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Quantitative similarities for amenable algebras of two-by-two matrix-valued functions

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Abstract

We study norm-closed subalgebras A of $\mathrm{C}(X, \mathrm{M}_2)$ containing the scalar functions, where X is a compact metrizable space. We characterize amenability of A by semisimplicity of its fibres and a uniform bound on the norms of the elements $2p - \mathrm{I}_2$ associated with their primitive idempotents. This bound is the exact similarity constant in the faithful representation given by the direct sum of all evaluations. In particular, an amenable algebra of this form admits a similarity to a nuclear C^*-algebra with cost at most $\sqrt{2\,\mathrm{AM}(A) - 1}$. The estimate is sharp already for a two-dimensional algebra in M_2. The similarity is constructed from positive matrix-valued functions which are continuous away from the scalar fibres. Although these functions need not be continuous everywhere, the induced algebra isomorphism preserves continuous sections. We also prove the corresponding results for locally trivial bundles of M_2-algebras, and give a noncommutative example for which no continuous matrix-valued similarity is available.

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1 Introduction

The amenability of a Banach algebra is defined without reference to an involution. Nevertheless, for C^*-algebras, it is equivalent to nuclearity by the work of Connes and Haagerup; see [BLM04, Section 7.4] and [Haa83]. This leads to the question of whether an amenable operator algebra must be isomorphic, as a Banach algebra, to a nuclear C^*-algebra. Here an operator algebra is a norm-closed, not necessarily selfadjoint, subalgebra of $\mathrm{B}(H)$ for a complex Hilbert space H.

Choi, Farah and Ozawa constructed a nonseparable amenable subalgebra of $\ell^\infty(\mathbb{N}, \mathrm{M}_2)$ which is not isomorphic to a C^*-algebra [CFO14, Theorem 1]. Their construction also shows that local similarities with uniformly bounded costs do not, by themselves, provide a global similarity. The question restricted to separable operator algebras is not settled by that example. Positive

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results are known for several classes. In particular, Gifford treated algebras of compact operators using the total reduction property [Gif06], Choi proved an internal similarity theorem for commutative operator amenable subalgebras of finite von Neumann algebras [Cho13, Theorem 1.1], and Marcoux and Popov proved that every commutative amenable operator algebra is similar to a C*-algebra [MP16, Theorem 2.10]. Clouâtre and Marcoux obtained a similarity theorem for weak* closed residually finite-dimensional operator algebras with the total reduction property [CM19, Theorem 4.5]. The weak* closedness assumption in that result cannot simply be replaced by norm closedness.

In this paper, we consider a different locality assumption. Let X be a nonempty compact metrizable space and let A be a norm-closed algebra satisfying

$$\mathrm{C}(X)\mathrm{I}_2 \subset A \subset \mathrm{C}(X, \mathrm{M}_2). \tag{1.1}$$

For any $x \in X$, we denote its evaluation algebra by $A_x \stackrel{\mathrm{def}}{=} \{a(x) : a \in A\}$. The inclusion of the scalar functions makes A a $\mathrm{C}(X)$-module. Consequently, membership in A can be checked fibrewise, provided that the section under consideration is continuous. This last qualification is essential: a bounded field of similarities need not preserve continuous sections.

We describe the invariant which occurs in our result. Suppose that each A_x is semisimple. A unital semisimple subalgebra of M_2 is either $\mathbb{C}\mathrm{I}_2$, M_2, or a two-dimensional algebra of the form $\mathbb{C}p_x + \mathbb{C}(\mathrm{I}_2 - p_x)$, where p_x is a rank-one idempotent. Put

$$\mathrm{K}(A) \stackrel{\mathrm{def}}{=} \sup\bigl(\{1\} \cup \{\|2p_x - \mathrm{I}_2\| : x \in X,\ \dim A_x = 2\}\bigr). \tag{1.2}$$

The value in (1.2) is unchanged if p_x is replaced by $\mathrm{I}_2 - p_x$.

We use a specified faithful representation when discussing the optimal cost of a spatial similarity. Set $H_X \stackrel{\mathrm{def}}{=} \ell^2_X \otimes \mathbb{C}^2$ and let $\pi_X(a) \stackrel{\mathrm{def}}{=} \bigoplus_{x\in X} a(x)$. We define

$$s_X(A) \stackrel{\mathrm{def}}{=} \inf\bigl\{\|S\|\,\bigl\|S^{-1}\bigr\| : S \in \mathrm{B}(H_X) \text{ invertible}, S\pi_X(A)S^{-1} \text{ selfadjoint}\bigr\}, \tag{1.3}$$

with infimum $+\infty$ when there is no such operator. The use of all evaluations in this definition removes any possible dependence on a measure on X.

