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COHESIVE 2-TOPOI: FOUNDATIONAL ASPECTS AND NONABELIAN DIFFERENTIAL COHOMOLOGY

YURI XIMENES LECCA AND WILLIAN GOULART GOMES VELASCO

Abstract. We provide a self-contained exposition of cohesive $(2,1)$-topos theory, its internal synthetic geometry, and its realization of intrinsic, modal, and differential nonabelian cohomology. Starting from the $(2,1)$-categorical hierarchy, Grothendieck $(2,1)$-topoi, effective internal groupoids, and $\mathbb{E}_k$-algebra objects over $(2,1)$-operads, we develop a detailed comparison between *internal* and *external* deloopings in $(2,1)$-presheaf and Grothendieck $(2,1)$-topoi. Specifically, we formalize external delooping by transporting the canonical delooping transformation along the group, discrete, and groupoid Yoneda 2-embeddings, proving that it is 2-naturally equivalent to the internal delooping in any Grothendieck $(2,1)$-topos. We then establish the 2-truncated Dunn–Lurie additivity, Baez–Dolan stabilization, and Eckmann–Hilton characterization of k-tuply groupal objects as groupal $\mathbb{E}_k$-algebras stabilizing at $k=3$, and clarify the truncation behavior of iterated deloopings $B^k_{\mathbf{H}}G$. Finally, we review Lawvere–Schreiber cohesive modalities, construct abstract principal k-bundles, the flat de Rham modality $\flat_{\mathrm{dR}}$, universal Maurer–Cartan forms θ_k, connection moduli stacks $B^k_{\mathrm{con}}G$, and sharp-adjusted internal connection moduli $G\mathrm{Conn}^k(X)$. The paper culminates in synthetic differential cohomology, where we discuss extensions of the Maurer–Cartan morphism to non-groupal coefficients via $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}$, and internal classifying structures.

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1. Introduction

The search for an axiomatic and synthetic foundation for continuum physics and differential geometry—originating in Hilbert's Sixth Problem [15] and Lawvere's program of *categorical dynamics* and *axiomatic cohesion* [23, 24, 25]—has found a powerful modern realization in higher topos theory [29, 43]. In Lawvere's insight, a category of spaces $\mathbf{H}$ is distinguished from an arbitrary category of sheaves by a string of adjoint functors relating it to a base category of discrete or homotopy types, capturing the dialectical tension between discrete points and continuous cohesion. When lifted to the setting of $(\infty,1)$-topoi by Schreiber [43, 42, 44], an adjoint quadruple of geometric functors

$$\Pi \dashv \mathrm{LCst} \dashv \Gamma \dashv \mathrm{coDisc} : \mathbf{H} \longrightarrow \infty\mathbf{Grpd}$$

induces an adjoint triple of internal modalities on $\mathbf{H}$—the *shape* modality $\int = \mathrm{LCst} \circ \Pi$, the *flat* modality $\flat = \mathrm{LCst} \circ \Gamma$, and the *sharp* modality $\sharp = \mathrm{coDisc} \circ \Gamma$:

$$\int \dashv \flat \dashv \sharp : \mathbf{H} \longrightarrow \mathbf{H}.$$

Remarkably, this purely categorical structure suffices to internalize the kinematic (and, with linear/infinitesimal modalities, dynamical) structures of classical and quantum gauge field theories [11, 21, 50]:

(i) The *shape modality* $\int$ governs geometric realization, covering spaces, and the classification of principal bundles;
(ii) The *flat modality* $\flat$ and its homotopy kernel, the *flat de Rham modality* $\flat_{\mathrm{dR}}$, govern constant coefficients, local systems, and flat differential forms;
(iii) The interaction $\int \dashv \flat$ produces universal *Maurer–Cartan forms* $\theta : G \to \flat_{\mathrm{dR}} B_{\mathbf{H}} G$ and, via action groupoids and homotopy pullbacks with closed differential forms, the *moduli stack of principal connections* $B_{\mathrm{con}} G$;
(iv) The full triple $\int \dashv \flat \dashv \sharp$ modifies the naive internal hom $[X, B_{\mathrm{con}} G]_{\mathbf{H}}$ to yield the true *internal moduli stack of gauge connections* $G\mathrm{Conn}(X) \in \mathbf{H}$ whose morphisms are gauge transformations rather than smooth concordances.

While the full $(\infty,1)$-categorical machinery is required for arbitrary higher gauge theories (such as B-fields, string structures, and M-theory fluxes [16]), ordinary gauge theories of Yang–Mills, Chern–Simons, and Einstein–Hilbert–Palatini type [51, 33, 26, 34] live naturally within the $(2,1)$-*categorical truncation*, where hom-spaces are 1-groupoids (capturing gauge configurations and gauge transformations between them). Working explicitly in the $(2,1)$-categorical setting of *cohesive 2-topoi* offers one advantage:

- **Concrete 2-categorical control:** Every non-strict $(2,1)$-category (bicategory in which all 2-cells are invertible) can be strictified up to equivalence, allowing diagrams, limits, colimits, and Kan extensions to be manipulated via concrete 2-categorical algebra [47, 22] rather than infinite coherence towers. This could help the reader interested in the subject to have a first overview of the theory.

Furthermore, when generalizing existing theories or results, it is more natural to start to reason in some particular concrete limit. The present work is the initial step of a program with such kind of scope, and the $(2,1)$-categorical truncation is the perfect concrete limit. Indeed, as initially described in [35], a natural strategy concerning the generalization of ordinary gauge theories (which are about Lie algebra valued entities) is to consider their analogues inside some sort of *Algebra-Valued Geometry*, where other flavors of algebras play the role of "entity of infinitesimal symmetries" (for initial applications of this strategy, see [33] and [26]).

In turn, in the same way as ordinary gauge theory admits a cohesive topoi presentation, one should expect to have an incarnation of Algebra-Valued Geometry in some sort of "Algebra-Valued Topoi", having cohesive $(2,1)$-topoi as a particular case. Therefore, the present work not only reviews the theory of cohesive $(2,1)$-topoi, but does so in a way that makes it natural to understand the origin and motivation for those "Algebra-Valued Topoi", whose theory will appear in subsequent works.

More precisely, in this paper, we provide a self-contained review of cohesive $(2,1)$-topoi, the geometric structures existing on them, and how they model intrinsic cohomology and its differential refinement. Beyond serving as a pedagogical survey and dictionary for the $(2,1)$-categorical truncation of [29, 28, 43], our exposition in Section 3 formalizes the 2-functorial comparison between the *internal* and *external* incarnations of delooping via Yoneda 2-embeddings, and establishes their 2-natural equivalence (Theorem 3.1). While this equivalence is mathematically straightforward in the cartesian setting due to objectwise limit computation and left-exact stackification, isolating its precise 2-categorical mechanics is essential for understanding where and why it breaks down in our non-cartesian Algebra-Valued Topoi theory. Furthermore, we clarify the exact $(2,1)$-truncation boundaries of higher deloopings $B^k_{\mathbf{H}}G$, emphasizing that internal quotient deloopings are non-trivial only for $k=1$, whereas higher deloopings ($k \geq 2$) require internal groupoids or the ambient ∞-topos. Finally, in Section 4.3.3, we analyze extensions of the abstract Maurer–Cartan morphism to non-groupal objects (Proposition 4.21, Corollary 4.22, and Problem 4.23), laying the groundwork for differential cohomology with non-groupal coefficients.

What is new and what is not. Most of the material is the $(2,1)$-truncation of known $(\infty,1)$-categorical results [29, 28, 43, 36], and we credit them accordingly. Our own contributions are: (a) the explicit 2-functorial comparison of internal and external delooping (Theorem 3.1), which is elementary but isolates the cartesian axioms that fail in the non-cartesian setting we aim at; (b) the bookkeeping of which deloopings $B^k_{\mathbf{H}}G$ live in $\mathbf{H}$, in its groupoid objects, or only in the ambient ∞-topos; (c) the discussion of extensions of the Maurer–Cartan morphism to non-groupal coefficients (Section 4.3.3), where the adjunction $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}$ itself is due to Schreiber [43]. We also point out a limitation of the truncated setting: the $(2,1)$-truncated shape modality forgets all homotopy above degree 1, so for instance it does not detect the first Chern class of a circle bundle (Remark 4.3).

Organization of the paper. In Section 2, we review the (n,r)-categorical hierarchy, $(2,1)$-presheaf and Grothendieck $(2,1)$-topoi via stackification at coverages, internal groups and effective internal groupoids, symmetric monoidal $(2,1)$-categories, and algebras over the little k-cubes $(2,1)$-operads $\mathbb{E}_k$, culminating in the 2-truncated Dunn–Lurie additivity and stabilization theorems at $k=3$. In Section 3, we study delooping in cartesian $(2,1)$-cosmoi. We construct both the *internal delooping* $\mathbf{B}^{\mathrm{int}}$ and the *external delooping* $\mathbf{B}^{\mathrm{ext}}$ (via transportation along Yoneda 2-embeddings), prove their 2-natural equivalence, characterize k-tuply groupal objects as groupal $\mathbb{E}_k$-algebras via the $(2,1)$-categorical Eckmann–Hilton argument, establish the looping–delooping equivalence $B_{\mathbf{H}} : \mathrm{Grp}(\tau_{\leq 0}\mathbf{H}) \simeq \mathrm{Conn}(*/\mathbf{H})$ in any Grothendieck $(2,1)$-topos, and clarify the truncation behavior of higher deloopings $B^k_{\mathbf{H}}G$. In Section 4, we introduce cohesive modalities ($\int \dashv \flat \dashv \sharp$), formalize principal k-bundles and their topological realization, construct the flat de Rham modality $\flat_{\mathrm{dR}}$, closed forms $\Omega^k_{\mathrm{cl}}(-;A)$, universal Maurer–Cartan forms θ_k, nonabelian and higher connection moduli stacks $B^k_{\mathrm{con}}G$, and the sharp-adjusted internal moduli objects $G\mathrm{Conn}^k(X)$, illustrated with a concrete worked example in $\mathbf{SmoothGrpd}_2$. Finally, in Section 4.3, we develop intrinsic, modal, and differential $(2,1)$-cohomology, investigate extensions of the Maurer–Cartan form to arbitrary pointed objects, and discuss internal classifying objects.

2. (2,1)-Topoi

In this section we recall the fundamentals of $(2,1)$-topos theory. We begin with the (n,r)-categorical hierarchy and the $(2,1)$-categorical context, followed by $(2,1)$-presheaf categories and Grothendieck $(2,1)$-topoi defined via localizations at coverages. Next, we examine internal group and groupoid objects and the precise scope of effectiveness of internal groupoids in a Grothendieck $(2,1)$-topos. Finally, we review symmetric monoidal $(2,1)$-categories and algebraic objects modeled as algebras over $(2,1)$-operads, specifically the little k-cubes operads $\mathbb{E}_k$.

2.1. (2,1)-Context.

For a given integer $k \geq 0$, let $(n,r)\mathbf{Cat}_{k+1}$ denote the $(k+1)$-category of (n,r)-categories, where $0 \leq r \leq n \leq k$ (or more generally $n, r \leq k$ with the convention below) [1, 29]. Recall that an n-category has only trivial (identity) j-morphisms for $j > n$, and an (n,r)-category is an n-category in which all j-morphisms for $j > r$ are invertible.

- If $n \leq r$, the invertibility condition for $j > r$ is vacuous since all j-morphisms with $j > r \geq n$ are already trivial (identities); thus we write simply $n\mathbf{Cat}_{k+1}$ for $(n,n)\mathbf{Cat}_{k+1}$.
- If $r = 0$, $(n,0)\mathbf{Cat}_{k+1}$ coincides with the $(k+1)$-category of n-groupoids, denoted $n\mathbf{Grpd}_{k+1}$.
- If $n = 0$, then $(0,r)\mathbf{Cat}_{k+1}$ is equivalent to $\mathbf{Set}_{k+1}$, i.e., the category of sets trivially regarded as a $(k+1)$-category.

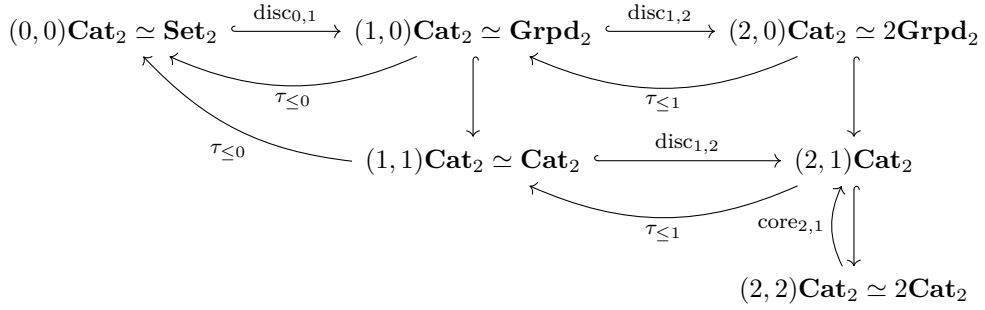
For every k, n, r, we have canonical $(k+1)$-embeddings

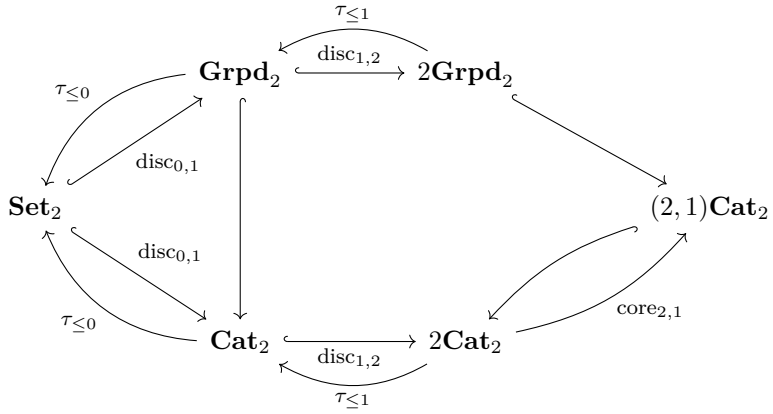
$$\mathrm{disc}^k_{n,n+1} : (n,r)\mathbf{Cat}_{k+1} \hookrightarrow (n+1,r)\mathbf{Cat}_{k+1},$$

which regard an (n,r)-category as a discrete $(n+1,r)$-category (adding only identity $(n+1)$-morphisms), as well as full subcategory inclusions $(n,r)\mathbf{Cat}_{k+1} \hookrightarrow (n,r+1)\mathbf{Cat}_{k+1}$. The discrete embedding $\mathrm{disc}^k_{n-1,n}$ admits a left adjoint

$$\tau^k_{\leq n-1} : (n,r)\mathbf{Cat}_{k+1} \longrightarrow (n-1,r)\mathbf{Cat}_{k+1}$$

given by *truncation* at $(n-1)$-morphisms (identifying $(n-1)$-morphisms that are connected by a zigzag of n-morphisms; when $r < n$ these are invertible, but for $r = n$ the identification is generated by *all* n-morphisms—e.g. $\tau_{\leq 0} : \mathbf{Cat} \to \mathbf{Set}$ is the set π_0 of connected components, not the set of isomorphism classes), whereas the inclusion $(n,r)\mathbf{Cat}_{k+1} \hookrightarrow (n,r+1)\mathbf{Cat}_{k+1}$ admits a right adjoint given by the *core construction* $\mathrm{core}^k_{r+1,r}$, which discards non-invertible $(r+1)$-morphisms [29, 39]. Because an (n,r)-category requires all j-morphisms for $j > r$ to be invertible and possesses only identity morphisms in dimensions $j > n$, the invertibility condition is vacuous for $r \geq n$. Consequently, $(n,r)\mathbf{Cat} \equiv (n,n)\mathbf{Cat}$ for all $r \geq n$, and the distinct classes of (n,r)-categories for $n \leq 2$ form the lower-triangular hierarchy $0 \leq r \leq n \leq 2$ displayed below (both as a lower-triangular grid and, equivalently, as a diamond separating the groupoid and category branches):





(2.1)

Since our primary goal is the synthetic formalization and generalization of ordinary gauge theories, in what follows we focus on the $k = 1$ case—that is, the 2-categorical setting, with special emphasis on the $(2,1)$-categorical context. Higher values of k are central to higher gauge theories (e.g., 2-groups, 3-groups, and string/fivebrane structures [3, 11]), which lie beyond the scope of this paper. Diagram (2.1) summarizes the fundamental adjunctions in the $k = 1$ case.

Remark 2.1. In what follows, if $\mathbf{C}$ is a 1-category or a 1-groupoid, we denote its image under $\mathrm{disc}_{1,2}$ by $\mathbf{C}_2$. Notice that this agrees with the notation used in (2.1): $\mathbf{Set}_2$, $\mathbf{Cat}_2$, and $\mathbf{Grpd}_2$ are precisely the images under $\mathrm{disc}_{1,2}$ (or $\mathrm{disc}_{0,2}$) of the 1-categories $\mathbf{Set}$ of sets, $\mathbf{Cat}$ of small 1-categories, and $\mathbf{Grpd}$ of small 1-groupoids.

Remark 2.2. Above we did not specify which model of "n-category" is assumed. Let us make a few foundational remarks on this choice:

(1) There exists the *strict* setting, the *fully weak* setting, and a spectrum of *semi-strict* intermediate settings [27, 46]. These settings are connected by *strictification functors* that commute with truncations $\tau_{\leq m}$ [22].
(2) In the strict setting, (n,r)-category theory can be formalized inductively via $\mathbf{Cat}$-enriched category theory [20], followed by invertibility conditions and truncation.
(3) For $n=2$, weak 2-categories are naturally presented by Bénabou's *bicategories* [4]. By the Mac Lane–Paré coherence and strictification theorem [32, 22], every bicategory is biequivalent to a strict 2-category; in particular, every weak $(2,1)$-category is equivalent to a strict $(2,1)$-category (i.e., a $\mathbf{Grpd}$-enriched category).[1]
(4) For $n>2$, there is no global strictification theorem: weak n-categories are not in general equivalent to strict n-categories (for $n=3$, they are equivalent only to semi-strict *Gray-categories* [14, 45]).
(5) The fully weak $(\infty,1)$-categorical case is intimately tied to abstract homotopy theory, with combinatorial models (such as quasi-categories, complete Segal spaces, and simplicial categories) extensively developed by Joyal, Rezk, Lurie, and others [18, 37, 29]. Presentations of weak $(2,1)$-category theory can thus be systematically obtained by 2-truncating combinatorial models of $(\infty,1)$-categories.

Remark 2.3. In the strict setting, for every k and every $m \leq n, r \leq k$, there are also forgetful $(k+1)$-functors $u^k_{n,n-m} : (n,r)\mathbf{Cat}_{k+1} \to (n-m,r)\mathbf{Cat}_{k+1}$ that discard i-morphisms for $n-m < i \leq n$. These forgetful functors are generally ill-defined in the weak setting because composition of $(n-1)$-morphisms is associative only up to coherent n-isomorphisms, so one cannot simply discard higher-order cells. When $n=m$, the assignment $\mathbf{C} \mapsto \mathrm{Ob}(\mathbf{C})$ of the underlying set of objects is defined on the underlying *1-category* of $(n,r)\mathbf{Cat}_{k+1}$ (strict functors), but it is *not* a $(k+1)$-functor to $\mathbf{Set}_{k+1}$ for $k \geq 1$: a 2-cell $\alpha : F \Rightarrow G$ between functors with $\mathrm{Ob}(F) \neq \mathrm{Ob}(G)$ has no image in the discrete 2-category $\mathbf{Set}_2$. On the underlying 1-category there is a natural quotient map $\mathrm{Ob}(\mathbf{C}) \twoheadrightarrow \mathrm{Iso}(\mathbf{C})$ onto isomorphism classes; note that $\mathrm{Iso}(\mathbf{C})$ agrees with $\tau_{\leq 0}\mathbf{C}$ only when $\mathbf{C}$ is a groupoid (for a general category, $\tau_{\leq 0}\mathbf{C} = \pi_0(\mathbf{C})$ is the set of connected components).

2.2. **(2,1)-Topos.** Recall that if $\mathbf{C}$ is a small 1-category and $\mathbf{A}$ is a $(2,1)$-category, we can form the $(2,1)$-category $(2,1)\mathrm{PShv}_{\mathbf{A}}(\mathbf{C})$ of 2-functors (pseudofunctors) $F : \mathbf{C}_2^{\mathrm{op}} \to \mathbf{A}$, pseudonatural transformations, and modifications [47]. Varying $\mathbf{C}$ contravariantly via precomposition f^* yields a 2-functor

$$(2,1)\mathrm{PShv}_{\mathbf{A}}(-) : \mathbf{Cat}_2^{\mathrm{op}} \longrightarrow (2,1)\mathbf{Cat}_2,$$

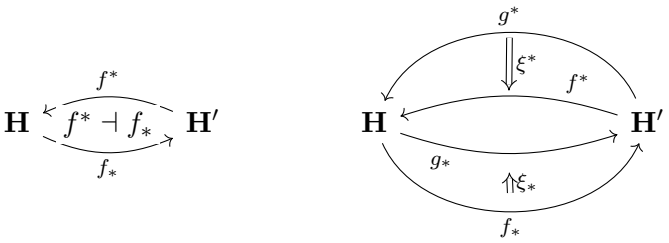
which factors through $(n,r)\mathbf{Cat}_2 \subset 2\mathbf{Cat}_2$ whenever $\mathbf{A}$ is an (n,r)-category for $0 \leq r \leq n \leq 2$. When $\mathbf{A}$ is complete and cocomplete, precomposition $f^* : (2,1)\mathrm{PShv}_{\mathbf{A}}(\mathbf{D}) \to (2,1)\mathrm{PShv}_{\mathbf{A}}(\mathbf{C})$ along $f : \mathbf{C} \to \mathbf{D}$ admits both a left 2-adjoint $f_! = \mathrm{Lan}_{f^{\mathrm{op}}}$ (left Kan extension along f^{op}) and a right 2-adjoint $f_* = \mathrm{Ran}_{f^{\mathrm{op}}}$ (right Kan extension along f^{op}); in particular, $f \mapsto f_!$ defines a covariant 2-functor $(2,1)\mathrm{PShv}^{\mathrm{cov}}_{\mathbf{A}}(-) : \mathbf{Cat}_2 \to (2,1)\mathbf{Cat}_2$. We call $(2,1)\mathrm{PShv}_{\mathbf{A}}(\mathbf{C})$ the $(2,1)$-category of $\mathbf{A}$-*valued* $(2,1)$-*presheaves* (or *prestacks*) on $\mathbf{C}$. In this work we focus primarily on the canonical case $\mathbf{A} = \mathbf{Grpd}_2$.

We say that a $(2,1)$-category $\mathbf{H}$ is a $(2,1)$-*presheaf* $(2,1)$-*category* if it is 2-equivalent to $(2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ for some small 1-category $\mathbf{C}$.[2] A *geometric morphism* $f : \mathbf{H} \to \mathbf{H}'$ between

[1] This, of course, does not mean that one can ignore weak 2-functors (pseudofunctors) and weak 2-limits. Rather, any construction or proof that is invariant under 2-equivalences (pseudonatural equivalences) may be performed on strict $(2,1)$-categories provided 2-limits and 2-colimits are understood in the weak (bilimit/bicolimit or pseudolimit) sense [47, 22].

[2] One could more generally take $\mathbf{C}$ to be a small $(2,1)$-category, as we will do when discussing the 2-Yoneda embedding in Section 3.2, but for classical and natural gauge theories a 1-categorical site of test spaces is sufficient.

two $(2,1)$-categories is an adjoint pair of 2-functors $(f^* \dashv f_*)$, with direct image $f_* : \mathbf{H} \to \mathbf{H}'$ and inverse image $f^* : \mathbf{H}' \to \mathbf{H}$, such that f^* is left exact, i.e., preserves finite $(2,1)$-limits (first diagram below). A *geometric transformation* $\xi : f \Rightarrow g$ between geometric morphisms $f, g : \mathbf{H} \to \mathbf{H}'$ is a mate pair (ξ_*, ξ^*) of 2-natural transformations $\xi_* : f_* \Rightarrow g_*$ and $\xi^* : g^* \Rightarrow f^*$ (where our convention for 2-cells in the 2-category of topoi follows the direction of $\xi_* : f_* \Rightarrow g_*$, with mate $\xi^* : g^* \Rightarrow f^*$; second diagram below). This defines the 2-category $(2,1)\mathbf{PShvTopos}_2$ of $(2,1)$-presheaf $(2,1)$-topoi, geometric morphisms, and geometric transformations [48, 29].



The 2-category $(2,1)\mathbf{PShvTopos}_2$ is a full sub-2-category of the 2-category $(2,1)\mathbf{PTopos}_2$ of (infinitary) $(2,1)$-*pretopoi* with geometric morphisms as 1-morphisms, i.e., $(2,1)$-categories that are 2-exact (effective internal groupoids and universal colimits) and infinitary 2-extensive (possessing disjoint, universal small coproducts) [48, 38]. Between $(2,1)\mathbf{PShvTopos}_2$ and $(2,1)\mathbf{PTopos}_2$ lies the full sub-2-category $(2,1)\mathbf{ShvTopos}_2$ of *Grothendieck* $(2,1)$-*topoi*, which are $(2,1)$-categories 2-equivalent to full sub-2-categories $\mathbf{H}_0 \subset \mathbf{H}$ of a $(2,1)$-presheaf category $\mathbf{H} \simeq (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ such that:

(1) $\mathbf{H}_0$ is *reflective*: the full inclusion 2-functor $\imath : \mathbf{H}_0 \hookrightarrow \mathbf{H}$ admits a left 2-adjoint $L_0 : \mathbf{H} \to \mathbf{H}_0$ (the *stackification* or *sheafification* functor);
(2) $\mathbf{H}_0$ is *left exact*: the left adjoint L_0 preserves finite $(2,1)$-limits;
(3) $\mathbf{H}_0$ is *accessible*: the composite $\imath \circ L_0$ is an accessible 2-functor (equivalently, $\mathbf{H}_0$ is presentable).[3]

Remark 2.4. The 2-category $(2,1)\mathbf{PTopos}_2$ (and its full sub-2-categories $(2,1)\mathbf{PShvTopos}_2$, $(2,1)\mathbf{ShvTopos}_2$) is not a sub-2-category of $(2,1)\mathbf{Cat}_2$ with respect to arbitrary 2-functors. However, there are two dual identity-on-objects forgetful 2-functors:

$$U_{\mathrm{topos}} = (-)_* : (2,1)\mathbf{PTopos}_2 \to (2,1)\mathbf{Cat}_2, \quad U^*_{\mathrm{topos}} = (-)^* : (2,1)\mathbf{PTopos}_2^{\mathrm{op}} \to (2,1)\mathbf{Cat}_2,$$

where U_{topos} sends a geometric morphism $f = (f^*, f_*)$ to its direct image 2-functor $f_* : \mathbf{H} \to \mathbf{H}'$ (which preserves all $(2,1)$-limits), while U^*_{topos} sends f to its inverse image 2-functor $f^* : \mathbf{H}' \to \mathbf{H}$ (which preserves finite $(2,1)$-limits and all $(2,1)$-colimits).

By the 2-categorical Giraud Theorem (due to Giraud [13] and Street [48, 49], and corresponding to the 2-truncation of the Rezk–Lurie $(\infty,1)$-Giraud theorem [38, 29]), the 2-category $(2,1)\mathbf{ShvTopos}_2$ is 2-equivalent to the 2-category $\mathbf{Stk}_2$ of *stack* $(2,1)$-*categories*. Explicitly, if $\mathbf{C}$ is a small 1-category endowed with a Grothendieck coverage (or pretopology) J, making $(\mathbf{C}, J)$ a site,[4] the $(2,1)$-category of stacks $\mathbf{Stk}(\mathbf{C}, J)$ is obtained as the left-exact reflective localization of $(2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ at the class W_J of *local* J-*equivalences*:

$$\mathbf{Stk}(\mathbf{C}, J) \simeq (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})[W_J^{-1}]. \tag{2.2}$$

Let $\mathbf{Site}_2$ denote the 2-category of small sites and cover-preserving functors. It comes equipped with a faithful forgetful 2-functor $U_{\mathrm{site}} : \mathbf{Site}_2 \to \mathbf{Cat}_2$ sending $(\mathbf{C}, J) \mapsto \mathbf{C}$. For a general coverage J, the underlying category $\mathbf{C}$ need not have all pullbacks. However, constructing the localization at W_J via Čech descent requires pullbacks along covering families. Let $\mathbf{Cat}_{2,\mathrm{pb}} \subset \mathbf{Cat}_2$ be the sub-2-category of small 1-categories with pullbacks and pullback-preserving functors,

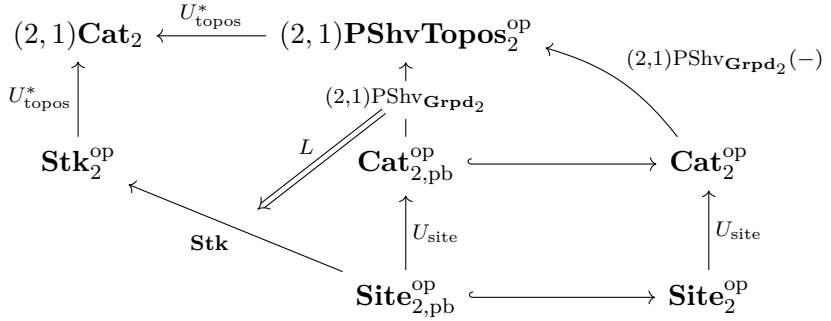
[3] Accessibility is part of Lurie's definition of n-topoi [29, §6.4]; it is what guarantees that $\mathbf{H}_0$ is presented by a site, cf. Street's 2-Giraud theorem [48, 49].

