hexagon:2611.00527v2

IMPROVING RAZMYSLOV'S (1974) BOUND: THE NAGATA–HIGMAN INDEX IS AT MOST n^2-2 FOR $n \geq 3$, AND OTHER MISCELLANEA

DAVE GOEL

First version: October 27, 2026. This version (1.96): November 24, 2026.

ABSTRACT. Let $d(n)$ be the least m such that every associative algebra over a field of characteristic zero satisfying $x^n = 0$ also satisfies $x_1 x_2 \cdots x_m = 0$ (the Nagata–Higman index). Razmyslov proved $d(n) \leq n^2$ in 1974, and the Kuzmin (1975) conjecture is $d(n) = n(n+1)/2$. We prove $d(n) \leq n^2 - 1$ for $n \geq 2$ and $d(n) \leq n^2 - 2$ for $n \geq 3$; in particular $d(6) \leq 34$.[1] The proof uses Formanek's criterion in cycle form and studies, one isotypic block of S_N at a time, the span of the N-cycles inside the block: a block with two corners meets this span exactly when its shape is a self-conjugate hook, and for every $k \geq 2$ the block of $(k+1, 2, 1^{k-1})$, which has three corners, meets it too. The next step, $d(n) \leq n^2 - 3$, reduces to one statement about the block of $(n^{n-2}, n-1, n-2)$; for every $n \geq 4$ we prove the part of it that concerns the constituents with at most two rows of length 3, and for $n = 5$ the whole statement (computer-assisted). Through the hook theorem the question becomes one about the polynomial identities of Kemer's superalgebras $M_{k,l}$ over the Grassmann algebra: lower bounds for their PI degrees give upper bounds for $d(n)$, and the lower half of the Samoilov (2010) conjecture on these PI degrees would give $d(n) < \frac{8}{9}n^2$ for $n \geq 3$.

1. INTRODUCTION

Let $d(n)$ be the Nagata–Higman index of nil algebras of index n in characteristic zero (defined in Subsection 1.1). Below, $\lambda \vdash N$, C_N is the span of the N-cycles in $F[S_N]$, I_λ is the isotypic block of λ and $U_\lambda := C_N \cap I_\lambda$; PIdeg is the least degree of a polynomial identity, here of Kemer's superalgebras $M_{k,l}(\Lambda)$ over the Grassmann algebra Λ (Section 8). Our main results, each with the best result before this paper:

New result	**Best result before this paper**
$d(n) \leq n^2 - 2$ for every $n \geq 3$ (Theorems A, B)	$d(n) \leq n^2$ (Razmyslov 1974 [Ra74]), still called the best known upper bound in 2022 [Ro22]
$d(6) \leq 34$	$d(6) \leq 36$ [Ra74]; Higman's recursion gives 43 [Hi56], from $d(5) \leq 21$
The images of the star elements, the conjugates of the Jucys–Murphy element J_N, generate $\mathrm{End}(S^\lambda)$ for every $\lambda \vdash N$ with at least two removable corners (Theorem C)	None known to us; $J_1, \ldots, J_N$ generate only the commutative Gelfand–Tsetlin algebra [OV96]
A block with two corners meets the span C_N of the N-cycles if and only if λ is a self-conjugate hook (Theorem D)	No classification known to us; the self-conjugate hooks meet C_N (standard polynomial; cf. Kostant [Ko58]), and $(3,3,2)$ does not, as $M_{1,2}(\Lambda)$ has no identity of degree 8 (Vishne [Vi02]; Lemma 8.2)

2020 *Mathematics Subject Classification.* Primary 16R10; Secondary 16N40, 20C30.

Key words and phrases. Nagata–Higman theorem, nil algebras, nilpotency index, Kuzmin's conjecture, polynomial identities, symmetric group.

[1] This is an expanded version of the following focused version [Go26]: https://hexagonmath.org/2611.00526

New result (continued)	**Best result before this paper**
The blocks $(k+1, 2, 1^{k-1})$, $k \geq 2$, with three corners, meet C_N (Theorem E; one finite computation)	For $k = 2$, $(3, 2, 1)$: classical ($[[x_1, x_2]^2, x_1]$ is Hall's identity of $M_2(F)$ and one of Popov's identities of $M_{1,1}(\Lambda)$ [Po82]; Lemma 8.2); no infinite family known to us
Every staircase block $\mathrm{St}_m = (m, m-1, \ldots, 1)$, $m \geq 2$, meets C_N; equivalently $\mathrm{PIdeg}\big(\bigoplus_{k+l=m-1} M_{k,l}(\Lambda)\big) = m(m+1)/2 - 1$ (Theorem 8.15)	Kuzmin's element behind $d(n) \geq n(n+1)/2$ [Ku75], not known to lie in a single block; for $m \leq 3$ classical (Amitsur–Levitzki [AL50]; Hall and Popov [Po82])
$\mathrm{PIdeg}(M_{k,l}(\Lambda)) \geq (k+1)(2l-k+1)$ for $1 \leq k \leq l$ (Proposition 8.5(d))	$(k+1)(l+1) - 1$, from the hook theorem [BR87]
The Samoilov (2010) conjecture holds for $M_{2,2}(\Lambda)$: $\mathrm{PIdeg}(M_{2,2}(\Lambda)) = 14$ (Theorem 11.11, computer-assisted)	$9 \leq \mathrm{PIdeg}(M_{2,2}(\Lambda)) \leq 14$: 9 because the subalgebra $M_{1,2}(\Lambda)$ has no identity of degree 8 (Vishne [Vi02]), 14 by Samoilov [Sa10]; exact values were known for $M_{1,1}(\Lambda)$ (5, Popov [Po82]) and $M_{1,2}(\Lambda)$ (9, Vishne [Vi02, Vi11])
For $n \geq 3$: $d(n) \leq n^2 - 3$ follows from one block statement, $U_{B(n,2)} = 0$ with $B(n,2) = (n^{n-2}, n-1, n-2)$, true for $n = 4, 5$ (through $d(4) = 10$ [VL93]); $d(n) \leq n^2 - k - 1$, $k \leq 5$, from finitely many (Theorems F, G)	Formanek's criterion [Fo90, Thm. 5] involves all blocks of the box jointly; no reduction to single blocks known to us
On $B(n,2)$, $n \geq 4$, $N = n^2 - 3$: the kernel $K_{B(n,2)}$ (Subsection 9.9) has no conjugation type with at most two columns, and every type of depth at most $n-1$ or with at most two rows of length 3 is $(3, 1^{N-3})$, once (by hand up to finite exact computations); partial results on the deeper types (Section 9)	None; the block and its kernel are introduced here
If the lower half of the Samoilov (2010) conjecture holds for $M_{a,b}(\Lambda)$, $b - a \leq 1$, then $d(n) < \frac{8}{9}n^2$ for $n \geq 3$ (Theorem H)	No link between these PI degrees and $d(n)$ known to us

In more detail, each with its place in the literature:

- $d(n) \leq n^2 - 2$ for every $n \geq 3$ (Theorems A and B); to our knowledge, the first improvement for general n of Razmyslov's bound $d(n) \leq n^2$ of 1974 [Ra74]. In particular $d(6) \leq 34$.
- The images of the star elements, the conjugates of the Jucys–Murphy element J_N [Ju74, Mu81], generate $\mathrm{End}(S^\lambda)$ whenever λ has at least two removable corners (Theorem C).
- A block with two corners meets the span of the N-cycles only for the self-conjugate hooks, and there in the image of the standard polynomial (Theorem D and Lemma 5.3; cf. Kostant [Ko58]); an infinite family of blocks with three corners meets it (Theorem E, through the Amitsur–Levitzki theorem [AL50] and one finite computation).
- Every staircase block meets the span of the N-cycles (Theorem 8.15): the element behind Kuzmin's lower bound $d(n) \geq n(n+1)/2$ [Ku75] lies in a single block.
- Steps toward the Kuzmin (1975) conjecture $d(n) = n(n+1)/2$ [Ku75]: for $n \geq 3$, $d(n) \leq n^2 - 3$ follows from the single statement $U_{B(n,2)} = 0$ (Theorem F), which holds for $n = 4, 5$ (through $d(4) = 10$ [VL93]); and $d(n) \leq n^2 - k - 1$, $k \leq 5$, follows from finitely many block statements (Theorem G).
- On the first open block $B(n,2)$, for every $n \geq 4$: its kernel has no conjugation type with at most two columns, and every type of depth at most $n-1$ or with at most two rows of length 3 is $(3, 1^{N-3})$, and it occurs once (Section 9; by hand up to finite exact computations). In its layers $b = 3$ ($a \leq (n^2-3)/2$) and $b = 4$ ($4 \leq a \leq (n^2-4)/2$), no eigenvector of the pencil at $\kappa = n$ has y-order 1 (Theorem 9.221, Corollary 9.228; finitely many cases computed); no earlier result of this kind is known to us. At $b = 3$ and $\kappa = n$, no φ_f with $f \in \mathrm{K}_n(n)_\tau$, $\tau = (n^2 - a - 3, a, 3)$, $3 \leq a \leq (n^2-3)/2$, has y-order 0 either (Theorem 9.233); with Theorem 9.201, $\mathrm{K}_n(n)_\tau = 0$ on the whole layer $b = 3$ (Corollary 9.258; new in type form for $n \geq 7$ and $a \geq n-1$, and for $n = 4, 5, 6$ a second proof of results computed here).
- A transfer from the polynomial identities of Kemer's superalgebras [Ke91]: the lower half of the Samoilov (2010) conjecture [Sa10] would give $d(n) < \frac{8}{9}n^2$ (Theorem H); and Samoilov's conjectured value $\mathrm{PIdeg}(M_{2,2}(\Lambda)) = 14$ holds (Theorem 11.11, computer-assisted).

1.1. **The Nagata–Higman theorem.** Let F be a field of characteristic zero. Dubnov and Ivanov [DI43], Nagata [Na52] and Higman [Hi56] proved that a nil algebra of bounded index is nilpotent: for every n there is a number m such that every associative (not necessarily unital) F-algebra satisfying $x^n = 0$ for all x also satisfies $x_1x_2\cdots x_m = 0$ for all $x_1, \dots, x_m$. The theorem is also called the Dubnov–Ivanov–Nagata–Higman theorem [Dr95, §5]; Higman's proof gives $m = 2^n - 1$. The least such m is the *Nagata–Higman index* $d(n)$. Equivalently, $d(n)$ is the least m such that $x_1\cdots x_m$ lies in the T-ideal of the free algebra generated by x^n [Ro22, §1.3]; this is the form we use (Section 2).

1.2. **Bounds and known values.** Razmyslov [Ra74, final remark] proved $d(n) \le n^2$, using the trace identities of the matrix algebra $M_n(F)$ and the Cayley–Hamilton theorem. Procesi [Pr76] and Formanek [Fo86, Fo87, Fo91] connected $d(n)$ with the degrees of generators of the invariants of several $n \times n$ matrices; see [Dr95, Thm. 5.6]. In the other direction, Kuzmin [Ku75] proved $d(n) \ge n(n+1)/2$, by the word $x_2x_1x_2x_1^2\cdots x_2x_1^{n-1}$ (of degree $n(n+1)/2-1$, which does not lie in the T-ideal of x^n) (see [Do18, Thm. 1.7]), and Kuzmin conjectured that $d(n) = n(n+1)/2$.

The exact values $d(1) = 1$, $d(2) = 3$ and $d(3) = 6$ are classical (Dubnov [Du35], see [Dr95]; for $n = 3$ also Higman [Hi56]), and $d(4) = 10$ was computed by Vaughan-Lee [VL93] with a computer. An independent computation (ranks of the relevant matrices modulo two primes; Appendix A) confirms $d(4) \le 10$, and so $d(4) = 10$ by Kuzmin's bound. Higman's argument gives $d(n) \le 2d(n-1)+1$ [Hi56] (the same recursion, for characteristic $p > n$, appears in [Lo12, proof of Lemma 4.4]); with $d(4) = 10$ this gives $d(5) \le 21$. For $n = 5$, Shestakov and Zhukavets [SZ04], using Vaughan-Lee's reduction [VL98] to superalgebras with at most 5 generators, confirmed Kuzmin's conjecture for 2-generator superalgebras satisfying $x^5 = 0$ (they are nilpotent of degree 15).

To our knowledge, in characteristic zero the bound n^2 has not been improved since 1974 for general n. Drensky [Dr95] gives $n(n+1)/2 \le d(n) \le n^2$ as "the best known bounds", Romano [Ro22, §1.3] writes that n^2 "to this day remains the best known upper bound", and Procesi [Pr26, Thm. 5.14], stating Razmyslov's degree bound n^2 for the matrix invariants, calls it "at the moment the best result". (For positive characteristic and for finitely generated nil algebras see Klein [Kl00] and Domokos [Do18].)

1.3. **Results.**

Theorem A. *For every $n \ge 2$, $d(n) \le n^2 - 1$.*

Theorem B. *For every $n \ge 3$, $d(n) \le n^2 - 2$.*

Theorems A and B improve Razmyslov's bound $d(n) \le n^2$ of 1974 [Ra74], which, to our knowledge, had not been improved for general n (Subsection 1.2). Once Formanek's criterion is available, the proof of Theorem A is short (Lemma 3.1), but we have not found Theorem A in print. With Kuzmin's bound, $n(n+1)/2 \le d(n) \le n^2 - 2$ for $n \ge 3$. Since $d(n)$ is known for $n \le 4$, and $d(5) \le 21$ (Subsection 1.2), the new values begin at $n = 6$: $d(6) \le 34$ (Higman's recursion gives $d(6) \le 43$).

The remaining results concern single isotypic blocks. For $\lambda \vdash N$ let $I_\lambda \subseteq F[S_N]$ be the isotypic block of λ, let C_N be the span of the N-cycles, and put $U_\lambda := C_N \cap I_\lambda$. Let K_λ be the set of elements of I_λ whose coefficient vanishes on every permutation with a fixed point; $K_\lambda \supseteq U_\lambda$ for $N \ge 2$. The *star elements* are $X^{(j)} := \sum_{i \ne j}(i\,j) \in F[S_N]$, $1 \le j \le N$.

Theorem C. *Let $\lambda \vdash N$ have at least two removable corners. Then the operators $\rho_\lambda(X^{(j)})$, $1 \le j \le N$, generate* $\mathrm{End}(S^\lambda)$ *as a unital algebra; equivalently, only the scalars commute with all of them. (If λ is a rectangle, every $\rho_\lambda(X^{(j)})$ is a scalar.)*

The star element $X^{(N)}$ is the Jucys–Murphy element $J_N = \sum_{i<N}(i\,N)$ [Ju74, Mu81], and each $X^{(j)}$ is conjugate to it; the Jucys–Murphy elements $J_1, \dots, J_N$ together generate only a commutative subalgebra of $F[S_N]$, the Gelfand–Tsetlin algebra [OV96].

Theorem D. *Let λ have exactly two removable corners. Then $U_\lambda \neq 0$ if and only if λ is a self-conjugate hook $(k, 1^{k-1})$, $k \geq 2$; in that case U_λ is one-dimensional, spanned by an explicit element f_Γ (Section 5). In particular, for $n \geq 3$ and $1 \leq j \leq n-2$, $U_\lambda = 0$ for $\lambda = (n^{n-j}, (n-j)^j)$, the $n \times n$ box minus a $j \times j$ square.*

For the hook $(k, 1^{k-1})$ the element f_Γ is, up to a nonzero scalar, $\Phi(s_{2k-2})$: the image of the standard polynomial s_{2k-2} under the bijection Φ from words onto N-cycles of Section 2 (Lemma 5.3; cf. Kostant [Ko58]). We also record which standard polynomials lie in the T-ideal of x^n: for $n \geq 2$, $s_m \in (x^n)^T$ if and only if $m \geq 2n-1$ (Proposition 5.4). We make no novelty claim here. That $s_{2n} \in (x^n)^T$ (and hence $s_m \in (x^n)^T$ for $m \geq 2n$) follows from the Amitsur–Levitzki theorem [AL50] and Procesi's theorem that the T-ideal of x^n contains every identity of $M_n(F)$ [Pr76, Cor. 4.8] (see also [Fo90]), and also directly from the Grassmann trick of Rosset's proof of the Amitsur–Levitzki theorem [Ros76]; the sharp case $m = 2n-1$ uses a sign test.

Beyond two corners we have one infinite family:

Theorem E. *For every $k \geq 2$, let $A_k := (k+1, 2, 1^{k-1})$, a self-conjugate partition of $N = 2k+2$ with three removable corners. Then $U_{A_k} \neq 0$; an explicit nonzero element is built from the standard polynomial s_{2k} with one extra letter inserted after an even number of letters (Theorem 5.12).*

It goes one corner beyond the self-conjugate hooks of Theorem D; to our knowledge, no earlier block with three corners was known to meet C_N.

The proof combines the Amitsur–Levitzki theorem, the supertrace identities of the matrix superalgebra $M_{1,1}(\Lambda)$ over the Grassmann algebra Λ, and a finite exact computation (1024 cases; step (4) of the proof of Theorem 5.12).

Theorem F (conditional)**.** *Let $n \geq 3$ and $B(n,2) := (n^{n-2}, n-1, n-2)$. If $U_{B(n,2)} = 0$, then $d(n) \leq n^2 - 3$. The hypothesis fails for $n = 3$, holds for $n = 4$ (by $d(4) = 10$ [VL93] and Proposition 2.1) and for $n = 5$ (by $d(5) \leq 21$, Subsection 1.2, and Proposition 2.1), and is open for $n \geq 6$.*

Its conclusion would be one more step from Razmyslov's bound n^2 [Ra74] toward Kuzmin's conjectured value $n(n+1)/2$ [Ku75].

Theorem G. *Let $n \geq 3$ and $k \leq 5$. The bound $d(n) \leq n^2 - k - 1$ follows from a finite, explicit list of statements about the blocks of finitely many shapes obtained from the $n \times n$ box by removing a small partition at its corner: kernel statements, and, where a shape has three parents, a level-two condition. For $k = 3$ and $n \geq 4$, all but three statements of the list are proved; the three that remain are each about one block or one conjugate pair of blocks; they hold for $n = 4$ (by $d(4) = 10$ [VL93]); for $n = 5$ the first of them (the hypothesis of Theorem F) holds and the other two are open; all three are open for $n \geq 6$.*

Like Theorem F, it rests on Formanek's criterion [Fo90, Thm. 5] (Proposition 2.1).

The same reduction holds for every k, with conditions of higher level where a shape has more than three parents, under a side condition that holds for $k \leq 12$ exactly when $k < n(n-1)$ (Corollary 6.8, Remarks 6.9 and 6.10).

For the kernels themselves we give a uniform tool (Section 7): for every n and every partition $\nu \neq (n^n)$ inside the $n \times n$ square, with $\lambda(\nu)$ the box with ν removed at its corner (a partition of $N = n^2 - |\nu|$), each irreducible constituent of $K_{\lambda(\nu)}$ under conjugation by S_N has at most $|\nu|$ columns, and the joint kernel of all box shapes of one size is determined exactly by Kronecker coefficients of the box (Theorems 7.5 and 7.7). This gives no new bound by itself.

Finally (Section 8) we translate the block problem into polynomial identities of Kemer's superalgebras $M_{k,l}(\Lambda)$ [Ke91]: an element of C_N lies in the block I_λ exactly when its polynomial is an identity of each $M_{r-1,c-1}(\Lambda)$, one for each corner (r,c) of λ (Lemma 8.2). In this language the results above give, among other things, the lower bound $\mathrm{PIdeg}(M_{k,l}(\Lambda)) \geq (k+1)(2l-k+1)$ for

$1 \le k \le l$. Conversely, lower bounds for these PI degrees give upper bounds for $d(n)$. Samoilov [Sa10] proved that $\mathrm{PIdeg}(M_{a,b}(\Lambda)) \le (a+1)(2b+1)-1$ for $a \le b$ and conjectured that this is sharp; the open lower half of Samoilov's conjecture is the input that this route needs:

Theorem H (conditional)**.** *Let $n \ge 2$, $1 \le a \le b \le n-1$ and $S := (n-a)(n-b)$. If $\mathrm{PIdeg}(M_{a,b}(\Lambda)) \ge n^2 - S + 1$, then $d(n) \le n^2 - S$. In particular:*

(a) if $\mathrm{PIdeg}(M_{a,a}(\Lambda)) \ge \gamma a^2$ for all large a, with a constant $\gamma > 1$, then $d(n) \le \big(1 - \big(\frac{\gamma-1}{\gamma+1}\big)^2 + o(1)\big)n^2$;

(b) if the lower half of Samoilov's conjecture holds for the algebras $M_{a,b}(\Lambda)$ with $b - a \le 1$, then for every $n \ge 3$, $d(n) \le n^2 - \lfloor (n+2)/3 \rfloor \lfloor (n+4)/3 \rfloor$, which is less than $\frac{8}{9}n^2$.

We prove $\mathrm{PIdeg}(M_{a,a}(\Lambda)) \ge (a+1)^2$ (Proposition 8.5(d); its order $(a+1)^2 - 1$ is classical, by the hook theorem, and (d) adds 1); this is the constant $\gamma = 1$ and gives only Theorem A. At $a = 2$ a computer-assisted computation gives Samoilov's value $\mathrm{PIdeg}(M_{2,2}(\Lambda)) = 14$ (Theorem 11.11), so the Samoilov (2010) conjecture holds for $a = b = 2$; it gives no new bound on $d(n)$. Samoilov's theorem gives $\gamma \le 2$, and Samoilov's conjecture is $\gamma = 2$. So any constant $\gamma > 1$ in a lower bound would give a saving quadratic in n. By Samoilov's theorem and Lemma 8.12, for $n \ge 3$ no single algebra $M_{a,b}(\Lambda)$ gives more than (b) through Theorem H, and no argument through one shape can give more (Corollary 8.17). The last statement rests on the staircases $\mathrm{St}_m = (m, m-1, \dots, 1)$: every staircase block meets the span of the N-cycles, that is, $U_{\mathrm{St}_m} \ne 0$ for every $m \ge 2$, or equivalently $\mathrm{PIdeg}\big(\bigoplus_{k+l=m-1} M_{k,l}(\Lambda)\big) = m(m+1)/2 - 1$ (Theorem 8.15). The element we use is the one behind Kuzmin's lower bound [Ku75]; what is new, as far as we know, is that it lies in a single block.

For the first open block $B(n,2)$, $n \ge 4$ (Section 9), and for the block $\lambda(3,1,1)$ of statement (iii) in the proof of Theorem G (Section 10) we prove the partial results listed below. For $B(n,2)$ only the types with three columns, $\kappa_3' \ge 3$ and depth at least n remain open, and only for $n \ge 6$ (for $n = 4$ and $n = 5$, $U_{B(n,2)} = 0$ holds, see Theorem F; for $n = 5$ the whole kernel is computed, with a computer-assisted step, Theorem 9.27).

What is proved, what is computer-assisted, and what is open. For the reader's convenience we list the status of the main statements. "By hand" means that no step uses a computer; the finite exact computations used by items of (a) are named there.

(a) *Proved by hand.* $d(n) \le n^2 - 1$ for $n \ge 2$ and $d(n) \le n^2 - 2$ for $n \ge 3$ (Theorems A and B); the star elements (Theorem C); the blocks with two corners (Theorem D); the staircase blocks (Theorem 8.15); the reductions of Theorems F and G and Corollary 6.8; the Casimir criterion (Theorems 7.5 and 7.7); the PI dictionary and the transfer (Lemma 8.2, Theorem H, Proposition 8.5(a), (d)). For the first open block $B(n,2)$ and every $n \ge 4$: $K_{B(n,2)}$ has no conjugation type with at most two columns (Corollary 9.42(b)), and every type of depth at most $n-1$ (Theorem 9.24) or with $\kappa_3' \le 2$, that is, with at most two rows of length 3 (Corollaries 9.174 and 9.189, and Corollary 9.258 for $\kappa_3' = 2$), is $(3, 1^{N-3})$, and it occurs once; Theorem 9.24, and through it Corollary 9.189, also use the small exact computations of step (e) in the proof of Theorem 9.22 (three matrices, of sizes 9, 44, 265) and of Theorem 9.9(a) (one rank, on 120 unknowns). Corollary 9.258 uses the finite exact computations named in Remark 9.259(2): the certificate of Lemma 9.237 (over $\mathbb{Q}(x,\kappa)$), the translation equivariance in Lemma 9.243, the base levels of Theorem 9.254, Lemma 9.248, (A3) and the case $(4,6)$ in Part II, the three cases of Remark 9.225(2), and, for $n = 4$, one exact rank computation. The finite-n form of Theorem 9.22 holds for every n at $b = 2$ ($a \ge 3$, $n \ge 3$; Corollary 9.81) and at $b = 3$ ($n \ge 4$; Corollary 9.91); at every other layer b with $3b \le n^2$ and every $n \ge 3$ it is reduced to the part of degree 0 in y of its kernel vectors (Corollary 9.197). Every eigenvector of the pencil has y-order at most 2 (Theorem 9.201), and y-order 2 does not occur for $4 \le b \le n+2$,

nor for $b = n + 3$ with $n \geq 5$ (Corollary 9.209). In every layer $b \geq 3$, the y-Hessian of an eigenvector of y-order 2 has only monomials of full support (Corollary 9.212). For $n \geq 6$, y-order 2 does not occur at $b = n + 4$ either (Corollary 9.217). At $b = 3$ and $\kappa = n$, y-order 1 does not occur for $a \leq (n^2 - 3)/2$ (Theorem 9.221; by hand except the three finite cases $(n, a) = (4, 5), (4, 6), (5, 11)$, Remark 9.225(2)). At $b = 4$ and $\kappa = n$, y-order 1 does not occur for any $n \geq 4$, so every eigenvector has y-order 0 (Corollary 9.228; by hand for $n \geq 5$, and one finite computation for $n = 4$). For the block $\lambda(3, 1)$ of statement (ii) in the proof of Theorem G ($k = 3$): six weights of its first layer for $n \geq 6$ (Theorem 10.28), and all its two-row types, at every depth, for $n \geq 3$ (Corollary 10.51).

(b) *Computer-assisted* (each step is an exact computation, described where it is used). The family of Theorem E (one finite check of 1024 cases), and through it Proposition 8.5(b), (c); the hypothesis of Theorem F for $n = 4, 5$ (through $d(4) = 10$ [VL93]); the independent check $d(4) \leq 10$ (Appendix A); the whole kernel $K_{B(5,2)}$, a single copy of $(3, 1^{19})$ (Theorem 9.27); at $n = 6$, 75 of the 108 possible conjugation types of $K_{B(6,2)}$, all with depth at most 12 or $\kappa'_3 \leq 4$: 74 do not occur, and $(3, 1^{30})$ occurs once (Theorem 11.9); at $n = 7$ and $n = 8$ some types with $\kappa'_3 \leq 1$, now contained in (a) (Corollaries 9.164 and 9.155); Theorem 9.22 at $b = 4$ for every $n \geq 5$ (one rank, Theorem 9.128); for $\lambda(3, 1, 1)$, statement (iii): no type of depth at most $n - 3$ for $n \geq 5$ (Theorem 10.15(b), through the computer-assisted Theorem 10.12) and no two-row type, at every depth, for $n \geq 3$ (Corollary 10.71); for $\lambda(3, 1)$: no three- or four-row type of depth at most 4, 5, 6 for $n \geq 7$, 8, 9 (Corollary 10.57); and $\mathrm{PIdeg}(M_{2,2}(\Lambda)) = 14$ (Theorem 11.11).

(c) *Open.* The types of $K_{B(n,2)}$ with $\kappa'_3 \geq 3$ and depth at least n, for $n \geq 6$ (Problem 11.1); so $d(n) \leq n^2 - 3$ for $n \geq 6$, and in particular $d(6) \leq 33$, is open; at $n = 6$, 33 types remain. Also open: the rest of statements (ii) and (iii) for $\lambda(3, 1)$ and $\lambda(3, 1, 1)$ (Problem 11.4) (with the hypothesis of Theorem F they give $d(n) \leq n^2 - 4$; Theorem G, $k = 3$), Conjecture 11.2 beyond the cases above, Conjecture 11.6, the lower half of the Samoilov (2010) conjecture (Problem 11.5(b)), and the Kuzmin (1975) conjecture itself.

1.4. **Method.** Our starting point is Formanek's criterion [Fo90, Thm. 5] (cf. [Ra74, final remark], [Pr76, Thm. 3.3], [St89]), which we use in the following cycle form (Proposition 2.1): $d(n) \leq m$ if and only if $C_{m+1} \cap B_n = 0$, where B_n is the sum of the blocks I_λ over the partitions $\lambda \vdash m + 1$ that fit in the $n \times n$ box. This is the nil analogue of the Helling–Razmyslov–Procesi description of the multilinear trace identities of matrix algebras [He74, Ra74, Pr76] (see [Dr95, Thm. 5.7]). The sign automorphism $\sigma \mapsto \mathrm{sgn}(\sigma)\sigma$ acts on C_N by a scalar and exchanges I_λ and $I_{\lambda'}$, so $U_\lambda = 0$ unless $\lambda = \lambda'$. On a block we restrict to S_{N-1}, where the Jucys–Murphy element J_N acts by the contents of the removable corners [Ju74, Mu81, OV96], and we use the star elements. The elements that survive are controlled by Frobenius' theorem on the characters of the alternating group [JK81, 2.5.13], the theorem that Kostant [Ko58] related to the Amitsur–Levitzki theorem [AL50]; for a self-conjugate hook it is the main input to Lemma 5.3. The deletion map of the decoupling step is the trace map $p \mapsto \mathrm{tr}(p)$ of the Razmyslov–Procesi correspondence between elements of $F[S_N]$ and multilinear trace polynomials (cf. the canonical projection of Kerov, Olshanski and Vershik [KOV93, KOV04]). In the background, the self-conjugate shapes in the $n \times n$ box are indexed by the subsets of $\{1, 3, \ldots, 2n - 1\}$ (Frobenius coordinates, [Ma95, I.1]), the degrees of the primitive classes of $\mathfrak{gl}_n$. A dual view is also useful: $K_\lambda = 0$ if and only if the images $\rho_\lambda(F[\mathrm{Stab}(j)])$, $1 \leq j \leq N$, of the point-stabilizer group algebras span $\mathrm{End}(S^\lambda)$ (for the trace form).

