} = \mathcal{G}_2 d_f$ with $\mathcal{G}_2 = \mathcal{J}(\mathsf{B}_2) + \psi_2(x)$ and $\mathsf{B}_2 = q(\mathsf{H}_0) + e_1\mathcal{N}_1[I]$. Here $[x, \mathsf{H}_0] = 2s\partial_x - 2x$, so $\mathcal{N}_1[I] = x\mathsf{H}_0 - \tau - s\partial_x + x$ (from $\pi_1(z) = z$, $\widetilde{\pi}_1 = 1$ in (302)). Expanding $\mathsf{H}_0 = -s\partial_x^2 + 2x\partial_x + V$ gives $\mathsf{B}_2 = b_0 + b_1\partial_x + b_2\partial_x^2$ with

$$b_2 = -s\,c_1, \qquad b_1 = \frac{x}{2AB} - (3x^2 - 1)\varrho_{AB}, \qquad b_0 = \frac{2V + 1 - \Sigma}{8AB} - \varrho_{AB}\,(xV - \tau + x), \tag{240}$$

and $\mathcal{J}(\mathsf{B}_2) = -b_0\partial_x - \frac{1}{2}b_1\partial_x^2 - \frac{1}{3}b_2\partial_x^3$. Expanding (239) with $\partial_x\mathsf{H}_0 = -s\partial_x^3 + 4x\partial_x^2 + (2+V)\partial_x + V'$ and comparing coefficients of $\partial_x^3, \partial_x^2, \partial_x$ shows that the right side of (239) differs from $\mathcal{J}(\mathsf{B}_2)$ by a multiplication operator, namely

$$U(x) = \frac{\varrho_{AB}}{2}\,\frac{(nx - m)^2 + (mx - n)^2}{s^2} + \frac{\chi_{mn}(x)}{4AB}. \tag{241}$$

It remains to see that $\psi_2 = U$. By Proposition B.2(b), $\psi_2 = \kappa_m^{[2]} - \mathcal{J}(\mathsf{B}_2)d_m/d_m$. The Wigner equation gives $\mathsf{H}_0 d_m = \lambda_m d_m$ and $\partial_x\mathsf{H}_0 d_m = \lambda_m d'_m$, so the right side of (239) acts on d_m as $P_m d'_m + Q_m d_m$, which equals $\kappa_m^{[2]} d_m$ by (237). Hence $U d_m = \kappa_m^{[2]} d_m - \mathcal{J}(\mathsf{B}_2)d_m$, that is $\psi_2 = U$, which proves (239) in the canonical sector with $m > 0$. Applying (239) to d_f and $\partial_x\mathsf{H}_0 d_f = \lambda_f d'_f$ gives (238). For $m = n = 0$, $U = 0$ and Proposition B.2(c) applies with $\psi_2 = 0$; then $G_{0,2} = 0$ and

$$G_{1,2} = \frac{x}{8}(A^{-2} + B^{-2}) - \frac{1}{8AB}. \tag{242}$$

The reductions preceding Lemma 8.1 multiply Φ_f and d^f_{nm} by the same sign $(-1)^{m-n}$ and leave x, AB, $A^{-2} + B^{-2}$, Σ and τ, hence P_f, Q_f and $\mathcal{G}_2$, unchanged (the column swap exchanges A and B, on which these depend only symmetrically); they replace d^f_{nm} by d^f_{mn} or $d^f_{-n,-m}$ with the same sign $(-1)^{m-n}$ as Φ_f, which gives the other chambers. □

9.2 All-order homogeneous local calculus and the degree -4 term

The closed first correction is the first instance of a general local inversion mechanism. If A is any differential operator independent of the recurrence rank f, then

$$\begin{aligned}(\mathcal{L}_{\mathrm{W}}y)_f &:= fR_{f+1}y_{f+1} + (f+1)R_f y_{f-1} - (2f+1)(\lambda_f x - mn)y_f,\\ \mathcal{L}_{\mathrm{W}}(\mathsf{A}d)_f &= -(2f+1)\lambda_f [x,\mathsf{A}]d_f,\end{aligned} \tag{243}$$

which is (298) for $(a,b) = (n,m)$. Since the x-dependence of the exact centered recurrence is confined to its diagonal term, every homogeneous source reduces to $(2f+1)\lambda_f$ times a finite-order local differential operator on d_f. An x-commutator antiderivative gives a local particular solution; the positive stretched seed fixes the scalar homogeneous mode.

Theorem 9.2 (All-order homogeneous local differential corrections in the stretched sector)**.** *Fix the canonical sector $m \geq |n|$, $m \geq 0$, and a compact interior strict-triangle cone. For $m = 0$ this means $m = n = 0$. For every $r \geq 0$ there is a rank-independent finite-order differential operator $\mathcal{G}_{2r}(A,B,C)$ whose coefficients are real analytic on the cone, rational in x, and homogeneous of degree $-2r$ in (A,B,C), such that for every fixed R and admissible f,*

$$\Phi_f = \sum_{r=0}^{R-1} \mathcal{G}_{2r}(A,B,C)d_f + O(H^{-2R}), \qquad \mathcal{G}_0 = I. \tag{244}$$

Appendix B.3 gives them by the same construction as for $3j$ (Proposition B.2). The only possible x-singularities are at $x = \pm 1$. In the other chambers, $\mathcal{G}_{2r}$ denotes the operator of the canonical representative obtained by the reductions preceding Lemma 8.1, which multiply Φ_f and d^f_{nm} by the same sign; the expansion (244) *then holds there by that exact symmetry. The first reduction exchanges A and B; this is immaterial, because x, q, η_k and the stretched seed are symmetric under $A \leftrightarrow B$, so by induction every $\mathcal{G}_{2r}(A,B,C)$ is.*

Proof. The recurrence (125) is an instance of (299), with $\eta(k) = \eta_k$, the diagonal shift q_f, and order equal to minus the homogeneous degree (Appendix B.3). Proposition B.2(a) gives the operators $\mathcal{G}_{2r} = \mathcal{J}(\mathsf{B}_{2r}) + \psi_{2r}$, whose coefficients are homogeneous and real analytic because $e_s(A,B)$, q and the earlier operators are. For $m > 0$ the multiplier ψ_{2r} is fixed at $f = m$ by Proposition B.2(b) and the degree $-2r$ coefficient of the positive stretched seed, which is globally real analytic on the cone by the $\Lambda = A + B + C$ construction of Theorem 8.7; the extremal d_m has no zeros on $-1 < x < 1$, and Lemma B.3 shows that ψ_{2r} is rational in x with poles only at ± 1. For $m = n = 0$, $\psi_{2r} = 0$ and Proposition B.2(c) reproduce both exact starts (154) to all orders. Forward uniqueness, with the link to the physical kernel in Appendix B.1 (paragraph 'Link to the physical kernels'), gives the expansion at every fixed rank, and Theorem 8.7 supplies the remainder. □

At degree -4 the recursion gives the constructive closed formula

$$\mathcal{G}_4 = \mathcal{J}(\mathsf{B}_4) + \psi_4(x), \tag{245}$$

where B_4 is a finite combination of $\mathcal{G}_2$, the Wigner operator and the first two homogeneous off-diagonal coefficients, while ψ_4 is fixed by the degree -4 stretched-seed coefficient. The construction is given in Appendix B.3. Consequently

$$\Phi_f = d_f + \mathcal{G}_2 d_f + \mathcal{G}_4 d_f + O(H^{-6}) \tag{246}$$

for every fixed rank in the stretched canonical sector, including the scalar $m = n = 0$ sector. In the latter case the two exact starts are satisfied to every order by Proposition B.2(c).

9.3 Why the usual Edmonds–Flude variables retain a Λ^{-1} defect

Flude's rigorous normalization and angle for the same family are [7, Thm. 3 and Eq. (6)]

$$\Psi_f := (-1)^{\xi}\sqrt{\omega_a\omega_b}\,W_f, \qquad \omega_a = 2a+n+f+1 = 2A+f, \quad \omega_b = 2B+f, \tag{247}$$

and

$$x_F := \cos\phi_F = \frac{(A+n/2)^2+(B+m/2)^2-C^2}{2(A+n/2)(B+m/2)}. \tag{248}$$

Theorem 8.7 allows the generic first-order discrepancy to be computed without re-expanding the Racah sum. Since

$$\frac{\sqrt{(2A+f)(2B+f)}}{2\sqrt{AB}} = 1+\frac{r_F}{\Lambda}+O(\Lambda^{-2}), \qquad r_F = \frac{f}{4}(\alpha^{-1}+\beta^{-1}), \tag{249}$$

and

$$x_F = x+\frac{\delta_F}{\Lambda}+O(\Lambda^{-2}), \tag{250}$$

with

$$\delta_F = \frac{1}{2}\left[\left(\frac{m}{\alpha}+\frac{n}{\beta}\right)-x\left(\frac{n}{\alpha}+\frac{m}{\beta}\right)\right], \tag{251}$$

we obtain

$$\Psi_f - d^f_{nm}(\arccos x_F) = \frac{1}{\Lambda}\left[r_F d_f - \delta_F\frac{\mathrm{d}d^f_{nm}(\arccos x)}{\mathrm{d}x}\right]+O(\Lambda^{-2}). \tag{252}$$

Thus the conventional formulation has a generic Λ^{-1} defect. The midpoint prescription removes it at the level of the exact recurrence, not merely by cancelling one computed coefficient.

10 Uniform expansions up to the hard edges

The bulk theorems are stated on compact interior sets and are not uniform at $x=\pm1$. The $3j$ grid stops a distance $O(N^{-1})$ from either endpoint. The $6j$ grid has the same scale at a separated edge, but its distance from $x=1$ is $O((A+B)^{-2})$ when $A-B$ and the inward index are bounded. At these points the spatially varying part of the discrete weight can differ by an order-one factor from its continuum Jacobi limit. Extracting that factor and the endpoint envelope leaves a polynomial whose coefficients are analytic in the inverse square scale. This exact factorization proves the all-order uniform expansion below; the relative Laguerre laws then follow on neighborhoods free of Wigner zeros.