[4] More generally, $(\mathbf{C}, J)$ can be taken to be a $(2,1)$-site [48].

and let $\mathbf{Site}_{2,\mathrm{pb}} \subset \mathbf{Site}_2$ be the pullback sub-2-category of sites with pullbacks and cover- and pullback-preserving functors:

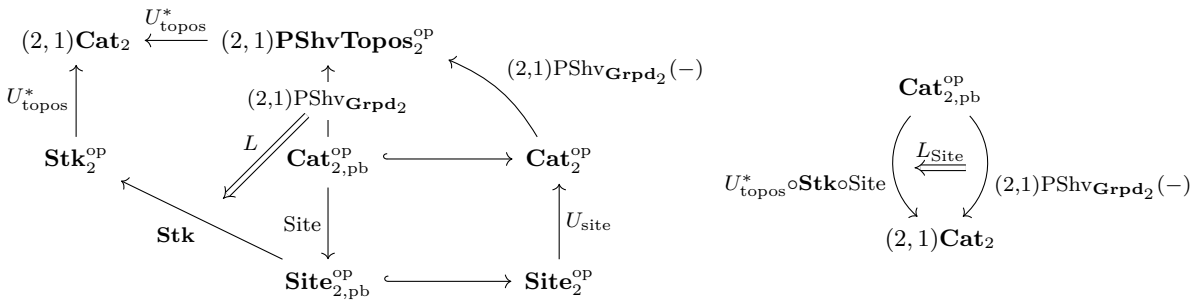
$$\begin{array}{ccc} \mathbf{Cat}_{2,\mathrm{pb}} & \hookrightarrow & \mathbf{Cat}_2 \\ {\scriptstyle U_{\mathrm{site}}}\uparrow & & \uparrow{\scriptstyle U_{\mathrm{site}}} \\ \mathbf{Site}_{2,\mathrm{pb}} & \hookrightarrow & \mathbf{Site}_2 \end{array}$$

Restricted to sites whose underlying categories have finite limits and to cover-preserving *left exact* functors (continuous morphisms of sites in the sense of SGA 4), the assignment $(\mathbf{C}, J) \mapsto \mathbf{Stk}(\mathbf{C}, J)$ is 2-functorial: for any such $f : (\mathbf{C}, J) \to (\mathbf{D}, K)$, precomposition $f^* : (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{D}) \to (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ sends K-stacks to J-stacks, and the restriction $f^* : \mathbf{Stk}(\mathbf{D}, K) \to \mathbf{Stk}(\mathbf{C}, J)$ is the *direct image* of a geometric morphism $\mathbf{Stk}(\mathbf{D}, K) \to \mathbf{Stk}(\mathbf{C}, J)$ whose *inverse image* is $f_{!s} := L_K \circ f_! \circ \imath_J$; the latter is left exact because $f_!$ is (left Kan extension along a left exact functor is computed by filtered colimits, see Section 3.2), and it satisfies $f_{!s} \circ L_J \simeq L_K \circ f_!$. (Precomposition f^* does *not* in general preserve local equivalences; that requires f to be cover-reflecting/cocontinuous. Mere preservation of pullbacks is also not enough for $f_{!s}$ to be left exact.) Accordingly, below $\mathbf{Site}_{2,\mathrm{pb}}$ should be read as the 2-category of finitely complete sites and continuous left exact functors, and $\mathbf{Stk}$ is *covariant* on it via the inverse images $f_{!s}$. Thus, passing to inverse image 2-functors, we obtain a 2-functor $\mathbf{C} \mapsto \mathbf{Stk}(\mathbf{C}, J)$ (with $f \mapsto f_{!s}$) into $(2,1)\mathbf{Cat}_2$, and the stackification localization extends to a 2-natural transformation L from the covariant presheaf 2-functor $f \mapsto f_!$ to it, since $f_{!s} \circ L_J \simeq L_K \circ f_!$. In the diagram below (kept in the notation of the original draft), the arrows labelled $(-)^{\mathrm{op}}$ and U^*_{topos} must be read with this covariant variance:



(2.3)

Furthermore, any section 2-functor $\mathrm{Site} : \mathbf{Cat}_{2,\mathrm{pb}} \to \mathbf{Site}_{2,\mathrm{pb}}$ endowing categories with a canonical coverage can be characterized by universal properties of the induced composite localization transformation $L_{\mathrm{Site}} : (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(-) \Rightarrow U^*_{\mathrm{topos}} \circ \mathbf{Stk} \circ \mathrm{Site}$:



For example:

(1) The *trivial coverage* $\mathrm{Site}_0(\mathbf{C})$, whose covering families consist solely of identity morphisms, is the universal coverage for which L_{Site_0} is a 2-natural equivalence. Consequently, every $(2,1)$-presheaf category $(2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ is canonically a Grothendieck $(2,1)$-topos equipped with the trivial coverage.

(2) A *subcanonical coverage* $\mathrm{Site}_{\mathrm{sub}}(\mathbf{C})$ is one for which the localization L_{Site} restricts to an equivalence on all representable $(2,1)$-presheaves $y_{\mathbf{C}}(X)$. Explicitly, if $\mathbf{C}$ has coproducts that are universal (preserved by pullbacks), a coverage J on $\mathbf{C}$ is subcanonical if and

only if for every covering family $\{U_i \to X\}_{i\in I}$ in J, the induced Čech diagram

$$\coprod_{i,j} U_i \times_X U_j \rightrightarrows \coprod_i U_i \longrightarrow X$$

is a colimit (coequalizer) diagram in $\mathbf{C}$.

Remark 2.5. Recall that in a $(2,1)$-category $\mathbf{H}$ with $(2,1)$-pullbacks, a 1-morphism $f : U \to X$ is an *effective epimorphism* if the diagram

$$U \times_X U \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} U \xrightarrow{f} X$$

(together with the 2-cell witnessing the homotopy pullback) exhibits X as the $(2,1)$-colimit (homotopy coequalizer) of the kernel pair [29]. If $\mathbf{C}$ is a 1-category with pullbacks and universal coproducts (for example, an extensive or cartesian closed 1-category with coproducts), pullbacks distribute over coproducts; hence, setting $U = \coprod_i U_i$, a coverage J is subcanonical if and only if the summed covering morphisms $U \to X$ are effective epimorphisms.

2.3. **Internal Groupoids.** Let $\mathbf{Cat}_{2,c} \subset \mathbf{Cat}_2$ be the sub-2-category of cartesian 1-categories (categories with finite products, i.e., binary products and a terminal object $*$) and cartesian (finite-product-preserving) functors. Let $\mathbf{Cat}_{2,cp} = \mathbf{Cat}_{2,c} \cap \mathbf{Cat}_{2,\mathrm{pb}}$ be the sub-2-category of categories with all finite limits (pullbacks and terminal object) and left-exact functors.

Recall that in every $\mathbf{C} \in \mathbf{Cat}_{2,c}$ there is an internal notion of *group object*. A group object is an object $G \in \mathbf{C}$ equipped with multiplication $m : G \times G \to G$, unit $e : * \to G$, and inversion $\mathrm{inv} : G \to G$ satisfying the standard associative, unital, and inverse commutative diagrams [30]. Homomorphisms of group objects form a category $\mathrm{Grp}(\mathbf{C})$ equipped with a faithful forgetful functor to $\mathbf{C}$ (written with a hooked arrow by abuse of notation). Denoting by $\mathrm{Grp}_2(\mathbf{C}) = \mathrm{disc}_{1,2}(\mathrm{Grp}(\mathbf{C}))$ the corresponding discrete $(2,1)$-category, and noting that cartesian functors preserve group objects, we obtain a 2-functor

$$\mathrm{Grp}_2 : \mathbf{Cat}_{2,c} \longrightarrow (2,1)\mathbf{Cat}_2.$$

Similarly, in every $\mathbf{C} \in \mathbf{Cat}_{2,cp}$ there is an internal notion of *groupoid object* (or *internal groupoid*) $\mathbf{G} = (G_1 \rightrightarrows G_0)$, consisting of an object of objects G_0, an object of morphisms G_1, source and target morphisms $s, t : G_1 \to G_0$, composition $\circ : G_1 \times_{s,G_0,t} G_1 \to G_1$, identity $\mathrm{id} : G_0 \to G_1$, and inversion $\mathrm{inv} : G_1 \to G_1$ satisfying associativity, unitality, and invertibility equations [17]. Together with *internal functors* and *internal natural transformations*, internal groupoids in $\mathbf{C}$ assemble into a $(2,1)$-category $\mathbf{Grpd}_2(\mathbf{C})$. Every object $X \in \mathbf{C}$ defines a discrete internal groupoid $\mathbf{E}_X = (X \rightrightarrows X)$ with $(\mathbf{E}_X)_0 = (\mathbf{E}_X)_1 = X$ and $s = t = \mathrm{id}_X$, while every group object $G \in \mathrm{Grp}(\mathbf{C})$ defines a one-object internal groupoid $\mathbf{B}_{\mathbf{C}}(G) = (G \rightrightarrows *)$ with object of objects $*$. Thus, for each $\mathbf{C} \in \mathbf{Cat}_{2,cp}$, we have canonical inclusions of $(2,1)$-categories (where $\mathrm{Grp}_2(\mathbf{C})$ embeds both via discrete groupoids on the underlying object and via one-object delooping groupoids; below we display the underlying-object inclusion):

$$\mathrm{Grp}_2(\mathbf{C}) \hookrightarrow \mathrm{disc}_{1,2}(\mathbf{C}) \hookrightarrow \mathbf{Grpd}_2(\mathbf{C}).$$

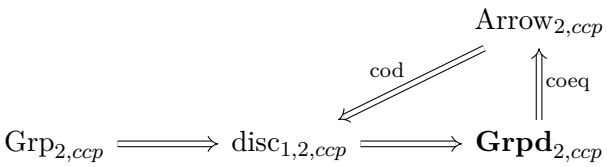
Since left-exact functors preserve pullbacks and hence internal groupoids, $\mathbf{Grpd}_2 : \mathbf{Cat}_{2,cp} \to (2,1)\mathbf{Cat}_2$ is a 2-functor, and we have a sequence of 2-natural transformations:

$$\mathrm{Grp}_2 \Longrightarrow \mathrm{disc}_{1,2} \Longrightarrow \mathbf{Grpd}_2.$$

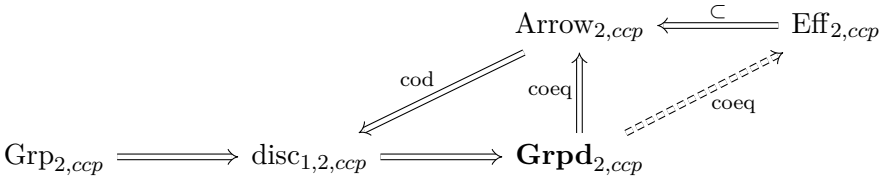
Now let $\mathbf{Cat}_{2,ccp} \subset \mathbf{Cat}_{2,cp}$ denote the sub-2-category of 1-categories that additionally possess coequalizers, with functors preserving finite limits and coequalizers. Fixing $\mathbf{C} \in \mathbf{Cat}_{2,ccp}$, to every internal groupoid $\mathbf{G} \in \mathbf{Grpd}_2(\mathbf{C})$ we associate its quotient morphism $\mathrm{coeq}_{\mathbf{C}}(\mathbf{G}) : G_0 \twoheadrightarrow G_0/\mathbf{G}$, obtained by taking the coequalizer of its source and target morphisms:

$$G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0 \xrightarrow{\mathrm{coeq}_{\mathbf{C}}(\mathbf{G})} \mathrm{cod}(\mathrm{coeq}_{\mathbf{C}}(\mathbf{G})) \simeq G_0/\mathbf{G}.$$

This construction extends 2-functorially to internal functors and internal natural transformations (which collapse to identities on the 1-categorical quotient in $\mathrm{disc}_{1,2}(\mathbf{C})$), defining a 2-functor $\mathrm{coeq}_{\mathbf{C}} : \mathbf{Grpd}_2(\mathbf{C}) \to \mathrm{Arrow}_{2,ccp}(\mathbf{C})$, where $\mathrm{Arrow}_{2,ccp}(\mathbf{C}) = \mathrm{Arrow}_2(\mathbf{C})$ is the arrow category viewed as a $(2,1)$-category (here and below, the subscript *ccp* on $\mathrm{Grp}_{2,ccp}$, $\mathrm{disc}_{1,2,ccp}$, $\mathbf{Grpd}_{2,ccp}$, $\mathrm{Arrow}_{2,ccp}$, and $\mathrm{Eff}_{2,ccp}$ indicates the restriction of the corresponding 2-functors to the domain 2-category $\mathbf{Cat}_{2,ccp}$). Varying $\mathbf{C} \in \mathbf{Cat}_{2,ccp}$ yields a 2-natural transformation coeq fitting into the (non-commutative) diagram below, where $\mathrm{cod} : \mathrm{Arrow}_{2,ccp} \Rightarrow \mathrm{disc}_{1,2,ccp}$ is the codomain projection transformation:



Because every coequalizer morphism is a regular (and hence effective) epimorphism, coeq factors objectwise through the full sub-2-category $\mathrm{Eff}_{2,ccp}(\mathbf{C}) \subset \mathrm{Arrow}_{2,ccp}(\mathbf{C})$ of effective epimorphisms:

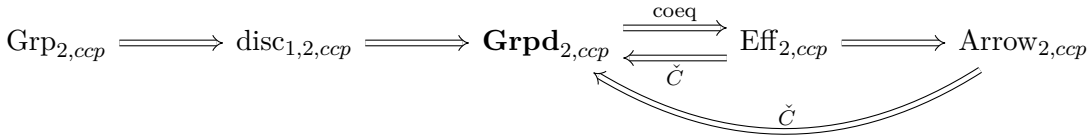


For an internal groupoid $\mathbf{G} \in \mathbf{Grpd}_{2,ccp}(\mathbf{C})$, we call the codomain $\mathrm{cod}(\mathrm{coeq}(\mathbf{G})) \simeq G_0/\mathbf{G} \in \mathbf{C}$ its *quotient object* (or *representing object*). Of fundamental importance are those internal groupoids that can be reconstructed from their quotient morphism—the *effective internal groupoids*:

Given $\mathbf{C} \in \mathbf{Cat}_{2,cp}$ and any morphism $f : U \to X$ in $\mathbf{C}$, we associate to f its *Čech groupoid* $\check{C}_{\mathbf{C}}(f) \in \mathbf{Grpd}_2(\mathbf{C})$, defined by $\check{C}_{\mathbf{C}}(f)_0 = U$, $\check{C}_{\mathbf{C}}(f)_1 = U \times_X U = \mathrm{pb}(f,f)$, with source, target, and composition induced by the universal property of the pullback:

$$\begin{array}{ccc} U \times_X U & \xrightarrow{\hat{f}_1} & U \\ {\scriptstyle \hat{f}_2}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[f]{} & X \end{array} \qquad\qquad U \times_X U \underset{\hat{f}_2}{\overset{\hat{f}_1}{\rightrightarrows}} U.$$

This assignment extends to a 2-functor $\check{C}_{\mathbf{C}} : \mathrm{Arrow}_{2,cp}(\mathbf{C}) \to \mathbf{Grpd}_2(\mathbf{C})$ and a 2-natural transformation $\check{C} : \mathrm{Arrow}_{2,cp} \Rightarrow \mathbf{Grpd}_2$. By Remark 2.5, a morphism $f : U \to X$ is an effective epimorphism if and only if $\mathrm{coeq}(\check{C}(f)) \cong f$. Consequently, the composite $\mathrm{coeq} \circ \check{C}$ is naturally isomorphic to the identity when restricted to $\mathrm{Eff}_{2,ccp}$:



Conversely, the opposite composition $\check{C} \circ \mathrm{coeq}$ takes an internal groupoid $\mathbf{G}$, forms its quotient morphism $G_0 \twoheadrightarrow \mathrm{cod}(\mathrm{coeq}(\mathbf{G}))$, and takes the Čech groupoid of that morphism. An internal groupoid $\mathbf{G}$ is called *effective* if the canonical comparison functor $\mathbf{G} \to \check{C}(\mathrm{coeq}(\mathbf{G}))$ is an equivalence in $\mathbf{Grpd}_2(\mathbf{C})$.

Remark 2.6 (Groupoid objects in a $(2,1)$-category and the quotient object). Two points must be made precise before passing to $(2,1)$-categories. First, an internal groupoid in a $(2,1)$-category $\mathbf{H}$ must be understood homotopically: as a simplicial object $G_\bullet : \Delta^{\mathrm{op}} \to \mathbf{H}$ satisfying the groupoidal Segal conditions with respect to $(2,1)$-pullbacks [29, §6.1.2], not as a strict diagram $(G_1 \rightrightarrows G_0)$ subject to equations. Correspondingly, the $(2,1)$-category $\mathbf{Grpd}_2(\mathbf{H})$ has to be localized at the

weakly invertible (fully faithful and essentially surjective in the internal sense) internal functors; for a 1-category with a pretopology this localization is computed by anafunctors [40]. Second, the *quotient object* of a groupoid object is the $(2,1)$-colimit of the whole simplicial diagram $G_\bullet$ (its geometric realization or codescent object), *not* the $(2,1)$-coequalizer of the pair s,t. These differ already for $\mathbf{H} = \mathbf{Grpd}_2$ and $\mathbf{B}(G) = (G \rightrightarrows *)$ with G a discrete group: the pseudo-coequalizer of the two maps $G \to *$ freely adjoins an isomorphism for each element of G, giving the delooping of the *free* group on the underlying set of G, whereas the realization of the nerve gives BG. In what follows, $\mathrm{coeq}(\mathbf{G})$ always denotes the realization $\varinjlim_{\Delta^{\mathrm{op}}} G_\bullet$, and "coequalizer" should be read in this sense.

The discussion above extends from 1-categories $\mathbf{Cat}_{2,ccp}$ to $(2,1)$-categories $(2,1)\mathbf{Cat}_{2,ccp}$, replacing 1-limits by $(2,1)$-limits (homotopy pullbacks / iso-comma objects, even when $\mathbf{H}$ is strictified to a $\mathbf{Grpd}$-enriched category as in Remark 2.2(3)) and 1-coequalizers by $(2,1)$-colimits (codescent objects / homotopy coequalizers of groupoid nerves, which preserve non-trivial 2-cells in the $(2,1)$-categorical arrow category $\mathrm{Arrow}_{2,ccp}(\mathbf{H})$ rather than collapsing them to identities as in a 1-category $\mathbf{C}$) [48, 38, 29]. In any such $(2,1)$-category $\mathbf{H} \in (2,1)\mathbf{Cat}_{2,ccp}$, $\check{C}$ and coeq restrict to an equivalence of $(2,1)$-categories between effective epimorphisms $\mathrm{Eff}_{2,ccp}(\mathbf{H})$ and the full sub-2-category $\mathbf{Grpd}^{\mathrm{eff}}_{2,ccp}(\mathbf{H}) \subset \mathbf{Grpd}_{2,ccp}(\mathbf{H})$ of effective internal groupoids (whereas for a 1-category $\mathbf{C}$, $\mathrm{Eff}_{2,ccp}(\mathbf{C})$ is 0-truncated and thus equivalent to the 1-truncation $\tau_{\leq 1}\mathbf{Grpd}^{\mathrm{eff}}_{2,ccp}(\mathbf{C})$).

In a $(2,1)$-category $\mathbf{H}$, mapping spaces $\mathbf{H}(U,X)$ are 1-groupoids (i.e., 1-truncated spaces); an object $Y \in \mathbf{H}$ is called *0-truncated* (or *discrete*) if $\mathbf{H}(U,Y)$ is a 0-groupoid (a set) for all $U \in \mathbf{H}$, forming the full reflective 1-subcategory $\mathrm{Disc}(\mathbf{H}) = \tau_{\leq 0}\mathbf{H} \subset \mathbf{H}$. Because the homotopy pullback $U \times_X U$ of any morphism $f : U \to X$ in $\mathbf{H}$ with 0-truncated domain U has 0-truncated fibers (in particular, for any pointed object $* \to X$, the loop object $\Omega X = * \times_X *$ is automatically 0-truncated), the Čech groupoid $\check{C}(f)$ of any morphism $f : U \to X$ in $\mathbf{H}$ has 0-truncated diagonal fibers $(s,t) : \check{C}(f)_1 \to U \times U$. By the n-topos version of the Giraud–Rezk–Lurie theorem for $n = 2$ [29, §6.4] (see also [48, 38]), every Grothendieck $(2,1)$-topos $\mathbf{H}$ satisfies:

(1) $\mathbf{H}$ is locally presentable as a $(2,1)$-category, and therefore possesses all small $(2,1)$-limits and $(2,1)$-colimits; hence $(2,1)\mathbf{ShvTopos}_2 \subset (2,1)\mathbf{Cat}_{2,ccp}$ on underlying categories;
(2) Every internal groupoid $\mathbf{G} = (G_1 \rightrightarrows G_0) \in \mathbf{Grpd}_2(\mathbf{H})$ with 0-truncated morphism fibers $(s,t) : G_1 \to G_0 \times G_0$—and in particular every internal groupoid in $\mathrm{Disc}(\mathbf{H}) = \tau_{\leq 0}\mathbf{H}$—is effective in $\mathbf{H}$:

$$\mathbf{Grpd}_2(\tau_{\leq 0}\mathbf{H}) \subset \mathbf{Grpd}^{\mathrm{eff}}_2(\mathbf{H}) \simeq \mathrm{Eff}_{2,ccp}(\mathbf{H}),$$

while an arbitrary internal groupoid $\mathbf{G} \in \mathbf{Grpd}_2(\mathbf{H})$ with 1-truncated morphism fibers is effective as a 2-stack in the ambient $(3,1)$-topos $2\mathbf{Stk}(\mathbf{C},J) \simeq \tau_{\leq 2}\mathbf{H}_\infty$ (which is equivalent to the *localization* of $\mathbf{Grpd}_2(\mathbf{H})$ at weak equivalences, cf. Remark 2.6, not to $\mathbf{Grpd}_2(\mathbf{H})$ itself) (where $\mathbf{H}_\infty$ is the ambient $(\infty,1)$-topos of which $\mathbf{H} \simeq \tau_{\leq 1}\mathbf{H}_\infty$ is the $(2,1)$-truncation), with $\mathrm{cod}(\mathrm{coeq}_{\mathbf{H}}(\mathbf{G})) \in \mathbf{H}$ computing its 1-truncation.

2.4. **Monoidal (2,1)-Categories.** Recall that a *monoidal* $(2,1)$*-category* is a $(2,1)$-category $\mathbf{H}$ equipped with a tensor 2-bifunctor $\otimes : \mathbf{H} \times \mathbf{H} \to \mathbf{H}$, a unit object $1 : *_2 \to \mathbf{H}$, and coherent associator and unitor 2-natural equivalences satisfying the pentagon and triangle coherence conditions up to coherent 2-isomorphisms [8, 41]. Equivalently, this is the 2-truncation of a monoidal $(\infty,1)$-category in the sense of Lurie [28]. A monoidal $(2,1)$-category is *braided* or *symmetric* if it is further equipped with a coherent braiding $X \otimes Y \xrightarrow{\sim} Y \otimes X$ (satisfying the hexagon identities, and for symmetric, being involutive up to coherent 2-isomorphism).

Let $(2,1)\mathbf{Mon}_2 \subset (2,1)\mathbf{Cat}_2$ denote the 2-category of monoidal $(2,1)$-categories, strong monoidal 2-functors, and monoidal 2-natural transformations, and let $(2,1)\mathbf{SymMon}_2 \subset (2,1)\mathbf{Mon}_2$ denote the sub-2-category of symmetric monoidal $(2,1)$-categories.

Example 2.7. For our purposes, the fundamental examples of symmetric monoidal $(2,1)$-categories are the *cartesian monoidal* $(2,1)$*-categories*, in which the tensor product $\otimes = \times$ is given by the categorical $(2,1)$-product and the unit is the terminal object $*$. In particular, every

Grothendieck $(2,1)$-topos $\mathbf{H}$ carries a canonical cartesian symmetric monoidal structure. More precisely, choosing products gives a canonical inclusion 2-functor $(2,1)\mathbf{Cat}_{2,c} \hookrightarrow (2,1)\mathbf{SymMon}_2$.

In any monoidal $(2,1)$-category $(\mathbf{H}, \otimes, 1)$, we have the $(2,1)$-category $\mathbf{Mon}_2(\mathbf{H})$ of *monoid objects* (or *pseudomonoids* [8]) in $\mathbf{H}$. An object of $\mathbf{Mon}_2(\mathbf{H})$ is an object $X \in \mathbf{H}$ equipped with multiplication $\mu : X \otimes X \to X$, unit $e : 1 \to X$, and associativity and unitality 2-isomorphisms satisfying the Mac Lane pentagon and triangle equations. This assignment defines a 2-functor $\mathbf{Mon}_2 : (2,1)\mathbf{Mon}_2 \to (2,1)\mathbf{Cat}_2$.

If $(\mathbf{H}, \otimes, 1)$ is symmetric monoidal, we further obtain $(2,1)$-categories $\mathrm{BrMon}_2(\mathbf{H})$ and $\mathrm{ComMon}_2(\mathbf{H})$ of *braided monoid objects* and *commutative (or symmetric) monoid objects* in $\mathbf{H}$, linked by forgetful functors:

$$\mathrm{ComMon}_2(\mathbf{H}) \hookrightarrow \mathrm{BrMon}_2(\mathbf{H}) \xrightarrow{\text{forget}} \mathbf{Mon}_2(\mathbf{H}) \xrightarrow{\text{forget}} \mathbf{H}.$$

Note that in a $(2,1)$-category, being commutative is a property of a braided monoid (the braiding 2-cell $\gamma_{X,X} : \mu \circ \tau \Rightarrow \mu$ satisfies $\gamma_{X,X}^2 = \mathrm{id}$), whereas being braided on a monoid or being a monoid on an object involves extra data (1-morphisms and 2-cells), though we write hooked arrows by abuse of notation to indicate faithful/subcategory-like forgetful functors.

When $\mathbf{H}$ is a cartesian monoidal $(2,1)$-category, a monoid object $G \in \mathbf{Mon}_2(\mathbf{H})$ is a *group object* (or *2-group / gr-object*) if the shear morphism $(p_1, \mu) : G \times G \to G \times G$ is an equivalence in $\mathbf{H}$ (equivalently, if $\tau_{\leq 0}\mathbf{H}(X, G)$ is a group for every $X \in \mathbf{H}$). Thus, the forgetful 2-functor $U : \mathrm{Grp}_2(\mathbf{H}) \hookrightarrow \mathbf{Mon}_2(\mathbf{H})$ is fully faithful. Defining the $(2,1)$-categories of *braided group objects* $\mathrm{BrGrp}_2(\mathbf{H})$ and *commutative group objects* $\mathrm{ComGrp}_2(\mathbf{H})$ as the homotopy pullbacks in the diagram below, we obtain a commutative ladder of $(2,1)$-categories:

$$\begin{array}{ccccccccc}
\mathrm{ComGrp}_2(\mathbf{H}) & \hookrightarrow & \mathrm{BrGrp}_2(\mathbf{H}) & \longrightarrow & \mathrm{Grp}_2(\mathbf{H}) & \longrightarrow & \mathbf{H} & \hookrightarrow & \mathbf{Grpd}_2(\mathbf{H}) \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathrm{ComMon}_2(\mathbf{H}) & \hookrightarrow & \mathrm{BrMon}_2(\mathbf{H}) & \longrightarrow & \mathbf{Mon}_2(\mathbf{H}) & \longrightarrow & \mathbf{H} & &
\end{array}$$

This construction is 2-functorial in $\mathbf{H} \in (2,1)\mathbf{Cat}_{2,cp}$, extending to a diagram of 2-natural transformations:

$$\begin{array}{ccccccccc}
\mathrm{ComGrp}_2 & \Longrightarrow & \mathrm{BrGrp}_2 & \Longrightarrow & \mathrm{Grp}_2 & \Longrightarrow & \mathrm{id} & \Longrightarrow & \mathbf{Grpd}_2 \\
\Downarrow & & \Downarrow & & \Downarrow & & \| & & \\
\mathrm{ComMon}_2 & \Longrightarrow & \mathrm{BrMon}_2 & \Longrightarrow & \mathbf{Mon}_2 & \Longrightarrow & \mathrm{id} & &
\end{array}$$

2.5. **Algebras Over (2,1)-Operads.** In a symmetric monoidal $(2,1)$-category $\mathbf{H}$, monoids, braided monoids, and commutative monoids are special cases of a unified notion: *algebras over a $(2,1)$-operad* [28]. Let $(2,1)\mathbf{Oper}_2$ denote the 2-category of $(2,1)$-operads (i.e., 2-truncated ∞-operads, or $\mathbf{Grpd}$-enriched operads), and let $(2,1)\mathbf{SymOper}_2 \subset (2,1)\mathbf{Oper}_2$ be the sub-2-category of symmetric $(2,1)$-operads, which is equipped with a symmetric monoidal structure $\odot$ given by the 2-truncated *Boardman–Vogt tensor product* [5, 28].

