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SMOOTH NUMBERS IN ARITHMETIC PROGRESSIONS WITH UNIFORMITY IN THE RESIDUE CLASS

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Abstract. We improve the level of distribution of smooth numbers in arithmetic progressions to $x^{5/8-\varepsilon}$, with uniformity in the residue class. In particular, for every $\varepsilon, A > 0$ and integer $k \geq 1$, there exist $\delta, C > 0$ such that

$$\sum_{\substack{q \leq x^{5/8-\varepsilon} \\ (q, a_1 a_2)=1}} \tau_k(q) \sup_{0 \leq z \leq x} \left| \Psi(z, y; a_1 \overline{a_2}, q) - \frac{1}{\varphi(q)} \Psi_q(z; y) \right| \ll_{\varepsilon, A, k} \frac{\Psi(x; y)}{(\log x)^A},$$

uniformly for $x \geq 2$, $(\log x)^C \leq y \leq x^{1/C}$, and $a_1, a_2 \in \mathbb{Z} \setminus \{0\}$ satisfying

$$|a_1| \leq x^{1+\delta}, \qquad |a_2| \leq x^{\delta}, \qquad (a_1, a_2) = 1.$$

Here $\Psi(z; y)$ counts the y-smooth integers up to z, $\Psi_q(z; y)$ counts those coprime to q, and $\Psi(z, y; a, q)$ counts those congruent to a modulo q; τ_k is the k-fold divisor function. The proof combines dispersion with second and fourth spectral moments.

1. Introduction

Let $P^+(n)$ denote the largest prime factor of a positive integer n, with $P^+(1) = 1$. We call n y-smooth if $P^+(n) \leq y$. For $x \geq 0$, $y \geq 2$, $q \in \mathbb{Z}_{>0}$ and $a \in \mathbb{Z}$, write

$$\begin{aligned} \Psi(x; y) &:= \left| \{ n \in \mathbb{Z}_{>0} : n \leq x,\ P^+(n) \leq y \} \right|, \\ \Psi_q(x; y) &:= \left| \{ n \in \mathbb{Z}_{>0} : n \leq x,\ P^+(n) \leq y,\ (n, q) = 1 \} \right|, \\ \Psi(x, y; a, q) &:= \left| \{ n \in \mathbb{Z}_{>0} : n \leq x,\ P^+(n) \leq y,\ n \equiv a \pmod q \} \right|. \end{aligned}$$

For $(a, q) = 1$, the expected count in the class $a \pmod q$ is $\Psi_q(x; y)/\varphi(q)$, where φ is Euler's totient function. We study the discrepancy from this count, averaged over the moduli, while allowing the residue class and the summation endpoint to vary.

Our main result gives exponent of distribution $5/8$, uniformly in residue classes of the form $a_1 \overline{a_2} \pmod q$. Here $\overline{a_2}$ denotes the inverse of a_2 modulo q, and $\tau_k(q)$ is the number of ordered factorisations of q into k positive integers.

Theorem 1. *For any $\varepsilon, A > 0$ and integer $k \geq 1$, there exist $\delta, C > 0$ such that the following holds. Let $x \geq 2$ and $a_1, a_2 \in \mathbb{Z} \setminus \{0\}$ satisfy*

$$|a_1| \leq x^{1+\delta}, \qquad |a_2| \leq x^{\delta}, \qquad (a_1, a_2) = 1.$$

Uniformly for $(\log x)^C \leq y \leq x^{1/C}$, we have

$$\sum_{\substack{q \leq x^{5/8-\varepsilon} \\ (q, a_1 a_2)=1}} \tau_k(q) \sup_{0 \leq z \leq x} \left| \Psi(z, y; a_1 \overline{a_2}, q) - \frac{1}{\varphi(q)} \Psi_q(z; y) \right| \ll_{\varepsilon, A, k} \frac{\Psi(x; y)}{(\log x)^A}.$$

Section 2 outlines the argument. The convolution estimate used in the proof is stated in Section 3. We deduce Theorem 1 in Section 4 and prove the convolution estimate in Section 5, using spectral moments and Kloosterman sums.

1.1. **AI Usage.** This manuscript was largely generated using ChatGPT/Codex, including much of the mathematical argumentation, exposition, and LaTeX code. It builds on an earlier draft by the author concerning Balog's additive-representation problem, which used the exponent of distribution 153/256. The spectral-moment strategy was suggested by ChatGPT in response to the author's prompts. The author takes responsibility for the final manuscript.

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1.3. **Notation.** The letter p, with or without subscripts, is reserved for primes; sums and products indexed by p are therefore over primes. All logarithms are natural. We write $\mathbb{Z}_{>0}$ for the positive integers, (a,b) for the greatest common divisor, and $[a,b]$ for the least common multiple of a and b; the same conventions apply to more than two integers. Divisors in sums and products are positive unless otherwise specified. For a summation variable, $n \sim N$ means $N < n \le 2N$. We write $\mathbf{1}_E$ for the indicator of a condition E.

We use Landau's notation $F = O(G)$ and Vinogradov's notation $F \ll G$ or $G \gg F$ to mean $|F| \le CG$ for some constant $C > 0$, with $G \ge 0$. Subscripts indicate permitted dependence of the implied constant, as in O_ε or $\ll_\varepsilon$, and constants may change from line to line. For positive quantities, $F \asymp G$ means $F \ll G \ll F$. In a specified limit, $F = o(G)$ means $F/G \to 0$. In particular, $F(x) \ll x^{o(1)}$ means $F(x) \ll_\varepsilon x^\varepsilon$ for every $\varepsilon > 0$.

We write μ for the Möbius function and $\tau = \tau_2$ for the divisor function. The radical is $\mathrm{rad}(n) = \prod_{p \mid n} p$, and $d \mid n^\infty$ means that every prime factor of d divides n. A star on a residue sum restricts it to reduced residues; a star on a character sum restricts it to primitive characters. Overbars on complex coefficients denote complex conjugation; overbars on integers in modular expressions denote multiplicative inverses modulo the indicated modulus.

For a bounded function F, write $\|F\|_\infty = \sup_u |F(u)|$. We use J_ν for the Bessel function of the first kind and K_ν for the modified Bessel function of the second kind. The spectral notation is introduced in Section 5 and Appendix A as needed.

2. Outline

We explain the exponent 5/8 and the uniformity in the residue class, following the order of the overview in [13, Section 2]. Throughout this sketch, estimates are understood up to factors $x^{o(1)}$, and we suppress smooth weights and divisor sums. To keep the notation simple, take $a_2 = 1$ and $a_1 = a$ with $1 \le a \le x$. Coprimality conditions needed for multiplicative inverses are understood. The full proof treats both signs and restores the small powers of x needed for the stated residue range.

2.1. **Reduction to exponential sums.** The factorisation of smooth numbers allows us to choose three factor lengths M, N, L with $MNL = x$. After treating characters of small conductor separately, we have to bound

$$\sum_{\substack{r\sim R\\(r,a)=1}} \left| \sum_{m\sim M} \sum_{n\sim N} \sum_{\ell\sim L} \alpha_m \beta_n \gamma_\ell \, \mathcal{E}_D(mn\ell\overline{a}; r) \right|,$$

where $\mathcal{E}_D$ is the character remainder defined in Section 3. We seek a saving over its natural size x, with R as large as possible. The limiting choice of lengths is

$$M = L = \frac{x}{R}, \qquad N = \frac{R^2}{x}, \qquad K = NL = R. \tag{2.1}$$

These lengths indicate the exponent; they will be adjusted slightly at the end of the outline.

Linnik's dispersion method begins with Cauchy–Schwarz in r, m. Opening the square introduces two copies of $n\ell$. Poisson summation in m produces a frequency of length

$$H \approx \frac{R}{M} = N.$$

The zero frequency combines with the other dispersion main terms. In the nonzero frequencies, write one of the products as $k \asymp K$ and pass from r to the complementary divisor

$$t = \frac{n\ell - k}{r}, \qquad |t| \ll \frac{K}{R} \approx 1.$$

Reciprocity and a further application of Cauchy–Schwarz lead to sums of the form

$$\mathcal{K} = \sum_{n,n'\sim N} \alpha_{n,n'} \sum_{\substack{h,h'\sim H\\ n'h\neq nh'}} \sum_{\substack{k\asymp K\\ \ell\asymp L}} e\left(\frac{a(n'h - nh')\overline{nn'\ell}}{k}\right), \qquad e(u) = e^{2\pi i u}. \tag{2.2}$$

Here the bounded complementary divisor is suppressed. The coefficients $\alpha_{n,n'}$ are bounded, and the h, h', k, ℓ sums have smooth weights. The diagonal $n'h = nh'$ is estimated separately. If $\mathcal{R}$ denotes the resulting off-diagonal dispersion error, the second Cauchy–Schwarz step gives schematically

$$|\mathcal{R}| \ll \frac{M}{R}(KL)^{1/2}|\mathcal{K}|^{1/2}.$$

The dispersion main term has size $MK^2/R = xK/R$, so it remains to save a power of x in

$$\left(\frac{|\mathcal{R}|}{MK^2/R}\right)^4 \ll \frac{(KL)^2|\mathcal{K}|^2}{K^8} = \frac{|\mathcal{K}|^2}{N^6L^4}. \tag{2.3}$$

2.2. **The spectral estimates.** For the spectral calculation, first take n, n' squarefree with $(n, n') = 1$ and $|n'h - nh'| \asymp N^2$. We display only the positive frequencies. Completing the ℓ-sum in (2.2) introduces a frequency m of length $K/L \approx N$ and classical Kloosterman sums

$$S(m\overline{q}, av; k), \qquad q = nn', \quad v = n'h - nh'.$$

Kuznetsov's formula then introduces forms of level q. Thus the relevant sizes are

$$q \asymp Q = N^2, \qquad m \asymp N, \qquad v \asymp Q.$$

The spectral level Q is different from the original modulus size R. Common factors of n, n' and smaller values of $|v|$ are handled in the full proof.

Write $\rho_{f,\mathfrak{a}}(n)$ for the Fourier coefficients in the normalisation of Section 5, and $\lambda_f(n)$ for the Hecke eigenvalues. We first consider the exceptional Maass contribution $\mathcal{K}_{\mathrm{exc}}$. Put $\theta_f = \sqrt{1 - 4\lambda_f}$ and

$\kappa_f = i\theta_f/2$, where λ_f without an argument is the Laplace eigenvalue. The Kuznetsov transform contributes the factor $\max(1, X)^{\theta_f}$, where

$$X^2 \approx \frac{K^2 Q}{aNQ} = \frac{x}{a}.$$

The inverse Hecke relation separates the residue a from the Fourier index av:

$$\rho_{f,0}(av) = \sum_{e|(a,v)} \mu(e)\lambda_f(a/e)\rho_{f,0}(v/e) \qquad ((a,q)=1).$$

Ignoring these divisor sums for the sketch, the two Fourier polynomials have the shapes

$$P_f = \sum_{m \asymp N} \psi(m/N)\rho_{f,\infty}(m), \qquad L_f = \sum_{v \asymp Q} c_v \rho_{f,0}(v),$$

where ψ is a fixed smooth cutoff and, for each pair n, n', the coefficients c_v come from the smooth sum over h, h' subject to $n'h - nh' = v$. The coefficients of P_f are common to all levels; those of L_f retain this additive structure.

The residue now occurs through $\lambda_f(a)$. A pointwise bound for this eigenvalue would lose a power of a. Instead, we apply Hölder with exponents $2, 4, 4$, keeping it in a second moment. Put

$$W = N, \qquad V = \max\left\{1, \frac{x}{a\sqrt{N}}\right\}.$$

Then $V^{\theta_f/2} W^{\theta_f/4} \geq \max(1, X)^{\theta_f}$ at the limiting sizes, up to a constant. Consequently,

$$|\mathcal{K}_{\mathrm{exc}}|^2 \ll L^2 Q \left(\sum_{q \asymp Q} \sum_f V^{\theta_f} |\lambda_f(a)|^2\right) \times \left(\sum_{q \asymp Q} \sum_f \frac{W^{\theta_f}|P_f|^4}{\cosh^2(\pi\kappa_f)}\right)^{1/2} \left(\sum_{q \asymp Q} \sum_f \frac{|L_f|^4}{\cosh^2(\pi\kappa_f)}\right)^{1/2}.$$

Here f runs over the exceptional Maass forms at level q, and the levels satisfy $(q, a) = 1$.

The fourth moment of P_f, obtained from Watt's estimate and the Hecke relation, admits the weight $W = Q^2/N^3 = N$. To bound the fourth moment of L_f, we use the additive structure of c_v, Poisson summation and Voronoi summation. At these lengths, Lemmas 9 and 11 give

$$\left(\sum_{q \asymp Q} \sum_f \frac{W^{\theta_f}|P_f|^4}{\cosh^2(\pi\kappa_f)}\right)^{1/2} \ll N, \qquad \left(\sum_{q \asymp Q} \sum_f \frac{|L_f|^4}{\cosh^2(\pi\kappa_f)}\right)^{1/2} \ll Q.$$

For the Hecke factor, with $\theta = 7/32$, the fixed-index second moment in Lemma 6 gives

$$\sum_{q \asymp Q} \sum_f V^{\theta_f} |\lambda_f(a)|^2 \ll Q(Q + \sqrt{a}) \max\left\{1, \frac{x}{\sqrt{N}\max\{a, Q^2\}}\right\}^{\theta}.$$

The factor after Q is nondecreasing in a, up to an absolute constant, since $\theta < 1/2$. For $a \leq x$ and $Q \leq \sqrt{x}$, it is therefore $O(\sqrt{x})$. Combining the three estimates yields

$$|\mathcal{K}_{\mathrm{exc}}|^2 \ll L^2 Q \cdot Q\sqrt{x} \cdot N \cdot Q = L^2 N^7 \sqrt{x}. \tag{2.4}$$

This is the step that allows the residue to grow with x. The exceptional weights are retained in the moments, and the residue stays fixed while its Hecke eigenvalue is averaged over forms and levels.

Proposition 14 proves the same bound for $\mathcal{K}$, including the real Maass, holomorphic and Eisenstein contributions and the suppressed common factors.

2.3. **The exponent** $5/8$**.** Substituting (2.4) into (2.3) gives

$$\left(\frac{|\mathcal{R}|}{MK^2/R}\right)^4 \ll \frac{N\sqrt{x}}{L^2} = \frac{R^4}{x^{5/2}}.$$

Thus a power saving is available when $R \le x^{5/8-\varepsilon}$. The condition $Q = N^2 \le \sqrt{x}$ used above gives the same endpoint.

At the exact lengths (2.1), however, the diagonal ratio $N^2L^3/(xR)$ equals 1. To obtain a saving there as well, choose a small fixed $\lambda > 0$ and take

$$M = \frac{x^{1-\lambda}}{R}, \qquad L = \frac{x^{1-3\lambda}}{R}, \qquad N = \frac{R^2}{x^{1-4\lambda}}.$$

Then $MNL = x$, $L/M = x^{-2\lambda}$ and $NL/R = x^{\lambda}$. Including the complementary-divisor cost from Proposition 2, the two relevant ratios become

$$\frac{N^2L^3}{xR} = x^{-\lambda}, \qquad \left(\frac{NL}{R}\right)^{C_*} \frac{N\sqrt{x}}{L^2} = R^4 x^{-5/2+(C_*+10)\lambda}.$$

Both save a fixed power for $R \le x^{5/8-\varepsilon}$ when λ is sufficiently small in terms of ε. Section 4 checks the remaining length conditions. The auxiliary squarefree and coprimality restrictions, the range of a_2, and the small extension $|a_1| \le x^{1+\delta}$ are accounted for in the convolution proof without changing this exponent.