Theorem 1.1 *Let X be a nonempty compact metrizable space and let A satisfy (1.1). Then the following conditions are equivalent:*

1. *A is amenable as a Banach algebra.*
2. *Every algebra A_x is semisimple and $\mathrm{K}(A) < \infty$.*

If these conditions hold and $K \stackrel{\mathrm{def}}{=} \mathrm{K}(A)$, there exists a bounded Borel function $h\colon X \to \mathrm{M}_2$ such that

$$\det h(x) = 1, \quad K^{-1}\mathrm{I}_2 \leqslant h(x) \leqslant K\mathrm{I}_2 \quad \text{for any } x \in X, \tag{1.4}$$

and the map

$$\Theta\colon A \to \mathrm{C}(X, \mathrm{M}_2), \quad a \mapsto h^{\frac{1}{2}} a h^{-\frac{1}{2}}, \tag{1.5}$$

is a $\mathrm{C}(X)$-linear algebra isomorphism onto a nuclear C^*-subalgebra. Moreover, h is continuous outside the closed set $\{x : A_x = \mathbb{C}\mathrm{I}_2\}$, and

$$s_X(A) = K, \quad \|\Theta\|_{\mathrm{cb}} \leqslant K, \quad \bigl\|\Theta^{-1}\bigr\|_{\mathrm{cb}} \leqslant K. \tag{1.6}$$

Finally,

$$\frac{1 + K^2}{2} \leqslant \mathrm{AM}(A) \leqslant K^2. \tag{1.7}$$

In particular, Theorem 1.1 gives the estimate

$$s_X(A) \leqslant \sqrt{2\,\mathrm{AM}(A) - 1}. \tag{1.8}$$

For any prescribed $K \geqslant 1$, a two-dimensional algebra in M_2 attains equality in (1.8); see Example 6.1. Thus this estimate cannot be improved as a function of the amenability constant alone in the class (1.1). We do not claim that the upper bound in (1.7) is an equality in general.

The locality assumption (1.1) is part of the theorem. We do not remove it, and the result does not address arbitrary separable amenable subalgebras of $\mathrm{C}(X, \mathrm{M}_2)$. Also, the similarity in (1.6) is taken in $\mathrm{B}(H_X)$, not in $\mathrm{C}(X, \mathrm{M}_2)$. Example 6.2 shows that this distinction is necessary, even for a noncommutative algebra in the class considered here.

Approach of the paper The two-dimensional fibres have an intrinsic pair of primitive idempotents, up to permutation. We first compute both their amenability constants and their optimal similarity constants. For such a pair p, q, the matrix $p^*p + q^*q$ defines a compatible positive metric. A trace normalization places these metrics in a fixed convex ball of selfadjoint matrices. We extend the normalized metrics through the full matrix fibres using Dugundji's theorem [Dug51, Theorem 4.1], and then normalize the determinant. At a scalar fibre, a continuous section is asymptotically scalar. This makes the discontinuity of the metric harmless for conjugation. A partition of unity then gives an adjoint in the original algebra, rather than merely in an ambient algebra of bounded functions.

Structure of the paper Section 2 records the localization and amenability facts used below. Section 3 contains the two-dimensional computations. We prove Theorem 1.1 in Section 4. Section 5 extends the result to locally trivial bundles of M_2-algebras. Section 6 gives examples and describes the distinction between continuous metrics and continuous transported adjoints.

2 Preliminaries and localization

We write M_d for $\mathrm{M}_d(\mathbb{C})$ with its operator norm. The symbol Tr denotes the unnormalized trace. An idempotent is an element p satisfying $p^2 = p$. The word projection is reserved for a selfadjoint idempotent. All algebras in the main results are unital, with the same identity as their ambient algebra. We denote by $A^\dagger \stackrel{\mathrm{def}}{=} \{a^* : a \in A\}$ the adjoint algebra, reserving A^* for the Banach dual when both notions occur.

For a unital Banach algebra A, let $A \mathbin{\widehat{\otimes}_\pi} A$ denote its Banach-space projective tensor product. We use the actions $a \cdot (x \otimes y) = ax \otimes y$ and $(x \otimes y) \cdot a = x \otimes ya$, and the multiplication map $\mu(x \otimes y) = xy$. A bounded approximate diagonal is a bounded net (m_α) such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \mu(m_\alpha) \to 1 \quad \text{for any } a \in A. \tag{2.1}$$

By Johnson's characterization, A is amenable if and only if it has a bounded approximate diagonal [Joh72]; see also [BLM04, Lemma 7.4.4]. The amenability constant $\mathrm{AM}(A)$ is the infimum of the bounds of such nets. We set $\mathrm{AM}(A) = +\infty$ when A is not amenable.

Lemma 2.1 *Let A and B be unital Banach algebras.*

1. *If $q\colon A \to B$ is a contractive surjective homomorphism, then $\mathrm{AM}(B) \leqslant \mathrm{AM}(A)$.*
2. *If $\theta\colon B \to A$ is a bounded algebra isomorphism and B is an amenable C^*-algebra, then $\mathrm{AM}(A) \leqslant \|\theta\|^2$.*

Proof : 1. Apply $q \otimes q$ to an approximate diagonal of A. For $b = q(a)$, its centrality defect is the image of $a \cdot m_\alpha - m_\alpha \cdot a$, and its product converges to 1_B. The norm of $q \otimes q$ is at most 1.