The terminal object of $(2,1)\mathbf{SymOper}_2$ is the commutative operad $\mathbb{E}_\infty$, which is filtered by the sequence of *little k-cubes $(2,1)$-operads* $\mathbb{E}_k$ (for $k \geq 0$), obtained by 2-truncating the topological little k-cubes operads:

$$\mathbb{E}_0 \longrightarrow \mathbb{E}_1 \longrightarrow \mathbb{E}_2 \longrightarrow \cdots \longrightarrow \mathbb{E}_{k-1} \longrightarrow \mathbb{E}_k \longrightarrow \mathbb{E}_{k+1} \longrightarrow \cdots \longrightarrow \mathbb{E}_\infty = \varinjlim_k \mathbb{E}_k. \tag{2.4}$$

A foundational property of the operads $\mathbb{E}_k$ is Dunn–Lurie additivity:

Theorem 2.8 (Dunn–Lurie Additivity [9, 28])**.** *For every $k, l \geq 0$, the Boardman–Vogt tensor product of $\mathbb{E}_k$ and $\mathbb{E}_l$ satisfies*

$$\mathbb{E}_k \odot \mathbb{E}_l \simeq \mathbb{E}_{k+l}.$$

Proof. For the purposes of this paper the statement is only used through its consequence on algebras: for a symmetric monoidal $(2,1)$-category $\mathbf{H}$, regarded as a symmetric monoidal ∞-category with 1-truncated mapping spaces, we *define* $\mathbb{E}_k\mathrm{Alg}_2(\mathbf{H}) := \mathrm{Alg}_{\mathbb{E}_k}(\mathbf{H})$ using the ∞-operads $\mathbb{E}_k$ of [28]. Lurie's additivity theorem [28, Theorem 5.1.2.2] gives $\mathbb{E}_k \otimes \mathbb{E}_l \simeq \mathbb{E}_{k+l}$ as ∞-operads, hence $\mathrm{Alg}_{\mathbb{E}_{k+l}}(\mathbf{H}) \simeq \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Alg}_{\mathbb{E}_l}(\mathbf{H}))$, with no truncation of operads involved. A genuinely $(2,1)$-operadic statement (with $\mathbb{E}_k$ replaced by $\tau_{\leq 1}\mathbb{E}_k$ and $\odot$ by a $(2,1)$-categorical Boardman–Vogt product) would additionally require showing that $\tau_{\leq 1}$ is symmetric monoidal for the Boardman–Vogt product; we do not need this. □

For any $(2,1)$-operad $\mathcal{O}$ and any symmetric monoidal $(2,1)$-category $\mathbf{H}$, we have the $(2,1)$-category $\mathcal{O}\mathrm{Alg}_2(\mathbf{H})$ of *$\mathcal{O}$-algebra objects* in $\mathbf{H}$. Equivalently, associating to $\mathbf{H}$ its endomorphism $(2,1)$-operad $\mathcal{E}(\mathbf{H})$, $\mathcal{O}\mathrm{Alg}_2(\mathbf{H})$ is the $(2,1)$-category of $(2,1)$-operad morphisms $\mathcal{O} \to \mathcal{E}(\mathbf{H})$. Every morphism of $(2,1)$-operads $\phi : \mathcal{O} \to \mathcal{O}'$ induces a forgetful 2-functor $\phi^* : \mathcal{O}'\mathrm{Alg}_2(\mathbf{H}) \to \mathcal{O}\mathrm{Alg}_2(\mathbf{H})$, which admits a left 2-adjoint ϕ_* whenever $\mathbf{H}$ is presentable and the tensor product preserves colimits in each variable:

$$\mathcal{O}\mathrm{Alg}_2(\mathbf{H}) \underset{\phi^*}{\overset{\phi_*}{\perp}} \mathcal{O}'\mathrm{Alg}_2(\mathbf{H}).$$

Example 2.9. Recall that $\mathbb{E}_0$-algebras in $\mathbf{H}$ are pointed objects $1 \to X$ in $\mathbf{H}$ (or, forgetting the basepoint as in our convention for $\mathrm{Del}_2^0(\mathbf{H})$ below, objects of $\mathbf{H}$), $\mathbb{E}_1$-algebras are associative monoid objects, $\mathbb{E}_2$-algebras are braided monoid objects, and $\mathbb{E}_3$-algebras in a $(2,1)$-category are symmetric/commutative monoid objects [19, 1, 28]:

$$\begin{aligned}
\mathbb{E}_0\mathrm{Alg}_2^{\mathrm{unpt}}(\mathbf{H}) &\simeq \mathbf{H} \quad (\text{or } \mathbb{E}_0\mathrm{Alg}_2(\mathbf{H}) \simeq 1/\mathbf{H}),\\
\mathbb{E}_1\mathrm{Alg}_2(\mathbf{H}) &\simeq \mathbf{Mon}_2(\mathbf{H}),\\
\mathbb{E}_2\mathrm{Alg}_2(\mathbf{H}) &\simeq \mathrm{BrMon}_2(\mathbf{H}),\\
\mathbb{E}_3\mathrm{Alg}_2(\mathbf{H}) &\simeq \mathrm{ComMon}_2(\mathbf{H}).
\end{aligned}$$

Hence, the filtration (2.4) induces a commutative ladder of forgetful 2-functors:

$$\begin{array}{ccccccc}
\mathbb{E}_3\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbb{E}_2\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbb{E}_1\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbf{H} \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \| \\
\mathrm{ComMon}_2(\mathbf{H}) & \longrightarrow & \mathrm{BrMon}_2(\mathbf{H}) & \longrightarrow & \mathbf{Mon}_2(\mathbf{H}) & \longrightarrow & \mathbf{H}
\end{array} \tag{2.5}$$

Now suppose $\mathbf{H}$ is a cartesian monoidal $(2,1)$-category. Given a $(2,1)$-operad $\mathcal{O}$ equipped with a morphism $\iota : \mathbb{E}_1 \to \mathcal{O}$, we define the full sub-2-category $\mathcal{O}\mathrm{Alg}_{2,\iota}^{\mathrm{grp}}(\mathbf{H}) \subset \mathcal{O}\mathrm{Alg}_2(\mathbf{H})$ of *groupal (or grouplike) $\mathcal{O}$-algebra objects* relative to ι as the homotopy pullback of $\mathrm{Grp}_2(\mathbf{H}) \hookrightarrow \mathbf{Mon}_2(\mathbf{H})$ along $\iota^* : \mathcal{O}\mathrm{Alg}_2(\mathbf{H}) \to \mathbb{E}_1\mathrm{Alg}_2(\mathbf{H}) \simeq \mathbf{Mon}_2(\mathbf{H})$:

$$\begin{array}{ccccc}
\mathcal{O}\mathrm{Alg}_2(\mathbf{H}) & \xrightarrow{\iota^*} & \mathbb{E}_1\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbf{H} \\
\uparrow & & \simeq\downarrow & & \| \\
 & & \mathbf{Mon}_2(\mathbf{H}) & \longrightarrow & \mathbf{H} \\
 & \overset{\iota^*}{\nearrow} & \uparrow & & \\
\mathcal{O}\mathrm{Alg}_{2,\iota}^{\mathrm{grp}}(\mathbf{H}) & \xrightarrow[\iota^*]{} & \mathrm{Grp}_2(\mathbf{H}) & &
\end{array}$$

By direct inspection, the groupal $\mathbb{E}_k$-algebras for $k = 1,2,3$ (relative to the canonical inclusion $\mathbb{E}_1 \to \mathbb{E}_k$) are precisely group objects, braided group objects, and commutative group objects in

H, extending diagram (2.5) to:

$$\begin{array}{ccccccc}
\mathbb{E}_3\mathrm{Alg}_2^{\mathrm{grp}}(\mathbf{H}) & \longrightarrow & \mathbb{E}_2\mathrm{Alg}_2^{\mathrm{grp}}(\mathbf{H}) & \longrightarrow & \mathbb{E}_1\mathrm{Alg}_2^{\mathrm{grp}}(\mathbf{H}) & & \\
\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \\
\mathrm{ComGrp}_2(\mathbf{H}) & \longrightarrow & \mathrm{BrGrp}_2(\mathbf{H}) & \longrightarrow & \mathrm{Grp}_2(\mathbf{H}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{ComMon}_2(\mathbf{H}) & \longrightarrow & \mathrm{BrMon}_2(\mathbf{H}) & \longrightarrow & \mathbf{Mon}_2(\mathbf{H}) & \longrightarrow & \mathbf{H} \\
\uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} & & \| \\
\mathbb{E}_3\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbb{E}_2\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbb{E}_1\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbf{H}
\end{array}$$

Finally, although the sequence (2.4) of $(2,1)$-operads continues beyond $\mathbb{E}_3$, the Baez–Dolan Stabilization Theorem guarantees that in a $(2,1)$-category, $\mathbb{E}_k$-algebras stabilize at $k=3$:

Theorem 2.10 (Stabilization of $\mathbb{E}_k$-Algebras in $(2,1)$-Categories [1, 28])**.** *For every symmetric monoidal $(2,1)$-category* $\mathbf{H}$ *and every $k \geq 3$, the canonical forgetful functor $\mathbb{E}_k\mathrm{Alg}_2(\mathbf{H}) \to \mathbb{E}_3\mathrm{Alg}_2(\mathbf{H}) \simeq \mathrm{ComMon}_2(\mathbf{H})$ is an equivalence of $(2,1)$-categories:*

$$\begin{array}{ccccccccc}
\mathbb{E}_k\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \cdots & \longrightarrow & \mathbb{E}_3\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbb{E}_2\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbb{E}_1\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbf{H} \\
& \searrow^{\simeq} & & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \| \\
& & & & \mathrm{ComMon}_2(\mathbf{H}) & \longrightarrow & \mathrm{BrMon}_2(\mathbf{H}) & \longrightarrow & \mathbf{Mon}_2(\mathbf{H}) & \longrightarrow & \mathbf{H}
\end{array}$$

Proof. Since **H** is a $(2,1)$-category, its hom-spaces are 1-groupoids (i.e., 1-truncated spaces). By [28, §5.1.1] (the corollary stating that $\mathbb{E}_{n+1}$-algebras in a symmetric monoidal n-category are automatically $\mathbb{E}_\infty$; the numbering should be checked against the current version of [28]), in a symmetric monoidal ∞-category whose mapping spaces are $(n-1)$-truncated, the forgetful functor $\mathbb{E}_\infty\mathrm{Alg}(\mathbf{H}) \to \mathbb{E}_k\mathrm{Alg}(\mathbf{H})$ is an equivalence for all $k \geq n+1$. Setting $n=2$ yields stabilization at $k=3$. □

Remark 2.11. Of course, Theorem 2.10 does not imply that the only algebraic structures in a symmetric monoidal $(2,1)$-category are monoids, braided monoids, and commutative monoids (one may consider algebras over arbitrary operads or Lawvere theories, such as Lie algebras, Poisson algebras, or racks). Rather, it asserts that within the $\mathbb{E}_k$-hierarchy of weakening commutativity, there is no further weakening beyond $k=3$ in a $(2,1)$-category. In general, in a symmetric monoidal $(n,1)$-category, the sequence of $\mathbb{E}_k$-algebras stabilizes for $k \geq n+1$.

3. Delooping

In this section we examine delooping groupoids and delooping objects in $(2,1)$-topoi. In our broader program, deloopings are required to formulate an internal and parameterized definition of principal G-bundles and nonabelian 1-cohomology. As we demonstrate in Sections 3.1–3.3, the delooping construction can be realized either *internally* (via internal groupoid objects in a cartesian $(2,1)$-cosmos) or *externally* (by transporting the canonical delooping along Yoneda 2-embeddings on presheaf categories and stackifying), producing 2-naturally equivalent internal groupoids. The internal formulation has the advantage of making sense in any cartesian $(2,1)$-cosmos even when it is not a $(2,1)$-topos, whereas the external formulation directly bridges point-set/site-level algebraic symmetries with sheafified stacks and serves as the blueprint for plain and genuine delooping systems in algebra-valued cosmoi. We then study the representing delooping objects $B_{\mathbf{H}}G$, higher iterated deloopings $B^k_{\mathbf{H}}G$, their characterization via the $(2,1)$-categorical Eckmann–Hilton argument and $\mathbb{E}_k$-algebras, and the looping–delooping equivalence on connected pointed objects.

3.1. **Internal Delooping.** Let $(2,1)\mathbf{Cart}_2 \subset (2,1)\mathbf{Cat}_{2,cp} \subset (2,1)\mathbf{Cat}_2$ denote the 2-category of *cartesian* $(2,1)$-*cosmoi*, defined as the sub-2-category of complete and cocomplete $(2,1)$-categories with 2-functors preserving finite $(2,1)$-limits, and let $(2,1)\mathbf{Cart}_{2,\mathrm{coeq}} \subset (2,1)\mathbf{Cat}_{2,ccp}$ denote the subcategory whose 2-functors additionally preserve $(2,1)$-colimits (homotopy coequalizers). Let

$$\mathrm{Grp}_2, \mathbf{Grpd}_2 : (2,1)\mathbf{Cart}_2 \longrightarrow (2,1)\mathbf{Cat}_2$$

denote the restrictions to $(2,1)\mathbf{Cart}_2$ of the internal group and internal groupoid 2-functors from Section 2.3.

The *internal delooping* is the 2-natural transformation $\mathbf{B}^{\mathrm{int}} : \mathrm{Grp}_2 \Rightarrow \mathbf{Grpd}_2$ which assigns to every $\mathbf{H} \in (2,1)\mathbf{Cart}_2$ and every group object $G \in \mathrm{Grp}_2(\mathbf{H})$ the one-object internal groupoid $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) \in \mathbf{Grpd}_2(\mathbf{H})$ defined by

$$\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)_0 = *_{\mathbf{H}}, \qquad \mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)_1 = G,$$

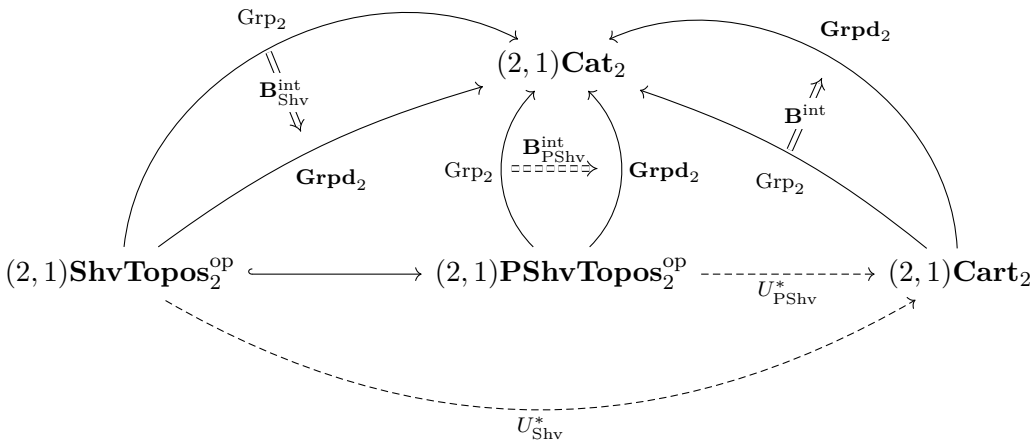
where $*_{\mathbf{H}}$ is the terminal object of $\mathbf{H}$, the source and target morphisms $s, t : G \to *_{\mathbf{H}}$ are the unique terminal morphisms, and composition $\circ : G \times_{*_{\mathbf{H}}} G \simeq G \times G \xrightarrow{m} G$, unit $e : *_{\mathbf{H}} \to G$, and inversion $\mathrm{inv} : G \to G$ are given by the group structure of G:

$$\begin{array}{ccc} & (2,1)\mathbf{Cat}_2 & \\ \mathrm{Grp}_2 & \overset{\mathbf{B}^{\mathrm{int}}}{\Longrightarrow} & \mathbf{Grpd}_2 \\ & (2,1)\mathbf{Cart}_2 & \end{array}$$

Notice that an arbitrary $(2,1)$-pretopos need not belong to $(2,1)\mathbf{Cart}_2$, as it may lack arbitrary small $(2,1)$-limits or $(2,1)$-colimits. However, every $(2,1)$-presheaf $(2,1)$-topos and every Grothendieck $(2,1)$-topos is locally presentable [29], and therefore is complete and cocomplete; moreover, direct image functors f_* of geometric morphisms are right 2-adjoints and thus preserve all $(2,1)$-limits (so $U_{\mathrm{topos}}(f) = f_*$ lands in $(2,1)\mathbf{Cart}_2$, which suffices for Grp_2, $\mathbf{Grpd}_2$, and $\mathbf{B}^{\mathrm{int}}$), while inverse image functors f^* preserve both finite $(2,1)$-limits and all $(2,1)$-colimits (so $U^*_{\mathrm{topos}}(f) = f^*$ lands in $(2,1)\mathbf{Cart}_{2,\mathrm{coeq}} \subset (2,1)\mathbf{Cart}_2$, which additionally commutes with taking quotient delooping objects $B_{\mathbf{H}} = \mathrm{cod} \circ \mathrm{coeq} \circ \mathbf{B}^{\mathrm{int}}_{\mathbf{H}}$ and stackification localizations L_J). Consequently, both forgetful 2-functors $U_{\mathrm{topos}} : (2,1)\mathbf{PTopos}_2 \to (2,1)\mathbf{Cat}_2$ and $U^*_{\mathrm{topos}} : (2,1)\mathbf{PTopos}_2^{\mathrm{op}} \to (2,1)\mathbf{Cat}_2$ from Remark 2.4 factor on $(2,1)\mathbf{PShvTopos}_2$ and $(2,1)\mathbf{ShvTopos}_2$ through $(2,1)\mathbf{Cart}_2$ (via $U_{\mathrm{PShv}}, U_{\mathrm{Shv}}$ and $U^*_{\mathrm{PShv}}, U^*_{\mathrm{Shv}}$, respectively):[5]

$$\begin{array}{ccccc} & & & \overset{U^*_{\mathrm{Shv}}}{\cdots\!\cdots\!\cdots\!

$(2,1)$-topoi:



(3.1)

3.2. **External Delooping.** The construction in Section 3.1 is *internal* in the sense that it operates directly on group objects inside $\mathbf{H}$ without reference to a presenting site $(\mathbf{C}, J)$. We now construct an *external* presentation of delooping by acting on presheaves of groups and groupoids via the 2-categorical Yoneda embedding.

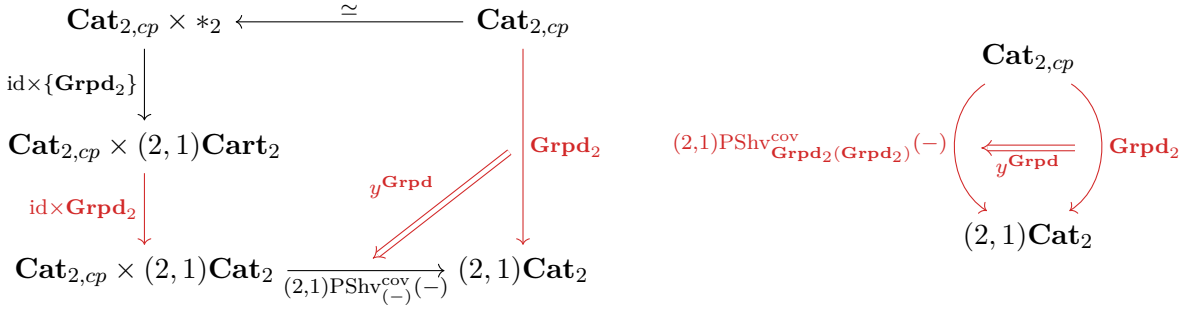
Consider the two-variable presheaf 2-functor

$$(2,1)\mathrm{PShv}_{(-)}(-) : \mathbf{Cat}_2^{\mathrm{op}} \times (2,1)\mathbf{Cat}_2 \longrightarrow (2,1)\mathbf{Cat}_2, \qquad (\mathbf{C}, \mathbf{A}) \longmapsto (2,1)\mathrm{PShv}_{\mathbf{A}}(\mathbf{C}),$$

together with its covariant realization $(2,1)\mathrm{PShv}^{\mathrm{cov}}_{(-)}(-) : \mathbf{Cat}_{2,cp} \times (2,1)\mathbf{Cart}_{2,\mathrm{filt}} \to (2,1)\mathbf{Cart}_2$ acting on 1-categories with finite 1-limits $\mathbf{C} \in \mathbf{Cat}_{2,cp}$ via left-exact left Kan extension $f_! = \mathrm{Lan}_{f^{\mathrm{op}}}$ (where $(2,1)\mathbf{Cart}_{2,\mathrm{filt}} \subset (2,1)\mathbf{Cart}_2$ denotes cartesian $(2,1)$-cosmoi—such as $\mathbf{Grpd}_2$, $\mathbf{Set}_2$, and all Grothendieck $(2,1)$-topoi—in which filtered $(2,1)$-colimits commute with finite $(2,1)$-limits; indeed, for any finite-limit-preserving functor $f : \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat}_{2,cp}$ and $d \in \mathbf{D}$, the comma category $(d \downarrow f)$ has all finite limits and is therefore cofiltered, so the indexing category $(f^{\mathrm{op}} \downarrow d) \simeq (d \downarrow f)^{\mathrm{op}}$ for the pointwise $(2,1)$-colimit $(\mathrm{Lan}_{f^{\mathrm{op}}}F)(d)$ is filtered, ensuring that $f_! = \mathrm{Lan}_{f^{\mathrm{op}}}$ preserves finite $(2,1)$-limits and hence internal groupoids). Recall the 2-categorical Yoneda embedding [47], which assigns to every small $(2,1)$-category (and in particular every $\mathbf{C} \in \mathbf{Cat}_{2,cp}$) the fully faithful 2-functor

$$y_{\mathbf{C}} : \mathbf{C}_2 \hookrightarrow (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C}), \qquad X \longmapsto \mathbf{C}_2(-, X).$$

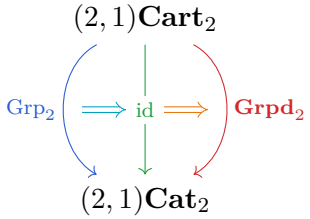
Observe that $\mathbf{Grpd}_2 \simeq \mathbf{Grpd}_2(\mathbf{Set}_2)$, where $\mathbf{Set}_2 = \mathrm{disc}_{1,2}(\mathbf{Set}) \in (2,1)\mathbf{Cart}_2$, and for $f : \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat}_{2,cp}$ we have $f_! \circ y_{\mathbf{C}} \simeq y_{\mathbf{D}} \circ f$. Denoting by $\mathrm{id} \times \{\mathbf{Grpd}_2\} : \mathbf{Cat}_{2,cp} \times *_2 \to \mathbf{Cat}_{2,cp} \times (2,1)\mathbf{Cart}_2$ the constant inclusion selecting the object $\mathbf{Grpd}_2 \in (2,1)\mathbf{Cart}_2$, the assignment $\mathbf{C} \mapsto y_{\mathbf{C}}$ (extended to internal groupoids $\mathbf{G} = (G_1 \rightrightarrows G_0)$ in $\mathbf{C}$ via $\mathbf{Grpd}_2(\mathbf{C}) \hookrightarrow (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2(\mathbf{Set}_2)}(\mathbf{C}) \hookrightarrow (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2(\mathbf{Grpd}_2)}(\mathbf{C})$ induced by $\mathbf{Set}_2 \hookrightarrow \mathbf{Grpd}_2$) is 2-natural in $\mathbf{C} \in \mathbf{Cat}_{2,cp}$, defining a 2-natural transformation $y^{\mathbf{Grpd}}$—the *groupoid Yoneda 2-embedding*—as in the diagrams below (where colors are used as visual aids to trace the corresponding 2-functors):



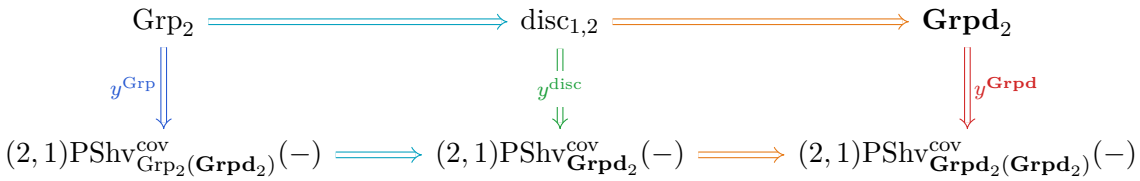
(3.2)

Next, recall from Section 2.3 that on $(2,1)\mathbf{Cart}_2$ (and on $\mathbf{Cat}_{2,cp}$) we have the sequence of inclusion 2-natural transformations between Grp_2, $\mathrm{disc}_{1,2}$ (or $\mathrm{id}_{(2,1)\mathbf{Cart}_2}$), and $\mathbf{Grpd}_2$. We

denote these 2-functors and their Yoneda embeddings by Grp_2, $\mathrm{disc}_{1,2}$, and $\mathbf{Grpd}_2$ (colored blue, green, and red below, with their intermediate inclusions colored cyan and orange):



Replacing the groupoid 2-functor $\mathbf{Grpd}_2$ in diagram (3.2) with $\mathrm{disc}_{1,2}$ (where $\mathrm{id}(\mathbf{Grpd}_2) = \mathbf{Grpd}_2$) or Grp_2, the Yoneda 2-embedding factors through the canonical inclusions, yielding the *discrete Yoneda 2-embedding* y^{disc} and the *group Yoneda 2-embedding* y^{Grp}:

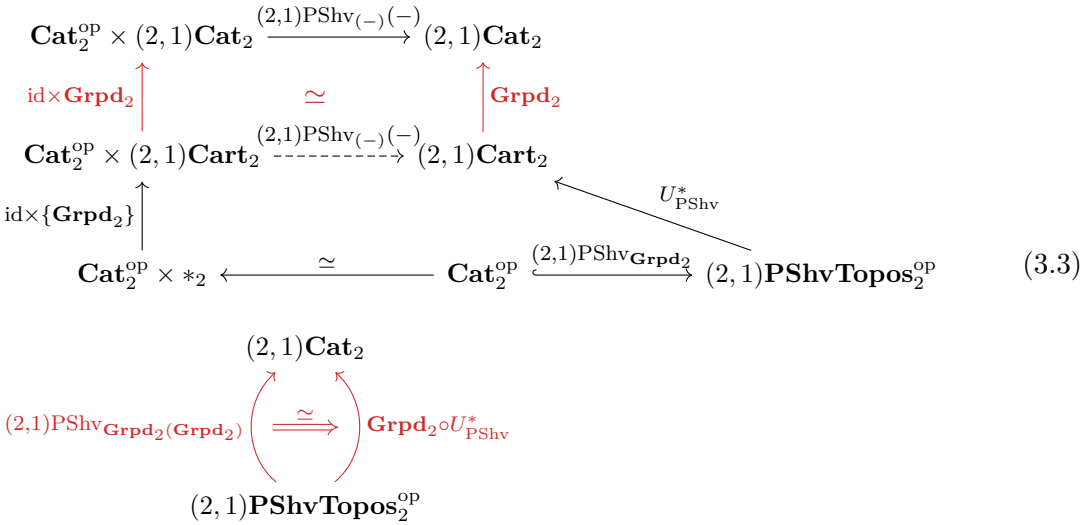


Now, taking the delooping transformation $\mathbf{B}_{\mathbf{Grpd}_2} : \mathrm{Grp}_2(\mathbf{Grpd}_2) \to \mathbf{Grpd}_2(\mathbf{Grpd}_2)$ (which restricts on 0-truncated objects $\mathbf{Set}_2 \subset \mathbf{Grpd}_2$ to $\mathbf{B}_{\mathbf{Set}_2} : \mathrm{Grp}_2(\mathbf{Set}_2) \to \mathbf{Grpd}_2(\mathbf{Set}_2)$) and applying $(2,1)\mathrm{PShv}_{(-)}(\mathbf{C})$ yields a 2-natural transformation (both contravariantly on $\mathbf{Cat}_2^{\mathrm{op}}$ via precomposition f^* and covariantly on $\mathbf{Cat}_{2,cp}$ via left-exact left Kan extension $f_!$)

$$\mathbf{B}^{\mathrm{ext}}_{\mathrm{PShv}} : (2,1)\mathrm{PShv}_{\mathrm{Grp}_2(\mathbf{Grpd}_2)}(-) \Longrightarrow (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2(\mathbf{Grpd}_2)}(-),$$

compatible with the Yoneda 2-embeddings $y^{\mathrm{Grp}}, y^{\mathrm{disc}}, y^{\mathbf{Grpd}}$ above. This is the *external delooping* for $(2,1)$-presheaf $(2,1)$-topoi. For Grothendieck $(2,1)$-topoi $\mathbf{Stk}(\mathbf{C}, J)$, the external delooping $\mathbf{B}^{\mathrm{ext}}_{\mathrm{Shv}}$ is obtained by composing $\mathbf{B}^{\mathrm{ext}}_{\mathrm{PShv}}$ with the stackification localization transformation L from (2.3).

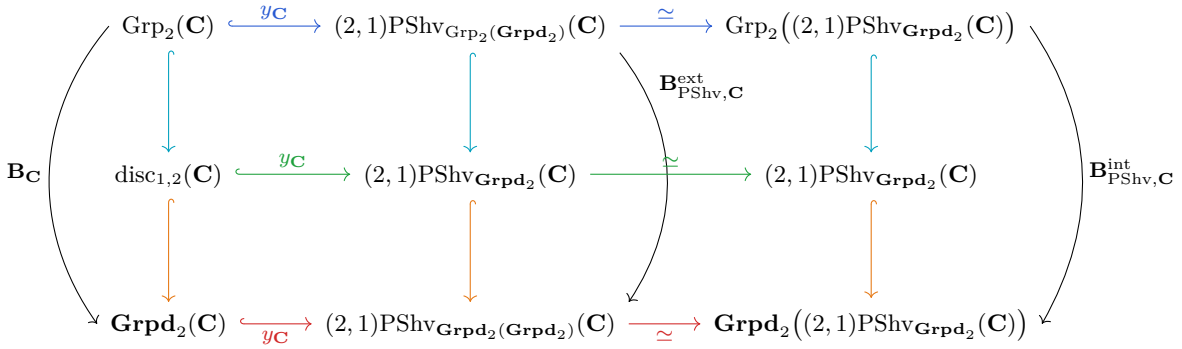
3.3. **Equivalence.** The equivalence between the internal and external delooping constructions rests on the fundamental property that internal algebraic or categorical structures in a $(2,1)$-presheaf category $(2,1)\mathrm{PShv}_{\mathbf{A}}(\mathbf{C})$ are computed objectwise in $\mathbf{A}$. Formally, because the 2-functor $(2,1)\mathrm{PShv}_{(-)}(\mathbf{C}) : (2,1)\mathbf{Cart}_2 \to (2,1)\mathbf{Cart}_2$ preserves $(2,1)$-limits (and both precomposition f^* along $f : \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat}_2^{\mathrm{op}}$, which is the inverse image functor U^*_{PShv} of the geometric morphism $(f^* \dashv f_*) : (2,1)\mathrm{PShv}(\mathbf{C}) \to (2,1)\mathrm{PShv}(\mathbf{D})$, and left Kan extension $f_!$ along $f \in \mathbf{Cat}_{2,cp}$, which is the inverse image functor of the opposite geometric morphism $(f_! \dashv f^*) : (2,1)\mathrm{PShv}(\mathbf{D}) \to (2,1)\mathrm{PShv}(\mathbf{C})$, preserve finite $(2,1)$-limits), it commutes up to canonical 2-natural equivalence with $\mathbf{Grpd}_2$, $\mathrm{id}_{(2,1)\mathbf{Cart}_2}$, and Grp_2:



(3.3)

and analogously (both contravariantly via f^* on $\mathbf{Cat}_2^{\mathrm{op}}$ and covariantly via $f_!$ on $\mathbf{Cat}_{2,cp}$) with $\mathbf{Grpd}_2$ replaced by id or Grp_2.

Pasting each Yoneda diagram (3.2) with the corresponding limit commutation diagram (3.3), and comparing with the definition (3.1) of the internal delooping $\mathbf{B}^{\mathrm{int}}_{\mathrm{PShv}}$, we see that $\mathbf{B}^{\mathrm{int}}_{\mathrm{PShv}}$ is precisely the transport of the external delooping $\mathbf{B}^{\mathrm{ext}}_{\mathrm{PShv}}$ along the canonical 2-equivalences of (3.3). Objectwise, for each $\mathbf{C} \in \mathbf{Cat}_{2,cp}$, this is expressed by the 2-commutativity of the following diagram (and for a Grothendieck $(2,1)$-topos $\mathbf{Stk}(\mathbf{C}, J)$, by appending the exact localization functor L_J from (2.3), which preserves finite $(2,1)$-limits and all $(2,1)$-colimits, and therefore commutes 2-naturally with Grp_2, $\mathbf{Grpd}_2$, $\mathbf{B}$, and quotient delooping objects $B = \mathrm{cod} \circ \mathrm{coeq} \circ \mathbf{B}$):



(3.4)

We summarize these results in the following theorem:

Theorem 3.1 (Equivalence of Internal and External Delooping)**.** *We have:*

(1) For every cartesian $(2,1)$-cosmos $\mathbf{H} \in (2,1)\mathbf{Cart}_2$, a 2-functor

$$\mathbf{B}^{\mathrm{int}}_{\mathbf{H}} : \mathrm{Grp}_2(\mathbf{H}) \longrightarrow \mathbf{Grpd}_2(\mathbf{H}),$$

the internal delooping*;*

(2) For every $(2,1)$-presheaf $(2,1)$-topos $\mathbf{H} \simeq (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$, a 2-functor

$$\mathbf{B}^{\mathrm{ext}}_{\mathbf{H}} : (2,1)\mathrm{PShv}_{\mathrm{Grp}_2(\mathbf{Grpd}_2)}(\mathbf{C}) \longrightarrow (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2(\mathbf{Grpd}_2)}(\mathbf{C}),$$

the external delooping.