The closed forms of Lemmas 5.4 and 8.5 already contain the whole argument: after division by the edge factor E of (64), respectively (204), and the endpoint envelope W, only the scalar factor $\Pi^{-1/2}$ remains to be expanded. Throughout this section $\varsigma = (1-x)/2$.

10.1 The uniform theorem

Theorem 10.1 (Uniform edge-factorized expansion to all orders)**.** (a) *Fix L, M, σ and put $\varsigma = (1-x)/2$, $d = L - L_0$. There are polynomials $s_r(\varsigma)$ of degree at most d, with $W_{3j}s_0 = d^L_{M\sigma}(\arccos x)$, such that for every $R \geq 1$, uniformly on the whole physical grid $\mathcal{X}_N$,*

$$\boxed{\Phi^{M,\sigma}_L(x;N^{-1}) = E_N(x)\,W_{3j}(x)\Big[\sum_{r=0}^{R-1}\frac{s_r(\varsigma)}{N^{2r}}+O_{L,M,\sigma,R}(N^{-2R})\Big].} \tag{253}$$

(b) *Fix admissible f, m, n and $0<\lambda<1$, and let $A, B \to \infty$ through admissible physical values with*

$$\lambda \leq \frac{A}{B} \leq \lambda^{-1}. \tag{254}$$

Put $\varsigma = (1-x)/2$, $\delta = (A-B)/(A+B)$ and $d = f - f_0$. There are polynomials $s_r(\varsigma;\delta)$ of degree at most d in ς, with coefficients rational in δ and poles only at $\delta = \pm 1$, with $W_{6j} s_0 = d^f_{nm}(\arccos x)$, such that for every $R \geq 1$, uniformly on the whole physical c-lattice,

$$\boxed{\Phi_f(A,B,C) = E_{A,B}(C)\, W_{6j}(x)\Big[\sum_{r=0}^{R-1} \frac{s_r(\varsigma;\delta)}{(A+B)^{2r}} + O_{f,m,n,\lambda,R}\big((A+B)^{-2R}\big)\Big].} \tag{255}$$

No separation of A from B is assumed. The series in (a) *converges for $N > L$, and the series in* (b) *converges at fixed δ for $2\min(A,B) > f$. These ranges contain every physical lattice that carries the rank ($N \geq L+1$, respectively $2\min(A,B) \geq f+1$), and for $L > 0$, respectively $f > 0$, they are the exact domains of convergence uniformly on the lattice (as series with coefficients polynomial in ς). For $R = 1$ both statements read $\Phi = E[d + O(S^{-2})W]$ with $S = N$ or $S = A + B$.*

Proof. By Lemmas 5.4 and 8.5, $\Phi = \varepsilon\, EW\, Y$ in every chamber and sector, with $\varepsilon = 1$ for $3j$ and $Y = \Pi^{-1/2}\widetilde{P}$, where $\widetilde{P}$ is already a polynomial in ς whose coefficients are polynomials in the inverse squared scales. Thus Y is analytic in (ς, t) for $3j$, and in (ς, δ, t) for $6j$ (below), and only the scalar factor $\Pi^{-1/2}$ has to be expanded; the expansion then follows from Lemma 3.2 on the whole lattice.

The scalar cases $L = 0$ and $f = 0$ have $E = W = \Pi = \widetilde{P} = 1$ and $\Phi = 1$, so their expansions are exact. We may therefore assume $L > 0$ in (a) and $f > 0$ in (b).

(a) For $L > 0$, $\Pi_L(t)^{-1/2}$ is real analytic for $|t| < 1/L^2$ and equals 1 at $t = 0$, so $Y(\varsigma, t) = \Pi_L(t)^{-1/2}\widetilde{P}_L(\varsigma; t)$ is real analytic on $\mathbb{R} \times (-L^{-2}, L^{-2})$ (it is polynomial in ς). Lemma 3.2 on $K = [0,1]$ gives (253), with s_r the coefficient of $t^r = h^{2r}$ in Y, a polynomial of degree at most d in ς, and a remainder uniform for $0 \leq \varsigma \leq 1$ and $N^{-2} \leq \rho$.

(b) Under (254) put $S = A+B$ and $t = S^{-2}$, so that $A^{-2} = 4t(1+\delta)^{-2}$, $B^{-2} = 4t(1-\delta)^{-2}$ and $(AB)^{-1} = 4t(1-\delta^2)^{-1}$ with $|\delta| \leq \delta_0 := (1-\lambda)/(1+\lambda) < 1$. Then $\widetilde{P}_f$ is a polynomial in ς and t with coefficients rational in δ and poles only at $\delta = \pm 1$. The factor $1 - q^2/(4A^2) = 1 - q^2 t(1+\delta)^{-2}$ vanishes only at $t = (1+\delta)^2/q^2$, and similarly for B, so $\Pi_f^{-1/2}$ is real analytic in t for $|t| < (1-\delta_0)^2/f^2$, uniformly for the admitted δ, and equals 1 at $t = 0$. Hence $Y(\varsigma, \delta, t)$ is real analytic near $[0,1] \times [-\delta_0, \delta_0] \times \{0\}$, and Lemma 3.2 with $z = (\varsigma, \delta)$ gives (255) with s_r the coefficient of t^r in $\varepsilon_{mn} Y$, uniformly for $|x| \leq 1$.

In both cases the bracket at $t = 0$ is the zero-scale value of $\varepsilon\widetilde{P}$, which is d/W by (74), (212) and (68), (207); this identifies $W s_0$. The domains of convergence follow from $2\min(A,B) = S(1-|\delta|)$. They are exact because the leading coefficient of the bracket is $k_{3j}\Pi^{-1/2}$, respectively $k_{6j}\Pi^{-1/2}$: for $L > 0$ it has a branch point at $|h| = 1/L$, and for $f > 0$ the factor with $q = f$ of the smaller of A, B gives a branch point at $2\min(A,B) = f$ (a pole if $\delta = 0$), which no other factor cancels. □

The edge factor E carries the spatially varying ratio of the discrete and continuum seed weights; their scalar normalization is included in $\Pi^{-1/2}$ and $\widetilde{P}$. For fixed small labels, $0 < c_0 \leq E \leq 1$ on the physical lattice: pair the shifts q and $-q$ in each factor, and use $X \geq (p+1)/2$. Thus the remainder in the theorem is a uniform bound after division by EW, and also an absolute bound of the stated order. It is not a relative estimate at zeros of the Wigner mode. Consequently every endpoint pole of the bulk coefficients f_{2r} of Theorem 5.6, and of the homogeneous $6j$ coefficients $G_{f,2r}$, comes from the expansion of E: for instance the terms $-r_\pm(r_\pm^2 - 1)/(48u_\pm^2)$ in (100) are the N^{-2} coefficients of $\log E_N$ by (63).

Remark 10.2 (Exact bulk radii at fixed rank)**.** *Lemmas 5.4 and 8.5 also give the complex radius of the bulk expansions. At fixed $x \in (-1,1)$ the analytic interpolation of Theorem 5.6 is the closed form* (84)*, $G_L^{M,\sigma}(x,t) = E_N(x) W_{3j}(x) \Pi_L(t)^{-1/2} \widetilde{P}_L$ with $t = N^{-2}$ the dilation variable*

of Section 3 and $\widetilde{P}_L$ evaluated at $\varsigma = (1-x)/2$, and pairing the shifts $\pm q$ in E_N gives

$$E_N(x)^2 = \prod_{q}\Big(1 - \frac{4q^2 t}{(1-x)^2}\Big)\prod_{q'}\Big(1 - \frac{4q'^2 t}{(1+x)^2}\Big), \tag{256}$$

with q and q' running over the positive elements of the symmetric shift sets of sizes p_+ and p_-. Hence G is holomorphic in t for

$$|t| < \rho_{3j}(x) := \min\Big\{\frac{1}{L^2}, \frac{(1-x)^2}{(p_+-1)^2}, \frac{(1+x)^2}{(p_--1)^2}\Big\}, \tag{257}$$

where a term is omitted when $L = 0$ or $p_\pm \le 1$. In the same way, at a fixed normalized shape $(\alpha,\beta,\gamma) = (A,B,C)/S$ with $t = S^{-2}$ for any positive homogeneous scale S, the $6j$ interpolation is holomorphic for

$$|t| < \rho_{6j} := \min\Big\{\frac{4\min(\alpha,\beta)^2}{f^2}, \frac{4(\gamma - |\alpha-\beta|)^2}{(p_{\min}-1)^2}, \frac{4(\alpha+\beta-\gamma)^2}{(p_{\max}-1)^2}\Big\}, \tag{258}$$

with the same conventions. When the linear factors of E^2 and Π have distinct zeros in t, each of these points is a simple square-root branch point, which the polynomial $\widetilde{P}$ cannot remove. The radius is therefore exactly ρ, except on a proper analytic subset where it can only be larger: where $G(x,\cdot)$ vanishes identically (for example $3j$ with $M = \sigma = 0$, L odd, at $x = 0$), or where zeros of two linear factors coincide, giving a polynomial factor or a pole that may cancel against $\widetilde{P}$. This happens for $3j$ at isolated x (for example $x = 0$ when $p_+ = p_-$), and for $6j$ for example on $\alpha = \beta$, where the two $p_{\min}$ factors of $E_{A,B}$ combine into a polynomial and the A and B factors of $\Pi_f^{-1/2}$ pair into poles; for $f = 1$, $m = n = 0$ at $\alpha = \beta = \gamma$ the pole cancels and $G_1 \equiv \frac{1}{2}$. They give explicit uniform lower bounds. For $|x| \le 1-\delta$ and every rank $L \le L_*$, $\rho_{3j}(x) \ge \delta^2/(4L_*^2)$, because $p_\pm - 1 < 2L_*$, so the interpolation through all these ranks is analytic for $|t| < \delta^2/(4L_*^2)$. For the cone $\mathcal{T}_\delta$ with $S = \Lambda = A+B+C$, every normalized edge and strict-triangle form is at least $\delta/3$ (Theorem 8.7), so $\rho_{6j} \ge \delta^2/(9f_*^2)$ for all ranks $f \le f_*$. The shrinking of ρ_{3j} toward $x = \pm 1$ reflects the branch points of E. At fixed rank, $\widetilde{P}_L$ (respectively $\widetilde{P}_f$) is, up to a constant and the affine change of variable, the polynomial whose zeros the companion paper [8] locates in the growing-rank limit. The first entries $1/L^2$ and $4\min(\alpha,\beta)^2/f^2$ of ρ_{3j} and ρ_{6j} are the branch points of the hopping square roots $\sqrt{1-k^2t}$ and η_k at the top rank. If $p_+, p_- \le 1$, that is $(M,\sigma) \in \{(0,0),(\pm\frac{1}{2},\pm\frac{1}{2}),(\pm1,0),(0,\pm1)\}$, respectively $p_{\min}, p_{\max} \le 1$, that is $(m,n) \in \{(0,0),(\pm\frac{1}{2},\pm\frac{1}{2}),(\pm1,0),(0,\pm1)\}$, then $E \equiv 1$: the bulk coefficients are W times polynomials, have no endpoint poles, and the bulk expansion holds uniformly on $[-1,1]$.