2. A nuclear C*-algebra has approximate diagonals with bounds arbitrarily close to 1, by Haagerup's theorem; see [BLM04, Section 7.4.17]. Apply $\theta \otimes \theta$ to such a net. For any $a \in A$, use $b = \theta^{-1}(a)$ in the centrality relation. Its norm is bounded by $\|\theta\|^2$ times the original bound. This proves the assertion. ∎

Lemma 2.2 *Let X be a compact Hausdorff space, let V be a finite-dimensional normed space, and let Y be a closed $\mathrm{C}(X)$-submodule of $\mathrm{C}(X, V)$. For any $x \in X$, set $Y_x \stackrel{\text{def}}{=} \{y(x) : y \in Y\}$. For any $b \in \mathrm{C}(X, V)$, we have*

$$\operatorname{dist}(b, Y) = \sup_{x \in X} \operatorname{dist}(b(x), Y_x). \tag{2.2}$$

Consequently, a continuous function b belongs to Y if and only if $b(x) \in Y_x$ for any $x \in X$.

Proof : Every evaluation is contractive, which gives one inequality. Put $d \stackrel{\text{def}}{=} \sup_x \operatorname{dist}(b(x), Y_x)$ and fix $\varepsilon > 0$. For any $x \in X$, choose $y_x \in Y$ with $\|b(x) - y_x(x)\| < d + \varepsilon$. This estimate remains valid on an open neighbourhood U_x of x. Choose a finite subcover $U_{x_1}, \dots, U_{x_N}$ and a continuous partition of unity (φ_i) subordinate to it. The function $y = \sum_i \varphi_i y_{x_i}$ belongs to Y, and

$$\|b(z) - y(z)\| \leqslant \sum_i \varphi_i(z)\, \|b(z) - y_{x_i}(z)\| \leqslant d + \varepsilon \quad (z \in X).$$

Taking the supremum over z and then letting $\varepsilon \to 0$ proves (2.2). The last assertion follows from the norm closedness of Y. ∎

Remark 2.3 The proof of Lemma 2.2 also applies to continuous sections of a finite-dimensional normed vector bundle: the norm of the difference of two sections is continuous, and scalar partitions of unity act on sections. We will use this formulation in Section 5.

Lemma 2.4 *Let D be a unital C*-subalgebra of $\mathrm{C}(X, \mathrm{M}_2)$ which contains $\mathrm{C}(X)\mathrm{I}_2$. Then every irreducible representation of D has dimension at most 2. In particular, D is nuclear.*

Proof : The central copy of $\mathrm{C}(X)$ acts by scalars in any unital irreducible representation ρ of D. Thus there is $x \in X$ such that $\rho(f\mathrm{I}_2) = f(x)I$ for any $f \in \mathrm{C}(X)$. If $a(x) = 0$, cutoff functions vanishing near x show that a is a norm limit of elements fa with $f(x) = 0$. Indeed, choose f with $0 \leqslant f \leqslant 1$, equal to 1 outside a neighbourhood on which $\|a(\cdot)\|$ is small. Hence ρ factors through the evaluation algebra D_x. This is a C*-subalgebra of M_2, so its irreducible representations have dimension at most 2. Finally, subhomogeneous C*-algebras are nuclear by [BO08, Proposition 2.7.7 p. 57]. ∎

3 Two-dimensional fibres and sharp constants

Lemma 3.1 *A unital semisimple subalgebra B of M_2 is either $\mathbb{C}\mathrm{I}_2$, M_2, or $\mathbb{C}p + \mathbb{C}(\mathrm{I}_2 - p)$ for a rank-one idempotent p. In particular, this conclusion holds when B is amenable.*

Proof : A finite-dimensional semisimple complex algebra is a direct sum of full matrix algebras. Its faithful unital representation on $\mathbb{C}^2$ is a direct sum of simple modules. Counting their dimensions leaves only a scalar representation, the sum of two distinct one-dimensional

representations, and the two-dimensional simple representation of M_2. In the middle case the two central primitive idempotents have rank one and sum to I_2. Finally, a finite-dimensional amenable operator algebra is semisimple; see [BLM04, Proposition 7.2.7 and Corollary 7.4.14]. ∎

For a rank-one idempotent $p \in \mathrm{M}_2$, put

$$q \stackrel{\text{def}}{=} \mathrm{I}_2 - p, \quad u \stackrel{\text{def}}{=} 2p - \mathrm{I}_2, \quad s \stackrel{\text{def}}{=} \|u\|, \quad k \stackrel{\text{def}}{=} p^*p + q^*q. \tag{3.1}$$

The two primitive idempotents of $B_p = \mathbb{C}p + \mathbb{C}q$ are p and q.