3. Convolution estimate

We introduce the character sums used to separate small and large conductors in the proof of Theorem 1. Let

$$\omega_D(k;r) := \sum_{\substack{\chi \in \mathcal{X}_D \\ \mathrm{cond}(\chi)\mid r}} \chi(k), \quad \text{where} \quad \mathcal{X}_D := \{\chi \text{ primitive} : \mathrm{cond}(\chi) \le D\}.$$

Note that

$$|\omega_D(k;r)| \le |\mathcal{X}_D| \le D^2.$$

Let

$$\mathcal{E}_D(k;r) := \mathbf{1}_{k\equiv 1 \pmod r} - \frac{\mathbf{1}_{(k,r)=1}}{\varphi(r)}\omega_D(k;r) = \frac{\mathbf{1}_{(k,r)=1}}{\varphi(r)} \sum_{\substack{\chi \pmod r \\ \mathrm{cond}(\chi) > D}} \chi(k).$$

The following convolution estimate controls the contribution of characters with large conductors in Theorem 1.

Proposition 2. *There is an absolute constant $C_* > 1$ with the following property. For every $\kappa > 0$, there is a $\delta > 0$ such that the following holds. Let $M, N, L, R \gg 1$, $x = MNL$, and let $a_1, a_2 \in \mathbb{Z} \setminus \{0\}$ satisfy*

$$|a_1| \le x^{1+\delta}, \qquad |a_2| \le x^{\delta}, \qquad (a_1, a_2) = 1.$$

Let $(\alpha_m)_{m\sim M}$, $(\beta_n)_{n\sim N}$ and $(\gamma_\ell)_{\ell\sim L}$ be 1-bounded complex sequences. Suppose

$$\begin{aligned} &N \ge x^{\kappa}, \qquad NL \le x^{2/3-5\kappa}, \qquad L \le x^{-\kappa}M, \qquad M \le R \le x^{-\kappa}NL, \\ &N^2 \le x^{1/2-\kappa}, \qquad \frac{N^2L^3}{xR} \le x^{-\kappa}, \qquad \left(\frac{NL}{R}\right)^{C_*} \frac{N\sqrt{x}}{L^2} \le x^{-\kappa}. \end{aligned} \tag{3.1}$$

Then, for every $1 \le D \le x^{\kappa}$,

$$\sum_{\substack{r\sim R\\(r,a_1a_2)=1}} \left| \sum_{m\sim M}\sum_{n\sim N}\sum_{\ell\sim L} \alpha_m\beta_n\gamma_\ell\, \mathcal{E}_D(mn\ell\overline{a_1}a_2; r)\right| \ll_\kappa \frac{x(\log x)^4}{\sqrt{\min\{x^\delta, D\}}}.$$

The proof is given in Section 5. We follow the dispersion arguments of Drappeau [5, Théorème 3] and Pascadi [13, Proposition 6.3]. The dependence on the residue is controlled by a second moment for its Hecke eigenvalue and fourth moments for the two remaining spectral polynomials. We first assume that β_n is supported on squarefree integers and that the auxiliary coprimality condition (5.26) below holds. These restrictions are removed at the end of the proof by separating powerful parts and common factors, and applying the multiplicative large sieve.

4. Proof of Theorem 1

We use the standard smooth-number estimate

$$\Psi(x; (\log x)^C) = x^{1-1/C+o(1)} \qquad (x \to \infty) \tag{4.1}$$

for each fixed $C > 1$, which follows from [8, Theorems 1 and 2(i)]. By monotonicity in y, for every fixed $c > 0$ we therefore have $\Psi(x; y) \ge x^{1-c}$ when $y \ge (\log x)^C$, provided C and then x are sufficiently large.

Proof. We use Proposition 2 and give the factor lengths explicitly.

Write $\Delta(z; q) := \Psi(z, y; a_1\overline{a_2}, q) - \Psi_q(z; y)/\varphi(q)$. We first prove the estimate without the supremum over z, that is, for any fixed $B > 0$,

$$\sum_{\substack{q\le x^{5/8-\varepsilon}\\(q,a_1a_2)=1}} \tau_k(q)\,|\Delta(x; q)| \ll_{\varepsilon,B,k} \frac{\Psi(x; y)}{(\log x)^B}. \tag{4.2}$$

Following the proof of [13, Theorem 7.1], with $q_0 = 1$, $y_1 = y$, $y_2 = x$, and the given k, split the characters at conductor $D = x^\eta$, where $\eta > 0$ is sufficiently small. The argument in [13, p. 40, paragraph beginning "For S_{small}"], with $A = B + 1$ and summation over dyadic ranges of moduli, gives, for sufficiently large C,

$$\sum_{q\le x^{5/8-\varepsilon}} \frac{\tau_k(q)}{\varphi(q)} \sum_{\substack{\chi \pmod q\\1<\mathrm{cond}(\chi)\le D}} \left| \sum_{\substack{1\le n\le x\\P^+(n)\le y}} \chi(n)\right| \ll_{\varepsilon,B,k} \frac{\Psi(x; y)}{(\log x)^B}.$$

This bound is uniform in the residue: after the triangle inequality, the factors $\overline{\chi(a_1)}\chi(a_2)$ disappear as they have modulus one for $(q, a_1a_2) = 1$.

For $D \le R \le x^{1/2}$, Harper's estimate [7, Proposition 2] gives

$$\sum_{q\le R}\frac{1}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \mathrm{cond}(\chi)>D}}\left|\sum_{\substack{1\le n\le x\\ P^+(n)\le y}}\chi(n)\right| \ll (\log x)^{7/2}\sqrt{\Psi(x;y)}\left(R+x^{1/2-\eta}(\log x)^2\right),$$

provided η is sufficiently small and C is sufficiently large. Taking $R = x^{1/2-c}$, with $c > 0$ sufficiently small and fixed, and using $\Psi(x;y) \le x$, gives $O(x^{1-c_1})$ for some $c_1 > 0$. This bound is independent of the residue class.

For the remaining moduli we use Proposition 2. We give the length calculation, since it determines the improved exponent. We may assume $0 < \varepsilon < 1/100$. Choose

$$0 < \lambda < \min\left\{\frac{1}{1000}, \frac{\varepsilon}{100(C_*+100)}\right\}, \qquad \kappa = \frac{\lambda}{4},$$

and take the constant c in the preceding paragraph smaller than $\lambda/1000$. Let $\delta_* > 0$ be supplied by Proposition 2 with parameter κ. Choose $\zeta, \delta > 0$ sufficiently small in terms of $\varepsilon, \lambda, \delta_*$, and then choose the conductor exponent $\eta > 0$ sufficiently small in terms of ζ, κ, δ_*. The constant C is increased last.

First discard the integers $n \le x^{1-\zeta}$. The kernel $\mathcal{E}_D$ defined in Section 3 satisfies

$$\sum_{\substack{q\le x^{5/8-\varepsilon}\\ (q,a_1a_2)=1}}\left|\sum_{\substack{n\le x^{1-\zeta}\\ P^+(n)\le y}}\mathcal{E}_D(na_2\overline{a_1};q)\right| \ll x^{1-\zeta+2\eta+o(1)} + x^{5/8-\varepsilon+o(1)}.$$

Indeed, the progression term is bounded by summing the divisors of $a_2n - a_1$; the possible equality $a_2n = a_1$ contributes at most one term for each modulus. The other term is bounded using $|\omega_D| \le D^2$ and $\sum_{q\le x} 1/\varphi(q) \ll \log x$. Since $2\eta < \zeta$, this is a power saving.

Split the remaining integers into intervals $(X, 2X]$ with $X \gg x^{1-\zeta}$, and split the moduli into dyadic intervals $q \sim R$. By the choices above, the retained blocks satisfy

$$X^{1/2-\lambda/100} \le R \le X^{5/8-\varepsilon/2}, \qquad |a_1| \le X^{1+\delta_*}, \quad |a_2| \le X^{\delta_*}, \quad D \le X^{\min\{\kappa,\delta_*\}}.$$

For each such block put

$$M_0 = \frac{X^{1-\lambda}}{R}, \qquad L_0 = \frac{X^{1-3\lambda}}{R}, \qquad N_0 = \frac{R^2}{X^{1-4\lambda}}, \qquad K_0 = N_0L_0.$$

Then

$$M_0N_0L_0 = X, \qquad \frac{K_0}{R} = X^{\lambda}, \qquad \frac{L_0}{M_0} = X^{-2\lambda}, \qquad \frac{N_0^2L_0^3}{XR} = X^{-\lambda}.$$

Writing $R = X^\rho$, the two critical inequalities are

$$\begin{aligned}\log(N_0^2) &= (4\rho - 2 + 8\lambda)\log X \\ &\le \left(\frac{1}{2} - 2\varepsilon + 8\lambda\right)\log X < \left(\frac{1}{2} - \kappa\right)\log X,\\ \log\left\{\left(\frac{K_0}{R}\right)^{C_*}\frac{N_0\sqrt{X}}{L_0^2}\right\} &= \left(4\rho - \frac{5}{2} + (C_*+10)\lambda\right)\log X \\ &\le \big(-2\varepsilon + (C_*+10)\lambda\big)\log X < -\varepsilon\log X.\end{aligned}$$

The remaining conditions in (3.1) follow from

$$\log N_0 = (2\rho - 1 + 4\lambda)\log X > 3\lambda\log X,$$
$$M_0 \le R, \qquad K_0 = RX^\lambda \le X^{2/3-5\kappa}.$$

All these inequalities retain a fixed positive margin under sufficiently small fixed-power changes in the lengths.

Now use the smooth-number decomposition in the proof of [13, Theorem 6.4], with these target lengths. Its separated, 1-bounded coefficient sequences have dyadic lengths differing from M_0, N_0, L_0 by powers of y of bounded degree, and their product is comparable to X. The separation integrals and dyadic sums have polylogarithmic total mass, as in [5, Section 3.7]. Taking C sufficiently large makes these length changes smaller than the margins above. The residue and cutoff inequalities likewise hold at the actual product of the lengths. Proposition 2 therefore gives $O(x^{1-\kappa_1})$ for some $\kappa_1 > 0$, after summing the blocks and the separation integrals.

Combining the two ranges gives, for some $c_2 > 0$,

$$\sum_{\substack{q\le x^{5/8-\varepsilon}\\(q,a_1a_2)=1}}\frac{1}{\varphi(q)}\left|\sum_{\substack{\chi \pmod q\\ \mathrm{cond}(\chi)>D}}\overline{\chi(a_1)}\chi(a_2)\sum_{\substack{1\le n\le x\\ P^+(n)\le y}}\chi(n)\right| = O(x^{1-c_2}).$$

For fixed k, multiplication by $\max_{q\le x}\tau_k(q) = x^{o(1)}$ leaves $O(x^{1-c_2/2})$, which is $O(\Psi(x;y)(\log x)^{-B})$ by (4.1) once C is sufficiently large. Thus, with suitable constants depending on ε, B, k, we have established (4.2).

We now introduce the supremum, following the argument in the proof of [5, Corollaire 1]. Put $Q = x^{5/8-\varepsilon}$ and $t_0 = x^{1-\gamma}$, where $\gamma > 0$ will be chosen sufficiently small in terms of ε, A, k. We first consider $0 \le z < t_0$. For $(q, a_1a_2) = 1$, the triangle inequality gives

$$\begin{aligned}\sup_{0\le z<t_0}|\Delta(z;q)| &\le \Psi(t_0,y;a_1\overline{a_2},q) + \frac{\Psi_q(t_0;y)}{\varphi(q)}\\ &\le \sum_{\substack{1\le n\le t_0\\ q\mid a_2n-a_1}} 1 + \frac{t_0}{\varphi(q)}.\end{aligned}$$

For $a_2n \ne a_1$, the divisor bound gives

$$\sum_{q\mid\mid a_2n-a_1|}\tau_k(q) = \tau_{k+1}(|a_2n-a_1|) \ll_k x^{o(1)}.$$

There is at most one integer n with $a_2 n = a_1$. Summing over q therefore gives

$$\begin{aligned}
&\sum_{\substack{q\le Q\\(q,a_1a_2)=1}} \tau_k(q) \sup_{0\le z<t_0} |\Delta(z;q)| \\
&\le \sum_{\substack{1\le n\le t_0\\ a_2n\neq a_1}} \tau_{k+1}(|a_2n-a_1|) + \sum_{q\le Q}\tau_k(q) + t_0\sum_{q\le Q}\frac{\tau_k(q)}{\varphi(q)} \\
&\ll_k x^{1-\gamma+o(1)} + Q(\log x)^{k-1}.
\end{aligned}$$

Here we used $\sum_{q\le Q}\tau_k(q)/\varphi(q) \ll_k (\log x)^k$. For sufficiently small γ and sufficiently large C, the last bound is $O_{\varepsilon,A,k}(\Psi(x;y)(\log x)^{-A})$, by (4.1), since $\Psi(x;y)\ge x^{1-\gamma/2}$ for sufficiently large x.

We now consider the remaining range $t_0\le z\le x$. Apply (4.2) with $\varepsilon/2$ and B sufficiently large in terms of A,k, and denote its constants by δ_0, C_0. Taking γ,δ sufficiently small and C sufficiently large ensures that, for $t_0\le t\le x$,

$$Q\le t^{5/8-\varepsilon/2},\qquad |a_1|\le t^{1+\delta_0},\qquad |a_2|\le t^{\delta_0},\qquad (\log t)^{C_0}\le y\le t^{1/C_0}.$$

Thus (4.2) applies at every such t. Choose a partition

$$t_0<t_1<\cdots<t_J=x,\qquad J\ll(\log x)^{A+k+1},$$

such that

$$\Psi(t_{j+1};y)-\Psi(t_j;y)\le 1+\frac{\Psi(x;y)}{(\log x)^{A+k+1}}\qquad (0\le j<J).$$

For $t_j\le z\le t_{j+1}$, the number of smooth integers $n\le z$ in the residue class $a_1\overline{a_2} \pmod q$ lies between the corresponding counts up to t_j and t_{j+1}. Hence

$$\begin{aligned}
\Delta(z;q) &\le \Delta(t_{j+1};q)+\frac{\Psi_q(t_{j+1};y)-\Psi_q(z;y)}{\varphi(q)},\\
\Delta(z;q) &\ge \Delta(t_j;q)-\frac{\Psi_q(z;y)-\Psi_q(t_j;y)}{\varphi(q)}.
\end{aligned}$$

Each numerator on the right counts smooth integers coprime to q in a subinterval of $(t_j,t_{j+1}]$, so it is at most $\Psi(t_{j+1};y)-\Psi(t_j;y)$. Thus

$$|\Delta(z;q)|\le \max\{|\Delta(t_j;q)|,|\Delta(t_{j+1};q)|\}+\frac{\Psi(t_{j+1};y)-\Psi(t_j;y)}{\varphi(q)}.$$

Summing over q and using (4.2), together with $\sum_{q\le Q}\tau_k(q)/\varphi(q)\ll_k(\log x)^k$, we obtain

$$\begin{aligned}
&\sum_{\substack{q\le Q\\(q,a_1a_2)=1}} \tau_k(q)\sup_{t_0\le z\le x}|\Delta(z;q)| \\
&\le \sum_{j=0}^{J}\sum_{\substack{q\le Q\\(q,a_1a_2)=1}}\tau_k(q)\,|\Delta(t_j;q)| + \left(1+\frac{\Psi(x;y)}{(\log x)^{A+k+1}}\right)\sum_{q\le Q}\frac{\tau_k(q)}{\varphi(q)} \\
&\ll_{\varepsilon,A,k} \frac{(J+1)\Psi(x;y)}{(\log x)^B} + \left(1+\frac{\Psi(x;y)}{(\log x)^{A+k+1}}\right)(\log x)^k \\
&\ll_{\varepsilon,A,k} \frac{\Psi(x;y)}{(\log x)^{A+1}} + (\log x)^k \ll_{\varepsilon,A,k} \frac{\Psi(x;y)}{(\log x)^A}.
\end{aligned}$$

Combining the two ranges completes the proof. □

5. Triple convolution estimate with uniformity in residue class

We prove Proposition 2 by establishing the required spectral moments, passing to Kloosterman sums, and then applying the dispersion argument. We treat the exceptional Maass spectrum here. The real Maass, holomorphic and continuous spectra involve no exceptional spectral weight; their treatment is given in Appendix A. Deligne's bound for holomorphic forms and the divisor bound for Eisenstein eigenvalues simplify the latter two cases further.