In practice the second order of Theorem 10.1 is the most useful, and its coefficient is already known from the bulk formulas.

Corollary 10.3 (Uniform next-to-leading form)**.** *Let $N^{-2}e_2^{3j}$ and e_2^{6j} be the second-order terms of the interior expansion of $\log E$ given by (63):*

$$e_2^{3j}(x) = -\frac{p_+(p_+^2-1)}{12(1-x)^2} - \frac{p_-(p_-^2-1)}{12(1+x)^2}, \tag{259}$$

$$\begin{aligned} e_2^{6j} = &-\frac{p_{\min}(p_{\min}^2-1)}{48}\Big[\frac{1}{(A-B+C)^2} + \frac{1}{(-A+B+C)^2}\Big] \\ &-\frac{p_{\max}(p_{\max}^2-1)}{48}\Big[\frac{1}{(A+B-C)^2} + \frac{1}{(A+B+C)^2}\Big]. \end{aligned} \tag{260}$$

Then, uniformly on the whole physical lattice (for (262) *under the ratio bound* (254)*),*

$$\Phi_L^{M,\sigma} = E_N\Big[d_{M\sigma}^L + N^{-2}(f_2 - e_2^{3j}\, d_{M\sigma}^L)\Big] + O(N^{-4})\, E_N W_{3j}, \tag{261}$$

$$\Phi_f = E_{A,B}\Big[d_{nm}^f + G_{f,2} - e_2^{6j}\, d_{nm}^f\Big] + O((A+B)^{-4})\, E_{A,B} W_{6j}, \tag{262}$$

with f_2 from Theorem 6.1 and $G_{f,2}$ from (238)*. The combinations $f_2 - e_2^{3j} d$ and $G_{f,2} - e_2^{6j} d$ equal W times a polynomial of degree at most $L - L_0$, respectively $f - f_0$. Their apparent endpoint poles cancel exactly. The remaining endpoint powers are those of W; evaluating the polynomial form avoids numerical cancellation near an edge.*

Proof. On a compact interior set $E_N = 1 + N^{-2} e_2^{3j} + O(N^{-4})$ and $E_{A,B} = 1 + e_2^{6j} + O((A+B)^{-4})$, so the case $R = 2$ of Theorem 10.1 and the bulk expansions give $W s_1 = f_2 - e_2^{3j} d$, respectively $G_{f,2} - e_2^{6j} d$, there. Both sides are analytic in x on $(-1, 1)$, so the identity holds on the whole interval, and Theorem 10.1 with $R = 2$ gives the uniform statements. □

10.2 Laguerre hard edges

Corollary 10.4 ($3j$ hard edge)**.** *Fix L, M, σ. There is $\theta_0 > 0$ such that, for all sufficiently large admissible N and uniformly for $0 \le n \le \theta_0 N$,*

$$\boxed{\frac{\Phi(x_n^{\pm})}{d_{M\sigma}^L(\arccos x_n^{\pm})} = \mathfrak{R}_{p_{\pm},n}\,[1 + O(N^{-2})].} \tag{263}$$

In particular, with $C_+^{L,M,\sigma} := \lim_{x\uparrow 1}((1-x)/2)^{-p_+/2} d_{M\sigma}^L(\arccos x) \neq 0$,

$$\Phi(x_n^+) = C_+^{L,M,\sigma} N^{-p_+/2} \sqrt{(n+1)_{p_+}}\,[1 + O((n+1)/N)], \tag{264}$$

and likewise $\Phi(x_n^-) = C_-^{L,M,\sigma} N^{-p_-/2} \sqrt{(n+1)_{p_-}}\,[1 + O((n+1)/N)]$ with $C_-^{L,M,\sigma} := \lim_{x\downarrow -1}((1+x)/2)^{-p_-/2} d_{M\sigma}^L(\arccos x) \neq 0$, where the first-order remainder now comes only from the Taylor expansion of $d_{M\sigma}^L$ itself.

Proof. Since $\widehat{P}_d(0) \neq 0$ and $\widehat{P}_d(1) \neq 0$, (68) gives $|d_{M\sigma}^L| \ge c_0 W_{3j}$ on $(1 \mp x)/2 \le 2\theta_0$ for small θ, so (253) becomes $\Phi/d = E_N[1 + O(N^{-2})]$ there. At x_n^+ the first factor of E_N is $\mathfrak{R}_{p_+,n}$ and the second has argument $\ge N/2$, hence equals $1 + O(N^{-2})$; similarly at the lower edge. □

Corollary 10.5 (One-small $6j$ hard edges and the lower corner)**.** *Under* (254) *there is $\theta_0 > 0$ such that, for all sufficiently large $A + B$ and uniformly for all lattice points with $(1 + x)/2 \le \theta_0$,*

$$\boxed{\frac{\Phi_f(C_r^{\max})}{d_{nm}^f(\arccos x_r^{\max})} = \mathfrak{R}_{p_{\max},r}\,[1 + O((A+B)^{-2})],} \tag{265}$$

and, uniformly for all lattice points with $(1 - x)/2 \le \theta_0$,

$$\boxed{\frac{\Phi_f(C_r^{\min})}{d_{nm}^f(\arccos x_r^{\min})} = \mathfrak{R}_{p_{\min},r}\,\mathcal{R}_{p_{\min}}\left(2|A-B| + r + \frac{p_{\min}+1}{2}\right)[1 + O((A+B)^{-2})].} \tag{266}$$

If moreover $|A - B|/(A + B) \ge \delta > 0$, the second factor in (266) *is $1 + O((A+B)^{-2})$ and the lower edge has the same universal form as the upper edge and the $3j$ edges.*

Proof. As for Corollary 10.4, using (255) and (206). At $C_r^{\max}$ the arguments $A - B + C$, $-A + B + C$, $A + B + C$ are $\asymp A + B$ by (254). At $C_r^{\min}$ the two upper-edge arguments are $\asymp A + B$, one lower argument is $r + (p_{\min} + 1)/2$ and the other is $2|A - B| + r + (p_{\min} + 1)/2$. □

Remark 10.6 (The lower corner)**.** *At fixed rank the lower corner* $|A-B| = o(A+B)$ *needs no separate scaling analysis:* (266) *holds uniformly through it, and the corner appears only as the second explicit Pochhammer factor, which is of order one for bounded* $|A-B|$ *and* $1 + O(|A-B|^{-2})$ *for* $1 \ll |A-B| \ll A+B$*. A rank growing with the large lengths is not covered.*

Remark 10.7 (The Laguerre recurrence behind the edge factor)**.** *The edge factor also follows directly from the recurrences. Near the upper* $3j$ *edge write* $x = 1 - hy$*,* $\Phi(x) = h^{p/2}\Xi(y;h)$*,* $p = p_+$*; at* $y = p + 1 + 2n$ *the leading part of* (30) *is*

$$\sqrt{n(n+p)}\,\Xi_{n-1} + \sqrt{(n+1)(n+p+1)}\,\Xi_{n+1} - (2n+p+1)\Xi_n = 0, \tag{267}$$

the orthonormal Laguerre Jacobi recurrence at spectral parameter zero, whose solution $\sqrt{(n+1)_p}$*, the numerator of* $\mathfrak{R}_{p,n}$*, satisfies the* $n = 0$ *row* $\sqrt{p+1}\,\Xi_1 = (p+1)\Xi_0$ *and is therefore the unique solution of the one-sided recurrence up to scale. The c-lattice Hamiltonian of Theorem 8.2 gives the same recurrence at the separated* $6j$ *edges: at* $c = c_{\min} + r$ *with* $A > B$ *and* $p = p_{\min}$ *the two small factors of* (161) *are* $r[2(A-B)+p+r]$ *and* $(p+r)[2(A-B)+r]$*, and in the lower corner both are small, which is the origin of the second factor in* (266)*.*

11 Numerical accuracy

We compare the expansions with exact Wigner symbols evaluated at high precision, covering the convention-sensitive cases: both large-spin parity classes, the all-chamber Hahn phase, integer and half-integer $6j$ sectors, integer $3j$ sectors and the exact half-integer spin–orbit ray (85), the removable $c = 0$ Racah row, the exact Racah orthogonality and completeness relations, and the homogeneous degree -2 and degree -4 centered $6j$ operators. These comparisons illustrate the practical accuracy of the expansions at moderate sizes; every statement they illustrate is proved above.

Two exact physical $6j$ rays make the degree -4 operator test reproducible without fitting asymptotic slopes. For their labels $E_{A,B} = 1$ ($p_{\min}, p_{\max} \le 1$), so Lemma 8.5 gives the displayed closed forms directly, the denominators being $\Pi_f^{1/2}$. In the integer-label family $m = 1$, $n = 0$, $f = 2$ with $(A,B,C) = (3\Lambda, 2\Lambda, 2\Lambda)$ and integer Λ (so that $a = 3\Lambda - \frac{1}{2}$ and $c = 2\Lambda - \frac{1}{2}$ are half-integers), one obtains at $x = 3/4$

$$\begin{aligned}\frac{\Phi_2}{d^2_{0,1}(\arccos(3/4))} &= \frac{1 - \Lambda^{-2}/9}{\sqrt{(1-\Lambda^{-2}/36)(1-\Lambda^{-2}/16)(1-\Lambda^{-2}/9)(1-\Lambda^{-2}/4)}} \\ &= 1 + \frac{11}{96\Lambda^2} + \frac{1121}{55296\Lambda^4} + O(\Lambda^{-6}).\end{aligned} \tag{268}$$

For half-integer offsets $m = n = \frac{1}{2}$, $f = \frac{3}{2}$, use $(A,B,C) = (3\Lambda, 3\Lambda, 2\Lambda)$ with $\Lambda = k + \frac{1}{4}$ and integer k. At $x = 7/9$,

$$\frac{\Phi_{3/2}}{d^{3/2}_{1/2,1/2}(\arccos(7/9))} = \frac{1}{1-(144\Lambda^2)^{-1}} = 1 + \frac{1}{144\Lambda^2} + \frac{1}{20736\Lambda^4} + O(\Lambda^{-6}). \tag{269}$$

Evaluating $\mathcal{G}_2$ of (239), and $\mathcal{G}_4$ by an exact symbolic implementation of the recursion of Appendix B.3, on these rays reproduces both coefficients.