Furthermore, under the canonical equivalences

$$\mathrm{Grp}_2(\mathbf{H}) \simeq (2,1)\mathrm{PShv}_{\mathrm{Grp}_2(\mathbf{Grpd}_2)}(\mathbf{C}) \quad \textit{and} \quad \mathbf{Grpd}_2(\mathbf{H}) \simeq (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2(\mathbf{Grpd}_2)}(\mathbf{C}),$$

the 2-functors $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}$ and $\mathbf{B}^{\mathrm{ext}}_{\mathbf{H}}$ are 2-naturally equivalent. Moreover, if $\mathbf{H}_0 = \mathbf{Stk}(\mathbf{C}, J) \subset \mathbf{H}$ is a Grothendieck $(2,1)$-topos with left-exact stackification $L_J : \mathbf{H} \to \mathbf{H}_0$, then both deloopings descend along L_J and remain 2-naturally equivalent on $\mathbf{H}_0$.

Proof. In $(2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$, finite $(2,1)$-limits are computed objectwise in $\mathbf{Grpd}_2$. Given a group object $G \in \mathrm{Grp}_2((2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C}))$, evaluating at each $U \in \mathbf{C}$ yields a group object $G(U) \in \mathrm{Grp}_2(\mathbf{Grpd}_2)$ whose one-object delooping groupoid in $\mathbf{Grpd}_2(\mathbf{Grpd}_2)$ is given by $\mathbf{B}_{\mathbf{Grpd}_2}(G(U)) = (G(U) \rightrightarrows *)$. By definition of $\mathbf{B}^{\mathrm{ext}}_{\mathbf{H}}$, the presheaf $U \mapsto \mathbf{B}_{\mathbf{Grpd}_2}(G(U))$ is $\mathbf{B}^{\mathrm{ext}}_{\mathbf{H}}(G)$. Since the terminal presheaf $*$ has $*(U) = *$, the internal groupoid associated to $\mathbf{B}^{\mathrm{ext}}_{\mathbf{H}}(G)$ has object of objects $*$ and object of morphisms G, which is literally $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)$. Finally, since $L_J : \mathbf{H} \to \mathbf{H}_0$ preserves finite $(2,1)$-limits, it preserves terminal objects and products, hence commutes with both $\mathrm{Grp}_2(-)$, $\mathbf{Grpd}_2(-)$, and the construction $G \mapsto (G \rightrightarrows *)$. □

Remark 3.2 (Foundational Role and Bifurcation in Non-Cartesian Algebra-Valued Cosmoi). While the equivalence between internal and external delooping in Theorem 3.1 is mathematically immediate in the cartesian setting (following directly from objectwise limits and left-exact stackification), its explicit 2-functorial formalization serves a vital purpose for higher geometry and mathematical physics: it pinpoints the exact structural axioms responsible for the coincidence of internal and external delooping in $\mathbf{Grpd}_2$-valued topoi. Specifically, $(2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$ is cartesian monoidal ($\otimes = \times$), its monoidal unit is the terminal presheaf $*_{\mathbf{H}}(U) = *$, and finite pullbacks over $*_{\mathbf{H}}$ coincide with binary products $G \times_{*_{\mathbf{H}}} G \simeq G \times G$.

To see how this connects directly to physics, recall from [33, 26] that in an *algebra-valued gauge theory* (such as algebra-valued Yang–Mills or Einstein–Hilbert–Palatini gravity), one replaces the Lie algebra $\mathfrak{g}$ of a Lie group G with a general (associative or non-associative) algebra A equipped with a bilinear product $\cdot_A : A \times A \to A$. Locally over a chart U, a connection is an A-valued 1-form $\omega \in \Omega^1(U; A)$ whose curvature $F_\omega = d\omega + \omega \wedge_A \omega$ (and higher polynomial invariants) is governed by the algebraic multiplication $\cdot_A$ rather than a Lie bracket integrated to an invertible group G. Passing from local infinitesimal data to global finite symmetries encounters two structural departures from ordinary gauge theory: (i) because a general algebra A (or its multiplicative monoid M) lacks global group inverses g^{-1}, the classical Maurer–Cartan formula $g^{-1}dg$ is unavailable (motivating the non-groupal de Rham unit η^{dR} developed in Section 4.3.3); and (ii) in the sequel to this work [35], when the base cosmos $\mathbf{Grpd}_2$ is replaced by a non-cartesian symmetric monoidal Bénabou 2-cosmos $(\mathbb{A}, \otimes_{\mathbb{A}}, I_{\mathbb{A}})$ (e.g., differential graded algebras, $\mathcal{O}$-algebras, or modules over a ring stack) equipped with the Day convolution or objectwise tensor product, the monoidal unit $I_{\mathbb{A}}$ is no longer the terminal object $*$, and $\mathbf{B}^{\mathrm{int}}$ (defined via internal $\otimes_{\mathbb{A}}$-categories over $I_{\mathbb{A}}$) bifurcates from $\mathbf{B}^{\mathrm{ext}}$ (defined via Yoneda transport of point-set deloopings), giving rise to distinct *plain* and *genuine* delooping systems.

3.4. **Delooping Objects.** By Theorem 3.1, to every group object $G \in \mathrm{Grp}_2(\mathbf{H})$ in a cartesian $(2,1)$-cosmos $\mathbf{H}$ there corresponds an internal delooping groupoid $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) \in \mathbf{Grpd}_2(\mathbf{H})$. Taking its quotient object in $\mathbf{H}$ via the homotopy coequalizer functor $\mathrm{cod} \circ \mathrm{coeq} : \mathbf{Grpd}_2(\mathbf{H}) \to \mathbf{H}$ from Section 2.3, we obtain the representing object

$$B^{\mathrm{int}}_{\mathbf{H}} G := \mathrm{cod}\big(\mathrm{coeq}(\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G))\big) = \varinjlim_{[n]\in\Delta^{\mathrm{op}}} G^{\times n} \in \mathbf{H},$$

the realization of the bar construction (nerve) of G, cf. Remark 2.6. We call $B^{\mathrm{int}}_{\mathbf{H}} G$ the *internal delooping object* (or *classifying object*) of G.

If $\mathbf{H}$ is a $(2,1)$-presheaf $(2,1)$-topos or a Grothendieck $(2,1)$-topos, we also have the external delooping groupoid $\mathbf{B}^{\mathrm{ext}}_{\mathbf{H}}(G)$ and its representing object, the *external delooping object*

$$B^{\mathrm{ext}}_{\mathbf{H}} G := \mathrm{cod}\big(\mathrm{coeq}(\mathbf{B}^{\mathrm{ext}}_{\mathbf{H}}(G))\big) \in \mathbf{H}.$$

By Theorem 3.1, the canonical 2-natural equivalence $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) \simeq \mathbf{B}^{\mathrm{ext}}_{\mathbf{H}}(G)$ in $\mathbf{Grpd}_2(\mathbf{H})$ induces an equivalence of delooping objects in $\mathbf{H}$:

$$B^{\mathrm{int}}_{\mathbf{H}} G \simeq B^{\mathrm{ext}}_{\mathbf{H}} G,$$

which we henceforth denote simply by $B_{\mathbf{H}} G$ (or BG when $\mathbf{H}$ is clear from context).

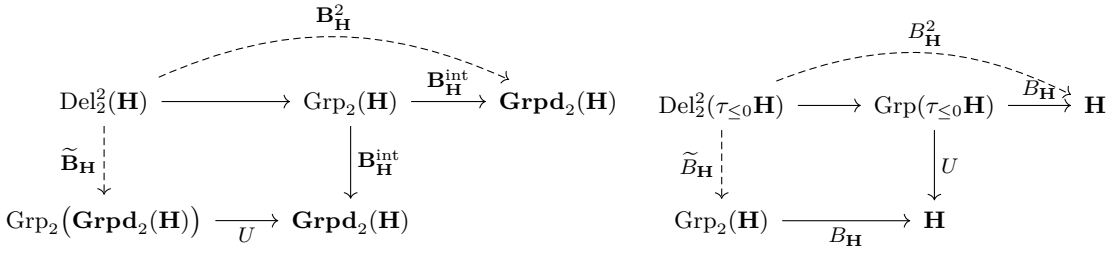
Crucially, because in a Grothendieck $(2,1)$-topos $\mathbf{H}$ every internal groupoid with 0-truncated morphism fibers is effective in $\mathbf{H}$, for any 0-truncated group object $G \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H}) \subset \mathrm{Grp}_2(\mathbf{H})$ the delooping groupoid $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)$ can be losslessly reconstructed from its delooping object $B_{\mathbf{H}} G$ together with its canonical basepoint $* \twoheadrightarrow B_{\mathbf{H}} G$ as the Čech groupoid $\check{C}(* \twoheadrightarrow B_{\mathbf{H}} G)$, whereas for a general 1-truncated 2-group $G \in \mathrm{Grp}_2(\mathbf{H})$, $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) \in \mathbf{Grpd}_2(\mathbf{H})$ retains the full 2-group data prior to 1-truncation by $\mathrm{cod} \circ \mathrm{coeq}$.

3.5. **Higher Delooping.** For every cartesian $(2,1)$-cosmos $\mathbf{H}$, composing the internal delooping 2-functor $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}} : \mathrm{Grp}_2(\mathbf{H}) \to \mathbf{Grpd}_2(\mathbf{H})$ with the quotient object 2-functor $\mathrm{cod} \circ \mathrm{coeq} : \mathbf{Grpd}_2(\mathbf{H}) \to \mathbf{H}$ yields the delooping object 2-functor

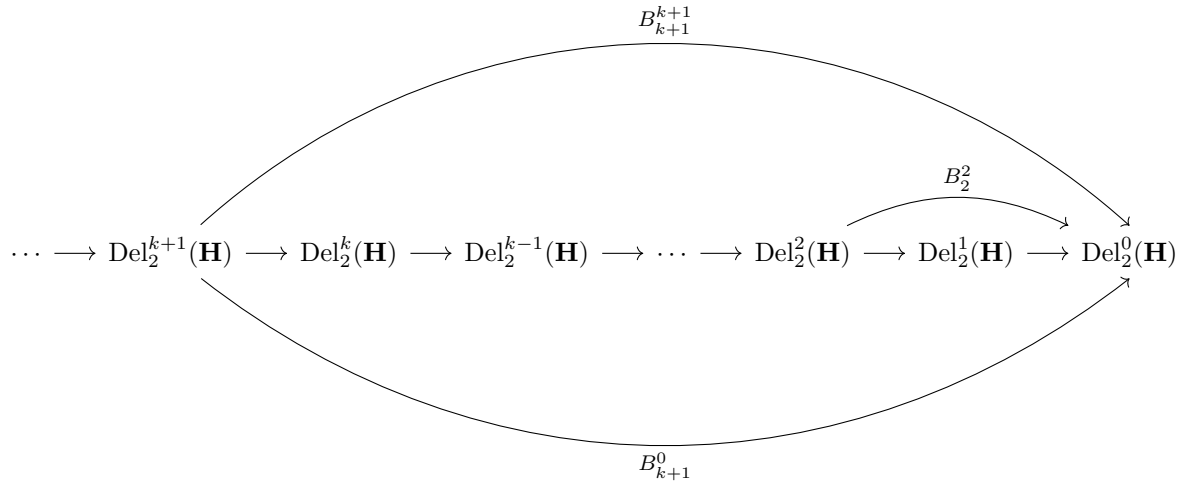
$$B_{\mathbf{H}} = \mathrm{cod} \circ \mathrm{coeq} \circ \mathbf{B}^{\mathrm{int}}_{\mathbf{H}} : \mathrm{Grp}_2(\mathbf{H}) \longrightarrow \mathbf{H}.$$

It is natural to ask when the delooping construction can be iterated. Observe that a second delooping $B^2_{\mathbf{H}} G := B_{\mathbf{H}}(B_{\mathbf{H}} G)$ (realized losslessly via the internal groupoid $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) \in \mathbf{Grpd}_2(\mathbf{H})$ as explained in Remark 3.3 below, or via the quotient $B_{\mathbf{H}} G \in \mathbf{H}$ when $G \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$ is 0-truncated) is defined if and only if $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) \in \mathbf{Grpd}_2(\mathbf{H})$ (resp. $B_{\mathbf{H}} G \in \mathbf{H}$) itself admits a lift to a group object in $\mathrm{Grp}_2(\mathbf{Grpd}_2(\mathbf{H}))$ (resp. $\mathrm{Grp}_2(\mathbf{H})$). That is, let $\mathrm{Del}^2_2(\mathbf{H})$ denote the $(2,1)$-category (fitting into the lossless homotopy pullback square on the left below, which for

0-truncated groups in a Grothendieck $(2,1)$-topos is equivalent to the quotient pullback square on the right) of group objects G whose delooping is equipped with a compatible group structure:



As we will prove in Section 3.6, the forgetful functor $\mathrm{Del}^2_2(\mathbf{H}) \to \mathrm{Grp}_2(\mathbf{H})$ is faithful on 1-morphisms (1-faithful), while $\mathrm{Del}^3_2(\mathbf{H}) \to \mathrm{Del}^2_2(\mathbf{H})$ is fully faithful and $\mathrm{Del}^k_2(\mathbf{H}) \to \mathrm{Del}^3_2(\mathbf{H})$ is an equivalence for all $k \geq 3$. Recursively, we define $(2,1)$-categories $\mathrm{Del}^k_2(\mathbf{H})$ for $k \geq 0$, with forgetful functors $\mathrm{Del}^k_2(\mathbf{H}) \to \mathrm{Del}^l_2(\mathbf{H})$ for $l \leq k$, consisting of objects that are *k-tuply deloopable*, i.e., for which the k-fold delooping $B^k_{\mathbf{H}}G$ exists. Setting $\mathrm{Del}^1_2(\mathbf{H}) := \mathrm{Grp}_2(\mathbf{H})$ and $\mathrm{Del}^0_2(\mathbf{H}) := \mathbf{H}$, we obtain the tower of iterated delooping 2-functors:



where $B^j_m : \mathrm{Del}^m_2(\mathbf{H}) \to \mathrm{Del}^0_2(\mathbf{H})$ denotes the j-fold delooping $B^j_{\mathbf{H}}$ (and B^0_m is the underlying object functor). Taking the inverse $(2,1)$-limit of the forgetful tower in $(2,1)\mathbf{Cat}_2$ yields the $(2,1)$-category

$$\mathrm{Del}^\infty_2(\mathbf{H}) := \varprojlim_k \mathrm{Del}^k_2(\mathbf{H})$$

of *infinitely deloopable objects* (or connective spectrum objects) in $\mathbf{H}$.

Remark 3.3 (1-Truncation of Quotient Deloopings vs. Higher Ambient Deloopings). In a $(2,1)$-category $\mathbf{H}$ (where every object is 1-truncated), one must carefully distinguish between:

(1) The *internal delooping groupoid* $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) = (G \rightrightarrows *) \in \mathbf{Grpd}_2(\mathbf{H})$, which is lossless for any group object $G \in \mathrm{Grp}_2(\mathbf{H})$ (since $\Omega_* \mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) = G$ in $\mathbf{Grpd}_2(\mathbf{H})$); and
(2) The *quotient delooping object* $B^{\mathrm{quot}}_{\mathbf{H}}G := \mathrm{cod}(\mathrm{coeq}_{\mathbf{H}}(G \rightrightarrows *)) \in \mathbf{H}$, which reflects $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)$ into the 1-truncated category $\mathbf{H}$.

Because $\Omega X = * \times_X *$ in a $(2,1)$-category $\mathbf{H}$ is always 0-truncated ($\Omega X \in \tau_{\leq 0}\mathbf{H}$), the quotient $B^{\mathrm{quot}}_{\mathbf{H}}G \in \mathbf{H}$ satisfies $\Omega B^{\mathrm{quot}}_{\mathbf{H}}G \simeq \tau_{\leq 0}G$. In particular, if $A \in \mathrm{ComGrp}(\tau_{\leq 0}\mathbf{H})$ is a 0-truncated abelian group sheaf (e.g., $U(1)$), its first delooping $B_{\mathbf{H}}A \in \mathbf{H}$ is 0-connected ($\tau_{\leq 0}B_{\mathbf{H}}A \simeq *$), so taking a second quotient coequalizer *inside* $\mathbf{H}$ would yield $\mathrm{cod}(\mathrm{coeq}_{\mathbf{H}}(B_{\mathbf{H}}A \rightrightarrows *)) \simeq \tau_{\leq 1}K(A,2) \simeq *_{\mathbf{H}}$. Instead, for $k \geq 2$, the k-fold delooping $B^k_{\mathbf{H}}G$ of a k-tuply groupal object G is losslessly realized as the $(k-1)$-fold internal groupoid $\mathbf{B}^{k-1}_{\mathbf{H}}(B_{\mathbf{H}}G) \in \mathbf{Grpd}^{k-1}_2(\mathbf{H})$ (equivalently, as the k-truncated object B^kG in the ambient $(\infty,1)$-topos $\mathbf{H}_\infty$ of which $\mathbf{H} = \tau_{\leq 1}\mathbf{H}_\infty$ is the $(2,1)$-truncation). Notice that for $k = 2$ and $A \in \mathrm{ComGrp}(\tau_{\leq 0}\mathbf{H})$, $B_{\mathbf{H}}A \in \mathrm{ComGrp}_2(\mathbf{H})$ is a 1-truncated symmetric 2-group in $\mathbf{H}$ and $B^2_{\mathbf{H}}A = \mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(B_{\mathbf{H}}A) = (B_{\mathbf{H}}A \rightrightarrows *) \in \mathbf{Grpd}_2(\mathbf{H})$ is

a single internal groupoid (a 2-stack) in $\mathbf{H}$ classifying A-bundle gerbes. Furthermore, since any finite-limit-preserving endofunctor (such as $\flat$ and $\sharp$; note that $\int$ in general preserves only finite *products*, and preserves pullbacks only in a totally 1-connected topos, cf. Remark 4.4) and any representable functor $\mathbf{H}(X,-)$ preserves internal groupoids, they prolong objectwise to $\mathbf{Grpd}_2^{k-1}(\mathbf{H})$, with $\mathbf{H}(X, B^k_{\mathbf{H}}G) := \tau_{\leq 1}\mathbf{H}_\infty(X, B^kG)$ denoting the 1-truncated hom-groupoid of $(k-1)$-bundles. Throughout what follows, whenever $k \geq 2$ (or when $G \in \mathrm{Grp}_2(\mathbf{H})$ is 1-truncated), $B^k_{\mathbf{H}}G$ denotes this uncollapsed delooping.

3.6. **Eckmann–Hilton.** To say that the delooping $B_{\mathbf{H}}G$ exists is to say that G is a group object in $\mathbf{H}$. In turn, to say that $B^2_{\mathbf{H}}G = B_{\mathbf{H}}(B_{\mathbf{H}}G)$ exists (in the lossless sense of Remark 3.3) is to say that both G and its first delooping $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)$ carry compatible group structures. Since finite products commute with $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}$ (and with $B_{\mathbf{H}}$), a group structure on $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)$ (or on $B_{\mathbf{H}}G$ when $\Omega B_{\mathbf{H}}G \simeq G$) induces a second group structure on G that is homomorphic with respect to the first.

More generally, define the $(2,1)$-category $\mathrm{Grp}^k_2(\mathbf{H})$ of *k-tuply group objects* in $\mathbf{H}$ inductively by

$$\mathrm{Grp}^0_2(\mathbf{H}) := \mathbf{H}, \qquad \mathrm{Grp}^1_2(\mathbf{H}) := \mathrm{Grp}_2(\mathbf{H}), \qquad \mathrm{Grp}^k_2(\mathbf{H}) := \mathrm{Grp}_2(\mathrm{Grp}^{k-1}_2(\mathbf{H})) \quad (k \geq 1).$$

That is, an object of $\mathrm{Grp}^k_2(\mathbf{H})$ is an object of $\mathbf{H}$ equipped with k pairwise compatible group structures.

Lemma 3.4. *For every cartesian $(2,1)$-cosmos $\mathbf{H}$ (with $\mathrm{Del}^k_2(\mathbf{H})$ defined via the lossless internal delooping $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}$ of Remark 3.3, or via $B_{\mathbf{H}}$ on $\tau_{\leq 0}\mathbf{H}$ in a Grothendieck $(2,1)$-topos) and every $k \geq 0$, there is a natural equivalence of $(2,1)$-categories*

$$\mathrm{Grp}^k_2(\mathbf{H}) \simeq \mathrm{Del}^k_2(\mathbf{H}).$$

Proof. We proceed by induction on k. For $k = 0, 1$, we have $\mathrm{Grp}^0_2(\mathbf{H}) = \mathbf{H} = \mathrm{Del}^0_2(\mathbf{H})$ and $\mathrm{Grp}^1_2(\mathbf{H}) = \mathrm{Grp}_2(\mathbf{H}) = \mathrm{Del}^1_2(\mathbf{H})$ by definition. For the inductive step, observe that $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}} : \mathrm{Grp}_2(\mathbf{H}) \to \mathbf{Grpd}_2(\mathbf{H})$ (and $B_{\mathbf{H}}$) preserves finite products, and recovering G as the automorphism/loop object of the basepoint $* \to \mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)$ preserves all $(2,1)$-limits. Consequently, a group object structure on the delooping of G corresponds to an internal group structure on G inside $\mathrm{Grp}_2(\mathbf{H})$, i.e., an object of $\mathrm{Grp}_2(\mathrm{Grp}_2(\mathbf{H})) = \mathrm{Grp}^2_2(\mathbf{H})$. Iterating this equivalence $k-1$ times yields $\mathrm{Del}^k_2(\mathbf{H}) \simeq \mathrm{Grp}_2(\mathrm{Del}^{k-1}_2(\mathbf{H})) \simeq \mathrm{Grp}_2(\mathrm{Grp}^{k-1}_2(\mathbf{H})) = \mathrm{Grp}^k_2(\mathbf{H})$. □

By the $(2,1)$-categorical *Eckmann–Hilton argument* [10, 19, 1], two compatible group (or monoid) structures on an object G in a cartesian $(2,1)$-category coincide up to canonical 2-isomorphism, and their compatibility data translates into a braiding on the single underlying multiplication; for $k \geq 3$ compatible structures, the braiding becomes symmetric and no further data can be added. Let us formalize this via $\mathbb{E}_k$-algebras.

Let $\mathbf{H}$ be a cartesian monoidal $(2,1)$-category. From Lemma 3.4, we have the tower of forgetful functors:

$$\cdots \longrightarrow \mathrm{Grp}^k_2(\mathbf{H}) \longrightarrow \cdots \longrightarrow \mathrm{Grp}^3_2(\mathbf{H}) \longrightarrow \mathrm{Grp}^2_2(\mathbf{H}) \longrightarrow \mathrm{Grp}^1_2(\mathbf{H}) \longrightarrow \mathrm{Grp}^0_2(\mathbf{H}). \tag{3.5}$$

Define the corresponding tower of *k-tuply monoid objects* $\mathbf{Mon}^k_2(\mathbf{H})$ inductively by

$$\mathbf{Mon}^0_2(\mathbf{H}) := \mathbf{H}, \quad \mathbf{Mon}^1_2(\mathbf{H}) := \mathbf{Mon}_2(\mathbf{H}), \quad \mathbf{Mon}^k_2(\mathbf{H}) := \mathbf{Mon}_2(\mathbf{Mon}^{k-1}_2(\mathbf{H})) \ \ (k \geq 1).$$

Since $\mathrm{Grp}_2(\mathbf{C}) \subset \mathbf{Mon}_2(\mathbf{C})$ is the full sub-2-category of grouplike monoids for any cartesian $(2,1)$-category $\mathbf{C}$, each $\mathrm{Grp}^k_2(\mathbf{H})$ is a full sub-2-category of $\mathbf{Mon}^k_2(\mathbf{H})$ fitting into the pullback squares:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathbf{Mon}^k_2(\mathbf{H}) & \longrightarrow & \cdots & \longrightarrow & \mathbf{Mon}^3_2(\mathbf{H}) & \longrightarrow & \mathbf{Mon}^2_2(\mathbf{H}) & \longrightarrow & \mathbf{Mon}^1_2(\mathbf{H}) & \longrightarrow & \mathbf{Mon}^0_2(\mathbf{H}) \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \| \\
\cdots & \longrightarrow & \mathrm{Grp}^k_2(\mathbf{H}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Grp}^3_2(\mathbf{H}) & \longrightarrow & \mathrm{Grp}^2_2(\mathbf{H}) & \longrightarrow & \mathrm{Grp}^1_2(\mathbf{H}) & \longrightarrow & \mathrm{Grp}^0_2(\mathbf{H})
\end{array} \tag{3.6}$$

Theorem 3.5 (Eckmann–Hilton / Dunn Identification for k-Tuply Monoids)**.** *Let* $\mathbf{H}$ *be a cartesian monoidal* $(2,1)$*-category. For every* $k \geq 0$*, there is a 2-natural equivalence*

$$\mathbf{Mon}_2^k(\mathbf{H}) \simeq \mathbb{E}_k\mathrm{Alg}_2(\mathbf{H})$$

between k*-tuply monoid objects and* $\mathbb{E}_k$*-algebra objects (with* $\mathbb{E}_0\mathrm{Alg}_2$ *understood in the unpointed convention* $\mathbb{E}_0\mathrm{Alg}_2(\mathbf{H}) \simeq \mathbf{H}$ *for* $k = 0$*), such that the following square of forgetful 2-functors commutes up to 2-natural equivalence:*

$$\begin{array}{ccc} \mathbb{E}_k\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbb{E}_{k-1}\mathrm{Alg}_2(\mathbf{H}) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathbf{Mon}_2^k(\mathbf{H}) & \longrightarrow & \mathbf{Mon}_2^{k-1}(\mathbf{H}) \end{array}$$

Proof. We proceed by induction on $k \geq 1$. For $k = 1$, $\mathbf{Mon}_2^1(\mathbf{H}) = \mathbf{Mon}_2(\mathbf{H}) \simeq \mathbb{E}_1\mathrm{Alg}_2(\mathbf{H})$ by Example 2.9. Suppose the equivalence $\mathbf{Mon}_2^{k-1}(\mathbf{H}) \simeq \mathbb{E}_{k-1}\mathrm{Alg}_2(\mathbf{H})$ holds for $k-1 \geq 1$. By the universal property of the Boardman–Vogt tensor product of ∞-operads [28, §2.2.5] (applied with the convention of the proof of Theorem 2.8), for any ∞-operads $\mathcal{O}_1, \mathcal{O}_2$ and any symmetric monoidal $(2,1)$-category $\mathbf{H}$, there is a canonical equivalence

$$(\mathcal{O}_1 \odot \mathcal{O}_2)\mathrm{Alg}_2(\mathbf{H}) \simeq \mathcal{O}_1\mathrm{Alg}_2(\mathcal{O}_2\mathrm{Alg}_2(\mathbf{H})).$$

Taking $\mathcal{O}_1 = \mathbb{E}_1$ and $\mathcal{O}_2 = \mathbb{E}_{k-1}$, and applying the Dunn–Lurie Additivity Theorem 2.8 ($\mathbb{E}_1 \odot \mathbb{E}_{k-1} \simeq \mathbb{E}_k$), we obtain

$$\mathbb{E}_k\mathrm{Alg}_2(\mathbf{H}) \simeq (\mathbb{E}_1 \odot \mathbb{E}_{k-1})\mathrm{Alg}_2(\mathbf{H}) \simeq \mathbb{E}_1\mathrm{Alg}_2(\mathbb{E}_{k-1}\mathrm{Alg}_2(\mathbf{H})) \simeq \mathbf{Mon}_2(\mathbf{Mon}_2^{k-1}(\mathbf{H})) = \mathbf{Mon}_2^k(\mathbf{H}),$$

compatibly with the forgetful functors induced by $\mathbb{E}_0 \to \mathbb{E}_1$ (followed by the basepoint-forgetting projection $1/\mathbf{H} \to \mathbf{H}$ at the bottom level $k = 0$). □

Combining Theorem 3.5 with diagram (3.6) and the definition of groupal $\mathbb{E}_k$-algebras from Section 2.5, the vertical equivalences restrict to groupal objects:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{E}_k\mathrm{Alg}_2^{\mathrm{grp}}(\mathbf{H}) & \longrightarrow & \cdots & \longrightarrow & \mathbb{E}_2\mathrm{Alg}_2^{\mathrm{grp}}(\mathbf{H}) & \longrightarrow & \mathbb{E}_1\mathrm{Alg}_2^{\mathrm{grp}}(\mathbf{H}) \\ & & \simeq\downarrow & & & & \simeq\downarrow & & \simeq\downarrow \\ \cdots & \longrightarrow & \mathrm{Grp}_2^k(\mathbf{H}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Grp}_2^2(\mathbf{H}) & \longrightarrow & \mathrm{Grp}_2^1(\mathbf{H}) \\ & & \downarrow & & & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathbf{Mon}_2^k(\mathbf{H}) & \longrightarrow & \cdots & \longrightarrow & \mathbf{Mon}_2^2(\mathbf{H}) & \longrightarrow & \mathbf{Mon}_2^1(\mathbf{H}) \longrightarrow \mathbf{Mon}_2^0(\mathbf{H}) \\ & & \uparrow\simeq & & & & \uparrow\simeq & & \uparrow\simeq \quad \| \\ \cdots & \longrightarrow & \mathbb{E}_k\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \cdots & \longrightarrow & \mathbb{E}_2\mathrm{Alg}_2(\mathbf{H}) & \longrightarrow & \mathbb{E}_1\mathrm{Alg}_2(\mathbf{H}) \longrightarrow \mathbf{H} \end{array}$$

Putting Lemma 3.4 and Theorem 3.5 together yields the complete algebraic characterization of k-tuply deloopable objects:

Theorem 3.6. *For every cartesian* $(2,1)$*-cosmos* $\mathbf{H}$ *and every* $k \geq 1$*, there are 2-natural equivalences of* $(2,1)$*-categories*

$$\mathrm{Del}_2^k(\mathbf{H}) \simeq \mathrm{Grp}_2^k(\mathbf{H}) \simeq \mathbb{E}_k\mathrm{Alg}_2^{\mathrm{grp}}(\mathbf{H}).$$

Joining Theorem 3.6 with the Stabilization Theorem 2.10 and Example 2.9, we obtain:

Corollary 3.7. *Let* $X \in \mathbf{H}$ *be an object of a Grothendieck* $(2,1)$*-topos (or cartesian* $(2,1)$*-cosmos as above). Then:*

(1) X *is deloopable* ($k = 1$) *if and only if* X *admits the structure of a group object;*
(2) X *is doubly deloopable* ($k = 2$) *if and only if* X *admits the structure of a braided group object;*

(3) X is triply deloopable ($k = 3$) if and only if X admits the structure of a commutative group object;
(4) X is k-tuply deloopable for $k > 3$ if and only if X is triply deloopable;
(5) X is infinitely deloopable ($X \in \mathrm{Del}_2^\infty(\mathbf{H})$) if and only if X admits the structure of a commutative group object.