5.1. **Exceptional spectral estimates.** We use the automorphic normalisations of [12, Section 3]. For $q \in \mathbb{Z}_{>0}$, let $\mathcal{B}_q$ be a Petersson-orthonormal basis of Maass cusp forms for $\Gamma_0(q)$ with trivial nebentypus. For a primitive form g of level $s \mid q$, its oldspace at level q is the span of $g(\delta z)$ for $\delta \mid q/s$. Choose simultaneous eigenforms for the Laplacian, the unramified Hecke operators and reflection, separately in each such oldspace. Thus

$$f(-\overline{z}) = \varepsilon_f f(z), \qquad \varepsilon_f \in \{-1, 1\}.$$

Write the Laplace eigenvalue as $\lambda_f = 1/4 + \kappa_f^2$ and let $\mathcal{B}_q^{\mathrm{exc}}$ denote the exceptional forms, for which $\kappa_f = i\theta_f/2$. All spectral sums in this subsection are over exceptional forms. Kim–Sarnak [10, Appendix 2] gives

$$\theta := \frac{7}{32}, \qquad \theta_f \le \theta, \qquad |\lambda_f(m)| \le \tau(m) m^{\theta/2} \quad ((m,q)=1), \tag{5.1}$$

where $\lambda_f(m)$ denotes the normalised Hecke eigenvalue.

At the cusps ∞ and 0 take

$$\sigma_\infty = I, \qquad \sigma_0 = \begin{pmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{pmatrix}.$$

Write $\rho_{f,\mathfrak{a}}(n)$ for the Fourier coefficients in the normalisation of [12, Section 3]. The spectral weights below are positive: on the exceptional spectrum, $\cosh(\pi\kappa_f) = \cos(\pi\theta_f/2) \ge \cos(\pi\theta/2) > 0$.

The matrix σ_0 acts by $z \mapsto -1/(qz)$. The corresponding operator $f(z) \mapsto f(-1/(qz))$ is the Fricke involution. It preserves each primitive oldspace and commutes with the operators above, so we choose the forms in $\mathcal{B}_q$ to be its eigenfunctions as well. Its eigenvalues are ± 1, and hence

$$\rho_{f,0}(n) = \pm \rho_{f,\infty}(n),$$

with a sign depending only on f. It therefore suffices to prove the Fourier-moment bounds at ∞; the bounds at 0 are identical.

The auxiliary coprimality condition (5.26) allows us to use the unramified Fourier-coefficient factorisation of [1, 11]. For $b, n \in \mathbb{Z}_{>0}$, $(b,q)=1$, and $\mathfrak{a} \in \{\infty, 0\}$, the inverse Hecke relation is

$$\rho_{f,\mathfrak{a}}(bn) = \sum_{d \mid (b,n)} \mu(d) \lambda_f(b/d) \rho_{f,\mathfrak{a}}(n/d). \tag{5.2}$$

At infinity this is Möbius inversion of the usual Hecke relation, and the relation $\rho_{f,0}(n) = \pm\rho_{f,\infty}(n)$ gives the same identity at 0. Reflection gives $\rho_{f,\mathfrak{a}}(-n) = \varepsilon_f \rho_{f,\mathfrak{a}}(n)$. The Hecke relation holds for oldforms in the chosen basis; no nonvanishing assumption on $\rho_{f,\mathfrak{a}}(1)$ is needed, since we do not divide by that coefficient.

Put $e(t) = e^{2\pi it}$ and

$$\widehat{F}(t) = \int_{\mathbb{R}} F(u)e(-tu)\,du.$$

Coefficient sequences are extended by zero outside their indicated support. A smooth function has complexity at most Z if its jth derivative is $O_j(Z^j)$ on its fixed compact support. Constants $C_0, C_1, \dots$ in this section are absolute after fixing the supports and the finitely many smoothness orders used.

Lemma 3 (Exceptional spectral large-sieve bounds)**.** *Let $q \in \mathbb{Z}_{>0}$, $\eta > 0$, and $\mathfrak{a} \in \{\infty, 0\}$.*

(i) *For $J \geq 1/2$ and any complex sequence (b_n) supported on $n \asymp J$,*

$$\sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} \frac{1}{\cosh(\pi\kappa_f)} \left| \sum_n b_n \rho_{f,\mathfrak{a}}(n) \right|^2 \ll_\eta (qJ)^\eta \left(1 + \frac{J}{q}\right) \sum_n |b_n|^2. \tag{5.3}$$

(ii) *For $b \in \mathbb{Z}_{>0}$ with $(b, q) = 1$ and $0 < X \ll \max\{1, q^2/b\}$,*

$$\sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} \frac{X^{\theta_f} |\rho_{f,\mathfrak{a}}(b)|^2}{\cosh(\pi\kappa_f)} \ll_\eta (qb)^\eta \left(1 + \frac{\sqrt{b}}{q}\right). \tag{5.4}$$

Proof. These are the exceptional cases of [12, Lemmas 3.8–3.9]; the factors $\cosh(\pi\kappa_f)$ are bounded above and below. A fixed partition of unity covers the stated supports and lengths between $1/2$ and 1. □

Lemma 4 (Fourier coefficients of newforms and oldforms)**.** *Let $\eta > 0$ and let g be a Petersson-normalised primitive exceptional Maass form of level s. For any fixed even nebentypus,*

$$s^{-1-\eta} \ll_\eta \frac{|\rho_{g,\infty}(1)|^2}{\cosh(\pi\kappa_g)} \ll_\eta s^{-1+\eta}. \tag{5.5}$$

If the nebentypus is trivial, $s \mid q$, and $f \in \mathcal{B}_q$ belongs to the oldspace generated by g, then

$$\frac{\rho_{f,\infty}(n)}{\sqrt{\cosh(\pi\kappa_f)}} = \sum_{\delta \mid q/s} c_f(\delta)\lambda_g(n/\delta), \qquad |c_f(\delta)| \ll_\eta q^{-1/2+\eta}\sqrt{\delta}. \tag{5.6}$$

Here $\lambda_g(u) = 0$ when u is not a positive integer.

Proof. Equation (5.5) follows from [9, (0.9)–(0.10), Corollary 0.3]: their Petersson norm is divided by the volume, whereas ours is not, and $\mathrm{vol}(\Gamma_0(s)\backslash\mathbb{H}) = s^{1+o(1)}$. The lower bound is the one recorded immediately before their (0.10). For the standard orthonormal oldform basis, [3, (3.8)–(3.11)] gives (5.6) with the stated coefficient bound, since $\theta < 1$. A unitary change to our chosen basis costs at most $\sqrt{\tau(q/s)}$ by Cauchy–Schwarz. This is absorbed into q^η. □

Lemma 5 (Watt's mean-value estimate)**.** *Let $\eta > 0$, $Q, Z \geq 1$, $J_1, J_2 \geq 1/2$, and*

$$0 < X \ll \frac{Q^2}{J_1^2 J_2}.$$

Let ψ_1, ψ_2 be smooth functions of complexity at most Z, supported in fixed compact subintervals of $(0, \infty)$ and independent of q. Then

$$\sum_{q \asymp Q} \sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} \frac{X^{\theta_f}}{\cosh(\pi\kappa_f)} \left| \sum_{u,v \geq 1} \psi_1(u/J_1)\psi_2(v/J_2)\rho_{f,\infty}(uv) \right|^2 \ll_\eta Q^\eta Z^5 (Q + J_1J_2) J_1 J_2. \tag{5.7}$$

Proof. This is [13, Proposition 3.5], with the bounded exceptional normalisation factors absorbed. A fixed partition of unity covers the stated supports and the bounded ranges of lengths below 1. □

Lemma 6. *For $b, q \in \mathbb{Z}_{>0}$ with $(b,q)=1$ and $V \geq 1$,*

$$\sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} V^{\theta_f} |\lambda_f(b)|^2 \ll_\eta (qbV)^\eta (q+\sqrt{b}) \max\left\{1, \frac{V}{\max\{1, q^2/b\}}\right\}^\theta.$$

Proof. For a primitive form g of level $s \mid q$, let $\mathcal{B}_q(g)$ be the basis of its oldspace at level q. The normalised lift $g/[\Gamma_0(s):\Gamma_0(q)]^{1/2}$ and Cauchy–Schwarz give

$$\sum_{f \in \mathcal{B}_q(g)} \frac{|\rho_{f,\infty}(1)|^2}{\cosh(\pi\kappa_f)} \geq \frac{|\rho_{g,\infty}(1)|^2}{[\Gamma_0(s):\Gamma_0(q)]\cosh(\pi\kappa_g)} \gg_\eta q^{-1-\eta}, \tag{5.8}$$

by (5.5) and $[\Gamma_0(s):\Gamma_0(q)] \ll_\eta (q/s)q^\eta$, with sufficiently small exponents in these inputs. Since $(b,q)=1$, we have $\rho_{f,\infty}(b) = \lambda_g(b)\rho_{f,\infty}(1)$ throughout this oldspace. Its dimension is $\tau(q/s)$, and $\theta_f = \theta_g$. Summing over the primitive forms therefore gives

$$\sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} V^{\theta_f} |\lambda_f(b)|^2 \ll_\eta q^{1+\eta} \sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} \frac{V^{\theta_f} |\rho_{f,\infty}(b)|^2}{\cosh(\pi\kappa_f)}.$$

Apply (5.4) at level q, first with $X = \min\{V, \max(1, q^2/b)\}$ and then using $\theta_f \leq \theta$ to increase the weight to V. Choosing the intermediate small exponents below η proves the claim. □

Lemma 7 (Fourth moments)**.** *For $|\beta_n| \leq 1$ supported on $n \asymp J$, independently of f,*

$$\sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} \frac{1}{\cosh^2(\pi\kappa_f)} \left| \sum_n \beta_n \rho_{f,\mathfrak{a}}(n) \right|^4 \ll_\eta (qJ)^\eta \left(\frac{J^2}{q} + \frac{J^4}{q^2} \right), \qquad \mathfrak{a} = \infty, 0. \tag{5.9}$$

For primitive exceptional forms of fixed level s and nebentypus ω of conductor m_ω, one also has

$$\left(\sum_{\substack{g \text{ primitive } (s,\omega) \\ g \text{ exceptional}}} \left| \sum_n \beta_n \lambda_g(n) \right|^4 \right)^{1/4} \ll_\eta (sJ)^\eta m_\omega^{C_0} \left(s^{1/4}\sqrt{J} + J \right). \tag{5.10}$$

Proof. The Hecke relation, with ω extended by zero at primes dividing s, gives

$$\left(\sum_{n \asymp J} \beta_n \lambda_g(n) \right)^2 = \sum_{k \ll J^2} c_k \lambda_g(k), \qquad c_k = \sum_{uv=k} \sum_{du, dv \asymp J} \omega(d) \beta_{du} \beta_{dv}.$$

Thus $|c_k| \ll J\tau(k)/\sqrt{k}$ and $\sum_k |c_k|^2 \ll_\eta J^{2+\eta}$. The nebentypus large sieve [6, Proposition 4.7] has factor $1 + m_\omega^{1/2} J/s$ at length J. Apply it at length $O(J^2)$ and use (5.5); the fourth moment of the Hecke polynomial is

$$\ll_\eta (sJ)^\eta \left(sJ^2 + m_\omega^{1/2} J^4 \right).$$

This proves (5.10). For trivial nebentypus, insert (5.6) and apply Hölder to its divisor sum. For each $s \mid q$ and $\delta \mid q/s$, the resulting contribution is bounded, apart from the displayed small powers, by

$$\frac{\delta^2}{q^2} \left(s + \frac{J^2}{\delta^2} \right) \frac{J^2}{\delta^2} = \frac{sJ^2}{q^2} + \frac{J^4}{\delta^2 q^2} \leq \frac{J^2}{q} + \frac{J^4}{q^2}.$$

Summing over s, δ and the $\tau(q/s)$ oldforms costs only a divisor factor, which is absorbed into the small power. □

We record the transform bounds used in the two fourth-moment arguments. For a primitive exceptional form g of level s and nebentypus conductor m_ω, suppose $(a,c)=1$, $(c,m_\omega)=1$, $h=(c,s)$ is an exact divisor of s, meaning $(h,s/h)=1$, and $(c/h,s)=1$. Exact-divisor Voronoi summation [2, Proposition 5.4, Corollary 6.3] expresses

$$\sum_n \lambda_g(n)e(an/c)F(n/J) = \frac{J}{\sqrt{\mathfrak{C}}}\sum_{\pm}\varepsilon_{g,\pm}\sum_{n\geq 1}\lambda_{g^*}(n)e(a_\pm n/c)[\mathcal{H}^{\pm}_{\kappa_g}F](nJ/\mathfrak{C}), \qquad \mathfrak{C}=\frac{sc^2}{h}. \tag{5.11}$$

Here $|\varepsilon_{g,\pm}|\leq 1$; g^* is the corresponding Atkin–Lehner partner, and the dual residues are fixed on each family of fixed level and nebentypus. The map on primitive forms is bijective. For $c=1$ and trivial nebentypus, we may take $g^*=g$, absorbing its eigenvalue under $g(z)\mapsto g(-1/(sz))$ into $\varepsilon_{g,\pm}$. The transforms, with the parity sign included in $\varepsilon_{g,-}$, are

$$\mathcal{H}^+_t F(v) = \frac{\pi i}{\sinh(\pi t)}\int_0^\infty \big(J_{2it}(4\pi\sqrt{vu})-J_{-2it}(4\pi\sqrt{vu})\big)F(u)\,du,$$
$$\mathcal{H}^-_t F(v) = 4\cosh(\pi t)\int_0^\infty K_{2it}(4\pi\sqrt{vu})F(u)\,du.$$

At $t=0$ the removable limits are understood. The conductor is $[c^2, m_\omega c, s] = sc^2/h$. To see the normalisation directly, put $s_1=s/h$ and choose $d_0, b_0\in\mathbb{Z}$ with $as_1d_0-b_0c=1$. Since $(s_1,c)=1$, the matrix

$$W=\frac{1}{\sqrt{s_1}}\begin{pmatrix} as_1 & b_0 \\ cs_1 & s_1d_0\end{pmatrix}$$

is an Atkin–Lehner matrix for the exact divisor s_1 of s, and

$$W^{-1}\left(\frac{a}{c}+iy\right) = -\frac{d_0}{c}+\frac{i}{c^2s_1y}.$$

Thus the dual conductor is $c^2s_1=\mathfrak{C}$ and the prefactor is $1/(c\sqrt{s_1})=\mathfrak{C}^{-1/2}$. The condition $(c,m_\omega)=1$ places the nebentypus conductor in s_1, so the exact-divisor cusp relation gives the stated partner family and a scalar of modulus one.