Proposition 3.2 *With the notation (3.1), the following assertions hold.*

1. *The least cost of a similarity transforming B_p into a selfadjoint subalgebra of M_2 is s.*
2. *The amenability constant is*

$$\mathrm{AM}(B_p) = \frac{1+s^2}{2}. \tag{3.2}$$

3. *The matrices*

$$g = \frac{2k}{\mathrm{Tr}(k)}, \quad h = \frac{g}{\sqrt{\det g}} \tag{3.3}$$

satisfy $hp = p^*h$, $\det h = 1$, and

$$\|g - \mathrm{I}_2\| = \frac{s^2-1}{s^2+1}, \quad \sigma(h) = \{s^{-1}, s\}. \tag{3.4}$$

The matrices g and h are unchanged when p and q are interchanged, and their construction commutes with $*$-automorphisms of M_2.

Proof : Choose an orthonormal basis whose first vector spans $\mathrm{Ran}\, p$, and change the phase of the second vector if necessary. In this basis,

$$p = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & -t \\ 0 & 1 \end{bmatrix}, \quad t \geqslant 0. \tag{3.5}$$

From (3.5), a direct computation gives

$$s = \sqrt{1+t^2} + t, \quad k = \begin{bmatrix} 1 & t \\ t & 1+2t^2 \end{bmatrix}, \quad \det k = 1 + t^2. \tag{3.6}$$

By (3.6), the eigenvalues of $g = \frac{k}{1+t^2}$ are $1 \pm \frac{t}{\sqrt{1+t^2}}$. Hence the eigenvalues of $h = \frac{k}{\sqrt{1+t^2}}$ are $\sqrt{1+t^2} \pm t$, which proves (3.4). Also $kp = p^*k$, by the identities $pq = qp = 0$. Scalar normalization preserves this relation. The last assertion follows because the formulas use only algebraic operations, the adjoint, trace and determinant, and because every $*$-automorphism of M_2 is unitary conjugation.

The relation $hp = p^*h$ implies that $h^{\frac{1}{2}}ph^{-\frac{1}{2}}$ is an orthogonal projection. The cost of this similarity is s, by (3.4). Conversely, if SB_pS^{-1} is selfadjoint, it is a commutative C*-algebra. Its idempotent SpS^{-1} is therefore an orthogonal projection, and SuS^{-1} is a selfadjoint unitary. Consequently,

$$s = \|u\| \leqslant \left\|S^{-1}\right\| \left\|SuS^{-1}\right\| \|S\| = \|S\| \left\|S^{-1}\right\|.$$

This proves assertion 1.

We now prove assertion 2. The only diagonal in $B_p \otimes B_p$ is

$$d = p \otimes p + q \otimes q = \frac{1}{2}(\mathrm{I}_2 \otimes \mathrm{I}_2 + u \otimes u). \tag{3.7}$$

Indeed, commutation with p eliminates the mixed terms in the basis $p \otimes p, p \otimes q, q \otimes p, q \otimes q$, and $\mu(d) = \mathrm{I}_2$ determines the other coefficients. Since $B_p \widehat{\otimes}_\pi B_p$ is finite-dimensional, every bounded approximate diagonal has a cluster point which is a diagonal. Hence $\mathrm{AM}(B_p) = \|d\|_\pi$. Equation (3.7) gives $\|d\|_\pi \leqslant \frac{1}{2}(1 + s^2)$.

To prove the reverse inequality, let

$$v = \frac{1}{\sqrt{1+s^2}} \begin{bmatrix} 1 \\ s \end{bmatrix}, \quad w = \frac{1}{\sqrt{1+s^2}} \begin{bmatrix} s \\ -1 \end{bmatrix}.$$

These vectors are orthonormal and $uv = sw$. Thus, for $a = \lambda \mathrm{I}_2 + \mu u$, we have

$$|\lambda|^2 + s^2|\mu|^2 = \|av\|^2 \leqslant \|a\|^2. \tag{3.8}$$

Define a complex-bilinear form on B_p by

$$\beta(\lambda \mathrm{I}_2 + \mu u, \lambda' \mathrm{I}_2 + \mu' u) = \lambda\lambda' + s^2 \mu\mu'.$$

The Cauchy–Schwarz inequality and (3.8) give $\|\beta\| \leqslant 1$. Therefore

$$\|d\|_\pi \geqslant |\beta(d)| \overset{(3.7)}{=} \frac{1+s^2}{2}.$$

This proves (3.2). ∎

Remark 3.3 The determinant normalization in (3.3) balances the extreme eigenvalues of the metric. The trace normalization preceding it is used for a different purpose: it places $g - \mathrm{I}_2$ in a convex norm ball of traceless selfadjoint matrices. This separation between extension and normalization will preserve the optimal constant in the global construction.

4 Global similarities and the proof of the main theorem

We first construct a similarity under the fibrewise condition in Theorem 1.1. No amenability assumption is used in this construction.