3.7. **Looping vs. Delooping.** Recall that a *based (or pointed) object* in a cartesian $(2,1)$-category $\mathbf{H}$ with terminal object $*$ is a 1-morphism $x : * \to X$, i.e., an object of the coslice $(2,1)$-category $*/\mathbf{H}$. Equivalently, it is an object X endowed with a global element. There is a canonical projection 2-functor $\pi : */\mathbf{H} \to \mathbf{H}$ that forgets the basepoint.

Assuming $\mathbf{H}$ has $(2,1)$-pullbacks, we define the *looping 2-functor* $\Omega : */\mathbf{H} \to \mathbf{H}$ by taking the homotopy self-intersection ($(2,1)$-pullback) of the basepoint $* \xrightarrow{x} X$ along itself, equipped with its universal loop 2-isomorphism $\lambda_X : x\circ! \Rightarrow x\circ!$:

$$\begin{array}{ccc} \Omega X & \xrightarrow{\ !\ } & * \\ {\scriptstyle !}\downarrow & \Downarrow \lambda_X & \downarrow{\scriptstyle x} \\ * & \xrightarrow[x]{} & X \end{array}$$

We call $\Omega X = * \times_X *$ the *looping object* of (X, x). Observe that the pullback diagram above is precisely the diagram defining the Čech groupoid $\check{C}_{\mathbf{H}}(x)$ of the basepoint morphism $x : * \to X$. Since the object of objects of $\check{C}_{\mathbf{H}}(x)$ is $\check{C}_{\mathbf{H}}(x)_0 = *$, the internal groupoid $\check{C}_{\mathbf{H}}(x) = (\Omega X \rightrightarrows *)$ is a one-object internal groupoid, and hence is the internal delooping $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(\Omega X)$ of a canonical group object structure on ΩX (where multiplication is induced by vertical pasting of the loop 2-cell λ_X in the pullback). Moreover, because $\mathbf{H}$ is a $(2,1)$-category (so its hom-groupoids are 1-truncated), the homotopy pullback $\Omega X = * \times_X *$ is automatically 0-truncated ($\Omega X \in \tau_{\leq 0}\mathbf{H}$). Consequently, the looping 2-functor Ω factors canonically through 0-truncated group objects $\mathrm{Grp}(\tau_{\leq 0}\mathbf{H}) \subset \mathrm{Grp}_2(\mathbf{H})$:

$$\begin{array}{ccc} & & \mathrm{Grp}(\tau_{\leq 0}\mathbf{H}) & \hookrightarrow & \mathrm{Grp}_2(\mathbf{H}) \\ & {\scriptstyle \Omega}\nearrow & & & \downarrow \\ */\mathbf{H} & & \xrightarrow[\Omega]{} & & \mathbf{H} \end{array}$$

Conversely, for every group object $G \in \mathrm{Grp}_2(\mathbf{H})$ in a cartesian $(2,1)$-cosmos, its internal delooping object $B_{\mathbf{H}}G = \mathrm{coeq}(G \rightrightarrows *)$ comes equipped with the universal quotient morphism $p_G : * \twoheadrightarrow B_{\mathbf{H}}G$, which is a canonical global element of $B_{\mathbf{H}}G$. Thus, $B_{\mathbf{H}}$ factors canonically through the pointed category $*/\mathbf{H}$:

$$\begin{array}{ccc} B_{\mathbf{H}}G & \xleftarrow{p_G} & * \\ {\scriptstyle p_G}\uparrow & & \uparrow \\ * & \xleftarrow{\quad} & G \end{array} \qquad \begin{array}{ccc} \mathbf{H} & \xleftarrow{B_{\mathbf{H}}} & \mathrm{Grp}_2(\mathbf{H}) \\ {\scriptstyle \pi}\uparrow & \swarrow_{B_{\mathbf{H}}} & \\ */\mathbf{H} & & \end{array}$$

Combining both factorizations yields the adjoint pair $B_{\mathbf{H}} \dashv \Omega$ between $\mathrm{Grp}_2(\mathbf{H})$ (or $\mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$) and $*/\mathbf{H}$:

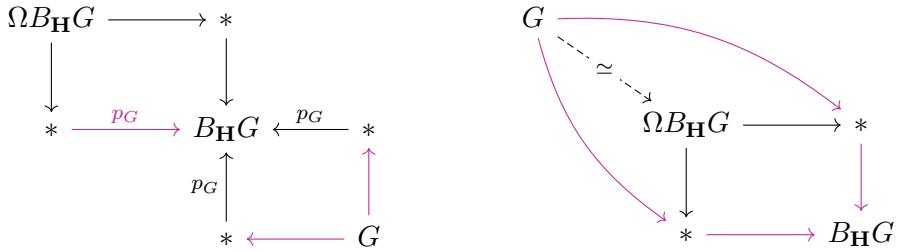
$$\begin{array}{ccc} \mathbf{H} & \xleftarrow{B_{\mathbf{H}}} & \mathrm{Grp}_2(\mathbf{H}) \\ {\scriptstyle \pi}\uparrow & {\scriptstyle B_{\mathbf{H}}}\swarrow\ \ \nearrow{\scriptstyle \Omega} & \downarrow \\ */\mathbf{H} & \xrightarrow[\Omega]{} & \mathbf{H} \end{array}$$

When $\mathbf{H}$ is a Grothendieck $(2,1)$-topos, every internal groupoid with 0-truncated morphism fibers is effective in $\mathbf{H}$. Thus, for every 0-truncated group object $G \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$ (while for a general 2-group $G \in \mathrm{Grp}_2(\mathbf{H})$, $\Omega B^{\mathrm{quot}}_{\mathbf{H}}G \simeq \tau_{\leq 0}G$ unless $B_{\mathbf{H}}G$ is taken in the ambient $(3,1)$-topos

$\mathbf{Grpd}_2(\mathbf{H})$ as in Remark 3.3), effectiveness means that $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G) \xrightarrow{\sim} \check{C}(* \twoheadrightarrow B_{\mathbf{H}}G) = (\Omega B_{\mathbf{H}}G \rightrightarrows *)$, so the unit of $B_{\mathbf{H}} \dashv \Omega$ is a 2-natural equivalence:

$$G \xrightarrow{\sim} \Omega B_{\mathbf{H}}G,$$

as illustrated by the universal pullback comparison:



On the other hand, for a pointed object $(X, x : * \to X) \in */\mathbf{H}$, the counit $B_{\mathbf{H}}\Omega X \to X$ is an equivalence if and only if the basepoint $x : * \to X$ is an *effective epimorphism* (so that X is the quotient of its Čech groupoid $\check{C}(x) = \mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(\Omega X)$). A pointed object (X, x) whose basepoint $* \twoheadrightarrow X$ is an effective epimorphism (equivalently, $\tau_{\leq 0}X \cong *$) is called *0-connected* (or simply *connected*). Thus, for an arbitrary pointed object $X \in */\mathbf{H}$ and group object $G \in \mathrm{Grp}_2(\mathbf{H})$, $B^{\mathrm{quot}}_{\mathbf{H}}$ and Ω inside $\mathbf{H}$ fail to be mutually inverse in general for two reasons:

(1) In an arbitrary cartesian $(2,1)$-cosmos $\mathbf{H}$, internal delooping groupoids $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)$ need not be effective (and even in a Grothendieck $(2,1)$-topos $\mathbf{H}$, the 1-truncated quotient $B^{\mathrm{quot}}_{\mathbf{H}}G \in \mathbf{H}$ is effective in $\mathbf{H}$ if and only if $G \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$ is 0-truncated);
(2) Even when all 0-truncated groupoids are effective, $B_{\mathbf{H}}\Omega X \to X$ is an equivalence only on the connected component of the chosen basepoint $* \to X$.

In a Grothendieck $(2,1)$-topos $\mathbf{H}$, restricting to 0-truncated group objects $\mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$ and to the full sub-2-category $\mathrm{Conn}(*/\mathbf{H}) \subset */\mathbf{H}$ of 0-connected pointed objects resolves both obstructions:

Theorem 3.8 (Looping–Delooping Equivalence [29, Lemma 7.2.2.11])**.** *For every Grothendieck $(2,1)$-topos $\mathbf{H}$, the internal delooping 2-functor $B_{\mathbf{H}}$ factors through connected pointed objects and establishes an equivalence of categories*

$$B_{\mathbf{H}} : \mathrm{Grp}(\tau_{\leq 0}\mathbf{H}) \xrightarrow{\simeq} \mathrm{Conn}(*/\mathbf{H}),$$

with quasi-inverse given by the looping 2-functor $\Omega : \mathrm{Conn}(*/\mathbf{H}) \xrightarrow{\simeq} \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$ (while on all of $\mathrm{Grp}_2(\mathbf{H})$, delooping gives an equivalence onto connected pointed objects of the $(3,1)$-topos $\tau_{\leq 2}\mathbf{H}_\infty$, i.e. of the localization of $\mathbf{Grpd}_2(\mathbf{H})$ at weak equivalences, cf. Remark 2.6).

Proof. This is the case $n = 2$ of the equivalence between group objects and pointed connected objects in an ∞-topos [29, Lemma 7.2.2.11], restricted to 1-truncated objects. Explicitly, for any $G \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$, the coequalizer morphism $* \twoheadrightarrow B_{\mathbf{H}}G$ is an effective epimorphism, so $(B_{\mathbf{H}}G, p_G) \in \mathrm{Conn}(*/\mathbf{H})$, and effectiveness of $\mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(G)$ gives $G \xrightarrow{\sim} \Omega B_{\mathbf{H}}G$. Conversely, for $(X, x) \in \mathrm{Conn}(*/\mathbf{H})$, $x : * \twoheadrightarrow X$ is an effective epimorphism with 0-truncated kernel pair $\Omega X \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$, so $X \simeq \mathrm{cod}(\mathrm{coeq}(\check{C}(x))) = B_{\mathbf{H}}(\Omega X)$. □

Remark 3.9 (Higher Looping–Delooping Equivalence). In the ambient $(\infty,1)$-topos $\mathbf{H}_\infty$ of which $\mathbf{H} = \tau_{\leq 1}\mathbf{H}_\infty$ is the 1-truncated subtopos (or via iterated internal groupoids $\mathbf{Grpd}^k_2(\mathbf{H})$ as in Remark 3.3), Theorem 3.8 extends by induction to k-tuply groupal objects for all $k \geq 1$ [28, Theorem 5.2.6.10]: defining a pointed object (X, x) to be $(k-1)$-*connected* if $* \to X$ is an effective epimorphism and ΩX is $(k-2)$-connected, the k-fold delooping B^k and k-fold looping Ω^k establish an equivalence

$$B^k : \mathrm{Del}^k_2(\mathbf{H}) \simeq \mathrm{Grp}^k_2(\mathbf{H}) \simeq \mathbb{E}_k\mathrm{Alg}^{\mathrm{grp}}_2(\mathbf{H}) \xrightarrow{\simeq} \mathrm{Conn}^{\leq k+1}_{k-1}(*/\mathbf{H}_\infty)$$

between groupal $\mathbb{E}_k$-algebras in $\mathbf{H}$ and $(k-1)$-connected, $(k+1)$-truncated pointed objects in $\mathbf{H}_\infty$ (note that inside $\mathbf{H} = \tau_{\leq 1}\mathbf{H}_\infty$, every object is 1-truncated, so a 1-connected object inside $\mathbf{H}$ is already contractible).

4. Cohesive 2-Topoi

Having established the general theory of Grothendieck $(2,1)$-topoi and delooping objects, we now introduce the axiomatic structure of *cohesion* [25, 43]. Whereas an arbitrary Grothendieck $(2,1)$-topos behaves like a category of generalized spaces on which one can do topology and sheaf theory, a *cohesive* $(2,1)$*-topos* carries an adjoint triple of internal modalities $\int \dashv \flat \dashv \sharp$ that enriches its internal logic with the synthetic contrast between discrete underlying sets of points, constant coefficients, and continuous/smooth geometric cohesion. In Section 4.1 we define cohesive geometric morphisms and their induced modalities. In Section 4.2 we show how the progressive combination of $\int$, $\flat$, and $\sharp$ internalizes principal bundles, flat and closed differential forms, principal connections, and internal moduli stacks of gauge fields. Finally, in Section 4.3 we synthesize these constructions into intrinsic, modal, and differential $(2,1)$-cohomology.

4.1. **Cohesive Modalities.** Let $f = (f_*, f^*) : \mathbf{H} \to \mathbf{H}'$ be a geometric morphism between $(2,1)$-pretopoi (or Grothendieck $(2,1)$-topoi).

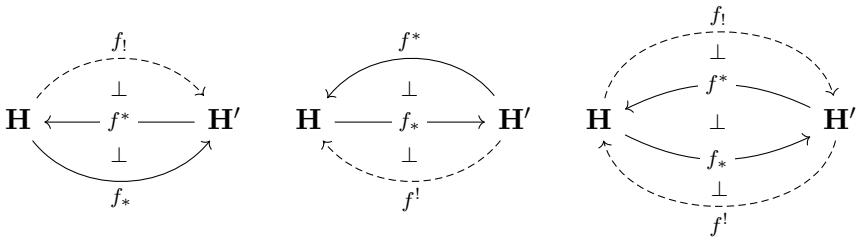
- We say that f is *1-connected* if the inverse image 2-functor $f^* : \mathbf{H}' \to \mathbf{H}$ is fully faithful.
- If f^* admits a further left 2-adjoint $f_! : \mathbf{H} \to \mathbf{H}'$, we say that f is *locally 1-connected* (first diagram below).
- If f is locally 1-connected and $f_!$ preserves finite $(2,1)$-products, we say that f is *strongly 1-connected* (or *product-preserving locally 1-connected*).

Every 1-connected locally 1-connected geometric morphism satisfies $f_! f^* \simeq \mathrm{id}_{\mathbf{H}'}$; in particular, a locally 1-connected geometric morphism $(f_!, f^*, f_*)$ is 1-connected if and only if the counit $f_! f^* \Rightarrow \mathrm{id}_{\mathbf{H}'}$ is an equivalence (and when $\mathbf{H}' = \mathbf{Grpd}_2$, if and only if $f_!$ preserves the terminal object $*$ [17, 43]). Thus, when $\mathbf{H}' = \mathbf{Grpd}_2$ (where preservation of the empty product, i.e. of the terminal object, characterizes 1-connectedness), we have the hierarchy of inclusions:

$$\text{strongly 1-connected} \subset \text{1-connected} \subset \text{locally 1-connected}. \tag{4.1}$$

Dually:

- If the direct image $f_* : \mathbf{H} \to \mathbf{H}'$ admits a further right 2-adjoint $f^! : \mathbf{H}' \to \mathbf{H}$, we say that f is *locally local* (second diagram below).
- If f is locally local and $f^!$ is fully faithful, we say that f is *local*. By general properties of adjoint triples $(f^* \dashv f_* \dashv f^!)$, $f^!$ is fully faithful if and only if f^* is fully faithful; hence a locally local geometric morphism is local if and only if it is 1-connected [17].
- Finally, we say that f is *locally cohesive* (third diagram below) if it is both locally 1-connected and locally local, so that we have an adjoint quadruple $f_! \dashv f^* \dashv f_* \dashv f^!$; and we say that f is *cohesive* if, in addition, it is strongly 1-connected and local.

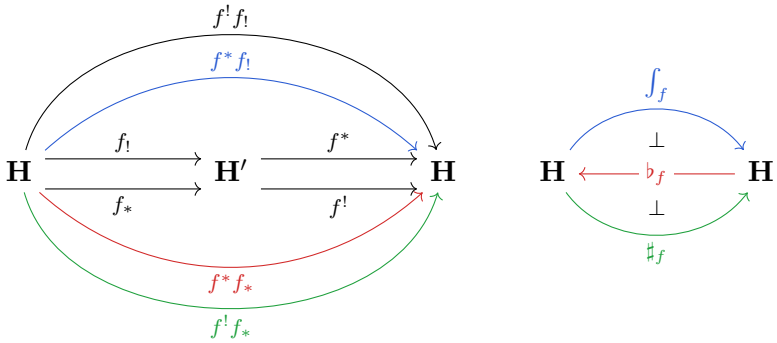


Given a $(2,1)$-pretopos $\mathbf{H}$, endo-2-functors $M : \mathbf{H} \to \mathbf{H}$ (typically idempotent monads or comonads) are called *internal modalities* on $\mathbf{H}$. For $X \in \mathbf{H}$, we say that $M(X) \in \mathbf{H}$ represents X in the modality M. Every adjoint pair $(L \dashv R) : \mathbf{C} \to \mathbf{D}$ between 2-categories defines an endofunctor $L \circ R : \mathbf{C} \to \mathbf{C}$ (a comonad on $\mathbf{C}$). In particular, every geometric morphism $f : \mathbf{H} \to \mathbf{H}'$ induces the *f-flat modality* (a lex idempotent comonad when f is 1-connected):

$$\flat_f := f^* \circ f_* : \mathbf{H} \longrightarrow \mathbf{H}.$$

Furthermore, an adjoint triple $L' \dashv L \dashv R$ with $L : \mathbf{A} \to \mathbf{B}$ induces adjoint pairs of endofunctors $(L' \circ L) \dashv (R \circ L)$ on $\mathbf{A}$ and $(L \circ L') \dashv (L \circ R)$ on $\mathbf{B}$. Consequently (applying the second form to $f_! \dashv f^* \dashv f_*$, with $L = f^*$, and the first to $f^* \dashv f_* \dashv f^!$, with $L = f_*$):

- Every locally 1-connected geometric morphism $(f_!, f^*, f_*) : \mathbf{H} \to \mathbf{H}'$ induces an adjoint pair of internal modalities $\int_f \dashv \flat_f$, where
$$\int_f := f^* \circ f_! : \mathbf{H} \longrightarrow \mathbf{H}$$
is called the *f-shape modality* (an idempotent monad when f is 1-connected).
- Every locally local geometric morphism $(f^*, f_*, f^!) : \mathbf{H} \to \mathbf{H}'$ induces an adjoint pair of internal modalities $\flat_f \dashv \sharp_f$, where
$$\sharp_f := f^! \circ f_* : \mathbf{H} \longrightarrow \mathbf{H}$$
is called the *f-sharp modality* (an idempotent monad when f is local).
- More generally, in a locally cohesive (and in particular a cohesive) geometric morphism $(f_! \dashv f^* \dashv f_* \dashv f^!) : \mathbf{H} \to \mathbf{H}'$, the adjoint quadruple induces an adjoint triple of internal modalities on $\mathbf{H}$:[6]
$$\int_f \dashv \flat_f \dashv \sharp_f : \mathbf{H} \longrightarrow \mathbf{H}.$$



Recall that $\mathbf{Grpd}_2 \simeq (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(*)$ is the terminal object in the 2-category of $(2,1)$-presheaf topoi $(2,1)\mathbf{PShvTopos}_2$. Specifically, for every $\mathbf{H} = (2,1)\mathrm{PShv}_{\mathbf{Grpd}_2}(\mathbf{C})$, there is an essentially unique *terminal geometric morphism*
$$(\Gamma, \mathrm{Cst}) = (\mathrm{El}, \mathrm{Cst}) : \mathbf{H} \longrightarrow \mathbf{Grpd}_2,$$
whose direct image is the *global sections* (or *global elements*) 2-functor $\Gamma = \mathrm{El} := \mathbf{H}(*_{\mathbf{H}}, -) \simeq \lim_{\mathbf{C}^{\mathrm{op}}} : \mathbf{H} \to \mathbf{Grpd}_2$, and whose inverse image is the *constant presheaf* 2-functor $\mathrm{Cst} : \mathbf{Grpd}_2 \to \mathbf{H}$ sending $K \in \mathbf{Grpd}_2$ to the constant presheaf $U \mapsto K$. Since the stackification localization $L_J : \mathbf{H} \to \mathbf{H}_0 = \mathbf{Stk}(\mathbf{C}, J)$ defining a Grothendieck $(2,1)$-topos is left exact, $\mathbf{Grpd}_2$ remains the terminal object in $(2,1)\mathbf{ShvTopos}_2$, with terminal geometric morphism $(\Gamma, \mathrm{LCst}) : \mathbf{H}_0 \to \mathbf{Grpd}_2$ given by
$$\Gamma := \mathbf{H}_0(*_{\mathbf{H}_0}, -) = \mathrm{El} \circ \imath, \qquad \mathrm{LCst} := L_J \circ \mathrm{Cst}.$$
A Grothendieck $(2,1)$-topos $\mathbf{H}$ is called *locally 1-connected*, *strongly 1-connected*, *locally local*, *local*, *locally cohesive*, or *cohesive* (over $\mathbf{Grpd}_2$) if its terminal geometric morphism $(\Gamma, \mathrm{LCst}) : \mathbf{H} \to \mathbf{Grpd}_2$ has the corresponding property, with extra adjoints denoted $\Pi \dashv \mathrm{LCst} \dashv \Gamma \dashv \mathrm{coDisc}$. In a cohesive Grothendieck $(2,1)$-topos $\mathbf{H}$, we work with the canonical modalities
$$\int = \mathrm{LCst} \circ \Pi \dashv \flat = \mathrm{LCst} \circ \Gamma \dashv \sharp = \mathrm{coDisc} \circ \Gamma$$
induced by the terminal geometric morphism.

[6] One can also form the fourth composite endofunctor $f^! \circ f_! : \mathbf{H} \to \mathbf{H}$ (indicated by the top curved arrow in the left diagram below), though unlike $\int_f \dashv \flat_f \dashv \sharp_f$ it is not part of the adjoint modality string and only $\int_f \dashv \flat_f \dashv \sharp_f$ is required for differential geometry.

4.2. **Geometric Structures.** In the archetypal model for classical gauge theories—the cohesive Grothendieck $(2,1)$-topos $\mathbf{H} = \mathbf{SmoothGrpd}_2 = \mathbf{Stk}(\mathbf{CartSp}, J_{\text{open}})$ of smooth 1-groupoids (stacks on the site of Cartesian spaces $\mathbb{R}^n$ and smooth maps with the open cover topology [43, 11])—the three cohesive modalities play the following distinct geometric roles:

- The *shape modality* $\int$ forgets smooth structure down to fundamental path groupoids (homotopy types) and is used to formalize *principal bundles* and their topological realization;
- The *flat modality* $\flat$ sends a smooth stack to the constant stack on its underlying discrete groupoid of points and is used to formalize *flat differential forms* and local systems;
- The *sharp modality* $\sharp$ sends a smooth stack to the codiscrete smooth stack on its points (into which any smooth plot maps uniquely once it maps on bare points) and is used to adjust internal mapping stacks into *differential moduli spaces*.

Consequently, their combinations yield:

- *Shape + Flat* ($\int \dashv \flat$): formalizes universal *Maurer–Cartan forms*, *connection forms*, and *curvature forms* on principal bundles;
- *Shape + Flat + Sharp* ($\int \dashv \flat \dashv \sharp$): formalizes the *internal differential moduli stacks of connection forms* $G\mathrm{Conn}^k(X) \in \mathbf{H}$.

Because this internal moduli stack of connection forms serves as the domain of the action functional $\exp(iS) : G\mathrm{Conn}^k(X) \to U(1)$ of a classical gauge theory on a closed n-dimensional X (obtained by transgression of a Lagrangian $B_{\text{con}}G \to B^n_{\text{con}}U(1)$), the synthetic dictionary above shows that the kinematics of gauge theories can be completely internalized inside any cohesive $(2,1)$-topos [43, 42]. (Adding infinitesimal/jet modalities further internalizes variational calculus and field dynamics [21]; here we focus on the kinematic and cohomological structures.) For a comprehensive $(\infty,1)$-categorical treatment, see [43, Sections 3.7–3.9, 5.1–5.2].

4.2.1. *Bundles:* $\int$. Let $\mathbf{Top}_2 = \mathrm{disc}_{1,2}(\mathbf{Top})$ denote the 1-category of (compactly generated weak Hausdorff) topological spaces and continuous maps viewed as a discrete $(2,1)$-category, and let $2\mathbf{Top}_2$ denote the $(2,1)$-category of topological spaces, continuous maps, and homotopy classes of homotopies between them; its 1-truncation $\tau_{\leq 1}(2\mathbf{Top}_2)$ is the classical homotopy 1-category $\mathrm{Ho}(\mathbf{Top})$. In classical bundle theory, isomorphism classes of principal G-bundles over a paracompact space X are in natural bijection $\mathrm{Iso}_G(X) \cong [X, BG]$ via pullback of the universal principal G-bundle $u_G : EG \twoheadrightarrow BG$.

This universal bundle admits a direct interpretation in terms of internal groupoids. Let $\mathbf{Grpd}_2(\mathbf{Top}_2)$ be the $(2,1)$-category of topological groupoids, equipped with the forgetful functor to $\mathbf{Grpd}_2(\mathbf{Set}_2) = \mathbf{Grpd}_2$. There is a classical *geometric realization 2-functor*

$$|-| : \mathbf{Grpd}_2 \longrightarrow 2\mathbf{Top}_2 \quad (\text{or } |-|_{\mathbf{Top}_2} : \mathbf{Grpd}_2(\mathbf{Top}_2) \longrightarrow 2\mathbf{Top}_2),$$

whose essential image lands in homotopy 1-types (CW complexes with $\pi_{>1}(-) = 0$, i.e., disjoint unions of Eilenberg–Mac Lane spaces of type $K(\pi,1)$). To every topological group $G \in \mathrm{Grp}_2(\mathbf{Top}_2)$, one associates the internal action (or pair) groupoid $\mathbf{E}G = (G \times G \rightrightarrows G)$ (which is equivalent to the terminal groupoid $*$, and should not be confused with the discrete groupoid $\mathbf{E}_X = (X \rightrightarrows X)$ of Section 2.3) and the one-object internal delooping groupoid $\mathbf{B}^{\text{int}}_{\mathbf{Top}_2}(G) = (G \rightrightarrows *)$, linked by the canonical quotient functor $\mu_G : \mathbf{E}G \to \mathbf{B}^{\text{int}}_{\mathbf{Top}_2}(G)$ (or, on object-of-objects inclusions, $\mathrm{disc}_{1,2}(*) \to \mathbf{B}^{\text{int}}_{\mathbf{Top}_2}(G)$). Its geometric realization $|\mu_G|$ in $2\mathbf{Top}_2$ is precisely the universal principal G-bundle $u_G : EG \to BG$.

On the other hand, in a Grothendieck $(2,1)$-topos $\mathbf{H}$, every internal groupoid with 0-truncated morphism fibers is effective and thus completely determined by its representing quotient object. Hence, for a 0-truncated group object G in a Grothendieck $(2,1)$-topos $\mathbf{H}$, the universal G-bundle is axiomatized internally simply by the canonical basepoint morphism $* \twoheadrightarrow B_{\mathbf{H}}G$, and principal G-bundles over $X \in \mathbf{H}$ are axiomatized as homotopy pullbacks of $* \twoheadrightarrow B_{\mathbf{H}}G$:

$$\begin{array}{ccc} P & \longrightarrow & * \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p_G} \\ X & \xrightarrow[f]{} & B_{\mathbf{H}}G \end{array}$$

Definition 4.1 (Abstract Principal G-Bundle)**.** Let $\mathbf{H}$ be a Grothendieck $(2,1)$-topos and let $G \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$ be a 0-truncated group object (for a general 2-group $G \in \mathrm{Grp}_2(\mathbf{H})$ one has only $\Omega B_{\mathbf{H}} G \simeq \tau_{\leq 0} G$ inside $\mathbf{H}$, so the fiber below does not carry a G-action; see Remark 3.3). A 1-morphism $f : X \to B_{\mathbf{H}}G$ in $\mathbf{H}$ is said to *classify an abstract principal G-bundle over X*. The *principal G-bundle classified by f* is the $(2,1)$-pullback morphism $\pi : P \to X$ in the diagram above (equipped with its canonical G-action inherited from $\Omega B_{\mathbf{H}} G \simeq G$).

To relate an "abstract bundle" $P \to X$ in $\mathbf{H}$ to a concrete topological bundle in $2\mathbf{Top}_2$, we require an *abstract geometric realization 2-functor* $|-|_{\mathbf{H}} : \mathbf{H} \to 2\mathbf{Top}_2$ that sends abstract universal G-bundles to concrete universal bundles. Thus, $|-|_{\mathbf{H}}$ must preserve group objects, terminal objects, and delooping objects (which are $(2,1)$-colimits of simplicial diagrams), and should factor through the classical geometric realization $|-| : \mathbf{Grpd}_2 \to 2\mathbf{Top}_2$ via a 2-functor $F : \mathbf{H} \to \mathbf{Grpd}_2$:

$$\mathbf{H} \xrightarrow[F]{} \mathbf{Grpd}_2 \xrightarrow[|-|]{} 2\mathbf{Top}_2 \qquad \text{with } |-|_{\mathbf{H}} : \mathbf{H} \to 2\mathbf{Top}_2 \tag{4.2}$$

Caution is needed here: the classical geometric realization $|-| : \mathbf{Grpd}_2 \to 2\mathbf{Top}_2$ is *right* adjoint (up to homotopy) to the fundamental groupoid Π_1, i.e. it is the inclusion of homotopy 1-types, $\Pi_1 = \tau_{\leq 1} \dashv |-|$. It preserves finite products and limits but *not* $(2,1)$-colimits computed in $\mathbf{Grpd}_2$: e.g. the pushout $* \sqcup_{B\mathbb{Z}} *$ is $*$ in $\mathbf{Grpd}_2$ but S^2 in spaces. Consequently, $|-|_{\mathbf{H}} = |-| \circ F$ preserves group objects and terminal objects as soon as F preserves finite products, and it sends delooping objects $B_{\mathbf{H}}G$ to $|F(B_{\mathbf{H}}G)| \simeq \tau_{\leq 1} B|F(G)|$ when F preserves finite products and $(2,1)$-colimits, which is the classifying space $B|F(G)|$ only when $|F(G)|$ is homotopy discrete.

In a *strongly 1-connected* Grothendieck $(2,1)$-topos $\mathbf{H}$, the extra left adjoint $\Pi : \mathbf{H} \to \mathbf{Grpd}_2$ to the terminal inverse image LCst preserves all $(2,1)$-colimits (being a left adjoint) and preserves finite products (by definition of strong 1-connectivity). This motivates setting $F = \Pi$:

$$\mathbf{H} \underset{\Gamma}{\overset{\Pi}{\leftarrow}} \mathbf{Grpd}_2 \xrightarrow[|-|]{} 2\mathbf{Top}_2 \qquad \text{with } \mathbf{H} \xleftarrow{\mathrm{LCst}} \mathbf{Grpd}_2 \text{ and } |-|_{\mathbf{H}} : \mathbf{H} \to 2\mathbf{Top}_2$$

Note that the internal endofunctor corresponding to Π inside $\mathbf{H}$ is the shape modality $\int = \mathrm{LCst} \circ \Pi$.