1.5. **Organization.** Section 2 fixes the setting and proves Formanek's criterion in cycle form. Section 3 proves Theorem A and Section 4 proves Theorem B. Section 5 treats single blocks and the star elements (Theorems C and D), the standard polynomials (Subsection 5.2), and the component reduction and the deletion map used later (Subsection 5.6). Section 6 restricts to S_{N-1} and proves Theorems F and G (the decoupling step), and Section 7 gives the Casimir criterion for the kernels

K_λ. Section 8 is the bridge to PI degrees and proves Theorem H. Section 9 studies the first open block $B(n,2)$: the part of each kernel $K_{\lambda(\nu)}$ near the sign representation is a finite problem that does not depend on n (Theorem 9.3), and for $B(n,2)$, $n \geq 4$, it is solved on every type of depth at most $n-1$ (Theorem 9.24), and for $n=5$ on every type (Theorem 9.27, computer-assisted); Subsection 9.11 relates it to rectangles, Subsection 9.12 solves the rectangle problem for $\tau_3 = 1$, and Subsection 9.13 for $\tau = (ns-5,3,2)$, $n \geq 4$; Subsection 9.14 shows that a nonzero invariant function of bidegree (a,b) killed by $\mathcal{E}$ lies in the m-th power of the ideal of the closure of $\{(X,Y) : X \text{ regular},\ Y \in \operatorname{Im}\operatorname{ad}_X\}$ only for $m \leq \lfloor a/4 \rfloor$; Subsection 9.15 applies the lowering operator of $\mathfrak{gl}_3$ to the line of Theorem 9.37 and finds the eigenvalue $(a+3)/n$ of the pencil in bidegree $(a,2)$ for all $a \geq 3$ and $n \geq a+2$; Subsection 9.16 shows that on the rectangle spaces either of the two tests alone cuts out the kernel, Subsection 9.17 proves the mirror symmetry $n \mapsto -n$ of the pencil at $b=2$, Subsection 9.18 proves Theorem 9.22 at $b=2$ and $a \geq 3$ for every $n \geq 3$; Subsection 9.19 shows that at $b=3$, for every $n \geq 4$, certain rows of $\mathcal{Z}$ at the points $(\operatorname{diag}(x), Z)$ with distinct x_p and Z off-diagonal (the slice rows) vanish on an element of $\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ exactly when it does not depend on Z at the points $(\operatorname{diag}(x), Z + \operatorname{diag}(y))$; with the rows at the points with $y \neq 0$, the subsection then proves Theorem 9.22 at $b=3$; Subsection 9.23 proves it at $b=4$ for every $n \geq 5$ (one step computer-assisted); and Subsection 9.20 shows that $\mu = 0$ occurs, and for $n \geq a+2$ with multiplicity at least the number of partitions of a into parts ≥ 2. Subsection 9.21 shows, by the linkage theorem of Bernstein, Gelfand and Gelfand, that an eigenvector in the $\mathfrak{gl}_3$-module generated by passing highest weight vectors of higher weight has $\mu \in \{0,3\}$ or comes from a smaller layer, and so reduces Conjecture 11.3, by induction on a, to one principle. Subsections 9.24 and 9.25 prove the conclusion of Conjecture 11.3 for every $a \geq 3$ and $n \geq a+6$ (Theorem 9.150); at $s=n$ this lies inside Theorem 9.24. Subsection 9.26 shows, with a computer, that $\kappa = n$ is not an eigenvalue at $b=2$ for odd a at 18 pairs (n,a) with $4 \leq n \leq 8$ and $n \leq a$; at $s=n$ these are types of depth $a \geq n$, beyond Theorem 9.24; Subsection 9.27 reduces the statement at $\kappa = n$, for every $n \geq 4$ and every dominant $a \geq n$, to one step of that proof. Subsection 9.28 shows, with a computer, that at $n=7$ the eigenvalue $\kappa = 7$ does not occur at $b=2$ for any $3 \leq a \leq 23$; so every conjugation type κ of $K_{B(7,2)}$ with $\kappa'_3 \leq 1$ is $(3,1^{43})$. Subsection 9.29 proves, by hand, at every n and for odd a, the one step to which Subsection 9.27 reduces the statement at $\kappa = n$; so $\kappa = n$ is not an eigenvalue at $b=2$ for every $n \geq 4$ and every odd a with $n \leq a \leq (n^2-2)/2$. Subsections 9.30 and 9.31 do the same for even a, by hand: there step (2) fails, and one more row, a triangle of order-one rows computed in closed form, takes its place; so, for every $n \geq 4$, every conjugation type κ of $K_{B(n,2)}$ with $\kappa'_3 \leq 1$ is $(3,1^{N-3})$. Section 10 extends the matrix-pair model to every self-conjugate ν (one test, built from a grid of rc rank-one matrices, for each outer corner (r,c) of ν with $rc \leq k$) and treats $\lambda(3,1,1)$; Subsection 10.6 proves three cases of the first layer for $\nu = (3,1,1)$ (Conjecture 10.18); Subsection 10.7 extends the corner dictionary to every ν (the tests are then the live corners of ν') and, for $n \geq 6$, excludes from $K_{\lambda(3,1)}$ the types of six weights of the first layer (Theorem 10.28), a first step on statement (ii) of Theorem G; Subsection 10.8 shows that the leading term is injective on the column $(\tau_2,1,1)$, although its pointwise kernel is not 0, and for $n \geq 4$ excludes the two-row types of depth $1,\dots,n-3$ (Corollary 10.36); Subsection 10.9 excludes all two-row types, at every depth and for every $n \geq 3$, using only the tests at diagonal points (Corollary 10.51); Subsection 10.10, by exact computer-assisted certificates, excludes the three- and four-row types of depth at most 4 for $n \geq 7$, at most 5 for $n \geq 8$ and at most 6 for $n \geq 9$ (Corollary 10.57); and Subsection 10.11 proves Conjecture 10.18 on the column $(\tau_2,1,1,1)$ and excludes from $K_{\lambda(3,1,1)}$ the two-row types, at every depth and for every $n \geq 3$ (Corollary 10.71; with exact symbolic identities checked by computer). Section 11 collects what is known for small n and the open problems, among them a conjecture on single blocks (Conjecture 11.6) that would give Kuzmin's conjecture block by block.

2. Setting and Formanek's criterion in cycle form

2.1. **Setting.** Throughout, F is a field of characteristic 0, and $F\langle X\rangle$ is the free *non-unitary* associative algebra on $X = \{x_1, x_2, \dots\}$. Its endomorphisms send each letter to a polynomial without constant term. For $n \geq 1$ let $T := (x^n)^T$ be the T-ideal generated by x^n (the smallest ideal that contains x^n and is closed under all endomorphisms), and

$$d(n) := \min\{m : x_1 \cdots x_m \in T\}.$$

Let V_m be the space of multilinear polynomials in $x_1, \dots, x_m$ (its basis is the $m!$ words in which each of $x_1, \dots, x_m$ occurs once) and $W_{n,m} := V_m \cap T$. We use the following standard facts.

(F0) $d(n) \leq m$ iff $x_1 \cdots x_m \in T$ iff $W_{n,m} = V_m$. [T is an ideal, and renaming the variables maps $x_1 \cdots x_m$ onto every word of V_m; [Ro22, Prop. 1.8].]

(F1) *Spanning set.* Let $L(y_1, \dots, y_n) := \sum_{\pi \in S_n} y_{\pi(1)} \cdots y_{\pi(n)}$. In characteristic 0, $T = (L)^T$: L is the multilinear component of $(y_1 + \cdots + y_n)^n$, so $L \in T$ (a T-ideal over an infinite field contains the multihomogeneous components of its elements), and $L(x, \dots, x) = n!\, x^n$. The T-ideal $(L)^T$ is spanned by the elements $p \cdot L(h_1, \dots, h_n) \cdot q$, where p, q are monomials or 1 and the h_i have no constant term. Indeed, this span is an ideal, contains L, and is closed under every endomorphism of the non-unitary algebra (φ maps $p\, L(h)\, q$ to $\varphi(p)\, L(\varphi(h_1), \dots, \varphi(h_n))\, \varphi(q)$; the $\varphi(h_i)$ have no constant term, and expanding $\varphi(p)$, $\varphi(q)$ into monomials puts this in the span). By multilinearity the h_i may be taken to be monomials. Each such $p \cdot L(h) \cdot q$ is then multihomogeneous, so $V_m \cap T$ is spanned by the spanning elements of multidegree $(1, \dots, 1)$. These are exactly the elements

$$a \cdot L(v_1, \dots, v_n) \cdot b, \qquad v_i \text{ nonempty words, } a, b \text{ words or empty,}$$

in which every letter $x_1, \dots, x_m$ occurs exactly once.

(F2) *Wedderburn and Young.* $F[S_N] = \bigoplus_{\lambda \vdash N} I_\lambda$ with $I_\lambda = F[S_N]e_\lambda$, where the element $e_\lambda = (f^\lambda/N!) \sum_\sigma \chi_\lambda(\sigma)\sigma$ is a central primitive idempotent and $f^\lambda = \dim S^\lambda$; ρ_λ restricts to an isomorphism $I_\lambda \cong \mathrm{End}(S^\lambda)$. Each I_λ is a simple algebra. The Specht modules are defined over $\mathbb{Q}$ and are absolutely irreducible. Writing $a(\sigma)$ for the coefficient of σ in a, Fourier inversion reads

$$a(\sigma) = \sum_\lambda \frac{f^\lambda}{N!} \operatorname{tr}\big(\rho_\lambda(\sigma^{-1})\rho_\lambda(a)\big).$$

[Ja78]

(F3) *Orthogonality.* Put $\langle a, b\rangle := \sum_\sigma a(\sigma)b(\sigma)$. The I_λ are mutually orthogonal, and $\langle\,,\rangle$ is non-degenerate. [$\langle a, b\rangle$ is the coefficient of 1 in $a \cdot b^\vee$, where $b^\vee := \sum_\sigma b(\sigma)\sigma^{-1}$. The map $\vee$ is an anti-automorphism fixing each e_λ (characters are real), so $a \cdot b^\vee \in I_\lambda I_\mu = 0$ for $a \in I_\lambda$, $b \in I_\mu$, $\lambda \neq \mu$.]

(F4) For $m \geq n$ regard $S_{n+1} \subseteq S_N$ (fixing $n+2, \dots, N$) and put $E := \sum_{\pi \in S_{n+1}} \mathrm{sgn}(\pi)\pi$. The two-sided ideal generated by E is $I := \bigoplus_{\ell(\lambda) \geq n+1} I_\lambda$, where $\ell(\lambda)$ is the number of rows. [Each I_λ is simple, so a two-sided ideal is the sum of the I_λ it meets; $Ee_\lambda \neq 0$ iff $\rho_\lambda(E) \neq 0$ iff $\mathrm{Res}_{S_{n+1}} S^\lambda$ contains sgn iff $(1^{n+1}) \subseteq \lambda$, by the iterated branching rule (F5).]

(F5) *Branching rule.* $\mathrm{Res}^{S_N}_{S_{N-1}} S^\lambda \cong \bigoplus S^\mu$, over the μ obtained from λ by removing one removable corner. [Ja78, Thm. 9.2]

(F6) *Burnside.* For an absolutely irreducible module V of a group G, $\rho(F[G]) = \mathrm{End}_F(V)$.

(F7) *Sign twist.* $\varepsilon(\sigma) := \mathrm{sgn}(\sigma)\sigma$ extends to an algebra automorphism of $F[S_N]$ with $\varepsilon(I_\lambda) = I_{\lambda'}$, where λ' is the conjugate partition. On C_N it is the scalar $(-1)^{N-1}$, because every N-cycle has sign $(-1)^{N-1}$.

2.2. **Notation.** We compose permutations right to left, $(\sigma\tau)(i)=\sigma(\tau(i))$. All of the following depends on N; we write $B_n^{(N)}$ where N must be shown. C_N is the span of the N-cycles in $F[S_N]$, and P is the coordinate projection onto C_N (orthogonal for $\langle\,,\rangle$). Put

$$A:=\bigoplus_{\ell(\lambda)\le n} I_\lambda,\qquad A':=\bigoplus_{\lambda_1\le n} I_\lambda,\qquad B_n=B_n^{(N)}:=A\cap A'=\bigoplus_{\lambda\vdash N,\ \lambda\subseteq(n^n)} I_\lambda,$$

the last sum being over the partitions of N inside the $n\times n$ box. For a sequence $\alpha=(a_1,\dots,a_r)$ of elements of $[m]$ we write $x_\alpha:=x_{a_1}\cdots x_{a_r}$ (the empty word if $r=0$).

2.3. **The criterion.**

Proposition 2.1 (Formanek's criterion, cycle form)**.** *Let $n\ge 1$, $m\ge n$ and $N=m+1$. Then*

$$\dim W_{n,m}=m!-\dim\bigl(C_N\cap B_n^{(N)}\bigr).$$

In particular, $d(n)\le m$ if and only if $C_N\cap B_n^{(N)}=0$. (For $1\le m<n$ the formula also holds, trivially; see the first line of the proof.)

Proposition 2.1 is our form of Formanek's criterion [Fo90, Thm. 5], from which Razmyslov's bound follows (Remark 2.4); Strickland [St89] gave a criterion in another form. We include a self-contained proof.

Proof. If $m<n$, every word has degree $m<n$, so $W_{n,m}=0$; every $\lambda\vdash N\le n$ fits in the box, so $B_n=F[S_N]$, and $C_N\cap B_n=C_N$ has dimension $m!$. From now on $m\ge n$, so $S_{n+1}\subseteq S_N$.

(1) *Words and N-cycles.* Let $\Phi(x_{j_1}\cdots x_{j_m})$ be the N-cycle $N\to j_1\to\cdots\to j_m\to N$. This is a bijection from the words onto the N-cycles, and a linear isomorphism $\Phi\colon V_m\to C_N$.

(2) $I=\operatorname{span}\{\tau E_X:\tau\in S_N,\ X\subseteq[N],\ |X|=n+1\}$, where $E_X:=\sum_{\pi\in\mathrm{Sym}(X)}\operatorname{sgn}(\pi)\pi$. [$gEh=(gh)E_{h^{-1}[n+1]}$, and conversely.]

(3) If no $\tau\pi$ ($\pi\in\mathrm{Sym}(X)$) is an N-cycle, then $P(\tau E_X)=0$. Otherwise $\tau E_X=\operatorname{sgn}(\pi_0)\tau_0E_X$ with $\tau_0=\tau\pi_0$ an N-cycle. So we may assume that τ is an N-cycle.

(4) *Arcs.* For $x\in X$ let $s(x):=\tau^k(x)$ for the least $k\ge1$ with $\tau^k(x)\in X$. Then s is an $(n+1)$-cycle on X. The *arc* of $x\in X$ is $\mathrm{Arc}(x):=(\tau(x),\dots,s(x))$. The $n+1$ arcs $\mathrm{Arc}(x)$, $x\in X$, are nonempty, they partition $[N]$, and $\mathrm{Arc}(x)$ ends at $s(x)$. For $\pi\in\mathrm{Sym}(X)$, $\tau\pi$ agrees with τ off X and sends x to $\tau(\pi(x))$. From x, $\tau\pi$ runs through $\mathrm{Arc}(\pi(x))$ and lands on $s(\pi(x))$. So $\tau\pi$ is an N-cycle iff $s\circ\pi$ is an $(n+1)$-cycle.

(5) *Constant sign.* Let $\Pi:=\{\pi\in\mathrm{Sym}(X):s\circ\pi\text{ is an }(n+1)\text{-cycle}\}$. Since s is itself an $(n+1)$-cycle, $\operatorname{sgn}(s)=(-1)^n$, and for $\pi\in\Pi$, $\operatorname{sgn}(\pi)=\operatorname{sgn}(s)\operatorname{sgn}(s\circ\pi)=(-1)^n(-1)^n=+1$. So $P(\tau E_X)=\sum_{\pi\in\Pi}\tau\pi$, with no sign (for an N-cycle τ).

(6) The map $\pi\mapsto\rho:=s\circ\pi$ is a bijection from Π onto the set of $(n+1)$-cycles on X. Since $\pi(\rho^kx)=s^{-1}\rho^{k+1}x$, and $\mathrm{Arc}(s^{-1}z)$ is the arc that ends at z, $\tau\pi$ runs through the arcs ending at $\rho(x),\rho^2(x),\dots$, in this order. So as π runs over Π, the cyclic order of the arcs in $\tau\pi$ runs over every cyclic order of the $n+1$ arcs exactly once, and each arc keeps its internal order. Let $\mathrm{Arc}_0=(\zeta,N,\xi)$ be the arc that contains N, with ξ (after N) and ζ (before N) possibly empty. Read from N, $\tau\pi$ is ξ, then the other n arcs $\mathrm{Arc}_1,\dots,\mathrm{Arc}_n$ in the linear order induced after Arc_0, then ζ. So

$$\Phi^{-1}\Bigl(\sum_{\pi\in\Pi}\tau\pi\Bigr)=x_\xi\cdot L(x_{\mathrm{Arc}_1},\dots,x_{\mathrm{Arc}_n})\cdot x_\zeta,$$

and the Arc_i ($i\ge1$) are nonempty and avoid N. By (F1) this gives the *forward half*, $P(I)\subseteq\Phi(W_{n,m})$.

Converse. Given $a\cdot L(v_1,\dots,v_n)\cdot b$ as in (F1), take $\tau:=\Phi(av_1\cdots v_nb)$, and let X consist of the last letters of the v_i together with z, where z is the last letter of a, or $z:=N$ if a is empty. The v_i are nonempty and all letters are distinct, so these are $n+1$ distinct points. Along τ

we meet $N, a, v_1, \dots, v_n, b, N$. So the arcs are $v_1, \dots, v_n$ and $\mathrm{Arc}_0 = (b, N, a)$, or (b, N) if a is empty, and $\sum_{\pi\in\Pi} \tau\pi = \Phi(a \cdot L(v) \cdot b)$. Hence $P(I) = \Phi(W_{n,m})$.

(7) *Duality.* For $c \in C_N$, $\langle c, a\rangle = \langle c, Pa\rangle$. So for $f \in C_N$: $f \perp \Phi(W_{n,m})$ iff $f \perp I$ (by (6)) iff $f \in I^\perp$. Here $I^\perp = A$: (F3) gives $A \subseteq I^\perp$, and equality holds because $\langle\,,\rangle$ is nondegenerate, so $\dim I^\perp = N! - \dim I = \dim A$ ((F2), (F4)). By (F7), ε maps $C_N \cap A$ into $C_N \cap A'$ (ε is $\pm$ the identity on C_N and sends A to A'), so $C_N \cap A \subseteq C_N \cap A \cap A' = C_N \cap B_n$; the reverse inclusion is trivial. So $C_N \cap \Phi(W_{n,m})^\perp = C_N \cap B_n$. The N-cycles are orthonormal, so $\dim W_{n,m} = m! - \dim(C_N \cap B_n)$.

Finally, by (F0), $d(n) \le m$ iff $\dim W_{n,m} = m!$ iff $C_N \cap B_n = 0$. □

Remark 2.2 (what the bounds use). Theorems A and B use only the direction "$C_N \cap B_n = 0 \Rightarrow d(n) \le m$". This direction needs only the forward half of step (6), $P(I) \subseteq \Phi(W_{n,m})$, together with $I^\perp = A$ and the sign twist in step (7). The converse in step (6) is needed only for the exact dimension formula and for the direction "$d(n) \le m \Rightarrow C_N \cap B_n = 0$".

Remark 2.3 (a direct proof of $C_N \cap A = C_N \cap A'$). By step (5), $\mathrm{sgn}(\pi) = +1$ on Π, so $P(\tau E_X) = P(\tau \cdot \sum_{\pi\in\mathrm{Sym}(X)} \pi) = P(\tau S_X)$ for an N-cycle τ, where $S_X := \sum_{\pi\in\mathrm{Sym}(X)} \pi$ (and $P(\tau E_X) = \pm P(\tau S_X)$ in general, by step (3)). Hence $P(I) = P(\varepsilon(I))$, where $\varepsilon(I) = \bigoplus_{\lambda_1 \ge n+1} I_\lambda$, and so $C_N \cap A = C_N \cap A'$ by the duality in step (7). This explains why (F7) works.

Remark 2.4 (Razmyslov's bound). For $N > n^2$ no partition of N fits in the $n \times n$ box, so $B_n^{(N)} = 0$, and $d(n) \le n^2$.

3. The one-corner step: Theorem A

Lemma 3.1 (one-corner step)**.** *For $n \ge 2$, $C_{n^2} \cap B_n^{(n^2)} = 0$.*

Proof. Let $N = n^2$. The only partition of n^2 in the box is the square $\beta := (n^n)$, so $B_n^{(N)} = I_\beta$. Let $f \in C_N \cap I_\beta$ and $M := \rho_\beta(f)$. Then $f(\sigma) = (f^\beta/N!)\,\mathrm{tr}(\rho_\beta(\sigma^{-1})M)$ by (F2). For $\sigma \in \mathrm{Stab}(N) = S_{N-1}$, σ is not an N-cycle, so $f(\sigma) = 0$. The square has *one* removable corner, so $\mathrm{Res}_{S_{N-1}} S^\beta = S^{(n^{n-1},n-1)}$ is absolutely irreducible ((F5), (F2)), and by (F6) the $\rho_\beta(\sigma)$, $\sigma \in S_{N-1}$, span $\mathrm{End}(S^\beta)$. Hence $\mathrm{tr}(YM) = 0$ for every $Y \in \mathrm{End}(S^\beta)$, so $M = 0$ and $f = 0$. □

Proof of Theorem A. Apply Proposition 2.1 with $m = n^2 - 1$ (here $m \ge n$ because $n \ge 2$); by Lemma 3.1 and the forward half of the criterion (Remark 2.2), $d(n) \le n^2 - 1$. □

Corollary 3.2. *$n(n+1)/2 \le d(n) \le n^2 - 1$ for every $n \ge 2$ (the lower bound is Kuzmin's* [Ku75]*). The bound is new only for $n \ge 6$: $d(6) \le 35$ (for $n = 5$, $d(5) \le 21$ is known, Subsection 1.2).*

4. The two-corner step: Theorem B

4.1. **Reduction to one block.** Let $N = n^2 - 1$. We work over $\mathbb{C}$: every space below (C_N, I_ν, their intersection) is defined over $\mathbb{Q}$, so its dimension is the same over $\mathbb{Q}$, over $\mathbb{C}$, and over any field of characteristic 0. Over $\mathbb{C}$ we may take square roots (to normalize $\Gamma^2 = I$ below), and the constituents of the restrictions to A_N are defined. The only partition of N inside the $n \times n$ box is

$$\nu := (n^{n-1}, n-1),$$

the box minus one cell, which must be the removable corner (n, n). So $B_n = I_\nu$, and by Proposition 2.1 it suffices to show $C_N \cap I_\nu = 0$. For $f \in I_\nu$ put $M := \rho_\nu(f)$; then

$$f(\sigma) = \frac{f^\nu}{N!}\,\mathrm{tr}\big(\rho_\nu(\sigma^{-1})M\big).$$

4.2. **Tools.** Besides (F2)–(F6) we use the following facts. For a standard tableau T and an entry k, $c_T(k)$ is the content (column minus row) of the cell of k in T.

(J) *Jucys–Murphy elements.* $J_k := \sum_{i<k}(i\,k)$. On the Young (Gelfand–Tsetlin) basis $\{v_T\}$ of S^λ, $J_k v_T = c_T(k) v_T$ [Ju74, OV96]. Consecutive entries never have equal content. In particular $\rho(J_k)$ is diagonal in the Young basis.

(R) $s := (k, k+1)$ satisfies $sJ_{k+1} - J_k s = 1$ [directly: $s(i,k+1) = (i,k)s$ for $i<k$]. Moreover $\rho(s)v_T \in \operatorname{span}(v_T, v_{sT})$, with the v_{sT} term only if sT is standard (Young's seminormal form [OV96], [CST10, Ch. 3]). Taking the v_T-coefficient of the identity $sJ_{k+1} - J_k s = 1$ gives

$$\rho(s)_{TT} = \frac{1}{c_T(k+1) - c_T(k)}.$$

(Γ) For a self-conjugate λ, $S^\lambda \otimes \operatorname{sgn} \cong S^\lambda$, so there is an invertible Γ with $\Gamma\rho(\sigma) = \operatorname{sgn}(\sigma)\rho(\sigma)\Gamma$. Γ^2 commutes with $\rho(S_N)$, so it is a scalar (Schur) (nonzero, since Γ is invertible), and over $\mathbb{C}$ we rescale to $\Gamma^2 = I$. Γ commutes with $\rho(A_N)$, where A_N is the alternating group, and the restriction of S^λ to A_N is the sum of two non-isomorphic irreducibles $V_+ \oplus V_-$ [JK81, §2.5]. $\Gamma \neq \pm I$, because Γ anticommutes with $\rho((1\,2)) \neq 0$. So (since $V_+ \ncong V_-$, Γ preserves each of them and, by Schur's lemma, is a scalar on each; as $\Gamma^2 = I$ these scalars are ± 1) $\Gamma = +1$ on one constituent and -1 on the other: the eigenspaces of Γ are the two A_N-constituents. Hence

$$\psi(\sigma) := \operatorname{tr}(\rho(\sigma)\Gamma)$$

vanishes for odd σ (as $\Gamma\rho(\sigma) = -\rho(\sigma)\Gamma$), and $\psi = \pm(\chi_+ - \chi_-)$ on A_N. *Frobenius' theorem* [JK81, Thm. 2.5.13]: let $h(\lambda) = (h_{11}, h_{22}, \dots, h_{dd})$ be the diagonal hook lengths, and $\varepsilon_\lambda := (-1)^{(N-d)/2}$. Then $\chi_+(\sigma) = \chi_-(\sigma)$ unless σ has cycle type $h(\lambda)$. On the two A_N-classes of that type, $\chi_\pm(\sigma) = \big(\varepsilon_\lambda \pm \sqrt{\varepsilon_\lambda \prod h_{ii}}\big)/2$, so $\chi_+ - \chi_- = \pm\sqrt{\varepsilon_\lambda \prod h_{ii}} \neq 0$ there. So ψ is supported *exactly* on the permutations of cycle type $h(\lambda)$.

4.3. **Fixed points and the star elements.** Recall $X^{(j)} = \sum_{i\neq j}(i\,j)$, and put $Y_j := \rho_\nu(X^{(j)})$.

Lemma 4.1 (fixed points and stars)**.** *An element $f \in I_\nu$ vanishes on every permutation with a fixed point if and only if $MY_j = -Y_j M$ for all j.*

Proof. ν has exactly two removable corners: $A = (n-1, n)$ with content $+1$ and $B = (n, n-1)$ with content -1. (The other row ends (i,n), $i < n-1$, are not removable: the next row has the same length.) By (F5), $\operatorname{Res}_{\operatorname{Stab}(N)} S^\nu = S^{\nu-A} \oplus S^{\nu-B}$, with two different shapes. So by (F2) and (F6), $\rho(F[\operatorname{Stab}(N)]) = \operatorname{End}(S^{\nu-A}) \oplus \operatorname{End}(S^{\nu-B})$, block-diagonal for this splitting. $Y_N = \rho(X^{(N)}) = \rho(J_N)$ acts by $+1$ on the first block and by -1 on the second (J), so the block projections are $P_\pm = (I \pm Y_N)/2$. Now "$f = 0$ on $\operatorname{Stab}(N)$" holds iff $M \perp \rho(F[\operatorname{Stab}(N)])$ for the trace form $(X, M) \mapsto \operatorname{tr}(XM)$ on $\operatorname{End}(S^\nu)$ (not the coefficient form of Section 2), iff $P_+ M P_+ = 0 = P_- M P_-$. Expanding, $(I+Y)M(I+Y) = 0$ and $(I-Y)M(I-Y) = 0$ iff $YM + MY = 0$ (use $Y^2 = I$), where $Y = Y_N$.

Transfer to $\operatorname{Stab}(j)$. Take $g \in S_N$ with $g(N) = j$. Then

$$f(g\sigma g^{-1}) = \frac{f^\nu}{N!} \operatorname{tr}\big(\rho(\sigma^{-1})\,\rho(g)^{-1} M \rho(g)\big),$$

and $g\operatorname{Stab}(N)g^{-1} = \operatorname{Stab}(j)$. So $f = 0$ on $\operatorname{Stab}(j)$ iff $\rho(g)^{-1}M\rho(g)$ anticommutes with Y_N iff M anticommutes with $\rho(g)Y_N\rho(g)^{-1} = Y_j$. Every permutation with a fixed point lies in some $\operatorname{Stab}(j)$. □

Lemma 4.2 (the star commutant for ν)**.** *For $n \geq 2$, the only operators on S^ν that commute with all Y_j are the scalars.*

Proof. Let $\mathcal{A}$ be the unital algebra generated by $Y_1, \dots, Y_N$. Conjugation by $\rho(g)$ permutes the Y_j, so $\mathcal{A}$ is stable under it.