Suppose that every A_x is semisimple and that $K = \mathrm{K}(A) < \infty$. Let

$$Z \overset{\text{def}}{=} \{x \in X : A_x = \mathbb{C}\mathrm{I}_2\}, \quad U \overset{\text{def}}{=} X \setminus Z, \quad F \overset{\text{def}}{=} \{x \in U : \dim A_x = 2\}. \tag{4.1}$$

For the sets in (4.1), we first observe that $x \mapsto \dim A_x$ is lower semicontinuous. Indeed, finitely many sections whose values are linearly independent at one point remain linearly independent on a neighbourhood. Hence Z is closed, U is open, and F is closed in U. Every fibre over $U \setminus F$ is M_2.

For $x \in F$, define g_x and h_x by (3.3), using either primitive idempotent of A_x. These functions are continuous on F. To check this locally, fix $x_0 \in F$ and choose $a \in A$ with $a(x_0)$ nonscalar. Its two eigenvalues are distinct. On a neighbourhood of x_0, they admit continuous branches λ_1, λ_2, and

$$p_x = \frac{a(x) - \lambda_2(x)\mathrm{I}_2}{\lambda_1(x) - \lambda_2(x)}, \quad x \in F, \tag{4.2}$$

is a primitive idempotent of A_x. Formula (3.3) is therefore locally continuous. Its invariance under permutation makes the local definitions agree.

Set

$$r \overset{\text{def}}{=} \frac{K^2-1}{K^2+1} < 1. \tag{4.3}$$

By Proposition 3.2, the map $x \mapsto g_x - \mathrm{I}_2$ takes its values in the convex set

$$\mathcal{V}_r \overset{\text{def}}{=} \{v \in (\mathrm{M}_2)_{\mathrm{sa}} : \mathrm{Tr}(v) = 0,\ \|v\| \leqslant r\}.$$

Dugundji's extension theorem [Dug51, Theorem 4.1], applied to the metric space U and its closed subset F, gives a continuous extension $v\colon U \to \mathcal{V}_r$. If F is empty, take $v = 0$. Put

$$g(x) \overset{\text{def}}{=} \mathrm{I}_2 + v(x), \quad h(x) \overset{\text{def}}{=} \frac{g(x)}{\sqrt{\det g(x)}} \quad (x \in U), \quad h(x) \overset{\text{def}}{=} \mathrm{I}_2 \quad (x \in Z). \tag{4.4}$$

The eigenvalues of $v(x)$ are λ and $-\lambda$, with $|\lambda| \leqslant r$. Consequently, $g(x)$ is positive invertible, and the eigenvalues of $h(x)$ are

$$\sqrt{\frac{1+\lambda}{1-\lambda}}, \quad \sqrt{\frac{1-\lambda}{1+\lambda}}.$$

Equations (4.3) and (4.4) imply (1.4). The function h is continuous on U and Borel on X.

For any $a \in A$, define $J(a)(x) = h(x)^{-1}a(x)^*h(x)$. This is continuous on U. If $z \in Z$, write $a(z) = \lambda \mathrm{I}_2$. Then

$$\big\|J(a)(x) - \overline{\lambda}\mathrm{I}_2\big\| \leqslant K^2 \,\|a(x) - \lambda \mathrm{I}_2\| \to 0 \quad \text{as } x \to z. \tag{4.5}$$

Thus $J(a)$ is continuous on X. Its value belongs to A_x at every point: this follows from Proposition 3.2 on F, and is immediate on scalar and full matrix fibres. Lemma 2.2 gives

$$J(a) \in A \quad \text{for any } a \in A.$$

The map J is conjugate-linear and antimultiplicative, and $J^2 = \mathrm{Id}_A$, because h is positive. It follows that $h^{-1}A^\dagger h = A$ as algebras of bounded functions.

Write $s(x) = h(x)^{\frac{1}{2}}$. The same argument as in (4.5), with bound K instead of K^2, shows that $\Theta(a) = sas^{-1}$ is continuous on X for any $a \in A$. The image $D \overset{\text{def}}{=} \Theta(A)$ is norm closed, since both Θ and its inverse on its range are bounded. Also,

$$\Theta(a)^* = sJ(a)s^{-1} = \Theta(J(a)),$$

so D is selfadjoint. The map Θ in (1.5) fixes $\mathrm{C}(X)\mathrm{I}_2$ pointwise, hence Lemma 2.4 shows that D is nuclear. At every matrix level, conjugation by $I_n \otimes s(x)$ gives

$$\|\Theta\|_{\mathrm{cb}} \leqslant K, \quad \big\|\Theta^{-1}\big\|_{\mathrm{cb}} \leqslant K.$$

In particular, A is amenable and Lemma 2.1 gives $\mathrm{AM}(A) \leqslant K^2$.

Conversely, suppose that A is amenable. Evaluation at x and Lemma 2.1 give $\mathrm{AM}(A_x) \leqslant \mathrm{AM}(A)$. Lemma 3.1 gives semisimplicity and the stated fibre types. For $x \in F$, Proposition 3.2 yields

$$\frac{1 + \|2p_x - \mathrm{I}_2\|^2}{2} \overset{(3.2)}{=} \mathrm{AM}(A_x) \leqslant \mathrm{AM}(A).$$

Taking the supremum gives $K < \infty$ and the lower bound in (1.7). If F is empty, this lower bound is simply $1 \leqslant \mathrm{AM}(A)$, which follows from (2.1) and the contractivity of multiplication.

