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Quantitative similarities for amenable algebras of two-by-two matrix-valued functions

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Abstract

We study norm-closed subalgebras A of $\mathrm{C}(X, \mathrm{M}_2)$ containing the scalar functions, where X is a compact metrizable space. We characterize amenability of A by semisimplicity of its fibres and a uniform bound on the norms of the elements $2p - \mathrm{I}_2$ associated with their primitive idempotents. This bound is the exact similarity constant in the faithful representation given by the direct sum of all evaluations. In particular, an amenable algebra of this form admits a similarity to a nuclear C^*-algebra with cost at most $\sqrt{2\,\mathrm{AM}(A) - 1}$. The estimate is sharp already for a two-dimensional algebra in M_2. The similarity is constructed from positive matrix-valued functions which are continuous away from the scalar fibres. Although these functions need not be continuous everywhere, the induced algebra isomorphism preserves continuous sections. We also prove the corresponding results for locally trivial bundles of M_2-algebras, and give a noncommutative example for which no continuous matrix-valued similarity is available. We prove a central saturation inequality and a finite-orbit descent result. For fields with only scalar and two-dimensional fibres, the transported adjoint and similarity preserve Hölder regularity with bounds independent of the spectral gap. An explicit smooth section shows that differentiability need not be preserved by any selfadjointizing similarity.

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1 Introduction

The amenability of a Banach algebra is defined without reference to an involution. Nevertheless, for C^*-algebras, it is equivalent to nuclearity by the work of Connes and Haagerup; see [BLM04, Section 7.4] and [Haa83]. This leads to the question of whether an amenable operator algebra

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must be isomorphic, as a Banach algebra, to a nuclear C*-algebra. Here an operator algebra is a norm-closed, not necessarily selfadjoint, subalgebra of $\mathrm{B}(H)$ for a complex Hilbert space H.

Choi, Farah and Ozawa constructed a nonseparable amenable subalgebra of $\ell^\infty(\mathbb{N}, \mathrm{M}_2)$ which is not isomorphic to a C*-algebra [CFO14, Theorem 1]. Their construction also shows that local similarities with uniformly bounded costs do not, by themselves, provide a global similarity. The question restricted to separable operator algebras is not settled by that example. Positive results are known for several classes. In particular, Gifford treated algebras of compact operators using the total reduction property [Gif06], Choi proved an internal similarity theorem for commutative operator amenable subalgebras of finite von Neumann algebras [Cho13, Theorem 1.1], and Marcoux and Popov proved that every commutative amenable operator algebra is similar to a C*-algebra [MP16, Theorem 2.10]. Clouâtre and Marcoux obtained a similarity theorem for weak* closed residually finite-dimensional operator algebras with the total reduction property [CM19, Theorem 4.5]. The weak* closedness assumption in that result cannot simply be replaced by norm closedness.

In this paper, we consider a different locality assumption. Let X be a nonempty compact metrizable space and let A be a norm-closed algebra satisfying

$$\mathrm{C}(X)\mathrm{I}_2 \subset A \subset \mathrm{C}(X, \mathrm{M}_2). \tag{1.1}$$

For any $x \in X$, we denote its evaluation algebra by $A_x \stackrel{\mathrm{def}}{=} \{a(x) : a \in A\}$. The inclusion of the scalar functions makes A a $\mathrm{C}(X)$-module. Consequently, membership in A can be checked fibrewise, provided that the section under consideration is continuous. This last qualification is essential: a bounded field of similarities need not preserve continuous sections.

We describe the invariant which occurs in our result. Suppose that each A_x is semisimple. A unital semisimple subalgebra of M_2 is either $\mathbb{C}\mathrm{I}_2$, M_2, or a two-dimensional algebra of the form $\mathbb{C}p_x + \mathbb{C}(\mathrm{I}_2 - p_x)$, where p_x is a rank-one idempotent. Put

$$\mathrm{K}(A) \stackrel{\mathrm{def}}{=} \sup\bigl(\{1\} \cup \{\|2p_x - \mathrm{I}_2\| : x \in X,\ \dim A_x = 2\}\bigr). \tag{1.2}$$

The value in (1.2) is unchanged if p_x is replaced by $\mathrm{I}_2 - p_x$.

We use a specified faithful representation when discussing the optimal cost of a spatial similarity. Set $H_X \stackrel{\mathrm{def}}{=} \ell^2_X \otimes \mathbb{C}^2$ and let $\pi_X(a) \stackrel{\mathrm{def}}{=} \bigoplus_{x\in X} a(x)$. We define

$$s_X(A) \stackrel{\mathrm{def}}{=} \inf\bigl\{\|S\|\,\bigl\|S^{-1}\bigr\| : S \in \mathrm{B}(H_X) \text{ invertible}, S\pi_X(A)S^{-1} \text{ selfadjoint}\bigr\}, \tag{1.3}$$

with infimum $+\infty$ when there is no such operator. The use of all evaluations in this definition removes any possible dependence on a measure on X.

Theorem 1.1 *Let X be a nonempty compact metrizable space and let A satisfy (1.1). Then the following conditions are equivalent:*

1. *A is amenable as a Banach algebra.*
2. *Every algebra A_x is semisimple and $\mathrm{K}(A) < \infty$.*

If these conditions hold and $K \stackrel{\mathrm{def}}{=} \mathrm{K}(A)$, there exists a bounded Borel function $h\colon X \to \mathrm{M}_2$ such that

$$\det h(x) = 1, \quad K^{-1}\mathrm{I}_2 \leqslant h(x) \leqslant K\mathrm{I}_2 \quad \textit{for any } x \in X, \tag{1.4}$$

and the map

$$\Theta\colon A \to \mathrm{C}(X, \mathrm{M}_2), \quad a \mapsto h^{\frac{1}{2}} a h^{-\frac{1}{2}}, \tag{1.5}$$

is a $\mathrm{C}(X)$*-linear algebra isomorphism onto a nuclear* C^**-subalgebra. Moreover,* h *is continuous outside the closed set* $\{x : A_x = \mathbb{C}\mathrm{I}_2\}$*, and*

$$s_X(A) = K, \quad \|\Theta\|_{\mathrm{cb}} \leqslant K, \quad \left\|\Theta^{-1}\right\|_{\mathrm{cb}} \leqslant K. \tag{1.6}$$

Finally,

$$\frac{1+K^2}{2} \leqslant \mathrm{AM}(A) \leqslant K^2. \tag{1.7}$$

In particular, Theorem 1.1 gives the estimate

$$s_X(A) \leqslant \sqrt{2\,\mathrm{AM}(A) - 1}. \tag{1.8}$$

For any prescribed $K \geqslant 1$, a two-dimensional algebra in M_2 attains equality in (1.8); see Example 7.1. Thus this estimate cannot be improved as a function of the amenability constant alone in the class (1.1). We do not claim that the upper bound in (1.7) is an equality in general.

The locality assumption (1.1) is part of the theorem. We do not remove it, and the result does not address arbitrary separable amenable subalgebras of $\mathrm{C}(X, \mathrm{M}_2)$. Also, the similarity in (1.6) is taken in $\mathrm{B}(H_X)$, not in $\mathrm{C}(X, \mathrm{M}_2)$. Example 7.2 shows that this distinction is necessary, even for a noncommutative algebra in the class considered here.

Approach of the paper The two-dimensional fibres have an intrinsic pair of primitive idempotents, up to permutation. We first compute both their amenability constants and their optimal similarity constants. For such a pair p, q, the matrix $p^*p + q^*q$ defines a compatible positive metric. A trace normalization places these metrics in a fixed convex ball of selfadjoint matrices. We extend the normalized metrics through the full matrix fibres using Dugundji's theorem [Dug51, Theorem 4.1], and then normalize the determinant. At a scalar fibre, a continuous section is asymptotically scalar. This makes the discontinuity of the metric harmless for conjugation. A partition of unity then gives an adjoint in the original algebra, rather than merely in an ambient algebra of bounded functions.

Structure of the paper Section 2 records the localization and amenability facts used below. Section 3 contains the two-dimensional computations. We prove Theorem 1.1 in Section 4. Section 5 extends the result to locally trivial bundles of M_2-algebras. Section 7 gives examples and describes the distinction between continuous metrics and continuous transported adjoints. Sections 6 and 8 contain the additional localization and regularity results.

2 Preliminaries and localization

We write M_d for $\mathrm{M}_d(\mathbb{C})$ with its operator norm. The symbol Tr denotes the unnormalized trace. An idempotent is an element p satisfying $p^2 = p$. The word projection is reserved for a selfadjoint idempotent. All algebras in the main results are unital, with the same identity as their ambient algebra. We denote by $A^\dagger \stackrel{\mathrm{def}}{=} \{a^* : a \in A\}$ the adjoint algebra, reserving A^* for the Banach dual when both notions occur.

For a unital Banach algebra A, let $A \widehat{\otimes}_\pi A$ denote its Banach-space projective tensor product. We use the actions $a \cdot (x \otimes y) = ax \otimes y$ and $(x \otimes y) \cdot a = x \otimes ya$, and the multiplication map $\mu(x \otimes y) = xy$. A bounded approximate diagonal is a bounded net (m_α) such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \mu(m_\alpha) \to 1 \quad \text{for any } a \in A. \tag{2.1}$$

By Johnson's characterization, A is amenable if and only if it has a bounded approximate diagonal [Joh72]; see also [Run02, Theorem 2.2.4 pp. 45–46]. The amenability constant $\mathrm{AM}(A)$ is the infimum of the bounds of such nets. We set $\mathrm{AM}(A) = +\infty$ when A is not amenable.

Lemma 2.1 *Let E be a unital Banach algebra, and suppose that $M \in (E \widehat{\otimes}_\pi E)^{**}$ satisfies*

$$a \cdot M = M \cdot a \quad (a \in E), \qquad \mu^{**}(M) = 1_E.$$

Then E has an approximate diagonal bounded by $\|M\|$. In particular, $\mathrm{AM}(E) \leqslant \|M\|$.

Proof : Put $c = \|M\|$. Goldstine's theorem [Meg98, Theorem 2.6.26 p. 232] supplies a net (n_β) in $E \widehat{\otimes}_\pi E$, bounded by c, converging weak* to M. Fix $a_1, \ldots, a_s \in E$. In the finite product

$$\left(E \widehat{\otimes}_\pi E\right)^s \times E,$$

the tuples

$$\left(a_1 \cdot n_\beta - n_\beta \cdot a_1, \ldots, a_s \cdot n_\beta - n_\beta \cdot a_s, \mu(n_\beta) - 1_E\right)$$

converge weakly to zero, by the two assumed identities. The weak closure of a convex subset of a Banach space equals its norm closure, by Hahn–Banach separation, equivalently [Meg98, Theorem 2.5.16 p. 216]. Hence, for every $\varepsilon > 0$, a finite convex combination of these tuples has norm less than ε. Take the same convex combination of the tensors n_β. Its norm is at most c, each of its displayed module defects has norm less than ε, and its product is within ε of 1_E. Directing these choices by finite subsets of E and positive tolerances gives the required approximate diagonal. ∎

Lemma 2.2 *Let A and B be unital Banach algebras.*

1. *If $q\colon A \to B$ is a contractive surjective homomorphism, then $\mathrm{AM}(B) \leqslant \mathrm{AM}(A)$.*
2. *If $\theta\colon B \to A$ is a bounded algebra isomorphism and B is a nuclear C^*-algebra, then $\mathrm{AM}(A) \leqslant \|\theta\|^2$.*

Proof : 1. Apply $q \otimes q$ to an approximate diagonal of A. For $b = q(a)$, its centrality defect is the image of $a \cdot m_\alpha - m_\alpha \cdot a$, and its product converges to 1_B. The norm of $q \otimes q$ is at most 1.

2. A nuclear C^*-algebra has approximate diagonals with bounds arbitrarily close to 1, by Haagerup's norm-one virtual diagonal theorem [Haa83, Theorem 3.1 p. 311] and Lemma 2.1; see also [BLM04, Section 7.4.17]. Apply $\theta \otimes \theta$ to such a net. For any $a \in A$, use $b = \theta^{-1}(a)$ in the centrality relation. Its norm is bounded by $\|\theta\|^2$ times the original bound. This proves the assertion. ∎

Lemma 2.3 *Let X be a compact metric space, let V be a finite-dimensional normed space, and let Y be a closed $\mathrm{C}(X)$-submodule of $\mathrm{C}(X, V)$. For any $x \in X$, set $Y_x \stackrel{\mathrm{def}}{=} \{y(x) : y \in Y\}$. For any $b \in \mathrm{C}(X, V)$, we have*

$$\operatorname{dist}(b, Y) = \sup_{x \in X} \operatorname{dist}(b(x), Y_x). \tag{2.2}$$

Consequently, a continuous function b belongs to Y if and only if $b(x) \in Y_x$ for any $x \in X$.