(a) $Y_N = \rho(J_N) \in \mathcal{A}$, so $P_\pm \in \mathcal{A}$.
(b) $Y_{N-1} = \rho(J_{N-1} + s)$ with $s = (N-1, N)$, since $X^{(N-1)} = \sum_{i<N-1}(i, N-1) + (N, N-1)$.
(c) $D := P_+ Y_{N-1} P_+ + P_- Y_{N-1} P_- \in \mathcal{A}$. Now $\rho(J_{N-1})$ is diagonal, hence block-diagonal. By (R), the off-diagonal entries of $\rho(s)$ join v_T to v_{sT}. When sT is standard, N sits at a corner of ν in sT, so $c_{sT}(N) = c_T(N-1)$ is ± 1, and it differs from $c_T(N)$ (J). So these entries join *different* J_N-eigenspaces. The only off-diagonal entries of $\rho(s)$ are the (sT, T) entries, and N lies in different corners in T and in sT, so the block-diagonal part of $\rho(s)$ is its diagonal. Hence, by (R) with $k = N-1$,

$$Dv_T = g\big(c_T(N-1), c_T(N)\big)v_T, \qquad g(x, y) := x + \frac{1}{y - x}.$$

(d) The possible pairs $(c_T(N-1), c_T(N))$: if N is at A, then $N-1$ is at a corner of $\nu - A = (n^{n-2}, n-1, n-1)$, whose corners have contents -1 (the cell B) and $+2$ (the cell $(n-2, n)$, present iff $n \geq 3$). If N is at B, then $N-1$ is at a corner of $\nu - B = (n^{n-1}, n-2)$, whose corners have contents $+1$ (the cell A) and -2 (the cell $(n, n-2)$, present iff $n \geq 3$). The pairs and their values $(c_T(N), g)$ are

$$(-1, 1) \mapsto (1, -\tfrac{1}{2}), \qquad (2, 1) \mapsto (1, 1), \qquad (1, -1) \mapsto (-1, \tfrac{1}{2}), \qquad (-2, -1) \mapsto (-1, -1).$$

They are all distinct. For $n = 2$ only $(-1, 1)$ and $(1, -1)$ occur, and they are still distinct. So $c_T(N-1)$ is a function of the joint eigenvalue of the commuting diagonal pair (Y_N, D), and $\rho(J_{N-1}) = p(Y_N, D)$ for a polynomial p (Lagrange interpolation). So $\rho(J_{N-1}) \in \mathcal{A}$, and $\rho(s) = Y_{N-1} - \rho(J_{N-1}) \in \mathcal{A}$.
(e) By (a)–(d), $\rho((N-1, N)) \in \mathcal{A}$; conjugating, $\rho(t) \in \mathcal{A}$ for every transposition t. So $\mathcal{A} \supseteq \rho(F[S_N]) = \mathrm{End}(S^\nu)$ (F6), and its commutant is $F \cdot I$. □

Remark 4.3 (scope). Lemma 4.2 uses the *two* corners of ν, and it fails for rectangles with at least two rows and two columns: for such a rectangle (one corner), $\rho(J_N)$ is a scalar, so every Y_j is a scalar and the commutant of the stars is all of $\mathrm{End}(S^\lambda)$. Lemma 4.2 is the case $\lambda = \nu$ of Theorem C.

4.4. **Proof of Theorem B.**

Theorem 4.4. *For every $n \geq 3$, $C_{n^2-1} \cap B_n^{(n^2-1)} = 0$. Hence $d(n) \leq n^2 - 2$.*

Proof. Let $f \in C_N \cap I_\nu$. A permutation with a fixed point is not an N-cycle, so by Lemma 4.1 M anticommutes with every Y_j. The shape ν is self-conjugate ($\nu' = \nu$: columns $1, \ldots, n-1$ have length n and column n has length $n-1$), so Γ exists (Γ). Each $X^{(j)}$ is a sum of transpositions, so Γ anticommutes with every Y_j. Then ΓM commutes with every Y_j, so $\Gamma M = cI$ by Lemma 4.2, and $M = c\Gamma$. So $f(\sigma) = c'\psi(\sigma^{-1})$, which by ($\Gamma$) is supported exactly on the cycle type $h(\nu)$. The diagonal cells of ν are (i, i) for $i = 1, \ldots, n-1$, each with arm = leg $= n - i$, so

$$h(\nu) = (2n-1, 2n-3, \ldots, 5, 3).$$

All these hooks are odd, so $h(\nu)$ is an even type and lies in A_N, where $\psi \neq 0$. For $n \geq 3$ the type has $n - 1 \geq 2$ cycles, so it is not an N-cycle, and $f \in C_N$ vanishes there. Hence $c' = 0$ and $f = 0$. Proposition 2.1 with $m = n^2 - 2$ (here $m \geq n$ because $n \geq 2$) gives $d(n) \leq n^2 - 2$. □

Theorem B is Theorem 4.4.

Remark 4.5 (where $n \geq 3$ enters). For $n = 2$, $h(\nu) = (3)$ *is* the N-cycle type, so Γ survives: $f = (1\,2\,3) - (1\,3\,2) \in C_3 \cap I_{(2,1)}$, and $d(2) = 3$.

Corollary 4.6. $n(n+1)/2 \leq d(n) \leq n^2 - 2$ *for $n \geq 3$ (lower bound: Kuzmin* [Ku75]*). The bound is new only for $n \geq 6$: $d(6) \leq 34$ (for $n = 5$, $d(5) \leq 21$ is known, Subsection 1.2).*

Remark 4.7 (Frobenius and Kostant). The theorem of Frobenius used in (Γ) is the one that Kostant [Ko58] connected with the Amitsur–Levitzki theorem [AL50]. For the self-conjugate hook $(k,1^{k-1})\vdash 2k-1$ it is the main input to Lemma 5.3: there f_Γ is a nonzero multiple of $\Phi(s_{2k-2})$, the image of the standard polynomial.

5. Single blocks and the star elements

Throughout this section $\lambda\vdash N$, $\rho=\rho_\lambda$, and for $f\in I_\lambda$ we put $M:=\rho(f)$, so that $f(\sigma)=(f^\lambda/N!)\operatorname{tr}(\rho(\sigma^{-1})M)$ by (F2). We put $Y_j:=\rho(X^{(j)})$, as in Section 4. Where square roots are needed we work over $\mathbb{C}$; as in Section 4, the spaces U_λ and K_λ are defined over $\mathbb{Q}$, so their dimensions do not depend on the field.

5.1. The sign twist and rectangles.

Proposition 5.1 (sign twist). *For every set S of partitions of N, with $S':=\{\lambda':\lambda\in S\}$,*

$$C_N\cap\bigoplus_{\lambda\in S}I_\lambda\subseteq\bigoplus_{\lambda\in S\cap S'}I_\lambda.$$

In particular $U_\lambda=0$ unless $\lambda=\lambda'$.

Proof. Let $f\in C_N\cap\bigoplus_S I_\lambda$, with components f_λ. Then $\varepsilon(f)=(-1)^{N-1}f$ by (F7), and the $I_{\lambda'}$-component of $\varepsilon(f)$ is $\varepsilon(f_\lambda)$. So $f_{\lambda'}=(-1)^{N-1}\varepsilon(f_\lambda)$ for every λ. If $\lambda\in S$ but $\lambda'\notin S$, the left side is 0, so $\varepsilon(f_\lambda)=0$ and $f_\lambda=0$. □

Remark 5.2 (rectangles). If λ is a rectangle and $N\geq 2$, the proof of Lemma 3.1 applies word for word (a rectangle has one removable corner): no nonzero $f\in I_\lambda$ vanishes on $\mathrm{Stab}(N)$. In particular $K_\lambda=U_\lambda=0$.

5.2. Self-conjugate hooks and the standard polynomial.

For Lemma 5.3 and the example after it we work over $\mathbb{C}$, as in Section 4; the rest of this subsection holds over any field F of characteristic 0. For a self-conjugate $\lambda\vdash N$ take Γ and ψ as in (Γ), and let $f_\Gamma\in I_\lambda$ be the element with $\rho_\lambda(f_\Gamma)=(N!/f^\lambda)\Gamma$; by Fourier inversion (F2), $f_\Gamma(\sigma)=\psi(\sigma^{-1})$. For $m\geq 1$ let $s_m:=\sum_{\pi\in S_m}\operatorname{sgn}(\pi)\,x_{\pi(1)}\cdots x_{\pi(m)}$ be the standard polynomial, and let $\Phi\colon V_m\to C_{m+1}$ be the isomorphism of step (1) of the proof of Proposition 2.1. For a sequence $j=(j_1,\dots,j_m)$ that lists $[m]$, $\operatorname{sgn}(j)$ is the sign of the permutation $i\mapsto j_i$. The *arrangement* $(a_1,\dots,a_N)$ of an N-cycle $(a_1\ \dots\ a_N)$ is read as the permutation $i\mapsto a_i$ of $[N]$.

Lemma 5.3 (the hook element; cf. Kostant [Ko58]). *Let $k\geq 2$, $\lambda=(k,1^{k-1})$, $N=2k-1$ and $m=2k-2$. Then $f_\Gamma=c\,\Phi(s_m)$ for a constant $c\neq 0$. In particular $\Phi(s_m)\in I_\lambda$; since this is a linear condition with rational coefficients ($e_\lambda\in\mathbb{Q}[S_N]$), it holds over every field of characteristic 0. With Theorem D, $U_\lambda=\operatorname{span}\Phi(s_{2k-2})$.*

Proof. (1) By (Γ), Γ is $+1$ on one A_N-constituent V_+ of S^λ and -1 on the other, V_-, so $\psi=\chi_+-\chi_-$ on A_N, and $\psi=0$ on the odd permutations. The character χ_- is χ_+ conjugated by an odd permutation, which swaps the two A_N-classes of a split class; so ψ takes opposite values on the two A_N-classes of N-cycles. Here $h(\lambda)=(N)$, so by Frobenius' theorem in (Γ), ψ vanishes off the N-cycles and is nonzero on them. So $f_\Gamma\in C_N\cap I_\lambda$.

(2) The two A_N-classes of N-cycles (N odd): the permutations that conjugate $(a_1\ \dots\ a_N)$ to $(b_1\ \dots\ b_N)$ form the coset $\pi\langle(a_1\ \dots\ a_N)\rangle$, where $\pi\colon a_i\mapsto b_i$, and $\langle(a_1\ \dots\ a_N)\rangle\subseteq A_N$. So the two cycles are A_N-conjugate iff their arrangements have equal sign. A change of starting point composes the arrangement with an N-cycle of positions, which is even, so this sign is well defined.

(3) $\Phi(x_{j_1}\cdots x_{j_m})=(N\,j_1\ \dots\ j_m)$ has arrangement sign $(-1)^{N-1}\operatorname{sgn}(j)=\operatorname{sgn}(j)$; its inverse $(N\,j_m\ \dots\ j_1)$ has arrangement sign $\operatorname{sgn}(j)(-1)^{m(m-1)/2}$. By (1) and (2), $f_\Gamma(\Phi(x_{j_1}\cdots x_{j_m}))=\psi\big(\Phi(x_{j_1}\cdots x_{j_m})^{-1}\big)=c\operatorname{sgn}(j)$ with one constant $c\neq 0$.

(4) Every N-cycle is Φ of exactly one word, and f_Γ vanishes off the N-cycles, so $f_\Gamma = c\,\Phi(s_m)$. $\square$

For $k=2$: $c^{-1}f_\Gamma = (1\,2\,3) - (1\,3\,2) = \Phi(x_1x_2 - x_2x_1) = \Phi(s_2)$ (here $c = \pm i\sqrt{3}$).

Rationality. ψ is not a class function of S_N: it takes opposite values on the two A_N-classes of type $h(\lambda)$. For $\lambda = (k, 1^{k-1})$ these values are $\pm\sqrt{(-1)^{k-1}N}$ by (Γ), so f_Γ has entries in $\mathbb{Q}(\sqrt{(-1)^{k-1}N})$, which is $\mathbb{Q}$ only when N is a perfect square. But U_λ is defined over $\mathbb{Q}$, so it always contains a nonzero rational element; Lemma 5.3 exhibits one, $\Phi(s_{2k-2})$.

Proposition 5.4 (standard polynomials in the T-ideal of x^n)**.** *Let $n \geq 2$. Then $s_m \in T$ if and only if $m \geq 2n-1$.*

Proof. Let $\mathcal{S} := \sum_{\sigma\in S_m} \operatorname{sgn}(\sigma)\sigma$ act on V_m by renaming the variables (σ sends x_i to $x_{\sigma(i)}$). For a word $\mu = x_{j_1}\cdots x_{j_m}$ put $\operatorname{sgn}(\mu) := \operatorname{sgn}(j)$; then $\mathcal{S}(\mu) = \operatorname{sgn}(\mu)s_m$, so $\mathcal{S}(g) = \langle g, s_m\rangle s_m$ with $\langle g, s_m\rangle = \sum_\mu g_\mu \operatorname{sgn}(\mu)$, where g_μ is the coefficient of the word μ in g. Renaming preserves T, so $\mathcal{S}(W_{n,m}) \subseteq W_{n,m}$; and $\langle s_m, s_m\rangle = m! \neq 0$. Hence $s_m \in W_{n,m}$ iff some $g \in W_{n,m}$ has $\langle g, s_m\rangle \neq 0$, iff some *spanning* element does. By (F1), $W_{n,m}$ is spanned by the $g = a\cdot L(w_1,\ldots,w_n)\cdot b$ (a, b words or empty, w_i nonempty words, every letter once). Reordering the blocks w_i changes the sign of the word by the sign of the induced reordering of the blocks of *odd* length. So $\langle g, s_m\rangle = \pm n!$ if at most one $|w_i|$ is odd, and $\langle g, s_m\rangle = 0$ otherwise (pair each order with the one that swaps two odd blocks). At most one odd $|w_i|$, with all $|w_i| \geq 1$, forces $\sum_i |w_i| \geq 1+2(n-1) = 2n-1$. So $s_m \notin T$ for $m \leq 2n-2$. For $m = 2n-1$, $g = L(x_1,\, x_2x_3, \ldots, x_{2n-2}x_{2n-1})$ has $\langle g, s_m\rangle = \pm n!$, so $s_{2n-1} \in T$. Finally, $s_m \in T$ implies $s_{m+1} \in T$, since $s_{m+1}(x_1,\ldots,x_{m+1}) = \sum_{i=1}^{m+1}(-1)^{i-1}x_i\, s_m(x_1,\ldots,\widehat{x_i},\ldots,x_{m+1})$. $\square$

We make no novelty claim for Proposition 5.4. Its case $m \geq 2n$ ($s_{2n} \in T$, hence $s_m \in T$ for $m \geq 2n$ by the last step of the proof) follows from the Amitsur–Levitzki theorem [AL50] and Procesi's theorem below [Pr76, Cor. 4.8], and also directly from the Grassmann trick of Rosset's proof of the Amitsur–Levitzki theorem [Ros76] (Remark 5.5); the sharp case $m = 2n-1$ uses the sign test above. Recall that the least degree of a standard identity of $M_n(F)$ is $2n$ [AL50].

Remark 5.5 (Rosset's trick). Let A satisfy $x^n = 0$, and let Λ be the Grassmann algebra on $e_1, e_2, \ldots$, with even part Λ_0. In characteristic 0, $x^n = 0$ is equivalent to the multilinear identity $L = 0$ of (F1), so $A \otimes \Lambda_0$ (Λ_0 commutative) also satisfies $x^n = 0$. For $a_1, \ldots, a_{2n} \in A$ put $w := \sum_i a_i \otimes e_i$. Then $w^2 = \sum_{i<j}[a_i, a_j] \otimes e_ie_j \in A \otimes \Lambda_0$, so $w^{2n} = (w^2)^n = 0$. The coefficient of $e_1\cdots e_{2n}$ in w^{2n} is $s_{2n}(a_1,\ldots,a_{2n})$. So s_{2n} vanishes on A; taking $A = F\langle X\rangle/T$ gives $s_{2n} \in T$.

Remark 5.6 (Lemma 5.3 and the Amitsur–Levitzki theorem). The hook $(n+1, 1^n)$ has $n+1$ rows. By Lemma 5.3 with $k = n+1$, $\Phi(s_{2n}) \in C_N \cap I$, where $N = 2n+1$ and I is as in (F3), and $C_N \cap I \subseteq P(I) \subseteq \Phi(W_{n,2n})$ by the forward half of step (6) of the proof of Proposition 2.1. So $s_{2n} \in T$. This is Kostant's route to the Amitsur–Levitzki theorem read through Formanek's criterion: Lemma 5.3 explains where $\Phi(s_{2n})$ sits, and Proposition 5.4 gives the sharp threshold.

Remark 5.7 (identities of M_n). Every multilinear identity of $M_n(F)$ lies in T: this is classical [Pr76, Cor. 4.8]; see also [Fo90]. In our setting: by the Razmyslov–Procesi correspondence $g \mapsto \operatorname{tr}(g\,x_N)$ [Ra74, Pr76] (conventions differ by $\sigma \mapsto \sigma^{-1}$, under which I and C_N are stable), Φ maps the multilinear identities of degree m of $M_n(F)$ onto $C_N \cap I$ (a nonzero identity of $M_n(F)$ has degree $\geq 2n$ [AL50], so $m \geq 2n > n$ and (F3) applies), and $C_N \cap I \subseteq P(I) = \Phi(W_{n,m})$.

5.3. **The star elements: Theorem C.** We work over $\mathbb{R}$, with Young's orthogonal form: S^λ has the orthonormal Young basis v_T (T standard), every $\rho(\sigma)$ is orthogonal, and so every Y_j is symmetric. The statement over any field of characteristic 0 follows: in Young's seminormal form the Y_j have rational entries, the seminormal form is diagonally conjugate to the orthogonal form, and the dimension of the algebra generated by rational matrices does not change under field extension. The Y_j generate a $*$-closed, hence semisimple, algebra, so by the double commutant theorem the two forms of Theorem C are equivalent. Let $\mathcal{A}$ be the unital algebra generated by $Y_1, \ldots, Y_N$.

Since $gX^{(j)}g^{-1} = X^{(g(j))}$, $\mathcal{A}$ is stable under conjugation by every $\rho(g)$. A corner is a removable cell; contents of distinct corners differ by at least 2. Put $s := (N-1, N)$ and $s' := (N-2, N-1)$. By (R), $\rho(s)v_T \in \operatorname{span}(v_T, v_{sT})$ with $\rho(s)_{TT} = 1/(c_T(N) - c_T(N-1))$, the v_{sT} term occurring only if sT is standard; if $N-1$ and N lie in one row (column) of T, then $\rho(s)v_T = +v_T$ ($-v_T$). The same holds for s' with $N-2$, $N-1$. (These facts hold in the orthogonal form as well as in the seminormal form, since the two are diagonally conjugate: same diagonal entries, same support.) If λ is a rectangle, $\rho(J_N)$ acts by the content of the one corner, so every Y_j is a scalar.

Proof of Theorem C. Steps (1)–(3) are those of Lemma 4.2, (a)–(d), for any number of corners; steps (4)–(6) settle the one case they leave open.

(1) $Y_N = \rho(J_N) \in \mathcal{A}$, so the projections P_X onto $V_X := \operatorname{span}\{v_T : N \text{ at } X\}$ (X a corner) lie in $\mathcal{A}$.

(2) $Y_{N-1} = \rho(J_{N-1}) + \rho(s)$. Put $D := \sum_X P_X Y_{N-1} P_X \in \mathcal{A}$. When sT is standard, N sits in sT at the cell of $N-1$ in T, a corner different from the cell of N in T. So the off-diagonal entries of $\rho(s)$ join different V_X, and $Dv_T = g(c_T(N-1), c_T(N))v_T$ with $g(x,y) := x + 1/(y-x)$.

(3) Fix a corner X with content c_X, and T with N at X. Then $N-1$ sits at a corner of $\lambda - X$: either at a corner $Y \neq X$ of λ (removing X cannot block it), or at X_-, the cell left of X, or at X_+, the cell above X (contents $c_X - 1$, $c_X + 1$), when these are corners of $\lambda - X$. The values are $g(c_Y, c_X) = c_Y + 1/(c_X - c_Y)$ and $g(c_X \pm 1, c_X) = c_X$. With $d_Y := c_X - c_Y$ ($|d_Y| \geq 2$),

$$g(c_Y, c_X) = g(c_Z, c_X) \iff (d_Z - d_Y)\Big(1 + \frac{1}{d_Y d_Z}\Big) = 0 \iff Y = Z \text{ or } d_Y d_Z = -1,$$

and $g(c_Y, c_X) = c_X$ iff $d_Y^2 = 1$. Both exceptions are impossible for $|d| \geq 2$. (Y_N and D commute: both are diagonal in the Young basis.) So the joint eigenvalue of the pair (Y_N, D) determines $c_T(N-1)$, except at a *collision corner* X, where both X_- and X_+ are corners of $\lambda - X$. There the joint eigenspace for (c_X, c_X) is $E := E^- \oplus E^+$, with $E^\mp := \operatorname{span}\{v_T : N \text{ at } X,\ N-1 \text{ at } X_\mp\}$.

(4) Let $X = (i, \lambda_i)$ be a collision corner, so $\lambda_{i-1} = \lambda_i$ ($i \geq 2$) and $\lambda_{i+1} \leq \lambda_i - 2$. The joint spectral projection Π_E is a polynomial in Y_N and D, so $\Pi_E \in \mathcal{A}$. On E, $N-1$ and N share a row (E^-) or a column (E^+), so $\rho(s) = +1$ on E^- and -1 on E^+. Hence $\rho(s)$ maps E into E; it is symmetric, so it also maps $E^\perp$ into $E^\perp$, and it commutes with Π_E. Put $\theta := \rho(s)\Pi_E = \Pi_{E^-} - \Pi_{E^+}$. Now $X^{(N-2)} = J_{N-2} + s' + ss's$, so

$$H := \Pi_E Y_{N-2} \Pi_E = \rho(J_{N-2})\Pi_E + R + \theta R \theta, \qquad R := \Pi_E \rho(s') \Pi_E.$$

An off-diagonal entry of R joins T and $s'T$, both in E. The tableau $s'T$ has $N-1$ at the cell of $N-2$ in T; this must be X_- or X_+, and it is not the cell of $N-1$ in T. So $s'T$ lies in the *other* summand: the off-diagonal entries of R only join E^- to E^+. Then $R + \theta R \theta = 2\operatorname{diag}(1/r'_T)$, with $r'_T = c_T(N-1) - c_T(N-2)$, and

$$Hv_T = h(T)v_T, \qquad h(T) := c_T(N-2) + \frac{2}{c_T(N-1) - c_T(N-2)} \qquad (T \in E).$$

(5) *The values of h.* Removing X and X_- changes only row i, whose new end X_{--} may become removable, and row $i-1$, whose end X_+ is removable; the cell above X_- is blocked by X_+. Symmetrically, in $\lambda - X - X_+$ the cell left of X_+ is blocked by X_-. The end of every other row is removable exactly when it was in λ (for $\lambda - X - X_+$, read "column" for "row"; this gives X_{++} below). So the following lists are complete. For $T \in E^-$, $N-2$ sits at a corner of $\lambda - X - X_-$:

- X_+, with $h = c_X + 1 - 1 = c_X$;
- $X_{--} := (i, \lambda_i - 2)$, if it is a corner, with $h = c_X - 2 + 2 = c_X$;
- a corner $Y \neq X$ of λ, with $h = h^-(Y) := c_Y + 2/(c_X - 1 - c_Y)$.

For $T \in E^+$, $N-2$ sits at a corner of $\lambda - X - X_+$: at X_-, with $h = c_X$; at $X_{++} := (i-2, \lambda_i)$, if it is a corner, with $h = c_X$; or at a corner $Y \neq X$, with $h = h^+(Y) := c_Y + 2/(c_X + 1 - c_Y)$.

Claim: every corner $Y \neq X$ has $|d_Y| \geq 3$. Every corner below row i lies in a column $\leq \lambda_{i+1} \leq \lambda_i - 2$ and in a row $\geq i+1$, so its content is $\leq c_X - 3$. Row $i-1$ has no corner ($\lambda_{i-1} = \lambda_i$), so every corner above X lies in a row $\leq i-2$ and a column $\geq \lambda_i + 1$, so its content is $\geq c_X + 3$.

So, for every corner $Y \neq X$, $h^-(Y)$ lies neither in $\{c_X\}$ nor in $\{h^+(Z) : Z \neq X \text{ a corner}\}$:

- $h^-(Y) = c_X$ iff $d_Y^2 - d_Y - 2 = 0$, i.e. $d_Y \in \{2, -1\}$. Excluded.
- $h^-(Y) = h^+(Z)$ iff $d_Z - d_Y = 2/(d_Z + 1) - 2/(d_Y - 1)$; the denominators are nonzero, as $|d| \geq 3$. For $Z = Y$ this needs $d + 1 = d - 1$, which is impossible. For $Z \neq Y$, the left side has absolute value $|c_Y - c_Z| \geq 2$. With $|d| \geq 3$, $|d_Z + 1| \geq 2$ and $|d_Y - 1| \geq 2$, so each term on the right has absolute value ≤ 1, with equality only at $d_Z = -3$, $d_Y = 3$; there the right side is -2 and the left side is -6. Otherwise the right side has absolute value < 2.

Since λ has a corner $Y \neq X$, and Y is still a corner of $\lambda - X - X_-$, the shape $\lambda - X - X_- - Y$ is a partition, so some $T \in E^-$ has $N-2$ at Y. So the $h^-(Y)$-eigenspace E_Y of H on E is nonzero and lies inside E^-. Then $\Pi_{E_Y} = \Pi_E\, p(H)\, \Pi_E$ for a Lagrange polynomial p (the outer Π_E removes the value $p(0)$ on $E^\perp$), so $\Pi_{E_Y} \in \mathcal{A}$.

(6) E^- is $\rho(S_{N-2})$-stable and irreducible, isomorphic to $S^{\lambda - X - X_-}$ (the Young vectors with $N-1$, N in fixed cells). Put

$$\text{Avg} := \frac{1}{(N-2)!} \sum_{g \in S_{N-2}} \rho(g)\, \Pi_{E_Y}\, \rho(g)^{-1} \in \mathcal{A}.$$

It commutes with $\rho(S_{N-2})$, maps E^- into E^- and kills its orthogonal complement (each $\rho(g)$ is orthogonal and preserves E^-). $S^{\lambda - X - X_-}$ is absolutely irreducible (F2), so Schur's lemma applies over $\mathbb{R}$, and $\text{Avg} = (\dim E_Y / \dim E^-)\, \Pi_{E^-}$ with a nonzero scalar (step (5)). So $\Pi_{E^-} \in \mathcal{A}$, and $\rho(J_{N-1})\Pi_E = (c_X - 1)\Pi_{E^-} + (c_X + 1)(\Pi_E - \Pi_{E^-}) \in \mathcal{A}$.

(7) The joint spectral projections of (Y_N, D) lie in $\mathcal{A}$ and sum to I. On every one except the collision spaces, $\rho(J_{N-1})$ is a known scalar (step (3)). Steps (4)–(6) for one collision corner X use only Π_E, Y_{N-2}, one other corner Y and the conjugation-stability of $\mathcal{A}$, so they apply to each collision corner separately. So $\rho(J_{N-1}) \in \mathcal{A}$, and $\rho(s) = Y_{N-1} - \rho(J_{N-1}) \in \mathcal{A}$. Conjugating, $\rho(t) \in \mathcal{A}$ for every transposition t, so $\mathcal{A} \supseteq \rho(F[S_N]) = \text{End}(S^\lambda)$ (F6). □

Remark 5.8. The separating value in step (5) comes from $N-2$ sitting at *another* corner Y. This is why a rectangle with at least two rows and two columns fails (one corner, which is a collision corner), and why two corners always suffice. For self-conjugate two-corner shapes $((u+v)^u, u^v)$, collision corners occur exactly when $u, v \geq 2$ (the smallest is $(4,4,2,2)$, $N = 12$); non-self-conjugate two-corner shapes can also have them, e.g. $(3,3,1)$, $(3,2,2)$, $(4,4,1,1)$, $(5,5,1)$.

5.4. **Two corners: Theorem D.**

Lemma 5.9 (two corners and fixed points)**.** *Let λ have exactly two removable corners A, B, with contents $a > b$, and put $Z_j := (2Y_j - (a+b))/(a-b)$. Then $f \in I_\lambda$ lies in K_λ if and only if $Z_j M = -M Z_j$ for all j. If $K_\lambda \neq 0$, then every nonzero element of $\rho(K_\lambda)$ is invertible, and $\dim K_\lambda = 1$.*

Proof. By (F5), $\text{Res}_{\text{Stab}(N)} S^\lambda = S^{\lambda - A} \oplus S^{\lambda - B}$, two different shapes, and $Y_N = \rho(J_N)$ acts on them by a and b (J); so Z_N acts by $+1$ and -1. The proof of Lemma 4.1, with Z_N in place of Y_N, gives the first claim. Now let $M \in \rho(K_\lambda)$ be nonzero. For $w \in \ker M$, $M Z_j w = -Z_j M w = 0$, so $\ker M$ is invariant under every Z_j, hence under the unital algebra they generate, which is that of the Y_j, i.e. $\text{End}(S^\lambda)$ by Theorem C. So $\ker M = 0$. If $M' \in \rho(K_\lambda)$ too, then $M^{-1}M'$ commutes with every Z_j, so it is a scalar (Theorem C). □

Proof of Theorem D. If $\lambda \neq \lambda'$, then $U_\lambda = 0$ by Proposition 5.1. Let $\lambda = \lambda'$ and write $\lambda = (p^u, q^v)$ with $p > q \geq 1$. Then $\lambda' = ((u+v)^q, u^{p-q})$, so $\lambda = \lambda'$ iff $q = u$ and $p = u + v$: $\lambda = ((u+v)^u, u^v)$.