Lemma 8. *Let F have fixed compact support in $(0,\infty)$ and complexity Z. For a fixed dyadic cutoff ψ put $F_{t,\rho}(u)=\psi(u)[\mathcal{H}^{\pm}_t F](\rho u)$. For $t\in i[-\theta/2,\theta/2]$, every fixed integer $j\geq 0$ and real $B\geq 0$ satisfy*

$$\|F^{(j)}_{t,\rho}\|_\infty \ll_{j,B} (1+Z)^{C_{j,B}}\begin{cases}\rho^{-\theta/2}(1+|\log\rho|), & 0<\rho\leq 1,\\ \rho^{-B}, & \rho\geq 1.\end{cases}$$

The same bounds hold for $\int_{\mathbb{R}}(1+|\zeta|)^j\sup_t|\widehat{F}_{t,\rho}(\zeta)|\,d\zeta$, after increasing $C_{j,B}$.

Proof. For small arguments, the Bessel expansions give powers $v^{\pm it}$ and a logarithm at $t=0$. The same bounds hold after applying $(v\,d/dv)^j$. For $v\geq 1$, the change of variables $z=4\pi\sqrt{vu}$ identifies the transform with the Bessel transforms in [12, Lemma 3.4], applied to

$$\varphi_v(z)=\frac{z^2}{8\pi^2 v}F\left(\frac{z^2}{16\pi^2 v}\right).$$

Their bounds give an initial polynomial estimate. Repeatedly move the Bessel differential operator onto F; each integration by parts contributes v^{-1}, with costs controlled by the derivatives of F.

This proves arbitrary decay, also after applying $(v\,d/dv)^j$. Integration by parts in the Fourier transform gives

$$|\widehat{F}_{t,\rho}(\zeta)| \ll_j (1+|\zeta|)^{-j-2}\sum_{r=0}^{j+2}\|F_{t,\rho}^{(r)}\|_\infty.$$

Multiplying by $(1+|\zeta|)^j$ and integrating proves the second assertion, after increasing $C_{j,B}$. Fourier inversion writes $F_{t,\rho}(u)$ as an integral of $e(\zeta u)$ with coefficient $\widehat{F}_{t,\rho}(\zeta)$. Minkowski's inequality and the uniform bound on this coefficient then allow us to apply the spectral moment to sums whose coefficients are independent of the form. □

Lemma 9. *Let $Q, U \geq 1$, $\zeta \in \mathbb{R}$, and let ψ be a fixed smooth function supported in a compact subinterval of $(0,\infty)$. Put*

$$P_f(U,\zeta) := \sum_n \psi(n/U)e(\zeta n/U)\rho_{f,\infty}(n), \qquad W := \max\left\{1, \frac{Q^2}{U^3}\right\}.$$

Then

$$\sum_{q\asymp Q}\sum_{f\in\mathcal{B}_q^{\mathrm{exc}}} \frac{W^{\theta_f}|P_f(U,\zeta)|^4}{\cosh^2(\pi\kappa_f)} \ll_{\eta,\psi} (QU)^\eta(1+|\zeta|)^{C_0}U^2. \tag{5.12}$$

The cutoff and the phase in P_f are the same for every level.

Proof. First take primitive exceptional forms at levels $s \asymp S$ and write

$$H_g(J,\zeta) = \sum_n \psi(n/J)e(\zeta n/J)\lambda_g(n).$$

Hecke multiplication, Minkowski, (5.5), and (5.7) give

$$\begin{aligned}\left(\sum_{s\asymp S}\sum_g X^{\theta_g}|H_g(J,\zeta)|^4\right)^{1/2} &\ll_\eta S^{1/2+\eta}(1+|\zeta|)^{C_0}\sum_{d\ll J}\left(\left(S+\frac{J^2}{d^2}\right)\frac{J^2}{d^2}\right)^{1/2}\\ &\ll_\eta (SJ)^\eta(1+|\zeta|)^{C_0}\left(SJ+\sqrt{S}\,J^2\right)\end{aligned}$$

for $X \ll \max(1, S^2/J^3)$. Indeed, $H_g^2 = \sum_{(d,s)=1}\sum_{u,v}\psi(du/J)\psi(dv/J)e(\zeta d(u+v)/J)\lambda_g(uv)$. For each fixed d, the restriction $(d,s)=1$ is dropped only after forming a nonnegative moment; Watt's estimate applies for $X \ll S^2d^3/J^3$. If $S^2d^3/J^3 < 1$, Lemma 7 gives the bound at weight 1.

For $J \leq \sqrt{S}$ the squared bound is $O((SJ)^\eta(1+|\zeta|)^{C_0}S^2J^2)$. For $J \geq \sqrt{S}$, use (5.11) with $c=1$. The dual indices $n \asymp \rho S/J$ have length $J_\rho = \rho S/J$ and prefactor $J/\sqrt{s}$. Lemma 8 separates both the spectral parameter and s/S before the moment is applied. The preceding estimate applies to these indices for

$$X \ll \max\left\{1, \frac{S^2}{J_\rho^3}\right\} = \max\left\{1, \frac{J^3}{S}\rho^{-3}\right\}.$$

At weight J^3/S, the fourth moment of this dual sum, before including the transform bound from Lemma 8, is therefore at most

$$\frac{J^4}{S^2}\left(S^2J_\rho^2 + SJ_\rho^4\right)\max(1,\rho^3)^\theta \leq S^2J^2(\rho^2+\rho^4)\max(1,\rho^3)^\theta.$$

For $\rho \leq 1$, multiplying the fourth root of this bound by the factor $\rho^{-\theta/2}(1+|\log\rho|)$ from Lemma 8 gives $O((S^2J^2)^{1/4}\rho^{(1-\theta)/2}(1+|\log\rho|))$; for $\rho \geq 1$, that lemma gives the factor $O_B(\rho^{-B})$ for any

fixed $B > 0$. Minkowski thus gives, at every $J \geq 1/2$,

$$\sum_{s \asymp S} \sum_{g} X^{\theta_g} |H_g(J,\zeta)|^4 \ll_\eta (SJ)^\eta (1+|\zeta|)^{C_0} S^2 J^2 \max\left\{1, \frac{X}{\max(1, S^2/J^3, J^3/S)}\right\}^\theta. \tag{5.13}$$

To include oldforms, put $q = sm$, fix $\delta \mid m$, and apply (5.6) and Hölder. With $S = Q/m$ and $J = U/\delta$, the size in (5.13) becomes

$$\frac{\delta^2}{Q^2} S^2 J^2 = \frac{U^2}{m^2}.$$

The denominator $\max(1, S^2/J^3, J^3/S)$ in (5.13) is at least $\max(1, Q^2\delta^3/(m^2U^3))$, so taking $X = W$ costs at most $m^{2\theta}$. The divisor sums and oldspace multiplicities give

$$\ll_\eta (QU)^\eta (1+|\zeta|)^{C_0} U^2 \sum_{m \geq 1} \frac{\tau(m)^{C_1}}{m^{2-2\theta}} \ll_\eta (QU)^\eta (1+|\zeta|)^{C_0} U^2.$$

The series converges since $2 - 2\theta > 1$; intermediate divisor exponents are chosen smaller than the final η. If $J = U/\delta < 1$, nonempty support forces $J \gg 1$, so replacing J by 1 changes the bound only by a constant; empty supports contribute nothing. □

Lemma 10. *Fix an integer $B \geq 1$. Suppose $q = Dq_0$, where $D, d, q_0 \in \mathbb{Z}_{>0}$, q_0 is squarefree, $(q_0, d) = 1$, $D \mid d^\infty$, and $D \leq d^B$. Let $c_0 \mid q_0$, $c_1 \mid d^B$, $a_0, a_1 \in \mathbb{Z}$, and $\alpha = a_0/c_0 + a_1/c_1$. For a fixed smooth compactly supported ψ and $V \geq 1/2$,*

$$\begin{aligned} &\left(\sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} \frac{1}{\cosh^2(\pi\kappa_f)} \left| \sum_n \psi(n/V) e(\alpha n) \rho_{f,\mathfrak{a}}(n) \right|^4 \right)^{1/4} \\ &\qquad \ll_{\eta,B,\psi} (qV)^\eta d^{C_B} \left(\frac{\sqrt{V}}{q^{1/4}} + \sqrt{c_0} \right), \quad \mathfrak{a} = \infty, 0. \end{aligned} \tag{5.14}$$

Replacing $\psi(u)$ by $\psi(u)e(\zeta u)$ multiplies the right-hand side by at most $(1+|\zeta|)^{C_B}$, after increasing C_B.

Proof. Reduce the two rational fractions first; replacing c_0, c_1 by divisors only improves the bound. Expand an oldvector generated by a primitive form g of level $s \mid q$ using (5.6). After dividing by $\sqrt{\cosh(\pi\kappa_g)}$, a term indexed by $\delta \mid q/s$ has coefficient $O_\eta(q^{-1/2+\eta}\sqrt{\delta})$ and length $J = V/\delta$. Let s_0, δ_0 denote the parts of s, δ coprime to d. Then $s_0\delta_0 \mid q_0$ and $(s_0, \delta_0) = 1$. Let c' be the reduced additive denominator away from d. Then

$$c' \mid \frac{c_0}{(c_0, \delta_0)}.$$

The remaining denominator, say b, divides a fixed power of d. For clarity, remove this latter denominator exactly. In a primitive polynomial $\sum_n F(n/J) e(An/c') e(un/b) \lambda_g(n)$, write $h = (n, b)$, $n = hv$, and $m = b/h$. The inverse Hecke relation gives

$$\lambda_g(hv) = \sum_{\substack{j \mid (h,v) \\ (j,s)=1}} \mu(j) \lambda_g(h/j) \lambda_g(v/j).$$

Put $v = jw$. Nonempty terms have $(j, m) = (w, m) = 1$, and

$$\mathbf{1}_{(w,m)=1} e(ujw/m) = \sum_{\chi \pmod m} \gamma_\chi(uj)\chi(w),$$

$$\gamma_\chi(v) = \frac{1}{\varphi(m)} \sideset{}{^*}\sum_{z \bmod m} e(vz/m)\overline{\chi(z)}, \qquad \sum_\chi |\gamma_\chi(v)| \le \sqrt{\varphi(m)}.$$

Let g_χ be the primitive twist by the primitive character inducing χ. Its level s' divides sm^2, its spectral parameter is unchanged, and its level away from d is s_0. The conductor bound follows from the principal-series conductor formula and [14, Proposition 4] for the essentially discrete-series cases. Remove the remaining unit restriction by

$$\mathbf{1}_{(w,m)=1} \lambda_{g_\chi}(w) = \sum_{\ell \mid \mathrm{rad}(m)^2} \mu_{g_\chi}(\ell)\lambda_{g_\chi}(w/\ell),$$

where μ_f is multiplicative, $\mu_f(p) = -\lambda_f(p)$, and $\mu_f(p^2) = \omega_f(p)$ for $p \nmid \mathrm{level}(f)$ and 0 otherwise; here ω_f is the nebentypus of f. These coefficients come from the inverse Euler factors of f. The three identities express the original polynomial as a linear combination of

$$\sum_z F(w_0 z/J) e(Aw_0 z/c')\lambda_{g_\chi}(z), \qquad w_0 = hj\ell \le d^{O_B(1)}.$$

The coefficients have total absolute value at most d^{C_B}: use the displayed bound for $\sum_\chi |\gamma_\chi(uj)|$ and the elementary estimate $|\lambda_f(n)| \le \tau(n)n^{1/2}$, also at ramified indices, since all remaining indices are bounded by $d^{O_B(1)}$. Twisting is injective for each fixed character. Partitioning by s' and its nebentypus creates only $d^{O_B(1)}$ families. For each fixed h, j, χ, ℓ, the polynomial coefficients $F(w_0 z/J)e(Aw_0 z/c')$ are common within each family. The scalar factors $\lambda_g(h/j)\mu_{g_\chi}(\ell)$ may depend on the form; we bound them uniformly by $d^{O_B(1)}$ before applying the moment estimate.

For each such family the length is $J' = V/(\delta w_0)$. The nebentypus conductor is supported on d, whereas $(c', d) = 1$ and (c', s') is an exact divisor of s'. Apply (5.11) with $\mathfrak{C} = s'c'^2/(c', s')$. The essential arithmetic bound is

$$\frac{\delta \mathfrak{C}}{q} \ll_B d^{C_B} c_0. \tag{5.15}$$

Indeed, write $q_0 = s_0\delta_0 d_0$ with pairwise coprime factors and put $v_0 = c'/(c', s_0)$. Then $v_0 \mid d_0$ and

$$\frac{\delta_0 s_0 c'^2}{q_0(c', s_0)} = c' \frac{v_0}{d_0} \le c' \le c_0.$$

The prime powers supported on d have polynomial size, proving (5.15). For dual indices $n \asymp \mathfrak{C}/J'$, (5.10), including the oldform coefficient and Voronoi prefactor, gives

$$q^{-1/2}\sqrt{\delta}\,\frac{J'}{\sqrt{\mathfrak{C}}}\left(s'^{1/4}\sqrt{\frac{\mathfrak{C}}{J'}} + \frac{\mathfrak{C}}{J'}\right) = q^{-1/2}\sqrt{\delta}\left(s'^{1/4}\sqrt{J'} + \sqrt{\mathfrak{C}}\right)$$

$$\ll_B d^{C_B}\left(\frac{\sqrt{V}}{q^{1/4}} + \sqrt{c_0}\right).$$

Here $s' \ll q d^{O_B(1)}$ and $\delta J' \le V$. For dual indices $n \asymp \rho\mathfrak{C}/J'$, the two terms acquire factors $\rho^{1/2}$ and ρ, respectively. The transform bound in Lemma 8 makes the resulting bounds summable over dyadic ρ. Summing the finite decompositions proves the estimate. □

Lemma 11. *Let $d, e, b, m \in \mathbb{Z}_{>0}$ and let $n, n' \asymp N$ be squarefree. Put*

$$g = (n, n'), \qquad n = gs, \quad n' = gs', \qquad S = N/g, \qquad q = dgss'.$$

Thus g, s, s' are squarefree and pairwise coprime. Suppose $1/2 \le H \le N$, $Y \ge 1/2$, $Y \ll HS$, and $(e, q) = 1$. For fixed smooth functions Φ, ψ_1, ψ_2 supported away from zero and phases $\omega_1, \omega_2 \in \mathbb{R}/\mathbb{Z}$, define

$$c_v = \sum_{s'h - sh' = v} \psi_1(h/H)\psi_2(h'/H)e(h\omega_1 + h'\omega_2).$$

Suppose $\omega_1 = A/m + \xi/H$, with $A \in \mathbb{Z}$ and $\xi \in \mathbb{R}$. Let $\mathfrak{d}$ be a positive integer divisible by d, b, m, and assume $d, b, m, 1 + |\xi| \le \mathfrak{d}^{B_0}$ for a fixed B_0. Then

$$\sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} \frac{1}{\cosh^2(\pi\kappa_f)} \left| \sum_v \Phi(ev/Y) c_{ev} \rho_{f,0}(bv) \right|^4 \ll_\eta \mathfrak{d}^{C_0} (qYe)^\eta g^4 \left(\frac{Y^2}{q} + \min\left\{ \frac{Y^4}{q^2}, q \right\} \right). \tag{5.16}$$

There is no fixed positive power of e in this bound. Replacing $\Phi(u)$ by $\Phi(u)e(\zeta u)$ multiplies the right-hand side by at most $(1 + |\zeta|)^{C_0}$, after increasing C_0.