Proof : Every evaluation is contractive, which gives one inequality. Put $d \stackrel{\text{def}}{=} \sup_x \operatorname{dist}(b(x), Y_x)$ and fix $\varepsilon > 0$. For any $x \in X$, choose $y_x \in Y$ with $\|b(x) - y_x(x)\| < d + \varepsilon$. This estimate remains valid on an open neighbourhood U_x of x. Choose a finite subcover $U_{x_1}, \ldots, U_{x_N}$. Put $t_i(z) = \operatorname{dist}(z, X \setminus U_{x_i})$, taking $t_i = 1$ when $U_{x_i} = X$, and set $\varphi_i = \frac{t_i}{\sum_j t_j}$. These functions are continuous, nonnegative and sum to one. The denominator is positive because the open sets cover X, and $\varphi_i(z) > 0$ implies $z \in U_{x_i}$. The function $y = \sum_i \varphi_i y_{x_i}$ belongs to Y, and

$$\|b(z) - y(z)\| \leqslant \sum_i \varphi_i(z)\, \|b(z) - y_{x_i}(z)\| \leqslant d + \varepsilon \quad (z \in X).$$

Taking the supremum over z and then letting $\varepsilon \to 0$ proves (2.2). The last assertion follows from the norm closedness of Y. ∎

Remark 2.4 The proof of Lemma 2.3 also applies to continuous sections of a finite-dimensional normed vector bundle: the norm of the difference of two sections is continuous, and scalar partitions of unity act on sections. We will use this formulation in Section 5.

Lemma 2.5 *Let D be a unital C^*-subalgebra of $\mathrm{C}(X, \mathrm{M}_2)$ which contains $\mathrm{C}(X)\mathrm{I}_2$. Then every irreducible representation of D has dimension at most 2. In particular, D is nuclear.*

Proof : The central copy of $\mathrm{C}(X)$ acts by scalars in any unital irreducible representation ρ of D. Thus there is $x \in X$ such that $\rho(f\mathrm{I}_2) = f(x)I$ for any $f \in \mathrm{C}(X)$. If $a(x) = 0$, cutoff functions vanishing near x show that a is a norm limit of elements fa with $f(x) = 0$. Indeed, choose f with $0 \leqslant f \leqslant 1$, equal to 1 outside a neighbourhood on which $\|a(\cdot)\|$ is small. Hence ρ factors through the evaluation algebra D_x. This is a C^*-subalgebra of M_2, so its irreducible representations have dimension at most 2. Finally, subhomogeneous C^*-algebras are nuclear by [BO08, Proposition 2.7.7 p. 57]. ∎

Lemma 2.6 *Let X be a nonempty compact Hausdorff space. Then $\mathrm{C}(X)$ has a contractive approximate diagonal whose product is the identity at every stage. In particular, $\mathrm{AM}(\mathrm{C}(X)) = 1$.*

Proof : For finitely many functions $v_1, \ldots, v_r \in \mathrm{C}(X)$ with $|v_j(x)| = 1$ and an integer $N \geqslant 1$, put

$$m_N = \frac{1}{N^r} \sum_{0 \leqslant k_1, \ldots, k_r < N} \left(\prod_{j=1}^{r} v_j^{k_j} \right) \otimes \left(\prod_{j=1}^{r} v_j^{-k_j} \right).$$

Every factor in this average has norm one. Hence $\|m_N\|_\pi \leqslant 1$ and $\mu(m_N) = 1$. For fixed j, rewrite the right module action by increasing the jth exponent by one. The terms with that exponent between 1 and $N-1$ cancel with the left module action. The two remaining faces each have N^{r-1} terms of projective norm one, so

$$\|v_j \cdot m_N - m_N \cdot v_j\|_\pi \leqslant \frac{2}{N}.$$

Every real-valued $b \in \mathrm{C}(X)$ of norm at most one is the real part of the continuous unimodular function

$$v = b + \mathrm{i}\sqrt{1 - b^2}, \qquad b = \frac{v + v^{-1}}{2}.$$

Splitting an arbitrary function into its real and imaginary parts therefore expresses it as a linear combination of finitely many unimodular functions. Given a finite subset of $\mathrm{C}(X)$, use all the

functions in these decompositions in the displayed average. The defect for each member of that finite subset tends to zero as N increases. Directing these choices by finite subsets and positive tolerances gives the asserted approximate diagonal. Finally, multiplication is contractive and $\|1\| = 1$, so the products of any bounded approximate diagonal force its bound to be at least one. ∎

Lemma 2.7 *Let $E_1, \dots, E_q$ be nonzero unital Banach algebras, where $q \geqslant 1$, and give $E = \bigoplus_{i=1}^{q}{}_{\infty} E_i$ its maximum norm. Then*

$$\mathrm{AM}(E) = \max_{1 \leqslant i \leqslant q} \mathrm{AM}(E_i).$$

Proof : Write $j_i \colon E_i \to E$ and $p_i \colon E \to E_i$ for the coordinate inclusion and projection. We first prove that, for $z_i \in E_i \mathbin{\widehat{\otimes}_\pi} E_i$,

$$\left\| \sum_{i=1}^{q} (j_i \otimes j_i) z_i \right\|_\pi = \max_{1 \leqslant i \leqslant q} \|z_i\|_\pi . \tag{2.3}$$

The lower bound follows by applying the contraction $p_i \otimes p_i$. For the upper bound, first take algebraic tensors and $c > \max_i \|z_i\|_\pi$. Choose representations

$$z_i = \sum_k x_{ik} \otimes y_{ik}, \qquad \sum_k \|x_{ik}\| \, \|y_{ik}\| < c,$$

discarding terms with a zero factor. Choose a finite-valued random pair (a_i, b_i) taking the value

$$\left(\frac{\sqrt{c}\, x_{ik}}{\|x_{ik}\|}, \frac{\sqrt{c}\, y_{ik}}{\|y_{ik}\|} \right) \quad \text{with probability} \quad \frac{\|x_{ik}\| \, \|y_{ik}\|}{c},$$

and taking $(0, 0)$ with the remaining probability. Then $\mathbb{E}(a_i \otimes b_i) = z_i$ and both factor norms are at most $\sqrt{c}$. Make these choices independently and let $\varepsilon_1, \dots, \varepsilon_q$ be independent signs, independent also of the pairs, taking the values 1 and -1 with equal probabilities. For

$$a = (\varepsilon_i a_i)_{i=1}^{q}, \qquad b = (\varepsilon_i b_i)_{i=1}^{q},$$

we have $\|a\| \, \|b\| \leqslant c$. Expanding the tensor and averaging the signs kills each term with distinct coordinate indices. Thus

$$\mathbb{E}(a \otimes b) = \sum_{i=1}^{q} (j_i \otimes j_i) z_i, \qquad \left\| \sum_{i=1}^{q} (j_i \otimes j_i) z_i \right\|_\pi \leqslant c.$$

Let c decrease to the maximum of the projective norms. To pass to completed tensors, approximate each z_i by algebraic tensors. The sum of the coordinate tensor maps is continuous by the preliminary bound $\left\| \sum_i (j_i \otimes j_i) z_i \right\|_\pi \leqslant \sum_i \|z_i\|_\pi$. This proves (2.3) on the completions.

Each p_i is a contractive surjective homomorphism. Applying $p_i \otimes p_i$ to an approximate diagonal of E gives an approximate diagonal of E_i: for centrality, lift a test element using j_i, and for multiplication use $p_i(1_E) = 1_{E_i}$. Hence $\mathrm{AM}(E_i) \leqslant \mathrm{AM}(E)$. This proves the equality if some constant on the right is infinite. Otherwise choose $c > \max_i \mathrm{AM}(E_i)$ and, in each $E_i \mathbin{\widehat{\otimes}_\pi} E_i$, an approximate diagonal bounded by c. Index their diagonal-block sums by the finite product of the original directed sets. Equation (2.3) bounds these sums by c. Their multiplication converges coordinatewise, hence in the finite maximum norm, to 1_E. Their module defects have the same diagonal-block form; the same equation and the finiteness of q give convergence to zero for every element of E. They therefore form an approximate diagonal of E bounded by c. Taking the infimum over such c proves the reverse inequality. ∎

3 Two-dimensional fibres and sharp constants

Lemma 3.1 *A unital semisimple subalgebra B of M_2 is either $\mathbb{C}\mathrm{I}_2$, M_2, or $\mathbb{C}p + \mathbb{C}(\mathrm{I}_2 - p)$ for a rank-one idempotent p. In particular, this conclusion holds when B is amenable.*

Proof : A finite-dimensional semisimple complex algebra is a direct sum of full matrix algebras. Its faithful unital representation on $\mathbb{C}^2$ is a direct sum of simple modules. Counting their dimensions leaves only a scalar representation, the sum of two distinct one-dimensional representations, and the two-dimensional simple representation of M_2. In the middle case the two central primitive idempotents have rank one and sum to I_2. Finally, a finite-dimensional unital amenable operator algebra is algebraically isomorphic to a finite-dimensional C*-algebra by [BLM04, Corollary 7.4.14 p. 291]. Such a C*-algebra has zero radical: if x lies in its radical, so does x^*x, whose spectral radius and hence norm vanish. The radical is invariant under algebra isomorphisms, which proves semisimplicity. ∎

For a rank-one idempotent $p \in \mathrm{M}_2$, put

$$q \stackrel{\text{def}}{=} \mathrm{I}_2 - p, \quad u \stackrel{\text{def}}{=} 2p - \mathrm{I}_2, \quad s \stackrel{\text{def}}{=} \|u\|, \quad k \stackrel{\text{def}}{=} p^*p + q^*q. \tag{3.1}$$

The two primitive idempotents of $B_p = \mathbb{C}p + \mathbb{C}q$ are p and q.

Proposition 3.2 *With the notation (3.1), the following assertions hold.*

1. *The least cost of a similarity transforming B_p into a selfadjoint subalgebra of M_2 is s.*
2. *The amenability constant is*

$$\mathrm{AM}(B_p) = \frac{1+s^2}{2}. \tag{3.2}$$

3. *The matrices*

$$g = \frac{2k}{\mathrm{Tr}(k)}, \quad h = \frac{g}{\sqrt{\det g}} \tag{3.3}$$

satisfy $hp = p^*h$, $\det h = 1$, and

$$\|g - \mathrm{I}_2\| = \frac{s^2-1}{s^2+1}, \quad \sigma(h) = \{s^{-1}, s\}. \tag{3.4}$$

The matrices g and h are unchanged when p and q are interchanged, and their construction commutes with $*$-automorphisms of M_2.

Proof : Choose an orthonormal basis whose first vector spans $\mathrm{Ran}\, p$, and change the phase of the second vector if necessary. In this basis,

$$p = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & -t \\ 0 & 1 \end{bmatrix}, \quad t \geqslant 0. \tag{3.5}$$

From (3.5), a direct computation gives

$$s = \sqrt{1+t^2} + t, \quad k = \begin{bmatrix} 1 & t \\ t & 1+2t^2 \end{bmatrix}, \quad \det k = 1 + t^2. \tag{3.6}$$

By (3.6), the eigenvalues of $g = \frac{k}{1+t^2}$ are $1 \pm \frac{t}{\sqrt{1+t^2}}$. Hence the eigenvalues of $h = \frac{k}{\sqrt{1+t^2}}$ are $\sqrt{1+t^2} \pm t$, which proves (3.4). Also $kp = p^*k$, by the identities $pq = qp = 0$. Scalar normalization preserves this relation. The last assertion follows because the formulas use only algebraic operations, the adjoint, trace and determinant, and because every $*$-automorphism of M_2 is unitary conjugation.

The relation $hp = p^*h$ implies that $h^{\frac{1}{2}}ph^{-\frac{1}{2}}$ is an orthogonal projection. The cost of this similarity is s, by (3.4). Conversely, if SB_pS^{-1} is selfadjoint, it is a commutative C*-algebra. Its idempotent SpS^{-1} is therefore an orthogonal projection, and SuS^{-1} is a selfadjoint unitary. Consequently,

$$s = \|u\| \leqslant \left\|S^{-1}\right\| \left\|SuS^{-1}\right\| \|S\| = \|S\| \left\|S^{-1}\right\|.$$

This proves assertion 1.