The corners are $A=(u,u+v)$, with content v, and $B=(u+v,u)$, with content $-v$; so $Z_j=Y_j/v$. Γ anticommutes with every Y_j (a sum of transpositions), and $\Gamma^2=I$; so M anticommutes with every Z_j iff ΓM commutes with every Y_j. By Lemma 5.9 and Theorem C, $\rho(K_\lambda)=\Gamma\cdot\{\text{scalars}\}$, i.e. $K_\lambda=\operatorname{span}(f_\Gamma)$. By ($\Gamma$), f_Γ is supported exactly on the cycle type $h(\lambda)$. Now $h(\lambda)=(N)$ iff λ has one diagonal cell, iff $u=1$. If $u\ge 2$, $h(\lambda)$ has $u\ge 2$ parts, so f_Γ vanishes on the N-cycles, and $U_\lambda\subseteq K_\lambda\cap C_N=0$ (an N-cycle has no fixed point, as $N\ge 2$). If $u=1$, $\lambda=(v+1,1^v)$ is a hook, $f_\Gamma\in C_N$ by step (1) of the proof of Lemma 5.3, and so $U_\lambda=K_\lambda=\operatorname{span}(f_\Gamma)$. Finally, for $n\ge 3$ and $1\le j\le n-2$, the shape $(n^{n-j},(n-j)^j)$ is $((u+v)^u,u^v)$ with $u=n-j\ge 2$, $v=j\ge 1$; it is self-conjugate with two corners and not a hook, so $U_\lambda=0$. □

For $j=1$ this is the block ν of Section 4 (Theorem 4.4). For $j\ge 2$ it is a single-block statement only: for instance, $d(n)\le n^2-5$ would also need the coupled blocks at $N=n^2-4$ (the decoupling step). The proof also shows $K_\lambda=\operatorname{span}(f_\Gamma)$ for every self-conjugate two-corner shape. For the other two-corner shapes:

Corollary 5.10. *Let λ have exactly two removable corners A, B, with contents $a>b$, and suppose $K_\lambda\neq 0$. Then $\dim K_\lambda=1$, $f^{\lambda-A}=f^{\lambda-B}$, and $4c(\lambda)=N(a+b)$, where $c(\lambda)$ is the sum of the contents of the cells of λ.*

Proof. $\dim K_\lambda=1$ is Lemma 5.9. Let $M\in\rho(K_\lambda)$ be nonzero; it is invertible and $MZ_NM^{-1}=-Z_N$, so $\operatorname{tr}Z_N=0$, i.e. $f^{\lambda-A}=f^{\lambda-B}$. Each transposition lies in exactly two stars, so $\sum_j X^{(j)}=2\mathcal{T}$ with $\mathcal{T}:=\sum_{i<j}(i\,j)=\sum_k J_k$, which acts on S^λ by $c(\lambda)$ (J). Hence $\sum_j Z_j=\kappa I$ with $\kappa=(4c(\lambda)-N(a+b))/(a-b)$. Summing $Z_jM+MZ_j=0$ over j gives $2\kappa M=0$, so $\kappa=0$. □

The last condition does not need Theorem C. The last two conditions are necessary for $K_\lambda\neq 0$, not sufficient: $(5,3)$ meets both ($4c=40=N(a+b)$, $f^{(4,3)}=f^{(5,2)}=14$), yet $K_{(5,3)}=0$ by direct computation.

Remark 5.11 (three or more corners). For three or more corners, "$f=0$ on $\operatorname{Stab}(N)$" says that the diagonal blocks of M for the eigenspaces of Y_N vanish, which is not one anticommutation relation; so Theorem C does not by itself control K_λ there.

5.5. **A family with three corners.** For $k\ge 2$ let $A_k:=(k+1,2,1^{k-1})\vdash N=2k+2$. It is self-conjugate, with diagonal hooks $(2k+1,1)$ and three removable corners. In the letters $x_1,\dots,x_{2k},y$ put

$$g_j:=\sum_{\pi\in S_{2k}}\operatorname{sgn}(\pi)\,x_{\pi(1)}\cdots x_{\pi(j)}\,y\,x_{\pi(j+1)}\cdots x_{\pi(2k)}\qquad(0\le j\le 2k),$$

so $g_0=ys_{2k}$ and $g_{2k}=s_{2k}y$, and

$$g_k^*:=(k-1)(g_0+g_{2k})-2(g_2+g_4+\cdots+g_{2k-2}).$$

We regard g_k^* as an element of V_{2k+1}, with $y=x_{2k+1}$.

Theorem 5.12. *For every $k\ge 2$, $f_k:=\Phi(g_k^*)$ is a nonzero element of $U_{A_k}=C_N\cap I_{A_k}$.*

For $k=2$, $A_2=(3,2,1)$, and $U_{(3,2,1)}$ is 15-dimensional (by direct computation). For $k=1$ the formula gives $g_1^*=0$. In the proof, Λ is the Grassmann algebra on $e_1,e_2,\dots$, with even part Λ_0 and odd part Λ_1, and

$$M_{1,1}(\Lambda):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}:a,d\in\Lambda_0,\ b,c\in\Lambda_1\right\}\subseteq M_2(\Lambda).$$

Proof. (1) *g_k^* is a consequence of s_{2k}, hence an identity of $M_k(F)$.* Let O be the sum of the g_j over odd j, and $Q:=g_2+g_4+\cdots+g_{2k-2}$. The substitution $x_i\mapsto x_iy$ is an endomorphism of the

free algebra, so $s_{2k}(x_1,\dots,x_iy,\dots,x_{2k}) \in (s_{2k})^T$, and

$$\sum_{i=1}^{2k} s_{2k}(x_1,\dots,x_iy,\dots,x_{2k}) = g_1+\cdots+g_{2k}$$

(each term puts y right after the letter x_i; summing over i puts y once after every position). Also $s_{2k+1}(y,x_1,\dots,x_{2k}) = \sum_j (-1)^j g_j$, and $s_{2k+1} \in (s_{2k})^T$ by the expansion $s_{2k+1}(z_1,\dots,z_{2k+1}) = \sum_i (-1)^{i-1} z_i\, s_{2k}(z_1,\dots,\widehat{z_i},\dots,z_{2k+1})$. So g_0, g_{2k}, $Q+O$ and $Q-O$ lie in $(s_{2k})^T$; hence so do Q and g_k^*. By the Amitsur–Levitzki theorem [AL50], g_k^* is an identity of $M_k(F)$.

(2) *Dictionary.* If $g \in V_{N-1}$ is an identity of $M_k(F)$, then $\Phi(g) \in \bigoplus_{\ell(\mu)\geq k+1} I_\mu$: this is the Razmyslov–Procesi description of the multilinear trace identities of M_k, applied to $\mathrm{tr}(g\,x_N)$ [Ra74, Pr76, Fo91], as in Remark 5.7 (the conventions differ by $\sigma \mapsto \sigma^{-1}$, which fixes every I_μ). So $f_k \in \bigoplus_{\ell(\mu)\geq k+1} I_\mu$.

(3) *Summation by parts.* Put $c_j := [x_{2j-1}, x_{2j}]$ and $P_i := \mathrm{Alt}(c_1\cdots c_i\, y\, c_{i+1}\cdots c_k)$, where $\mathrm{Alt} = \sum_{\tau\in S_{2k}} \mathrm{sgn}(\tau)\tau$ renames $x_1,\dots,x_{2k}$. Expanding the c_j gives 2^k words, each with coefficient $(-1)^{(\text{number of swapped pairs})}$ and with that same sign as a permutation; so $P_i = 2^k g_{2i}$. With $w_i := i(k-i)$ ($w_0 = w_k = 0$), the sum $\sum_{i=1}^{k-1} w_i(P_{i+1} - 2P_i + P_{i-1})$ has coefficient $w_1 = w_{k-1} = k-1$ at P_0 and P_k, and $w_{m-1} - 2w_m + w_{m+1} = -2$ at P_m, $1 \leq m \leq k-1$; so

$$g_k^* = 2^{-k} \sum_{i=1}^{k-1} i(k-i)\,(P_{i+1} - 2P_i + P_{i-1}).$$

Expanding $[c_i,[c_{i+1},y]]$ into four terms, and using that swapping the pairs i and $i+1$ is an even permutation of the x's, gives $P_{i+1} - 2P_i + P_{i-1} = \mathrm{Alt}\big(c_1\cdots c_{i-1}[c_i,[c_{i+1},y]]c_{i+2}\cdots c_k\big)$. Split Alt over the left cosets of the S_4 that permutes $x_{2i-1},\dots,x_{2i+2}$, with signs; the signed sum over this S_4 turns the word into $c_1\cdots c_{i-1}R(x_{2i-1},\dots,x_{2i+2},y)c_{i+2}\cdots c_k$, where

$$R(x_1,x_2,x_3,x_4,y) := \sum_{\pi\in S_4} \mathrm{sgn}(\pi)\,[[x_{\pi1},x_{\pi2}],[[x_{\pi3},x_{\pi4}],y]].$$

So $g_k^* \in (R)^T$. (For $k=2$ the same computation gives $R = 4g_2^*$.)

(4) *R is an identity of $M_{1,1}(\Lambda)$.* R is multilinear, so it suffices to substitute spanning elements $u\otimes\xi$: u a matrix unit and ξ a Grassmann monomial, even for $u = e_{11}, e_{22}$ and odd for $u = e_{12}, e_{21}$. Even monomials are central and factor out, so we may take $\xi = 1$ for the diagonal units; an odd monomial is an even monomial times one generator, so each off-diagonal variable may be taken as $u\otimes\eta$ with η one of the generators e_j; and if two variables share a generator, every term vanishes. So the worst case is distinct generators $\eta_1,\eta_2,\dots$, and then a word w contributes $u_{w_1}\cdots u_{w_5}\otimes \varepsilon_{\mathrm{odd}}(w)\,\eta_1\cdots\eta_r$, where $\varepsilon_{\mathrm{odd}}(w)$ is the sign of the order in which the odd letters occur in w. So R vanishes on $M_{1,1}(\Lambda)$ iff $\sum_w c_w \varepsilon_{\mathrm{odd}}(w) u_{w_1}\cdots u_{w_5} = 0$ in $M_2(F)$ for each of the $4^5 = 1024$ assignments of matrix units. This finite check was done by computer, in exact integer arithmetic, by independent programs: every one of the 1024 sums is 0. With step (3), g_k^* is an identity of $M_{1,1}(\Lambda)$.

(5) *Super dictionary.* Let $\mathcal{W} = Fw_0 \oplus Fw_1$ (w_0 even, w_1 odd), and let S_N act on $\mathcal{W}^{\otimes N}$ by permuting the factors with the Koszul sign.

(a) *The kernel of $F[S_N] \to \mathrm{End}(\mathcal{W}^{\otimes N})$ is $\bigoplus_{\mu_2\geq 2} I_\mu$* (the hook theorem of Berele–Regev [BR87] and Sergeev [Se85], here for $F^{1|1}$; see [CW12]). Indeed, let $\mathcal{W}_r$ be the span of the basis tensors with r odd factors. S_N permutes them up to sign, transitively, and the stabilizer $S_{N-r}\times S_r$ of $w_0^{\otimes(N-r)}\otimes w_1^{\otimes r}$ acts on it by $\mathrm{triv}\boxtimes\mathrm{sgn}$. So $\mathcal{W}_r \cong \mathrm{Ind}_{S_{N-r}\times S_r}^{S_N}(\mathrm{triv}\boxtimes\mathrm{sgn})$, which by Pieri's rule is $S^{(N-r,1^r)} \oplus S^{(N-r+1,1^{r-1})}$ for $1 \leq r \leq N-1$; $\mathcal{W}_0$ is trivial and $\mathcal{W}_N$ is the sign module. So exactly the hooks occur.

(b) *If $g \in V_{N-1}$ is an identity of $M_{1,1}(\Lambda)$, then $\Phi(g)$ acts as 0 on $\mathcal{W}^{\otimes N}$.* $M_{1,1}(\Lambda)$ is the even part of $\mathrm{End}(\mathcal{W})\otimes\Lambda$. For even $X_1,\dots,X_N$, acting Λ-linearly on $(\mathcal{W}\otimes\Lambda)^{\otimes N}$, the supertrace of $\sigma\circ(X_1\otimes\cdots\otimes X_N)$ is the product over the cycles $(i_1\ \dots\ i_r)$ of σ of $\mathrm{str}(X_{i_1}\cdots X_{i_r})$, up to orientation: this is the super form of the classical trace identity, with no Koszul signs because every X_i is even [Be88, Ra87]. So for $a := \Phi(g)$, $\mathrm{str}(a\circ(X_1\otimes\cdots\otimes X_N)) = \mathrm{str}(g(X_1,\dots,X_{N-1})X_N) = 0$ for all $X_i \in M_{1,1}(\Lambda)$. Take $X_i = A_i\otimes\xi_i$ with $A_i \in \mathrm{End}(\mathcal{W})$ homogeneous and $\xi_i = 1$ (A_i even) or a fresh generator (A_i odd). Then $X_1\otimes\cdots\otimes X_N = \pm(A_1\otimes\cdots\otimes A_N)\otimes\prod_i\xi_i$, with one sign for each parity pattern and a nonzero Grassmann product; so $\mathrm{str}(a\circ(A_1\otimes\cdots\otimes A_N)) = 0$ for all homogeneous A_i. These $A_1\otimes\cdots\otimes A_N$ span $\mathrm{End}(\mathcal{W}^{\otimes N})$, and the supertrace form is nondegenerate; so a acts as 0.

Neither the orientation convention (I_μ is stable under $\sigma\mapsto\sigma^{-1}$) nor the super-sign convention ($\varepsilon(I_\mu) = I_{\mu'}$, and the set of hooks is closed under $\mu\mapsto\mu'$) changes the kernel in (a). By (a), (b) and step (4), $f_k \in \bigoplus_{\mu_2\ge 2} I_\mu$.

(6) *Sign twist.* By steps (2) and (5), $f_k \in C_N \cap \bigoplus_{\mu\in S} I_\mu$ with $S = \{\mu : \ell(\mu)\ge k+1,\ \mu_2\ge 2\}$, so $f_k \in \bigoplus_{\mu\in S\cap S'} I_\mu$ by Proposition 5.1. A partition of $2k+2$ with $\ell(\mu)\ge k+1$ and $\mu_1\ge k+1$ uses at least $2k+1$ cells in its first row and column, so it is $(k+2,1^k)$, $(k+1,1^{k+1})$ or A_k; and $\mu_2\ge 2$ leaves only A_k. So $f_k \in C_N\cap I_{A_k}$.

(7) *Nonzero.* Φ is injective, the g_j have disjoint sets of words, and the coefficient of g_0 in g_k^* is $k-1\neq 0$. □

Remark 5.13. Polynomial identities of $M_{1,1}(\Lambda)$ and supertrace identities must not be confused: the ordinary cocharacter of $M_{1,1}(\Lambda)$ lies in the $(2,2)$-hook (for $M_{k,l}(\Lambda)$ it lies in the $(k^2+l^2, 2kl)$-hook [BR87]), while step (5) concerns the $(1,1)$-hook of the supertrace identities. By Leron [Le73], every multilinear identity of M_k of degree $2k+1$ ($k>2$) is a consequence of s_{2k}; by Remark 5.7 (both directions) and the partition count in step (6), these identities correspond to $C_N\cap\big(I_{A_k}\oplus I_{(k+2,1^k)}\oplus I_{(k+1,1^{k+1})}\big)$. Leron's theorem is not used above.

5.6. **Components and the deletion map.** These two facts will be used in the decoupling step (Section 6).

Proposition 5.14 (component reduction)**.** *Let $S = S'$ and $f \in C_N \cap \bigoplus_{\lambda\in S} I_\lambda$, with components f_λ and $M_\lambda := \rho_\lambda(f_\lambda)$.*

(i) For each pair $\lambda\neq\lambda'$ in S, $f_{\lambda'} = (-1)^{N-1}\varepsilon(f_\lambda)$.

(ii) For each $\lambda = \lambda'$ in S, $\Gamma M_\lambda\Gamma^{-1} = (-1)^{N-1}M_\lambda$, so the component f_λ (which need not lie in C_N) vanishes on every permutation of sign $-(-1)^{N-1}$.

(iii) For a pair, $f_\lambda + f_{\lambda'}$ takes the value $f_\lambda(\sigma)\big(1+(-1)^{N-1}\mathrm{sgn}(\sigma)\big)$ at σ.

So f is determined by its components on one shape of each conjugate pair and on the self-conjugate shapes.

Proof. (i) is the proof of Proposition 5.1. (ii): on I_λ with $\lambda=\lambda'$, ε acts as $M\mapsto\Gamma M\Gamma^{-1}$, because $\rho(\varepsilon(\sigma)) = \mathrm{sgn}(\sigma)\rho(\sigma) = \Gamma\rho(\sigma)\Gamma^{-1}$; and $\varepsilon(f_\lambda) = (-1)^{N-1}f_\lambda$ gives $\mathrm{sgn}(\sigma)f_\lambda(\sigma) = (-1)^{N-1}f_\lambda(\sigma)$. (iii): $(\varepsilon g)(\sigma) = \mathrm{sgn}(\sigma)g(\sigma)$. □

So f vanishes automatically on the permutations of sign $-(-1)^{N-1}$; on those of sign $(-1)^{N-1}$, $f(\sigma) = 2\sum f_\lambda(\sigma) + \sum_{\lambda=\lambda'} f_\lambda(\sigma)$, the first sum over one shape of each pair in S. The condition "f vanishes off the N-cycles" says that this sum vanishes on the non-N-cycles of sign $(-1)^{N-1}$. Only when S is a single pair is it a condition on one component alone. For example, at $N = n^2-2$ (the step $d(n)\le n^2-3$ of Theorem F) the two box shapes $(n^{n-1}, n-2)$ and $(n^{n-2},(n-1)^2)$ are conjugate to each other, so the problem reduces to one block, with the condition only on the permutations of sign $(-1)^{N-1}$.

Proposition 5.15 (deletion map)**.** *Let $\delta\colon F[S_{N+1}] \to F[S_N]$ send σ to the permutation of $[N]$ obtained by cutting the point $N+1$ out of its cycle: $\delta(\sigma)(i) = \sigma(i)$ if $\sigma(i) \neq N+1$, and $\delta(\sigma)(i) = \sigma(N+1)$ otherwise. Then δ is a map of S_N-bimodules, $\delta(C_{N+1}) \subseteq C_N$, and for $\lambda \vdash N+1$*

$$\delta(I_\lambda) = \bigoplus \{I_{\lambda - X} : X \text{ a corner of } \lambda,\ c_X \neq -1\},$$

where c_X is the content of X. Hence $\delta(C_{N+1} \cap B_n^{(N+1)}) \subseteq C_N \cap B_n^{(N)}$.

Proof. $\delta(g\sigma h) = g\,\delta(\sigma)h$ for $g, h \in S_N$: check the two cases $\sigma(h(i)) \neq N+1$ and $\sigma(h(i)) = N+1$ (h fixes $N+1$). Cutting a point out of an $(N+1)$-cycle leaves an N-cycle. The preimages of $\tau \in S_N$ are τ and the N permutations $\tau \circ (i, N+1)$, $i \leq N$, so $\delta(f) = \mathrm{Res}_{S_N}\big(f \cdot (1 + J_{N+1})\big)$, where Res_{S_N} keeps the coefficients on S_N. As an S_N-bimodule, I_λ is the sum of the $S^\mu \boxtimes S^{\tilde\mu}$ over the shapes $\mu, \tilde\mu$ obtained from λ by removing a corner, while $F[S_N]$ has only the types $S^\mu \boxtimes S^\mu$, inside I_μ; so $\delta(I_\lambda) \subseteq \bigoplus_X I_{\lambda - X}$. For $f \in I_\lambda$, J_{N+1} acts on the child $S^{\lambda - X}$ by c_X (J), so the $I_{\lambda-X}$-part of $\delta(f)$ is $(1 + c_X)$ times the $I_{\lambda-X}$-part of $\mathrm{Res}_{S_N}(f)$; and the latter runs over all of $I_{\lambda-X}$, since by (F2) and (F5) it corresponds to the $(\lambda - X)$-diagonal block of $\rho_\lambda(f)$, which is arbitrary. This gives the formula. Every partition inside a box shape is again in the box. □

Proposition 5.15 gives no injectivity or surjectivity between the spaces $C_{N+1} \cap B_n^{(N+1)}$ and $C_N \cap B_n^{(N)}$: δ kills whole blocks ($\delta(I_{(2,2,2)}) = 0$, as its one corner has content -1), and $\delta(I_\lambda)$ can miss a child block (δ kills the $I_{(N)}$-part of $I_{(N,1)}$).

6. Decoupling: Theorems F and G

By Proposition 2.1, $d(n) \leq n^2 - k - 1$ if and only if $C_N \cap B_n^{(N)} = 0$ for $N = n^2 - k$. For $k \geq 2$ the ideal $B_n^{(N)}$ is a sum of several blocks, and an element of $C_N \cap B_n^{(N)}$ need not lie in any one of them. In this section we restrict to S_{N-1}, where each block splits into the blocks of its children, and we use the central element $\mathcal{T}$ to separate the parents of a common child. This reduces $C_N \cap B_n^{(N)} = 0$, level by level, to statements about the blocks of a few shapes near the corner of the box.

6.1. **Restriction to S_{N-1}.** In this section $\mathrm{Res}\colon F[S_N] \to F[S_{N-1}]$ keeps the coefficients on $S_{N-1} = \mathrm{Stab}(N)$, as in the proof of Proposition 5.15. For $f \in F[S_N]$ and $\lambda \vdash N$, $f_\lambda := f e_\lambda$ is the I_λ-component of f. For a set S of partitions of a number M put

$$U(S) := C_M \cap \bigoplus_{\lambda \in S} I_\lambda,$$

so that $U(\{\lambda\}) = U_\lambda$, $U(\emptyset) = 0$, and $C_N \cap B_n^{(N)} = U(S)$ for the set S of the partitions of N inside the box. As in Corollary 5.10, $\mathcal{T} = \sum_{i<j}(i\,j)$ is central and acts on I_λ by the content sum $c(\lambda)$, so $f\mathcal{T}^i = \sum_\lambda c(\lambda)^i f_\lambda$. Let $\mathrm{cyc}(\sigma)$ be the number of cycles of σ, fixed points included, and for $M \geq 1$ and $i \geq 0$ let $\mathcal{D}^{(i)}(M)$ be the span of the $\sigma \in S_M$ with $\mathrm{cyc}(\sigma) \leq i$ and $\mathrm{cyc}(\sigma) \equiv i \pmod 2$. So $\mathcal{D}^{(0)}(M) = 0$, $\mathcal{D}^{(1)}(M) = C_M$, and $\mathcal{D}^{(2)}(M)$ is the span of the permutations with exactly two cycles.

Let S be a set of partitions of N. A *child* of S is a partition $\mu \vdash N-1$ with $\mu \subset \lambda$ for some $\lambda \in S$; these λ are the *parents* of μ, $p(\mu)$ is their number, and $p_{\max}(S)$ is the largest $p(\mu)$. For $i \geq 0$, $\mathrm{ch}_{>i}(S)$ is the set of the children with more than i parents. A child with one parent is *private*.

Lemma 6.1 (restriction)**.** *Let $N \geq 2$.*

(a) For $h \in I_\lambda$, $\mathrm{Res}\, h \in \bigoplus_X I_{\lambda - X}$, the sum over the removable corners X of λ.
(b) Two different parents of the same child have different content sums.
(c) For $f \in C_N$ and $i \geq 0$, $\mathrm{Res}(f\mathcal{T}^i) \in \mathcal{D}^{(i)}(N-1)$. In particular $\mathrm{Res}\, f = 0$ and $\mathrm{Res}(f\mathcal{T}) \in C_{N-1}$.
(d) For $f \in C_N$, $\mathrm{Res}(f\mathcal{T}) = \delta(f)$, where $\delta\colon F[S_N] \to F[S_{N-1}]$ is the deletion map of Proposition 5.15 (with $N-1$ in place of N).

Proof. (a) Let $M=\rho_\lambda(h)$, so $h(\sigma)=(f^\lambda/N!)\operatorname{tr}(\rho_\lambda(\sigma^{-1})M)$ by (F2). By (F5), in a suitable basis $\rho_\lambda(\sigma)=\bigoplus_X \rho_{\lambda-X}(\sigma)$ for $\sigma\in S_{N-1}$. So for $\sigma\in S_{N-1}$, $h(\sigma)=\sum_X (f^\lambda/N!)\operatorname{tr}(\rho_{\lambda-X}(\sigma^{-1})M_X)$, where M_X is the diagonal block of M for $S^{\lambda-X}$. By (F2) for S_{N-1}, the functions $\sigma\mapsto\operatorname{tr}(\rho_\mu(\sigma^{-1})A)$, $A\in\operatorname{End}(S^\mu)$, span I_μ.

(b) If $\lambda\neq\lambda'$ are parents of μ, then $\lambda=\mu+x$ and $\lambda'=\mu+x'$ for two different addable cells x,x' of μ, and $c(\lambda)-c(\lambda')=c(x)-c(x')$. The contents of the addable cells of a partition strictly decrease from the top row to the bottom row, so this is not 0.

(c) The coefficient of $\sigma\in S_{N-1}$ in $f\mathcal{T}^i$ is $\sum f(\sigma t_1\cdots t_i)$, over the ordered i-tuples of transpositions (each t is an involution, and reversing the order of a tuple is a bijection). Right multiplication by a transposition changes the number of cycles by exactly 1: it joins two cycles or splits one. Since f vanishes off the N-cycles, a nonzero term needs i such steps from $\operatorname{cyc}(\sigma)$ (counted in S_N) down to 1; so $\operatorname{cyc}(\sigma)-1\le i$ and $\operatorname{cyc}(\sigma)-1\equiv i \pmod 2$. Since σ fixes N, its number of cycles on $[N-1]$ is $\operatorname{cyc}(\sigma)-1$, and this is ≥ 1 because $N-1\ge 1$. Hence $\operatorname{Res}(f\mathcal{T}^i)\in\mathcal{D}^{(i)}(N-1)$; for $i=0,1$ this is the last claim.

(d) By the proof of Proposition 5.15, $\delta(f)=\operatorname{Res}(f(1+J_N))$. Write $\mathcal{T}=\mathcal{T}'+J_N$ with $\mathcal{T}':=\sum_{i<j<N}(i\,j)\in F[S_{N-1}]$. Res is a map of right $F[S_{N-1}]$-modules, so for $f\in C_N$, $\operatorname{Res}(f\mathcal{T}')=(\operatorname{Res} f)\mathcal{T}'=0$ by (c); also $\operatorname{Res} f=0$. So $\operatorname{Res}(f\mathcal{T})=\operatorname{Res}(fJ_N)=\delta(f)$. □

Under Φ (Section 2), the N-cycle $\Phi(w)$ of a word w corresponds to the trace monomial $\operatorname{tr}(w\,x_N)$, and cutting N out of the cycle gives $\operatorname{tr}(w)$, that is, "put $x_N=1$". So by (d) the map $f\mapsto\operatorname{Res}(f\mathcal{T})$ on C_N is the trace map of the Razmyslov–Procesi correspondence mentioned in the introduction.

6.2. **The restriction lemma.**

Lemma 6.2 (restriction lemma)**.** *Let $N\ge 2$ and let S be a set of partitions of N. Assume, for each i with $1\le i\le p_{\max}(S)-1$,*

$$(H_i)\qquad \mathcal{D}^{(i)}(N-1)\cap\bigoplus_{\mu\in\mathrm{ch}_{>i}(S)} I_\mu=0.$$

Then $f_\lambda\in K_\lambda$ for every $f\in U(S)$ and every $\lambda\in S$.

So (H_1) says $U(\mathrm{ch}_{>1}(S))=0$, a statement at size $N-1$ about the children with at least two parents; for $p_{\max}(S)=1$ there is no hypothesis. The assumption $N\ge 2$ is needed: for $S=\{(1)\}$, the identity lies in $U(S)$, but $K_{(1)}=0$, because the identity of S_1 has a fixed point.

Proof. Let $f\in U(S)$. For a child μ and a parent λ of μ, let $R_{\lambda,\mu}$ be the I_μ-component of $\operatorname{Res} f_\lambda$. By Lemma 6.1(a), $\operatorname{Res} f_\lambda=\sum_\mu R_{\lambda,\mu}$, over the children μ of λ. Since $f\mathcal{T}^i=\sum_\lambda c(\lambda)^i f_\lambda$, the I_μ-component of $\operatorname{Res}(f\mathcal{T}^i)$ is

$$\eta_i(\mu):=\sum_{\lambda\text{ a parent of }\mu} c(\lambda)^i\,R_{\lambda,\mu}.$$

We show by induction on i that $\eta_i(\mu)=0$ for every child μ and $0\le i\le p_{\max}(S)-1$. For $i=0$, $E_0(\mu)$ is a component of $\operatorname{Res} f=0$ (Lemma 6.1(c)). Let $1\le i\le p_{\max}(S)-1$ and assume $\eta_0=\cdots=\eta_{i-1}=0$ for every child. If $p(\mu)\le i$, the equations $\eta_0(\mu)=\cdots=\eta_{p(\mu)-1}(\mu)=0$ form a Vandermonde system in the $p(\mu)$ numbers $c(\lambda)$, λ a parent of μ, which are distinct by Lemma 6.1(b); so every $R_{\lambda,\mu}=0$, and $\eta_i(\mu)=0$. Hence $\operatorname{Res}(f\mathcal{T}^i)=\sum_\mu\eta_i(\mu)$ lies in $\bigoplus_{\mu\in\mathrm{ch}_{>i}(S)} I_\mu$, and by Lemma 6.1(c) also in $\mathcal{D}^{(i)}(N-1)$. By (H_i), $\operatorname{Res}(f\mathcal{T}^i)=0$, so $\eta_i(\mu)=0$ for every μ.