Table 1 reports maximum absolute errors in the normalized kernels Φ; no points are excluded near zeros. The five columns compare a one-sided leading formula, the centered Wigner mode d, the bulk first correction, the edge-factorized leading term Ed, and the uniform first correction of Corollary 10.3. In the bulk the observed orders are $S^{-1}, S^{-2}, S^{-4}, S^{-2}, S^{-4}$, with $S = N$ or

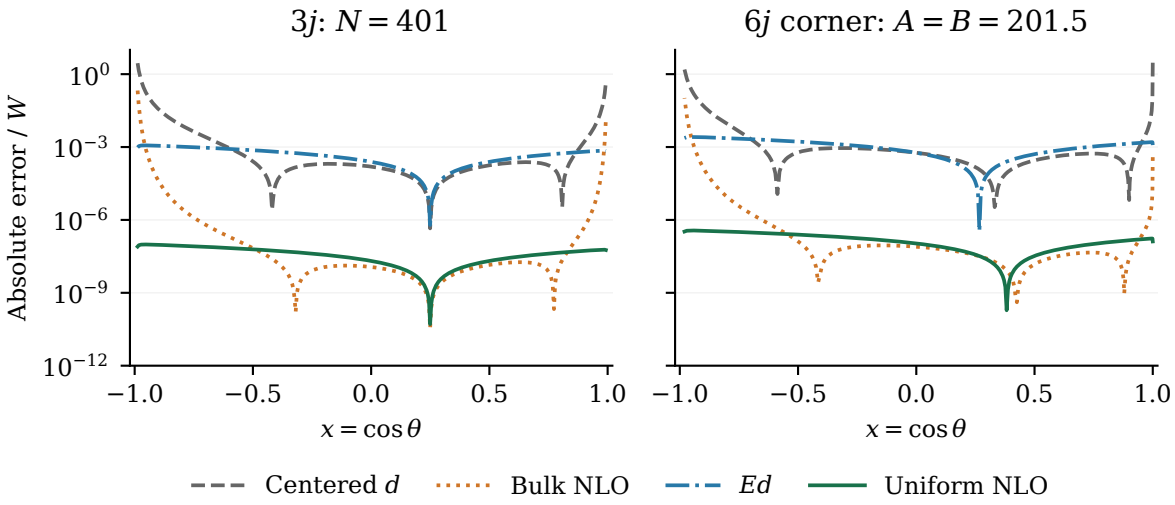


Figure 1: Errors across complete physical lattices, divided by the endpoint envelope W. Left: $3j$ with $(L, M, \sigma) = (4, 1, 3)$ and $N = 401$ (half-integer large spins $j_\pm = 403/2, 397/2$). Right: $6j$ with $(f, m, n) = (4, 3, 0)$ (an additional case) and $(a, b) = (201, 199.5)$, hence $A = B = 201.5$. The four centered approximations of Table 1 are compared with exact symbols at 50-digit precision; the lines join lattice values. The edge factor restores the second-order error envelope, and the uniform next-to-leading form improves it to fourth order. The finite plots illustrate the bounds; the proof is Theorem 10.1.

$A + B$. Only the last two approximations retain their stated envelope-weighted orders at the edges; Figure 1 shows this distinction across complete lattices, including the lower $6j$ corner.

For $3j$, the one-sided approximation uses the Ponzano–Regge/Flude scale $j_+ + \frac{1}{2}$ and the magnetic index m_+ of j_+; in Φ units it is

$$\sqrt{\frac{N}{2j_+ + 1}}\, d^L_{M\sigma}(\arccos x_*), \qquad x_* := -\frac{2m_+}{2j_+ + 1} = \frac{2\mu + M}{2j_+ + 1}. \tag{270}$$

For $6j$, the Flude approximation in the same units is

$$\frac{2\sqrt{AB}}{\sqrt{(2A + f)(2B + f)}}\, d^f_{nm}(\arccos x_F), \tag{271}$$

with x_F from (248). For these two comparisons, cosines outside $[-1, 1]$ are clipped to the nearest endpoint. This specifies a real-valued continuation of the leading formulas on the full physical lattice; it is not an extension of their interior asymptotic estimates. The bulk NLO columns use (4) and (238); the uniform NLO columns use (261) and (262).

The $3j$ test triples (L, M, σ) are

$$(1,1,0),\ (1,0,1),\ (2,1,1),\ (2,2,1),\ (3,0,2),\ (3,1,-2),\ (4,2,1),\ (4,1,3),\ (4,0,0),\ (2,0,0). \tag{272}$$

At nominal size S_0 we take $N = S_0$ or $S_0 + 1$ so that $j_\pm$ are integers. The $6j$ triples (f, m, n) are

$$(1,1,0),\ (2,1,0),\ (2,1,-1),\ (3,2,1),\ (3,1,-1),\ (2,0,0),\ (4,2,0),\ (\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}),\ (\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2}). \tag{273}$$

Each is evaluated on two shapes: $(a, b) = (0.6S_0 - 1, 0.4S_0 - 1)$ and $b = S_0/2 - 1$, $a = b + (m - n)/2$. The latter has exactly $A = B$, including for half-integer offsets. The nominal sizes are $S_0 = 10^2, 10^3, 10^4$. The table uses the specified physical subsequences.

(a) $3j$: ten triples (L, M, σ), $L \le 4$; scale $S = N$

$S \approx$	region	one-sided	centered d	bulk NLO	$E\,d$	uniform NLO
10^2	$\lvert x\rvert \le 0.8$	3.8×10^{-2}	1.1×10^{-3}	6.1×10^{-6}	9.8×10^{-4}	1.4×10^{-6}
	full lattice	1.1×10^{-1}	6.2×10^{-3}	3.4×10^{-4}	2.5×10^{-3}	3.5×10^{-6}
10^3	$\lvert x\rvert \le 0.8$	3.9×10^{-3}	1.1×10^{-5}	6.6×10^{-10}	9.8×10^{-6}	1.4×10^{-10}
	full lattice	3.9×10^{-2}	6.8×10^{-4}	1.9×10^{-5}	2.7×10^{-5}	3.9×10^{-10}
10^4	$\lvert x\rvert \le 0.8$	3.9×10^{-4}	1.1×10^{-7}	6.6×10^{-14}	9.8×10^{-8}	1.4×10^{-14}
	sampled lattice	1.2×10^{-2}	6.8×10^{-5}	1.9×10^{-6}	2.7×10^{-7}	3.9×10^{-14}

(b) One-small $6j$: nine triples (f, m, n), $f \le 4$; scale $S = A + B$

$S \approx$	region	Flude	centered d	bulk NLO	$E\,d$	uniform NLO
10^2	$\lvert x\rvert \le 0.8$	6.4×10^{-2}	3.0×10^{-3}	3.9×10^{-5}	3.6×10^{-3}	1.1×10^{-5}
	full lattice	1.1×10^{-1}	1.3×10^{-2}	4.8×10^{-4}	3.6×10^{-3}	1.1×10^{-5}
10^3	$\lvert x\rvert \le 0.8$	6.3×10^{-3}	3.0×10^{-5}	3.9×10^{-9}	3.6×10^{-5}	1.1×10^{-9}
	full lattice	2.4×10^{-2}	1.7×10^{-3}	4.8×10^{-5}	3.6×10^{-5}	1.1×10^{-9}
10^4	$\lvert x\rvert \le 0.8$	6.3×10^{-4}	3.0×10^{-7}	4.0×10^{-13}	3.6×10^{-7}	1.1×10^{-13}
	sampled lattice	7.9×10^{-3}	1.7×10^{-4}	4.8×10^{-6}	3.6×10^{-7}	1.1×10^{-13}

Table 1: Maximum absolute errors in the normalized kernels $\Phi_L^{M,\sigma}$ and Φ_f, using 30-digit evaluations of the exact Racah formulas. All methods use the same normalization; no points near zeros are excluded. The $6j$ test includes $A/B \approx 3/2$ and exactly $A = B$. The two smaller lattices are complete; at $S \approx 10^4$ we sample a uniform stride plus all points within 30 steps of either edge, so these entries are sampled maxima. Methods and test parameters are specified in the text.

12 Discussion and outlook

In both families Regge symmetry pairs the large labels symmetrically about a midpoint, and the Weyl shift places that midpoint where the generic first-order term of the exact recurrence vanishes and the centered recurrence becomes invariant under reversal of the large scale. Evenness is a separate fact: it holds because the physical initial data lie in the positive sector of that reversal (Remark 3.3), and it is made explicit by the square-root-free closed forms. Independently, the Hahn and Racah structures give exact finite spectral systems and unitaries to truncated Wigner windows, whose sampled bulk forms are local differential operators to all orders at fixed rank, printed in closed form through N^{-4} for $3j$ and through degree -2 for $6j$, with the degree -4 term of $6j$ given constructively by (245).

For the one-small $6j$ symbol the results are intrinsic on the physical strict-triangle cone: coefficient functions and local operators are homogeneous in (A, B, C), with no arbitrary large scale. At fixed rank and for A/B bounded away from 0 and ∞, Section 10 gives a single expansion on the whole physical lattice, including the lower corner $A \approx B$: after the explicit edge factor and the endpoint envelope are extracted, every coefficient is a polynomial, and the relative Laguerre laws are endpoint consequences of this factorization.