We now prove assertion 2. The only diagonal in $B_p \otimes B_p$ is

$$d = p \otimes p + q \otimes q = \frac{1}{2}(\mathrm{I}_2 \otimes \mathrm{I}_2 + u \otimes u). \tag{3.7}$$

Indeed, commutation with p eliminates the mixed terms in the basis $p \otimes p, p \otimes q, q \otimes p, q \otimes q$, and $\mu(d) = \mathrm{I}_2$ determines the other coefficients. Since $B_p \widehat{\otimes}_\pi B_p$ is finite-dimensional, every bounded approximate diagonal has a cluster point which is a diagonal. Hence $\mathrm{AM}(B_p) = \|d\|_\pi$. Equation (3.7) gives $\|d\|_\pi \leqslant \frac{1}{2}(1+s^2)$.

To prove the reverse inequality, let

$$v = \frac{1}{\sqrt{1+s^2}} \begin{bmatrix} 1 \\ s \end{bmatrix}, \quad w = \frac{1}{\sqrt{1+s^2}} \begin{bmatrix} s \\ -1 \end{bmatrix}.$$

These vectors are orthonormal and $uv = sw$. Thus, for $a = \lambda \mathrm{I}_2 + \mu u$, we have

$$|\lambda|^2 + s^2|\mu|^2 = \|av\|^2 \leqslant \|a\|^2. \tag{3.8}$$

Define a complex-bilinear form on B_p by

$$\beta(\lambda \mathrm{I}_2 + \mu u, \lambda' \mathrm{I}_2 + \mu' u) = \lambda\lambda' + s^2\mu\mu'.$$

The Cauchy–Schwarz inequality and (3.8) give $\|\beta\| \leqslant 1$. Therefore

$$\|d\|_\pi \geqslant |\beta(d)| \overset{(3.7)}{=} \frac{1+s^2}{2}.$$

This proves (3.2). ∎

Remark 3.3 The determinant normalization in (3.3) balances the extreme eigenvalues of the metric. The trace normalization preceding it is used for a different purpose: it places $g - \mathrm{I}_2$ in a convex norm ball of traceless selfadjoint matrices. This separation between extension and normalization will preserve the optimal constant in the global construction.

4 Global similarities and the proof of the main theorem

Proof of Theorem 1.1 : We first construct a similarity under the fibrewise condition in Theorem 1.1. No amenability assumption is used in this construction.

Suppose that every A_x is semisimple and that $K = \mathrm{K}(A) < \infty$. Let

$$Z \stackrel{\text{def}}{=} \{x \in X : A_x = \mathbb{C}\mathrm{I}_2\}, \quad U \stackrel{\text{def}}{=} X \setminus Z, \quad F \stackrel{\text{def}}{=} \{x \in U : \dim A_x = 2\}. \tag{4.1}$$

For the sets in (4.1), we first observe that $x \mapsto \dim A_x$ is lower semicontinuous. Indeed, finitely many sections whose values are linearly independent at one point remain linearly independent on a neighbourhood. Hence Z is closed, U is open, and F is closed in U. Every fibre over $U \setminus F$ is M_2.

For $x \in F$, define g_x and h_x by (3.3), using either primitive idempotent of A_x. These functions are continuous on F. To check this locally, fix $x_0 \in F$ and choose $a \in A$ with $a(x_0)$ nonscalar. Its two eigenvalues are distinct. On a neighbourhood of x_0, they admit continuous branches λ_1, λ_2, and

$$p_x = \frac{a(x) - \lambda_2(x)\mathrm{I}_2}{\lambda_1(x) - \lambda_2(x)}, \quad x \in F, \tag{4.2}$$

is a primitive idempotent of A_x. Formula (3.3) is therefore locally continuous. Its invariance under permutation makes the local definitions agree.

Set

$$r \stackrel{\text{def}}{=} \frac{K^2 - 1}{K^2 + 1} < 1. \tag{4.3}$$

By Proposition 3.2, the map $x \mapsto g_x - \mathrm{I}_2$ takes its values in the convex set

$$\mathcal{V}_r \stackrel{\text{def}}{=} \{v \in (\mathrm{M}_2)_{\mathrm{sa}} : \mathrm{Tr}(v) = 0,\ \|v\| \leqslant r\}.$$

Dugundji's extension theorem [Dug51, Theorem 4.1], applied to the metric space U and its closed subset F, gives a continuous extension $v\colon U \to \mathcal{V}_r$. If F is empty, take $v = 0$. Put

$$g(x) \stackrel{\text{def}}{=} \mathrm{I}_2 + v(x), \quad h(x) \stackrel{\text{def}}{=} \frac{g(x)}{\sqrt{\det g(x)}} \quad (x \in U), \quad h(x) \stackrel{\text{def}}{=} \mathrm{I}_2 \quad (x \in Z). \tag{4.4}$$

The eigenvalues of $v(x)$ are λ and $-\lambda$, with $|\lambda| \leqslant r$. Consequently, $g(x)$ is positive invertible, and the eigenvalues of $h(x)$ are

$$\sqrt{\frac{1+\lambda}{1-\lambda}}, \quad \sqrt{\frac{1-\lambda}{1+\lambda}}.$$

Equations (4.3) and (4.4) imply (1.4). The function h is continuous on U and Borel on X.

For any $a \in A$, define $J(a)(x) = h(x)^{-1}a(x)^*h(x)$. This is continuous on U. If $z \in Z$, write $a(z) = \lambda\mathrm{I}_2$. Then

$$\big\|J(a)(x) - \overline{\lambda}\mathrm{I}_2\big\| \leqslant K^2 \,\|a(x) - \lambda\mathrm{I}_2\| \to 0 \quad \text{as } x \to z. \tag{4.5}$$

Thus $J(a)$ is continuous on X. Its value belongs to A_x at every point: this follows from Proposition 3.2 on F, and is immediate on scalar and full matrix fibres. Lemma 2.3 gives

$$J(a) \in A \quad \text{for any } a \in A.$$

The map J is conjugate-linear and antimultiplicative, and $J^2 = \mathrm{Id}_A$, because h is positive. It follows that $h^{-1}A^\dagger h = A$ as algebras of bounded functions.

Write $s(x) = h(x)^{\frac{1}{2}}$. The same argument as in (4.5), with bound K instead of K^2, shows that $\Theta(a) = sas^{-1}$ is continuous on X for any $a \in A$. The image $D \overset{\text{def}}{=} \Theta(A)$ is norm closed, since both Θ and its inverse on its range are bounded. Also,

$$\Theta(a)^* = sJ(a)s^{-1} = \Theta(J(a)),$$

so D is selfadjoint. The map Θ in (1.5) fixes $\mathrm{C}(X)\mathrm{I}_2$ pointwise, hence Lemma 2.5 shows that D is nuclear. At every matrix level, conjugation by $I_n \otimes s(x)$ gives

$$\|\Theta\|_{\mathrm{cb}} \leqslant K, \quad \left\|\Theta^{-1}\right\|_{\mathrm{cb}} \leqslant K.$$

In particular, A is amenable and Lemma 2.2 gives $\mathrm{AM}(A) \leqslant K^2$.

Conversely, suppose that A is amenable. Evaluation at x and Lemma 2.2 give $\mathrm{AM}(A_x) \leqslant \mathrm{AM}(A)$. Lemma 3.1 gives semisimplicity and the stated fibre types. For $x \in F$, Proposition 3.2 yields

$$\frac{1 + \|2p_x - \mathrm{I}_2\|^2}{2} \overset{(3.2)}{=} \mathrm{AM}(A_x) \leqslant \mathrm{AM}(A).$$

Taking the supremum gives $K < \infty$ and the lower bound in (1.7). If F is empty, this lower bound is simply $1 \leqslant \mathrm{AM}(A)$, which follows from (2.1) and the contractivity of multiplication.

It remains to verify the exact spatial cost defined in (1.3). The diagonal operator $S \overset{\text{def}}{=} \bigoplus_{x \in X} s(x)$ on H_X has $\|S\|, \left\|S^{-1}\right\| \leqslant \sqrt{K}$, and it implements Θ in the evaluation representation. Hence $s_X(A) \leqslant K$.

For the reverse inequality, let $T \in \mathrm{B}(H_X)$ be invertible and suppose that $T\pi_X(A)T^{-1}$ is selfadjoint. For any real-valued $f \in \mathrm{C}(X)$, the operator $T\pi_X(f\mathrm{I}_2)T^{-1}$ is central in this C*-algebra. It is therefore normal, and its spectrum as an operator is real, by similarity invariance of the spectrum. Hence it is selfadjoint. Taking adjoints gives

$$T^*T\pi_X(f\mathrm{I}_2) = \pi_X(f\mathrm{I}_2)T^*T.$$

Since continuous scalar functions separate points, T^*T is block diagonal in the representation $H_X = \bigoplus_x \mathbb{C}^2$. Write $T^*T = \bigoplus_x k_x$. By the polar decomposition, conjugation by $(T^*T)^{\frac{1}{2}}$ also makes $\pi_X(A)$ selfadjoint. At the coordinate x, its image is the C*-algebra $k_x^{\frac{1}{2}} A_x k_x^{-\frac{1}{2}}$. Thus Proposition 3.2 gives

$$\|2p_x - \mathrm{I}_2\| \leqslant \left\|k_x^{\frac{1}{2}}\right\| \left\|k_x^{-\frac{1}{2}}\right\| \leqslant \|T\| \left\|T^{-1}\right\|, \quad x \in F.$$

Taking the supremum proves $s_X(A) \geqslant K$. This completes the proof of Theorem 1.1. ∎

Remark 4.1 A separable Hilbert space suffices to implement the similarity. Indeed, let X_0 be a countable dense subset of X. Evaluation on X_0 is completely isometric on $\mathrm{C}(X, \mathrm{M}_2)$, and $\bigoplus_{x \in X_0} h(x)^{\frac{1}{2}}$ has cost at most K. To retain the exact lower bound as well, choose X_0 to contain a countable subset dense in F for its relative topology. The function $x \mapsto \|2p_x - \mathrm{I}_2\|$ is continuous on F, by (4.2). Its supremum on $X_0 \cap F$ is therefore K when $K > 1$. The proof of the lower bound then applies to this separable representation.

5 Locally trivial bundles of matrix algebras

Let $\mathscr{M} \to X$ be a locally trivial bundle of unital C*-algebras with fibre M_2 and $*$-automorphisms as transition maps. We write $\Gamma(X, \mathscr{M})$ for its algebra of continuous sections, with the supremum norm. Its identity section is denoted by 1. Trace and determinant are defined fibrewise

independently of the trivialization. This follows from their invariance under $*$-automorphisms of M_2.

Let A be a norm-closed algebra with

$$\mathrm{C}(X)1 \subset A \subset \Gamma(X, \mathscr{M}). \tag{5.1}$$

Write $A_x \subset \mathscr{M}_x$ for the evaluation algebra. When all A_x are semisimple, define $\mathrm{K}(A)$ by (1.2), interpreted in the fibres. Choose, for each x, a faithful irreducible representation of $\mathscr{M}_x$ on $\mathbb{C}^2$. Their direct sum defines a faithful representation π_X of $\Gamma(X, \mathscr{M})$. The value of $s_X(A)$ is independent of these choices, since two such choices are related by a block diagonal unitary.

Theorem 5.1 *Let X be a nonempty compact metrizable space and let A satisfy (5.1). Then A is amenable if and only if all the algebras A_x are semisimple and $\mathrm{K}(A) < \infty$. If these conditions hold, there is a bounded Borel section h of $\mathscr{M}$, continuous outside the scalar fibres, such that*

$$\det h(x) = 1, \quad \mathrm{K}(A)^{-1} 1_x \leqslant h(x) \leqslant \mathrm{K}(A) 1_x, \tag{5.2}$$

and $a \mapsto h^{\frac{1}{2}} a h^{-\frac{1}{2}}$ is a $\mathrm{C}(X)$-linear algebra isomorphism of A onto a nuclear C^*-subalgebra of $\Gamma(X, \mathscr{M})$. The conclusions (1.6), (1.7) and (1.8) remain valid.

Proof : Suppose first that all fibres are semisimple and put $K \stackrel{\mathrm{def}}{=} \mathrm{K}(A) < \infty$. The fibrewise calculations of Section 3 are invariant under $*$-automorphisms. They therefore define, on the relatively closed set F of two-dimensional fibres inside the nonscalar locus U, a continuous section $v = g - 1$ of the real vector bundle of traceless selfadjoint elements of $\mathscr{M}$. Its norm is at most $r = \frac{K^2-1}{K^2+1}$.

We describe the extension step. Cover U by trivializing open sets and choose a locally finite refinement (U_i) with a subordinate continuous partition of unity (φ_i). Such a refinement exists because U is metrizable. In a trivialization over U_i, the section v on the relatively closed set $F \cap U_i$ is a continuous map into the closed norm ball of radius r in the traceless selfadjoint matrices. Dugundji's theorem extends it to that ball on all of U_i; if $F \cap U_i$ is empty, take the zero section. Let v_i denote the resulting local section. The sum

$$\widetilde{v} = \sum_i \varphi_i v_i$$

is well defined and continuous on U, since the sum is locally finite and each term is extended by zero outside the support of φ_i. It agrees with v on F and has norm at most r. Trace zero and selfadjointness are preserved by the transition maps and by real convex combinations.