It remains to verify the exact spatial cost defined in (1.3). The diagonal operator $S \overset{\text{def}}{=} \bigoplus_{x\in X} s(x)$ on H_X has $\|S\|, \|S^{-1}\| \leqslant \sqrt{K}$, and it implements Θ in the evaluation representation. Hence $s_X(A) \leqslant K$.

For the reverse inequality, let $T \in \mathrm{B}(H_X)$ be invertible and suppose that $T\pi_X(A)T^{-1}$ is selfadjoint. For any real-valued $f \in \mathrm{C}(X)$, the operator $T\pi_X(f\mathrm{I}_2)T^{-1}$ is central in this C*-algebra. It is therefore normal, and its spectrum as an operator is real, by similarity invariance of the spectrum. Hence it is selfadjoint. Taking adjoints gives

$$T^*T\pi_X(f\mathrm{I}_2) = \pi_X(f\mathrm{I}_2)T^*T.$$

Since continuous scalar functions separate points, T^*T is block diagonal in the representation $H_X = \bigoplus_x \mathbb{C}^2$. Write $T^*T = \bigoplus_x k_x$. By the polar decomposition, conjugation by $(T^*T)^{\frac{1}{2}}$ also makes $\pi_X(A)$ selfadjoint. At the coordinate x, its image is the C*-algebra $k_x^{\frac{1}{2}} A_x k_x^{-\frac{1}{2}}$. Thus Proposition 3.2 gives

$$\|2p_x - \mathrm{I}_2\| \leqslant \left\|k_x^{\frac{1}{2}}\right\| \left\|k_x^{-\frac{1}{2}}\right\| \leqslant \|T\| \left\|T^{-1}\right\|, \quad x \in F.$$

Taking the supremum proves $s_X(A) \geqslant K$. This completes the proof of Theorem 1.1.

Remark 4.1 A separable Hilbert space suffices to implement the similarity. Indeed, let X_0 be a countable dense subset of X. Evaluation on X_0 is completely isometric on $\mathrm{C}(X, \mathrm{M}_2)$, and $\bigoplus_{x\in X_0} h(x)^{\frac{1}{2}}$ has cost at most K. To retain the exact lower bound as well, choose X_0 to contain a countable subset dense in F for its relative topology. The function $x \mapsto \|2p_x - \mathrm{I}_2\|$ is continuous on F, by (4.2). Its supremum on $X_0 \cap F$ is therefore K when $K > 1$. The proof of the lower bound then applies to this separable representation.

5 Locally trivial bundles of matrix algebras

Let $\mathscr{M} \to X$ be a locally trivial bundle of unital C*-algebras with fibre M_2 and $*$-automorphisms as transition maps. We write $\Gamma(X, \mathscr{M})$ for its algebra of continuous sections, with the supremum norm. Its identity section is denoted by 1. Trace and determinant are defined fibrewise independently of the trivialization. This follows from their invariance under $*$-automorphisms of M_2.

Let A be a norm-closed algebra with

$$\mathrm{C}(X)1 \subset A \subset \Gamma(X, \mathscr{M}). \tag{5.1}$$

Write $A_x \subset \mathscr{M}_x$ for the evaluation algebra. When all A_x are semisimple, define $\mathrm{K}(A)$ by (1.2), interpreted in the fibres. Choose, for each x, a faithful irreducible representation of $\mathscr{M}_x$ on $\mathbb{C}^2$. Their direct sum defines a faithful representation π_X of $\Gamma(X, \mathscr{M})$. The value of $s_X(A)$ is independent of these choices, since two such choices are related by a block diagonal unitary.

Theorem 5.1 *Let X be a nonempty compact metrizable space and let A satisfy (5.1). Then A is amenable if and only if all the algebras A_x are semisimple and $\mathrm{K}(A) < \infty$. If these conditions hold, there is a bounded Borel section h of $\mathscr{M}$, continuous outside the scalar fibres, such that*

$$\det h(x) = 1, \quad \mathrm{K}(A)^{-1}1_x \leqslant h(x) \leqslant \mathrm{K}(A)1_x, \tag{5.2}$$

and $a \mapsto h^{\frac{1}{2}} a h^{-\frac{1}{2}}$ is a $\mathrm{C}(X)$-linear algebra isomorphism of A onto a nuclear C-subalgebra of $\Gamma(X, \mathscr{M})$. The conclusions (1.6), (1.7) and (1.8) remain valid.*

Proof : Suppose first that all fibres are semisimple and put $K \stackrel{\text{def}}{=} \mathrm{K}(A) < \infty$. The fibrewise calculations of Section 3 are invariant under $*$-automorphisms. They therefore define, on the relatively closed set F of two-dimensional fibres inside the nonscalar locus U, a continuous section $v = g - 1$ of the real vector bundle of traceless selfadjoint elements of $\mathscr{M}$. Its norm is at most $r = \frac{K^2-1}{K^2+1}$.