Definition 4.2 (Topological Realization)**.** Let $\mathbf{H}$ be a strongly 1-connected Grothendieck $(2,1)$-topos and let $G \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$. The *topological realization* of an abstract G-bundle $P \to X$ classified by $f : X \to B_{\mathbf{H}}G$ is the map

$$|f|_{\mathbf{H}} : |X|_{\mathbf{H}} \longrightarrow |B_{\mathbf{H}}G|_{\mathbf{H}} \simeq \tau_{\leq 1} B|G|_{\mathbf{H}}.$$

Remark 4.3 (Truncation of the shape). If $\mathbf{H} = \tau_{\leq 1}\mathbf{H}_\infty$ is the $(2,1)$-truncation of a cohesive ∞-topos with shape Π_∞, then the shape of $\mathbf{H}$ is $\Pi = \tau_{\leq 1} \circ \Pi_\infty$ (both are left adjoint to LCst on 1-types). Hence the realization above only remembers the 1-type of $B|G|$. For instance, in $\mathbf{SmoothGrpd}_2$ one has $\Pi_\infty(B_{\mathbf{H}}U(1)) \simeq K(\mathbb{Z},2)$, so $\Pi(B_{\mathbf{H}}U(1)) \simeq \tau_{\leq 1}K(\mathbb{Z},2) \simeq *$: the $(2,1)$-truncated shape cannot see the first Chern class of a circle bundle, and $H^1_{\int}(X;U(1))$ in the sense of Section 4.3.2 is trivial. Recovering characteristic classes of Lie-group bundles therefore requires the shape of the ambient ∞-topos (or at least of $\tau_{\leq n}\mathbf{H}_\infty$ for n large enough). This is an intrinsic limitation of the $(2,1)$-categorical setting and should be weighed against the advantages listed in the Introduction.

Remark 4.4. Observe that, a priori, the topological bundle classified by $|f|_{\mathbf{H}}$ (which is the pullback of $E|G|_{\mathbf{H}} \to B|G|_{\mathbf{H}}$ along $|f|_{\mathbf{H}}$) need not coincide with $|P|_{\mathbf{H}} \to |X|_{\mathbf{H}}$ for an arbitrary classifying map f. Indeed, while $|-| : \mathbf{Grpd}_2 \to 2\mathbf{Top}_2$ preserves finite $(2,1)$-limits, the shape functor $\Pi : \mathbf{H} \to \mathbf{Grpd}_2$ of a strongly 1-connected $(2,1)$-topos is only guaranteed to preserve finite products, not arbitrary $(2,1)$-pullbacks. However, on smooth manifolds and locally trivial

fiber bundles (where Π commutes with pullbacks along submersion/fibration projections by the Seifert–van Kampen theorem for Π), $|P|_{\mathbf{H}} \to |X|_{\mathbf{H}}$ does agree with the pullback bundle [43, Proposition 4.3.13]. A $(2,1)$-topos for which Π preserves all finite $(2,1)$-limits is called *totally 1-connected* (or Π-*lex*).

Replacing group objects with k-tuply deloopable objects (with $B^k_{\mathbf{H}}G$ understood in the lossless sense of Remark 3.3) yields higher bundles (such as bundle gerbes for $k=2$):

Definition 4.5 (Abstract k-Bundles)**.** Let $\mathbf{H}$ be a cartesian $(2,1)$-cosmos and let $G \in \mathrm{Del}^k_2(\mathbf{H})$ be a k-tuply deloopable object ($k \geq 1$). With respect to $(2,1)$-truncation, for $k=1$, $B_{\mathbf{H}}G \in \mathbf{H}$ is internal to $\mathbf{H}$ and classifies ordinary principal G-bundles; for $k=2$ and $A \in \mathrm{ComGrp}(\tau_{\leq 0}\mathbf{H})$, $B^2_{\mathbf{H}}A = \mathbf{B}^{\mathrm{int}}_{\mathbf{H}}(B_{\mathbf{H}}A) \in \mathbf{Grpd}_2(\mathbf{H})$ is an internal 2-stack in $\mathbf{H}$ classifying A-bundle gerbes; while for $k \geq 3$, $B^k_{\mathbf{H}}G$ is an object of the ambient $(\infty,1)$-topos $\mathbf{H}_\infty$ with 1-truncated mapping groupoid $\mathbf{H}(X, B^k_{\mathbf{H}}G) := \tau_{\leq 1}\mathbf{H}_\infty(X, B^k G)$. An *abstract G-k-bundle* over $X \in \mathbf{H}$ is a morphism $P_k \to X$ obtained as the $(2,1)$-pullback of $* \to B^k_{\mathbf{H}}G$ along a classifying morphism $f : X \to B^k_{\mathbf{H}}G$:

$$\begin{array}{ccc} P_k & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & B^k_{\mathbf{H}}G \end{array}$$

When $\mathbf{H}$ is a strongly 1-connected $(2,1)$-topos, the *topological realization* of $P_k \to X$ requires the shape of the ambient ∞-topos, $|f| : |X| \to |\Pi_\infty B^k G| \simeq B^k |\Pi_\infty G|$; the $(2,1)$-truncated shape Π only retains its 1-truncation, which is contractible for $k \geq 2$ and G 0-truncated (Remark 4.3).

4.2.2. *Differential Forms: $\flat$.* Let $(\Gamma, \mathrm{LCst}) : \mathbf{H} \to \mathbf{Grpd}_2$ be a Grothendieck $(2,1)$-topos and let $\flat = \mathrm{LCst} \circ \Gamma : \mathbf{H} \to \mathbf{H}$ be its flat modality. The counit of the adjunction $\mathrm{LCst} \dashv \Gamma$ defines a canonical 2-natural transformation $\epsilon : \flat \Rightarrow \mathrm{id}_{\mathbf{H}}$, assigning to every object $A \in \mathbf{H}$ the canonical inclusion of constant points $\epsilon_A : \flat A \to A$.

Whenever A is a pointed object $* \to A$ (and in particular whenever $A \in \mathrm{Grp}_2(\mathbf{H})$ or $A = B^k_{\mathbf{H}}G$), let $i_A : \flat_{\mathrm{dR}} A \to \flat A$ denote the *homotopy kernel* (homotopy fiber over the basepoint) of $\epsilon_A : \flat A \to A$, defined by the $(2,1)$-pullback square:

$$\begin{array}{ccc} \flat_{\mathrm{dR}} A & \xrightarrow{i_A} & \flat A \\ \downarrow & & \downarrow{\scriptstyle \epsilon_A} \\ *_{\mathbf{H}} & \longrightarrow & A \end{array}$$

We call $\flat_{\mathrm{dR}} A$ the *de Rham refinement* (or *de Rham coefficient object*) of A. Varying $A \in */\mathbf{H}$ defines a 2-functor

$$\flat_{\mathrm{dR}} : */\mathbf{H} \longrightarrow */\mathbf{H} \quad (\text{or } \mathbf{H} \longrightarrow \mathbf{H} \text{ for canonically pointed coefficients}),$$

called the *flat de Rham modality* [43]. The terminology stems from the fact that for a group object G, by pasting the pullback square for $\flat_{\mathrm{dR}} BG$ with the looping pullback $G \simeq \Omega BG = * \times_{BG} *$, $\flat_{\mathrm{dR}} BG$ is equivalently the homotopy cofiber (quotient action groupoid) $G//\flat G$:

$$\flat_{\mathrm{dR}} B_{\mathbf{H}} G \simeq B_{\mathbf{H}}(\flat G \to G),$$

which in $\mathbf{SmoothGrpd}_2$, for G a Lie group, is equivalent via the Maurer–Cartan map $g \mapsto g^{-1}dg$ to the *sheaf* (0-truncated) of flat $\mathfrak{g}$-valued 1-forms $\Omega^1_{\mathrm{flat}}(-;\mathfrak{g})$: the action of $\flat G$ on G is free, and on a contractible test space every flat form is $g^{-1}dg$ for a g unique up to a constant. (The action groupoid $\Omega^1_{\mathrm{flat}}(-;\mathfrak{g})//G$, on the other hand, presents $\flat B_{\mathbf{H}}G$, the moduli of flat connections.) More generally, for abelian Lie A the object $\flat_{\mathrm{dR}} B^k_{\mathbf{H}} A$ is presented by the truncated de Rham complex $\Omega^1(-;\mathfrak{a}) \to \cdots \to \Omega^k_{\mathrm{cl}}(-;\mathfrak{a})$ [43]; its global sections in degree 0 are closed k-forms modulo exact ones.

To pass from *flat* forms to *closed* forms (which serve as curvature targets for non-flat connections), we observe that on any test object X, every flat form is a closed form, and locally (on contractible test spaces) every closed k-form is the exterior derivative/curvature of a $(k-1)$-form potential:

Definition 4.6 (Moduli Stack of Closed Differential k-Forms [43, Section 3.9.6])**.** Let $\mathbf{H}$ be a cohesive Grothendieck $(2,1)$-topos and let $A \in \mathrm{Del}_2^k(\mathbf{H})$ be a k-tuply deloopable object (where $\Omega^k_{\mathrm{cl}}(-;A)$ and $\flat_{\mathrm{dR}} B^k_{\mathbf{H}} A$ live internal to $\mathbf{H}$ for $k=1$, as internal 2-stacks in $\mathbf{Grpd}_2(\mathbf{H})$ for $k=2$, and in the ambient $(\infty,1)$-topos $\mathbf{H}_\infty$ for $k \geq 3$). A *moduli stack of closed A-valued k-forms* (or a *differential coefficients specification*) is an object $\Omega^k_{\mathrm{cl}}(-;A)$ equipped with a morphism $\sigma_k : \Omega^k_{\mathrm{cl}}(-;A) \to \flat_{\mathrm{dR}} B^k_{\mathbf{H}} A$ such that:

(1) $\Omega^k_{\mathrm{cl}}(-;A)$ is 0-truncated (a sheaf of sets) for abelian A (or a 1-truncated action groupoid $\Omega^1_{\mathrm{cl}}(-;\mathfrak{a})//A$ when twisting by an adjoint action on the linearized/infinitesimal coefficient object $\mathfrak{a} = \mathrm{Lie}(A)$ in concrete models such as $\mathbf{SmoothGrpd}_2$) and shape-contractible ($\int \Omega^k_{\mathrm{cl}}(-;A) \simeq *$);

(2) The homotopy fiber of $\sigma_k : \Omega^k_{\mathrm{cl}}(-;A) \to \flat_{\mathrm{dR}} B^k_{\mathbf{H}} A$ over the basepoint classifies $(k-1)$-form potentials $\Omega^{k-1}(-;\mathfrak{a})$ (via the de Rham differential $d : \Omega^{k-1}(-;\mathfrak{a}) \to \Omega^k_{\mathrm{cl}}(-;A)$), and on every test object U in a generating family of contractible test spaces (where $\mathbf{H}(U, \flat B^k_{\mathbf{H}} A) \simeq *$), pulling back along σ_k exhibits local exactness of closed forms (the Poincaré lemma).

4.2.3. *Connection Forms: $\int \dashv \flat$.* In classical differential geometry, a principal connection on a principal G-bundle $P \to X$ is specified either globally as a G-equivariant vertical $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P;\mathfrak{g})^G$, or locally over a cover $\{U_i \to X\}$ by local gauge potentials $A_i \in \Omega^1(U_i;\mathfrak{g})$ transforming on overlaps $U_i \cap U_j$ via

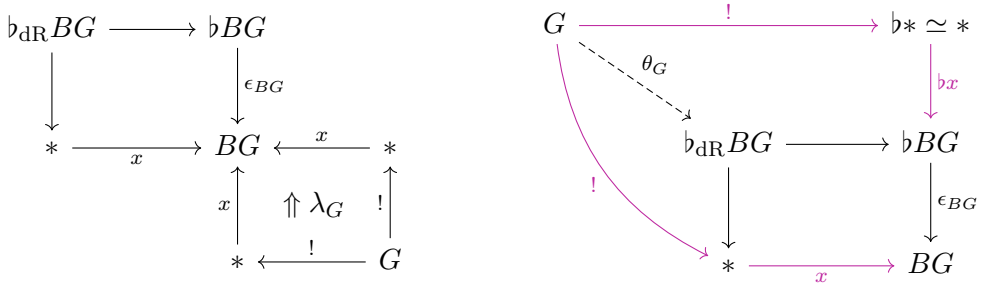
$$A_j = \mathrm{Ad}_{g_{ij}^{-1}}(A_i) + g_{ij}^* \theta_G, \qquad \theta_G = g^{-1} dg \in \Omega^1_{\mathrm{flat}}(G;\mathfrak{g}). \tag{4.3}$$

Here $\theta_G = g^{-1}dg$ is the canonical left-invariant *Maurer–Cartan form* on the Lie group G associated with the left action $g \cdot x$ (under the convention where gauge transformations act on potentials as $A \mapsto g^{-1}Ag + g^{-1}dg$).

Remarkably, in any Grothendieck $(2,1)$-topos $\mathbf{H}$, every group object $G \in \mathrm{Grp}_2(\mathbf{H})$ carries a canonical *abstract Maurer–Cartan morphism*

$$\theta_G : G \longrightarrow \flat_{\mathrm{dR}} B_{\mathbf{H}} G,$$

constructed purely from the universal property of the $(2,1)$-pullback defining $\flat_{\mathrm{dR}} B_{\mathbf{H}} G$. Indeed, since Γ is a right adjoint and LCst is left exact, $\flat = \mathrm{LCst} \circ \Gamma$ preserves all finite $(2,1)$-limits and, in particular, preserves the terminal object: $\flat * \simeq *$. Applying $\flat$ to the basepoint $x = p_G : * \to BG$ gives $\flat x : * \simeq \flat * \to \flat BG$. By naturality of $\epsilon : \flat \Rightarrow \mathrm{id}_{\mathbf{H}}$, we have $\epsilon_{BG} \circ \flat x \simeq x \circ \epsilon_* = x : * \to BG$. Since $G \simeq \Omega BG = * \times_{BG} *$ is the homotopy pullback of $x : * \to BG$ along itself equipped with the universal loop 2-cell $\lambda_G : x \circ ! \Rightarrow x \circ !$, pasting λ_G with the naturality 2-isomorphism $x \simeq \epsilon_{BG} \circ \flat x$ fills the outer square on the right below with a non-trivial 2-isomorphism $x \circ ! \Rightarrow \epsilon_{BG} \circ \flat x \circ !$, inducing the universal Maurer–Cartan morphism $\theta_G : G \to \flat_{\mathrm{dR}} BG$:



This first Maurer–Cartan morphism $\theta_1 = \theta_G : G \to \flat_{\mathrm{dR}} B_{\mathbf{H}} G$ exists for *every* (generally nonabelian) group object G, without requiring G to be braided or $B^2_{\mathbf{H}} G$ to exist. Let us see how θ_G directly governs nonabelian principal connections:

(1) **General Nonabelian Group Objects $G \in \mathrm{Grp}_2(\mathbf{H})$:** For an arbitrary nonabelian group object G (such as $G = SU(N)$ or $SO(p,q)$), G is not braided and $B^2_{\mathbf{H}} G$ does not exist; correspondingly, the curvature $F_A = dA + \frac{1}{2}[A \wedge A]$ of a nonabelian connection is not an invariant closed 2-form on the base X, but rather transforms in the adjoint representation ($d_A F_A = 0$); its natural home is the automorphism 2-group $\mathrm{AUT}(G) = (G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G))$,

while the *inner* automorphism 2-group $\mathrm{INN}(G) = (G \xrightarrow{\mathrm{id}} G)$ is contractible and plays the role of a Weil algebra [3, 2, 43]. Below we keep the symbol $\mathrm{INN}(G)$ of the original draft for the 2-group through which the curvature is meant to be classified; the authors must decide whether $\mathrm{AUT}(G)$ or $\mathrm{INN}(G)$ is intended, since the two choices give different constructions. *The pullback description* (4.4) *is a proposal that is not proved here*; the established definition, used in the rest of the paper, is the presentation (4.5) [12, 43]. Synthetically in any cohesive $(2,1)$-topos $\mathbf{H}$, this 2-group deloops in $\tau_{\leq 2}\mathbf{H}_\infty$ with universal adjoint Maurer–Cartan morphism $\theta_{\mathrm{adj}} : B_{\mathbf{H}}G \to \flat_{\mathrm{dR}} B_{\mathbf{H}}\mathrm{INN}(G)$, and a *nonabelian differential 1-form coefficient specification* for G consists of a closed adjoint curvature 2-form stack $\sigma_2^{\mathrm{adj}} : \Omega^2_{\mathrm{cl}}(-;\mathrm{INN}(G)) \to \flat_{\mathrm{dR}} B_{\mathbf{H}}\mathrm{INN}(G)$ together with a local 1-form potential stack $\Omega^1_{\mathrm{con}}(-;G) \to \Omega^2_{\mathrm{cl}}(-;\mathrm{INN}(G))$ whose $(2,1)$-pullback defines the *nonabelian moduli stack of principal G-connections* $B_{\mathrm{con}}G$:

$$\begin{array}{ccc} B_{\mathrm{con}}G & \xrightarrow{\mathrm{curv}} & \Omega^2_{\mathrm{cl}}(-;\mathrm{INN}(G)) \\ {\scriptstyle p_{\mathrm{con}}}\downarrow & & \downarrow{\scriptstyle \sigma_2^{\mathrm{adj}}} \\ B_{\mathbf{H}}G & \xrightarrow[\theta_{\mathrm{adj}}]{} & \flat_{\mathrm{dR}} B_{\mathbf{H}}\mathrm{INN}(G) \end{array} \tag{4.4}$$

such that the homotopy fiber of curv over 0 is $\flat_{\mathrm{dR}} B_{\mathbf{H}}G \simeq G//\flat G \to B_{\mathrm{con}}G$. Concretely, at the level of connection 1-forms in $\mathbf{SmoothGrpd}_2$, the first Maurer–Cartan morphism $\theta_G : G \to \flat_{\mathrm{dR}} B_{\mathbf{H}}G \simeq G//\flat G$ induces the affine gauge action (4.3) of G on local 1-form potentials $\Omega^1(-;\mathfrak{g})$, presenting $B_{\mathrm{con}}G$ as the stackified quotient action groupoid:

$$B_{\mathrm{con}}G \;\simeq\; L_J(\Omega^1(-;\mathfrak{g})//G) \xrightarrow{p_{\mathrm{con}}} B_{\mathbf{H}}G. \tag{4.5}$$

(2) **Braided/Abelian and Higher Deloopable Objects $G \in \mathrm{Del}_2^{k+1}(\mathbf{H})$:** When $G \in \mathrm{Del}_2^k(\mathbf{H})$ is k-tuply deloopable ($k \geq 1$), applying the abstract Maurer–Cartan construction to the group object $B_{\mathbf{H}}^{k-1}G$ yields the *k-th universal Maurer–Cartan / universal curvature characteristic morphism*:

$$\theta_k = \theta_{B^{k-1}G} : B_{\mathbf{H}}^{k-1}G \longrightarrow \flat_{\mathrm{dR}} B_{\mathbf{H}}^k G.$$

In particular, for $k = 2$, when $G \in \mathrm{BrGrp}_2(\mathbf{H}) \simeq \mathrm{Del}_2^2(\mathbf{H})$ is braided/abelian, the adjoint action is trivial, the curvature is an honest closed 2-form, and the relevant coefficient object is $B_{\mathbf{H}}G$, which deloops to $B_{\mathbf{H}}^2 G$ in the sense of Remark 3.3 (or when $B_{\mathbf{H}}G$ is mapped to an abelian target via a Chern–Weil invariant polynomial); then θ_{adj} reduces to $\theta_2 : B_{\mathbf{H}}G \to \flat_{\mathrm{dR}} B_{\mathbf{H}}^2 G$, and (4.4) reduces directly to the $(2,1)$-pullback of θ_2 along $\sigma_2 : \Omega^2_{\mathrm{cl}}(-;G) \to \flat_{\mathrm{dR}} B_{\mathbf{H}}^2 G$, as formalized in the next definition.

Definition 4.7 (Moduli Stack of Principal Connections)**.** Let $\mathbf{H}$ be a Grothendieck $(2,1)$-topos. For a Lie group G in $\mathbf{H} = \mathbf{SmoothGrpd}_2$ (or, more generally, whenever a sheaf of $\mathfrak{g}$-valued 1-forms with affine gauge action is available), the *moduli stack of principal G-connections* $B_{\mathrm{con}}G = B^1_{\mathrm{con}}G \in \mathbf{H}$ is the stackified action groupoid (4.5), with $p_{\mathrm{con}} : B_{\mathrm{con}}G \to B_{\mathbf{H}}G$ forgetting the forms (whether it is also the pullback (4.4) is left open); when $G \in \mathrm{BrGrp}_2(\mathbf{H}) \simeq \mathrm{Del}_2^2(\mathbf{H})$ is braided (or equipped with a Chern–Weil curvature characteristic $\theta_2 : B_{\mathbf{H}}G \to \flat_{\mathrm{dR}} B_{\mathbf{H}}^2 G$ and closed curvature 2-forms $\Omega^2_{\mathrm{cl}}(-;G)$), (4.4) simplifies to the $(2,1)$-pullback:

$$\begin{array}{ccc} B_{\mathrm{con}}G & \xrightarrow{\mathrm{curv}} & \Omega^2_{\mathrm{cl}}(-;G) \\ {\scriptstyle p_{\mathrm{con}}}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ B_{\mathbf{H}}G & \xrightarrow[\theta_2]{} & \flat_{\mathrm{dR}} B_{\mathbf{H}}^2 G \end{array}$$

Given an abstract principal G-bundle $P \to X$ classified by $f : X \to B_{\mathbf{H}}G$, an *abstract principal G-connection* on P is a lift $\hat{f} : X \to B_{\mathrm{con}}G$ of f along p_{con} (depicted below for the braided/Chern–Weil case, and with (4.4) as the right-hand square for general nonabelian G):

$$\begin{array}{ccccc} & & B_{\mathrm{con}}G & \xrightarrow{\mathrm{curv}} & \Omega^2_{\mathrm{cl}}(-;G) \\ & \overset{\hat{f}}{\nearrow} & \downarrow{\scriptstyle p_{\mathrm{con}}} & & \downarrow{\scriptstyle \sigma_2} \\ X & \xrightarrow[f]{} & B_{\mathbf{H}}G & \xrightarrow[\theta_2]{} & \flat_{\mathrm{dR}} B^2_{\mathbf{H}}G \end{array}$$

More generally, replacing θ_2 by θ_{k+1} for a $(k+1)$-tuply deloopable object $G \in \mathrm{Del}_2^{k+1}(\mathbf{H})$ defines connections on higher k-bundles:

Definition 4.8 (Moduli Stack of k-Connections)**.** Let $\mathbf{H}$ be a Grothendieck $(2,1)$-topos and let $G \in \mathrm{Del}_2^{k+1}(\mathbf{H})$ be a $(k+1)$-tuply deloopable object ($k \geq 0$). At the level of $(2,1)$-truncation, $B_{\mathrm{con}}G = B^1_{\mathrm{con}}G \in \mathbf{H}$ is internal to $\mathbf{H}$; for $k = 2$, $B^2_{\mathrm{con}}G \in \mathbf{Grpd}_2(\mathbf{H})$ is an internal 2-stack; for $k \geq 3$, $B^k_{\mathrm{con}}G$ is an object of the ambient $(\infty,1)$-topos $\mathbf{H}_\infty$, with mapping groupoid $\mathbf{H}(X, B^k_{\mathrm{con}}G) := \tau_{\leq 1}\mathbf{H}_\infty(X, B^k_{\mathrm{con}}G)$. The *moduli stack of G-connection k-forms* $B^k_{\mathrm{con}}G$ is the $(2,1)$-pullback on the left below; a *connection* on a G-k-bundle $P_k \to X$ classified by $f : X \to B^k_{\mathbf{H}}G$ is a lift $\hat{f} : X \to B^k_{\mathrm{con}}G$ as on the right:

$$\begin{array}{ccc} B^k_{\mathrm{con}}G & \xrightarrow{\mathrm{curv}_k} & \Omega^{k+1}_{\mathrm{cl}}(-;G) \\ \downarrow{\scriptstyle p^k_{\mathrm{con}}} & & \downarrow{\scriptstyle \sigma_{k+1}} \\ B^k_{\mathbf{H}}G & \xrightarrow[\theta_{k+1}]{} & \flat_{\mathrm{dR}} B^{k+1}_{\mathbf{H}}G \end{array} \qquad \begin{array}{ccc} & & B^k_{\mathrm{con}}G \\ & \overset{\hat{f}}{\nearrow} & \downarrow{\scriptstyle p^k_{\mathrm{con}}} \\ X & \xrightarrow[f]{} & B^k_{\mathbf{H}}G \end{array}$$

4.2.4. *Moduli of Connection Forms: $\int \dashv \flat \dashv \sharp$.* In Section 4.2.3, we constructed the classifying object $B^k_{\mathrm{con}}G \in \mathbf{H}$ whose external hom-groupoid $\mathbf{H}(X, B^k_{\mathrm{con}}G) \in \mathbf{Grpd}_2$ is the groupoid of G-connections on k-bundles over X and gauge transformations between them. However, in the study of classical and quantum gauge field theories, action functionals, symplectic structures, and path integrals are defined on the *smooth moduli space of connections on X*, which must exist as an *internal object* $G\mathrm{Conn}^k(X) \in \mathbf{H}$ rather than merely an external groupoid in $\mathbf{Grpd}_2$.

Since every Grothendieck $(2,1)$-topos $\mathbf{H}$ is cartesian closed, it possesses an internal hom 2-bifunctor $[-,-]_{\mathbf{H}} : \mathbf{H}^{\mathrm{op}} \times \mathbf{H} \to \mathbf{H}$ characterized by the adjunction $\mathbf{H}(U \times X, Y) \simeq \mathbf{H}(U, [X,Y]_{\mathbf{H}})$. One might naively attempt to define the internal moduli stack of k-connections on X simply as the internal hom $[X, B^k_{\mathrm{con}}G]_{\mathbf{H}}$. However, as analyzed in detail in [43, Sections 5.2.13.4, 6.4.16], the naive internal hom $[X, B^k_{\mathrm{con}}G]_{\mathbf{H}}$ fails to reproduce the physical gauge groupoid even for $G = U(1)$ and $k = 1$ in $\mathbf{SmoothGrpd}_2$!

To see why, evaluate $[X, B^1_{\mathrm{con}}U(1)]_{\mathbf{H}}$ on a test Cartesian space $U \in \mathbf{CartSp}$:

$$[X, B^1_{\mathrm{con}}U(1)]_{\mathbf{H}}(U) \simeq \mathbf{H}(U \times X, B^1_{\mathrm{con}}U(1)).$$

Every object of $\mathbf{H}(U \times X, B^1_{\mathrm{con}}U(1))$ is a $U(1)$-bundle with connection on the product manifold $U \times X$, which includes not only U-parameterized families of connections on X (components along T^*X), but also spurious connection components along T^*U! Equivalently, a U-parameterized family of gauge configurations on X should be a *smooth family of differential forms on X* (parameterized smoothly by U), whereas the underlying topological bundle and gauge transformations along U may vary only via bare U-indexed families of gauge transformations on X.

This discrepancy is resolved systematically in any locally local Grothendieck $(2,1)$-topos $\mathbf{H}$ using the *sharp modality* $\sharp = \mathrm{coDisc} \circ \Gamma$ (right adjoint to $\flat$). Recall that for any object $Y \in \mathbf{H}$, the unit of the adjunction $\flat \dashv \sharp$ (or the canonical morphism $\eta^\sharp_Y : Y \to \sharp Y$) embeds Y into its codiscrete version $\sharp Y$, where U-plots $U \to \sharp Y$ correspond via $\mathbf{H}(U, \sharp Y) \simeq \mathbf{H}(\flat U, Y)$ to bare point-indexed maps $\Gamma(U) \to \Gamma(Y)$.

To ensure that U-plots of the internal moduli stack of connections have smoothly U-parameterized local connection k-form potentials (without T^*U-legs) while gauge transformations

and bundle transitions are parameterized via $\sharp$ (i.e., by $\flat U$), we form the $(2,1)$-pullback over the curvature:

Definition 4.9 (Internal Moduli Stack of Connections $G\mathrm{Conn}^k(X)$ [43, Sections 5.2.13.4, 6.4.16])**.** Let $\mathbf{H} = \mathbf{Stk}(\mathbf{C}, J)$ be a locally local Grothendieck $(2,1)$-topos (so that $\flat \dashv \sharp$ exists) and let $G \in \mathrm{Del}_2^{k+1}(\mathbf{H})$ (or $G \in \mathrm{Grp}_2(\mathbf{H})$ for $k=1$). The *internal moduli stack of G-k-connections on X*, denoted $G\mathrm{Conn}^k(X)$, is defined synthetically by the curvature $(2,1)$-pullback square in $\mathbf{H}$:

$$\begin{array}{ccc} G\mathrm{Conn}^k(X) & \longrightarrow & \big[X, \Omega^{k+1}_{\mathrm{cl}}(-;G)\big]_{\mathbf{H}} \\ \downarrow & & \downarrow{\scriptstyle [X,\eta^\sharp]_{\mathbf{H}}} \\ \sharp\big[X, B^k_{\mathrm{con}}G\big]_{\mathbf{H}} & \xrightarrow[\sharp[X,\mathrm{curv}_k]_{\mathbf{H}}]{} & \sharp\big[X, \Omega^{k+1}_{\mathrm{cl}}(-;G)\big]_{\mathbf{H}} \end{array} \tag{4.6}$$

At the level of $(2,1)$-truncation, $G\mathrm{Conn}^1(X) \in \mathbf{H}$ is an internal 1-stack in $\mathbf{H}$; for $k=2$, $G\mathrm{Conn}^2(X)$ is an internal 2-stack in $\mathbf{Grpd}_2(\mathbf{H})$; for $k \geq 3$, it is an object of $\mathbf{H}_\infty$.

Remark 4.10 (Caveat on Definition 4.9). As stated, the pullback (4.6) makes the *curvature* forms smoothly U-parameterized while the connections themselves are only point-indexed; e.g. any point-indexed (non-smooth in u) family of flat connections defines a U-plot. It therefore does *not* produce smooth families of connection 1-forms. The construction of [11, 43] ("differential concretification") instead uses the 1-image of $[X, B_{\mathrm{con}}G]_{\mathbf{H}} \to \sharp[X, B_{\mathrm{con}}G]_{\mathbf{H}}$ relative to the sheaf of connection *potentials*, and yields, for $k=1$ and a Lie group G in $\mathbf{SmoothGrpd}_2$,

$$G\mathrm{Conn}^1(X) \;\simeq\; L_J\big([X, \Omega^1(-;\mathfrak{g})]_{\mathbf{H}} /\!\!/ [X,G]_{\mathbf{H}}\big), \tag{4.7}$$

whose U-plots are smooth U-families of connection 1-forms on X (without T^*U-legs) and whose morphisms are *smooth* U-families of gauge transformations. Note that the action of a merely point-indexed family $\sharp[X,G]_{\mathbf{H}}$ on smooth families of forms would not be well defined, since $A \mapsto \mathrm{Ad}_{g(u)^{-1}}A(u) + g(u)^{-1}dg(u)$ destroys smoothness in u. Definition 4.9 should be replaced by the concretification construction; Proposition 4.11 remains valid for it, since Γ inverts the comparison with $\sharp$.