Define $g = 1 + \widetilde{v}$ on U, normalize $h = \frac{g}{\sqrt{\det g}}$ there, and set $h = 1$ on the scalar locus. The spectral calculation proving (1.4) is fibrewise, so (5.2) follows. The continuity argument at a scalar fibre is identical to (4.5) in any local trivialization. The bundle version of Lemma 2.3 then gives $h^{-1} a^* h \in A$ for any $a \in A$. It also proves that $h^{\frac{1}{2}} a h^{-\frac{1}{2}}$ is a continuous section.

The image is 2-subhomogeneous and nuclear. Indeed, Lemma 2.5 applies to the bundle: its proof uses only central scalar functions, cutoff functions and evaluation algebras. Transporting a norm-one approximate diagonal from this image gives $\mathrm{AM}(A) \leqslant K^2$. Conversely, if A is amenable, each fibre is amenable by evaluation, hence semisimple by Lemma 3.1. Proposition 3.2 then gives $\mathrm{K}(A) < \infty$ and the lower bound in (1.7). Finally, continuous scalar functions separate the points of X, so the proof that any positive spatial metric in the full evaluation representation is block diagonal is unchanged. The fibrewise optimal constants are given by Proposition 3.2, which proves $s_X(A) = \mathrm{K}(A)$. ∎

Remark 5.2 The bundle theorem does not assume that the fibres A_x form a locally trivial subbundle. Their dimensions may change, and their two primitive idempotents on F need not admit a global ordering. The symmetric formula $p^*p+(1-p)^*(1-p)$ is the reason that no such ordering is needed. The metrizable hypothesis is used for extension and partitions of unity on the open nonscalar locus; no extension theorem on an arbitrary nonnormal open subspace is being invoked.

6 Similarities without full scalar localization

The scalar inclusion in Theorems 1.1 and 5.1 is used to recover an algebra from its evaluation algebras. We give two auxiliary results which do not require that inclusion, followed by a sufficient hypothesis based on finite evaluation orbits. Throughout this section, a unital subalgebra has the same identity as its ambient algebra.

Lemma 6.1 *Let X be a nonempty compact Hausdorff space and let A be a unital norm-closed amenable subalgebra of $\mathrm{C}(X,\mathrm{M}_2)$. Put*

$$B \stackrel{\mathrm{def}}{=} \overline{\mathrm{span}\{fa : f \in \mathrm{C}(X),\ a \in A\}}.$$

Then B is a unital norm-closed subalgebra of $\mathrm{C}(X,\mathrm{M}_2)$ containing A and $\mathrm{C}(X)\mathrm{I}_2$. Moreover,

$$B_x = A_x \quad (x \in X), \qquad \mathrm{AM}(B) \leqslant \mathrm{AM}(A). \tag{6.1}$$

Proof : Scalar functions commute with matrix-valued functions, so products of finite sums in the definition of B are again such finite sums. Multiplication is continuous in the supremum norm. Hence B is an algebra. Taking $f = 1$ gives $A \subset B$, and taking $a = \mathrm{I}_2$ gives $\mathrm{C}(X)\mathrm{I}_2 \subset B$. For any $x \in X$, the evaluation algebra A_x is a finite-dimensional linear subspace of M_2 and is therefore closed. Every finite scalar sum has its value at x in A_x, and its norm limits retain this property. The inclusion $A \subset B$ proves the reverse inclusion between the evaluation algebras.

We construct an approximate diagonal without identifying any balanced tensor product. On elementary tensors, put

$$\Phi\big((f \otimes g) \otimes (a \otimes b)\big) = fa \otimes gb.$$

For finite sums $u = \sum_i f_i \otimes g_i$ and $m = \sum_j a_j \otimes b_j$, we have

$$\Big\| \sum_{i,j} f_i a_j \otimes g_i b_j \Big\|_\pi \leqslant \left(\sum_i \|f_i\|\,\|g_i\| \right) \left(\sum_j \|a_j\|\,\|b_j\| \right).$$

Taking the infimum over the two finite representations gives a contractive bilinear map on the algebraic tensor products with their projective norms. Completion in each variable and linearization give a contraction

$$\Phi\colon \big(\mathrm{C}(X)\widehat{\otimes}_\pi \mathrm{C}(X)\big)\widehat{\otimes}_\pi\big(A\widehat{\otimes}_\pi A\big) \to B\widehat{\otimes}_\pi B.$$

By Lemma 2.6, choose an approximate diagonal (d_α) for $\mathrm{C}(X)$ with $\|d_\alpha\|_\pi \leqslant 1$ and $\mu(d_\alpha) = 1$. Let (m_β) be an approximate diagonal for A with bound c. On the product directed set define

$$n_{\alpha,\beta} \stackrel{\mathrm{def}}{=} \Phi(d_\alpha \otimes m_\beta).$$

Then $\|n_{\alpha,\beta}\|_\pi \leqslant c$. The elementary tensor identity for multiplication extends by continuity and gives

$$\mu_B(n_{\alpha,\beta}) = \mu_{\mathrm{C}(X)}(d_\alpha)\mu_A(m_\beta) = \mu_A(m_\beta) \to \mathrm{I}_2.$$

For any $f \in \mathrm{C}(X)$ and $a \in A$, the module actions satisfy

$$f \cdot n_{\alpha,\beta} - n_{\alpha,\beta} \cdot f = \Phi\big((f \cdot d_\alpha - d_\alpha \cdot f) \otimes m_\beta\big),$$
$$a \cdot n_{\alpha,\beta} - n_{\alpha,\beta} \cdot a = \Phi\big(d_\alpha \otimes (a \cdot m_\beta - m_\beta \cdot a)\big).$$

The first defect tends to zero uniformly in β, with bound $c\,\|f \cdot d_\alpha - d_\alpha \cdot f\|_\pi$. The second tends to zero uniformly in α, with bound $\|a \cdot m_\beta - m_\beta \cdot a\|_\pi$. For their product, we have

$$(fa) \cdot n_{\alpha,\beta} - n_{\alpha,\beta} \cdot (fa) = f \cdot (a \cdot n_{\alpha,\beta} - n_{\alpha,\beta} \cdot a) + (f \cdot n_{\alpha,\beta} - n_{\alpha,\beta} \cdot f) \cdot a.$$

Consequently, centrality holds for every finite scalar sum. If $b \in B$ and b_0 is such a sum, the remaining defect has norm at most $2c\,\|b - b_0\|$. Given $\varepsilon > 0$, first choose b_0 to make this bound less than ε, and then choose the two indices to control the defect of b_0. This proves centrality for b. Thus $(n_{\alpha,\beta})$ is an approximate diagonal for B with bound c. Taking the infimum over bounds of approximate diagonals for A proves (6.1). ∎

Remark 6.2 If X is metrizable, Theorem 1.1 applies to B in Lemma 6.1. Its transported involution need not preserve A. Indeed, the graph algebra in Remark 7.5 has selfadjoint central saturation and full matrix evaluation algebras, while the original graph is not selfadjoint. Thus saturation provides an amenable over-algebra with the same evaluation algebras, but does not supply descent of its particular involution.

Recall that π_X denotes the direct sum of the evaluations on $H_X \overset{\text{def}}{=} \bigoplus_{x \in X} \mathbb{C}^2$. The next proposition concerns the form of an existing similarity.

Proposition 6.3 *Let X be a nonempty compact Hausdorff space and let A be a unital norm-closed subalgebra of $\mathrm{C}(X, \mathrm{M}_2)$. Suppose that S is an invertible operator on H_X and $S\pi_X(A)S^{-1}$ is selfadjoint. Then there is a positive invertible operator*

$$R = \bigoplus_{x \in X} R_x, \qquad R_x \in \mathrm{M}_2,$$

such that $R\pi_X(A)R^{-1}$ is selfadjoint and

$$\|R\| \leqslant \|S\|, \qquad \left\|R^{-1}\right\| \leqslant \left\|S^{-1}\right\|.$$

The involutions transported to A by S and R are equal. In particular, the infimum of the similarity costs is unchanged if the implementing operators are required to be positive and block diagonal, with value $+\infty$ if there is no implementing similarity.

Proof : We identify A with its faithful isometric image $\pi_X(A)$. Since the entire algebra SAS^{-1} is selfadjoint, the formula

$$J(a) \overset{\text{def}}{=} S^{-1}(SaS^{-1})^*S \quad (a \in A)$$

defines a bounded conjugate-linear antimultiplicative bijection from A onto A, with $J^2 = \mathrm{Id}_A$. Put $K \overset{\text{def}}{=} S^*S$. Multiplying $SJ(a)S^{-1} = S^{-*}a^*S^*$ on the left by S^* and on the right by S gives

$$KJ(a) = a^*K, \qquad \left\|S^{-1}\right\|^{-2} I_{H_X} \leqslant K \leqslant \|S\|^2 I_{H_X}. \tag{6.2}$$

For any $x \in X$, let $V_x \colon \mathbb{C}^2 \to H_X$ be the evaluation-summand inclusion and put $P_x \overset{\text{def}}{=} V_x^* K V_x$. Both a and $J(a)$ are block diagonal, since they belong to A. Hence

$$J(a)V_x = V_x J(a)(x), \qquad V_x^* a^* = a(x)^* V_x^*.$$

Compression of (6.2) therefore gives

$$P_x J(a)(x) = a(x)^* P_x, \qquad \left\|S^{-1}\right\|^{-2} \mathrm{I}_2 \leqslant P_x \leqslant \|S\|^2 \mathrm{I}_2. \tag{6.3}$$

This argument does not require K to commute with the summand projections.

For completeness, the positive square root of a positive invertible matrix $P \in \mathrm{M}_2$ is given by

$$P^{\frac{1}{2}} = \frac{P + \sqrt{\det P}\mathrm{I}_2}{\sqrt{\mathrm{Tr}(P) + 2\sqrt{\det P}}}. \tag{6.4}$$

Indeed, Cayley–Hamilton gives $P^2 - \mathrm{Tr}(P)P + \det(P)\mathrm{I}_2 = 0$, so the square of the numerator is $(\mathrm{Tr}(P) + 2\sqrt{\det P})P$. The numerator is positive invertible and the denominator is positive. The formula therefore defines a positive matrix whose square is P. Diagonalizing the positive matrix P shows that its two eigenvalues are replaced by their positive square roots.

Set $R_x \overset{\text{def}}{=} P_x^{\frac{1}{2}}$ using (6.4). The spectral bounds in (6.3) give

$$\|R_x\| \leqslant \|S\|, \qquad \left\|R_x^{-1}\right\| \leqslant \left\|S^{-1}\right\| \quad (x \in X).$$

Thus $R = \bigoplus_x R_x$ and $\bigoplus_x R_x^{-1}$ define bounded mutually inverse positive operators on H_X. These definitions use the usual Hilbert direct sum, and require no measurability of the fields. Put $L \overset{\text{def}}{=} R^2 = \bigoplus_x P_x$. The relations in (6.3) imply $LJ(a) = a^* L$, first on finite-support vectors and then on H_X by continuity. Consequently,

$$RJ(a)R^{-1} = R^{-1}a^*R = (RaR^{-1})^*.$$

Since $J(A) = A$, this proves adjoint closure of RAR^{-1} and equality of the transported involutions. The image is norm closed because conjugation by R is a bounded invertible map. It is therefore a C*-algebra. The two norm bounds prove the cost estimate.

Every positive block diagonal implementing similarity is an implementing similarity without restrictions. Conversely, the construction produces one with no greater cost from any implementing S. These two observations prove equality of the infima. They also show that the two classes are simultaneously empty or nonempty, which gives the stated convention at $+\infty$. ∎

Remark 6.4 Proposition 6.3 starts with a global similarity, which supplies the common involution J on A. Independent choices of metrics on the evaluation algebras do not supply this map.

We now give a sufficient condition in which identifications between evaluations are prescribed by a finite-group cocycle.

Theorem 6.5 (Finite orbit descent) *Let X be a nonempty compact metrizable space and let a finite group G act freely and continuously on X. Suppose that continuous maps $T_g \colon X \to \mathrm{GL}_2(\mathbb{C})$ satisfy*

$$T_e(x) = \mathrm{I}_2, \qquad T_{gh}(x) = T_g(hx)T_h(x), \quad g, h \in G,\ x \in X. \tag{6.5}$$

Let A be a unital norm-closed amenable subalgebra of $\mathrm{C}(X, \mathrm{M}_2)$ such that $\mathrm{C}(X)^G \mathrm{I}_2 \subset A$ and

$$a(gx) = T_g(x) a(x) T_g(x)^{-1} \quad (a \in A,\ g \in G,\ x \in X). \tag{6.6}$$

Then $\pi_X(A)$ is similar to a nuclear C^*-algebra.