We describe the extension step. Cover U by trivializing open sets and choose a locally finite refinement (U_i) with a subordinate continuous partition of unity (φ_i). Such a refinement exists because U is metrizable. In a trivialization over U_i, the section v on the relatively closed set $F \cap U_i$ is a continuous map into the closed norm ball of radius r in the traceless selfadjoint matrices. Dugundji's theorem extends it to that ball on all of U_i; if $F \cap U_i$ is empty, take the zero section. Let v_i denote the resulting local section. The sum

$$\widetilde{v} = \sum_i \varphi_i v_i$$

is well defined and continuous on U, since the sum is locally finite and each term is extended by zero outside the support of φ_i. It agrees with v on F and has norm at most r. Trace zero and selfadjointness are preserved by the transition maps and by real convex combinations.

Define $g = 1 + \widetilde{v}$ on U, normalize $h = \frac{g}{\sqrt{\det g}}$ there, and set $h = 1$ on the scalar locus. The spectral calculation proving (1.4) is fibrewise, so (5.2) follows. The continuity argument at a scalar fibre is identical to (4.5) in any local trivialization. The bundle version of Lemma 2.2 then gives $h^{-1}a^*h \in A$ for any $a \in A$. It also proves that $h^{\frac{1}{2}}ah^{-\frac{1}{2}}$ is a continuous section.

The image is 2-subhomogeneous and nuclear. Indeed, Lemma 2.4 applies to the bundle: its proof uses only central scalar functions, cutoff functions and evaluation algebras. Transporting a norm-one approximate diagonal from this image gives $\mathrm{AM}(A) \leqslant K^2$. Conversely, if A is amenable, each fibre is amenable by evaluation, hence semisimple by Lemma 3.1. Proposition 3.2 then gives $\mathrm{K}(A) < \infty$ and the lower bound in (1.7). Finally, continuous scalar functions separate the points of X, so the proof that any positive spatial metric in the full evaluation representation is block diagonal is unchanged. The fibrewise optimal constants are given by Proposition 3.2, which proves $s_X(A) = \mathrm{K}(A)$. ∎

Remark 5.2 The bundle theorem does not assume that the fibres A_x form a locally trivial subbundle. Their dimensions may change, and their two primitive idempotents on F need not admit a global ordering. The symmetric formula $p^*p + (1-p)^*(1-p)$ is the reason that no such ordering is needed. The metrizable hypothesis is used for extension and partitions of unity on the open nonscalar locus; no extension theorem on an arbitrary nonnormal open subspace is being invoked.

6 Examples, optimality and continuity of the metric

Example 6.1 For $t \geqslant 0$, let $p_t \stackrel{\text{def}}{=} \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$ and $A_t \stackrel{\text{def}}{=} \mathbb{C}p_t + \mathbb{C}(\mathrm{I}_2 - p_t)$. Take X to be a one-point space. Proposition 3.2 gives

$$s_X(A_t) = \sqrt{1+t^2} + t, \quad \mathrm{AM}(A_t) = \frac{1 + (\sqrt{1+t^2}+t)^2}{2}.$$

Hence equality holds in (1.8). Conversely, any $K \geqslant 1$ is obtained by taking $t = \frac{1}{2}(K - K^{-1})$.

Example 6.2 There need not be a continuous matrix-valued similarity, even for a noncommutative algebra covered by Theorem 1.1. Let $X = \{0\} \cup \{\frac{1}{n} : n \geqslant 1\}$ and fix $t > 0$. Put $B_0 \stackrel{\text{def}}{=} \mathbb{C}p_0 + \mathbb{C}(\mathrm{I}_2 - p_0)$ and $B_t \stackrel{\text{def}}{=} \mathbb{C}p_t + \mathbb{C}(\mathrm{I}_2 - p_t)$, with p_t as in Example 6.1. Define

$$B_n = \begin{cases} \mathrm{M}_2, & n \equiv 0 \pmod 3, \\ B_0, & n \equiv 1 \pmod 3, \\ B_t, & n \equiv 2 \pmod 3, \end{cases}$$

and let A consist of the continuous sections a with $a(0) \in \mathbb{C}\mathrm{I}_2$ and $a(\frac{1}{n}) \in B_n$. This is a norm-closed algebra containing $\mathrm{C}(X)\mathrm{I}_2$. Its evaluation algebras are the prescribed ones: a section supported at an isolated point realizes any element of B_n. The algebra is noncommutative because some fibres are M_2, and Theorem 1.1 shows that it is amenable with $s_X(A) = \sqrt{1+t^2} + t$.