Now every child μ has $\eta_0(\mu)=\cdots=\eta_{p(\mu)-1}(\mu)=0$, since $p(\mu)\le p_{\max}(S)$, and the Vandermonde argument gives $R_{\lambda,\mu}=0$ for all λ and μ. So $\operatorname{Res} f_\lambda=0$: f_λ vanishes on $\operatorname{Stab}(N)$. For $g\in S_N$, $g^{-1}fg$ again lies in $U(S)$ (C_N and every block are stable under conjugation), and $(g^{-1}fg)_\lambda=g^{-1}f_\lambda g$, since e_λ is central; the hypotheses do not involve f. So $g^{-1}f_\lambda g$ vanishes on $\operatorname{Stab}(N)$, that is, f_λ vanishes on $g\operatorname{Stab}(N)g^{-1}=\operatorname{Stab}(g(N))$. Every permutation with a fixed point lies in some $\operatorname{Stab}(j)$, so $f_\lambda\in K_\lambda$. □

Remark 6.3 (the hypotheses). (1) (H_1) cannot be dropped. Let $N = 6$ and $S = \{(4,1,1), (3,1,1,1)\}$. The only common child is the hook $(3,1,1)$, and $U_{(3,1,1)} \neq 0$ (Theorem D), so (H_1) fails. The shape $(4,1,1)$ has two corners, with contents 3 and -2, and $4c(\lambda) = 12 \neq 6 = N(a+b)$; so $K_{(4,1,1)} = 0$ by Corollary 5.10, and $K_{(3,1,1,1)} = \varepsilon(K_{(4,1,1)}) = 0$. So the conclusion of Lemma 6.2 would give $U(S) = 0$. But $\dim U(S) = 5$, by direct computation.

(2) (H_2) cannot be dropped either. By direct computation over all 2^{11} sets S of partitions of 6: exactly 12 of them have $p_{\max}(S) = 3$ and satisfy (H_1) but not (H_2), and for each of these 12 the conclusion of Lemma 6.2 fails.

(3) A hypothesis (H_i) can fail for small N by counting alone: two subspaces of $F[S_{N-1}]$ whose dimensions add up to more than $(N-1)!$ meet, so (H_i) needs $\sum_{\mu \in \mathrm{ch}_{>i}(S)} (f^\mu)^2 \leq (N-1)! - \dim \mathcal{D}^{(i)}(N-1)$.

6.3. **Two parents: Theorem F.**

Proposition 6.4 (a conjugate pair)**.** *Let $N \geq 2$ and let $\lambda \vdash N$ with $\lambda \neq \lambda'$, where λ has exactly two removable corners, with contents $a > b$, $4c(\lambda) \neq N(a+b)$, and $c(\lambda) \neq 0$. Let $\mathcal{M}$ be the set of the common children of λ and λ', and choose $g_1, \ldots, g_N \in S_N$ with $g_j(N) = j$. Then $\mathrm{Res}(g^{-1} f_\lambda g) = (2c(\lambda))^{-1} \mathrm{Res}(g^{-1} f g\, \mathcal{T}) \in U(\mathcal{M})$ for every $f \in U(\{\lambda, \lambda'\})$ and $g \in S_N$, and*

$$f \longmapsto \big(\mathrm{Res}(g_j^{-1} f_\lambda g_j)\big)_{j=1,\ldots,N}$$

is an injective linear map from $U(\{\lambda, \lambda'\})$ into the direct sum of N copies of $U(\mathcal{M})$. In particular $\dim U(\{\lambda, \lambda'\}) \leq N \dim U(\mathcal{M})$, and $U(\{\lambda, \lambda'\}) = 0$ if $U(\mathcal{M}) = 0$ (for instance, if λ and λ' have no common child).

Proof. Put $S = \{\lambda, \lambda'\}$. Conjugation negates contents, so $c(\lambda') = -c(\lambda) \neq c(\lambda)$. Let $f \in U(S)$, with $R_{\lambda,\mu}$ as in the proof of Lemma 6.2. Since $\mathrm{Res}\, f = 0$ (Lemma 6.1(c)), $R_{\lambda,\mu} = 0$ for a private child μ, and $R_{\lambda',\mu} = -R_{\lambda,\mu}$ for $\mu \in \mathcal{M}$. So the I_μ-component of $\mathrm{Res}(f\mathcal{T})$ is 0 for a private μ, and $c(\lambda) R_{\lambda,\mu} + c(\lambda') R_{\lambda',\mu} = 2c(\lambda) R_{\lambda,\mu}$ for $\mu \in \mathcal{M}$. Hence $\mathrm{Res}\, f_\lambda = \sum_{\mu \in \mathcal{M}} R_{\lambda,\mu} = (2c(\lambda))^{-1} \mathrm{Res}(f\mathcal{T})$, which lies in C_{N-1} (Lemma 6.1(c)) and in $\bigoplus_{\mu \in \mathcal{M}} I_\mu$, that is, in $U(\mathcal{M})$. The same holds for $g^{-1} f g \in U(S)$, whose λ-component is $g^{-1} f_\lambda g$. If all $\mathrm{Res}(g_j^{-1} f_\lambda g_j)$ vanish, then f_λ vanishes on every $\mathrm{Stab}(j)$, as in the proof of Lemma 6.2; so $f_\lambda \in K_\lambda$, and $K_\lambda = 0$ by Corollary 5.10, since $4c(\lambda) \neq N(a+b)$. Then $f_{\lambda'} = (-1)^{N-1} \varepsilon(f_\lambda) = 0$ by Proposition 5.14(i), and $f = 0$. □

Theorem 6.5. *Let $n \geq 3$, $N = n^2 - 2$, $\lambda := (n^{n-1}, n-2)$, and let $B := B(n,2) = (n^{n-2}, n-1, n-2) \vdash N-1$, the $n \times n$ box minus the hook $(2,1)$ at its corner; B is self-conjugate. For every $f \in C_N \cap B_n^{(N)}$ and every $g \in S_N$,*

$$\mathrm{Res}\big((g^{-1} f g)_\lambda\big) = \tfrac{1}{2} \mathrm{Res}\big(g^{-1} f g\, \mathcal{T}\big) \in U_B,$$

and $\dim(C_N \cap B_n^{(N)}) \leq N \dim U_B$. Hence, if $U_B = 0$, then $d(n) \leq n^2 - 3$; this is Theorem F.

Proof. The partitions of N inside the box are the box minus two cells at its corner: λ and $\lambda' = (n^{n-2}, (n-1)^2)$, and $\lambda \neq \lambda'$. So $C_N \cap B_n^{(N)} = U(\{\lambda, \lambda'\})$. The box has content sum 0, and λ lacks the cells $(n, n-1)$ and (n,n), with contents -1 and 0; so $c(\lambda) = 1$. The partition λ has two corners: $(n-1, n)$, with content 1 and $\lambda - (n-1, n) = B$, and $(n, n-2)$, with content -2 and $\lambda - (n, n-2) = (n^{n-1}, n-3) =: \mu_b$; here $n \geq 3$ makes μ_b a partition. The children of λ' are the conjugates $B' = B$ and μ_b'. Now $\mu_b \not\subset \lambda'$ (μ_b has $n-1$ rows of length n, λ' only $n-2$), and so $\mu_b' \not\subset \lambda$. So B is the only common child. Finally $a + b = -1$, and $4c(\lambda) = 4 \neq -N = N(a+b)$. Proposition 6.4 applies, with $\mathcal{M} = \{B\}$ and $c(\lambda) = 1$. If $U_B = 0$, then $C_N \cap B_n^{(N)} = 0$, and $d(n) \leq N - 1 = n^2 - 3$ by Proposition 2.1. □

The hypothesis $U_{B(n,2)} = 0$. For $n = 3$, $B = (3,2,1)$ is the staircase, and $\dim U_{(3,2,1)} = 15$ (Subsection 5.5), so the hypothesis fails; Theorem F is sufficient, not necessary, since $d(3) = 6$. For

$n=4$, $B=(4,4,3,2)\vdash 13$ lies in the box, so $U_B\subseteq C_{13}\cap B_4^{(13)}$, which is 0 by $d(4)=10$ [VL93] and Proposition 2.1; since the conclusion $d(4)\le 13$ is weaker than this input, the case $n=4$ is a consistency check, not evidence for the hypothesis. For $n=5$ it holds for the same reason, as $d(5)\le 21=n^2-4$ (Subsection 1.2), and again the conclusion $d(5)\le 22$ is weaker than the input. For $n\ge 6$ the hypothesis is open. It would follow from $d(n)\le n^2-4$ (as $U_B\subseteq C_{n^2-3}\cap B_n^{(n^2-3)}$), and hence from Kuzmin's conjecture. It is a statement about one self-conjugate block: no coupling between shapes is left.

Remark 6.6 (the shapes $B(n,2)$). The partitions whose removable corners have contents exactly $\{2,0,-2\}$ are exactly the $B(n,2)$, $n\ge 3$, one for each size, and all of them are self-conjugate. Indeed, the removable contents $x_1<\cdots<x_r$ and the addable contents $y_0<\cdots<y_r$ of a partition interlace, $y_0<x_1<y_1<\cdots<x_r<y_r$, satisfy $\sum y_i=\sum x_i$, and determine the partition (they are the minima and the maxima of its profile). For $x=(-2,0,2)$ this forces $y_1=-1$, $y_2=1$ and $y_0+y_3=0$, so $(y_0,y_3)=(-n,n)$ with $n\ge 3$; that is one partition for each n, and $B(n,2)$ has addable contents $\{-n,-1,1,n\}$. The profile is symmetric, so the partition is self-conjugate. One step down, the shapes with corner contents $\{1,-1\}$ are the $B(n,1)=\nu=(n^{n-1},n-1)$ of Section 4, and $U_\nu\neq 0$ only for $n=2$ (Theorem 4.4 and the remark after it). The hypothesis of Theorem F says the same for $B(n,2)$ (open for $n\ge 6$), with the exception $n=3$; in both cases the exception is the staircase.

Remark 6.7 (toward the hypothesis). In this remark $N:=|B|=n^2-3$ (not n^2-2 as in Theorem 6.5). B has three corners, so Theorem C applies: the operators $Y_j=\rho_B(X^{(j)})$, $1\le j\le N$, generate $\mathrm{End}(S^B)$. Let $f\in I_B$ and $M=\rho_B(f)$. The operator $Y_N=\rho_B(J_N)$ has the eigenvalues $2,0,-2$ (the contents of the corners), and f vanishes on $\mathrm{Stab}(N)$ if and only if the diagonal blocks of M for these eigenspaces vanish (Remark 5.11). Since $\alpha^2+\alpha\beta+\beta^2=4$ for $\alpha\neq\beta$ in $\{2,0,-2\}$, and $3\alpha^2\neq 4$, this says $Y_N^2M+Y_NMY_N+MY_N^2=4M$. So $f\in K_B$ if and only if $Y_j^2M+Y_jMY_j+MY_j^2=4M$ for all j, and an element of U_B also satisfies $\Gamma M\Gamma^{-1}=(-1)^nM$ (Proposition 5.14(ii): $(-1)^{N-1}=(-1)^n$). Unlike in Section 4, this is not an (anti)commutation relation, so Theorem C alone does not decide U_B.

6.4. **The program: Theorem G.** Fix n. For a partition ν inside the $n\times n$ square let $\mathrm{rot}(\nu):=\{(n+1-i,n+1-j):(i,j)\in\nu\}$, and let $\lambda(\nu)$ be the box minus $\mathrm{rot}(\nu)$, a partition of $n^2-|\nu|$: the box with ν removed at its corner. The partitions of N inside the box are exactly the $\lambda(\nu)$ with $\nu\vdash n^2-N$ inside the square, and the cell $\mathrm{rot}(x)$ has content $-c(x)$. Hence:

(a) the corners of $\lambda(\nu)$ are the cells $\mathrm{rot}(x)$, for x an addable cell of ν inside the square; so the children of $\lambda(\nu)$ are the $\lambda(\nu+x)$, and the parents of $\lambda(\nu^+)$ are the $\lambda(\nu^+-y)$, for y a removable corner of ν^+;
(b) $c(\lambda(\nu))=c(\nu)$, as the box has content sum 0; and $\lambda(\nu)'=\lambda(\nu')$.

Levels. For $k\ge 0$ let $\mathcal{N}^{(0)}$ be the set of the partitions of k inside the square, and let $\mathcal{N}^{(i+1)}$ be the set of the partitions $\nu^+\vdash k+i+1$ inside the square that have at least two removable corners y with $\nu^+-y\in\mathcal{N}^{(i)}$. Let q be the least i with $\mathcal{N}^{(i)}=\emptyset$, and put $\mathcal{L}^{(i)}:=\{\lambda(\nu):\nu\in\mathcal{N}^{(i)}\}$, a set of partitions of n^2-k-i. By (a), the children of $\mathcal{L}^{(i)}$ with at least two parents in $\mathcal{L}^{(i)}$ form exactly $\mathcal{L}^{(i+1)}$, and $p(\lambda(\nu^+))$ is the number of y with $\nu^+-y\in\mathcal{N}^{(i)}$. The set $\mathcal{L}^{(0)}$ consists of all partitions of n^2-k inside the box, so $U(\mathcal{L}^{(0)})=C_{n^2-k}\cap B_n^{(n^2-k)}$. For a set S of partitions of M, the *kernel statement* $\mathrm{KF}(S)$ says: no nonzero $f\in U(S)$ has $f_\lambda\in K_\lambda$ for every $\lambda\in S$. For one shape and $M\ge 2$, $\mathrm{KF}(\{\lambda\})$ is $U_\lambda=0$, as $U_\lambda\subseteq K_\lambda$.

Corollary 6.8 (the program)**.** *Let $n\ge 1$ and $k\ge 0$, and assume that every nonempty level has size at least 2, that is, $n^2-k-(q-1)\ge 2$. Then $d(n)\le n^2-k-1$, provided that for every level $i<q$:*

(a) the hypotheses (H_j) of Lemma 6.2 hold for $S=\mathcal{L}^{(i)}$, for $2\le j\le p_{\max}(\mathcal{L}^{(i)})-1$; and

(b) $\mathrm{KF}(\mathcal{L}^{(i)})$ *holds.*

Proof. We show $U(\mathcal{L}^{(i)}) = 0$ by downward induction on i, starting from $U(\mathcal{L}^{(q)}) = U(\emptyset) = 0$. At a level $i < q$, the hypothesis (H_1) of Lemma 6.2 for $\mathcal{L}^{(i)}$ is $U(\mathcal{L}^{(i+1)}) = 0$, which holds by induction, and the other (H_j) are (a). The lemma applies, since the size is at least 2; so every $f \in U(\mathcal{L}^{(i)})$ has $f_\lambda \in K_\lambda$ for all λ, and (b) gives $f = 0$. At $i = 0$ this says $C_{n^2-k} \cap B_n^{(n^2-k)} = 0$, so $d(n) \le n^2 - k - 1$ by Proposition 2.1. □

For $k = 0$ the only level is $\mathcal{L}^{(0)} = \{(n^n)\}$, a rectangle with $K = 0$ (Remark 5.2): this is Theorem A. For $k = 1$ it is $\{\lambda(1)\}$, where $\lambda(1) = (n^{n-1}, n-1)$ is the shape ν of Section 4, and KF is $U_{\lambda(1)} = 0$: Theorem B. For $k = 2$ the levels are $\{\lambda, \lambda'\}$ and $\{B(n,2)\}$ of Theorem 6.5; KF holds at the first level because $K_\lambda = 0$, and at the second it is $U_{B(n,2)} = 0$: this is Theorem F again.

Remark 6.9 (the side condition). The levels inside the $n \times n$ square are the levels for large n intersected with the square, with the same parent counts, because every parent of a shape inside the square lies inside it. The side condition cannot be dropped: for $(n,k) = (2,2)$ the levels are $\mathcal{L}^{(0)} = \{(2),(1,1)\}$ and $\mathcal{L}^{(1)} = \{(1)\}$, and (a), (b) hold ($K_{(2)} = K_{(1,1)} = 0$, and $K_{(1)} = 0$), but $d(2) = 3$. A level of size 1 has $U = C_1 \cap I_{(1)} = F \neq 0$, so (H_1) fails one level up, and the method cannot work there anyway. By enumeration of the levels, for $2 \le n \le 7$ the side condition fails exactly when $k \ge n(n-1)$; this is far outside the range $k \le n(n-1)/2 - 1$ in which $d(n) \le n^2 - k - 1$ is possible at all, by Kuzmin's bound. For $k \le 12$ the same enumeration (for large n) gives $|\nu| \le 35$ for every ν in every level, so every shape has size at least $n^2 - 35 \ge 14$ for $n \ge 7$; so for $k \le 12$ the side condition holds for all $n \ge 7$ as well, and hence, for $k \le 12$, exactly when $k < n(n-1)$. In particular it holds for $k \le 5$ and $n \ge 3$: for $n \ge 4$ because then $|\nu| \le 10 < n^2 - 1$, and for $n = 3$ by the enumeration.

Remark 6.10 (how many hypotheses). The following table, for large n, is computed by enumerating the levels. "Levels" is the number of nonempty levels $\mathcal{L}^{(i)}$ with $i \ge 1$; "parents" is the largest $p_{\max}(\mathcal{L}^{(i)})$; "hypotheses" lists the (H_j), $j \ge 2$, needed in (a) at some level; the last column counts the shapes with at least four parents, over all levels.

k	1	2	3	4	5	6	7	8	9	10	11	12
levels	0	1	2	4	5	8	9	12	14	17	18	23
parents	1	2	2	3	3	4	4	4	5	5	5	6
hypotheses	–	–	–	H_2	H_2	H_2, H_3	H_2, H_3	H_2, H_3	H_2–H_4	H_2–H_4	H_2–H_4	H_2–H_5
≥ 4 parents	0	0	0	0	0	9	16	81	216	642	867	3670

The first shape with four parents is $\lambda(\nu)$ for $\nu = (4,3,2,1)$, at $k = 6$; its parents are the $\lambda(\nu - y)$ for the four corners y of the staircase. For fixed k the reduction is to finitely many patterns ν, each giving one statement for every n, and the number of patterns grows fast with k. A saving that grows with n ($k = k(n) \to \infty$) therefore needs statements that are proved uniformly in the shape, not one shape at a time.

For a conjugate pair the kernel statement is about one block:

Lemma 6.11 (pair criterion)**.** *Let $\lambda \vdash N$, $N \ge 2$, with $\lambda \neq \lambda'$. Then* $\mathrm{KF}(\{\lambda, \lambda'\})$ *holds if and only if no nonzero $h \in K_\lambda$ vanishes on every permutation of sign $(-1)^{N-1}$ without fixed points that is not an N-cycle.*

Proof. Let $f \in U(\{\lambda, \lambda'\})$ with $f_\lambda \in K_\lambda$. By Proposition 5.14, $f_{\lambda'} = (-1)^{N-1}\varepsilon(f_\lambda)$ and $f(\sigma) = f_\lambda(\sigma)(1 + (-1)^{N-1}\operatorname{sgn}(\sigma))$; so $h := f_\lambda$ vanishes on the permutations of sign $(-1)^{N-1}$ that are not N-cycles, and $h = 0$ only if $f = 0$. Conversely, for such an h, $f := h + (-1)^{N-1}\varepsilon(h)$ takes the value $h(\sigma)(1 + (-1)^{N-1}\operatorname{sgn}(\sigma))$, which vanishes off the N-cycles; so $f \in U(\{\lambda, \lambda'\})$, its components h and $(-1)^{N-1}\varepsilon(h)$ lie in K_λ and $K_{\lambda'}$, and $f \neq 0$ if $h \neq 0$. □

$K_\lambda = 0$ is sufficient for $\mathrm{KF}(\{\lambda, \lambda'\})$; the criterion asks less.

Proof of Theorem G. Let $n \geq 3$ and $k \leq 5$. By the table in Remark 6.10 (the parent counts inside the square are those for large n, by Remark 6.9), every shape in the levels has at most three parents, so (a) of Corollary 6.8 asks only for (H_2), the level-two condition $\mathcal{D}^{(2)}(N-1) \cap \bigoplus I_\mu = 0$ over the shapes μ with three parents, and only for $k = 4, 5$; and the side condition holds (Remark 6.9). So Corollary 6.8 gives the finite list.

Now let $k = 3$ and $n \geq 4$. The levels are $\mathcal{N}^{(0)} = \{(3), (2,1), (1,1,1)\}$, $\mathcal{N}^{(1)} = \{(3,1), (2,1,1)\}$ and $\mathcal{N}^{(2)} = \{(3,1,1)\}$, every child having two parents, and $\mathcal{N}^{(3)} = \emptyset$; there is no (H_j) with $j \geq 2$. At level 0, $\lambda(3) = (n^{n-1}, n-3)$ has two corners ($n \geq 4$), with contents 1 and -3, and $c(\lambda(3)) = c((3)) = 3$, so $4c = 12 \neq -2N = N(a+b)$; hence $K_{\lambda(3)} = 0$ (Corollary 5.10) and $K_{\lambda(1,1,1)} = \varepsilon(K_{\lambda(3)}) = 0$. So an $f \in U(\mathcal{L}^{(0)})$ with all components in the K_λ equals $f_{\lambda(2,1)} \in U_{\lambda(2,1)}$, and $\mathrm{KF}(\mathcal{L}^{(0)})$ follows from (i) below. Level 1 is the conjugate pair $\lambda(3,1)$, $\lambda(2,1,1)$, handled by Lemma 6.11, and level 2 is one shape. So $d(n) \leq n^2 - 4$ follows from:

(i) $U_{\lambda(2,1)} = 0$, where $\lambda(2,1) = B(n,2) \vdash n^2 - 3$ (the hypothesis of Theorem F);
(ii) for $N = n^2 - 4$, no nonzero $h \in K_{\lambda(3,1)}$ vanishes on every permutation of sign $(-1)^{N-1}$ without fixed points that is not an N-cycle;
(iii) $U_{\lambda(3,1,1)} = 0$, at size $n^2 - 5$.

Each is about one block or one conjugate pair. They hold for $n = 4$: they are implied by $C_{13} \cap B_4^{(13)} = 0$, $C_{12} \cap B_4^{(12)} = 0$ and $C_{11} \cap B_4^{(11)} = 0$, which follow from $d(4) = 10$ [VL93] and Proposition 2.1. For $n = 5$, (i) holds by $d(5) \leq 21$ (Subsection 1.2) and Proposition 2.1; (ii) and (iii), at the sizes 21 and 20, would follow from $d(5) \leq 20$ and $d(5) \leq 19$ and are open. All three are open for $n \geq 6$. The shapes $\lambda(2,1)$, $\lambda(3,1)$ and $\lambda(3,1,1)$ have three corners, so none of Theorem D, Corollary 5.10 or Remark 5.2 applies to them. □

Remark 6.12 ($k = 4$). For $k = 4$ and $n \geq 4$ the levels $\mathcal{N}^{(0)}, \ldots, \mathcal{N}^{(4)}$ are (the shapes are the $\lambda(\nu)$; a superscript [3] marks three parents, all other shapes in $\mathcal{N}^{(i)}$, $i \geq 1$, have two)

$$\{(4),(3,1),(2,2),(2,1,1),(1^4)\}, \qquad \{(4,1),(3,2),(3,1,1),(2,2,1),(2,1^3)\},$$
$$\{(4,2),(4,1,1),(3,2,1)^{[3]},(3,1^3),(2,2,1,1)\}, \quad \{(4,2,1)^{[3]},(4,1^3),(3,2,1,1)^{[3]}\}, \quad \{(4,2,1,1)^{[3]}\}.$$

Three steps need (H_2): $\mathcal{D}^{(2)} \cap I_{\lambda(3,2,1)} = 0$, $\mathcal{D}^{(2)} \cap (I_{\lambda(4,2,1)} \oplus I_{\lambda(3,2,1,1)}) = 0$ and $\mathcal{D}^{(2)} \cap I_{\lambda(4,2,1,1)} = 0$, at the sizes n^2-6, n^2-7, n^2-8. At level 0 the kernel statement is joint: $\lambda(2,2)$ is self-conjugate with two corners, so $K_{\lambda(2,2)} = \mathrm{span}(f_\Gamma) \neq 0$ (proof of Theorem D), and the pair $\lambda(3,1)$, $\lambda(2,1,1)$ may have $K \neq 0$; the two can meet on the class $h(\lambda(2,2))$. This happens only for odd n. For even n, $N = n^2 - 4$ is even, and Proposition 5.14(ii) gives $\Gamma M \Gamma^{-1} = -M$ on every self-conjugate block of an element of C_N, while $M = \Gamma$ for f_Γ; so $f_{\lambda(2,2)} = 0$, and the statement at level 0 is the pair statement of Lemma 6.11 for $\lambda(3,1)$, $\lambda(2,1,1)$.

Remark 6.13 ($n = 3$). For $n = 3$ and $k = 3$ ($N = 6$) the method proves a structural fact. The levels are $\{(3,3),(3,2,1),(2,2,2)\}$, $\{(3,2),(2,2,1)\}$ and $\{(2,2)\}$. Here $U_{(2,2)} = 0$ (a rectangle), and $K_{(3,2)} = 0$ by Corollary 5.10 (corner contents 2 and 0, $c = 2$, and $8 \neq 10$), so $U(\{(3,2),(2,2,1)\}) = 0$ by Lemma 6.2. At level 0, Lemma 6.2 and $K_{(3,3)} = K_{(2,2,2)} = 0$ give $C_6 \cap B_3^{(6)} = U_{(3,2,1)}$: at $N = 6$ everything lives on the staircase block, which has dimension 15. So statement (i) of the proof of Theorem G fails for $n = 3$ (there $\lambda(2,1) = (3,2,1)$), as it must, since $d(3) = 6 > 5$.

7. A Casimir criterion for the kernels K_λ

The kernel statements of Section 6 concern the spaces K_λ of the shapes $\lambda = \lambda(\nu)$ near the corner of the box. In this section we put $\lambda(\nu)$ back into the box, using the $|\nu|$ cells that are left over. Inside the single irreducible module of the box, the N conditions "f vanishes on $\mathrm{Stab}(j)$" add

up to one Casimir identity. This gives a uniform bound on the conjugation types that can occur in $K_{\lambda(\nu)}$ (Theorem 7.5) and an exact description of the joint kernel of all box shapes of one size (Theorem 7.7). Nothing in this section needs the rectangle to be a square. No new bound on $d(n)$ is claimed here.

7.1. **Setting.** Let R be a rectangle, $k \geq 0$ and $N = |R| - k \geq 1$ (in this section k is $|\nu|$, not the k of a fixed program in Section 6). Split $[|R|] = I \sqcup O$ with $I = [N]$ (a set, not the ideal of (F4)) and $O = \{N+1, \dots, |R|\}$, so $|O| = k$. For a partition $\alpha \subseteq R$, $\overline{\alpha}$ is the complement of α in R, rotated by 180 degrees; it is a partition. For $\nu \vdash k$ inside R put $\lambda(\nu) := \overline{\nu}$, the rectangle with ν removed at its corner, as in Subsection 6.4. For a partition κ with $\kappa_1 \leq k$, κ^{+k} is κ with one cell added at the bottom of each of the columns $1, \dots, k$; it has exactly k columns.

In this section $F = \mathbb{R}$; the spaces K_λ and JK_N (below) are defined over $\mathbb{Q}$, so their dimensions and their decompositions do not depend on the field of characteristic 0. Each ρ_λ is Young's orthogonal form [CST10, Ch. 3], so $\rho_\lambda(\sigma^{-1}) = \rho_\lambda(\sigma)^T$. On $\mathrm{End}(S^\lambda)$ we use $\langle A, B\rangle := \mathrm{tr}(A^T B)$ (not the coefficient form of (F3)) and the conjugation action $\mathrm{Ad}(s)A := \rho_\lambda(s) A \rho_\lambda(s)^T$, extended linearly to $F[S_N]$. For $M \in \mathrm{End}(S^\lambda)$ let $f_M \in F[S_N]$ be the element with $f_M(\sigma) = \langle M, \rho_\lambda(\sigma)\rangle$. By (F2), $f_M = (N!/f^\lambda)\, f$ for the $f \in I_\lambda$ with $\rho_\lambda(f) = M$; so $M \mapsto f_M$ is a linear bijection $\mathrm{End}(S^\lambda) \to I_\lambda$, and $f_{\mathrm{Ad}(s)M}(\sigma) = f_M(s^{-1}\sigma s)$. K_λ is stable under conjugation by S_N (vanishing on the permutations with a fixed point is a conjugation-invariant condition), so it is the direct sum of its κ-isotypic parts; κ is a *conjugation type* of K_λ if its κ-part is nonzero. For $j \in [N]$ let Δ_j be the orthogonal projection of $\mathrm{End}(S^\lambda)$ onto $\rho_\lambda(F[\mathrm{Stab}(j)])$; then f_M vanishes on $\mathrm{Stab}(j)$ iff $\Delta_j M = 0$.

$g(\alpha, \beta, \gamma)$ is the Kronecker coefficient. Let $\mathrm{Mult}(\tau) := \mathrm{Hom}_{S_{|R|}}(S^\tau, \mathrm{End}(S^R))$ for the action Ad. Under Ad, $\mathrm{End}(S^R) \cong S^R \otimes S^R$, so the τ-isotypic part $\mathrm{End}(S^R)^\tau$ is $S^\tau \otimes \mathrm{Mult}(\tau)$ (an element x of $\mathrm{Mult}(\tau)$ applied to $v \in S^\tau$), with $\dim \mathrm{Mult}(\tau) = g(R, R, \tau)$. $\mathcal{T} = \sum_t t$ (over all transpositions of $S_{|R|}$) and $\mathcal{T}_I$ (over those of S_I); $\mathrm{Ad}(\mathcal{T})$ acts on $\mathrm{End}(S^R)^\tau$ by the scalar $c(\tau)$. Let $\Omega(O)$ be the commutant of $\rho_R(S_O)$ in $\mathrm{End}(S^R)$, and for $j \in I$ let Ω_j be the commutant of $\rho_R(S_{O\cup\{j\}})$.