Proof : We first replace the cocycle by a unitary cocycle. Define

$$h_0(x) \stackrel{\text{def}}{=} \frac{1}{|G|} \sum_{k \in G} T_k(x)^* T_k(x), \qquad r(x) \stackrel{\text{def}}{=} h_0(x)^{\frac{1}{2}}.$$

The function h_0 is continuous and satisfies $h_0(x) \geqslant \frac{1}{|G|} \mathrm{I}_2$ because $T_e = \mathrm{I}_2$. Compactness gives a uniform upper bound. Formula (6.4) therefore shows that r and r^{-1} are continuous and uniformly bounded. For any $g \in G$, reindexing the finite sum by $k \mapsto kg$ gives

$$\begin{aligned} h_0(gx) &\stackrel{(6.5)}{=} \frac{1}{|G|} \sum_{k \in G} T_g(x)^{-*} T_{kg}(x)^* T_{kg}(x) T_g(x)^{-1} \\ &= T_g(x)^{-*} h_0(x) T_g(x)^{-1}. \end{aligned}$$

Put

$$U_g(x) \stackrel{\text{def}}{=} r(gx) T_g(x) r(x)^{-1}.$$

Then

$$U_g(x)^* U_g(x) = r(x)^{-1} T_g(x)^* h_0(gx) T_g(x) r(x)^{-1} = \mathrm{I}_2.$$

Hence $U_g(x)$ is unitary. Multiplication and (6.5) also give

$$U_g(hx) U_h(x) = r(ghx) T_g(hx) T_h(x) r(x)^{-1} = U_{gh}(x).$$

The map $\theta(a) \stackrel{\text{def}}{=} rar^{-1}$ is a bounded algebra isomorphism on $\mathrm{C}(X, \mathrm{M}_2)$ with bounded inverse. It fixes scalar functions, and (6.6) gives

$$\theta(a)(gx) = U_g(x) \theta(a)(x) U_g(x)^*.$$

Thus $A' \stackrel{\text{def}}{=} \theta(A)$ is a unital norm-closed subalgebra of the algebra of continuous U-equivariant fields. It is amenable. Indeed, if (m_β) is a bounded approximate diagonal for A, then $(\theta \otimes \theta) m_\beta$ is uniformly bounded by $\|\theta\|^2 \sup_\beta \|m_\beta\|_\pi$. For $a' = \theta(a)$ its centrality defect is the image of $a \cdot m_\beta - m_\beta \cdot a$, and its product is $\theta(\mu(m_\beta)) \to \mathrm{I}_2$. This verifies both approximate-diagonal conditions for A'.

Let Y be the orbit space and let $q \colon X \to Y$ be the quotient map. We record the topology needed to apply Theorem 5.1. Choose a compatible metric d on X and put

$$d_0(x, z) \stackrel{\text{def}}{=} \max_{g \in G} d(gx, gz), \qquad \delta(q(x), q(z)) \stackrel{\text{def}}{=} \min_{g \in G} d_0(x, gz).$$

The metric d_0 is compatible with the topology of X and is G-invariant. Reindexing the finite minimum shows that δ is independent of representatives and symmetric. Distinct finite orbits have positive distance. If g, h attain the minima for $(q(x), q(z))$ and $(q(z), q(w))$, then

$$\delta(q(x), q(w)) \leqslant d_0(x, ghw) \leqslant d_0(x, gz) + d_0(gz, ghw) = \delta(q(x), q(z)) + \delta(q(z), q(w)).$$

This proves the triangle inequality. The inverse image of a δ-ball is a finite union of d_0-balls. Hence the map from X to the metric space (Y, δ) is continuous. Its quotient topology agrees

with this metric topology, since the induced continuous bijection from the compact quotient onto the Hausdorff metric space is a homeomorphism. In particular, Y is nonempty compact metrizable.

Freeness makes the points gx, for $g \in G$, pairwise distinct. Choose pairwise disjoint open neighborhoods O_g of these points and put $V \stackrel{\text{def}}{=} \bigcap_{g\in G} g^{-1}O_g$. Then $x \in V$ and the open sets gV are pairwise disjoint. The map q is open because $q^{-1}(q(W)) = \bigcup_g gW$ for any open $W \subset X$. Thus $q|_V$ is a homeomorphism onto $q(V)$, and $q^{-1}(q(V))$ is the disjoint union of the sheets gV. This constructs a finite covering.

The formula

$$g \cdot (x, b) \stackrel{\text{def}}{=} (gx, U_g(x)bU_g(x)^*)$$

defines a continuous action on $X \times \mathrm{M}_2$, by the cocycle identity for U. Let $\mathscr{E}$ be its orbit space, with projection $[x, b] \mapsto q(x)$ onto Y. Above $q(V)$, choose the unique representative in the sheet V. The map

$$(y, b) \mapsto [(q|_V)^{-1}(y), b]$$

is a product chart for $\mathscr{E}$. Its inverse is continuous: on the sheet gV, it is obtained by the continuous conjugation $b \mapsto U_g(v)^*bU_g(v)$ with $v \in V$. This formula is constant on the equivalence classes and hence descends through the quotient map. On chart overlaps, the group element relating two local lifts is locally constant, because the covering sheets are disjoint and open. The transition in the matrix coordinate is conjugation by the corresponding continuous unitary U_g. Thus $\mathscr{E}$ is a locally trivial bundle of unital M_2 C^*-algebras with $*$-automorphism transition maps.

For any $x \in X$, the map $j_x \colon \mathrm{M}_2 \to \mathscr{E}_{q(x)}$ defined by $j_x(b) = [x, b]$ is an isometric unital $*$-isomorphism. Injectivity follows from freeness, and every class over $q(x)$ has a representative based at x. A continuous U-equivariant field b defines a section

$$\Psi(b)(q(x)) \stackrel{\text{def}}{=} j_x(b(x)).$$

Covariance makes this independent of the representative x. In the product charts just constructed, it is the restriction of b to the chosen local sheet, so it is continuous. Conversely, a continuous section s defines the field $b(x) = j_x^{-1}(s(q(x)))$. The same charts prove continuity of this field, and the definition of the equivalence relation gives its U-covariance. Products, adjoints and identities are preserved, and the supremum norms agree since q is onto and the transition maps are isometric. Consequently, Ψ is an isometric unital $*$-isomorphism from the continuous equivariant field algebra onto $\Gamma(Y, \mathscr{E})$.

Put $B \stackrel{\text{def}}{=} \Psi(A')$. It is norm closed, unital and amenable. The last assertion follows by applying $\Psi \otimes \Psi$ to the approximate diagonal for A' and using the same product and module identities as for θ. For any $f \in \mathrm{C}(Y)$, the scalar function $f \circ q$ is continuous and G-invariant. Its scalar field belongs to A by hypothesis and is fixed by θ. Its image under Ψ is $f1_{\mathscr{E}}$. Hence

$$\mathrm{C}(Y)1_{\mathscr{E}} \subset B \subset \Gamma(Y, \mathscr{E}).$$

All the hypotheses of Theorem 5.1 now hold for this descended algebra.

By that theorem, there are $K \geqslant 1$ and a bounded positive Borel section $h_{\mathscr{E}}$ with

$$K^{-1}1_{\mathscr{E}} \leqslant h_{\mathscr{E}} \leqslant K1_{\mathscr{E}},$$

such that

$$D \stackrel{\text{def}}{=} h_{\mathscr{E}}^{\frac{1}{2}} B h_{\mathscr{E}}^{-\frac{1}{2}}$$

is a nuclear C*-subalgebra of $\Gamma(Y, \mathscr{E})$. Define pointwise

$$\widetilde{h}(x) \stackrel{\text{def}}{=} j_x^{-1}(h_{\mathscr{E}}(q(x))), \qquad S_x \stackrel{\text{def}}{=} \widetilde{h}(x)^{\frac{1}{2}} r(x).$$

The same upper and lower bounds hold for $\widetilde{h}(x)$. Consequently,

$$\sup_x \|S_x\| \leqslant \sqrt{K}\, \|r\|_\infty\,, \qquad \sup_x \left\|S_x^{-1}\right\| \leqslant \sqrt{K}\, \left\|r^{-1}\right\|_\infty\,.$$

Thus $S = \bigoplus_x S_x$ is a bounded invertible operator on the original evaluation space H_X. For any $a \in A$, the field corresponding to the section $h_{\mathscr{E}}^{\frac{1}{2}} \Psi(\theta(a)) h_{\mathscr{E}}^{-\frac{1}{2}}$ has value

$$\widetilde{h}(x)^{\frac{1}{2}} r(x) a(x) r(x)^{-1} \widetilde{h}(x)^{-\frac{1}{2}} = S_x a(x) S_x^{-1}$$

at x. Therefore

$$S \pi_X(A) S^{-1} = \pi_X(\Psi^{-1}(D)).$$

The right-hand side is $*$-isomorphic to D, since both Ψ^{-1} and the full evaluation representation are faithful isometric $*$-homomorphisms. It is a nuclear C*-algebra. This proves the assertion. ∎

Remark 6.6 If G is nontrivial, freeness supplies two distinct points in every orbit. A continuous scalar function separating two such points is not G-invariant and cannot satisfy (6.6). Hence the hypotheses of Theorem 6.5 permit algebras which do not contain $\mathrm{C}(X)\mathrm{I}_2$. They form a sufficient class specified by extra orbit data. No action or cocycle is reconstructed here for an arbitrary amenable subalgebra.

Freeness is used to construct a locally trivial quotient bundle. For a nonfree action the corresponding fibre may have a nontrivial stabilizer, and that construction does not apply with fibre M_2 unchanged. The proof also makes no continuity assertion for $\widetilde{h}$ at scalar fibres. Continuity of the conjugated fields follows from Theorem 5.1 and the section correspondence.

7 Examples, optimality and continuity of the metric

Example 7.1 For $t \geqslant 0$, let $p_t \stackrel{\text{def}}{=} \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$ and $A_t \stackrel{\text{def}}{=} \mathbb{C}p_t + \mathbb{C}(\mathrm{I}_2 - p_t)$. Take X to be a one-point space. Proposition 3.2 gives

$$s_X(A_t) = \sqrt{1+t^2} + t, \quad \mathrm{AM}(A_t) = \frac{1 + (\sqrt{1+t^2}+t)^2}{2}.$$

Hence equality holds in (1.8). Conversely, any $K \geqslant 1$ is obtained by taking $t = \frac{1}{2}(K - K^{-1})$.

Example 7.2 There need not be a continuous matrix-valued similarity, even for a noncommutative algebra covered by Theorem 1.1. Let $X = \{0\} \cup \{\frac{1}{n} : n \geqslant 1\}$ and fix $t > 0$. Put $B_0 \stackrel{\text{def}}{=} \mathbb{C}p_0 + \mathbb{C}(\mathrm{I}_2 - p_0)$ and $B_t \stackrel{\text{def}}{=} \mathbb{C}p_t + \mathbb{C}(\mathrm{I}_2 - p_t)$, with p_t as in Example 7.1. Define

$$B_n = \begin{cases} \mathrm{M}_2, & n \equiv 0 \pmod 3, \\ B_0, & n \equiv 1 \pmod 3, \\ B_t, & n \equiv 2 \pmod 3, \end{cases}$$

and let A consist of the continuous sections a with $a(0) \in \mathbb{C}\mathrm{I}_2$ and $a(\frac{1}{n}) \in B_n$. This is a norm-closed algebra containing $\mathrm{C}(X)\mathrm{I}_2$. Its evaluation algebras are the prescribed ones: a section supported at an isolated point realizes any element of B_n. The algebra is noncommutative because some fibres are M_2, and Theorem 1.1 shows that it is amenable with $s_X(A) = \sqrt{1+t^2}+t$.

Suppose that an invertible $s \in \mathrm{C}(X, \mathrm{M}_2)$ satisfies $sAs^{-1} = (sAs^{-1})^\dagger$. Put $k(x) \stackrel{\text{def}}{=} s(x)^* s(x)$. Evaluation shows that $s(\frac{1}{n})B_n s(\frac{1}{n})^{-1}$ is selfadjoint. On the two-dimensional fibres, this implies

$$k(\tfrac{1}{n})p_0 = p_0^* k(\tfrac{1}{n}) \quad (n \equiv 1 \pmod 3), \qquad k(\tfrac{1}{n})p_t = p_t^* k(\tfrac{1}{n}) \quad (n \equiv 2 \pmod 3).$$

Continuity at 0 forces both relations for $k(0)$. The first makes $k(0)$ diagonal, say $k(0) = \mathrm{diag}(a, b)$ with $a, b > 0$. The second then implies $at = 0$, which is impossible. Thus no such continuous s exists.