Proposition 4.11 (Global Sections of the Internal Connection Moduli)**.** *Let $\mathbf{H}$ be a local Grothendieck $(2,1)$-topos. For every $X \in \mathbf{H}$ and $G \in \mathrm{Del}_2^{k+1}(\mathbf{H})$ (or $G \in \mathrm{Grp}_2(\mathbf{H})$ for $k=1$), the global sections groupoid of $G\mathrm{Conn}^k(X)$ recovers the external groupoid of G-k-connections on X:*

$$\Gamma\big(G\mathrm{Conn}^k(X)\big) = \mathbf{H}\big(*, G\mathrm{Conn}^k(X)\big) \;\simeq\; \mathbf{H}\big(X, B^k_{\mathrm{con}}G\big).$$

Proof. Since $\mathbf{H}$ is local, the global sections 2-functor $\Gamma = \mathbf{H}(*,-) : \mathbf{H} \to \mathbf{Grpd}_2$ is both a right 2-adjoint to LCst and a left 2-adjoint to coDisc, so it preserves all $(2,1)$-limits, $(2,1)$-colimits, and stackification localizations (since $*_{\mathbf{H}}$ is locally contractible/projective on local covers). Moreover, since $\sharp = \mathrm{coDisc} \circ \Gamma$ with $\Gamma \circ \mathrm{coDisc} \simeq \mathrm{id}_{\mathbf{Grpd}_2}$, we have $\Gamma(\sharp Y) \simeq \Gamma(Y)$ naturally in $Y \in \mathbf{H}$, under which $\Gamma(\eta^\sharp_Y)$ becomes the identity equivalence. Applying Γ to the defining curvature pullback square (4.6) replaces every $\sharp[X,-]_{\mathbf{H}}$ by $\Gamma([X,-]_{\mathbf{H}}) \simeq \mathbf{H}(X,-)$, under which $\Gamma(\eta^\sharp)$ becomes an equivalence. Thus, $\Gamma(G\mathrm{Conn}^k(X))$ is computed as the pullback along an equivalence, canonically recovering $\mathbf{H}(X, B^k_{\mathrm{con}}G)$ (which also directly matches $\Gamma([X,\Omega^1]_{\mathbf{H}} /\!\!/ [X,G]_{\mathbf{H}}) \simeq \mathbf{H}(X,\Omega^1)//\mathbf{H}(X,G) \simeq \mathbf{H}(X, B_{\mathrm{con}}G)$ via (4.7) for $k=1$, granted that Γ commutes with stackification as asserted above; this step uses that the terminal object is a point of the site, which holds for **CartSp** and should be stated as a hypothesis in general). □

Example 4.12 (Concrete Realization in $\mathbf{SmoothGrpd}_2$)**.** Consider the cohesive 2-topos $\mathbf{H} = \mathbf{SmoothGrpd}_2 = \mathbf{Stk}(\mathbf{CartSp}, J_{\mathrm{open}})$ and let G be a Lie group with Lie algebra $\mathfrak{g}$, viewed as a 0-truncated group sheaf $U \mapsto C^\infty(U,G)$.

(1) *Delooping and Flat de Rham Coefficient:* The external/internal delooping prestack sends $U \in \mathbf{CartSp}$ to the one-object groupoid $\mathbf{B}^{\mathrm{ext}}G(U) = (C^\infty(U,G) \rightrightarrows *)$, and its stackification $B_{\mathbf{H}}G = L_{J_{\mathrm{open}}}(\mathbf{B}^{\mathrm{ext}}G)$ classifies smooth principal G-bundles: for a smooth

manifold X with good open cover $\{U_i \to X\}$, $\mathbf{H}(X, B_{\mathbf{H}}G)$ is the groupoid of Čech 1-cocycles $g_{ij} : U_i \cap U_j \to G$ ($g_{ij}g_{jk} = g_{ik}$) and coboundaries $h_i : U_i \to G$ ($g'_{ij} = h_i^{-1} g_{ij} h_j$). Since $\flat G(U) = G^\delta(U)$ consists of locally constant maps $U \to G$, the homotopy fiber $\flat_{\mathrm{dR}} B_{\mathbf{H}}G \simeq G//G^\delta$ is equivalent via the Maurer–Cartan map $g \mapsto g^{-1}dg$ to the sheaf $\Omega^1_{\mathrm{flat}}(-;\mathfrak{g})$ of flat $\mathfrak{g}$-valued 1-forms (while the action groupoid $\Omega^1_{\mathrm{flat}}(-;\mathfrak{g})//G$ under gauge transformations $A \mapsto \mathrm{Ad}_{g^{-1}} A + g^{-1}dg$ presents $\flat B_{\mathbf{H}}G$).

(2) *Moduli Stack of Connections $B_{\mathrm{con}}G$:* Dropping the flatness constraint $dA + \frac{1}{2}[A \wedge A] = 0$ on $U \in \mathbf{CartSp}$ defines $B_{\mathrm{con}}G(U) = \Omega^1(U;\mathfrak{g})//C^\infty(U,G)$. By Čech descent over $\{U_i \to X\}$, an object of $\mathbf{H}(X, B_{\mathrm{con}}G)$ is a pair (g_{ij}, A_i) with $A_i \in \Omega^1(U_i;\mathfrak{g})$ satisfying $A_j = \mathrm{Ad}_{g_{ij}^{-1}}(A_i) + g_{ij}^{-1} dg_{ij}$ on $U_i \cap U_j$, i.e., a principal G-bundle with connection $(P, \nabla) \to X$, and a morphism $(g_{ij}, A_i) \to (g'_{ij}, A'_i)$ is a gauge transformation $h_i : U_i \to G$.

(3) *Sharp-Adjusted Internal Moduli $G\mathrm{Conn}^1(X)$:* Evaluating the pullback (4.6) on a test space $U \in \mathbf{CartSp}$, an object of $G\mathrm{Conn}^1(X)(U)$ is a smoothly U-parameterized family of connection 1-forms $A_i(u) \in \Omega^1(U_i;\mathfrak{g})$ along T^*X only (no du-legs along T^*U, because $[X, \Omega^1(-;\mathfrak{g})]_{\mathbf{H}}(U) = \Omega^1(X; C^\infty(U,\mathfrak{g}))$), together with transition functions and gauge transformations $h_i \in C^\infty(U \times U_i, G)$ depending smoothly on $u \in U$ (see Remark 4.10; point-indexed families $u \mapsto h_i(u)$, as given by $\sharp[X, B_{\mathrm{con}}G]_{\mathbf{H}}(U) \simeq \mathbf{H}(\flat U \times X, B_{\mathrm{con}}G)$, would not act on smooth families of forms).

4.3. **Geometric Cohomology.** We now show how the categorical and geometric structures developed in the preceding sections unify under the umbrella of *intrinsic, modal, and differential cohomology* internal to a cohesive $(2,1)$-topos.

4.3.1. *Intrinsic Cohomology.* A vast array of cohomology theories across mathematics—including abelian sheaf cohomology, singular cohomology, group cohomology, Čech cohomology, and nonabelian 1-cohomology—are manifestations of a single homotopical principle first axiomatized by Brown [6]: they are connected components ($\pi_0 = \tau_{\leq 0}$) of derived mapping spaces into a classifying or Eilenberg–Mac Lane object. Lurie [29, Section 7.2.2, Definition 7.2.2.14] observed that this definition is completely intrinsic to any $(\infty,1)$-topos (and indeed any $(\infty,1)$-category [43]). In our $(2,1)$-categorical context, it takes the following concise form:

Definition 4.13 (Intrinsic Cohomology)**.** Let $\mathbf{H}$ be a $(2,1)$-category and let $A \in \mathbf{H}$ be an object. For every $X \in \mathbf{H}$, define the *0-th intrinsic cohomology* of X with coefficients in A as the set of equivalence classes of 1-morphisms $X \to A$:

$$H^0(X; A) := \tau_{\leq 0}\mathbf{H}(X, A) \in \mathbf{Set}.$$

Suppose now that $\mathbf{H}$ is a cartesian $(2,1)$-cosmos and that $A \in \mathrm{Del}_2^k(\mathbf{H})$ is k-tuply deloopable ($k \geq 1$; formed internal to $\mathbf{H}$ for $k = 1$, in $\mathbf{Grpd}_2(\mathbf{H})$ for $k = 2$, and in $\mathbf{H}_\infty$ for $k \geq 3$, cf. Remark 3.3). Define the *k-th intrinsic cohomology* of X with coefficients in A as

$$H^k(X; A) := \tau_{\leq 0}\mathbf{H}(X, B^k_{\mathbf{H}}A) = H^0(X; B^k_{\mathbf{H}}A).$$

Combining Definition 4.13 with Corollary 3.7 immediately yields:

Lemma 4.14. *Let $\mathbf{H}$ be a Grothendieck $(2,1)$-topos (or cartesian $(2,1)$-cosmos as in Corollary 3.7) and let $A \in \mathbf{H}$ be an object. Then:*

(1) A admits a group object structure if and only if $H^k(X;A)$ for $k = 0,1$ is canonically defined via delooping for every $X \in \mathbf{H}$;
(2) A admits a braided group object structure if and only if $H^k(X;A)$ for $k = 0,1,2$ is canonically defined for every $X \in \mathbf{H}$;
(3) A admits a commutative group object structure if and only if $H^k(X;A)$ for $k = 0,1,2,3$ is canonically defined for every $X \in \mathbf{H}$;
(4) A admits a commutative group object structure if and only if $H^k(X;A)$ is canonically defined for every $X \in \mathbf{H}$ and every $k \geq 0$.

Proof. Immediate from the characterization of k-tuply deloopable objects in Corollary 3.7. □

For a fixed object $X \in \mathbf{H}$, 0-th cohomology defines a 2-functor $H^0(X;-) : \mathbf{H} \to \mathbf{Set}_2$ given by the composite

$$\mathbf{H} \xrightarrow[\mathbf{H}(X,-)]{} \mathbf{Grpd}_2 \xrightarrow[\tau_{\leq 0}]{} \mathbf{Set}_2. \qquad H^0(X;-) : \mathbf{H} \to \mathbf{Set}_2$$

Because the representable 2-functor $\mathbf{H}(X,-) : \mathbf{H} \to \mathbf{Grpd}_2$ preserves all $(2,1)$-limits and $\tau_{\leq 0} : \mathbf{Grpd}_2 \to \mathbf{Set}_2$ preserves finite products, the composite $H^0(X;-) : \mathbf{H} \to \mathbf{Set}_2$ is a cartesian (finite-product-preserving) 2-functor. Consequently, $H^0(X;-)$ preserves group objects, braided group objects, and commutative group objects:

$$\begin{array}{ccc}
\mathrm{ComGrp}_2(\mathbf{H}) & \xrightarrow{H^0(X;-)} & \mathrm{ComGrp}_2(\mathbf{Set}_2) \\
\downarrow & & \| \\
\mathrm{BrGrp}_2(\mathbf{H}) & \xrightarrow{H^0(X;-)} & \mathrm{BrGrp}_2(\mathbf{Set}_2) \\
\downarrow & & \downarrow \\
\mathrm{Grp}_2(\mathbf{H}) & \xrightarrow{H^0(X;-)} & \mathrm{Grp}_2(\mathbf{Set}_2) \\
\downarrow & & \downarrow \\
\mathbf{H} & \xrightarrow{\mathbf{H}(X,-)} \mathbf{Grpd}_2 \xrightarrow{\tau_{\leq 0}} & \mathbf{Set}_2
\end{array}$$

$$H^0(X;-) : \mathbf{H} \to \mathbf{Set}_2$$

Since $\mathbf{Set}_2$ is a discrete 2-category (a 1-category), any braided group object in $\mathbf{Set}_2$ is already an abelian group ($\mathrm{BrGrp}_2(\mathbf{Set}_2) = \mathrm{ComGrp}_2(\mathbf{Set}_2) = \mathbf{Ab}_2$), and for $A \in \mathrm{Del}_2^k(\mathbf{H})$, $B^k_{\mathbf{H}}A$ retains 0 group structures when $k = 1$ on a general group A, 1 group structure when $A \in \mathrm{Del}_2^{k+1}(\mathbf{H})$, and at least 2 group structures (hence braided/abelian on H^k) when $A \in \mathrm{Del}_2^{k+2}(\mathbf{H})$. More precisely:

Lemma 4.15. *Let $\mathbf{H}$ be a cartesian $(2,1)$-cosmos and let $A \in \mathbf{H}$ be an object.*

(1) If $H^1(X;A)$ is defined for every $X \in \mathbf{H}$ (so $A \in \mathrm{Grp}_2(\mathbf{H})$), then $H^0(X;A)$ is a group and $H^1(X;A)$ is a pointed set; if $H^2(X;A)$ is also defined (so $A \in \mathrm{BrGrp}_2(\mathbf{H})$), then $H^0(X;A)$ is an abelian group and $H^1(X;A)$ is a group;

(2) Let $k \geq 1$. If $A \in \mathrm{Del}_2^{k+1}(\mathbf{H})$ (i.e. $H^{k+1}(X;A)$ is defined), then $H^k(X;A)$ is a group; if moreover $A \in \mathrm{Del}_2^{k+2}(\mathbf{H})$, then $H^k(X;A)$ is an abelian group. In particular, if $A \in \mathrm{ComGrp}_2(\mathbf{H})$ then $H^k(X;A)$ is an abelian group for all $k \geq 0$ (since then $B^k_{\mathbf{H}}A$ is again a commutative group object). (For a braided but non-symmetric 2-group A, $H^1(X;A)$ is a group that need not be abelian.)

This justifies referring to $H^1(X;A)$ as *nonabelian 1-cohomology* whenever $A \in \mathrm{Grp}_2(\mathbf{H})$ is not braided (so $H^k(X;A)$ is undefined for $k > 1$).

4.3.2. *Modal Cohomologies.* Each internal modality $F : \mathbf{H} \to \mathbf{H}$ on a $(2,1)$-topos induces a corresponding modal cohomology theory by twisting the coefficient object:

Definition 4.16 (Modal Cohomology)**.** Let $\mathbf{H}$ be a cartesian $(2,1)$-cosmos and let $F : \mathbf{H} \to \mathbf{H}$ be an internal modality on $\mathbf{H}$.

(1) For any object $A \in \mathbf{H}$, define the *0-th F-cohomology* of $X \in \mathbf{H}$ with coefficients in A as

$$H^0_F(X;A) := H^0(X;F(A)) = \tau_{\leq 0}\mathbf{H}(X,F(A)).$$

(2) If $F(A) \in \mathrm{Del}_2^k(\mathbf{H})$ is k-tuply deloopable, define the *k-th F-cohomology* of $X \in \mathbf{H}$ with coefficients in A as

$$H^k_F(X;A) := H^0(X;B^k_{\mathbf{H}}F(A)).$$

Whenever the modality $F : \mathbf{H} \to \mathbf{H}$ preserves finite products and $(2,1)$-coequalizers (for instance, when $F = \flat = \mathrm{LCst} \circ \Gamma$ in a locally local $(2,1)$-topos where Γ is a left adjoint to coDisc and LCst is both a left adjoint and left exact, or when $F = \int = \mathrm{LCst} \circ \Pi$ in a strongly 1-connected $(2,1)$-topos), F commutes with the delooping object construction:

Lemma 4.17. *Let $F : \mathbf{H} \to \mathbf{H}$ be a modality on a cartesian $(2,1)$-cosmos $\mathbf{H}$ that preserves finite products and $(2,1)$-coequalizers. If $A \in \mathrm{Del}_2^k(\mathbf{H})$ is k-tuply deloopable, then $F(A) \in \mathrm{Del}_2^k(\mathbf{H})$ is k-tuply deloopable and there is a canonical equivalence*

$$B^k_{\mathbf{H}} F(A) \simeq F(B^k_{\mathbf{H}} A).$$

Consequently, $H^k_F(X; A)$ is defined and satisfies $H^k_F(X; A) \cong H^0_F(X; B^k_{\mathbf{H}} A) = H^0(X; F(B^k_{\mathbf{H}} A))$.

Proof. Since F preserves finite products, it preserves k-tuply group objects, so $F(A) \in \mathrm{Grp}_2^k(\mathbf{H}) \simeq \mathrm{Del}_2^k(\mathbf{H})$. Since $B_{\mathbf{H}} G = \varinjlim_{\Delta^{\mathrm{op}}} G^{\times \bullet}$ is the realization of the bar construction (Remark 2.6) and F preserves finite products and these $(2,1)$-colimits, induction on k gives $B^k_{\mathbf{H}} F(A) \simeq F(B^k_{\mathbf{H}} A)$. □

Specializing to the cohesive modalities on a Grothendieck $(2,1)$-topos $\mathbf{H}$:

- The *flat modality* $\flat$ induces *flat cohomology* $H^0_\flat(X; A) = H^0(X; \flat A)$ (measuring local systems / flat bundles), and its refinement $\flat_{\mathrm{dR}}$ induces *flat de Rham cohomology* $H^0_{\mathrm{dR}}(X; A) = H^0(X; \flat_{\mathrm{dR}} A)$;
- In a locally 1-connected $(2,1)$-topos, the *shape modality* $\int$ induces *shape cohomology* $H^0_\int(X; A) = H^0(X; \int A)$;
- In a locally local $(2,1)$-topos, the *sharp modality* $\sharp$ induces *sharp cohomology* $H^0_\sharp(X; A) = H^0(X; \sharp A)$.

Proposition 4.18. *Let $\mathbf{H}$ be a Grothendieck $(2,1)$-topos and let $A \in \mathrm{Del}_2^k(\mathbf{H})$ be a k-tuply deloopable object.*

(1) If $\mathbf{H}$ is strongly 1-connected, then $\int A$ is k-tuply deloopable and, for $k = 1$, $B_{\mathbf{H}} \int A \simeq \int B_{\mathbf{H}} A$; in particular $H^1_\int(X; A) \cong H^0_\int(X; B_{\mathbf{H}} A)$. (For $k \geq 2$ the deloopings live outside $\mathbf{H}$ and the statement must be formulated with the shape of $\tau_{\leq k+1}\mathbf{H}_\infty$; note also Remark 4.3.)

(2) If $\mathbf{H}$ is locally local, then $\flat A$ is k-tuply deloopable, $B^k_{\mathbf{H}} \flat A \simeq \flat B^k_{\mathbf{H}} A$, and $H^k_\flat(X; A) \cong H^0_\flat(X; B^k_{\mathbf{H}} A)$, while $H^k_{\mathrm{dR}}(X; A) := H^0(X; \flat_{\mathrm{dR}} B^k_{\mathbf{H}} A)$ classifies flat A-valued k-forms on X.

Proof. In (1), Π preserves $(2,1)$-colimits and finite products (by strong 1-connectivity), and LCst preserves $(2,1)$-colimits (being left adjoint to Γ) and finite $(2,1)$-limits (being an inverse image functor), so $\int = \mathrm{LCst} \circ \Pi$ satisfies Lemma 4.17. In (2), Γ is left adjoint to coDisc (by local locality) and right adjoint to LCst, so $\flat = \mathrm{LCst} \circ \Gamma$ preserves both finite $(2,1)$-limits and all $(2,1)$-colimits, again satisfying Lemma 4.17. □

4.3.3. *Differential Cohomology.* In Section 4.2.3 we constructed the moduli stack of G-connections on k-bundles $B^k_{\mathrm{con}} G \in \mathbf{H}$ whenever $G \in \mathrm{Del}_2^{k+1}(\mathbf{H})$ is $(k+1)$-tuply deloopable (as well as $B_{\mathrm{con}} G$ for nonabelian group objects $G \in \mathrm{Grp}_2(\mathbf{H})$). We now define *intrinsic differential cohomology* with coefficients in G, and investigate whether the assignment $G \mapsto B^k_{\mathrm{con}} G$ can be extended to a genuine internal modality $(-)_{\mathrm{con}} : */\mathbf{H} \to */\mathbf{H}$ defined on arbitrary (non-groupal) pointed objects $A \in */\mathbf{H}$.

Definition 4.19 (Differential Cohomology)**.** Let $\mathbf{H}$ be a Grothendieck $(2,1)$-topos and let $G \in \mathrm{Del}_2^{k+1}(\mathbf{H})$ be a $(k+1)$-tuply deloopable object ($k \geq 0$, with $B^k_{\mathrm{con}} G$ formed internal to $\mathbf{H}$ for $k = 0, 1$, in $\mathbf{Grpd}_2(\mathbf{H})$ for $k = 2$, and in $\mathbf{H}_\infty$ for $k \geq 3$, cf. Remark 3.3) equipped with closed differential $(k+1)$-form coefficients $\sigma_{k+1} : \Omega^{k+1}_{\mathrm{cl}}(-; G) \to \flat_{\mathrm{dR}} B^{k+1}_{\mathbf{H}} G$ (or $G \in \mathrm{Grp}_2(\mathbf{H})$ for $k = 1$ via Definition 4.7 and (4.5)). The *k-th differential cohomology* of $X \in \mathbf{H}$ with coefficients in G is defined as

$$H^k_{\mathrm{diff}}(X; G) := H^0(X; B^k_{\mathrm{con}} G) = \tau_{\leq 0}\mathbf{H}(X, B^k_{\mathrm{con}} G).$$

By the defining $(2,1)$-pullback square of $B^k_{\mathrm{con}}G$ (Definition 4.8), $H^k_{\mathrm{diff}}(X;G)$ comes equipped with the canonical *characteristic class map* and *curvature map*:

$$\mathrm{char}_k = (p^k_{\mathrm{con}})_* : H^k_{\mathrm{diff}}(X;G) \longrightarrow H^k(X;G), \qquad \mathrm{curv}_k : H^k_{\mathrm{diff}}(X;G) \longrightarrow H^0(X;\Omega^{k+1}_{\mathrm{cl}}(-;G)),$$

fitting into a commutative square over the de Rham characteristic map $(\theta_{k+1})_* : H^k(X;G) \to H^{k+1}_{\mathrm{dR}}(X;G)$.

Remark 4.20 (Fundamental Fiber Sequences of Differential Cohomology). Pasting the defining $(2,1)$-pullback square of $B^k_{\mathrm{con}}G$ (Definition 4.8) with the homotopy fiber sequence $\flat B^k_{\mathbf{H}}G \to B^k_{\mathbf{H}}G \xrightarrow{\theta_{k+1}} \flat_{\mathrm{dR}} B^{k+1}_{\mathbf{H}}G$ and the potential fiber $\Omega^k(-;\mathfrak{g}) = \mathrm{fib}(\sigma_{k+1})$ immediately yields the two fundamental homotopy fiber sequences of differential cohomology:

$$\flat B^k_{\mathbf{H}}G \longrightarrow B^k_{\mathrm{con}}G \xrightarrow{\mathrm{curv}_k} \Omega^{k+1}_{\mathrm{cl}}(-;G), \qquad \Omega^k(-;\mathfrak{g}) \longrightarrow B^k_{\mathrm{con}}G \xrightarrow{p^k_{\mathrm{con}}} B^k_{\mathbf{H}}G.$$

Passing to $H^0(X;-) = \tau_{\leq 0}\mathbf{H}(X,-)$ recovers the classical exact sequences of Simons–Sullivan / Hopkins–Singer differential cohomology: the kernel of the curvature map $\mathrm{curv}_k : H^k_{\mathrm{diff}}(X;G) \to H^0(X;\Omega^{k+1}_{\mathrm{cl}}(-;G))$ is the flat cohomology $H^k_\flat(X;G)$ (classifying flat G-k-bundles), while the fiber of the characteristic class map $\mathrm{char}_k : H^k_{\mathrm{diff}}(X;G) \to H^k(X;G)$ over the trivial k-bundle is the quotient of global connection k-forms $\Omega^k(X;\mathfrak{g})$ by gauge transformations.

Now let us ask whether $B^k_{\mathrm{con}}G$ can be viewed as the modal cohomology of an internal modality $(-)_{\mathrm{con}} : */\mathbf{H} \to */\mathbf{H}$ satisfying $(B^k_{\mathbf{H}}G)_{\mathrm{con}} \simeq B^k_{\mathrm{con}}G$. Observe that for $k=0$, $B^0_{\mathrm{con}}A$ is defined by the $(2,1)$-pullback square:

$$\begin{array}{ccc} B^0_{\mathrm{con}}A & \xrightarrow{\mathrm{curv}_0} & \Omega^1_{\mathrm{cl}}(-;A) \\ {\scriptstyle p^0_{\mathrm{con}}}\downarrow & & \downarrow{\scriptstyle \sigma_1} \\ A & \xrightarrow[\theta_A]{} & \flat_{\mathrm{dR}} B_{\mathbf{H}}A \end{array} \tag{4.8}$$

Thus, the only place where the group structure of A is used in (4.8) is in the target $\flat_{\mathrm{dR}}B_{\mathbf{H}}A$ of the Maurer–Cartan morphism $\theta_A : A \to \flat_{\mathrm{dR}}B_{\mathbf{H}}A$! To extend (4.8) to an arbitrary pointed object $A \in */\mathbf{H}$, we must reformulate the Maurer–Cartan morphism without assuming that A is deloopable.

We construct two canonical reformulations of the Maurer–Cartan morphism for arbitrary pointed objects $A \in */\mathbf{H}$: the first is valid in any Grothendieck $(2,1)$-topos and produces a morphism with domain ΩA, while the second uses the shape modality in a locally 1-connected $(2,1)$-topos to produce both a looped comparison map with codomain A and an unlooped unit morphism $\eta^{\mathrm{dR}}_A : A \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A$ with domain A itself.

(1) **The Flat/Looping Extension ($\tilde{\theta}^\flat_A : \Omega A \to \flat_{\mathrm{dR}}A$):** Recall from Theorem 3.8 that looping Ω and delooping $B_{\mathbf{H}}$ form an equivalence between 0-truncated group objects $\mathrm{Grp}(\tau_{\leq 0}\mathbf{H})$ and 0-connected pointed objects $\mathrm{Conn}(*/\mathbf{H})$ (and on all of $\mathrm{Grp}_2(\mathbf{H})$ with connected pointed 2-stacks $\mathrm{Conn}(*/\mathbf{Grpd}^{\mathrm{eff}}_2(\mathbf{H}))$). Applying the looping functor Ω to the Maurer–Cartan morphism $\theta_G : G \to \flat_{\mathrm{dR}}B_{\mathbf{H}}G$ of a group object G yields a morphism $\Omega\theta_G : \Omega G \to \Omega\flat_{\mathrm{dR}}B_{\mathbf{H}}G$. Now observe that both $\flat_{\mathrm{dR}}X = * \times_X \flat X$ and $\Omega X = * \times_X *$ are defined as $(2,1)$-pullbacks, and $(2,1)$-pullbacks commute with one another and with the right-exact/left-exact flat modality $\flat$:

$$\Omega\flat_{\mathrm{dR}}B_{\mathbf{H}}G \;\simeq\; \flat_{\mathrm{dR}}\Omega B_{\mathbf{H}}G \;\simeq\; \flat_{\mathrm{dR}}G,$$

where in the second step we used the equivalence $\Omega B_{\mathbf{H}}G \simeq G$. More directly, for *any* pointed object $(A, x : * \to A) \in */\mathbf{H}$, since $\flat * \simeq *$ and $\epsilon_A \circ \flat x \simeq x$, the universal property of the pullback $\flat_{\mathrm{dR}}A = * \times_A \flat A$ immediately induces a canonical morphism

$$\tilde{\theta}^\flat_A : \Omega A = * \times_A * \longrightarrow * \times_A \flat A = \flat_{\mathrm{dR}}A, \tag{4.9}$$

which makes sense for *every* pointed object $A \in */\mathbf{H}$ without any group structure![7] Note, however, that $\tilde{\theta}^\flat_A$ is simply the Maurer–Cartan morphism of the *group* ΩA, so it does not by itself extend the theory to non-groupal coefficients. Notice carefully the degree shift in the domain of (4.9): $\tilde{\theta}^\flat_A$ has domain ΩA rather than A. Consequently, pulling back a closed 1-form coefficient $\sigma_1 : \Omega^1_{\mathrm{cl}}(-;\Omega A) \to \flat_{\mathrm{dR}} A$ along $\tilde{\theta}^\flat_A : \Omega A \to \flat_{\mathrm{dR}} A$ produces the 0-connection object $B^0_{\mathrm{con}}(\Omega A) \to \Omega A$ over ΩA: when $A = B_{\mathbf{H}} G$, $\tilde{\theta}^\flat_{B_{\mathbf{H}} G} : G \simeq \Omega B_{\mathbf{H}} G \to \flat_{\mathrm{dR}} B_{\mathbf{H}} G$ is literally θ_G, so this pullback recovers $B^0_{\mathrm{con}} G \to G$ (whereas when $A = B^2_{\mathbf{H}} G$ for $G \in \mathrm{BrGrp}_2(\mathbf{H})$, $\tilde{\theta}^\flat_{B^2_{\mathbf{H}} G} : B_{\mathbf{H}} G \simeq \Omega B^2_{\mathbf{H}} G \to \flat_{\mathrm{dR}} B^2_{\mathbf{H}} G$ is θ_2, whose pullback with $\Omega^2_{\mathrm{cl}}(-;G) \to \flat_{\mathrm{dR}} B^2_{\mathbf{H}} G$ recovers $B^1_{\mathrm{con}} G \to B_{\mathbf{H}} G$).