Proof. The solutions of $s'h - sh' = ev$ form one arithmetic progression, so $|c_{ev}| \ll 1 + H/S \ll g$. Insert zeros at the other indices bv and apply (5.9); this gives the bound with $Y^2/q + Y^4/q^2$.

For the other bound, solving for h and applying Poisson summation to $h' \equiv -e\overline{s}\, v \pmod{s'}$ gives the exact identity

$$\begin{aligned} c_{ev} = \frac{H}{s'} e(ev\omega_1/s') \sum_{k \in \mathbb{Z}} e(-ke\overline{s}\, v/s') \\ \times \int_{\mathbb{R}} \psi_1\left(\frac{ev + sHu}{s'H} \right) \psi_2(u) e\left(H\left(\omega_2 + \frac{s}{s'}\omega_1 - \frac{k}{s'} \right) u \right) du. \end{aligned} \tag{5.17}$$

After multiplication by $\Phi(ev/Y)$, normalised derivatives in ev/Y cost only fixed powers of $1 + |\xi|$, since $Y/(Hs') \ll 1$. Integration by parts in u shows that the sum over k of the supremum norms of any fixed number of derivatives with respect to ev/Y is $O((1 + |\xi|)^{C_0}(1 + S/H))$, uniformly in the centre $s\omega_1 + s'\omega_2$ of the k-range.

Put $w = bv$ and insert $\mathbf{1}_{b|w} = b^{-1} \sum_{\nu \bmod b} e(\nu w/b)$. The rational part of each phase has denominator dividing bms'. Its part coprime to $\mathfrak{d}$ divides the corresponding part of s' and is at most $O(S)$. The remaining smooth phase is $e(\xi ew/(bHs'))$, whose normalised frequency is $\xi Y/(Hs')$; in particular it has no loss in e. The part of q supported on $\mathfrak{d}$ is at most $d\,\mathrm{rad}(\mathfrak{d})$, because gss' is squarefree. Lemma 10 therefore applies, with polynomial cost in $\mathfrak{d}$. Minkowski and (5.17) give

$$\begin{aligned} &\left(\sum_f \frac{1}{\cosh^2(\pi\kappa_f)} \left| \sum_v \Phi(ev/Y) c_{ev} \rho_{f,0}(bv) \right|^4 \right)^{1/4} \\ &\ll_\eta \mathfrak{d}^{C_0} (qYe)^\eta \frac{H}{S} \left(1 + \frac{S}{H} \right) \left(\frac{\sqrt{Y}}{q^{1/4}} + \sqrt{S} \right). \end{aligned}$$

Now $1 + H/S \ll g$, $Y \ll HS \le gS^2 \ll q$, and $\sqrt{S} \ll q^{1/4}$, so the last display is $O_\eta(\mathfrak{d}^{C_0}(qYe)^\eta g q^{1/4})$. Combining its fourth power with the first bound proves (5.16). □

5.2. **Kloosterman sums and the exponential convolution.** For integers m, n and $c \geq 1$, put

$$S(m,n;c) := \sum_{\substack{z \bmod c \\ (z,c)=1}} e\left(\frac{mz+n\overline{z}}{c}\right).$$

For $a \in \mathbb{Z} \setminus \{0\}$, $K, L > 0$, and finitely supported coefficients $(u_{q,v})$ with $q, v \in \mathbb{Z}_{>0}$ and $(q,a)=1$, put

$$\mathcal{T} = \sum_{q,v} u_{q,v} \sum_{\substack{k,\ell \geq 1 \\ (k,q\ell)=1}} W_1(\ell/L) W_2(k/K) e\left(\frac{av\overline{q\ell}}{k}\right),$$

where W_1, W_2 are smooth functions supported in fixed compact subintervals of $(0,\infty)$. Poisson summation in ℓ gives the identity

$$\mathcal{T} = L \sum_{q,v} u_{q,v} \sum_{(k,q)=1} \frac{W_2(k/K)}{k} \sum_{m \in \mathbb{Z}} \widehat{W}_1(mL/k) S(m\overline{q}, av; k). \tag{5.18}$$

Fix a smooth function ψ with compact support in $(0,\infty)$. For $U > 0$ and $\sigma \in \{-1, 1\}$, define the smooth nonzero-frequency block

$$\mathcal{T}_\sigma(U) := L \sum_{q,v} u_{q,v} \sum_{(k,q)=1} \frac{W_2(k/K)}{k} \sum_{m \geq 1} \psi(m/U) \widehat{W}_1(\sigma mL/k) S(\sigma m\overline{q}, av; k).$$

Let $\mathcal{T}_\sigma^{\mathrm{exc}}(U)$ denote its exceptional Maass contribution in Kuznetsov's formula at the cusps ∞ and 0. Write

$$P_f(U,\zeta) := \sum_{m \geq 1} \psi(m/U) e(\zeta m/U) \rho_{f,\infty}(m).$$

The coefficients of this polynomial are independent of the level q and the form f.

Lemma 12 (Spectral estimate for a completed block)**.** *With the notation above, let $x \geq 2$, $0 < \eta < 1$, and $K, L, Q, Y, U \geq 1/2$. Suppose $K, L, Q, Y, |a|$ are bounded by fixed powers of x, that $(u_{q,v})$ is supported on $q \asymp Q$, $v \asymp Y$, and that W_1, W_2 have complexity at most x^η. Assume*

$$U \ll \frac{K}{L} x^{2\eta}, \qquad X := \frac{K\sqrt{Q}}{\sqrt{|a|UY}} \geq x^{-C\eta},$$

where $C > 0$ is fixed. For every fixed $B > 0$ and sufficiently large fixed J, there is a constant $C_1 > 0$ such that

$$\begin{aligned} |\mathcal{T}_\sigma^{\mathrm{exc}}(U)| \ll L x^{C_1\eta} \int_{\mathbb{R}^2} \frac{d\zeta\, d\xi}{(1+|\zeta|+|\xi|)^J} \sum_{q \asymp Q} \sqrt{q} \sum_{f \in \mathcal{B}_q^{\mathrm{exc}}} \frac{\max(1,X)^{\theta_f}}{\cosh(\pi\kappa_f)} |P_f(U,\zeta)| \\ \times \left| \sum_v u_{q,v} e(\xi v/Y) \rho_{f,0}(|a|v) \right| + x^{-B} \sum_{q,v} |u_{q,v}|. \end{aligned} \tag{5.19}$$

The constants may depend on B, J, C, the fixed polynomial bounds, supports and smoothness orders; the implicit constant may also depend on η.

Proof. With the scaling matrices σ_∞, σ_0 chosen above, the Kloosterman sum $S_{\infty,0}$ between these cusps satisfies

$$S(\sigma m\overline{q}, av; k) = S_{\infty,0}(\sigma m, av; k\sqrt{q}) \qquad ((k,q)=1).$$

Apply the geometric-test-function Kuznetsov formula, as in [12, Proposition 3.5 and the proof of Corollary 5.11]. For $a > 0$ and $\sigma = 1$, keep $\psi(m/U)$ outside the transform and use the test function

$$\varphi_{m,v,q}(z) = W_2\left(\frac{4\pi\sqrt{amv}}{K\sqrt{q}\,z}\right)\widehat{W}_1\left(\frac{Lm\sqrt{q}\,z}{4\pi\sqrt{amv}}\right), \qquad z \asymp X^{-1}.$$

The factor $L/k = L\sqrt{q}/(k\sqrt{q})$ accounts for $L\sqrt{q}$ in (5.19). The Bessel-transform bounds in [12, Lemma 3.4] give $x^{O(\eta)}\max(1,X)^{\theta_f}$ on the exceptional spectrum. Extend the remaining dependence on m/U smoothly from the support of ψ, and Fourier-separate it from v/Y before taking absolute values. The resulting kernel is bounded by $x^{O(\eta)}(1+|\zeta|+|\xi|)^{-J}$, uniformly in q and f. Keeping $\psi(m/U)$ outside this separation gives exactly $P_f(U,\zeta)$, with coefficients common to all levels.

The other signs of σ and a follow from the opposite-sign Kuznetsov formula and reflection. There is no diagonal term in this geometric-test-function formula. □

Lemma 13. *For $Q \geq 1$, $1 \leq Y \leq Q$ and $W \geq 1$, put*

$$B(Y;Q) = \frac{Y}{Q} + \min\left\{\frac{Y^2}{Q^{3/2}}, 1\right\}.$$

Then

$$B(Y;Q)\max\left\{1, \frac{Q}{Y\sqrt{W}}\right\}^{\theta} \ll \max\{1, Q^{\theta/4}W^{-\theta/2}\}. \tag{5.20}$$

Proof. The contribution of Y/Q is at most 1, since, when $Q > Y\sqrt{W}$,

$$\frac{Y}{Q}\left(\frac{Q}{Y\sqrt{W}}\right)^{\theta} = \left(\frac{Y}{Q}\right)^{1-\theta}W^{-\theta/2} \leq 1.$$

The other contribution increases up to $Y = Q^{3/4}$ and decreases thereafter. Its value at that point is $\max\{1, Q^{\theta/4}W^{-\theta/2}\}$. □

Proposition 14. *Let $x \geq 2$, $\eta, \delta > 0$, and $K, L, N, H \gg 1$ be of at most fixed polynomial size in x. Put*

$$\Lambda = \max\left\{1, \frac{K}{NL}, \frac{H}{N}, \frac{N^2}{\sqrt{x}}, \frac{K^2}{xN}, \frac{xH}{K^2}\right\}. \tag{5.21}$$

Let $a, t \in \mathbb{Z}\setminus\{0\}$, $d \in \mathbb{Z}_{>0}$, $(a,d) = 1$, and $|a| \leq x^{1+\delta}$. Suppose

$$\omega_1 = A/m + \xi/H, \qquad A \in \mathbb{Z}, \quad m \in \mathbb{Z}_{>0},$$

while $\omega_2 \in \mathbb{R}/\mathbb{Z}$ is arbitrary. Let $\mathfrak{d}$ be a positive integer divisible by $d, |t|, m$, with $1+|\xi| \leq \mathfrak{d}^{B_0}$. Let $|\alpha_{n,n'}| \leq 1$ be supported on squarefree $n, n' \asymp N$ satisfying $(nn', at) = 1$. For fixed smooth weights of complexity at most x^{η}, define

$$\mathcal{K} = \sum_{n,n'}\alpha_{n,n'}\sum_{\substack{h,h'\in\mathbb{Z}\\ n'h\neq nh'}}\psi_1(h/H)\psi_2(h'/H)e(h\omega_1 + h'\omega_2)$$
$$\times \sum_{(k,dnn'\ell)=1} W_1(\ell/L)W_2(k/K)e\left(\frac{at(n'h-nh')\overline{dnn'\ell}}{k}\right).$$

The h, h' supports are bounded away from zero. There is an absolute constant C_0, depending only on B_0, the fixed supports and the required smoothness orders, such that

$$|\mathcal{K}|^2 \ll (\mathfrak{d}\Lambda)^{C_0}x^{C_0(\eta+\delta)}L^2N^7\sqrt{x}. \tag{5.22}$$

Additional conditions on n, n' may be included in $\alpha_{n,n'}$.

Proof. Write

$$g = (n, n'), \qquad n = gs, \quad n' = gs', \qquad (g, ss') = (s, s') = 1.$$

Squarefreeness gives the coprimalities in this display. Cancelling g in the reciprocal phase gives level $q = dgss'$ and additive index $at(s'h - sh')$. This preserves the condition on k, since nn' and gss' have the same prime divisors.

First suppose $\Lambda = 1$, and temporarily suppress factors bounded by fixed powers of $\mathfrak{d}$. Group $g \asymp G$ and $|s'h - sh'| \asymp Y$, and put

$$Q = N^2/G, \qquad 1 \le Y \ll HN/G \le Q.$$

For a fixed q, there are at most $\tau_3(q/d)$ triples (g, s, s'). Hölder over these triples costs $x^{O(\eta)}$, and the coefficients $\alpha_{n,n'}$ are included in $u_{q,v}$ and hence do not alter $P_f(U, \zeta)$. Rapid decay of $\widehat{W}_1$ restricts the nonzero completion frequencies, with negligible error, to

$$|m| \ll \frac{K}{L} x^{2\eta} \ll N x^{2\eta}.$$

Split the signs of $s'h - sh'$ and the nonzero completion frequency, absorbing the sign of $t(s'h - sh')$ into a. Apply Lemma 12, with $|t|Y$ in place of Y, to each block $\mathcal{T}_\sigma(U)$ with $U \ll Nx^{2\eta}$.

We first estimate the exceptional contribution. In (5.2), a divisor $e \mid |a|$ gives

$$e_0 = (e, t), \qquad e_1 = e/e_0, \qquad c_{e_1 v}\rho_{f,0}((t/e_0)v), \qquad (e_1, q) = 1,$$

where c_v is the coefficient defined in Lemma 11. Write the resulting polynomial as

$$L_{f,e}(Y) := \sum_v \Phi(e_1 v/Y) c_{e_1 v}\rho_{f,0}((|t|/e_0)v),$$

with its smooth cutoff and normalised Fourier twist included in Φ. Lemma 11 applies with $e \leftarrow e_1$ and $b \leftarrow |t|/e_0$. All divisor sums and Fourier integrations below cost $x^{O(\eta)}\mathfrak{d}^{O(1)}$. We write $P_f(U)$ for $P_f(U, \zeta)$ and suppress the dependence on the Fourier variables in $L_{f,e}(Y)$.