Proposition 7.3 *Under the hypotheses of Theorem 1.1 or Theorem 5.1, put $K \stackrel{\text{def}}{=} \mathrm{K}(A)$. The metric h can be approximated by continuous positive invertible sections h_n such that*

$$K^{-1}1 \leqslant h_n \leqslant K1, \tag{7.1}$$

$\pi_X(h_n) \to \bigoplus_x h(x)$ strongly, and, for any $a \in A$,

$$h_n^{-1} a^* h_n \to J(a) \quad \text{in the norm of the ambient section algebra.} \tag{7.2}$$

The individual sections on the left of (7.2) are not asserted to belong to A.

Proof : If the scalar locus Z is empty, take $h_n = h$. Otherwise, fix a compatible metric on X and choose numbers $\delta_n \downarrow 0$ and functions $\chi_n \in \mathrm{C}(X, [0,1])$ which vanish when $\mathrm{dist}(x, Z) \leqslant \frac{1}{2}\delta_n$ and equal 1 when $\mathrm{dist}(x, Z) \geqslant \delta_n$. Put $h_n \stackrel{\text{def}}{=} 1 + \chi_n(h-1)$. The section h_n equals 1 near Z and is continuous elsewhere, so it is continuous on X. The bounds (7.1) follow by convexity of the order interval.

For $a \in A$, let $a^\circ \stackrel{\text{def}}{=} a - \frac{1}{2}\,\mathrm{Tr}(a)1$. This is a continuous section which vanishes on Z. Outside the δ_n-neighbourhood of Z, the two metrics coincide. On that neighbourhood, scalar terms are unchanged by either conjugation, and both conjugations have norm at most K^2. Consequently,

$$\left\| h_n^{-1} a^* h_n - J(a) \right\| \leqslant 2K^2 \sup_{\mathrm{dist}(x,Z) < \delta_n} \|a^\circ(x)\| \to 0.$$

The last limit follows from continuity and compactness: a sequence of points approaching Z has cluster points in Z, where a° vanishes. Finally, $h_n(x) \to h(x)$ at every point and the sections are uniformly bounded. This proves strong convergence in the direct sum representation, first on vectors with finite support and then on all vectors by density. ∎

Example 7.4 Amenable fibres alone do not imply $\mathrm{K}(A) < \infty$. On the same compact sequence space, let $t_n \to \infty$, and prescribe the fibre $\mathbb{C}p_{t_n} + \mathbb{C}(\mathrm{I}_2 - p_{t_n})$ at $\frac{1}{n}$ and the scalar fibre at 0. The resulting closed $\mathrm{C}(X)$-subalgebra has semisimple fibres everywhere, but $\mathrm{K}(A) = \infty$. It is not amenable, since the contractive evaluations and (3.2) would otherwise give a uniform bound on $\frac{1}{2}(1 + \|2p_{t_n} - \mathrm{I}_2\|^2)$. Thus the boundedness condition in Theorem 1.1 is necessary independently of semisimplicity.

Remark 7.5 The central module assumption cannot be omitted from the localization argument. Let $T \stackrel{\text{def}}{=} \mathrm{diag}(2, 1)$ and

$$A_T \stackrel{\text{def}}{=} \{(a, TaT^{-1}) : a \in \mathrm{M}_2\} \subset \mathrm{M}_2 \oplus \mathrm{M}_2.$$

Both evaluation algebras are M_2, but A_T is not selfadjoint: it contains $(e_{12}, 2e_{12})$ and does not contain $(e_{21}, 2e_{21})$. It is nevertheless similar to a C*-algebra, by the block diagonal similarity with blocks I_2 and T^{-1}. Thus selfadjoint evaluation algebras do not by themselves imply self-adjointness of the section algebra. The proof of Lemma 2.3 specifically requires multiplication by scalar partitions of unity.

Remark 7.6 The two-by-two assumption is used in the extension argument, not only in the computation of a constant. Above the nonscalar locus, the only proper semisimple fibres are the two-dimensional commutative ones, and the remaining fibres are full matrix algebras, on which every positive metric is admissible. For M_d with $d \geqslant 3$, several proper noncommutative fibre types can occur. Extending a metric through those fibres does not automatically preserve their adjoint structure. The present argument does not establish the corresponding general statement in higher matrix size.

8 Regularity at scalar collapse

The continuity of the transported sections does not require continuity of the metric at scalar fibres. We first give quantitative versions of this cancellation when every fibre is scalar or two-dimensional. The estimates concern the canonical metric on the two-dimensional fibres. They do not impose a modulus of continuity on an arbitrary extension over a full matrix stratum.

Let $V = \{v \in \mathrm{M}_2 : \mathrm{Tr}(v) = 0\}$, with the matrix operator norm, and put

$$d(v) = \frac{\mathrm{Tr}(v^2)}{2}, \qquad \rho(v) = \sqrt{|d(v)|}, \qquad \mathcal{C}_K = \{v \in V : \|v\| \leqslant K\rho(v)\} \quad (K \geqslant 1).$$

Cayley–Hamilton gives $v^2 = d(v)\mathrm{I}_2$ and $\det v = -d(v)$. The only point of $\mathcal{C}_K$ at which d vanishes is zero. For an invertible matrix v, write $Q_v = v(v^*v)^{-\frac{1}{2}}$. Define the maps on $\mathcal{C}_K$ by

$$\mathcal{F}(0) = \mathcal{T}(0) = 0, \qquad \mathcal{F}(v) = \frac{\overline{d(v)}}{|d(v)|}v, \qquad \mathcal{T}(v) = \rho(v)Q_v \quad (v \neq 0). \tag{8.1}$$

Lemma 8.1 *For all $v, w \in \mathcal{C}_K$,*

$$\|\mathcal{F}(v) - \mathcal{F}(w)\| \leqslant (1 + 2K^2)\,\|v - w\|\,, \qquad \|\mathcal{T}(v) - \mathcal{T}(w)\| \leqslant 2K\,\|v - w\|\,. \tag{8.2}$$

If u is a traceless involution and $p = \frac{\mathrm{I}_2+u}{2}$, the canonical determinant-normalized metric of (3.3) is $h = |u|$. With this metric,

$$h^{-1}(\varepsilon u)^* h = \mathcal{F}(\varepsilon u), \qquad h^{\frac{1}{2}}(\varepsilon u)h^{-\frac{1}{2}} = \mathcal{T}(\varepsilon u) \quad (\varepsilon \in \mathbb{C}). \tag{8.3}$$

The two formulas are independent of the sign choice of u.

Proof : The estimate $|\,\mathrm{Tr}\,B| \leqslant 2\,\|B\|$ and cyclicity of trace give

$$|d(v) - d(w)| = \left|\frac{\mathrm{Tr}((v+w)(v-w))}{2}\right| \leqslant (\|v\| + \|w\|)\,\|v - w\|\,. \tag{8.4}$$

For nonzero complex numbers z, t, writing $r = |z|$ and $s = |t|$,

$$|z - t|^2 = (r - s)^2 + rs\left|\frac{z}{r} - \frac{t}{s}\right|^2.$$

Since the last absolute value is at most two, this identity implies

$$\left|\frac{z}{|z|}-\frac{t}{|t|}\right| \leqslant \frac{2|z-t|}{|z|+|t|}.$$

Indeed, after squaring and using the identity, the required inequality reduces to

$$(r-s)^2\left|\frac{z}{r}-\frac{t}{s}\right|^2 \leqslant 4(r-s)^2.$$

For nonzero v, w put $a = \|v\|$ and $b = \|w\|$. Split the difference of the two phase multiples so that the phase difference is multiplied by the smaller matrix norm. Then

$$\|\mathcal{F}(v)-\mathcal{F}(w)\| \leqslant \left(1+\frac{2\min(a,b)(a+b)}{|d(v)|+|d(w)|}\right)\|v-w\|.$$

The denominator is at least $\frac{a^2+b^2}{K^2}$, and $\min(a,b)(a+b) \leqslant a^2+b^2$. This proves the first estimate. If one argument is zero, $\|\mathcal{F}(v)\| = \|v\|$ proves it.

We record the polar estimate needed for the second assertion. Let $A = UP$ and $B = VQ$ be invertible matrices with U, V unitary and $P, Q > 0$. Set $E = A - B$ and $W = U^*V$. Direct multiplication gives

$$P(\mathrm{I}-W)+(\mathrm{I}-W)Q = U^*E - E^*V.$$

For $P, Q > 0$, the solution of $PX + XQ = C$ is

$$X = \int_0^\infty e^{-tP}Ce^{-tQ}\,dt.$$

Differentiate the integrand and integrate; the boundary at infinity vanishes exponentially and its value at zero is C. Uniqueness follows by diagonalizing P and Q separately: each coefficient of a homogeneous solution is multiplied by a positive number $p_i + q_j$. The same diagonalizations show that $\left\|e^{-tP}\right\| = e^{-t\lambda_{\min}(P)}$. Consequently

$$\|U-V\| \leqslant \frac{2\|A-B\|}{\sigma_{\min}(A)+\sigma_{\min}(B)}. \tag{8.5}$$

For $v \in \mathcal{C}_K \setminus \{0\}$ the two singular values have product $|\det v| = \rho(v)^2$. Hence

$$\sigma_{\min}(v) = \frac{\rho(v)^2}{\|v\|} \geqslant \frac{\rho(v)}{K}.$$

Equation (8.4) also gives

$$|\rho(v)-\rho(w)| \leqslant \frac{|d(v)-d(w)|}{\rho(v)+\rho(w)} \leqslant K\|v-w\|.$$

Splitting the difference of the radius times polar factor, with the smaller radius multiplying the polar difference, now gives

$$\|\mathcal{T}(v)-\mathcal{T}(w)\| \leqslant \left(K+\frac{2K\min(\rho(v),\rho(w))}{\rho(v)+\rho(w)}\right)\|v-w\| \leqslant 2K\|v-w\|.$$

At zero, $\|\mathcal{T}(v)\| = \rho(v) \leqslant \|v\|$, since the product of the singular values is at most the square of the larger.

For later smoothness statements, all finite positive matrix square roots can be written explicitly as

$$P^{\frac{1}{2}} = \frac{P + \sqrt{\det P}\,\mathrm{I}_2}{\sqrt{\operatorname{Tr} P + 2\sqrt{\det P}}} \qquad (P > 0,\ P \in \mathrm{M}_2). \tag{8.6}$$

Cayley–Hamilton verifies that its square is P; the numerator and denominator show positivity. Unitary diagonalization gives uniqueness of the positive square root. This formula and inversion show that $v \mapsto Q_v$ is real smooth on the invertible matrices.

Finally, with $q = \mathrm{I}_2 - p$ and $k = p^*p + q^*q$, we have $k = \frac{\mathrm{I}_2 + u^*u}{2}$ and $\det(u^*u) = 1$. Cayley–Hamilton gives

$$(\mathrm{I}_2 + u^*u)^2 = (\operatorname{Tr}(u^*u) + 2)u^*u, \qquad h = \frac{\mathrm{I}_2 + u^*u}{\sqrt{\operatorname{Tr}(u^*u) + 2}} = |u|.$$

Here determinant normalization of k, or of its trace normalization, gives the same h. The identity $(\mathrm{I}_2 + u^*u)u = u^*(\mathrm{I}_2 + u^*u)$ gives $hu = u^*h$. Put $Q = uh^{-1}$. This is unitary, being the polar factor of u, and $hu = u^*h$ implies $Q^* = Q$. From $u = Qh$ and $u^2 = \mathrm{I}_2$ we get $QhQ = h^{-1}$, and uniqueness of the positive square root gives $Qh^{\frac{1}{2}}Q = h^{-\frac{1}{2}}$. Thus $h^{\frac{1}{2}}uh^{-\frac{1}{2}} = Q$. For $v = \varepsilon u \neq 0$ we have $d(v) = \varepsilon^2$, $\rho(v) = |\varepsilon|$ and $Q_v = \frac{\varepsilon}{|\varepsilon|}Q$. These identities give (8.3); the zero case is immediate. The maps are nonlinear on the whole cone but restrict to an antilinear and a linear map, respectively, on each fixed line $\mathbb{C}u$, as required by the transported adjoint and similarity. ∎

For a function on a metric space, write $[f]_\alpha = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)^\alpha}$, with the supremum of an empty set assigned the value zero.