7.2. **Four lemmas.**

Lemma 7.1 (rectangles)**.** *For $a + b = |R|$, $\alpha \vdash a$ and $\beta \vdash b$, the Littlewood–Richardson coefficient $c^R_{\alpha,\beta}$ is 1 if $\alpha \subseteq R$ and $\beta = \overline{\alpha}$, and 0 otherwise. Hence:*

(i) $\mathrm{Res}_{S_I \times S_O} S^R = \bigoplus_{\nu \vdash k,\ \nu \subseteq R} S^{\lambda(\nu)} \boxtimes S^\nu$*, with pairwise different ν and pairwise different $\lambda(\nu)$; the same holds for $S_{I\setminus\{j\}} \times S_{O\cup\{j\}}$, with the blocks $S^{\overline{\mu}} \boxtimes S^\mu$, $\mu \vdash k+1$;*

(ii) $\Omega(O) = \bigoplus_\nu \mathrm{End}(S^{\lambda(\nu)}) \otimes 1_\nu$*;*

(iii) $\Omega_j = \rho_R(F[S_{I\setminus\{j\}}])$*;*

(iv) in Young's orthogonal basis $\{v_T\}$ of S^R, ordered by the pair (the restriction of T to I, the restriction to O), the block of ν is $S^{\lambda(\nu)} \otimes S^\nu$ with the product orthonormal basis: S_I acts by $\rho_{\lambda(\nu)} \otimes 1$, and S_O by $1 \otimes \rho'$, with ρ' orthogonally equivalent to ρ_ν.

Proof. Here and in the proof of Proposition 7.8, s_α, $s_{R/\alpha}$ and h_a are the Schur, skew Schur and complete symmetric functions (not the standard polynomials of Subsection 5.2), and $\langle\,,\rangle$ on symmetric functions is the Hall inner product [Ma95, I]. R/α rotated by 180 degrees is the straight shape $\overline{\alpha}$, and skew Schur functions do not change under this rotation, so $s_{R/\alpha} = s_{\overline{\alpha}}$ and $c^R_{\alpha,\beta} = \langle s_{R/\alpha}, s_\beta\rangle = [\beta = \overline{\alpha}]$. The map $\alpha \mapsto \overline{\alpha}$ is injective; this gives (i). (ii): $\mathrm{Res}_{S_O} S^R$ is the sum of the pairwise different S^ν, each with multiplicity space $S^{\lambda(\nu)}$; by Schur's lemma the commutant is $\bigoplus \mathrm{End}(S^{\lambda(\nu)}) \otimes 1_\nu$. (iii): in the same way the commutant of $S_{O\cup\{j\}}$ is $\bigoplus_\mu \mathrm{End}(S^{\overline{\mu}}) \otimes 1_\mu$. The $\overline{\mu}$ are pairwise different irreducibles of $S_{I\setminus\{j\}}$, so by (F6) and (F2) $\rho_R(F[S_{I\setminus\{j\}}])$ is this sum too. (iv): in Young's orthogonal form, $s_i = (i, i+1)$ acts on v_T through the contents of the cells of i and $i+1$ only, and changes T only in these two entries. For $i < N$ both lie in $T|_I$, for $i > N$ both lie in $T|_O$. So s_i $(i < N)$ acts by the orthogonal form of the shape $\lambda(\nu)$ of $T|_I$, and s_i $(i > N)$ by the orthogonal

form of the skew shape $R/\lambda(\nu)$. A rotation of $R/\lambda(\nu)$ by 180 degrees with reversed entries keeps every axial distance, so the latter is the orthogonal form of ν, relabelled by conjugation with the longest element of S_O. □

Lemma 7.2. *For $j \in I$ let $\Pi_j := \frac{1}{(k+1)!}\sum_{\pi\in S_{O\cup\{j\}}} \mathrm{Ad}(\pi)$, and for $Z \in \mathrm{End}(S^R)$ let $\phi_Z(\sigma) := \langle Z, \rho_R(\sigma)\rangle$ for $\sigma \in S_I$.*

(a) Π_j is the orthogonal projection of $\mathrm{End}(S^R)$ onto Ω_j, and for every Z: $\Pi_j Z = 0$ iff ϕ_Z vanishes on $S_{I\setminus\{j\}} = \mathrm{Stab}_{S_I}(j)$.

(b) For $Z = \sum_\nu M_\nu \otimes 1_\nu \in \Omega(O)$ with $M_\nu \in \mathrm{End}(S^{\lambda(\nu)})$: $\phi_Z = \sum_\nu f^\nu f_{M_\nu}$, with $f_{M_\nu} \in I_{\lambda(\nu)}$. The map $Z \mapsto \phi_Z$ is a linear bijection $\Omega(O) \to \bigoplus_\nu I_{\lambda(\nu)}$, and $\phi_{\mathrm{Ad}(s)Z}(\sigma) = \phi_Z(s^{-1}\sigma s)$ for $s \in S_I$.

(c) In particular, for one block $Z = M \otimes 1_\nu$: $\Pi_j Z = 0$ iff $\Delta_j M = 0$.

Proof. (a) Ad is orthogonal for $\langle\,,\rangle$, as ρ_R is orthogonal, so averaging over $S_{O\cup\{j\}}$ is the orthogonal projection onto the $\mathrm{Ad}(S_{O\cup\{j\}})$-fixed space, which is Ω_j. By Lemma 7.1(iii), Ω_j is spanned by the $\rho_R(\sigma)$, $\sigma \in S_{I\setminus\{j\}}$, so $\Pi_j Z = 0$ iff $Z \perp \rho_R(\sigma)$ for all these σ. (b) By Lemma 7.1(iv), $\rho_R(\sigma)$ ($\sigma \in S_I$) acts on the block of ν as $\rho_{\lambda(\nu)}(\sigma)\otimes 1$, so $\langle M_\nu \otimes 1_\nu, \rho_R(\sigma)\rangle = f^\nu \langle M_\nu, \rho_{\lambda(\nu)}(\sigma)\rangle$, and different blocks are orthogonal. The $\lambda(\nu)$ are pairwise different, $f^\nu \neq 0$, and $M \mapsto f_M$ is a bijection onto I_λ; so $Z \mapsto \phi_Z$ is a bijection. Equivariance: $\langle \rho(s) Z \rho(s)^T, \rho(\sigma)\rangle = \langle Z, \rho(s^{-1}\sigma s)\rangle$. (c) $\phi_Z = f^\nu f_M$, and by (a), $\Pi_j Z = 0$ iff $f_M = 0$ on $\mathrm{Stab}(j)$ iff $\Delta_j M = 0$. □

Lemma 7.3 (Casimir identity)**.** *For every $Z \in \Omega(O)$,*

$$(k+1)\sum_{j\in I} \Pi_j Z = \Big(N - \binom{k}{2}\Big) Z + \mathrm{Ad}(\mathcal{T})Z - \mathrm{Ad}(\mathcal{T}_I)Z.$$

Proof. $S_{O\cup\{j\}} = \bigsqcup_{x\in O\cup\{j\}} t_{xj} S_O$, where t_{xj} is the transposition $(x\,j)$ and $t_{jj} := \mathrm{id}$ (the left cosets of S_O are told apart by the image of j). Since $\mathrm{Ad}(s)Z = Z$ for $s \in S_O$, $(k+1)\Pi_j Z = Z + \sum_{i\in O} \mathrm{Ad}(t_{ij})Z$. Sum over $j \in I$. The transpositions t_{ij} with $i \in O$, $j \in I$ are all transpositions of $S_{|R|}$ except those of S_I and the $\binom{k}{2}$ of S_O, and each of the latter fixes Z. □

Lemma 7.4 (a minimal horizontal strip)**.** *Let $\kappa \vdash N$ and $\tau \supseteq \kappa$ with τ/κ a horizontal strip of k cells, and let $s(\tau/\kappa)$ be the sum of the contents of the cells of τ/κ. Then $s(\tau/\kappa) \geq \binom{k}{2} - N$, with equality iff $\kappa_1 \leq k$ and $\tau = \kappa^{+k}$.*

Proof. The cells lie in different columns $b_1 < \cdots < b_k$; the cell in column b sits in row $\kappa'_b + 1$ and has content $b - 1 - \kappa'_b$ (also for an empty column, where $\kappa'_b = 0$). So $s = \sum_r (b_r - 1) - \sum_r \kappa'_{b_r}$. The first sum is at least $0 + 1 + \cdots + (k-1)$, with equality iff $\{b_r\} = \{1, \ldots, k\}$; and $\sum_r \kappa'_{b_r} \leq \sum_b \kappa'_b = N$, with equality iff κ has no cell outside the columns b_r. □

7.3. **One block.**

Theorem 7.5 (Casimir criterion)**.** *Let $\nu \vdash k$ lie inside R, $\lambda = \lambda(\nu)$, and let $M \in \mathrm{End}(S^\lambda)$ lie in the κ-isotypic part for the conjugation action of S_N. Then $f_M \in K_\lambda$ if and only if $M \otimes 1_\nu$ lies in $\mathrm{End}(S^R)^{\kappa^{+k}}$. If $\kappa_1 > k$, then $f_M \in K_\lambda$ only for $M = 0$.*

Proof. $Z := M \otimes 1_\nu \in \Omega(O)$, and $\mathrm{Ad}(\mathcal{T}_I)Z = c(\kappa)Z$, as $\mathcal{T}_I$ is central in $F[S_I]$ and Z has S_I-type κ. Write $Z = \sum_\tau Z_\tau$ with $Z_\tau \in \mathrm{End}(S^R)^\tau$; these parts are mutually orthogonal. The isotypic projections are Ad of central idempotents of $F[S_{|R|}]$, so they commute with $\mathrm{Ad}(S_I \times S_O)$: each Z_τ lies in $\Omega(O)$ and has S_I-type κ. If $Z_\tau \neq 0$, then $\mathrm{Res}_{S_I\times S_O} S^\tau$ contains $S^\kappa \boxtimes (\text{trivial})$, so τ/κ is a horizontal strip of k cells (Pieri's rule), and $c(\tau) - c(\kappa) = s(\tau/\kappa)$. Since each Π_j is an orthogonal projection, Lemma 7.3 gives

$$\sum_{j\in I} \|\Pi_j Z\|^2 = \sum_{j\in I} \langle Z, \Pi_j Z\rangle = \frac{1}{k+1}\sum_\tau \Big(s(\tau/\kappa) - \Big(\binom{k}{2} - N\Big)\Big)\|Z_\tau\|^2.$$

By Lemma 7.4 every coefficient is ≥ 0, with equality iff $\tau = \kappa^{+k}$; if $\kappa_1 > k$, then κ^{+k} does not exist and every coefficient is > 0. By Lemma 7.2(c), $f_M \in K_\lambda$ iff every $\Pi_j Z = 0$, iff the left side is 0, iff $Z_\tau = 0$ for every $\tau \neq \kappa^{+k}$ (for $\kappa_1 > k$: iff $Z = 0$). □

Corollary 7.6. *(a) Every conjugation type κ of $K_{\lambda(\nu)}$ has $\kappa_1 \leq |\nu|$, and its multiplicity is at most $g(R, R, \kappa^{+k})$.*

(b) Let $\Psi\colon S^\kappa \otimes \mathrm{Mult}(\kappa^{+k}) \to \Omega(O)$ be the S_I-map that sends $v \otimes x$ to x applied to v, where S^κ is the $\kappa \boxtimes$ (trivial) component of $\mathrm{Res}_{S_I \times S_O} S^{\kappa^{+k}}$ (it occurs once, since $c^{\kappa^{+k}}_{\kappa,(k)} = 1$), and let $\pi_{\nu'}\colon \Omega(O) \to \mathrm{End}(S^{\lambda(\nu')})$ be the block projections. Put $X_\nu := \{x \in \mathrm{Mult}(\kappa^{+k}) : \pi_{\nu'}(\Psi(v \otimes x)) = 0$ for all $v \in S^\kappa$ and all $\nu' \neq \nu\}$. Then the multiplicity space $\mathrm{Hom}_{S_N}(S^\kappa, K_{\lambda(\nu)})$ is isomorphic to X_ν.

Proof. Ψ is injective, with image the S_O-fixed, S_I-type-κ part of $\mathrm{End}(S^R)^{\kappa^{+k}} \cong S^{\kappa^{+k}} \otimes \mathrm{Mult}(\kappa^{+k})$ (by Pieri's rule, multiplicity 1). By Theorem 7.5, $f_M \mapsto M \otimes 1_\nu$ maps the κ-part of $K_{\lambda(\nu)}$ injectively and S_I-equivariantly onto the part of $\Psi(S^\kappa \otimes \mathrm{Mult}(\kappa^{+k}))$ that lies in the single block ν. Its preimage is an S_I-submodule of $S^\kappa \otimes \mathrm{Mult}(\kappa^{+k})$ (S_I acting on the first factor only), hence of the form $S^\kappa \otimes X$, and $X = X_\nu$. This gives (b), and (a) since $\dim X_\nu \leq g(R, R, \kappa^{+k})$. □

For example (by direct computation), for $n = 3$ and $\nu = (2, 1)$, $K_{\lambda(\nu)} = K_{(3,2,1)}$ has the conjugation types $(3, 3)$ and $(3, 1, 1, 1)$, each once and each with $3 = |\nu|$ columns.

7.4. **The joint kernel.**

Theorem 7.7 (joint kernel)**.** *Let $\mathcal{F}_N \subseteq F[S_N]$ be the span of the permutations of $[N]$ without fixed points, and $JK_N := \big(\bigoplus_{\lambda \vdash N,\ \lambda \subseteq R} I_\lambda\big) \cap \mathcal{F}_N$. Under the conjugation action of S_N,*

$$JK_N \cong \bigoplus_{\kappa \vdash N,\ \kappa_1 \leq k} S^\kappa \otimes F^{\,g(R,R,\kappa^{+k})}.$$

Proof. (1) The shapes of size N inside R are exactly the $\lambda(\nu)$, $\nu \vdash k$, $\nu \subseteq R$. By Lemma 7.2(b), $Z \mapsto \phi_Z$ is an S_I-equivariant linear bijection $\Omega(O) \to \bigoplus_\nu I_{\lambda(\nu)}$, and by Lemma 7.2(a), $\phi_Z \in JK_N$ iff ϕ_Z vanishes on $\mathrm{Stab}_{S_I}(j)$ for every $j \in I$, iff $\Pi_j Z = 0$ for every $j \in I$. (2) The space $\{Z \in \Omega(O) : \Pi_j Z = 0$ for all $j \in I\}$ is $\mathrm{Ad}(S_I)$-stable: for $s \in S_I$, $\mathrm{Ad}(s)\Pi_j \,\mathrm{Ad}(s)^{-1} = \Pi_{s(j)}$, because s fixes O pointwise and $sS_{O \cup \{j\}}s^{-1} = S_{O \cup \{s(j)\}}$. So it is the direct sum of its κ-parts. (3) For $Z \in \Omega(O)$ of S_I-type κ (not necessarily in one block), the proof of Theorem 7.5 applies word for word, since it uses only $Z \in \Omega(O)$ and $\mathrm{Ad}(\mathcal{T}_I)Z = c(\kappa)Z$: so $\Pi_j Z = 0$ for all j iff $Z \in \mathrm{End}(S^R)^{\kappa^{+k}}$, and $Z = 0$ if $\kappa_1 > k$. (4) Hence the κ-part of JK_N is isomorphic to the S_O-fixed, S_I-type-κ part of $\mathrm{End}(S^R)^{\kappa^{+k}}$, which is $S^\kappa \otimes \mathrm{Mult}(\kappa^{+k})$, as in Corollary 7.6. □

For the square $R = (n^n)$ and $N \geq 2$, $U(S) \subseteq JK_N$ for the set S of the box shapes of size N, since N-cycles have no fixed points; so $C_N \cap B_n^{(N)} \subseteq JK_N$. For $k = 0$ no κ has $\kappa_1 \leq 0$, so $JK_{n^2} = 0$, which gives Theorem A again. For $k = 1$ the only $\kappa \vdash n^2 - 1$ with $\kappa_1 \leq 1$ is (1^{n^2-1}), with $\kappa^{+1} = (1^{n^2})$ and $g(R, R, (1^{n^2})) = 1$ (as $R = R'$). So JK_{n^2-1} is one copy of the sign representation. Over $\mathbb{C}$ it is $\mathrm{span}(f_\Gamma)$ for the shape $\lambda(1) = (n^{n-1}, n-1)$ (the shape ν of Section 4): by (Γ) and the proof of Theorem D, $K_{\lambda(1)} = \mathrm{span}(f_\Gamma)$ with $f_\Gamma \neq 0$; f_Γ has conjugation type sign and is supported on $h(\lambda(1)) = (2n-1, 2n-3, \ldots, 3)$, which has no fixed point. For $n \geq 3$, $h(\lambda(1))$ has $n - 1 \geq 2$ parts, so $C_{n^2-1} \cap B_n^{(n^2-1)} = 0$: Theorem B again.

Proposition 7.8 (two-row Kronecker coefficients of a rectangle)**.** *For every rectangle R and $1 \leq u \leq |R|/2$,*

$$g\big(R, R, (|R| - u, u)\big) = \#_u(R) - \#_{u-1}(R),$$

where $\#_i(R)$ is the number of partitions of i inside R.

This is a lemma of Pak and Panova [PP13, Lemma 3], stated there for every rectangle (see also [PPV16]); we include the short proof from Lemma 7.1.

Proof. By Jacobi–Trudi, $s_{(a,b)} = h_a h_b - h_{a+1}h_{b-1}$ for $a \geq b \geq 1$. Let $*$ be the Kronecker product of symmetric functions. Then $\langle s_R * s_R, h_a h_b\rangle = \dim(S^R \otimes S^R)^{S_a \times S_b} = \sum_{\alpha\vdash a,\beta\vdash b}(c^R_{\alpha,\beta})^2$, which by Lemma 7.1 is the number of $\alpha \vdash a$ inside R, that is $\#_a(R) = \#_b(R)$ (via $\alpha \mapsto \overline{\alpha}$). So $g(R,R,(a,b)) = \#_b(R) - \#_{b-1}(R)$. □

For the square and $k=2$, Theorem 7.7 and Proposition 7.8 give the joint kernel explicitly. The types are $\kappa = (2^i, 1^{N-2i})$, $0 \leq i \leq N/2$, $N = n^2-2$, and $\kappa^{+2} = (2^{i+1}, 1^{N-2i})$ has conjugate $(n^2-i-1, i+1)$. Since $g(R,R,\tau') = g(R,R',\tau) = g(R,R,\tau)$ for $R = R'$,

$$JK_{n^2-2} \cong \bigoplus_{0\leq i\leq N/2} S^{(2^i,1^{N-2i})} \otimes F^{m_i}, \qquad m_i = \#_{i+1}(n^n) - \#_i(n^n).$$

For $n=3$, $(m_0,\dots,m_3) = (0,1,1,0)$; for $n=4$, $(m_0,\dots,m_7) = (0,1,1,2,0,2,0,1)$.

Remark 7.9 (scope). For $\lambda(3,1,1)$ (here $|\nu| = 5$; this is statement (iii) of the proof of Theorem G, where $k=3$), Theorem 7.5 says that every conjugation type of $K_{\lambda(3,1,1)}$ has at most 5 columns. This does not decide (iii). At $n=4$ the joint kernel JK_{11} is large (by direct computation, the type $(4,3,2,1,1)$ occurs 28 times), while $K_{(4,3,3,1)} = 0$, also by direct computation; so the single-block condition of Corollary 7.6(b) carries the real content. Lemma 7.1 needs the outer shape to be a rectangle: only then is the restriction of S^R to $S_I \times S_O$ multiplicity-free with one block for each ν. The smallest rectangle containing $\lambda(3,1,1)$ is the square, so 5 columns is the best bound this method gives for $\lambda(3,1,1)$.

8. A bridge to PI degrees

The blocks I_μ correspond, through the hook theorem of Berele–Regev and Sergeev, to Kemer's superalgebras $M_{k,l}(\Lambda)$. In this section we show that an element of C_N lies in one block I_λ exactly when its polynomial is an identity of each $M_{r-1,c-1}(\Lambda)$, one for each corner (r,c) of λ (Lemma 8.2), and we translate our results into statements about PI degrees. Conversely, a lower bound for the PI degree of one algebra $M_{a,b}(\Lambda)$ gives an upper bound for $d(n)$ (Theorems 8.6 and 8.9). Samoilov [Sa10] proved an upper bound for these PI degrees and conjectured that it is sharp; Corollaries 8.7 and 8.13 record what the lower half of Samoilov's conjecture gives, and Subsection 8.5 shows that every staircase block meets the N-cycles, and Subsection 8.6 that, therefore, one-shape arguments stop at $k = S(n) - 1$.

8.1. **Kemer's algebras and the dictionary.** Λ is the Grassmann algebra of Subsection 5.5, with even part Λ_0 and odd part Λ_1. For $k,l \geq 0$, $M_{k,l}(\Lambda)$ is the algebra of $(k+l)\times(k+l)$ matrices over Λ with entries in Λ_0 in the two diagonal blocks of sizes k and l and entries in Λ_1 off them; $M_{1,1}(\Lambda)$ is the algebra of Subsection 5.5, $M_{k,l}(\Lambda) \cong M_{l,k}(\Lambda)$, and $M_{k,0}(\Lambda) = M_k(\Lambda_0)$ has the same multilinear identities as $M_k(F)$. For an algebra $\mathcal{A}$, $\mathrm{Id}(\mathcal{A}) \subseteq F\langle X\rangle$ is the T-ideal of its identities, and $\mathrm{PIdeg}(\mathcal{A})$ is the least degree of a nonzero identity; it is attained by a multilinear identity (take a nonzero multihomogeneous component, as in (F1), and linearize it). Let $\mathcal{H}(k,l) := \{\mu : \mu_{k+1} \leq l\}$, the (k,l)-hook. A *corner* of λ is a cell (r,c) with $\lambda_r = c > \lambda_{r+1}$, and λ is the union of the rectangles (c^r) over its corners. The *fat cross* of (k,l) is $\Xi_{k,l} := ((l+1)^{k+1}) \cup ((k+1)^{l+1})$; it is self-conjugate, $\Xi_{k,l} = \Xi_{l,k}$, and $|\Xi_{k,l}| = (l+1)^2 - (l-k)^2 = (k+1)(2l-k+1)$ for $k \leq l$.

Let $\mathcal{W} = F^{k|l}$, with k even and l odd basis vectors, and let S_N act on $\mathcal{W}^{\otimes N}$ by permuting the factors with the Koszul sign. The *hook theorem* of Berele–Regev [BR87] and Sergeev [Se85] says that the kernel of $F[S_N] \to \mathrm{End}(\mathcal{W}^{\otimes N})$ is exactly $\bigoplus_{\mu\notin\mathcal{H}(k,l)} I_\mu$; for $(k,l) = (1,1)$ this is step (5)(a) of the proof of Theorem 5.12. We use both directions: "no module outside the hook occurs" ($\bigoplus_{\mu\notin\mathcal{H}} I_\mu$ lies in the kernel) and "every hook module occurs" (the kernel lies in $\bigoplus_{\mu\notin\mathcal{H}} I_\mu$).

Lemma 8.1 (supertrace duality)**.** *Let $N \geq 2$, $g \in V_{N-1}$ and $f = \Phi(g)$. Then f acts as 0 on $\mathcal{W}^{\otimes N}$ if and only if $g \in \mathrm{Id}(M_{k,l}(\Lambda))$.*

Proof. "If" is step (5)(b) of the proof of Theorem 5.12, with $\mathcal{W} = F^{k|l}$ in place of $F^{1|1}$: the supertrace formula for even $X_1, \dots, X_N \in \mathrm{End}(\mathcal{W}) \otimes \Lambda$ [Be88, Ra87] holds for every (k, l), and the rest of the argument does not use $k = l = 1$. "Only if": if f acts as 0, it acts as 0 on the Λ-linear extension to $(\mathcal{W} \otimes \Lambda)^{\otimes N}$, so $\mathrm{str}(g(X_1, \dots, X_{N-1})X_N) = 0$ for all $X_i \in M_{k,l}(\Lambda)$. Fix $X_1, \dots, X_{N-1}$ and put $Y := g(X_1, \dots, X_{N-1})$. For a position (i, j) take $X_N := e_{ji} \otimes \xi$, with $\xi = 1$ if (j, i) is an even position and ξ a Grassmann generator that does not occur in Y if it is odd. Then $\mathrm{str}(YX_N) = \pm Y_{ij}\xi$, which is 0 only if $Y_{ij} = 0$. So $Y = 0$.

Conventions: $\Phi(g^{\mathrm{rev}}) = \Phi(g)^{\vee}$, where $\vee$ is the map $\sigma \mapsto \sigma^{-1}$ of (F3) and g^{rev} reverses every monomial. The kernel of the action is a two-sided ideal, hence a sum of blocks, and $\vee$ fixes each e_μ, so every block is $\vee$-stable; so the orientation of the supertrace formula does not matter. Another super-sign convention replaces σ by $\varepsilon(\sigma)$, which turns $\mathcal{H}(k, l)$ into $\mathcal{H}(l, k)$; on C_N this does not change the condition, by (F7) ($\varepsilon(f) = (-1)^{N-1}f$, so f acts as 0 iff $\varepsilon(f)$ does). □

Lemma 8.2 (the Kemer dictionary)**.** *Let $N \geq 2$, let $\lambda \vdash N$ have the corners $(r_1, c_1), \dots, (r_s, c_s)$, and let $g \in V_{N-1}$. Then*

$$\Phi(g) \in I_\lambda \iff g \in \mathrm{Id}\big(M_{r_i-1,c_i-1}(\Lambda)\big) \text{ for every } i = 1, \dots, s.$$

Proof. By the hook theorem and Lemma 8.1, $g \in \mathrm{Id}(M_{r-1,c-1}(\Lambda))$ iff $\Phi(g) \in \bigoplus_{\mu \notin \mathcal{H}(r-1,c-1)} I_\mu$, and $\mu \notin \mathcal{H}(r-1, c-1)$ iff $\mu_r \geq c$ iff $\mu \supseteq (c^r)$. The block decomposition is direct, so the intersection over i is $\bigoplus I_\mu$ over the $\mu \vdash N$ that contain every $(c_i^{r_i})$, that is, $\mu \supseteq \lambda$; and the only such μ is λ. □

For one corner this is the super form of the Razmyslov–Procesi duality [Ra87, Be88]; the intersection over several corners is a short observation. The direction "⇒" of Lemma 8.2 uses "no module outside the hook occurs" and "only if" of Lemma 8.1; the direction "⇐" uses "every hook module occurs" and "if".

Corollary 8.3 (one block)**.** *$U_\lambda \neq 0$ if and only if the* corner algebra $\bigoplus_i M_{r_i-1,c_i-1}(\Lambda)$ *satisfies a nonzero identity of degree at most $N - 1$.*

Proof. "Only if": Φ^{-1} of a nonzero element of U_λ is such an identity, by Lemma 8.2. "If": take a nonzero multilinear identity of degree $d \leq N - 1$ (as above), multiply it on the right by new letters $x_{d+1} \cdots x_{N-1}$ (identities form an ideal) and rename the letters; Φ of the result is a nonzero element of $C_N \cap I_\lambda$, by Lemma 8.2. □

Corollary 8.4 (PI degrees)**.** *Let $(k_1, l_1), \dots, (k_t, l_t)$ be pairs of nonnegative integers, with some $k_j + l_j \geq 1$, and $\Xi := \bigcup_j \Xi_{k_j,l_j}$. For $N \geq 2$,*

$$\mathrm{Id}\Big(\bigoplus_j M_{k_j,l_j}(\Lambda)\Big) \cap V_{N-1} = \Phi^{-1}\Big(C_N \cap \bigoplus_{\mu \vdash N,\ \mu \supseteq \Xi} I_\mu\Big).$$

Hence $\mathrm{PIdeg}\big(\bigoplus_j M_{k_j,l_j}(\Lambda)\big) = \min\{N : C_N \cap \bigoplus_{\mu \vdash N,\ \mu \supseteq \Xi} I_\mu \neq 0\} - 1 \geq |\Xi| - 1$*, with equality if and only if $U_\Xi \neq 0$.*

Proof. By the hook theorem and Lemma 8.1, $\mathrm{Id}(M_{k,l}(\Lambda)) \cap V_{N-1} = \Phi^{-1}(C_N \cap \bigoplus_{\mu \notin \mathcal{H}(k,l)} I_\mu)$. By Proposition 5.1 we may intersect the index set with its conjugate, and $\mu' \notin \mathcal{H}(k, l)$ iff $\mu \notin \mathcal{H}(l, k)$; and $\mu \notin \mathcal{H}(k, l) \cup \mathcal{H}(l, k)$ iff $\mu \supseteq ((l + 1)^{k+1})$ and $\mu \supseteq ((k + 1)^{l+1})$, iff $\mu \supseteq \Xi_{k,l}$. The identities of a direct sum are the intersection of the identities of the summands, and the blocks are direct; this gives the formula. An identity of degree d gives identities of every degree $N - 1 \geq d$ (as in the proof of Corollary 8.3), so PIdeg is the least $N - 1$ for which the right side is nonzero. For $N < |\Xi|$ no $\mu \vdash N$ contains Ξ, and for $N = |\Xi|$ only $\mu = \Xi$ does. □

Every self-conjugate λ is the union of the fat crosses $\Xi_{r-1,c-1}$ over its corners (r,c) (its corners come in pairs (r,c), (c,r), and λ is the union of the rectangles (c^r) over its corners), so Corollary 8.4 covers the corner algebra of every self-conjugate shape.

8.2. **Our results in the language of PI degrees.**

Proposition 8.5. *(a) For $l \geq 1$,* $\mathrm{PIdeg}(M_l(F)) = 2l$.
(b) $\mathrm{PIdeg}(M_{1,1}(\Lambda)) = 5$.
(c) For $k \geq 2$, $\mathrm{PIdeg}(M_k(F) \oplus M_{1,1}(\Lambda)) = 2k+1$.
(d) For $1 \leq k \leq l$, $\mathrm{PIdeg}(M_{k,l}(\Lambda)) \geq (k+1)(2l-k+1)$.