Put

$$W = \max\{1, Q^2/U^3\}, \qquad X^2 = \frac{K^2 Q}{|at|UY}, \qquad V = \max\{1, X^2/\sqrt{W}\}.$$

Let $\mathcal{K}_G(U, Y)$ denote the exceptional contribution with $g \asymp G$, $|m| \asymp U$ and $|s'h - sh'| \asymp Y$. Then $V^{\theta_f/2} W^{\theta_f/4} \ge \max(1, X)^{\theta_f}$. Hölder with exponents $2, 4, 4$, applied over both levels and forms in (5.19), gives

$$|\mathcal{K}_G(U, Y)|^2 \ll x^{O(\eta)} L^2 Q \left(\sum_{q \asymp Q} \sum_f V^{\theta_f} |\lambda_f(|a|/e)|^2 \right) \times \left(\sum_{q \asymp Q} \sum_f \frac{W^{\theta_f} |P_f(U)|^4}{\cosh^2(\pi\kappa_f)} \right)^{1/2} \left(\sum_{q \asymp Q} \sum_f \frac{|L_{f,e}(Y)|^4}{\cosh^2(\pi\kappa_f)} \right)^{1/2}.$$

Lemmas 6, 9, and 11 bound the three factors, respectively, by

$$Q(Q + \sqrt{|a|})\mathcal{F}^\theta, \qquad U, \qquad G^2 Q B(Y; Q),$$

up to $x^{O(\eta)}$, where

$$\mathcal{F} = \max\left\{1, \frac{K^2 Q}{UY \max\{|a|, Q^2\}\sqrt{W}}\right\}.$$

Indeed, the exact denominator for $|a|/e$ is $\max\{|a|, eQ^2\}$, and $|t| \geq 1$ can only decrease this factor. Consequently,

$$|\mathcal{K}_G(U,Y)|^2 \ll x^{O(\eta)} L^2 G^2 U Q^3 (Q + \sqrt{|a|}) B(Y;Q) \mathcal{F}^\theta. \tag{5.23}$$

The residue-dependent factor in (5.23) is nondecreasing in $|a|$, up to an absolute constant: below Q^2 its denominator is constant; above Q^2, use $Q + \sqrt{|a|} \asymp \sqrt{|a|}$ and $\theta < 1/2$. We may therefore replace $|a|$ by x, at cost $x^{O(\delta)}$. Since $Q \leq \sqrt{x}$ and $K^2 \leq xN$, the U-dependent part is at most

$$U \max\left\{1, \frac{NQ}{UY\sqrt{\max\{1, Q^2/U^3\}}}\right\}^\theta = \begin{cases} U \max\{1, N\sqrt{U}/Y\}^\theta, & U \leq Q^{2/3}, \\ U \max\{1, NQ/(UY)\}^\theta, & U \geq Q^{2/3}. \end{cases}$$

Both expressions are nondecreasing. Substituting $U = N$, and absorbing the Fourier truncation into $x^{O(\eta)}$, gives, with $W_G = \max\{1, N/G^2\}$,

$$|\mathcal{K}_G(Y)|^2 \ll x^{O(\eta+\delta)} \frac{L^2 N^7 \sqrt{x}}{G} B(Y;Q) \max\left\{1, \frac{Q}{Y\sqrt{W_G}}\right\}^\theta.$$

Lemma 13 applies, and

$$Q^{\theta/4} W_G^{-\theta/2} = G^{3\theta/4} \quad (G \leq \sqrt{N}), \qquad Q^{\theta/4} \leq G^{3\theta/4} \quad (G \geq \sqrt{N}).$$

Thus the exceptional contribution is

$$|\mathcal{K}_{G,\mathrm{exc}}|^2 \ll x^{O(\eta+\delta)} L^2 N^7 \sqrt{x}\, G^{-1+3\theta/4}.$$

The dyadic G-sum is convergent, even after taking square roots, since $-1 + 3\theta/4 = -107/128 < 0$.

The regular contributions satisfy the required bound by Proposition A.4 in Appendix A.

The zero Poisson frequency, still with nonzero additive index, is bounded using

$$|S(0, atv; k)| \leq (atv, k), \qquad \sum_{k \asymp K} \frac{(atv, k)}{k} \ll \tau(|atv|).$$

Its size is $O(x^{O(\eta)} L N^2 H^2)$, whose square is at most $x^{O(\eta)} L^2 N^8$, also admissible.

Finally restore $\mathfrak{d}$ and Λ. The levels satisfy $q \asymp dQ$; replacing dQ by Q in the displayed powers costs $\mathfrak{d}^{O(1)}$, while $B(Y; dQ) \leq B(Y; Q)$. The weight available for the fourth moment of P_f satisfies $\max\{1, (dQ)^2/U^3\} \geq \max\{1, Q^2/U^3\}$. The parameter X in Lemma 12 satisfies

$$X^2 \gg \frac{K^2 N}{|at| UH} \gg (\mathfrak{d}\Lambda)^{-O(1)} x^{-O(\eta+\delta)},$$

since $U \ll \Lambda N x^{2\eta}$ and $xH/K^2 \leq \Lambda$. Thus the Fourier integrals in (5.19) contribute at most $(\mathfrak{d}\Lambda)^{C_0} x^{C_0(\eta+\delta)}$, after increasing C_0. Similarly, $Q \leq \Lambda\sqrt{x}$, $K^2 \leq \Lambda x N$, and $H \leq \Lambda N$ introduce fixed powers of Λ in the preceding estimates. For $Q < Y \ll \Lambda Q$, the left side of (5.20) is at most $1 + Y/Q \ll \Lambda$. These observations also cover the enlarged completion range. Summing the dyadic blocks proves (5.22). □

For $H > 0$ and $\omega \in \mathbb{R}/\mathbb{Z}$, put

$$T_H(\omega) := \min_{m \in \mathbb{Z}_{>0}} \{m + 2H\|m\omega\|\}, \qquad \|u\| := \min_{j \in \mathbb{Z}} |u - j|.$$

The following estimate transfers the Kloosterman bound to the exponential sum arising from dispersion.

Lemma 15. *Let $x = MNL \geq 2$, $0 < \eta < 1$, $a \in \mathbb{Z} \setminus \{0\}$, and $v, d_1, d_2 \in \mathbb{Z}_{>0}$, with $(a, vd_1d_2) = 1$. Suppose $1 \ll M, K, N, L, H, R \ll x$, $K \geq 1$, and*

$$|a| \leq x^{1+\eta}, \qquad vd_1d_2 \leq x^{\eta},$$

$$x^{-\eta}NL \leq K \leq x^{\eta}NL, \quad L \leq x^{\eta}M, \quad R \leq x^{\eta}K, \quad H \leq x^{\eta}R/M, \quad N^2 \leq x^{1/2+\eta}.$$

Let $(u_k)_{K<k\leq 4K}$, $(\beta_n)_{n\sim N}$ and $(\lambda_\ell)_{\ell\sim L}$ be complex sequences with $|u_k| \leq \tau(k)$, $|\beta_n|, |\lambda_\ell| \leq 1$, and

$$(k, avd_1d_2) > 1 \implies u_k = 0, \qquad (n\ell, avd_1) > 1 \implies \beta_n\lambda_\ell = 0.$$

Suppose also that $\beta_n = 0$ unless n is squarefree. Let Φ, Ψ be fixed smooth functions supported in compact subintervals of $(0, \infty)$, and let $\omega \in \mathbb{R}/\mathbb{Z}$ satisfy $T_H(\omega) \leq x^{\eta}$. Put

$$\mathcal{R} := \sum_{\substack{r\sim R\\(r, avd_1d_2)=1}} \frac{M}{r}\Psi(r/R) \sum_{\substack{k,n,\ell\\ d_1k\equiv d_2n\ell \pmod r\\ (d_1k, d_2n\ell)=1}} u_k\beta_n\lambda_\ell \times \sum_{h\in\mathbb{Z}} e(h\omega)\Phi(h/H)e\left(-\frac{ha\overline{vd_1d_2k}}{r}\right).$$

There is an absolute constant $C_0 > 0$ such that

$$\frac{|\mathcal{R}|}{MK^2/R} \ll x^{C_0\eta}\left\{\left(\frac{N^2L^3}{xR}\right)^{1/2} + \max\left(1, \frac{K}{R}\right)^{C_0}\left(\frac{N\sqrt{x}}{L^2}\right)^{1/4}\right\}. \tag{5.24}$$

The constant C_0 is independent of η; the implicit constant may depend on η and the fixed smooth weights.

Proof. We follow the complementary-divisor argument in [5, Proposition 1], with the smooth h-sum of [13, Lemma 6.2]. Put

$$\nu = vd_1d_2, \qquad T = \max\left\{1, 4\frac{\max(d_1K, d_2NL)}{R}\right\} \ll x^{2\eta}\max(1, K/R).$$

Write $rt = d_2n\ell - d_1k$. The case $t = 0$ would force $d_1k = d_2n\ell = 1$, which is excluded by the supports. Reciprocity gives

$$-\frac{ah\overline{\nu k}}{r} \equiv \frac{aht\overline{\nu d_2n\ell}}{k} + \frac{ah\overline{rk}}{\nu} - \frac{ah}{\nu kr} \pmod 1.$$

We retain the last exponential. Put

$$D = 1 + \frac{|a|H}{\nu KR}, \qquad \nu D = \nu + \frac{|a|H}{KR} \ll x^{3\eta}.$$

For fixed k, ℓ, t, write $r(u) = (d_2u\ell - d_1k)/t$ and $b(u) = aH/(\nu kr(u))$. Choose a fixed nonnegative smooth cutoff Φ_0 equal to 1 on $[1, 4]$ and on the support of Φ, and supported away from zero. Fourier inversion gives

$$\Phi(h/H)e\left(-\frac{ah}{\nu kr(n)}\right) = \Phi_0(h/H)\int_{\mathbb{R}} \widehat{\Phi}(s + b(n))e(sh/H)\,ds.$$

On the interval $[u_0, u_1]$ defined by $N \leq u \leq 2N$ and $R \leq r(u) \leq 2R$, set

$$W_s(u) = \frac{R}{r(u)}\Psi(r(u)/R)\widehat{\Phi}(s + b(u)).$$

Since $b(u)$ is monotone and $|b(u)| \ll D$, differentiation and rapid decay give

$$\int_{\mathbb{R}} \sup_{k,\ell,t}\left(|W_s(u_1)| + \int_{u_0}^{u_1} |W_s'(u)|\,du\right) ds \ll D.$$

The supremum is over contributing triples. Partial summation in n, followed by the interval separation, character decomposition and Cauchy–Schwarz argument in [5, Section 3.5, (3.22)], now gives

$$|\mathcal{R}| \ll x^{C_1\eta}\frac{M}{R}\sqrt{KLT}\sup_{\substack{|s|\le x^\eta D\\ \rho\in\mathbb{R}/\mathbb{Z},\ \nu\rho=0}}\mathcal{B}(s,\rho)^{1/2}+O(x^{-100}). \tag{5.25}$$

Here, after applying the triangle inequality to the t-sum, we may take

$$\mathcal{B}(s,\rho):=\sum_{1\le|t|\le T}\sum_{\substack{n,n'\sim N\ \text{squarefree}\\ (nn',at)=1\\ t\mid n-n'}}\Bigg|\sum_{h,h'\in\mathbb{Z}}\Phi_0(h/H)\Phi_0(h'/H)e(h\alpha_1+h'\alpha_2)$$
$$\times\sum_{\substack{k,\ell\\ (k,dnn'\ell)=1}}\Phi_0(k/K)\Phi_0(\ell/L)e\left(\frac{at(n'h-nh')\overline{dnn'\ell}}{k}\right)\Bigg|,$$

where

$$d=\nu d_2,\qquad \alpha_1=-\alpha_2=\omega+\rho+s/H.$$

The squarefree support is retained when the square is opened. Further restrictions on n,n',t can be dropped after the displayed absolute value. The divisor and separation losses have total cost $x^{O(\eta)}\nu D\ll x^{C_1\eta}$. The Fourier tails are negligible. Moreover,

$$T_H(\alpha_1)\le\nu(1+2|s|)T_H(\omega)\ll x^{5\eta},$$

so we can write $\alpha_1=A/m+\xi/H$ with $m,1+|\xi|\ll x^{5\eta}$ by choosing m attaining the minimum in $T_H(\alpha_1)$ and A nearest to $m\alpha_1$.

The part of $\mathcal{B}$ with $n'h=nh'$ is at most

$$\mathcal{B}_0\ll x^\eta KLTNH.$$

Indeed, if $g=(n,n')$, its nonzero solutions are $h=(n/g)j$, $h'=(n'/g)j$, numbering $O(Hg/N)$. There are no solutions when this quantity is too small, and $\sum_{n,n'\sim N}(n,n')\ll N^2\log(2N)$.

For the remaining terms fix t, linearise the outer absolute values, and apply Proposition 14. The support gives $(a,dnn')=1$, and the condition $t\mid n-n'$ is included in its coefficients. The quantity Λ defined in (5.21) satisfies $\Lambda\ll x^{C_2\eta}$, since

$$\frac{K}{NL}\le x^\eta,\quad \frac{H}{N}\le x^{4\eta},\quad \frac{N^2}{\sqrt{x}}\le x^\eta,\quad \frac{K^2}{xN}\le x^{3\eta},\quad \frac{xH}{K^2}\le x^{3\eta}.$$

The auxiliary integer $d|t|m\lceil 1+|\xi|\rceil$ is at most $x^{C_3\eta}\max(1,K/R)$. Summing the resulting estimate over t gives

$$\mathcal{B}_1\ll x^{C_4\eta}\max(1,K/R)^{C_4}TLN^{7/2}x^{1/4}.$$

The fourth power of its contribution in (5.25), divided by $(MK^2/R)^4$, is at most

$$x^{C_5\eta}\max(1,K/R)^{C_5}\frac{(KLT)^2T^2L^2N^7\sqrt{x}}{K^8}\ll x^{C_6\eta}\max(1,K/R)^{C_6}\frac{N\sqrt{x}}{L^2}.$$

For the diagonal, the corresponding squared ratio is

$$x^{C_7\eta}\frac{KLT\cdot KLTNH}{K^4}\ll x^{C_8\eta}\frac{N^2L^3}{xR}.$$

Here $K\asymp NL$ up to x^η, $T\ll x^{2\eta}\max(1,K/R)$ and $H\le x^\eta R/M$; when $K<R$, the factor $R/K\le x^\eta$ is absorbed. Taking roots proves (5.24) after increasing C_0. □

We now prove Proposition 2.