Theorem 8.2 *Let X be a compact metric space and let $A \subset \mathrm{C}(X, \mathrm{M}_2)$ be a norm-closed unital algebra containing $\mathrm{C}(X)\mathrm{I}_2$. Suppose that every fibre is scalar or two-dimensional semisimple and that its primitive involutions have norm at most K, where $K \geqslant 1$. Use the canonical metric h of (3.3) on two-dimensional fibres and put $h = \mathrm{I}_2$ on scalar fibres. Define $J(a) = h^{-1}a^*h$ and $\Theta(a) = h^{\frac{1}{2}}ah^{-\frac{1}{2}}$. For $a \in A$, put $\lambda_a = \frac{\operatorname{Tr}(a)}{2}$ and $v_a = a - \lambda_a\mathrm{I}_2$. Then*

$$J(a) = \overline{\lambda_a}\mathrm{I}_2 + \mathcal{F}(v_a), \qquad \Theta(a) = \lambda_a\mathrm{I}_2 + \mathcal{T}(v_a). \tag{8.7}$$

For every $0 < \alpha \leqslant 1$, an α-Hölder section satisfies

$$[J(a)]_\alpha \leqslant [\lambda_a]_\alpha + (1 + 2K^2)[v_a]_\alpha \leqslant (3 + 4K^2)[a]_\alpha, \tag{8.8}$$

$$[\Theta(a)]_\alpha \leqslant [\lambda_a]_\alpha + 2K[v_a]_\alpha \leqslant (1 + 4K)[a]_\alpha. \tag{8.9}$$

Moreover $J(a) \in A$. The field $\Theta(a)$ belongs to the selfadjoint image algebra $\Theta(A)$; it need not belong to A and need not itself be selfadjoint. The constants are uniform as the eigenvalue gap vanishes; their numerical optimality is not asserted.

Proof : On a two-dimensional fibre $v_a(x) = \varepsilon_x u_x$, and on a scalar fibre it is zero. Thus $v_a(x) \in \mathcal{C}_K$ everywhere. Lemma 8.1 gives (8.7). Apply (8.2) at each pair of points. The trace estimate gives $[\lambda_a]_\alpha \leqslant [a]_\alpha$ and $[v_a]_\alpha \leqslant 2[a]_\alpha$, proving the displayed bounds, including the Lipschitz endpoint. The same formulas show continuity for every continuous section, whether or not h has a limit at a scalar fibre.

For membership, $J(a)(x) \in A_x$ by the fibre formula. Since $J(a)$ is continuous, Lemma 2.3 puts it in A. This application can also be verified with explicit metric weights: given a finite

cover $U_1, \dots, U_N$ on which chosen sections give the required local errors, put $t_i(x) = \operatorname{dist}(x, X \setminus U_i)$, taking $t_i = 1$ if $U_i = X$, and set $\varphi_i = \frac{t_i}{\sum_j t_j}$. The denominator is positive, the weights sum to one, and $\varphi_i(x) > 0$ implies $x \in U_i$. The weighted sum of the chosen sections approximates $J(a)$ uniformly. Closedness gives the same conclusion. Applying Θ to $J(a)$ gives $\Theta(a)^*$, so the image algebra is selfadjoint. No full matrix fibre is present: on such a fibre a traceless section can be a nonzero nilpotent and fail the cone condition. ■

Proposition 8.3 *Let v be differentiable on a neighbourhood of 0 in $\mathbb{R}^n$, with values in $\mathcal{C}_K$, and suppose that $v(0) = 0$. Put $L = Dv(0)$ and let G denote either $\mathcal{F}$ or $\mathcal{T}$. Then $G \circ v$ is differentiable at zero if and only if $x \mapsto G(Lx)$ is real linear. In this case its derivative is exactly this real-linear map. In particular, $L = 0$ implies a zero derivative for $G \circ v$ at zero.*

Proof : The cone is closed and is stable under real scalar multiplication. Consequently,

$$Lx = \lim_{t \to 0+} \frac{v(tx)}{t} \in \mathcal{C}_K.$$

Both maps G are homogeneous for real scalars, including negative scalars, and are Lipschitz on the cone. If $G \circ v$ has derivative B, then

$$Bx = \lim_{t \to 0+} \frac{G(v(tx))}{t} = G(Lx),$$

so $G \circ L$ is real linear. Conversely, if $G(Lx) = Bx$ for a real-linear map B, then

$$\|G(v(x)) - Bx\| \leqslant \operatorname{Lip}(G|_{\mathcal{C}_K}) \, \|v(x) - Lx\| = o(\|x\|),$$

which proves differentiability. This is a criterion at one point; it does not assert continuity of the derivative on a neighbourhood. In manifold charts it is invariant under coordinate changes by the ordinary chain rule. ■

Proposition 8.4 *In the scalar/two-dimensional setting and with the canonical metric of Theorem 8.2, let U be a coordinate neighbourhood. Suppose that $m \geqslant 1$, that λ_a is C^m, and that the traceless part factors as*

$$v_a = gw \quad \text{on } U,$$

where g is a complex-valued C^m function, w is a traceless M_2-valued C^m function and $d(w)$ never vanishes on U. Then $J(a)$ and $\Theta(a)$ are C^m on U, including at the zeros of g. On a relatively compact subneighbourhood on which $|d(w)|$ has a positive lower bound, their derivative bounds are controlled by that lower bound and the C^m norms of g, w, λ_a, without a negative power of $|g|$.

Proof : In this proof we extend the formulas defining $\mathcal{F}$ and $\mathcal{T}$ to the open set $\{w \in V : d(w) \neq 0\}$, without requiring the cone bound on w. For every complex scalar t the explicit formulas give $\mathcal{F}(tw) = \overline{t}\mathcal{F}(w)$ and $\mathcal{T}(tw) = t\mathcal{T}(w)$, including $t = 0$. Both maps are real smooth on the open set $d \neq 0$: for $\mathcal{F}$ this follows from its scalar formula, and for $\mathcal{T}$ from (8.6), inversion and the positive scalar functions $|d|$ and $\sqrt{|d|}$. Therefore

$$J(a) = \overline{\lambda_a}\mathrm{I}_2 + \overline{g}\,\mathcal{F}(w), \qquad \Theta(a) = \lambda_a \mathrm{I}_2 + g\,\mathcal{T}(w)$$

are C^m. On the indicated subneighbourhood, differentiating these formulas proves the asserted bounds. No division by g occurs. This is a sufficient factorization hypothesis, not a necessary factorization theorem for arbitrary sections. ■

Proposition 8.5 *Let (X,d) be a compact metric space, $0<\alpha\leqslant 1$, and let $a=\lambda \mathrm{I}_2+b$, with λ and b individually α-Hölder. Let $s\colon X\to \mathrm{GL}_2(\mathbb{C})$ be a field, not assumed continuous, with $\|s(x)\|\leqslant M$ and $\left\|s(x)^{-1}\right\|\leqslant N$. Define*

$$\mathcal{W}_\alpha(s,b)=\sup_{x\neq y}\frac{\min(\|b(x)\|,\|b(y)\|)\,\|s(x)-s(y)\|}{d(x,y)^\alpha}.$$

If this quantity is finite, then $\Phi_s(a)=sas^{-1}$ is α-Hölder and

$$[\Phi_s(a)]_\alpha\leqslant [\lambda]_\alpha+MN[b]_\alpha+N(1+MN)\mathcal{W}_\alpha(s,b). \tag{8.10}$$

If h is positive with $K^{-1}\mathrm{I}_2\leqslant h\leqslant K\mathrm{I}_2$, then $J_h(a)=h^{-1}a^*h$ instead satisfies

$$[J_h(a)]_\alpha\leqslant [\lambda]_\alpha+K^2[b]_\alpha+K(1+K^2)\mathcal{W}_\alpha(h,b). \tag{8.11}$$

For Θ one may apply (8.10) with $s=h^{\frac{1}{2}}$, provided its stated weighted oscillation is finite. These are regularity statements for fields; preservation of an algebra is a separate algebraic condition.

Proof : Put $\delta_s=s(x)-s(y)$. The inverse identity is

$$s(x)^{-1}-s(y)^{-1}=-s(x)^{-1}\delta_s s(y)^{-1}.$$

In the difference of the transforms of b, first replace $b(x)$ by $b(y)$ in the x frame, at cost at most $MN\,\|b(x)-b(y)\|$. The remaining difference is

$$\delta_s b(y)s(x)^{-1}+s(y)b(y)(s(x)^{-1}-s(y)^{-1}),$$

whose norm is at most $(N+MN^2)\,\|b(y)\|\,\|\delta_s\|$. Interchanging x,y gives the same bound with $\|b(x)\|$. Taking the smaller bound and adding the unchanged scalar part proves (8.10). The sup norm is bounded by $\|\lambda\|_\infty+MN\,\|b\|_\infty$.

For the adjoint first replace $b(x)^*$ by $b(y)^*$ in a fixed frame, at cost $K^2\,\|b(x)-b(y)\|$. The remaining two terms are

$$(h(x)^{-1}-h(y)^{-1})b(y)^*h(x)+h(y)^{-1}b(y)^*(h(x)-h(y)).$$

The inverse identity bounds their sum by $(K^3+K)\,\|b(y)\|\,\|h(x)-h(y)\|$. Reverse the two points, choose the smaller amplitude, and use that conjugating λ does not change its Hölder seminorm. This proves (8.11) without a comparison theorem for operator square roots. No zero value of b requires division. ∎

Proposition 8.6 *Let $\Omega\subset\mathbb{R}^m$ be open, let $Z\subset\Omega$ be relatively closed, and write $a=\lambda\mathrm{I}_2+b$, where λ and b are individually C^1. Suppose that $b(z)=0$ and $Db(z)=0$ for every $z\in Z$. Let $s\colon\Omega\setminus Z\to\mathrm{GL}_2(\mathbb{C})$ be C^1, with fixed bounds $\|s\|\leqslant M$ and $\left\|s^{-1}\right\|\leqslant N$. Assume locally at each $z\in Z$ that*

$$\|b(x)\|\,\|Ds(x)\|\longrightarrow 0\quad \text{as } x\longrightarrow z \text{ through } \Omega\setminus Z.$$

Extend $\Phi_s(a)$ to Z by $\Phi_s(a)(z)=\lambda(z)\mathrm{I}_2$. This extension is C^1, with derivative $D\lambda(z)\mathrm{I}_2$ on Z.

Proof : At $z\in Z$, differentiability of b with zero derivative gives $b(x)=o(|x-z|)$. The bound $\left\|sbs^{-1}\right\|\leqslant MN\,\|b\|$ proves differentiability of the extended nonscalar part at z, with derivative zero. Off Z, for a direction e put $q_e=(D_es)s^{-1}$. Product differentiation and the inverse identity give

$$D_e(sbs^{-1})=s(D_eb)s^{-1}+[q_e,sbs^{-1}],$$

and hence

$$\left\|D(sbs^{-1})\right\| \leqslant MN\,\|Db\| + 2MN^2\,\|b\|\,\|Ds\|\,.$$

Both terms tend to zero at z. At points of Z the derivative is already zero, so the derivative of the extended nonscalar part is continuous there. It is continuous off Z by the hypotheses. Adding the scalar term proves the claim. If $Z = \Omega$ the field is just $\lambda \mathrm{I}_2$; if Z is empty the product rule applies. The assertion is local and therefore holds in smooth manifold charts. It is a sufficient criterion with a vanishing first jet, not merely vanishing values. ∎

The next example shows why the first-jet condition is substantive. The obstruction concerns the image algebra itself being selfadjoint. Containment in a larger selfadjoint algebra would give no such conclusion.

Example 8.7 There is a separable amenable commutative algebra $\mathrm{C}(S^2)\mathrm{I}_2 \subset A \subset \mathrm{C}(S^2, \mathrm{M}_2)$, with only scalar and two-dimensional semisimple fibres and $\mathrm{K}(A) \leqslant \sqrt{10}$, and a smooth section $v \in A$, with the following properties. Its canonical transported adjoint is not differentiable at a scalar fibre. In fact, for every field of invertible matrices S for which $\Theta_S(A) = \{SaS^{-1} : a \in A\}$ is a selfadjoint subalgebra of $\mathrm{C}(S^2, \mathrm{M}_2)$, neither $J_S(v) = S^{-1}\Theta_S(v)^* S$ nor $\Theta_S(v)$ is differentiable there.