Proof. (a) $\Xi_{0,l} = (l+1, 1^l)$ is a self-conjugate hook, so $U_{\Xi_{0,l}} \neq 0$ (Theorem D), and Corollary 8.4 gives $\mathrm{PIdeg} = |\Xi_{0,l}| - 1 = 2l$. This is the Amitsur–Levitzki theorem [AL50], and by Lemma 5.3 the identity found is s_{2l}; nothing here is new. (b) $\Xi_{1,1} = (2,2)$. At $N=4$ only $(2,2)$ contains it, and $U_{(2,2)} = 0$ (Remark 5.2). At $N=5$ the shapes containing $(2,2)$ are $(3,2)$ and $(2,2,1)$, and $U(\{(3,2),(2,2,1)\}) = 0$ (Remark 6.13). At $N=6$ the shape $(3,2,1) = A_2$ contains $(2,2)$, and $U_{(3,2,1)} \neq 0$ (Theorem 5.12). So $\mathrm{PIdeg}(M_{1,1}(\Lambda)) = 6-1 = 5$. (c) $\Xi_{0,k} \cup \Xi_{1,1} = (k+1, 1^k) \cup (2,2) = A_k = (k+1, 2, 1^{k-1})$, of size $2k+2$, and $U_{A_k} \neq 0$ (Theorem 5.12); so Corollary 8.4 gives $2k+1$. (d) By Corollary 8.4 it suffices that $U_{\Xi_{k,l}} = 0$. If $k = l$, $\Xi_{k,l} = ((k+1)^{k+1})$ is a square (Remark 5.2). If $k < l$, $\Xi_{k,l}$ has exactly two corners, $(k+1, l+1)$ and $(l+1, k+1)$; it is self-conjugate and, as $k \geq 1$, not a hook, so $U_{\Xi_{k,l}} = 0$ by Theorem D. □

Part (b) agrees with Popov's description of the identities of $\Lambda \otimes \Lambda$ [Po82], which is PI-equivalent to $M_{1,1}(\Lambda)$ [Ke91]. Part (c) is Theorem E in the language of PI degrees: the least degree of a common identity of $M_k(F)$ and $M_{1,1}(\Lambda)$ is $2k+1$. The bound (d) gives 4 for $M_{1,1}(\Lambda)$ (true value 5), 8 for $M_{1,2}(\Lambda)$ (true value 9, by Vishne: the lower bound [Vi02, Prop. 5.3], the identities of degree 9 [Vi11]), 9 for $M_{2,2}(\Lambda)$ (true value 14, Theorem 11.11, computer-assisted), and $(k+1)^2$ for $M_{k,k}(\Lambda)$. Its order is classical: the hook theorem and Lemma 8.1 alone give $\mathrm{PIdeg}(M_{k,l}(\Lambda)) \geq (k+1)(l+1)-1$, since no $\mu \vdash N$ lies outside $\mathcal{H}(k,l)$ when $N < (k+1)(l+1)$. The bound (d) adds 1 for $k = l$ (Remark 5.2; at $k = l = 2$ the value 9 also follows from [Vi02], through the corner $M_{1,2}(\Lambda)$) and $(k+1)(l-k)+1$ for $k < l$ (the fat cross).

Samoilov [Sa10] proved that $M_{k,l}(\Lambda)$ satisfies an identity of degree $2(kl+k+l) - \min\{k,l\}$, and conjectured that this is its PI degree. For $k \leq l$ the conjectured value is $(k+1)(2l+1) - 1$, which exceeds the bound (d) by $k(k+1)-1$. Its upper half is Samoilov's theorem; its lower half is open in general. It holds for $(k,l) = (1,1)$ and $(1,2)$ (the true values above) and for $(2,2)$ (Theorem 11.11, computer-assisted).

8.3. **The transfer theorem.**

Theorem 8.6 (transfer)**.** *Let $n \geq 2$, $0 \leq k \leq n-2$, $a := n-k-1$ and $b := n-1$. If* $\mathrm{PIdeg}(M_{a,b}(\Lambda)) \geq n^2 - k$, *then $d(n) \leq n^2 - k - 1$.*

Proof. Let $N = n^2 - k$. Every $\lambda \vdash N$ inside the box is the box minus $\mathrm{rot}(\nu)$ for some $\nu \vdash k$ (Subsection 6.4); its first $n - \ell(\nu)$ rows and its first $n - \nu_1$ columns are full. As $\ell(\nu), \nu_1 \leq k$, λ contains $(n^{n-k}) \cup ((n-k)^n) = \Xi_{a,b}$. So $C_N \cap B_n^{(N)} \subseteq C_N \cap \bigoplus_{\mu \vdash N,\ \mu \supseteq \Xi_{a,b}} I_\mu$, which is $\Phi(\mathrm{Id}(M_{a,b}(\Lambda)) \cap V_{N-1})$ by Corollary 8.4, and this is 0 by the hypothesis. So $d(n) \leq N-1$ by Proposition 2.1. □

The proof uses only "no module outside the hook occurs" and "only if" of Lemma 8.1. For $k = n-1$ ($a = 0$) the hypothesis never holds, since $\mathrm{PIdeg}(M_{n-1}(F)) = 2n-2 < n^2 - n + 1$. For $k = 0, 1$ the hypothesis follows from Proposition 8.5(d), but these cases are Theorems A and B, which need no PI language; so Theorem 8.6 goes beyond Theorems A and B only for $k \geq 2$, where it needs the bound $(b+1)^2 - k$, which is $k^2 - k$ more than Proposition 8.5(d) gives.

Corollary 8.7 (a ceiling, and the lower half of Samoilov's conjecture)**.** *Put* $k^*(n) := \lfloor (n^2-n-1)/(2n-2) \rfloor$*; since* $(n^2-n-1)/(2n-2) = n/2 - 1/(2n-2)$*,* $k^*(n) = \lfloor (n-1)/2 \rfloor$*.*

(a) The hypothesis of Theorem 8.6 can hold only for $k \le k^*(n)$*.*
(b) If the lower half of Samoilov's conjecture holds for $M_{n-k^*(n)-1,n-1}(\Lambda)$*, then* $d(n) \le n^2 - k^*(n) - 1$*.*

Proof. By Samoilov's theorem, $M_{a,b}(\Lambda)$ has an identity of degree $(a+1)(2b+1)-1 = (n-k)(2n-1)-1$, and $(n-k)(2n-1)-1 \ge n^2-k$ iff $k(2n-2) \le n^2-n-1$ iff $k \le k^*(n)$. This gives (a). For (b), $k^*(n) = \lfloor (n-1)/2 \rfloor \le n-2$ for $n \ge 2$, and for $k = k^*(n)$ the conjectured value $(n-k)(2n-1)-1$ is at least n^2-k; so Theorem 8.6 applies. □

The values of $n^2 - k^*(n) - 1$ are $14, 22, 33, 45$ for $n = 4, 5, 6, 7$, and 95, 390, 2475, 9950 for $n = 10, 20, 50, 100$. Through Theorem 8.6, the full conjecture gives a saving of about $n/2$; for $n = 4$ it gives $d(4) \le 14$, which is Theorem B, and the first gain over Theorem B is at $n = 5$ ($d(5) \le 22$, still weaker than $d(5) \le 21$ of Subsection 1.2; the first gain over every known bound is at $n = 6$: $d(6) \le 33$). This ceiling concerns Theorem 8.6 only, that is, the fat cross $\Xi_{n-k-1,n-1}$; the nearly square fat crosses of Subsection 8.4 reach much further. The largest shape contained in all box shapes of size n^2-k gives the following variant:

Theorem 8.8. *Let* $n \ge 2$ *and* $0 \le k \le n^2-n-1$*, let* $\Xi_k^{\cap}$ *be the box minus* $\mathrm{rot}\{(i,j) \in [n]^2 : ij \le k\}$*, and let* $\mathcal{A}_k^{\cap} := \bigoplus M_{r-1,c-1}(\Lambda)$ *over the corners* (r,c) *of* $\Xi_k^{\cap}$*. If* $\mathcal{A}_k^{\cap}$ *satisfies no nonzero identity of degree at most* n^2-k-1*, then* $d(n) \le n^2-k-1$*.*

Proof. Every cell (i,j) of a partition $\nu \vdash k$ has $ij \le k$. Conversely, every cell $(i,j) \in [n]^2$ with $ij \le k$ lies in some $\nu \vdash k$ inside the square: the rectangle (j^i) together with $k-ij$ further cells, added in the first row up to length n, then in the next rows (possible since $k < n^2$). So $\Xi_k^{\cap}$ is the intersection of all box shapes of size n^2-k. It is self-conjugate, since the set $\{(i,j) : ij \le k\}$ is symmetric and rot commutes with transposition; hence it is the union of the fat crosses of its corners, and Corollary 8.4 for $\mathcal{A}_k^{\cap}$ replaces the one for $M_{a,b}(\Lambda)$ in the proof of Theorem 8.6. Proposition 2.1 applies, as $N = n^2-k \ge n+1$. □

For $k \le n-2$, since $(n-k,n)$ is a corner of $\Xi_k^{\cap}$ (rows $1, \dots, n-k$ are full, and row $n-k+1$ lacks its last cell), $M_{a,b}(\Lambda)$ with $a = n-k-1$, $b = n-1$ is a summand of $\mathcal{A}_k^{\cap}$, so Theorem 8.8 implies Theorem 8.6. By Corollary 8.4, its hypothesis says that $C_N \cap \bigoplus_{\mu \vdash N,\ \mu \supseteq \Xi_k^{\cap}} I_\mu = 0$ for every $N \le n^2-k$. For $k = 2$, $\Xi_2^{\cap} = B(n,2)$, and the first of these conditions is $U_{B(n,2)} = 0$, the hypothesis of Theorem F. How far Theorem 8.8 can reach is described in Subsections 8.4 and 8.6: under the lower half of Samoilov's conjecture it reaches $k = S(n)-1$ ($n \ge 3$), and no further (Corollary 8.17).

8.4. **Two parameters: a quadratic saving.** Theorem 8.6 uses one fat cross for each k, the one with $b = n-1$. Every fat cross $\Xi_{a,b}$ with $(n-a)(n-b) > k$ lies in every box shape of size n^2-k, and the nearly square ones reach much further.

Theorem 8.9 (two-parameter transfer)**.** *Let* $n \ge 2$ *and* $1 \le a \le b \le n-1$*, and put* $S := (n-a)(n-b)$*. If* $\mathrm{PIdeg}(M_{a,b}(\Lambda)) \ge n^2 - S + 1$*, then* $d(n) \le n^2 - S$*.*

Proof. Put $k := S-1$ and $N := n^2-k$; as $1 \le S \le (n-1)^2$, $k \ge 0$ and $N \ge 2n$. Every $\lambda \vdash N$ inside the box is the box minus $\mathrm{rot}(\nu)$ for some $\nu \vdash k$, and λ contains the cell (r,c) if and only if ν does not contain the cell $(n+1-r, n+1-c)$. A partition that contains the cell (i,j) contains the rectangle (j^i), so it has at least ij cells. For $(r,c) = (a+1, b+1)$ and $(r,c) = (b+1, a+1)$ this number is $(n-a)(n-b) = S > k$; so λ contains both cells, hence the rectangles $((b+1)^{a+1})$ and $((a+1)^{b+1})$, that is, $\lambda \supseteq \Xi_{a,b}$. So $C_N \cap B_n^{(N)} \subseteq C_N \cap \bigoplus_{\mu \vdash N,\ \mu \supseteq \Xi_{a,b}} I_\mu = \Phi(\mathrm{Id}(M_{a,b}(\Lambda)) \cap V_{N-1})$ (Corollary 8.4), which is 0 by the hypothesis. So $d(n) \le N-1 = n^2-S$ by Proposition 2.1. □

The condition $S > k$ is also necessary for $\Xi_{a,b}$ to lie in every box shape of size n^2-k: if $S \le k < n^2$, some $\nu \vdash k$ inside the square contains the rectangle $((n-b)^{n-a})$ (add $k-S$ cells as in the proof of Theorem 8.8), and then $\lambda(\nu)$ misses the cell $(a+1, b+1)$. For $b = n-1$, Theorem 8.9 is Theorem 8.6 ($S = k+1$). By Proposition 8.5(d), the pair $a = b = n-1$ gives Theorem A, and $(a,b) = (n-2, n-1)$ gives Theorem B. The case $a = 0$ is excluded, since $\mathrm{PIdeg}(M_b(F)) = 2b < n^2 - n(n-b) + 1$.

Corollary 8.10 (squares)**.** *Let $a \ge 1$ and $t \ge 0$. If $\mathrm{PIdeg}(M_{a,a}(\Lambda)) \ge (a+1)^2 + 2at$, then $d(a+t+1) \le (a+t+1)^2 - (t+1)^2$.*

Proof. Theorem 8.9 with $n = a+t+1$ and $b = a$: $S = (t+1)^2$ and $n^2 - S + 1 = a^2 + 2a(t+1) + 1 = (a+1)^2 + 2at$. □

Write $\mathrm{PIdeg}(M_{a,a}(\Lambda)) = (a+1)^2 + e(a)$. Proposition 8.5(d) says $e(a) \ge 0$, the case $t = 0$ of Corollary 8.10, which gives Theorem A and nothing more. (The bound $(a+1)^2 - 1$ already follows from the hook theorem: no $\mu \vdash N$ lies outside $\mathcal{H}(a,a)$ when $N < (a+1)^2$.) By Corollary 8.10, $e(a) \ge 2ta$ gives the saving $(t+1)^2$ at $n = a+t+1$. So $e(a) \ge 2a$ for all large a gives $d(n) \le n^2 - 4$ for all large n, and $e(a)/a \to \infty$ gives $n^2 - d(n) \to \infty$. Samoilov's conjectured value has $e(a) = a^2 + a - 1$. For a saving quadratic in n, the leading constant of $\mathrm{PIdeg}(M_{a,a}(\Lambda))/a^2$ is the whole question:

Corollary 8.11. *If there are $\gamma > 1$ and a_0 with $\mathrm{PIdeg}(M_{a,a}(\Lambda)) \ge \gamma a^2$ for all $a \ge a_0$, then $d(n) \le \big(1 - \big(\frac{\gamma-1}{\gamma+1}\big)^2 + o(1)\big)n^2$.*

Proof. For large n put $a := \lceil 2n/(\gamma+1) \rceil + 1$; then $a_0 \le a \le n-1$, as $\gamma > 1$. Since $(\gamma+1)a - 2n \ge \gamma + 1$, we have $(\gamma+1)a^2 - 2an \ge (\gamma+1)a \ge 1$, that is, $\gamma a^2 \ge 2an - a^2 + 1 = n^2 - (n-a)^2 + 1$. So $\mathrm{PIdeg}(M_{a,a}(\Lambda)) \ge n^2 - (n-a)^2 + 1$, and Theorem 8.9 with $b = a$ gives $d(n) \le n^2 - (n-a)^2$, where $n - a \ge cn - 2$ with $c := \frac{\gamma-1}{\gamma+1}$; so $d(n) \le n^2 - (cn-2)^2$ once $cn \ge 2$. □

By Samoilov's theorem only $\gamma \le 2$ is possible, and $\gamma = 2$ is Samoilov's conjecture. For a *fixed* saving the route $b = n-1$ asks less: with $p = n-a$, $q = n-b$, Theorem 8.9 needs $2n(q-1) + p^2 - 3pq + 2q$ more than Proposition 8.5(d) gives, which is $(p-1)(p-2)$ for $q = 1$, independent of n. For example, $d(n) \le n^2 - 4$ also follows from $\mathrm{PIdeg}(M_{n-4,n-1}(\Lambda)) \ge |\Xi_{n-4,n-1}| + 6$, while the square route needs $e(n-2) \ge 2n - 4$.

Lemma 8.12 (reach under Samoilov's value)**.** *Let $n \ge 3$ and $S(n) := \lfloor (n+2)/3 \rfloor \lfloor (n+4)/3 \rfloor$. Among the pairs $1 \le a \le b \le n-1$ with $(a+1)(2b+1) - 1 \ge n^2 - (n-a)(n-b) + 1$, the largest value of $(n-a)(n-b)$ is $S(n)$, attained at $(a,b) = (n - \lfloor (n+4)/3 \rfloor,\ n - \lfloor (n+2)/3 \rfloor)$.*

Proof. Put $p := n-a \ge q := n-b \ge 1$. The condition reads $\Theta(p,q) := n^2 - 2n(p+q) + 3n + 3pq - p - 2q - 1 \ge 0$, and we maximise pq. (a) If $p \ge q+2$ and $\Theta(p,q) \ge 0$, then $(p-1, q+1)$ is again admissible ($p-1 \ge q+1 \ge 2$), $\Theta(p-1, q+1) = \Theta(p,q) + 3(p-q) - 4 > 0$, and $(p-1)(q+1) = pq + p - q - 1 > pq$. So every optimum has $p - q \in \{0, 1\}$. (b) $p = q = s$: $\Theta = (n-s)(n-3s+3) - 1$, and $n - s \ge 1$; so $\Theta \ge 0$ iff $s \le (n+2)/3$. The best is s^2 with $s = \lfloor (n+2)/3 \rfloor$. (c) $p = s+1$, $q = s$: $\Theta = (n-s)(n-3s) + n - 2$. If $n \ge 3s$, $\Theta > 0$. If $n = 3s-1$, $\Theta = s - 2$, which is ≥ 0 iff $s \ge 2$. If $v := 3s - n \ge 2$, then $p = s+1 \le n-1$ gives $s \ge (v+2)/2$, and $\Theta = s(3-2v) + (v-2)(v+1) \le 1 - 3v/2 < 0$. So $\Theta \ge 0$ iff $s \le (n+1)/3$ (with $s \ge 2$ when $n = 3s-1$), and the best is $s(s+1)$; here $p = s+1 \le n-1$ holds for $s = \lfloor (n+1)/3 \rfloor$ as $n \ge 3$. (d) For $n = 3j$, $3j+1$, $3j+2$ the better of (b) and (c) is $j(j+1)$, $(j+1)^2$, $(j+1)(j+2)$, which is $S(n)$ in each case, and the optimal $p = \lfloor (n+4)/3 \rfloor$ is at most $n-1$ because $n \ge 3$. □

Corollary 8.13 (the lower half of Samoilov's conjecture)**.** *Let $n \ge 3$ and $(a,b) = (n - \lfloor (n+4)/3 \rfloor,\ n - \lfloor (n+2)/3 \rfloor)$. If $\mathrm{PIdeg}(M_{a,b}(\Lambda)) \ge n^2 - S(n) + 1$ (which the lower half of Samoilov's conjecture gives, by Lemma 8.12), then $d(n) \le n^2 - S(n)$; in particular $d(n) < \frac{8}{9}n^2$, since $\lfloor (n+2)/3 \rfloor \ge n/3$ and $\lfloor (n+4)/3 \rfloor \ge (n+2)/3$ give $S(n) > n^2/9$.*

Proof. Theorem 8.9 and Lemma 8.12. □

Here $b-a\le 1$. At $n=2$ the formula would need $a=0$ and would say $d(2)\le 2$, which is false; so $n\ge 3$ is needed. The values of $n^2-S(n)$ are 12, 19, 30, 40 for $n=4,5,6,7$, and 84, 344, 2194, 8844 for $n=10,20,50,100$ (Kuzmin's $n(n+1)/2$: 10, 15, 21, 28, 55, 210, 1275, 5050; Corollary 8.7: 14, 22, 33, 45, 95, 390, 2475, 9950). The needed bound and Samoilov's value at the optimal pair are 13 and 14 for $n=4$, 20 and 20 for $n=5$, 41 and 44 for $n=7$, 8845 and 8910 for $n=100$. For small n these conditional bounds are weaker than the known values $d(3)=6$ and $d(4)=10$ (for $n=5$ the value $n^2-S(n)=19$ would be below the known $d(5)\le 21$); the method is asymptotic. Since Samoilov's *theorem* bounds every $\mathrm{PIdeg}(M_{a,b}(\Lambda))$ by $(a+1)(2b+1)-1$, Lemma 8.12 also shows (unconditionally) that no single algebra $M_{a,b}(\Lambda)$ gives more than $d(n)\le n^2-S(n)$ through Theorem 8.9.

8.5. **The staircase blocks.** Let $\mathrm{St}_n:=(n,n-1,\dots,1)$, a self-conjugate partition of $n(n+1)/2$ whose corners are the cells $(r,n{+}1{-}r)$, $1\le r\le n$; so by Lemma 8.2 its corner algebra (Corollary 8.3) is $\bigoplus_{k+l=n-1}M_{k,l}(\Lambda)$. For $n\ge 2$ put

$$k_n(x,y):=\sum_{\pi}\operatorname{sgn}(\pi)\,x^{\pi(0)}yx^{\pi(1)}y\cdots yx^{\pi(n-1)}\in F\langle x,y\rangle,$$

the sum over the permutations π of $\{0,1,\dots,n-1\}$, with x^0 the empty word. It has degree $n(n+1)/2-1$. Kuzmin's word $yxyx^2\cdots yx^{n-1}$ [Do18, Thm. 1.7] is its term for $\pi=\mathrm{id}$, with coefficient 1 (the reversed word is also a term). For example $k_2=[y,x]$ and $k_3=[[x,y]^2,x]$.

Lemma 8.14. *If $k,l\ge 0$ and $k+l\le n-1$, then k_n is an identity of $M_{k,l}(\Lambda)$. For $k+l=n$ it is not, in general: in $M_n(F)\subseteq M_{n,0}(\Lambda)$, with rows and columns indexed $0,\dots,n-1$, take $x=\mathrm{diag}(0,1,\dots,n-1)$ and $y=\sum_i e_{i,i+1}$.*

Proof. For elements $z_0,\dots,z_{n-1},y$ of a unital algebra R put

$$\mathcal{C}(z_0,\dots,z_{n-1}):=\sum_{\pi}\operatorname{sgn}(\pi)\,z_{\pi(0)}yz_{\pi(1)}y\cdots yz_{\pi(n-1)}.$$

It is multilinear over the centre of R and alternating: if $z_i=z_j$ with $i\ne j$, then π and $(i\,j)\circ\pi$ give the same product with opposite signs. Also $k_n(x,y)=\mathcal{C}(1,x,\dots,x^{n-1})$. If $x^d=\sum_{i<d}c_ix^i$ with c_i central in R and $d\le n-1$, then x^{n-1} is a central combination of $1,x,\dots,x^{n-2}$, and $k_n(x,y)=0$.

Every element of $M_{k,l}(\Lambda)$ is an even supermatrix. Let $x,y\in M_{k,l}(\Lambda)$, let $\Lambda^{(r)}\subseteq\Lambda$ be the Grassmann algebra on the finitely many generators that occur in them, $J\subseteq\Lambda^{(r)}$ its augmentation ideal (so $J^{r+1}=0$), and $\Lambda_0^{(r)}$ its even part, which is commutative and central in $\Lambda^{(r)}$. Let t be an indeterminate, $R:=M_{k,l}(\Lambda^{(r)}\otimes F(t))$ and $x(t):=x+t\,\mathrm{diag}(1,2,\dots,k+l)$. Let $\bar A$, $\bar D$ be the images modulo J of the two diagonal blocks of x, so that $A_0:=\bar A+t\,\mathrm{diag}(1,\dots,k)$ and $D_0:=\bar D+t\,\mathrm{diag}(k{+}1,\dots,k{+}l)$ are those of $x(t)$. The resultant of their characteristic polynomials is t^{kl} times the resultant for $\mathrm{diag}(1,\dots,k)+t^{-1}\bar A$ and $\mathrm{diag}(k+1,\dots,k+l)+t^{-1}\bar D$, so it is a polynomial in t with leading coefficient $\prod_{i\le k<j}(i-j)\ne 0$. Hence the maps $\mathrm{Syl}\colon Z\mapsto A_0Z-ZD_0$ and $\mathrm{Syl}'\colon W\mapsto D_0W-WA_0$, whose determinants are $\pm$ this resultant, are invertible over $F(t)$. Suppose the off-diagonal blocks B, C of a conjugate of $x(t)$ have entries in J^s, and conjugate by $\omega=1+Y$, $Y=\left(\begin{smallmatrix}0&Z\\W&0\end{smallmatrix}\right)$, with odd Z,W with entries in $J^s\otimes F(t)$. The new upper-right block is $B+(A_0Z-ZD_0)+(A_+Z-ZD_+)$ plus terms with at least two factors from Y, where A_+, D_+, the parts of the diagonal blocks with zero image modulo J, are even and so have entries in J^2. So $Z:=-\mathrm{Syl}^{-1}$(the Grassmann-degree-s part of B), and W chosen in the same way with Syl', move the off-diagonal blocks into J^{s+1}; the diagonal blocks change only by even elements of J, so A_0, D_0 do not change, and ω is even and invertible (Y is nilpotent). After at most r steps we get an even invertible $\omega\in R$ with $\omega^{-1}x(t)\omega=\mathrm{diag}(A',D')$, where A', D' have entries in the commutative ring $\Lambda_0^{(r)}\otimes F(t)$. By the Cayley–Hamilton theorem over this ring, the monic polynomial $P:=p_{A'}p_{D'}$ of

degree $k+l$, whose coefficients lie in $\Lambda_0^{(r)}\otimes F(t)$ and so are central in R, satisfies $P(x(t))=0$. By the first paragraph, $k_n(x(t),y)=0$ in R. Its entries lie in $\Lambda^{(r)}\otimes F[t]\subseteq\Lambda^{(r)}\otimes F(t)$, so $k_n(x(t),y)=0$ as a polynomial in t, and $t=0$ gives $k_n(x,y)=0$.

For the example, the $(0,n-1)$ entry of $k_n(x,y)$ is $\det[s^p]_{s,p=0}^{n-1}=\prod_{i<j}(j-i)\neq 0$. □

The coefficients of P have denominators (from the resultant); degree $k+l$ suffices because the alternation needs only some monic central relation in a ring that contains $x(t)$ and y. That an even supermatrix satisfies a relation of degree $k+l$ of this kind is known [KT99, KV05]; polynomial super Cayley–Hamilton identities without denominators have higher degree.

Theorem 8.15 (staircases)**.** *Let $n\geq 2$, $N:=n(n+1)/2$, and let $g_n\in V_{N-1}$ be the full linearization of k_n. Then $\Phi(g_n)$ is a nonzero element of U_{St_n}. Equivalently,* $\mathrm{PIdeg}\big(\bigoplus_{k+l=n-1}M_{k,l}(\Lambda)\big)=n(n+1)/2-1$.

Proof. By Lemma 8.14, g_n is an identity of every $M_{r-1,n-r}(\Lambda)$, $1\leq r\leq n$ (the linearization of an identity is an identity), so $\Phi(g_n)\in I_{\mathrm{St}_n}$ by Lemma 8.2 (direction "⇐"); and $\Phi(g_n)\in C_N$. The words of k_n are distinct for distinct π, so $k_n\neq 0$; in characteristic zero $g_n\neq 0$; and Φ is injective. For the equivalence: k_n is a common identity of degree $N-1$, and by Corollary 8.4 the PI degree is at least $N-1$, with equality iff $U_{\mathrm{St}_n}\neq 0$, since the union of the fat crosses $\Xi_{k,n-1-k}$, $0\leq k\leq n-1$, is St_n. □

The element $\Phi(g_n)$ is Kuzmin's: the proof of $d(n)\geq n(n+1)/2$ in [Ku75] (written out in [Do18, §3]) shows that the alternating functional with the coefficients of k_n vanishes on the T-ideal of x^n, which is the statement $\Phi(g_n)\in C_N\cap B_n^{(N)}$ in the form of Proposition 2.1. Theorem 8.15 says more: $\Phi(g_n)$ lies in the single block of St_n. In the PI form the lower bound is free, and the content is the common identity of degree $N-1$. For $n\leq 3$ it is classical: k_3 is Hall's identity for $M_2(F)$ and one of Popov's two basic identities of $\Lambda\otimes\Lambda$ [Po82], which is PI-equivalent to $M_{1,1}(\Lambda)$ [Ke91]. For $n=4$ the summand $M_{1,2}(\Lambda)$ alone already has PI degree $9=N-1$ [Vi02, Vi11], and the common identity is k_4. One summand of the corner algebra is the middle algebra, $M_{m,m}(\Lambda)$ for $n=2m+1$ and $M_{m-1,m}(\Lambda)$ for $n=2m$, and Samoilov's conjectured PI degree for it is exactly $N-1$; so the corner algebra attains the value that Samoilov conjectures for its middle summand alone. Theorem 8.15 gives no lower bound for any single $M_{k,l}(\Lambda)$. Lemma 8.14 also gives the explicit identity k_{a+b+1} of $M_{a,b}(\Lambda)$, $a\leq b$, of degree $(a+b+1)(a+b+2)/2-1$; this never beats Samoilov's bound $(a+1)(2b+1)-1$ and ties it exactly when $b-a\in\{0,1\}$.

8.6. **The ceiling of one-shape arguments.** Formanek's condition involves all box shapes of size N jointly; we fix the form of the argument. A *one-shape argument* for $d(n)\leq N-1$ consists of a shape Ξ with $B_n^{(N)}\subseteq\bigoplus_{\mu\vdash N,\ \mu\supseteq\Xi}I_\mu$ and a proof that $C_N\cap\bigoplus_{\mu\vdash N,\ \mu\supseteq\Xi}I_\mu=0$. We may take Ξ self-conjugate (replace it by $\Xi\cup\Xi'$, by Proposition 5.1); then, by Corollary 8.4, this is the same as a lower bound for the PI degree of the corner algebra of Ξ, and Theorems 8.6, 8.8 and 8.9 are of this form. Kuzmin's problem itself is joint, not of this form. With $N=n^2-k$ the first condition says $\Xi\subseteq\Xi_k^{\cap}$. If a staircase St_m contains $\Xi_k^{\cap}$ and $U_{\mathrm{St}_m}\neq 0$, then $C_{N'}\cap\bigoplus_{\mu\supseteq\Xi}I_\mu\neq 0$ for $N'=|\mathrm{St}_m|$, hence (Corollary 8.4) for every $N\geq|\mathrm{St}_m|$, and the argument fails.