(2) **The Shape/De Rham Adjunction Extension ($\eta^{\mathrm{dR}}_A : A \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A$ and $\tilde{\theta}^{\int}_A : \Omega \int_{\mathrm{dR}} A \to A$):** Suppose $\mathbf{H}$ is 1-connected and locally 1-connected, so that we have the adjunction $\int \dashv \flat$ with unit $\eta^{\int}_A : A \to \int A$ and $\flat \int A \xrightarrow{\sim} \int A$. Dual to the homotopy kernel $\flat_{\mathrm{dR}} X = \mathrm{fib}(\flat X \to X)$, define for any pointed object $(A, x : * \to A)$ its *de Rham shape* $\int_{\mathrm{dR}} A$ as the homotopy cofiber ($(2,1)$-coequalizer / pushout) of $A \to \int A$:

$$\begin{array}{ccc} A & \xrightarrow{\eta^{\int}_A} & \int A \\ \downarrow & & \downarrow \\ *_{\mathbf{H}} & \longrightarrow & \int_{\mathrm{dR}} A \end{array}$$

By the universal property of pushouts/pullbacks under the adjunction $\int \dashv \flat$, for any pointed objects $A, Y \in */\mathbf{H}$ there is a natural adjunction equivalence

$$(*/\mathbf{H})\Big(\int_{\mathrm{dR}} A, Y\Big) \;\simeq\; (*/\mathbf{H})(A, \flat_{\mathrm{dR}} Y),$$

exhibiting $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}$ on $*/\mathbf{H}$. This adjunction provides a universal unit morphism with domain A itself for *every* pointed object $A \in */\mathbf{H}$:

$$\eta^{\mathrm{dR}}_A : A \longrightarrow \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A, \tag{4.10}$$

as well as a universal counit $\varepsilon^{\mathrm{dR}}_Y : \int_{\mathrm{dR}} \flat_{\mathrm{dR}} Y \to Y$. For a general pointed object $A \in */\mathbf{H}$, the canonical comparison between the fiber and the cofiber of $\eta^{\int}_A$ goes in the direction $\mathrm{fib}(\eta^{\int}_A) \to \Omega \int_{\mathrm{dR}} A$ (adjoint to $\Sigma\, \mathrm{fib}(\eta^{\int}_A) \to \int_{\mathrm{dR}} A$); there is *no* canonical morphism $\Omega \int_{\mathrm{dR}} A \to \mathrm{fib}(\eta^{\int}_A)$ in general. The companion morphism with codomain A is therefore only defined for group objects (see below):

$$\tilde{\theta}^{\int}_A := \Omega\vartheta_A : \Omega \int_{\mathrm{dR}} A \longrightarrow \Omega B_{\mathbf{H}} A \simeq A \qquad (A \in \mathrm{Grp}(\tau_{\leq 0}\mathbf{H})). \tag{4.11}$$

When $A \in \mathrm{Grp}_2(\mathbf{H})$ is a group object, the Maurer–Cartan morphism $\theta_A : A \to \flat_{\mathrm{dR}} B_{\mathbf{H}} A$ transposes uniquely under $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}$ to a comparison morphism $\vartheta_A : \int_{\mathrm{dR}} A \to B_{\mathbf{H}} A$, through which θ_A factors as $\theta_A = \flat_{\mathrm{dR}}(\vartheta_A) \circ \eta^{\mathrm{dR}}_A : A \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A \to \flat_{\mathrm{dR}} B_{\mathbf{H}} A$; applying the looping functor Ω to ϑ_A and using $\Omega B_{\mathbf{H}} A \simeq A$ (or the adjunction $B_{\mathbf{H}} \dashv \Omega$) then recovers $\tilde{\theta}^{\int}_A \simeq \Omega\vartheta_A : \Omega \int_{\mathrm{dR}} A \to \Omega B_{\mathbf{H}} A \simeq A$.

Proposition 4.21 (Adjunction of Extended Maurer–Cartan Constructions)**.** *In any 1-connected locally 1-connected Grothendieck $(2,1)$-topos $\mathbf{H}$, the de Rham shape and flat de Rham functors form an adjoint pair $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}} : */\mathbf{H} \to */\mathbf{H}$ induced by $\int \dashv \flat$. Under this adjunction and the loop–delooping adjunction, the universal morphisms $\tilde{\theta}^\flat$ of (4.9), η^{dR} of (4.10), and $\tilde{\theta}^{\int}$ of (4.11) are mutual mates.*

[7] This strategy was implicitly used by Bunke–Nikolaus–Völkl [7] and Schreiber [43] in the $(\infty,1)$-categorical setting (the precise numbered statement in [7] should be checked by the authors).

Proof. Since $\mathbf{H}$ is 1-connected and locally 1-connected, $\Gamma \circ \mathrm{LCst} \simeq \mathrm{id}_{\mathbf{Grpd}_2}$, the unit at the terminal object $\eta^{\int}_{*} : * \xrightarrow{\sim} \int *$ is an equivalence, and $\epsilon_{\int A} : \flat \int A \xrightarrow{\sim} \int A$ is a 2-natural equivalence. Explicitly, under the unpointed 2-adjunction equivalence $\mathbf{H}(\int A, Y) \simeq \mathbf{H}(A, \flat Y)$ sending $g : \int A \to Y$ to $\tilde{g} = \flat(g) \circ \epsilon^{-1}_{\int A} \circ \eta^{\int}_{A} : A \to \flat Y$, naturality of $\epsilon : \flat \Rightarrow \mathrm{id}_{\mathbf{H}}$ yields $\epsilon_Y \circ \tilde{g} = \epsilon_Y \circ \flat(g) \circ \epsilon^{-1}_{\int A} \circ \eta^{\int}_{A} \simeq g \circ \epsilon_{\int A} \circ \epsilon^{-1}_{\int A} \circ \eta^{\int}_{A} = g \circ \eta^{\int}_{A}$; hence precomposition $(\eta^{\int}_{A})^* : \mathbf{H}(\int A, Y) \to \mathbf{H}(A, Y)$ is canonically equivalent to postcomposition $(\epsilon_Y)_* : \mathbf{H}(A, \flat Y) \to \mathbf{H}(A, Y)$. For pointed objects $(A, x_A), (Y, y_0) \in */\mathbf{H}$, since $\eta^{\int}_{*} : * \xrightarrow{\sim} \int *$ is an equivalence, the pushout $\int_{\mathrm{dR}} A = * \sqcup_A \int A$ in $\mathbf{H}$ coincides with the cofiber of $\eta^{\int}_{A} : (A, x_A) \to (\int A, \int x_A \circ \eta^{\int}_{*})$ in $*/\mathbf{H}$. Thus, the pointed hom-groupoid $(*/\mathbf{H})(\int_{\mathrm{dR}} A, Y)$ is the homotopy fiber $\{y_0 \circ !_A\} \times_{\mathbf{H}(A,Y)} \mathbf{H}(\int A, Y)$ of $(\eta^{\int}_{A})^*$ (whose trivialization along A restricts along $x_A : * \to A$ to the basepoint datum on $\int * \simeq *$), which under $\mathbf{H}(\int A, Y) \simeq \mathbf{H}(A, \flat Y)$ is canonically equivalent to the pointed homotopy pullback $(*/\mathbf{H})(A, * \times_Y \flat Y) = (*/\mathbf{H})(A, \flat_{\mathrm{dR}} Y)$. This establishes $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}$, with unit $\eta^{\mathrm{dR}}_{A} : A \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A$ corresponding to $\mathrm{id}_{\int_{\mathrm{dR}} A}$. Finally, taking $Y = A$ for $\tilde{\theta}^{\flat}_{A} : \Omega A \to \flat_{\mathrm{dR}} A$, its left transpose under $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}$ is $\int_{\mathrm{dR}} \Omega A \to A$, which on connected pointed objects $A = B_{\mathbf{H}} G$ gives $\vartheta_G : \int_{\mathrm{dR}} G \to B_{\mathbf{H}} G$ satisfying $\theta_G = \flat_{\mathrm{dR}}(\vartheta_G) \circ \eta^{\mathrm{dR}}_{G}$. By definition (4.11), $\tilde{\theta}^{\int}_{G} = \Omega \vartheta_G$, i.e. the loop of the transpose ϑ_G of θ_G under $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}$; for non-groupal A only $\tilde{\theta}^{\flat}_{A}$ and η^{dR}_{A} are defined, and the statement about $\tilde{\theta}^{\int}$ must be restricted to group objects. □

Because the unit $\eta^{\mathrm{dR}}_{A} : A \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A$ of (4.10) has domain A for every pointed object $A \in */\mathbf{H}$, Proposition 4.21 immediately allows us to factor the universal Maurer–Cartan morphism θ_{k+1} and its connection pullback through the group-independent unit $\eta^{\mathrm{dR}}_{B^k_{\mathbf{H}} G}$:

Corollary 4.22 (Factorization of Connection Pullbacks via η^{dR})**.** *In any 1-connected locally 1-connected Grothendieck $(2,1)$-topos $\mathbf{H}$, for every $G \in \mathrm{Del}^{k+1}_2(\mathbf{H})$ $(k \geq 0)$, the universal Maurer–Cartan morphism $\theta_{k+1} : B^k_{\mathbf{H}} G \to \flat_{\mathrm{dR}} B^{k+1}_{\mathbf{H}} G$ factors via the unit $\eta^{\mathrm{dR}}_{B^k_{\mathbf{H}} G}$ of $\int_{\mathrm{dR}} \dashv \flat_{\mathrm{dR}}$ as*

$$\theta_{k+1} \;\simeq\; \flat_{\mathrm{dR}}(\vartheta_{B^k_{\mathbf{H}} G}) \circ \eta^{\mathrm{dR}}_{B^k_{\mathbf{H}} G} : B^k_{\mathbf{H}} G \longrightarrow \flat_{\mathrm{dR}} \int_{\mathrm{dR}} B^k_{\mathbf{H}} G \longrightarrow \flat_{\mathrm{dR}} B^{k+1}_{\mathbf{H}} G.$$

Consequently, defining the pulled-back closed form coefficient object over $\flat_{\mathrm{dR}} \int_{\mathrm{dR}} B^k_{\mathbf{H}} G$ by

$$\tilde{\Omega}^{k+1}_{\mathrm{cl}}(-; G) := \flat_{\mathrm{dR}} \int_{\mathrm{dR}} B^k_{\mathbf{H}} G \times_{\flat_{\mathrm{dR}} B^{k+1}_{\mathbf{H}} G} \Omega^{k+1}_{\mathrm{cl}}(-; G),$$

the moduli stack of k-connections $B^k_{\mathrm{con}} G$ is canonically presented by the pasted $(2,1)$-pullback along the unit $\eta^{\mathrm{dR}}_{B^k_{\mathbf{H}} G}$:

$$\begin{array}{ccccc}
B^k_{\mathrm{con}} G & \longrightarrow & \tilde{\Omega}^{k+1}_{\mathrm{cl}}(-; G) & \longrightarrow & \Omega^{k+1}_{\mathrm{cl}}(-; G) \\
\downarrow{\scriptstyle p^k_{\mathrm{con}}} & & \downarrow{\scriptstyle \tilde{\sigma}_{k+1}} & & \downarrow{\scriptstyle \sigma_{k+1}} \\
B^k_{\mathbf{H}} G & \xrightarrow[\eta^{\mathrm{dR}}_{B^k_{\mathbf{H}} G}]{} & \flat_{\mathrm{dR}} \int_{\mathrm{dR}} B^k_{\mathbf{H}} G & \xrightarrow[\flat_{\mathrm{dR}}(\vartheta_{B^k_{\mathbf{H}} G})]{} & \flat_{\mathrm{dR}} B^{k+1}_{\mathbf{H}} G
\end{array}$$

Proof. Immediate from Proposition 4.21, Definition 4.8, and the pasting law for $(2,1)$-pullbacks. □

Motivated by Corollary 4.22, pulling back a closed form coefficient morphism $\sigma_A : \Omega^{\bullet}_{\mathrm{cl}}(-; A) \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A$ along η^{dR}_{A} produces a pointed connection object $(A)_{\mathrm{con},\flat} := A \times_{\flat_{\mathrm{dR}} \int_{\mathrm{dR}} A} \Omega^{\bullet}_{\mathrm{cl}}(-; A)$ directly over A for an arbitrary pointed object $A \in */\mathbf{H}$, while dually using the counit $\varepsilon^{\mathrm{dR}}_{A} : \int_{\mathrm{dR}} \flat_{\mathrm{dR}} A \to A$ (or $\tilde{\theta}^{\int}_{A} : \Omega \int_{\mathrm{dR}} A \to A$) produces a companion endofunctor $(A)_{\mathrm{con},\int}$:

Problem 4.23 (Pointed Connection Endofunctors for Non-Groupal Coefficients)**.** *(*Stated as an open problem rather than a conjecture: the notion of "functorial differential form assignment" still has to be made precise, and without such a definition the statement can be satisfied trivially, e.g. by setting $\Omega^{\bullet}_{\mathrm{cl}}(-;A) := \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A$ with $\sigma_A = \mathrm{id}$ away from deloopings.*) In a 1-connected locally 1-connected Grothendieck $(2,1)$-topos $\mathbf{H}$ equipped with differential form coefficients, there exists a functorial differential form assignment $A \mapsto (\sigma_A : \Omega^{\bullet}_{\mathrm{cl}}(-;A) \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A)$ on $*/\mathbf{H}$ restricting on $A = B^k_{\mathbf{H}} G$ to $\tilde{\sigma}_{k+1} : \tilde{\Omega}^{k+1}_{\mathrm{cl}}(-;G) \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} B^k_{\mathbf{H}} G$ of Corollary 4.22, such that the pullback of σ_A along $\eta^{\mathrm{dR}}_A : A \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A$ defines a pointed connection endofunctor $(-)_{\mathrm{con},\flat} : */\mathbf{H} \to */\mathbf{H}$ (and dually via $\varepsilon^{\mathrm{dR}}_A$ and $\tilde{\theta}^{\int}_A$ a companion endofunctor $(-)_{\mathrm{con},\int} : */\mathbf{H} \to */\mathbf{H}$) satisfying $(B^k_{\mathbf{H}} G)_{\mathrm{con},\flat} \simeq B^k_{\mathrm{con}} G$ for all $k \geq 0$, with loop fiber $\Omega(A_{\mathrm{con},\flat}) \simeq \mathrm{fib}(\Omega\eta^{\mathrm{dR}}_A : \Omega A \to \flat_{\mathrm{dR}}\Omega \int_{\mathrm{dR}} A)$ whenever $\Omega^{\bullet}_{\mathrm{cl}}(-;A)$ is 0-truncated (which on connected deloopings $A = B^k_{\mathbf{H}} G$ with $\tilde{\sigma}_{k+1}$ as in Corollary 4.22 reduces to $\mathrm{fib}(\tilde{\theta}^{\flat}_A) \simeq \flat\Omega A$), while pulling back $\Omega^1_{\mathrm{cl}}(-;\Omega A) \to \flat_{\mathrm{dR}} A$ along $\tilde{\theta}^{\flat}_A : \Omega A \to \flat_{\mathrm{dR}} A$ recovers $B^0_{\mathrm{con}}(\Omega A)$.

Remark 4.24 (Third Approach: Universal Group Completion). A third approach to extending the Maurer–Cartan morphism to non-groupal objects (such as monoids $M \in \mathbf{Mon}_2(\mathbf{H})$ or general pointed objects $A \in */\mathbf{H}$) is via a *group replacement* (or *group completion*) left 2-adjoint $\mathrm{gr}(-) : \mathbf{Mon}_2(\mathbf{H}) \to \mathrm{Grp}_2(\mathbf{H})$ (or $\Omega B^{\mathrm{cat}}_{\mathbf{H}}(-) : */\mathbf{H} \to \mathrm{Grp}_2(\mathbf{H})$) to the inclusion $\mathrm{Grp}_2(\mathbf{H}) \hookrightarrow \mathbf{Mon}_2(\mathbf{H})$, with unit $\eta^{\mathrm{gr}}_M : M \to \mathrm{gr}(M)$, which is an equivalence whenever M is already a group object [28]. Composing η^{gr}_M with the Maurer–Cartan morphism of $\mathrm{gr}(M)$ yields a canonical Maurer–Cartan morphism

$$\theta^{\mathrm{gr}}_M := \theta_{\mathrm{gr}(M)} \circ \eta^{\mathrm{gr}}_M : M \longrightarrow \mathrm{gr}(M) \longrightarrow \flat_{\mathrm{dR}} B_{\mathbf{H}} \mathrm{gr}(M).$$

However, passing to $\mathrm{gr}(M)$ collapses non-invertible algebraic symmetries into invertible ones; avoiding this collapse while preserving a genuine Maurer–Cartan pullback (so as to encompass non-associative and general algebra-valued gauge symmetries as in [33, 26]) is one of the primary motivations for developing *parameterized enriched $\mathbb{A}$-topoi* in the sequel to this paper.

Example 4.25 (Extended Maurer–Cartan Unit in $\mathbf{SmoothGrpd}_2$)**.** In $\mathbf{H} = \mathbf{SmoothGrpd}_2$, let $(A, x_0 : * \to A)$ be a pointed 0-truncated smooth sheaf (for instance, a pointed smooth manifold or a smooth monoid $M \in \mathbf{Mon}_2(\mathbf{SmoothGrpd}_2)$ pointed at its identity 1_M). Its shape $\int A = \mathrm{LCst}(\Pi_1(A))$ is the locally constant stack associated to the fundamental groupoid $\Pi_1(A)$ (paths up to homotopy rel endpoints; not the thin-homotopy path groupoid used for parallel transport), and the unit $\eta^{\int}_A : A \to \int A$ sends a smooth plot $f : U \to A$ to the path-component/constant-path family at f. Consequently, the de Rham shape $\int_{\mathrm{dR}} A = * \sqcup_A \int A$ collapses the points of A in each plot to the basepoint while retaining the non-trivial smooth path-homotopy loops generated by smooth variation across A, and the flat de Rham fiber $\flat_{\mathrm{dR}} \int_{\mathrm{dR}} A = * \times_{\int_{\mathrm{dR}} A} \flat \int_{\mathrm{dR}} A$ is the stack of smooth A-plots trivialized inside the discrete fundamental groupoid $\flat \int_{\mathrm{dR}} A$. The universal unit $\eta^{\mathrm{dR}}_A : A \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} A$ of (4.10) sends a smooth plot $f : U \to A$ to the canonical path-parallel-transport discrepancy between f and its locally constant flattening $\flat f$, without requiring A to have inverses or group multiplication. When $A = M \in \mathbf{Mon}_2(\mathbf{SmoothGrpd}_2)$ is a non-groupal smooth monoid (such as $(M_n(\mathbb{R}), \cdot, I_n)$), $\eta^{\mathrm{dR}}_M : M \to \flat_{\mathrm{dR}} \int_{\mathrm{dR}} M$ retains the full pointed smooth structure of M before comparison with the group completion $\mathrm{gr}(M)$ of Remark 4.24 (which for $(M_n(\mathbb{R}), \cdot)$ is *trivial*: the zero matrix is absorbing, and in any group $0 \cdot m = 0$ forces $m = 1$; the group of units $GL_n(\mathbb{R})$ is a right adjoint construction, not the group completion).

4.3.4. *Classified Structures.* Recall that in classical Grothendieck 1-topos theory [31, 17], a geometric theory $\mathbb{T}$ possesses a *classifying 1-topos* $\mathbf{Set}[\mathbb{T}]$ equipped with a universal $\mathbb{T}$-model $M_{\mathbb{T}} \in \mathrm{Mod}_{\mathbf{Set}[\mathbb{T}]}(\mathbb{T})$ such that for every Grothendieck 1-topos $\mathcal{E}$, pulling back $M_{\mathbb{T}}$ along geometric morphisms induces an equivalence of categories

$$\mathrm{Mod}_{\mathcal{E}}(\mathbb{T}) \;\simeq\; 1\mathbf{ShvTopos}_2(\mathcal{E}, \mathbf{Set}[\mathbb{T}]).$$

Passing to isomorphism classes of models ($\tau_{\leq 0}$), we recognize the right-hand side as the 0-th intrinsic cohomology of $\mathcal{E}$ with coefficients in $\mathbf{Set}[\mathbb{T}]$ inside the 2-category $1\mathbf{ShvTopos}_2$:

$$\mathrm{Iso}(\mathrm{Mod}_{\mathcal{E}}(\mathbb{T})) \;\cong\; \tau_{\leq 0}(1\mathbf{ShvTopos}_2(\mathcal{E}, \mathbf{Set}[\mathbb{T}])) = H^0_{1\mathbf{ShvTopos}_2}(\mathcal{E}; \mathbf{Set}[\mathbb{T}]).$$

Crucially, this classifying paradigm can be reproduced *internally* within any cohesive $(2,1)$-topos $\mathbf{H}$ (replacing $1\mathbf{ShvTopos}_2$ by $\mathbf{H}$, test topoi $\mathcal{E}$ by objects $X \in \mathbf{H}$, and $\mathbf{Set}[\mathbb{T}]$ by a classifying stack $A \in \mathbf{H}$ with universal structure $E \to A$; see [35, Section 9.3] and [43, Section 5.1.10.1]). Specifically, given a 2-functor $\mathrm{Struct}_A : \mathbf{H}^{\mathrm{op}} \to \mathbf{Grpd}_2$ assigning to each $X \in \mathbf{H}$ the groupoid of geometric A-structures over X, an object $A_{\mathrm{class}} \in \mathbf{H}$ together with a universal structure $u_A \in \mathrm{Struct}_A(A_{\mathrm{class}})$ *classifies* Struct_A if pulling back u_A induces a 2-natural equivalence of groupoids

$$\mathbf{H}(X, A_{\mathrm{class}}) \xrightarrow{\simeq} \mathrm{Struct}_A(X),$$

and hence on equivalence classes a bijection with intrinsic cohomology:

$$\tau_{\leq 0}\mathrm{Struct}_A(X) \;\cong\; H^0(X; A_{\mathrm{class}}).$$

Table 1 summarizes the dictionary between the classifying objects constructed in Sections 3 and 4, the corresponding intrinsic/modal/differential cohomologies, and the geometric structures they classify over an object $X \in \mathbf{H}$.

Classifying Object in H	**Cohomology Set**	**Classified Geometric Structure on X**
$G \in \mathrm{Grp}_2(\mathbf{H})$	$H^0(X; G)$	Smooth/cohesive G-valued functions $X \to G$
$\flat G$	$H^0_\flat(X; G)$	Locally constant G-valued functions on X
$B_{\mathbf{H}} G$	$H^1(X; G)$	Principal G-bundles $P \to X$
$\int B_{\mathbf{H}} G \simeq B_{\mathbf{H}} \int G$	$H^1_{\int}(X; G)$	Topological/homotopy reductions of G-bundles
$\flat B_{\mathbf{H}} G \simeq B_{\mathbf{H}} \flat G$	$H^1_\flat(X; G)$	Flat principal G-bundles (G-local systems) on X
$\flat_{\mathrm{dR}} B_{\mathbf{H}} G$	$H^1_{\mathrm{dR}}(X; G)$	Flat Lie(G)-valued differential 1-forms on X
$\Omega^{k+1}_{\mathrm{cl}}(-; G)$ (or $\Omega^2_{\mathrm{cl}}(-; \mathrm{INN}(G))$)	$H^0(X; \Omega^{k+1}_{\mathrm{cl}}(-; G))$	Closed G-valued curvature $(k+1)$-forms on X
$B_{\mathrm{con}} G = B^1_{\mathrm{con}} G$	$H^1_{\mathrm{diff}}(X; G)$	Principal G-bundles with connection $(P, \nabla) \to X$
$B^k_{\mathbf{H}} G \;\; (G \in \mathrm{Del}^k_2(\mathbf{H}))$	$H^k(X; G)$	Principal G-k-bundles (G-gerbes for $k=2$) on X
$B^k_{\mathrm{con}} G \;\; (G \in \mathrm{Del}^{k+1}_2(\mathbf{H}))$	$H^k_{\mathrm{diff}}(X; G)$	Principal G-k-bundles with k-connection on X

TABLE 1. Summary of classifying objects, cohomologies, and classified geometric structures in a cohesive Grothendieck $(2,1)$-topos $\mathbf{H}$.

REFERENCES

[1] J. C. Baez and J. Dolan, *Categorification*, in *Higher Category Theory*, Contemp. Math. **230**, Amer. Math. Soc., Providence, RI (1998), 1–36.

[2] J. C. Baez and A. D. Lauda, *Higher-dimensional algebra V: 2-groups*, Theory Appl. Categ. **12** (2004), no. 14, 423–491.

[3] J. C. Baez and U. Schreiber, *Higher gauge theory*, in *Categories in Algebra, Geometry and Mathematical Physics*, Contemp. Math. **431**, Amer. Math. Soc., Providence, RI (2007), 7–30.

[4] J. Bénabou, *Introduction to bicategories*, in *Reports of the Midwest Category Seminar*, Lecture Notes in Math. **47**, Springer, Berlin (1967), 1–77.

[5] J. M. Boardman and R. M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Math. **347**, Springer, Berlin (1973).

[6] K. S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419–458.

[7] U. Bunke, T. Nikolaus, and M. Völkl, *Differential cohomology theories as sheaves of spectra*, J. Homotopy Relat. Struct. **11** (2016), no. 1, 1–66.

[8] B. Day and R. Street, *Monoidal bicategories and Hopf algebroids*, Adv. Math. **129** (1997), no. 1, 99–157.

[9] G. Dunn, *Tensor product of operads and iterated loop spaces*, J. Pure Appl. Algebra **50** (1988), no. 3, 237–258.

[10] B. Eckmann and P. J. Hilton, *Group-like structures in general categories. I. Multiplications and comultiplications*, Math. Ann. **145** (1962), 227–255.

[11] D. Fiorenza, C. L. Rogers, and U. Schreiber, *Higher $U(1)$-gerbe connections in geometric prequantization*, Rev. Math. Phys. **28** (2016), no. 6, 1650012, arXiv:1304.0236.

[12] D. Fiorenza, U. Schreiber, and J. Stasheff, *Čech cocycles for differential characteristic classes: an ∞-Lie theoretic construction*, Adv. Theor. Math. Phys. **16** (2012), no. 1, 149–250.

[13] J. Giraud, *Cohomologie non abélienne*, Grundlehren der mathematischen Wissenschaften **179**, Springer, Berlin (1971).

[14] R. Gordon, A. J. Power, and R. Street, *Coherence for tricategories*, Mem. Amer. Math. Soc. **117** (1995), no. 558.

[15] D. Hilbert, *Mathematische Probleme*, Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. (1900), 253–297; English transl. in Bull. Amer. Math. Soc. **8** (1902), no. 10, 437–479.

[16] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J. Differential Geom. **70** (2005), no. 3, 329–452.

[17] P. T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, Vols. 1–2, Oxford Logic Guides **43**–**44**, Oxford Univ. Press, Oxford (2002).

[18] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1–3, 207–222.

[19] A. Joyal and R. Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78.

[20] G. M. Kelly, *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lecture Note Ser. **64**, Cambridge Univ. Press, Cambridge (1982); reprinted in Repr. Theory Appl. Categ. **10** (2005), 1–136.

[21] I. Khavkine and U. Schreiber, *Synthetic geometry of differential equations: I. Jets and comonad structure*, preprint, arXiv:1701.06238 (2017).

[22] S. Lack, *A 2-categories companion*, in *Towards Higher Categories*, IMA Vol. Math. Appl. **152**, Springer, New York (2010), 105–191.

[23] F. W. Lawvere, *Categorical dynamics*, in *Topos Theoretic Methods in Geometry*, Various Publ. Ser. **30**, Aarhus Univ. (1979), 1–28.

[24] F. W. Lawvere, *Cohesive toposes and Cantor's "lauter Einsen"*, Philos. Math. (3) **2** (1994), no. 1, 5–15.

[25] F. W. Lawvere, *Axiomatic cohesion*, Theory Appl. Categ. **19** (2007), no. 3, 41–49.

[26] Y. X. Lecca and W. G. G. Velasco, *Algebra-valued Yang–Mills-type theories: polynomial obstructions and holonomy reductions*, preprint (2026), hexagon:2611.00413v1.

[27] T. Leinster, *Higher Operads, Higher Categories*, London Math. Soc. Lecture Note Ser. **298**, Cambridge Univ. Press, Cambridge (2004).

[28] J. Lurie, *Higher Algebra*, monograph available at https://www.math.ias.edu/~lurie/papers/HA.pdf (version of September 2017).

[29] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies **170**, Princeton Univ. Press, Princeton, NJ (2009).

[30] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Grad. Texts in Math. **5**, Springer, New York (1998).

[31] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Universitext, Springer, New York (1992).

[32] S. Mac Lane and R. Paré, *Coherence for bicategories and indexed categories*, J. Pure Appl. Algebra **37** (1985), 59–80.

[33] Y. X. Martins and R. J. Biezuner, *Topological and geometric obstructions on Einstein–Hilbert–Palatini theories*, J. Geom. Phys. **142** (2019), 229–239, arXiv:1808.09249.

[34] Y. X. Martins and R. J. Biezuner, *Higher category theory and Hilbert's sixth problem*, preprint, hal-02909681 (2021).

[35] Y. X. Martins, *Pursuing Abstraction: Problems and Ideas on Mathematics and Physics*, Ph.D. Thesis, Universidade Federal de Minas Gerais (UFMG), Belo Horizonte (2022).

[36] T. Nikolaus, U. Schreiber, and D. Stevenson, *Principal ∞-bundles: general theory*, J. Homotopy Relat. Struct. **10** (2015), no. 4, 749–801.

[37] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007.

[38] C. Rezk, *Toposes and homotopy toposes*, monograph available at https://faculty.math.illinois.edu/~rezk/homotopy-topos-sketch.pdf (2010).

[39] E. Riehl and D. Verity, *Elements of ∞-Category Theory*, Cambridge Studies in Advanced Mathematics **194**, Cambridge Univ. Press, Cambridge (2022).

[40] D. M. Roberts, *Internal categories, anafunctors and localisations*, Theory Appl. Categ. **26** (2012), no. 29, 788–829.

[41] C. J. Schommer-Pries, *The classification of two-dimensional extended topological field theories*, Ph.D. Thesis, Univ. of California, Berkeley, arXiv:1112.1000 (2011).

[42] U. Schreiber, *Classical field theory via cohesive homotopy types*, preprint, arXiv:1311.1172 (2013).

[43] U. Schreiber, *Differential cohomology in a cohesive ∞-topos*, monograph, arXiv:1310.7930 (2013).

[44] U. Schreiber and M. Shulman, *Quantum gauge field theory in cohesive homotopy type theory*, in *Proceedings 9th Workshop on Quantum Physics and Logic (QPL 2012)*, Electron. Proc. Theor. Comput. Sci. **158** (2014), 109–126, arXiv:1408.0054.

[45] C. Simpson, *Homotopy types of strict 3-groupoids*, preprint, `arXiv:math/9810059` (1998).
[46] C. Simpson, *Homotopy Theory of Higher Categories*, New Mathematical Monographs **19**, Cambridge Univ. Press, Cambridge (2011).
[47] R. Street, *Fibrations in bicategories*, Cah. Topol. Géom. Différ. Catég. **21** (1980), no. 2, 111–160.
[48] R. Street, *Two-dimensional sheaf theory*, J. Pure Appl. Algebra **23** (1982), no. 3, 251–270.
[49] R. Street, *Characterization of bicategories of stacks*, in *Category Theory (Gummersbach, 1981)*, Lecture Notes in Math. **962**, Springer, Berlin (1982), 282–291.
[50] F. Wellen, *Formalizing Cartan Geometry in Modal Homotopy Type Theory*, Ph.D. Thesis, Karlsruher Institut für Technologie (KIT) (2017), `doi:10.5445/IR/1000073164`; see also *Cartan geometry in modal homotopy type theory*, `arXiv:1806.05966`.
[51] C.-N. Yang and R. L. Mills, *Conservation of isotopic spin and isotopic gauge invariance*, Phys. Rev. **96** (1954), 191–195.

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