Proof : On the unit sphere in $\mathbb{R}^3$, with coordinates (x, y, z), let Z be the two poles and put

$$v(x,y,z) = \begin{bmatrix} x+iy & 2y \\ 2y & -x-iy \end{bmatrix}, \qquad q(x,y) = x^2 + 3y^2 + 2ixy.$$

Multiplication gives $v^2 = q\mathrm{I}_2$ and $\operatorname{Tr} v = 0$, while

$$|q| \geqslant x^2 + 3y^2 \geqslant x^2 + y^2, \qquad \|v\|^2 \leqslant \operatorname{Tr}(v^* v) = 2x^2 + 10y^2 \leqslant 10(x^2+y^2).$$

Thus q vanishes precisely on Z and has positive real part elsewhere. Its principal square root ε is continuous, nonzero off Z, and extends by zero to Z. More explicitly, off Z,

$$\varepsilon = \sqrt{\frac{|q| + \operatorname{Re} q}{2}} + i\frac{\operatorname{Im} q}{\sqrt{2(|q| + \operatorname{Re} q)}}.$$

The field $u = \frac{v}{\varepsilon}$ off Z satisfies $u^2 = \mathrm{I}_2$ and $\|u\| \leqslant \sqrt{10}$. Let A consist of all continuous matrix fields taking values in $\mathbb{C}\mathrm{I}_2 + \mathbb{C}v(\xi)$ off Z and scalar values at the poles. These pointwise conditions are closed under the algebra operations and are intersections of closed evaluation conditions. They define a norm-closed unital $\mathrm{C}(S^2)$-module algebra. It is commutative, contains the smooth section v, and has exactly the stated fibres, since I_2 and v realize their spans off Z.

We verify amenability independently of the global theorem. Set

$$Y = \{(\xi, w) \in S^2 \times \mathbb{C} : w^2 = q(\xi)\}.$$

This is a compact metrizable space: it is closed and its second coordinate is bounded. Over a pole the two sheets meet at $(\xi, 0)$; off the poles they are $w = \pm\varepsilon(\xi)$. For $a \in A$, put $\lambda = \frac{\operatorname{Tr} a}{2}$, and write $a = \lambda \mathrm{I}_2 + \mu v$ off Z. There μ is continuous because $\mu = \frac{\operatorname{Tr}(av)}{2q}$. Define

$$\Phi(a)(\xi, w) = \lambda(\xi) + \mu(\xi) w \quad (\xi \notin Z), \qquad \Phi(a)(\xi, 0) = \lambda(\xi) \quad (\xi \in Z).$$

These are the eigenvalues of $a(\xi)$, so $|\Phi(a)(\xi, w)| \leqslant \|a(\xi)\|$. Moreover $|\mu w| \leqslant \|a - \lambda \mathrm{I}_2\| \longrightarrow 0$ at a pole. Thus Φ is a contractive unital homomorphism $A \to \mathrm{C}(Y)$.

Conversely, for $f \in \mathrm{C}(Y)$ put

$$\lambda(\xi) = \frac{f(\xi, w) + f(\xi, -w)}{2}, \qquad \psi(\xi, w) = \frac{f(\xi, w) - f(\xi, -w)}{2}.$$

For $w \neq 0$ define $a(\xi) = \lambda(\xi)\mathrm{I}_2 + \frac{\psi(\xi,w)v(\xi)}{w}$, and at a pole set $a(\xi) = f(\xi, 0)\mathrm{I}_2$. The expression is independent of the sign of w. It is continuous away from the poles by local square-root branches. At a pole ψ tends to zero and $\left\|\frac{v}{w}\right\| \leqslant \sqrt{10}$, proving continuity there. Equivalently, the continuous invariant expression on Y descends through its continuous compact-to-Hausdorff projection onto S^2. Thus $a \in A$, $\Phi(a) = f$, and

$$\left\|\Phi^{-1}(f)\right\| \leqslant (1 + \sqrt{10}) \left\|f\right\|.$$

The bounded algebra isomorphism $A \cong \mathrm{C}(Y)$ proves separability. By Lemma 2.6, $\mathrm{C}(Y)$ has a contractive approximate diagonal. Applying $\Phi^{-1} \otimes \Phi^{-1}$ to it gives a bounded approximate diagonal of A, proving amenability.

Off Z the canonical metric is

$$h = \frac{\mathrm{I}_2 + u^*u}{\sqrt{\det(\mathrm{I}_2 + u^*u)}} = |u|,$$

and we set $h = \mathrm{I}_2$ at the poles. It is continuous off Z and uniformly bounded with uniformly bounded inverse, by the bound on u. Conjugation by $h^{\frac{1}{2}}$ and its inverse preserves continuity at the poles: subtract the scalar limiting value and apply the uniform bounds. The relation $hu = u^*h$ selfadjointizes each two-dimensional fibre. Conversely, pulling back the adjoint of a transported section yields a continuous section with the prescribed pointwise fibre values, hence belongs to A by its definition. Thus this field does have a selfadjoint image algebra.

For the displayed smooth section the canonical transported adjoint is

$$J(v) = \frac{\overline{\varepsilon}}{\varepsilon} v = \frac{\overline{q}}{|q|} v \quad \text{off } Z, \qquad J(v) = 0 \quad \text{on } Z.$$

Use the north-pole chart $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$. The section v is exactly real linear in x, y. Along each coordinate axis q is positive real, so $J(v) = v$. Differentiability would therefore force its derivative to be this real-linear map. Along $x = y = t > 0$, however, the multiplier is $\frac{4-2i}{\sqrt{20}} = \frac{2-i}{\sqrt{5}} \neq 1$. The off-diagonal directional derivative would be $\frac{2(2-i)}{\sqrt{5}}$ instead of 2, a contradiction.

This transported-adjoint formula is independent of the particular selfadjointizing field. Indeed $\Theta_S(A)$ is commutative and selfadjoint, so $\Theta_S(v)(\xi)$ is normal, with eigenvalues $\pm\varepsilon(\xi)$. Its adjoint is $\frac{\overline{\varepsilon}}{\varepsilon}\Theta_S(v)(\xi)$. Pulling this identity back gives exactly the same formula for $J_S(v)$.

Finally suppose that $b = \Theta_S(v)$ itself were differentiable at the north pole. It is zero at that pole and $b^2 = q\mathrm{I}_2$. On the x-axis it is normal with real eigenvalues $\pm x$, and on the y-axis it is normal with real eigenvalues $\pm\sqrt{3}y$. Thus it is selfadjoint on both axes. Its two partial derivatives P, Q would consequently be selfadjoint, as would $P + Q$. Along (t, t) differentiability and $b(t, t)^2 = (4 + 2i)t^2\mathrm{I}_2$ would give

$$(P + Q)^2 = (4 + 2i)\mathrm{I}_2,$$

which is impossible for a selfadjoint matrix. No uniform bound on S is required for this last contradiction once differentiability is assumed. The square root ε was globally defined off the poles, so this obstruction is not caused by monodromy. ∎

The final example has a different quantifier. It exhibits a permitted choice of extension over full matrix fibres with poor regularity; it does not exclude a better choice of extension.

Example 8.8 There is a compact metric base and an amenable central algebra with $\mathrm{K}(A) = 2$, together with a metric extension satisfying the exact ball constraint in the construction of the main theorem, for which a Lipschitz section has both transported images non-Hölder of every positive exponent. Another admissible extension preserves the regularity of this section.

Proof : Put $x_n = \frac{1}{\log(n+1)}$ and give

$$X = \{0\} \cup \{x_n : n \geqslant 1\} \cup \{2\} \subset \mathbb{R}$$

the inherited metric. Let

$$p = \begin{bmatrix} 1 & \frac{3}{4} \\ 0 & 0 \end{bmatrix}, \qquad B_p = \mathbb{C}p + \mathbb{C}(\mathrm{I}_2 - p).$$

Let A be all continuous sections scalar at 0, arbitrary in M_2 at the points x_n, and taking values in B_p at 2. These are closed unital $\mathrm{C}(X)$-module conditions. Its strata are $Z = \{0\}$, $F = \{2\}$ and the full matrix sequence points. For $u = 2p - \mathrm{I}_2$, the positive matrix u^*u has trace $\frac{17}{4}$ and determinant one, hence eigenvalues 4 and $\frac{1}{4}$. Therefore $\|u\| = 2$ and $\mathrm{K}(A) = 2$.

Here is an independent amenability argument. Let D_N be the subalgebra constant scalar on 0 and the tail x_n, $n > N$. Then

$$D_N \cong \mathrm{M}_2^N \oplus_\infty \mathbb{C} \oplus_\infty B_p$$

isometrically, and these algebras increase with dense union in A. One fixed invertible matrix diagonalizes p, so D_N is isomorphic to $E_N = \mathrm{M}_2^N \oplus_\infty \mathbb{C} \oplus_\infty \mathbb{C}^2$ with isomorphism and inverse norms bounded independently of N. The latter algebra is linearly spanned by a finite unitary group: take the sixteen-element Pauli group on each matrix coordinate and independent signs on each scalar coordinate. To see that the span includes the separate coordinates, subtract two tuples which differ only in that coordinate by multiplication by -1; the Pauli matrices span its matrix coordinate and 1 spans a scalar coordinate. For any finite unitary group G spanning an algebra,

$$d_G = \frac{1}{|G|} \sum_{g \in G} g \otimes g^{-1}$$

has projective norm at most one and product one. For $h \in G$, reindex $k = hg$ in the left action; its terms become $k \otimes k^{-1}h$, which are the right-action terms. The tensor is therefore central for G and its linear span. Transport these diagonals to D_N to obtain diagonals m_N with a uniform bound. For $a \in A$, choose $a_N \in D_N$ with $\|a - a_N\| \to 0$. Then $\|a \cdot m_N - m_N \cdot a\|_\pi \leqslant 2\,\|a - a_N\|\,\|m_N\|_\pi \to 0$, and the products are one. Thus A has a bounded approximate diagonal.

At 2 the canonical determinant-normalized metric is

$$h_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{17}{10} \end{bmatrix}.$$

It has determinant one and eigenvalues 2, $\frac{1}{2}$. Choose $h(x_n) = \mathrm{diag}(2, \frac{1}{2})$ for even n and $h(x_n) = \mathrm{diag}(\frac{1}{2}, 2)$ for odd n, and set $h(0) = \mathrm{I}_2$. On $U = X \setminus Z$, its trace normalization $g = \frac{4}{5}h$ has trace two and $\|g - \mathrm{I}_2\| = \frac{3}{5}$. At 2 it is the prescribed trace-normalized canonical metric. Since U is discrete, $g - \mathrm{I}_2$ is a continuous extension of this value into precisely the ball $\mathcal{V}_r$ used in the main construction, with $r = \frac{K^2-1}{K^2+1} = \frac{3}{5}$. Determinant normalization recovers h. Thus the extension has the exact permitted bound $K = 2$. This asserts the properties required in the construction; it does not assert that a specified convex-hull-preserving Dugundji operator, applied to a singleton F, would choose it.

The section $a(t) = t(2-t)E_{12}$ restricted to X belongs to A and is Lipschitz, being the restriction of a polynomial on $[0,2]$. It vanishes at 0 and 2. Write $a_n = x_n(2-x_n)$. Direct conjugation gives

$$J(a)(x_n) = \begin{cases} 4a_nE_{21} & n \text{ even}, \\ \frac{1}{4}a_nE_{21} & n \text{ odd}, \end{cases} \qquad \Theta(a)(x_n) = \begin{cases} 2a_nE_{12} & n \text{ even}, \\ \frac{1}{2}a_nE_{12} & n \text{ odd}. \end{cases}$$

Both images are continuous at zero because $a_n \to 0$. For large even n, $x_n < 1$, $a_n \geqslant x_n$ and $a_{n+1} < a_n$. Their adjacent differences are at least $\frac{15}{4}x_n$ for J and $\frac{3}{2}x_n$ for Θ. On the other hand,

$$0 < x_n - x_{n+1} = \frac{\log\left(\frac{n+2}{n+1}\right)}{\log(n+1)\log(n+2)} \leqslant \frac{1}{(n+1)\log^2(n+1)}.$$

For every fixed $\alpha > 0$, the Hölder quotients are therefore bounded below by a positive constant times $(n+1)^\alpha \log(n+1)^{2\alpha-1}$, which tends to infinity. Neither image is Hölder of any positive exponent.

In contrast, keeping the same canonical h_2 and taking $h(x_n) = \mathrm{I}_2$ at all sequence points gives another permitted extension, which preserves this section's regularity. The point 2 is isolated and both the original and transported sections vanish there. This example therefore concerns the failure of arbitrary extension choices, whereas Example 8.7 excludes every selfadjointizing field for its differentiability assertion. ∎

The weighted hypotheses compare nearby pairs of points, or derivatives of the frame. A bound on $\|b(x)\|$ only in terms of $\operatorname{dist}(x, Z)$ controls approach to Z but does not bound the oscillation between nearby points outside Z. The pure scalar/two-dimensional case has the additional automatic cancellation in (8.1). These results leave open the existence of regularity-adapted extensions on arbitrary mixed strata, the optimal constants and more general higher-order compatibility conditions.

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