Suppose that an invertible $s \in \mathrm{C}(X, \mathrm{M}_2)$ satisfies $sAs^{-1} = (sAs^{-1})^\dagger$. Put $k(x) \stackrel{\text{def}}{=} s(x)^* s(x)$. Evaluation shows that $s(\frac{1}{n})B_n s(\frac{1}{n})^{-1}$ is selfadjoint. On the two-dimensional fibres, this implies

$$k(\tfrac{1}{n})p_0 = p_0^* k(\tfrac{1}{n}) \quad (n \equiv 1 \pmod 3), \qquad k(\tfrac{1}{n})p_t = p_t^* k(\tfrac{1}{n}) \quad (n \equiv 2 \pmod 3).$$

Continuity at 0 forces both relations for $k(0)$. The first makes $k(0)$ diagonal, say $k(0) = \mathrm{diag}(a, b)$ with $a, b > 0$. The second then implies $at = 0$, which is impossible. Thus no such continuous s exists.

Proposition 6.3 *Under the hypotheses of Theorem 1.1 or Theorem 5.1, put $K \stackrel{\text{def}}{=} \mathrm{K}(A)$. The metric h can be approximated by continuous positive invertible sections h_n such that*

$$K^{-1}1 \leqslant h_n \leqslant K1, \tag{6.1}$$

$\pi_X(h_n) \to \bigoplus_x h(x)$ strongly, and, for any $a \in A$,

$$h_n^{-1} a^* h_n \to J(a) \quad \text{in the norm of the ambient section algebra.} \tag{6.2}$$

The individual sections on the left of (6.2) are not asserted to belong to A.

Proof : If the scalar locus Z is empty, take $h_n = h$. Otherwise, fix a compatible metric on X and choose numbers $\delta_n \downarrow 0$ and functions $\chi_n \in \mathrm{C}(X, [0,1])$ which vanish when $\mathrm{dist}(x, Z) \leqslant \frac{1}{2}\delta_n$ and equal 1 when $\mathrm{dist}(x, Z) \geqslant \delta_n$. Put $h_n \stackrel{\text{def}}{=} 1 + \chi_n(h - 1)$. The section h_n equals 1 near Z and is continuous elsewhere, so it is continuous on X. The bounds (6.1) follow by convexity of the order interval.

For $a \in A$, let $a^\circ \stackrel{\text{def}}{=} a - \frac{1}{2}\,\mathrm{Tr}(a)1$. This is a continuous section which vanishes on Z. Outside the δ_n-neighbourhood of Z, the two metrics coincide. On that neighbourhood, scalar terms are unchanged by either conjugation, and both conjugations have norm at most K^2. Consequently,

$$\left\| h_n^{-1} a^* h_n - J(a) \right\| \leqslant 2K^2 \sup_{\mathrm{dist}(x,Z) < \delta_n} \|a^\circ(x)\| \to 0.$$

The last limit follows from continuity and compactness: a sequence of points approaching Z has cluster points in Z, where a° vanishes. Finally, $h_n(x) \to h(x)$ at every point and the sections are uniformly bounded. This proves strong convergence in the direct sum representation, first on vectors with finite support and then on all vectors by density. ∎

Example 6.4 Amenable fibres alone do not imply $\mathrm{K}(A) < \infty$. On the same compact sequence space, let $t_n \to \infty$, and prescribe the fibre $\mathbb{C}p_{t_n} + \mathbb{C}(\mathrm{I}_2 - p_{t_n})$ at $\frac{1}{n}$ and the scalar fibre at 0. The resulting closed $\mathrm{C}(X)$-subalgebra has semisimple fibres everywhere, but $\mathrm{K}(A) = \infty$. It is not amenable, since the contractive evaluations and (3.2) would otherwise give a uniform bound on $\frac{1}{2}(1+\|2p_{t_n} - \mathrm{I}_2\|^2)$. Thus the boundedness condition in Theorem 1.1 is necessary independently of semisimplicity.

Remark 6.5 The central module assumption cannot be omitted from the localization argument. Let $T \stackrel{\text{def}}{=} \mathrm{diag}(2,1)$ and

$$A_T \stackrel{\text{def}}{=} \{(a, TaT^{-1}) : a \in \mathrm{M}_2\} \subset \mathrm{M}_2 \oplus \mathrm{M}_2.$$

Both evaluation algebras are M_2, but A_T is not selfadjoint: it contains $(e_{12}, 2e_{12})$ and does not contain $(e_{21}, 2e_{21})$. It is nevertheless similar to a C^*-algebra, by the block diagonal similarity with blocks I_2 and T^{-1}. Thus selfadjoint evaluation algebras do not by themselves imply selfadjointness of the section algebra. The proof of Lemma 2.2 specifically requires multiplication by scalar partitions of unity.

Remark 6.6 The two-by-two assumption is used in the extension argument, not only in the computation of a constant. Above the nonscalar locus, the only proper semisimple fibres are the two-dimensional commutative ones, and the remaining fibres are full matrix algebras, on which every positive metric is admissible. For M_d with $d \geqslant 3$, several proper noncommutative fibre types can occur. Extending a metric through those fibres does not automatically preserve their adjoint structure. The present argument does not establish the corresponding general statement in higher matrix size.

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