Practical use. The settings of the Introduction (squeezed $3j$ configurations on the sphere, with half-integer small labels covering spin–orbit couplings for which (85) is exact at $L = \frac{1}{2}$, and Edmonds-type tensor-operator recoupling) share one practical rule: use the offset-dependent Weyl-centered lengths, after which the first correction is quadratic in the inverse scale, and near the edges multiply by the explicit weight ratio. The one-small $6j$ symbol is also the one-small-spin limit of spin-network evaluations, whose all-large asymptotics is the Ponzano–Regge

formula [33, 18, 19]. For exact numerical values at small rank, recursions and prime-factorization codes are already fast and exact [9, 34, 35]; the formulas here add closed forms for the leading corrections with explicit dependence on the midpoint variables, explicit derivatives, and uniform error control (Table 1).

At fixed rank the construction is complete, down to the exact complex radii of the expansions away from a proper analytic exceptional set (Remark 10.2). Growing rank is treated in the companion paper [8]. Two further questions are open here. First, the quantum $U_q(\mathfrak{sl}_2)$ $6j$ symbol with one small entry: in its rank recurrence the coefficient of x is itself deformed, so Proposition B.2 does not apply as stated, and an exact finite spectral system, all-order local calculus and edge factorization for it are not established. Second, the centered finite-spectral architecture should extend to other bounded-width recoupling families; there one expects a reflection-graded normal form rather than universal entrywise evenness, since a finite channel space may carry a nontrivial involution under scale reversal.

Use of AI

All results, documents and scripts were generated by AI agents (Claude Opus and Sonnet 5.5, OpenAI's GPT 6.1 sol and Gemini 4 Argon/Carbon) with human supervision and many cross-checks, including exact symbolic and high-precision numerical regression checks of the stated formulas. I thank DeepMind and Anthropic for access to their scientist programs.

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A Hahn–Jacobi contraction and polar realization

This appendix records two classical structures of the finite $3j$ problem of Section 5 that the main results do not use: the exact contraction of the finite Hahn quadratic algebra to the Jacobi algebra, and a realization of the unitary S_N of Corollary 5.3 as the polar factor of a conditional expectation.

A.1 Exact Hahn-to-Jacobi algebra contraction

Let X denote multiplication by x, put

$$\Sigma = M^2 + \sigma^2, \qquad \tau = M\sigma, \qquad \mathsf{C}_h = [\mathsf{H}_h, \mathsf{X}]. \tag{274}$$

Proposition A.1 (Exact finite Hahn-to-Jacobi algebra contraction)**.** *The finite operators satisfy exactly*

$$[\mathsf{C}_h, \mathsf{X}] = 2(\mathsf{X}^2 - \mathbf{1}) + 4h^2 \left(\mathsf{H}_h - \frac{\Sigma - 1}{2}\mathbf{1}\right), \tag{275}$$

$$[\mathsf{H}_h, \mathsf{C}_h] = 2\{\mathsf{H}_h, \mathsf{X}\} - 4\tau\mathbf{1}. \tag{276}$$

Hence the finite Hahn-type quadratic relations contract directly to the Jacobi quadratic algebra (277) *of the Wigner equation as $h \to 0$. The explicit deformation term in* (275) *is proportional to $4h^2 = 4/N^2$.*

Proof. For grid points x_i, x_j, one has $(\mathsf{C}_h)_{ij} = (\mathsf{H}_h)_{ij}(x_j - x_i)$. Commuting once more with X multiplies each nearest-neighbor link by $(2h)^2$; inserting the diagonal of (41) gives (275). In $[\mathsf{H}_h, \mathsf{C}_h]$ the second-neighbor terms cancel, and the remaining diagonal and nearest-neighbor terms reduce to (276). □

The $h \to 0$ limit of these relations is the Jacobi algebra of the Wigner operator.
Put $\mathsf{X} = x$, let H_0 be (57), and set $\mathsf{C} = [\mathsf{H}_0, \mathsf{X}]$.

Proposition A.2 (Jacobi algebra and right-ordered normal form)**.** *On smooth interior functions,*

$$\begin{aligned} [\mathsf{H}_0, \mathsf{X}] &= \mathsf{C}, \\ [\mathsf{C}, \mathsf{X}] &= 2(\mathsf{X}^2 - \mathbf{1}), \\ [\mathsf{H}_0, \mathsf{C}] &= 2\{\mathsf{H}_0, \mathsf{X}\} - 4\tau, \end{aligned} \tag{277}$$

and the representation identity

$$\mathsf{C}^2 = -4(1 - \mathsf{X}^2)\mathsf{H}_0 + 4\mathsf{X}\mathsf{C} + 4(\Sigma - 2\tau\mathsf{X} - 1) \tag{278}$$

holds. Consequently every differential operator generated algebraically by $\mathsf{X}, \mathsf{H}_0, \mathsf{C}$ *can be reduced to the right-ordered form*

$$\mathcal{A} = \sum_{p \geq 0} [p_p(\mathsf{X}) + q_p(\mathsf{X})\mathsf{C}]\mathsf{H}_0^p \tag{279}$$

with rational coefficient functions whenever the original operator has rational coefficients.

Proof. The commutators follow by direct calculation from (57); explicitly $\mathsf{C} = 2(x^2 - 1)\partial_x + 2x$. Squaring this expression and eliminating second derivatives with H_0 gives (278). The commutator relations move every factor H_0 to the right, while (278) reduces every power C^r with $r \geq 2$. Iteration gives (279). □

A.2 Beta–binomial polar realization of the transmutation

Corollary 5.3 characterizes the exact unitary spectrally. The same map is also the polar factor of a natural finite-to-continuum contraction, a Bernstein–Durrmeyer operator with Jacobi weights of the kind studied by Derriennic and by Berens–Xu [36, 37]. This structure is classical and off the main line; we state what is used and leave the singular-system computation and the physical gauges to Appendix A.3.

Lemma A.3 (Polar unitary of a finite intertwiner)**.** *Let* $\mathcal{V}, \mathcal{W}$ *be finite-dimensional Hilbert spaces of the same dimension, let* $\mathsf{H}_0, \mathsf{H}_1$ *be self-adjoint, and let* $\mathsf{T} : \mathcal{V} \to \mathcal{W}$ *be invertible with* $\mathsf{H}_1\mathsf{T} = \mathsf{T}\mathsf{H}_0$*. Then* $\mathsf{T}^*\mathsf{T}$ *commutes with* H_0*, and*

$$\mathsf{U} = \mathsf{T}(\mathsf{T}^*\mathsf{T})^{-1/2} \tag{280}$$

is unitary and still intertwines.

In the chamber (46), put $\varsigma = (1 - x)/2$ and let ν_H be the beta measure and p_D the beta–binomial mass function,

$$\mathrm{d}\nu_H(\varsigma) = \frac{\varsigma^{\alpha_H}(1 - \varsigma)^{\beta_H}}{B(\alpha_H + 1, \beta_H + 1)}\,\mathrm{d}\varsigma, \tag{281}$$

$$p_D(u) = \binom{D}{u} \frac{B(u + \alpha_H + 1, D - u + \beta_H + 1)}{B(\alpha_H + 1, \beta_H + 1)}. \tag{282}$$

With $b_u^D(\varsigma) = \binom{D}{u}\varsigma^u(1 - \varsigma)^{D-u}$, the conditional expectation of $U \mid T \sim \mathrm{Binomial}(D, T)$, $T \sim \mathrm{Beta}(\alpha_H + 1, \beta_H + 1)$, is

$$(\mathsf{C}_D f)(u) = \frac{1}{p_D(u)} \int_0^1 b_u^D(\varsigma) f(\varsigma)\,\mathrm{d}\nu_H(\varsigma). \tag{283}$$

Proposition A.4 (Beta–binomial polar realization)**.** *Let $\mathcal{P}_D = \operatorname{span}\{J_0, \dots, J_D\} \subset L^2(\mathrm{d}\nu_H)$, where J_n and H_n are the unit-normalized Jacobi and Hahn modes in $L^2(\mathrm{d}\nu_H)$ and $\ell^2(p_D)$, signed as positive multiples of ${}_2F_1(-n, n+\alpha_H+\beta_H+1; \alpha_H+1; \varsigma)$ and of Q_n (so $Q_n(0) = 1$; Appendix A.3), and restrict conditional expectation to $\mathsf{T}_D := \mathsf{C}_D|_{\mathcal{P}_D} : \mathcal{P}_D \to \ell^2(p_D)$. For $0 \le n \le D$,*

$$\mathsf{T}_D J_n = s_n H_n, \qquad s_n^2 = \frac{D^{\underline{n}}}{(D+\alpha_H+\beta_H+2)_n}. \tag{284}$$

All s_n are positive, so T_D is invertible between two $(D+1)$-dimensional Hilbert spaces, and its polar factor $\mathsf{U}_D = \mathsf{T}_D(\mathsf{T}_D^*\mathsf{T}_D)^{-1/2}$ is a unitary $\mathcal{P}_D \to \ell^2(p_D)$ with $\mathsf{U}_D J_n = H_n$. (The polar factor of the unrestricted C_D on $L^2(\mathrm{d}\nu_H)$ is only a partial isometry, with kernel containing $\mathcal{P}_D^\perp$.) Under the physical continuum/discrete gauges $\mathsf{G}_c, \mathsf{G}_d$ defined in Appendix A.3, the Wigner and finite Hahn modes carry the same phase, so

$$\mathsf{S}_N = \mathsf{G}_d \mathsf{U}_D \mathsf{G}_c^{-1}. \tag{285}$$

Moreover, for $L = M + n$,

$$s_{L-M}^2 = \prod_{r=M+1}^{L} \frac{N-r}{N+r}. \tag{286}$$

Proof. The Jacobi–Hahn Bernstein identity and singular-value calculation are given in Appendix A.3; Lemma A.3 then gives the polar unitary. The physical gauge identities follow by comparing the exact Hahn kernel (52) with the standard Jacobi representation of the Wigner d matrix. The common $(-1)^{\alpha_H} = (-1)^{M-\sigma}$ phase, which is also the sign of the Jacobi representation of $d^L_{M\sigma}$, cancels, and substitution of $D = N - M - 1$, $\alpha_H + \beta_H = 2M$ and $n = L - M$ gives (286). □

Thus the exact finite-to-continuum map is both a spectral basis matching and the polar whitening of a smoothing channel. Before whitening,

$$\log s_{L-M}^2 = -\frac{L(L+1) - M(M+1)}{N} + O(N^{-3}), \tag{287}$$

so the positive contraction carries an N^{-1} heat-like attenuation even though the whitened midpoint unitary has an all-even local expansion; the whitening formula is in Appendix A.3.