Proof of Proposition 2. Write

$$\mathcal{D}(D) := \sum_{\substack{r\sim R\\(r,a_1a_2)=1}} \Bigg| \sum_{\substack{m\sim M\\n\sim N\\\ell\sim L}} \alpha_m\beta_n\gamma_\ell \times \left(\mathbf{1}_{mn\ell\equiv a_1\overline{a_2} \pmod r} - \frac{\mathbf{1}_{(mn\ell,r)=1}}{\varphi(r)}\omega_D(mn\ell\overline{a_1}a_2;r)\right)\Bigg|.$$

We first assume that β_n is supported on squarefree integers and that

$$(n\ell,a_1)>1 \implies \beta_n\gamma_\ell = 0. \tag{5.26}$$

Both assumptions will be removed below. We follow the dispersion argument in [5, Sections 3.1–3.6] and [13, Proposition 6.3]. Put

$$K=NL, \qquad u_k=\sum_{n\ell=k}\beta_n\gamma_\ell, \qquad f(m)=\Phi_0(m/M),$$

where $\Phi_0\geq 0$ is smooth, supported on $[1/2,3]$, and equal to 1 on $[1,2]$. Thus $|u_k|\leq\tau(k)$ and $u_k=0$ when $(k,a_1)>1$. Define

$$\mathcal{S}_1 := \sum_{\substack{r\sim R\\(r,a_1a_2)=1}} \sum_{(m,r)=1} f(m)\left|\sum_{k\equiv a_1\overline{a_2m} \pmod r} u_k\right|^2,$$

$$\mathcal{S}_2 := \sum_{\substack{r\sim R\\(r,a_1a_2)=1}} \frac{1}{\varphi(r)} \sum_{(m,r)=1} f(m) \sum_{\substack{(k_1,r)=1\\k_2\equiv a_1\overline{a_2m} \pmod r}} u_{k_1}\overline{u_{k_2}}\omega_D(mk_1\overline{a_1}a_2;r),$$

$$\mathcal{S}_3 := \sum_{\substack{r\sim R\\(r,a_1a_2)=1}} \frac{1}{\varphi(r)^2} \sum_{(m,r)=1} f(m)\left|\sum_{(k,r)=1} u_k\omega_D(mk\overline{a_1}a_2;r)\right|^2.$$

Cauchy–Schwarz gives

$$\mathcal{D}(D)^2 \ll MR(\mathcal{S}_1 - 2\Re\mathcal{S}_2 + \mathcal{S}_3). \tag{5.27}$$

The corresponding main terms are

$$X_1 := \sum_{\substack{r\sim R\\(r,a_1a_2)=1}} \frac{1}{r} \sum_{\substack{0<b<r\\(b,r)=1}} \left|\sum_{k\equiv b \pmod r} u_k\right|^2,$$

$$X_2 := \sum_{\substack{r\sim R\\(r,a_1a_2)=1}} \frac{1}{r\varphi(r)} \sum_{\substack{(k_1,r)=1\\(k_2,r)=1}} u_{k_1}\overline{u_{k_2}}\omega_D(k_1\overline{k_2};r),$$

$$X_3 := \sum_{\substack{r\sim R\\(r,a_1a_2)=1}} \frac{1}{r} \sum_{\substack{0<b<r\\(b,r)=1}} \left|\frac{1}{\varphi(r)}\sum_{(k,r)=1} u_k\omega_D(k\bar{b};r)\right|^2.$$

The estimates in [5, Sections 3.2–3.3] give, for $i=2,3$,

$$\mathcal{S}_i = \widehat{f}(0)X_i + O_\kappa(x^{1-c}K/R)$$

for some $c > 0$, provided $|a_1| \le x^{1+\delta_1}$, $|a_2| \le x^{\delta_1}$ and $D \le x^{s_1}$, with $\delta_1, s_1 > 0$ sufficiently small in terms of κ. These arguments are uniform in a_1: the argument for $\mathcal{S}_3$ is independent of its size, and the one for $\mathcal{S}_2$ allows $|a_1| \le x^{1+\delta_1}$. The inequalities $K \le x^{2/3-5\kappa}$ and $M = x/K$ leave a fixed power-saving margin in both estimates.

For $\mathcal{S}_1$, retain coprimality to a_1 throughout Drappeau's decomposition [5, Section 3.4]. In its notation,

$$v = (k_1, k_2), \quad d_1 \mid v^\infty, \quad e_1 \mid a_2^\infty, \quad e_2 \mid a_2, \quad d_2 = (k_2/(ve_2), v),$$

Thus $(a_1, vd_1e_1e_2d_2) = 1$. Under the substitution

$$(v, d_1, d_2) \longleftarrow \left(\frac{|a_2|v}{e_2d_2}, e_1d_1, e_2d_2\right), \qquad (vd_1d_2)_{\text{new}} = |a_2|vd_1e_1,$$

the new parameters remain coprime to a_1; the sign of a_2 is absorbed into the residue parameter. Dividing the coefficient indices by these factors preserves (5.26) and squarefreeness of the n-variable.

Use cutoffs x^η in this decomposition, where $\eta > 0$ is sufficiently small in terms of κ, and apply Lemma 15 to the retained ranges, with a sufficiently large constant multiple of $\eta + \delta_1$ as its small parameter, as in [13, proof of Proposition 6.3]. The new lengths differ from K, N, L by factors between $x^{-C\eta - C\delta_1}$ and $x^{C\eta + C\delta_1}$. The conditions $N^2 \le x^{1/2-\kappa}$, $L/M \le x^{-\kappa}$ and $R/K \le x^{-\kappa}$ therefore imply the size hypotheses of that lemma. Its diagonal and nonzero additive frequencies give

$$\left(\frac{|\mathcal{S}_1 - \widehat{f}(0)X_1|}{xK/R}\right)^4 \ll x^{C_1(\eta+\delta_1)} \left\{\left(\frac{N^2L^3}{xR}\right)^2 + \left(\frac{K}{R}\right)^{C_*} \frac{N\sqrt{x}}{L^2}\right\} + x^{-c_2\eta}.$$

Here C_* is fixed large enough to include the polynomial costs of the complementary divisors and the other auxiliary integers. The last term, with $c_2 > 0$, bounds the discarded gcd ranges, using $R/K \le x^{-\kappa}$ and choosing the divisor losses sufficiently small after η. By (3.1), choosing η and then δ_1 sufficiently small gives a power saving. Consequently

$$\mathcal{S}_i = \widehat{f}(0)X_i + O_\kappa(x^{1-c}K/R) \qquad (i = 1, 2, 3)$$

for some $c > 0$.

It remains to estimate the main terms. Parseval on the unit group gives

$$\begin{aligned} X_1 - 2\Re X_2 + X_3 &\le \frac{1}{R} \sum_{\substack{r \sim R \\ (r, a_1a_2) = 1}} \frac{1}{\varphi(r)} \sum_{\substack{\chi \bmod r \\ \text{primitive} \\ \text{cond}(\chi) > D}} \left| \sum_{(k,r)=1} u_k \chi(k) \right|^2 \\ &\le \frac{1}{R} \sum_{D < s \le 2R} \sum_{\chi}^{*} \pmod s \sum_{\substack{r \le 2R \\ s \mid r}} \frac{\tau(r/s)}{\varphi(r)} \sum_{d \mid r/s} \left| \sum_{d \mid k} u_k \chi(k) \right|^2 \\ &\ll \frac{(\log R)^2}{R} \sum_{d \le R} \frac{\tau(d)}{\varphi(d)} \sum_{D < s \le 2R} \frac{1}{\varphi(s)} \sum_{\chi}^{*} \pmod s \left| \sum_{k'} u_{dk'} \chi(k') \right|^2. \end{aligned}$$

The second line follows by Möbius inversion and Cauchy–Schwarz. Since $s > D \ge 1$, one has $s \ge 2$, and therefore $d \mid r/s$ implies $d \le R$. Partial summation and the multiplicative large sieve [5,

Lemme 6], with $|u_k| \le \tau(k)$, yield

$$X_1 - 2\Re X_2 + X_3 \ll \frac{(\log R)^2(\log K)^3}{R} \sum_{d \le R} \frac{\tau(d)^3}{\varphi(d)} \left(\frac{K}{dD} + R \right) \frac{K}{d}$$
$$\ll (\log x)^5 \left(\frac{K^2}{RD} + K \right).$$

Decrease s_1 so that $s_1 \le \kappa$. Then $RD \le K$, and substitution in (5.27), using $\widehat{f}(0) \ll M$ and $MK = x$, proves

$$\mathcal{D}(D) \ll_\kappa \frac{x(\log x)^3}{\sqrt{D}} + x^{1-c_1} \qquad (1 \le D \le x^{s_1}) \tag{5.28}$$

for some $0 < c_1 < 1/2$, under the two auxiliary support conditions.

We record a bound used to remove both conditions. Let (b_v) be supported on $v \ll x$, with $|b_v| \le \tau_3(v)$, and put $B = \sum_v |b_v|$. By the triangle inequality its contribution to $\mathcal{D}(D)$ is at most $T_1 + T_2$, where

$$T_1 := \sum_{\substack{r \sim R \\ (r, a_1 a_2) = 1}} \sum_{a_2 v \equiv a_1 \pmod r} |b_v|,$$

$$T_2 := \sum_{\substack{r \sim R \\ (r, a_1 a_2) = 1}} \frac{1}{\varphi(r)} \left| \sum_{(v,r)=1} b_v \omega_D(v\overline{a_1}a_2; r) \right|.$$

For $a_2 v \ne a_1$, each modulus in T_1 divides $a_2 v - a_1$. The divisor bound and the possible single equality $a_2 v = a_1$ give

$$T_1 \ll x^{o(1)}(B + R), \qquad \sum_v |b_v|^2 \ll x^{o(1)} B.$$

For T_2, expand into primitive characters of conductor d and write $r = dk$. The restriction left after using the vanishing of $\chi(v)$ at $(v, d) > 1$ is $(v, k) = 1$. Since $\varphi(dk) \ge \varphi(d)\varphi(k)$,

$$T_2 \ll x^{o(1)} \sum_{k \le 2R} \frac{1}{k} \sum_{d \le D} \frac{1}{d} \sideset{}{^*}\sum_{\chi \pmod d} \left| \sum_{(v,k)=1} b_v \chi(v) \right|.$$

For each fixed k, the coefficients are independent of d. Cauchy–Schwarz and the multiplicative large sieve give

$$\sum_{E < d \le 2E} \frac{1}{d} \sideset{}{^*}\sum_{\chi \pmod d} \left| \sum_{(v,k)=1} b_v \chi(v) \right|$$
$$\le \left(\sum_{E < d \le 2E} \frac{\varphi(d)}{d^2} \right)^{1/2} \left(\sum_{E < d \le 2E} \sideset{}{^*}\sum_{\chi \pmod d} \left| \sum_{(v,k)=1} b_v \chi(v) \right|^2 \right)^{1/2}$$
$$\ll (E^2 + x)^{1/2} \left(\sum_{(v,k)=1} |b_v|^2 \right)^{1/2} \ll x^{o(1)}(E + \sqrt{x})\sqrt{B}.$$

Summing the dyadic conductor blocks and bounding conductor 1 trivially yields

$$T_1 + T_2 \ll x^{o(1)} \{ B + R + (D + \sqrt{x})\sqrt{B} \}. \tag{5.29}$$

We now remove squarefreeness, retaining (5.26). Every positive integer has a unique factorisation $n = bn'$, where b is powerful, n' is squarefree and $(b, n') = 1$. Here powerful means that every prime divisor has exponent at least two. The representation $b = u^2v^3$, with v squarefree, gives, for every $B \geq 1$,

$$\#\{b \leq B : b \text{ powerful}\} \ll \sqrt{B}, \qquad \sum_{\substack{b>B \\ b \text{ powerful}}} \frac{1}{b} \ll B^{-1/2}, \qquad \sum_{b \text{ powerful}} b^{-1+c_1} < \infty. \tag{5.30}$$

Take δ_1, s_1, c_1 from (5.28) with $\kappa/2$ in place of κ, choose $0 < \eta_0 \ll \min\{\kappa, \delta_1, s_1\}$, and then choose $\delta_0, s_0 > 0$ sufficiently small in terms of η_0, with $s_0 \leq \eta_0/8$. For the terms whose powerful part exceeds x^{η_0}, set

$$b_v := \sum_{\substack{mn\ell=v \\ m\sim M,\ n\sim N,\ \ell\sim L \\ \text{powerful part of } n>x^{\eta_0}}} \alpha_m\beta_n\gamma_\ell.$$

Then $|b_v| \leq \tau_3(v)$ and

$$B = \sum_v |b_v| \ll x^{1-\eta_0/2+o(1)}.$$

Equation (5.29) bounds this contribution by

$$x^{1-\eta_0/2+o(1)} + Rx^{o(1)} + (D + \sqrt{x})x^{1/2-\eta_0/4+o(1)},$$

which saves a power for $D \leq x^{s_0}$.

For a retained powerful $b \leq x^{\eta_0}$, put

$$\beta_{n'}^{(b)} = \beta_{bn'}\mathbf{1}_{n' \text{ squarefree}}\mathbf{1}_{(n',b)=1}.$$

The coprimality assumption gives $(b, a_1) = 1$. Moduli sharing a prime factor with b contribute zero to both terms of the kernel. On the other moduli the block is the squarefree problem with

$$x' = x/b, \qquad N' = N/b, \qquad a_1' = a_1, \qquad a_2' = a_2b,$$

and M, L, R unchanged. For sufficiently small η_0, δ_0, s_0, the residue and cutoff conditions hold at x', and the length conditions hold with $\kappa/2$. In particular,

$$\frac{(N')^2L^3}{x'R} = \frac{1}{b}\frac{N^2L^3}{xR}, \qquad \left(\frac{N'L}{R}\right)^{C_*}\frac{N'\sqrt{x'}}{L^2} = b^{-C_*-3/2}\left(\frac{NL}{R}\right)^{C_*}\frac{N\sqrt{x}}{L^2}.$$

Sum (5.28) over these powerful b. The convergent sums in (5.30) give, for some $c_0 > 0$,

$$\mathcal{D}(D) \ll_\kappa \frac{x(\log x)^3}{\sqrt{D}} + x^{1-c_0} \qquad (1 \leq D \leq x^{s_0}), \tag{5.31}$$

uniformly for $|a_1| \leq x^{1+\delta_0}$, $|a_2| \leq x^{\delta_0}$, with no squarefree restriction.

We finally remove (5.26) and restore the full cutoff range. Take δ_0, s_0, c_0 from (5.31) with $\kappa/2$ in place of κ, and choose

$$0 < \delta \ll \eta \ll \min\{\kappa, \delta_0, s_0, c_0\}.$$

If $D > x^{\eta/4}$, expanding the difference of the character sums gives

$$|\mathcal{D}(D) - \mathcal{D}(x^{\eta/4})| \leq \sum_{r\sim R}\frac{1}{\varphi(r)}\sum_{\substack{\chi \pmod r \\ x^{\eta/4}<\mathrm{cond}(\chi)\leq D}}\left|\sum_{m\sim M}\alpha_m\chi(m)\sum_{n\sim N}\beta_n\chi(n)\sum_{\ell\sim L}\gamma_\ell\chi(\ell)\right|.$$

Bound the ℓ-sum by $O(L)$ and write $r = dk$, where d is the conductor. The inducing primitive character χ^* (mod d) satisfies

$$\chi(m) = \chi^*(m)\mathbf{1}_{(m,k)=1}.$$

For each fixed k, the restricted coefficients are 1-bounded and independent of d. Cauchy–Schwarz and the multiplicative large sieve bound a block $E < d \le 2E$ by

$$\begin{aligned} &\ll \frac{Lx^{o(1)}}{E} \sum_{k \le 2R/E} \frac{1}{k} \big(M(M+E^2)N(N+E^2)\big)^{1/2} \\ &\ll \frac{x^{1+o(1)}}{E} \left(1 + \frac{E}{\sqrt{M}}\right)\left(1 + \frac{E}{\sqrt{N}}\right). \end{aligned}$$

Summing the blocks gives

$$|\mathcal{D}(D) - \mathcal{D}(x^{\eta/4})| \ll x^{1+o(1)} \left(x^{-\eta/4} + \frac{1}{\sqrt{M}} + \frac{1}{\sqrt{N}} + \frac{D}{\sqrt{MN}}\right). \tag{5.32}$$

This saves a power, since $M \ge x^{1/3+5\kappa}$, $N \ge x^{\kappa}$ and $D \le x^{\kappa}$. We may therefore assume $D \le x^{\eta/4}$.