A.3 Singular system and polar whitening

The Bernstein–Hahn–Jacobi identity below is classical [38], as is the polynomial singular-system viewpoint for beta/binomial conditional expectation [39]; the physical parameter and gauge identification is the paper-specific step. The adjoint of (283) is the Bernstein averaging operator

$$(\mathsf{C}_D^* q)(\varsigma) = \sum_{u=0}^{D} b_u^D(\varsigma) q(u). \tag{288}$$

In the Hahn convention (49),

$$\sum_{u=0}^{D} b_u^D(\varsigma) Q_n(u; \alpha_H, \beta_H, D) = \frac{n!}{(\alpha_H+1)_n} P_n^{(\alpha_H, \beta_H)}(1-2\varsigma). \tag{289}$$

This follows by inserting the terminating ${}_3F_2$ series and using $\sum_u b_u^D(\varsigma)(-u)_k = (-D)_k \varsigma^k$. Fix the signs of the unit-normalized modes by taking J_n a positive multiple of ${}_2F_1(-n, n+\alpha_H+\beta_H+1; \alpha_H+1; \varsigma)$ and H_n a positive multiple of Q_n (note $Q_n(0) = 1$). By (289), $\mathsf{C}_D^* H_n$ is then a positive multiple of J_n. For $k \ne n$, $\langle H_k, \mathsf{C}_D J_n\rangle = \langle \mathsf{C}_D^* H_k, J_n\rangle = 0$ by Jacobi orthogonality, and

$\mathsf{C}_D J_n$ lies in the span of the H_k, $k \le D$, so the normalized Hahn and Jacobi modes are singular pairs, $\mathsf{C}_D J_n = s_n H_n$ with $s_n = \langle \mathsf{C}_D^* H_n, J_n\rangle > 0$ under this sign convention. The posterior moment $\mathsf{C}_D(\varsigma^n)$ has leading coefficient $(D+\alpha_H+\beta_H+2)_n^{-1}$ as a polynomial in u, while binomial averaging sends u^n to $D^{\underline{n}}\varsigma^n$ plus lower-degree terms. Hence $\mathsf{C}_D^*\mathsf{C}_D$ has the eigenvalue s_n^2 of (284) on the degree-n Jacobi mode.

The physical gauges are

$$(\mathsf{G}_c p)(x) = \frac{1}{\sqrt{2}}\left[\frac{\varsigma^{\alpha_H}(1-\varsigma)^{\beta_H}}{B(\alpha_H+1,\beta_H+1)}\right]^{1/2} p(\varsigma), \qquad \varsigma = \frac{1-x}{2}, \tag{290}$$

$$(\mathsf{G}_d q)(x_u) = \sqrt{\frac{N}{2}}\, p_D(u)^{1/2} q(u). \tag{291}$$

The standard Jacobi formula for $d^L_{M\sigma}$ and the exact Hahn kernel (52) give

$$u_L = (-1)^{\alpha_H}\mathsf{G}_c J_n, \qquad v_L = (-1)^{\alpha_H}\mathsf{G}_d H_n, \tag{292}$$

which proves the phase cancellation in (285).

For the unwhitened physical contraction $\widetilde{\mathsf{C}}_N = \mathsf{G}_d\mathsf{C}_D\mathsf{G}_c^{-1}$, define $\widehat{L} = (\sqrt{1+4\mathsf{H}_0}-1)/2$ on $\mathcal{V}_N$. Then

$$\widetilde{\mathsf{C}}_N^*\widetilde{\mathsf{C}}_N = \Gamma_{N,M}(\mathsf{H}_0), \tag{293}$$

$$\Gamma_{N,M}(\mathsf{H}_0) = \frac{\Gamma(N-M)\Gamma(N+M+1)}{\Gamma(N-\widehat{L})\Gamma(N+\widehat{L}+1)}, \tag{294}$$

and

$$\mathsf{S}_N = \widetilde{\mathsf{C}}_N\Gamma_{N,M}(\mathsf{H}_0)^{-1/2}. \tag{295}$$

So the conditional-expectation map already intertwines the continuum and finite Hamiltonians, and polar whitening removes only its known degree-dependent attenuation.

B Local calculus for centered Wigner rank recurrences

The all-order constructions for $3j$ (Theorem 6.5) and $6j$ (Theorem 9.2) are one construction applied to two recurrences of the same shape. We give it once and then specialize.

B.1 The general construction

Fix $a, b \in \frac{1}{2}\mathbb{Z}$ in one integer/half-integer class, put $k_0 = \max(|a|,|b|)$, and for ranks $k \ge k_0$ in the same class let

$$\lambda_k = k(k+1), \qquad R_k = \sqrt{(k^2-a^2)(k^2-b^2)}, \qquad \Sigma = a^2+b^2, \qquad \tau = ab, \tag{296}$$

$d_k = d^k_{ab}(\arccos x)$, and let $\mathsf{H}_0 = -\partial_x(1-x^2)\partial_x + (\Sigma - 2\tau x)/(1-x^2)$ be the Wigner operator, so that $\mathsf{H}_0 d_k = \lambda_k d_k$. The undeformed Wigner rank operator

$$\begin{aligned}(\mathcal{L}_{\mathrm{W}}Y)_k = {} & kR_{k+1}Y_{k+1} + (k+1)R_kY_{k-1} \\ & - (2k+1)(\lambda_k x - \tau)Y_k\end{aligned} \tag{297}$$

annihilates d. If A is any x-differential operator independent of k, then

$$\mathcal{L}_{\mathrm{W}}(\mathsf{A}d)_k = -(2k+1)\lambda_k[x,\mathsf{A}]d_k, \tag{298}$$

since the x-dependence of $\mathcal{L}_{\mathrm{W}}$ is confined to its diagonal term and $\mathcal{L}_{\mathrm{W}}(\mathsf{A}d) = \mathsf{A}\mathcal{L}_{\mathrm{W}}d + [\mathcal{L}_{\mathrm{W}},\mathsf{A}]d$.

The deformed recurrences are

$$(2k+1)\big[\lambda_k(x-q(\lambda_k))-\tau\big]\Phi_k = kR_{k+1}\eta(k+1)\Phi_{k+1} + (k+1)R_k\eta(k)\Phi_{k-1}, \tag{299}$$

with an even off-diagonal factor and a diagonal shift

$$\eta(k)=\sum_{s\geq 0} e_s k^{2s}, \quad e_0=1, \qquad q(\lambda)=\text{polynomial in } \lambda, \tag{300}$$

where e_s is of pure order $2s$ and q is homogeneous of order 2 in an inverse large scale (made precise in the two instances below, which are of this form). The undeformed case $\eta=1$, $q=0$ is annihilated by the operator $\mathcal{L}_{\mathrm{W}}$ of (297). Here a, b are generic labels, not the large $6j$ spins. The neighboring-moment reduction rests on an elementary polynomial fact.

Lemma B.1 (Moment polynomial)**.** *For every integer $s \geq 1$ there is a unique polynomial $\pi_s(z)$ of degree s satisfying*

$$\pi_s(\lambda_k)+\pi_s(\lambda_{k+1}) = 2(k+1)^{2s}, \tag{301}$$

and it obeys $\pi_s(0)=0$.

Proof. Put $y=k+1$, so $\lambda_k = y^2-y$ and $\lambda_{k+1}=y^2+y$. The linear map $T[p](y) := p(y^2-y)+p(y^2+y)$ sends polynomials of degree at most s in z to even polynomials of degree at most $2s$ in y, and it is triangular in the monomial bases, since $T[z^r]=2y^{2r}$ plus lower even powers. Hence T is invertible and there is a unique π_s with $T[\pi_s]=2y^{2s}$. Setting $y=0$ gives $2\pi_s(0)=0$. □

Thus $\pi_1(z)=z$ and $\pi_2(z)=z(z-1)$. Put $\widetilde{\pi}_s=\pi_s/z$ and, for a k-independent operator A,

$$\mathcal{N}_s[\mathsf{A}] := x\mathsf{A}\pi_s(\mathsf{H}_0) - \tau\mathsf{A}\widetilde{\pi}_s(\mathsf{H}_0) - \frac{1}{2}\Big([x,\mathsf{A}\pi_s(\mathsf{H}_0)] + [x,\mathsf{A}]\pi_s(\mathsf{H}_0)\Big). \tag{302}$$

Then

$$kR_{k+1}(k+1)^{2s}(\mathsf{A}d)_{k+1} + (k+1)R_k k^{2s}(\mathsf{A}d)_{k-1} = (2k+1)\lambda_k\mathcal{N}_s[\mathsf{A}]d_k. \tag{303}$$

Indeed, (301) writes $(k+1)^{2s}=\frac{1}{2}\{\pi_s(\lambda_k)+\pi_s(\lambda_{k+1})\}$ and $k^{2s}=\frac{1}{2}\{\pi_s(\lambda_{k-1})+\pi_s(\lambda_k)\}$. Call $(\mathcal{L}_{\mathrm{W}}Y)_k+(2k+1)(\lambda_k x-\tau)Y_k$ the off-diagonal part of $\mathcal{L}_{\mathrm{W}}$. Since $\pi_s(\lambda_j)(\mathsf{A}d)_j=(\mathsf{A}\pi_s(\mathsf{H}_0)d)_j$, the left side is $\frac{1}{2}\pi_s(\lambda_k)$ times the off-diagonal part of $\mathcal{L}_{\mathrm{W}}$ applied to $\mathsf{A}d$, plus $\frac{1}{2}$ times the off-diagonal part applied to $\mathsf{A}\pi_s(\mathsf{H}_0)d$. Adding back the diagonal terms and using (298), each off-diagonal part equals $(2k+1)\{(\lambda_k x-\tau)\mathsf{A}'-\lambda_k[x,\mathsf{A}']\}d_k$ with $\mathsf{A}'=\mathsf{A}$, respectively $\mathsf{A}'=\mathsf{A}\pi_s(\mathsf{H}_0)$; in the first, the prefactor $\pi_s(\lambda_k)$ is moved onto d_k as $\pi_s(\mathsf{H}_0)$. So the left side of (303) is

$$\tfrac{2k+1}{2}\Big\{2\lambda_k x\mathsf{A}\pi_s(\mathsf{H}_0) - 2\tau\mathsf{A}\pi_s(\mathsf{H}_0) - \lambda_k[x,\mathsf{A}]\pi_s(\mathsf{H}_0) - \lambda_k[x,\mathsf{A}\pi_s(\mathsf{H}_0)]\Big\}d_k,$$

and $\pi_s(\lambda_k)=\lambda_k\widetilde{\pi}_s(\lambda_k)$ turns this into (302). The all-order step therefore contains no hidden rank summation.

For a left-normal differential operator $\mathsf{B}=\sum_{j=0}^{J} b_j(x)\partial_x^j$ set

$$\mathcal{J}(\mathsf{B}) := -\sum_{j=0}^{J}\frac{b_j(x)}{j+1}\partial_x^{j+1}, \qquad [x,\mathcal{J}(\mathsf{B})]=\mathsf{B}. \tag{304}$$

Write $q_2:=q$ to mark its order. Starting with $\mathcal{G}_0=I$, define recursively

$$\mathsf{B}_{2r} := \mathcal{G}_{2r-2}\,q_2(\mathsf{H}_0) + \sum_{s=1}^{r} e_s\mathcal{N}_s[\mathcal{G}_{2(r-s)}], \tag{305}$$

$$\mathcal{G}_{2r} := \mathcal{J}(\mathsf{B}_{2r}) + \psi_{2r}(x), \tag{306}$$

with a multiplication operator ψ_{2r} still to be fixed, and grouping terms by their order.

Proposition B.2 (Local calculus)**.** (a) *The formal series* $\Phi_k = \sum_{r\ge 0} \mathcal{G}_{2r} d_k$ *solves* (299) *order by order for every choice of the* ψ_{2r}*.*

(b) *If* $k_0 > 0$*, the recurrence is one-sided at* k_0 *(because* $R_{k_0} = 0$*), and the formal solution with a given start* $\Phi_{k_0} = d_{k_0} \sum_r \kappa_r$*, with* κ_r *of order* $2r$*, is obtained with*

$$\psi_{2r} = \kappa_r - \frac{\mathcal{J}(\mathsf{B}_{2r}) d_{k_0}}{d_{k_0}}, \tag{307}$$

a function of x *that does not depend on* k*;* B_{2r} *involves only* $\psi_{2r'}$ *with* $r' < r$*, so the recursion is not circular.*

(c) *If* $a = b = 0$*, then* $\psi_{2r} = 0$ $(r \ge 1)$ *gives the starts* $\Phi_0 = 1$ *and* $\Phi_1 = (x - q(0))/\eta(1)$ *to all orders.*

Proof. (a) For a k-independent operator A, $q(\lambda_k)\mathsf{A} d_k = \mathsf{A} q(\mathsf{H}_0) d_k$. Inserting the series into (299) and using (298) and (303), the terms of order $2r$ are $(2k+1)\lambda_k$ times $(-[x, \mathcal{G}_{2r}] + \mathsf{B}_{2r}) d_k$, which vanishes by (304), since a multiplication operator commutes with x.

(b) At $k = k_0$ the order-$2r$ term of Φ_{k_0}/d_{k_0} is $\mathcal{J}(\mathsf{B}_{2r}) d_{k_0}/d_{k_0} + \psi_{2r}$, which equals κ_r by (307). The one-sided recurrence determines Φ_k for $k > k_0$ from Φ_{k_0} order by order, since its forward coefficients kR_{k+1} are nonzero.

(c) Here $d_0 = 1$, $d_1 = x$, $\tau = 0$, and $\pi_s(\mathsf{H}_0)1 = \pi_s(0) = 0$, $\pi_s(2) = 2$ by (301) at $k = 0$, and $\mathsf{H}_0 x = 2x$; so (302) gives the exact identity

$$\mathcal{N}_s[\mathsf{A}]1 = \mathsf{A} x \tag{308}$$

for every k-independent operator A. Since $\mathcal{J}(\mathsf{B})$ contains only positive derivative orders, $\mathcal{G}_{2r} 1 = 0$ for $r \ge 1$, which is the first start. Put $a_r := \mathcal{G}_{2r} x$, $a_0 = x$. Since $\mathcal{J}(\mathsf{B}) x = -\mathsf{B}1$ and $q(\mathsf{H}_0)1 = q(0)$, applying (305) to 1 and using (308) gives

$$a_r = -q(0)\, \delta_{r1} - \sum_{s=1}^{r} e_s a_{r-s}, \tag{309}$$

so the sum $\sum_r a_r$ of all orders satisfies $\eta(1) \sum_r a_r = x - q(0)$. This is the second start: since $R_k = k^2$ when $a = b = 0$, (299) divided by k and evaluated at $k = 0$ reads $(x - q(0))\Phi_0 = \eta(1)\Phi_1$, as in (154). Forward uniqueness of the recurrence with the two starts gives the all-order result. □

Link to the physical kernels. Proposition B.2 concerns formal series. The physical kernels enter as follows: the closed forms of Theorems 5.6 and 8.7 satisfy their rank recurrences identically in the continuous variables, because (72) and (210) are polynomial identities, so their Taylor coefficients satisfy (299) order by order. They have the same start, so by forward uniqueness they coincide with the series of Proposition B.2. For $6j$ the identity holds pointwise in (A, B, C); hence the fixed-shape Λ^{-2} expansion and the expansion in homogeneous degree at fixed (A, B) define the same coefficients. The proposition covers recurrences whose x-dependence sits only in the diagonal term $(2k+1)\lambda_k x$, with an x-independent diagonal shift $q(\lambda_k)$ and an even off-diagonal factor η; a recurrence in which the coefficient of x is itself deformed, such as the one-small quantum $6j$ recurrence, needs a different reduction.

B.2 The $3j$ instance

For $3j$ take $(a, b) = (M, \sigma)$, $k = L$, $q = 0$ and $\eta(k) = \sqrt{1 - k^2 t}$, $t = N^{-2}$, so that $e_s = c_s t^s$ with $\sqrt{1 - z} = \sum_s c_s z^s$, $c_0 = 1$, $c_1 = -\frac{1}{2}$, $c_2 = -\frac{1}{8}$; then (299) is (18), and $\mathcal{G}_{2r} = t^r \widehat{\mathcal{D}}_{2r}$, with

$\mathsf{B}_{2r}^{(3j)} = \sum_{s=1}^{r} c_s \mathcal{N}_s[\widehat{\mathcal{D}}_{2(r-s)}]$. In the canonical chamber $M \geq |\sigma|$, $M > 0$, put $u_\pm = (1 \pm x)/2$ and, extending the notation of Appendix C,

$$S_{2r}(p) := \sum_{q \in \{-\frac{p-1}{2}, -\frac{p-3}{2}, \ldots, \frac{p-1}{2}\}} q^{2r}, \qquad S_{2r}(p) = \frac{2B_{2r+1}\left(\frac{p+1}{2}\right)}{2r+1} \quad (r \geq 1,\ p \in \mathbb{Z}_{\geq 0}), \tag{310}$$

with B_n the Bernoulli polynomials. The exact extremal seed gives

$$\log \frac{\Phi_M^{M,\sigma}}{d_{M\sigma}^M} = \sum_{r \geq 1} \ell_r(x) t^r, \qquad \ell_r = \frac{1}{4r}\Big[S_{2r}(2M+1) - \frac{S_{2r}(M+\sigma)}{u_+^{2r}} - \frac{S_{2r}(M-\sigma)}{u_-^{2r}}\Big], \tag{311}$$

and κ_r in (307) is the coefficient of t^r in $\exp(\sum_{s \geq 1} \ell_s t^s)$. The series in (311) is not only formal: at every interior physical lattice point it converges, because the radius conditions $(M+\sigma)^2 t \leq 4u_+^2$, $(M-\sigma)^2 t \leq 4u_-^2$ and $(2M+1)^2 t \leq 4$ of the Mercator expansions of the finite products hold automatically there, and the formula needs only $M \pm \sigma \in \mathbb{Z}$ (for $M = \sigma = 0$ all ℓ_r vanish and $\Phi_0 = 1$). This ψ_{2r} is rational on the interior, and exact Regge and transposition covariance extends the same operator to every nondegenerate sign/order chamber. For $M = \sigma = 0$, Proposition B.2(c) with $q = 0$ and $\eta(1) = \sqrt{1-t}$ gives

$$\widehat{\mathcal{D}}_{2r} x = \frac{\binom{2r}{r}}{4^r}\, x, \tag{312}$$

the Taylor coefficients of the second start $\Phi_1 = x/\sqrt{1-t}$. At $r = 1$ the recursion gives (105), and at $r = 2$, after reduction with (39), the operator (323)–(324). This proves Theorem 6.5 in all physical sectors.

B.3 The $6j$ instance

For $6j$ take $(a, b) = (n, m)$, $k = f$, $\eta(k) = \eta_k$ of (122) and $q(\lambda) = (2\lambda + 1 - \Sigma)/(8AB)$, so that (299) is (125) and $\mathcal{L}_{\mathrm{W}}$ is the Wigner rank operator of (243). The order is the homogeneous degree in (A, B, C), counted negatively: the coefficients $e_s(A, B)$ defined by

$$\sum_{s \geq 0} e_s(A,B) z^s = \sqrt{\Big(1 - \frac{z}{4A^2}\Big)\Big(1 - \frac{z}{4B^2}\Big)}, \qquad e_0 = 1, \tag{313}$$

are homogeneous of degree $-2s$ and q of degree -2, so all objects in (305)–(306) are intrinsic homogeneous functions of (A, B, C) and the bookkeeping scale (114) does not appear. In the canonical sector, ψ_{2r} is fixed at $f = m$ by (307) with κ_r the degree $-2r$ coefficient of the positive stretched seed; for $m = n = 0$, Proposition B.2(c) with $q(0) = (8AB)^{-1}$ reproduces the exceptional starts (154) to every homogeneous order.