For the terms with $(n\ell, a_1) > x^{\eta}$, the coefficients

$$b_v := \sum_{\substack{mn\ell = v \\ m\sim M,\ n\sim N,\ \ell\sim L \\ (n\ell, a_1) > x^{\eta}}} \alpha_m \beta_n \gamma_\ell$$

satisfy $|b_v| \le \tau_3(v)$ and

$$B \ll x^{1+o(1)} \sum_{\substack{g \mid\mid a_1| \\ g > x^{\eta}}} \frac{1}{g} \ll x^{1-\eta+o(1)}.$$

By (5.29), their contribution is $O(x^{1-\eta/2+o(1)} + Rx^{o(1)})$, which saves a power. For the remaining terms put

$$d = (n, a_1), \qquad e = (\ell, a_1/d), \qquad g = de = (n\ell, a_1) \le x^{\eta},$$

and write $n = dn'$, $\ell = e\ell'$. The coefficients

$$\beta_{n'}^{(d)} = \beta_{dn'}\mathbf{1}_{(n', a_1/d)=1}, \qquad \gamma_{\ell'}^{(d,e)} = \gamma_{e\ell'}\mathbf{1}_{(\ell', a_1/g)=1}$$

satisfy (5.26) with a_1/g. Division by g modulo r gives the parameters

$$x' = x/g, \qquad N' = N/d, \qquad L' = L/e, \qquad a_1' = a_1/g,$$

with M, R, a_2 unchanged. The original moduli form a subset of those coprime to $a_1' a_2$. The length conditions hold with $\kappa/2$; the critical ratios are

$$\frac{(N')^2(L')^3}{x'R} = \frac{1}{de^2}\frac{N^2L^3}{xR}, \qquad \left(\frac{N'L'}{R}\right)^{C_*} \frac{N'\sqrt{x'}}{(L')^2} = d^{-C_*-3/2}e^{-C_*+3/2}\left(\frac{NL}{R}\right)^{C_*}\frac{N\sqrt{x}}{L^2}.$$

Take $C_* \ge 2$, so both improve. All other length, residue and cutoff conditions retain their margins because $g \le x^{\eta}$ and η, δ are sufficiently small. Apply (5.31) and sum the blocks, using

$$\sum_{de \mid\mid a_1|} \frac{1}{de} = \sum_{g \mid\mid a_1|} \frac{\tau(g)}{g} \le \left(\frac{|a_1|}{\varphi(|a_1|)}\right)^2 \ll (\log\log x)^2.$$

The resulting estimate is $x(\log x)^4/\sqrt{D} + O(x^{1-c})$ for some $c > 0$, with the number of blocks absorbed in the power-saving error. Decrease c to include the saving in (5.32). Restoring the original cutoff and decreasing δ so that $\delta \le \eta/4$ and $\delta < 2c$ gives

$$\mathcal{D}(D) \ll_\kappa \frac{x(\log x)^4}{\sqrt{\min\{x^\delta, D\}}} \qquad (1 \le D \le x^\kappa).$$

□

Appendix A. The regular spectrum

By the regular spectrum we mean the real Maass, holomorphic and continuous spectra. For real Maass forms we take $\kappa_f \ge 0$. We record the estimates needed to complete the proof of Proposition 14. The argument follows the exceptional case, but there is no factor X^{θ_f}. For the holomorphic and continuous spectra we use the Petersson and Plancherel weights, respectively, with the Fourier normalisations of [12, Section 3].

Lemma A.1 (Ordinary spectral large sieve)**.** *Let $q \in \mathbb{Z}_{>0}$, $\eta > 0$, $T \ge 1$, and $\mathfrak{a} \in \{\infty, 0\}$. For a complex sequence (b_n) supported on $n \asymp J$, $J \ge 1/2$,*

$$\sum_{\substack{f \in \mathcal{B}_q \\ 0 \le \kappa_f \le T}} \frac{1}{\cosh(\pi\kappa_f)} \left| \sum_n b_n \rho_{f,\mathfrak{a}}(n) \right|^2 \ll_\eta (qJT)^\eta \left(T^2 + \frac{J}{q} \right) \sum_n |b_n|^2. \tag{A.1}$$

The same bound holds for the holomorphic and continuous spectra, replacing the Maass sum and its weight by the corresponding Petersson sum or Plancherel integral, with spectral parameters truncated at T. For $b \in \mathbb{Z}_{>0}$ with $(b, q) = 1$, one also has

$$\sum_{\substack{f \in \mathcal{B}_q \\ 0 \le \kappa_f \le T}} \frac{|\rho_{f,\mathfrak{a}}(b)|^2}{\cosh(\pi\kappa_f)} \ll_\eta (qbT)^\eta \left(T^2 + \frac{\sqrt{b}}{q} \right). \tag{A.2}$$

Proof. These are [12, Lemmas 3.8–3.9]. As in Lemma 3, fixed compact supports and bounded lengths are covered by a fixed partition of unity. □

Lemma A.2 (Real Maass moments)**.** *Let $\eta > 0$ and $T \ge 1$. For $b, q \in \mathbb{Z}_{>0}$ with $(b, q) = 1$,*

$$\sum_{\substack{f \in \mathcal{B}_q \\ 0 \le \kappa_f \le T}} |\lambda_f(b)|^2 \ll_\eta (qbT)^\eta (1+T)^{C_0} (q + \sqrt{b}). \tag{A.3}$$

For $J \ge 1/2$ and $|\beta_n| \le 1$ supported on $n \asymp J$, independently of f,

$$\sum_{\substack{f \in \mathcal{B}_q \\ 0 \le \kappa_f \le T}} \frac{1}{\cosh^2(\pi\kappa_f)} \left| \sum_n \beta_n \rho_{f,\mathfrak{a}}(n) \right|^4 \ll_\eta (qJT)^\eta (1+T)^{C_0} \left(\frac{J^2}{q} + \frac{J^4}{q^2} \right), \quad \mathfrak{a} = \infty, 0. \tag{A.4}$$

Under the hypotheses and notation of Lemma 11,

$$\sum_{\substack{f\in\mathcal{B}_q\\ 0\le\kappa_f\le T}} \frac{1}{\cosh^2(\pi\kappa_f)}\left|\sum_v \Phi(ev/Y)c_{ev}\rho_{f,0}(bv)\right|^4 \ll_\eta \mathfrak{d}^{C_0}(qYeT)^\eta(1+T)^{C_0}g^4\left(\frac{Y^2}{q}+\min\left\{\frac{Y^4}{q^2},q\right\}\right). \tag{A.5}$$

Here C_0 may be increased between the three estimates.

Proof. For a primitive real Maass form g of level s and fixed even nebentypus, the first-coefficient estimate used in Lemma 4 takes the form

$$s^{-1-\eta}(1+T)^{-C_0} \ll_\eta \frac{|\rho_{g,\infty}(1)|^2}{\cosh(\pi\kappa_g)} \ll_\eta s^{-1+\eta}(1+T)^{C_0} \qquad (0\le\kappa_g\le T).$$

The oldspace mass bound (5.8) now acquires a factor $(1+T)^{-C_0}$. The argument of Lemma 6, applied directly at level q with (A.2), proves (A.3). The expansion (5.6) also holds, with an additional factor $(1+T)^{C_0}$ in the bound for $c_f(\delta)$.

The Hecke multiplication argument in Lemma 7 and the nebentypus large sieve [6, Proposition 4.7] give, for primitive forms of level s and even nebentypus ω of conductor m_ω,

$$\left(\sum_{\substack{g \text{ primitive } (s,\omega)\\ 0\le\kappa_g\le T}}\left|\sum_n \beta_n\lambda_g(n)\right|^4\right)^{1/4} \ll_\eta (sJT)^\eta(1+T)^{C_0}m_\omega^{C_0}\left(s^{1/4}\sqrt{J}+J\right).$$

Inserting (5.6) and summing over the primitive levels and divisors as in Lemma 7 proves (A.4).

For (A.5), the rational-twist argument of Lemma 10 also applies to real Maass forms. The Voronoi identity (5.11) has the same normalisation for real spectral parameters. The twists and the Voronoi partner preserve the spectral parameter, and the preceding primitive fourth moment applies at the resulting levels and nebentypus. The only additional dependence comes from the transforms. With the notation of Lemma 8,

$$\sup_{|t|\le T}\|F_{t,\rho}^{(j)}\|_\infty + \int_{\mathbb{R}}(1+|\zeta|)^j \sup_{|t|\le T}|\widehat{F}_{t,\rho}(\zeta)|\,d\zeta \ll_{j,B} (1+T+Z)^{C_{j,B}}\begin{cases}1+|\log\rho|, & 0<\rho\le 1,\\ \rho^{-B}, & \rho\ge 1.\end{cases}$$

Indeed, the small-argument powers $v^{\pm it}$ have modulus one, with a logarithm at $t=0$; the hyperbolic and gamma factors cancel their exponential dependence on t. The differential-operator and Fourier-inversion arguments are otherwise identical. Thus Lemma 10 holds for the real spectrum truncated at T, with an additional factor $(1+T)^{C_0}$. Applying it to (5.17), and using (A.4) for the other bound, gives (A.5). The divisor parameter e still has no fixed positive power. □

Let $\mathcal{T}_\sigma^{\mathrm{real}}(U)$, $\mathcal{T}_\sigma^{\mathrm{hol}}(U)$ and $\mathcal{T}_\sigma^{\mathrm{Eis}}(U)$ denote the real Maass, holomorphic and Eisenstein contributions to the block $\mathcal{T}_\sigma(U)$ defined before Lemma 12.

Lemma A.3 (Completed blocks on the regular spectrum)**.** *Under the hypotheses of Lemma 12, the estimate (5.19) holds for $\mathcal{T}_\sigma^{\mathrm{real}}(U)$ upon replacing the exceptional sum by*

$$\sum_{\substack{f\in\mathcal{B}_q\\ 0\le\kappa_f\le x^{C_1\eta}}} \frac{1}{\cosh(\pi\kappa_f)}|P_f(U,\zeta)|\left|\sum_v u_{q,v}e(\xi v/Y)\rho_{f,0}(|a|v)\right|.$$

For $\mathcal{T}_\sigma^{\mathrm{hol}}(U)$ and $\mathcal{T}_\sigma^{\mathrm{Eis}}(U)$, use the corresponding Petersson sum over weights at most $x^{C_1\eta}$ or Plancherel integral over $|t|\le x^{C_1\eta}$, with no exceptional factor. The remainder and the dependence of constants are as in that lemma.

Proof. Use the same Kuznetsov test function and Fourier separation. Since $X^{-1}\le x^{C\eta}$, the Bessel-transform estimates in [12, Lemma 3.4] allow the spectral parameters to be truncated at $x^{C_1\eta}$. Integration by parts, followed by Lemma A.1, bounds the tails by $O_{B,\eta}(x^{-B}\sum_{q,v}|u_{q,v}|)$. On the remaining spectra the transform costs $x^{O(\eta)}$, with no factor X^{θ_f}. Keeping the cutoff ψ outside the Fourier separation again makes the coefficients of P_f common to all levels. □

Proposition A.4. *Under the hypotheses of Proposition 14, let $\mathcal{K}_{\mathrm{reg}}$ be the sum of the real Maass, holomorphic and Eisenstein contributions obtained by completing the ℓ-sum and applying Kuznetsov to the nonzero completion frequencies. Then*

$$|\mathcal{K}_{\mathrm{reg}}|^2 \ll (\mathfrak{d}\Lambda)^{C_0}x^{C_0(\eta+\delta)}L^2N^7\sqrt{x}.$$

Proof. Use the reductions in the proof of Proposition 14: first take $\Lambda=1$, suppress fixed powers of $\mathfrak{d}$, and group

$$g\asymp G,\qquad Q=N^2/G,\qquad U\ll Nx^{2\eta},\qquad Y\ll HN/G\le Q.$$

Apply Lemma A.3. The factors $(1+T)^{C_0}$ at the cutoff $T=x^{C_1\eta}$ are absorbed into $x^{O(\eta)}$. For real Maass forms, apply the inverse Hecke relation and Hölder with exponents $2,4,4$ as before. Lemma A.2 gives the three factors

$$Q(Q+\sqrt{|a|}),\qquad U\left(1+\frac{U^2}{Q}\right)^{1/2},\qquad G^2QB(Y;Q),$$

up to $x^{O(\eta)}$, with $B(Y;Q)$ as in Lemma 13. Since $Q\le\sqrt{x}$, $|a|\le x^{1+\delta}$ and $B(Y;Q)\ll 1$, we obtain

$$\begin{aligned}|\mathcal{K}_{G,\mathrm{real}}(U,Y)|^2 &\ll x^{O(\eta+\delta)}L^2Q\cdot Q\sqrt{x}\cdot U\left(1+\frac{U^2}{Q}\right)^{1/2}\cdot G^2Q\\ &\ll x^{O(\eta+\delta)}L^2N^7\sqrt{x}\,G^{-1}\sqrt{1+G}\ll x^{O(\eta+\delta)}L^2N^7\sqrt{x}\,G^{-1/2}.\end{aligned}$$

Here the harmless enlargement of $U\le N$ is included in $x^{O(\eta)}$.

For holomorphic forms, Deligne's bound [4] gives $|\lambda_f(b)|\le\tau(b)$ at unramified indices. For Eisenstein series, the same bound follows from

$$\lambda_{\chi,t}(b)=\sum_{uv=b}\chi(u)\overline{\chi(v)}(u/v)^{it}.$$

Fix a divisor $e\mid|a|$ in the inverse Hecke relation and put $e_1=e/(e,t)$, as in the main proof. Restriction to multiples of e_1 does not increase the coefficient energy:

$$\sum_v|c_{e_1v}|^2\le\sum_w|c_w|^2\ll H^2(1+HG/N).$$

The last bound follows by counting equal differences $s'h - sh'$. The divisor sums and the Hecke eigenvalues cost $x^{O(\eta)}$, after allowing fixed powers of $\mathfrak{d}$. There are $O(N^2/G)$ pairs (n, n') with $(n, n') \asymp G$. Cauchy–Schwarz and the two ordinary large sieves from Lemma A.1 therefore give, using $U \ll Nx^{2\eta}$, $N \ll Q$ and $Y \ll Q$,

$$\begin{aligned}|\mathcal{K}_{G,\mathrm{hol}}(U,Y)|^2 + |\mathcal{K}_{G,\mathrm{Eis}}(U,Y)|^2 &\ll x^{O(\eta)} L^2 Q \cdot QU \cdot \frac{N^2H^2}{G}(1 + HG/N) \\ &\ll x^{O(\eta)} L^2 N^9 G^{-2} \leq x^{O(\eta)} L^2 N^7 \sqrt{x}\, G^{-2}.\end{aligned}$$

The last inequality uses $N^2 \leq \sqrt{x}$. The fixed powers of $\mathfrak{d}$ also absorb the small multipliers in the Fourier indices.

The dyadic G-sums converge after taking square roots, and the U, Y sums cost only $x^{O(\eta)}$. Finally, restore $\mathfrak{d}$ and Λ. As in the length comparisons in the main proof, the level becomes dQ, while the inequalities $H \leq N$, $U \leq N$, $Y \leq Q$ and $Q \leq \sqrt{x}$ are enlarged by fixed powers of $\mathfrak{d}\Lambda$ and $x^{O(\eta)}$. The spectral cutoff has the same type of enlargement. All preceding bounds therefore lose at most $(\mathfrak{d}\Lambda)^{C_0} x^{C_0(\eta+\delta)}$, proving the result. □

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