At degree -4, put

$$\zeta_{AB} := \frac{(A^{-2} - B^{-2})^2}{128}, \tag{314}$$

so $e_1 = -\varrho_{AB}$ and $e_2 = -\zeta_{AB}$. Then

$$\mathsf{B}_4 = \mathcal{G}_2\, q(\mathsf{H}_0) - \varrho_{AB} \mathcal{N}_1[\mathcal{G}_2] - \zeta_{AB} \mathcal{N}_2[I]. \tag{315}$$

To fix the homogeneous term define

$$\mu_4(r) = \frac{r(r^2-1)(3r^2-7)}{240}, \qquad \nu_4(m) = \frac{m(m+1)(2m+1)(3m^2+3m-1)}{30}, \tag{316}$$

and the triangle forms Δ_i of Section 9. The degree -4 logarithmic seed coefficient is

$$\ell_m^{[4]} = \frac{\nu_4(m)}{64}(A^{-4}+B^{-4}) - \frac{\mu_4(m-n)}{8}(\Delta_1^{-4}+\Delta_2^{-4}) - \frac{\mu_4(m+n)}{8}(\Delta_3^{-4}+\Delta_4^{-4}), \tag{317}$$

$$\kappa_m^{[4]} = \ell_m^{[4]} + \frac{1}{2}(\kappa_m^{[2]})^2, \tag{318}$$

and

$$\psi_4(x) = \kappa_m^{[4]} - \frac{\mathcal{J}(\mathsf{B}_4)d_m}{d_m}. \tag{319}$$

This is the constructive closed formula used in (245); no unevaluated rank recursion remains.

Lemma B.3 (Symmetric triangle-factor pairs)**.** *For any integer $r \geq 1$, let $\Delta_1 = A - B + C$, $\Delta_2 = -A + B + C$, $\Delta_3 = A + B - C$, $\Delta_4 = A + B + C$, and set $C^2 = A^2 + B^2 - 2ABx$. Then*

$$\Delta_1^{-2r} + \Delta_2^{-2r} = \frac{2\sum_{k=0}^{r}\binom{2r}{2k}(C^2)^{r-k}(A-B)^{2k}}{[2AB(1-x)]^{2r}}, \tag{320}$$

$$\Delta_3^{-2r} + \Delta_4^{-2r} = \frac{2\sum_{k=0}^{r}\binom{2r}{2k}(A+B)^{2r-2k}(C^2)^{k}}{[2AB(1+x)]^{2r}}. \tag{321}$$

Every homogeneous coefficient of the positive stretched seed is rational in x, with possible poles only at $x = \pm 1$, and so is the multiplier ψ_{2r}.

Proof. Use $\Delta_1\Delta_2 = 2AB(1-x)$ and $\Delta_3\Delta_4 = 2AB(1+x)$, and expand the sums of even powers of $C \pm (A-B)$ and of $(A+B) \pm C$. The resulting numerators are polynomials in C^2, hence in x. The Taylor coefficients of the logarithm of the centered finite products in the positive seed are finite combinations of these symmetric inverse-power pairs and A^{-2r}, B^{-2r}. Coefficientwise exponentiation preserves rationality, homogeneity and the allowed poles. For $m > 0$, the extremal Wigner mode obeys $d_m'/d_m = (n - mx)/(1-x^2)$; applying finite-order rational differential operators to d_m and dividing by d_m again yields a rational function with the same allowed poles, which applies to ψ_{2r} by (307). □

C Explicit fourth-order Regge coefficients

Put

$$s = 1 - x^2, \qquad J = L(L+1), \qquad \Sigma = M^2 + \sigma^2, \qquad \tau = M\sigma. \tag{322}$$

Then $\mathcal{D}_4 = A_4(x)\partial_x + B_4(x)$ with

$$\begin{aligned} A_4(x) = {} & \frac{\tau(3J-1) + J(3J-1)x}{60} \\ & + \frac{1}{s}\left[\frac{\tau(2J-\Sigma+1)}{20} + \frac{x(4J^2 - J\Sigma + J - 2\tau^2)}{60}\right] \\ & + \frac{1}{s^2}\left[-\frac{\tau(2J-7\Sigma+11)}{5} + \frac{x(3J\Sigma - 7J - 3\Sigma^2 + 10\Sigma - 6\tau^2 - 7)}{15}\right] \\ & + \frac{1}{s^3}\left[-\frac{8\tau(3\Sigma-5)}{15} + \frac{2x(3\Sigma^2 - 10\Sigma + 12\tau^2 + 7)}{15}\right], \end{aligned} \tag{323}$$

and

$$
\begin{aligned}
B_4(x) = {} & \frac{J(20J^2+41J-12)}{1440} \\
& + \frac{1}{s}\left[-\frac{5J^3-15J^2\Sigma+17J^2-6J\Sigma-15J\tau^2+18\tau^2}{360} - \frac{\tau(3J-1)x}{60}\right] \\
& + \frac{1}{s^2}\left[-\frac{5J^2\Sigma-21J^2+26J\Sigma+20J\tau^2-42J-15\Sigma\tau^2+27\tau^2}{120}\right. \\
& \qquad \left. + \frac{\tau(15J^2+15J\Sigma-21J+8\Sigma-5\tau^2-8)x}{180}\right] \\
& + \frac{1}{s^3}\left[\frac{18J\Sigma-42J-5\Sigma^3+10\Sigma^2-60\Sigma\tau^2-5\Sigma+76\tau^2}{90}\right. \\
& \qquad \left. - \frac{\tau(36J-15\Sigma^2+38\Sigma-20\tau^2-47)x}{90}\right] \\
& + \frac{1}{s^4}\left[\frac{\Sigma^3-2\Sigma^2+12\Sigma\tau^2+\Sigma-8\tau^2}{18} - \frac{\tau(3\Sigma^2-4\Sigma+4\tau^2+1)x}{9}\right].
\end{aligned} \tag{324}
$$

For the normalization check, in the extremal channel $M = L$ put

$$
r_\pm = L \pm \sigma, \qquad u_\pm = \frac{1 \pm x}{2}. \tag{325}
$$

For the symmetric central factors define the power sums over the shift set of $\mathfrak{C}_r$,

$$
S_2(r) = \sum_q q^2 = \frac{r(r^2-1)}{12}, \qquad S_4(r) = \sum_q q^4 = \frac{r(r^2-1)(3r^2-7)}{240}. \tag{326}
$$

Since $\Pi_L(h^2) = \mathfrak{C}_{2L+1}(1,h)$, the same sums at $r = 2L+1$ govern the denominator of (27), for integer and half-integer L alike. Writing the exact seed ratio as

$$
\frac{\Phi_L^{L,\sigma}}{d_{L\sigma}^L} = \exp\{\zeta_2 t + \zeta_4 t^2 + O(t^3)\}, \qquad t = h^2, \tag{327}
$$

gives

$$
\zeta_2 = \frac{1}{4}\Big[S_2(2L+1) - \frac{S_2(r_+)}{u_+^2} - \frac{S_2(r_-)}{u_-^2}\Big], \tag{328}
$$

$$
\zeta_4 = \frac{1}{8}\Big[S_4(2L+1) - \frac{S_4(r_+)}{u_+^4} - \frac{S_4(r_-)}{u_-^4}\Big], \tag{329}
$$

so the canonical extremal coefficient is

$$
\frac{f_4}{d_{L\sigma}^L} = \zeta_4 + \frac{\zeta_2^2}{2}. \tag{330}
$$

Substituting $M = L$ in (323)–(324) reduces to (330); the t^2 coefficient of the exact rank recurrence then propagates this normalization to every fixed rank, with the $M = \sigma = 0$ start checked from the two exact initial values.

D Algebra behind the exact $6j$ recurrence

This is the simplification used in Lemma 7.1. In DLMF 34.5.11 choose

$$
j_1 = f, \quad j_2 = b+m, \quad j_3 = b, \qquad l_1 = c, \quad l_2 = a, \quad l_3 = a+n. \tag{331}
$$

Then

$$j_2 - j_3 = m, \quad j_2 + j_3 + 1 = 2B, \qquad l_2 - l_3 = -n, \quad l_2 + l_3 + 1 = 2A, \tag{332}$$

which immediately gives (121). Writing $J_r = j_r(j_r+1)$ and $L_r = l_r(l_r+1)$, direct substitution of

$$a = A - \frac{n+1}{2}, \quad b = B - \frac{m+1}{2}, \quad c = C - \frac{1}{2} \tag{333}$$

into the DLMF diagonal factor yields

$$\begin{aligned}&(J_3 + J_2 - J_1)(L_3 + L_2 - J_1) - 2(J_3L_3 + J_2L_2 - J_1L_1)\\ &\quad = -4AB\left[f(f+1)\left(\frac{A^2+B^2-C^2}{2AB} - \frac{2f(f+1)+1-m^2-n^2}{8AB}\right) - mn\right].\end{aligned} \tag{334}$$

The unphased recurrence has the opposite sign on the right after division by $-4AB$, and $\Phi_{f\pm1}$ carries the opposite $(-1)^f$ phase from Φ_f, so the signs reverse once more, giving (125).

E Expansion of the stretched seed

Under the formal dilation $A = \Lambda\alpha$, $B = \Lambda\beta$, $C = \Lambda\gamma$, the central product has

$$\log \mathfrak{C}_r(\Lambda s) = r\log(\Lambda s) - \frac{r(r^2-1)}{24s^2\Lambda^2} + O(\Lambda^{-4}). \tag{335}$$

For either parity class of m,

$$\log \mathfrak{D}_m(\Lambda\alpha) = 2|Q_m|\log(2\Lambda\alpha) - \frac{m(m+1)(2m+1)}{24\alpha^2\Lambda^2} + O(\Lambda^{-4}). \tag{336}$$

Insert these into (147), include the pure monomial $4AB$ in the half-integer case, and halve the logarithmic coefficient because (147) is Φ_m^2. Restoring $A = \Lambda\alpha$, $B = \Lambda\beta$, $\Delta_i = \Lambda s_i$ gives (234).

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