Lemma 8.16 (staircases above $\Xi_k^{\cap}$)**.** *Let $n\geq 3$ and $S(n)\leq k\leq n^2-n-1$. The smallest staircase St_m containing $\Xi_k^{\cap}$ has $|\mathrm{St}_m|\leq n^2-k$ and $m\leq\lfloor 4n/3\rfloor-1$.*

Proof. Row $r=n+1-i$ of $\Xi_k^{\cap}$ has length $n-\min(n,\lfloor k/i\rfloor)$, and St_m contains a shape λ iff $m\geq\lambda_r+r-1$ for every nonempty row r. So $m\leq 2n-g(k)$, with $g(k):=\min_{i\geq 1}(i+\lfloor k/i\rfloor)$. Next, $g(k)\leq t$ iff $k<i(t+1-i)$ for some i, iff $k\leq\lfloor(t+1)^2/4\rfloor-1$. With $t_1:=\lceil 2n/3\rceil$ one checks $\lfloor(t_1+1)^2/4\rfloor=S(n)$ (for $n=3j,3j+1,3j+2$); so $k\geq S(n)$ gives $t:=g(k)\geq t_1+1\geq(2n+3)/3$. Taking $i=n$ shows $t\leq 2n-1$; put $u:=2n-t\geq 1$. Then $m\leq u$ and $k\leq(t+1)^2/4-1$, so

$$k+|\mathrm{St}_m|-n^2\leq\tfrac{(t+1)^2}{4}-1+\tfrac{u(u+1)}{2}-n^2=\tfrac{(u-1)(3u-4n+3)}{4}\leq 0,$$

as $1 \le u \le (4n-3)/3$. Finally $m \le u \le 2n - t_1 - 1 = \lfloor 4n/3 \rfloor - 1$. □

Corollary 8.17 (ceiling)**.** *Let $n \ge 3$. No one-shape argument proves $d(n) \le n^2 - S(n) - 1$.*

Proof. Such an argument works at some $N = n^2 - k$ with $k \ge S(n)$, and $k \le n^2 - n - 1$ (for $N \le n$, $B_n^{(N)} = F[S_N]$ and $C_N \ne 0$). By Lemma 8.16 and the paragraph before it, it fails, since $U_{\mathrm{St}_m} \ne 0$ for every $m \ge 2$ (Theorem 8.15) and $U_{(1)} = C_1 \ne 0$. In fact the failure is witnessed by k_m: if $\mathrm{St}_m \supseteq \Xi_k^{\cap}$, every corner (r, c) of $\Xi_k^{\cap}$ has $(r-1)+(c-1) \le m-1$, so by Lemma 8.14 k_m is a common identity of the corner algebra $\mathcal{A}_k^{\cap}$, hence of the corner algebra of every $\Xi \subseteq \Xi_k^{\cap}$, of degree $|\mathrm{St}_m| - 1 \le N - 1$ (Lemma 8.16). □

Since $\Xi_k^{\cap} \supseteq \Xi_{a,b}$ whenever $(n-a)(n-b) > k$, the hypothesis of Theorem 8.8 is weaker than that of Theorem 8.9; so under the lower half of Samoilov's conjecture Theorem 8.8 reaches $k = S(n)-1$, and, by Corollary 8.17, no one-shape argument goes further: every such argument stops at about $\frac{8}{9}n^2$ and cannot reach Kuzmin's $n^2/2$. For example, for $n = 5$ and $k = 6, 7, 8$, $\Xi_k^{\cap} \subseteq (4, 4, 3, 2) \subseteq \mathrm{St}_5$, so k_5, of degree 14, is a common identity of $\mathcal{A}_k^{\cap}$, while Theorem 8.8 would need none of degree at most $24 - k$.

9. The first open block $B(n, 2)$

By Theorem F, $d(n) \le n^2 - 3$ follows from $U_{B(n,2)} = 0$, and $B(n, 2) = \lambda(2, 1)$ is statement (i) in the proof of Theorem G. The shape has three corners, so Theorem D does not apply, and Theorem 7.5 says only that the conjugation types of $K_{B(n,2)}$ have at most 3 columns. In this section we show that the part of $K_{\lambda(\nu)}$ near the sign representation, the conjugation types with many rows, is one finite problem for all large n, posed in the group algebra of only $|\nu|+m$ letters (Theorem 9.3), and we show that this holds for $n \ge m + |\nu|/2 + 1$ (Theorem 9.7). For $B(n, 2)$ it gives one explicit element of $K_{B(n,2)}$ for every $n \ge 4$ and determines $K_{B(n,2)}$ near the sign exactly (Theorem 9.9); so $U_{B(n,2)} = 0$ cannot be proved by showing $K_{B(n,2)} = 0$. For $\lambda(3, 1, 1)$, statement (iii), it shows by computation that there is no type near the sign (Proposition 9.11). Subsection 9.11 writes the test as a pencil in the exponent of $\det(P_1)$ and, for integer exponents, as a problem on $n \times s$ rectangles, with a duality $n \leftrightarrow s$ (Theorem 9.33). Subsection 9.12 shows that for $\tau_3 = 1$ the pencil has no eigenvalue other than $(d-2)/n$, for every n: then $\mathrm{K}_s(n)_\tau = 0$, and $K_{B(n,2)}$ has no type with at most two columns (Corollary 9.42). Subsection 9.13 writes the test with $\tau_3 = 2$ in closed form at diagonal base points and shows that every eigenvalue of the pencil at bidegree $(3, 2)$ is $3/n$ or $6/n$; so $\mathrm{K}_s(n)_{(ns-5,3,2)} = 0$ for every $n \ge 4$ and every integer $s \ge 1$ with $ns \ge 8$ (Corollary 9.49). Subsection 9.14 shows that a nonzero invariant function of bidegree (a, b) killed by $\mathcal{E}$ lies in the m-th power of the ideal of the closure of $\{(X, Y) : X \text{ regular}, \ Y \in \operatorname{Im} \operatorname{ad}_X\}$ only for $m \le \lfloor a/4 \rfloor$ (Theorem 9.55). Subsection 9.15 applies the lowering operator of $\mathfrak{gl}_3$ to the line of Theorem 9.37 and shows that the pencil has the eigenvalue $\kappa = (a+3)/n$ on the invariant functions of bidegree $(a, 2)$ killed by $\mathcal{E}$, for all $a \ge 3$ and $n \ge a + 2$ (Corollary 9.63). Subsection 9.16 shows that (S1) and (S2) are equivalent on $\mathrm{R}_s(n)_\tau$ (Theorem 9.69). Subsection 9.17 proves the mirror symmetry $(\kappa, n) \mapsto (-\kappa, -n)$ of the pencil at $b = 2$ (Theorem 9.75) and that $\kappa = -2$ is an eigenvalue for every even $a \ge 4$ and $n \ge a + 6$ (Corollary 9.77). Subsection 9.18 shows that $\mathcal{Z}$ is injective on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ for every $a \ge 3$ and $n \ge 3$ (Corollary 9.81), Subsection 9.19 that it is injective on $\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ for every $n \ge 4$ (Corollary 9.91) and that, for every (a, b) and $n \ge 3$, an invariant function killed by $\mathcal{Z}$ is determined by its parts of degree 0 and 1 in the diagonal of Y (Corollary 9.94), Subsection 9.23 that $\ker \mathcal{Z} \cap \mathcal{F}^{\mathcal{E}}_{(a,4)}(n) = 0$ for every $n \ge 5$ and $a \ge 4$ (Corollary 9.130; one step computer-assisted), and Subsection 9.20 that $\mu = 0$ is an eigenvalue for every $a \ge 3$ and $n \ge a + 2$ (Theorem 9.102). Subsection 9.22 shows that, at $b = 2$, $n \ge a + 6$ and $\kappa \ne 0$, the coefficient of the top power-sum monomial in the data of an eigenvector has degree at most 3 in the two points (Corollary 9.123). Subsection 9.24 shows that, at $b = 2$ and $n \ge a + 6$, an eigenvector

with $\mu \neq 0$ lies on one explicit line, and that for odd a every eigenvalue has $\mu \in \{0,3,5,\dots,a\}$ (Corollary 9.144); with Subsection 9.25 this gives the conclusion of Conjecture 11.3 for every $a \geq 3$ and $n \geq a+6$ (Theorem 9.150). Subsection 9.26 shows that $\kappa = n$ is not an eigenvalue at $b = 2$ for odd a at 18 pairs with $4 \leq n \leq 8$, $n \leq a \leq (n^2-2)/2$ (at $n = 7, 8$ only $a \leq n+5$; Theorem 9.154, computer-assisted). Subsection 9.27 proves its step (5) by hand for every n and every dominant $a \geq n$ (Theorem 9.159); so, for these (n,a) with $n \geq 4$, $\kappa = n$ is excluded wherever step (2) holds (Corollary 9.160). Subsection 9.28 shows that at $n = 7$, $\kappa = 7$ is not an eigenvalue at $b = 2$ for any $3 \leq a \leq 23$, of either parity (Theorem 9.163, computer-assisted); so every type κ of $K_{B(7,2)}$ with $\kappa'_3 \leq 1$ is $(3,1^{43})$ (Corollary 9.164). Subsection 9.29 counts the solutions of $(\star_n)$ by hand at every n (Theorem 9.173); so step (2) holds for odd dominant a, and $\kappa = n$ is not an eigenvalue at $b = 2$ for every $n \geq 4$ and every odd a with $n \leq a \leq (n^2-2)/2$ (Corollary 9.174). Subsection 9.30 reduces the even case to one polynomial, a triangle of order-one rows on the stable line (Theorem 9.180), and Subsection 9.31 computes it in closed form and shows that it is not 0 (Theorem 9.188); so $\kappa = n$ is not an eigenvalue at $b = 2$ for every $n \geq 4$ and every even a with $n \leq a \leq (n^2-2)/2$, and every type κ of $K_{B(n,2)}$ with $\kappa'_3 \leq 1$ is $(3,1^{N-3})$ (Corollary 9.189). At $b = 3$, for φ_f with $f \in \mathrm{K}_n(n)_\tau$ and $3 \leq a \leq (n^2-3)/2$, Subsections 9.33, 9.35 and 9.37 exclude every y-order at $\kappa = n$ (Theorems 9.201, 9.221 and 9.233; y-order 2 cannot occur at $b = 3$, Lemma 9.84), so no type with $\kappa'_3 = 2$ occurs (Corollary 9.258).

9.1. **The twisted window.** Throughout this section $R = (n^n)$, and we use the setting of Subsection 7.1: $\nu \vdash k$ lies inside R, $k \geq 1$, $\lambda = \lambda(\nu)$, $N = n^2 - k$, $[n^2] = I \sqcup O$ with $|O| = k$, and $F = \mathbb{R}$; all spaces below are defined over $\mathbb{Q}$. For $\kappa \vdash N$ we call $N - \ell(\kappa)$ the *depth* of κ; the sign representation (1^N) has depth 0. Since $R = R'$, (Γ) gives an invertible $\Gamma_R \in \mathrm{End}(S^R)$ with $\Gamma_R \rho_R(\sigma) = \mathrm{sgn}(\sigma)\rho_R(\sigma)\Gamma_R$ for all $\sigma \in S_{n^2}$. Let $e_\nu \in F[S_O]$ be the central idempotent of ν (F2). Fix $D \subseteq I$ with $|D| = m < N$, and put $Q := O \sqcup D$. For $j \in I$ and $y \in F[S_Q]$ let $\vartheta_j(y) := y - \sum_{o \in O} (o\,j)\, y\, (o\,j) \in F[S_{Q \cup \{j\}}]$.

Definition 9.1. The *depth-m twisted window* of λ is

$$K_\lambda^{[m]} := \{ f \in K_\lambda : \ f(s\sigma s^{-1}) = \mathrm{sgn}(s) f(\sigma) \text{ for all } s \in S_{I \setminus D},\ \sigma \in S_N \}.$$

Let $\mathcal{Y}_m(\nu)$ be the space of the $y \in F[S_Q]$ with

(Tw) $y(u\pi u^{-1}) = \mathrm{sgn}(u)\, y(\pi)$ for all $u \in S_O$;
(BL) $y\, e_\nu = y$;
(B0) $y(\pi) = 0$ unless π has exactly one fixed point in O;
(Aa) $\vartheta_a(y) = 0$, that is, $y = \sum_{o \in O} (o\,a)\, y\, (o\,a)$, for every $a \in D$.

The space $\mathcal{Y}_m(\nu)$ depends only on ν and m, not on n. By (Tw), $uy = \mathrm{sgn}(u)\, yu$ for $u \in S_O$, so $e_\nu y = y\, e_{\nu'}$; the side in (BL) matters only when $\nu \neq \nu'$, and every ν used below is self-conjugate, except $\nu = (3)$, where ν and $\nu' = (1,1,1)$ both give $\mathcal{Y}_m(\nu) = 0$.

Since K_λ is stable under conjugation (Subsection 7.1), $K_\lambda \cong \bigoplus_\kappa (S^\kappa)^{\oplus r_\kappa}$, and $K_\lambda^{[m]}$ is the part on which $S_{I \setminus D}$ acts by the sign. By (F5), applied m times, the sign representation (1^{N-m}) occurs in $\mathrm{Res}_{S_{I \setminus D}} S^\kappa$ with multiplicity $f^{\kappa/(1^{N-m})}$, the number of ways to reach (1^{N-m}) from κ by removing m corners one at a time. So

$$\dim K_\lambda^{[m]} = \sum_\kappa r_\kappa\, f^{\kappa/(1^{N-m})},$$

and $K_\lambda^{[m]} \neq 0$ if and only if K_λ has a conjugation type of depth at most m. As an S_D-module, the κ-part of $K_\lambda^{[m]}$ is a sum of copies of the S^θ with $c^\kappa_{(1^{N-m}),\theta} \neq 0$ (Littlewood–Richardson).

Lemma 9.2 (box locality)**.** *Let $[n^2] = X \sqcup X^c$. The commutant of $\rho_R(S_X)$ in $\mathrm{End}(S^R)$ is $\rho_R(F[S_{X^c}])$. If $|X| \leq n$, then ρ_R is injective on $F[S_X]$. If $|X| = n+1$, its kernel on $F[S_X]$ is spanned by $s^+ := \sum_{\pi \in S_X} \pi$ and $s^- := \sum_{\pi \in S_X} \mathrm{sgn}(\pi)\pi$.*

Proof. By Lemma 7.1, applied to the split X, X^c, $\mathrm{Res}_{S_X\times S_{X^c}} S^R = \bigoplus_\alpha S^\alpha \boxtimes S^{\overline{\alpha}}$, over the $\alpha \vdash |X|$ inside R, and the $\overline{\alpha}$ are pairwise different. As in the proof of Lemma 7.1(iii), the commutant of S_X is $\bigoplus_\alpha 1 \otimes \mathrm{End}(S^{\overline{\alpha}})$, which is $\rho_R(F[S_{X^c}])$ by (F6) and (F2). By the same decomposition, ρ_R kills exactly the blocks I_α of $F[S_X]$ with $\alpha \not\subseteq R$. Every partition of at most n fits in R; for $|X| = n+1$ only $(n+1)$ and (1^{n+1}) do not, and their blocks are Fs^+ and Fs^-. □

Theorem 9.3 (window lemma)**.** *Let $\nu \vdash k$ lie inside R, $k \geq 1$, $\lambda = \lambda(\nu)$, and $|D| = m < N$.*

(a) For every $y \in \mathcal{Y}_m(\nu)$ there is a unique $M_y \in \mathrm{End}(S^\lambda)$ with $M_y \otimes 1_\nu = \Gamma_R \rho_R(y)$, and $f_{M_y} \in K_\lambda^{[m]}$.

(b) If $n \geq m+k$, the linear map $y \mapsto f_{M_y}$ is injective; if $n \geq m+k+1$, it is a bijection $\mathcal{Y}_m(\nu) \to K_\lambda^{[m]}$.

(c) If $k \geq 2$, it is injective for $n \geq m+k-1$ and bijective for $n \geq m+k$.

(d) S_D acts on $\mathcal{Y}_m(\nu)$ by $y \mapsto sys^{-1}$, and $\mathrm{Ad}(s)M_y = M_{y'}$ with $y' = \mathrm{sgn}(s)\, sys^{-1}$. Where the map is injective, every irreducible constituent S^η of this S_D-module has $\ell(\eta) \leq k$.

Proof. Step 1 (dictionary). For $Z \in \mathrm{End}(S^R)$ put $Y := \Gamma_R^{-1} Z$. Since $\rho_R(\pi)\Gamma_R\rho_R(\pi)^{-1} = \mathrm{sgn}(\pi)\Gamma_R$, we have $\mathrm{Ad}(\pi)Z = \mathrm{sgn}(\pi)\,\Gamma_R\,\mathrm{Ad}(\pi)Y$. Hence: (i) $\mathrm{Ad}(u)Z = Z$ for $u \in S_O$ iff $\mathrm{Ad}(u)Y = \mathrm{sgn}(u)Y$; (ii) $Z = Z\rho_R(e_\nu)$ iff $Y = Y\rho_R(e_\nu)$; (iii) for $s \in S_I$, $\mathrm{Ad}(s)Z = \mathrm{sgn}(s)Z$ iff $\mathrm{Ad}(s)Y = Y$; (iv) $\Pi_j Z = \Gamma_R \Pi_j^- Y$ with $\Pi_j^- := \frac{1}{(k+1)!}\sum_{\pi \in S_{O\cup\{j\}}} \mathrm{sgn}(\pi)\,\mathrm{Ad}(\pi)$. If $Y = \rho_R(y)$ and y satisfies (Tw), then $\mathrm{sgn}(u)\,\mathrm{Ad}(u)Y = Y$ for $u \in S_O$, and the left cosets $t_{xj}S_O$ of the proof of Lemma 7.3 give $(k+1)\Pi_j^- Y = \rho_R(\vartheta_j(y))$. By Lemma 7.1(i), (ii) and (iv), $\rho_R(e_\nu)$ is the projection onto the block of ν, so a $Z \in \Omega(O)$ with $Z = Z\rho_R(e_\nu)$ is $M \otimes 1_\nu$ for a unique $M \in \mathrm{End}(S^\lambda)$. For such Z and $s \in S_I$, $\mathrm{Ad}(s)Z = (\mathrm{Ad}(s)M) \otimes 1_\nu$ (Lemma 7.1(iv)), and $f_M \in K_\lambda$ iff $\Pi_j Z = 0$ for every $j \in I$ (Lemma 7.2(c)). Recall $f_{\mathrm{Ad}(s)M}(\sigma) = f_M(s^{-1}\sigma s)$.

Step 2 (a). Let $y \in \mathcal{Y}_m(\nu)$, $Y := \rho_R(y)$ and $Z := \Gamma_R Y$. By (Tw), $\mathrm{Ad}(u)Y = \rho_R(uyu^{-1}) = \mathrm{sgn}(u)Y$, so $Z \in \Omega(O)$ by (i), and (BL) gives (ii); so $Z = M_y \otimes 1_\nu$. Since $y \in F[S_Q]$, Y commutes with $\rho_R(S_{I\setminus D})$, and (iii) gives $\mathrm{Ad}(s)M_y = \mathrm{sgn}(s)M_y$ for $s \in S_{I\setminus D}$: f_{M_y} satisfies the window condition. For $a \in D$, (Aa) and (iv) give $\Pi_a Z = 0$. Let $b \in I \setminus D$. We show $\vartheta_b(y) = 0$ in $F[S_{Q\cup\{b\}}]$; then $\Pi_b Z = 0$ by (iv), so $f_{M_y} \in K_\lambda$. The term o of $\sum_o (o\,b)y(o\,b)$ contributes $y((o\,b)\pi(o\,b))$ at π, which can be nonzero only if $(o\,b)\pi(o\,b) \in S_Q$, that is, $\pi(o) = o$.

- If π fixes b, then $(o\,b)\pi(o\,b) = \pi$ for each such o, so the coefficient of $\vartheta_b(y)$ at π is $(1 - \#\mathrm{Fix}_O(\pi))\, y(\pi)$, which is 0 by (B0).
- If π moves b, then $y(\pi) = 0$, and for each o with $\pi(o) = o$ the permutation $(o\,b)\pi(o\,b)$ has the fixed points of π in O except o. By (B0) only $\#\mathrm{Fix}_O(\pi) = 2$ contributes, with fixed points o, o'; then $(o'\,b)\pi(o'\,b) = (o\,o')\,[(o\,b)\pi(o\,b)]\,(o\,o')$, and the two terms cancel by (Tw).

Step 3 (b). If $n \geq m+k$, then $|Q| \leq n$ and ρ_R is injective on $F[S_Q]$ (Lemma 9.2); so $y \neq 0$ gives $Z \neq 0$, $M_y \neq 0$ and $f_{M_y} \neq 0$. Now let $n \geq m+k+1$ and $f_M \in K_\lambda^{[m]}$; put $Z := M \otimes 1_\nu$ and $Y := \Gamma_R^{-1} Z$. By (iii), Y commutes with $\rho_R(S_{I\setminus D})$, so $Y = \rho_R(y)$ for some $y \in F[S_Q]$ (Lemma 9.2, $X = I \setminus D$), unique by injectivity. By injectivity again, (i) and (ii) give (Tw) and (BL), and $\Pi_a Z = 0$ gives (Aa). The set $I \setminus D$ is not empty; for b in it, $\Pi_b Z = 0$ gives $\rho_R(\vartheta_b(y)) = 0$ with $|Q \cup \{b\}| \leq n$, so $\vartheta_b(y) = 0$. Its coefficient at $\pi \in S_Q$ is $(1 - \#\mathrm{Fix}_O(\pi))\, y(\pi)$ (Step 2), which gives (B0). So $y \in \mathcal{Y}_m(\nu)$ and $M = M_y$.

Step 4 (c). Let $k \geq 2$. If $X \supseteq O$, $|X| = n+1$, and $d \in F[S_X]$ satisfies (Tw) and $\rho_R(d) = 0$, then $d = 0$: by Lemma 9.2, $d \in \mathrm{span}(s^+, s^-)$, which is fixed by conjugation, while an odd $u \in S_O$ (it exists, as $k \geq 2$) gives $udu^{-1} = -d$. Injectivity for $n \geq m+k-1$: then $|Q| \leq n+1$, and y satisfies (Tw). Bijectivity for $n \geq m+k$: then $|Q| \leq n$, and Step 3 runs unchanged except at (B0), where $|Q \cup \{b\}| \leq n+1$; there $\vartheta_b(y)$ satisfies (Tw), because $u\,\vartheta_b(y)\,u^{-1} = \mathrm{sgn}(u)\vartheta_b(y)$ (conjugation by u permutes the terms $(o\,b)y(o\,b)$), so $\vartheta_b(y) = 0$.

Step 5 (d). An $s \in S_D$ fixes O pointwise and commutes with S_O, so $y \mapsto sys^{-1}$ keeps (Tw), (BL) and (B0) and turns the condition (Aa) for a into the one for $s(a)$. Also $\mathrm{Ad}(s)(\Gamma_R\rho_R(y)) = \mathrm{sgn}(s)\Gamma_R\rho_R(sys^{-1})$, which gives $\mathrm{Ad}(s)M_y = M_{y'}$. Where $y \mapsto f_{M_y}$ is injective, it therefore maps the S_D-module $\mathcal{Y}_m(\nu)$, twisted by the sign, isomorphically into $K_\lambda^{[m]}$. A constituent S^η of $\mathcal{Y}_m(\nu)$ thus gives a constituent $S^{\eta'}$ of the κ-part of $K_\lambda^{[m]}$ for some conjugation type κ, so $0 \neq c^{\kappa}_{(1^{N-m}),\eta'} = c^{\kappa'}_{(N-m),\eta}$, and $\eta \subseteq \kappa'$. Hence $\ell(\eta) \le \ell(\kappa') = \kappa_1 \le k$ by Corollary 7.6(a). □

The ranges in (b) and (c) are sufficient, not sharp; Theorem 9.7 below lowers them to about $n \ge m + k/2$. The stabilization of the window *multiplicities* also follows from Manivel's stabilization of rectangular Kronecker coefficients [Man11]; Theorem 9.3 gives the kernel itself, with an explicit model that does not depend on n.

Computations. The dimensions of $\mathcal{Y}_m(\nu)$ below were computed from the linear system (Tw), (BL), (B0), (Aa), with one unknown for each S_O-orbit of permutations with exactly one fixed point in O and with even stabilizer (on the other orbits (Tw) forces $y = 0$), as ranks modulo the prime $p = 1000003$ (for the larger cases one system for each irreducible representation of S_D, in Young's seminormal form). The coefficients are p-integral: the denominators come from e_ν (they divide $k!$) and from the seminormal form (axial distances smaller than $2m$). For a p-integral matrix the rank modulo p is at most the rank over $\mathbb{Q}$. So a dimension computed modulo p is an upper bound for $\dim \mathcal{Y}_m(\nu)$, and it is exact when it is 0 or equals a lower bound proved otherwise.

9.2. **The window range.** For a set $X \subseteq [n^2]$ let $J_n(X)$ be the kernel of ρ_R on $F[S_X]$. By the proof of Lemma 9.2, $J_n(X)$ is the sum of the blocks I_ζ of $F[S_X]$ with $\zeta \not\subseteq R$; it is a two-sided ideal, and $J_n(X) \subseteq J_n(X')$ for $X \subseteq X'$.

Proposition 9.4 (the exact window)**.** *Let $\nu \vdash k$ lie inside R, $\lambda = \lambda(\nu)$, $|D| = m < N$, and fix $b \in I \setminus D$. Then $K_\lambda^{[m]}$ is isomorphic, by $y \mapsto f_{M_y}$ as in Theorem 9.3(a), to the space $\mathcal{S}_m(n)$ of the classes $y \in F[S_Q]/J_n(Q)$ such that (Tw), (BL) and (Aa) hold modulo $J_n(Q)$ and $\vartheta_b(y) \in J_n(Q \cup \{b\})$. These conditions do not depend on the representative y.*

Proof. Let $f_M \in K_\lambda^{[m]}$, $Z := M \otimes 1_\nu$ and $Y := \Gamma_R^{-1}Z$. As in Step 3 of the proof of Theorem 9.3, Y commutes with $\rho_R(S_{I\setminus D})$, so $Y = \rho_R(y)$ with $y \in F[S_Q]$, now unique only modulo $J_n(Q)$. Steps 1 and 2 of that proof use only $Y = \rho_R(y)$ and $\mathrm{sgn}(u)\,\mathrm{Ad}(u)Y = Y$ for $u \in S_O$; so each condition on M is the corresponding condition on y modulo the kernel of ρ_R. For the letters of $I \setminus D$ one b is enough: every $\beta \in S_{I\setminus D}$ centralizes $F[S_Q]$, so $\beta\,\vartheta_b(y)\,\beta^{-1} = \vartheta_{\beta(b)}(y)$ and $\beta J_n(Q\cup\{b\})\beta^{-1} = J_n(Q\cup\{\beta(b)\})$. The conditions are well defined on classes because $J_n(Q) \subseteq J_n(Q \cup \{b\})$ and $J_n(Q \cup \{b\})$ is a two-sided ideal. □

For $X \supseteq O$ call $z \in F[S_X]$ *twisted* if $uzu^{-1} = \mathrm{sgn}(u)z$ for all $u \in S_O$, that is, if z satisfies (Tw), and let $\Pi^{\mathrm{tw}}(z) := \frac{1}{k!}\sum_{u\in S_O} \mathrm{sgn}(u)\,uzu^{-1}$, the projection onto the twisted elements. It commutes with every linear map that commutes with conjugation by S_O; in particular, the twisted part of a sum of two subspaces that are stable under conjugation by S_O is the sum of their twisted parts. Put $\mathrm{sgn}\cdot z := \sum_\pi \mathrm{sgn}(\pi)z(\pi)\pi$. Then $\mathrm{sgn}\cdot I_\zeta = I_{\zeta'}$, and $\mathrm{sgn}\cdot z$ is twisted if and only if z is.

For $d \ge 0$ let $E_{\le d}(X) \subseteq F[S_X]$ be the span of the elements z with $z(\pi) = \prod_{s=1}^{j}[\pi(i_s) = t_s]$ for some $j \le d$ and $i_s, t_s \in X$ (for $j = 0$ the empty product is 1), and put $E_{\le d}(X) := 0$ for $d < 0$. A subspace $J \subseteq F[S_X]$ has *degree at most e* if $J \subseteq E_{\le e}(X) + \mathrm{sgn}\cdot E_{\le e}(X)$.

Lemma 9.5 (degree)**.** *Let $|X| = q$. Then $E_{\le d}(X) = \bigoplus_{\zeta_1 \ge q-d} I_\zeta$, and $J_n(X) = E_{\le d}(X) + \mathrm{sgn}\cdot E_{\le d}(X)$ for $d = q - n - 1$. In particular $J_n(X)$ has degree at most $q - n - 1$.*

The first statement is the description of the functions of degree at most d on a symmetric group in [EFP11].

Proof. A product $\prod_s [\pi(i_s) = t_s]$ with two equal i_s is 0 or has fewer factors, and with distinct i_s and two equal t_s it is 0. So $E_{\le d}(X)$ is spanned, for $j \le d$, by the functions $\pi \mapsto \langle e_{\mathbf{t}}, \pi e_{\mathbf{i}} \rangle$, where $\mathbf{i}, \mathbf{t}$ run over the injective j-tuples in X and S_X permutes the basis $e_{\mathbf{i}}$ of the permutation module M_j on these tuples. By (F2), and as ρ_ζ is orthogonal, I_ζ is the space of the elements whose coefficient functions are the matrix coefficients $\pi \mapsto \langle w, \rho_\zeta(\pi) v \rangle$ of S^ζ; so the matrix coefficients of a module span the sum of the I_ζ over its irreducible constituents S^ζ. The stabilizer of an injective j-tuple is the symmetric group of the other $q - j$ letters, so by Young's rule [Ja78] M_j contains S^ζ if and only if ζ dominates $(q-j, 1^j)$, that is, $\zeta_1 \ge q - j$. This gives the first statement. A partition $\zeta \vdash q$ does not fit in R if and only if $\zeta_1 \ge n+1 = q-d$ or $\zeta'_1 \ge n+1$, and in the second case $I_\zeta = \operatorname{sgn} \cdot I_{\zeta'}$ with $\zeta'_1 \ge q - d$. □

Lemma 9.6 (sign multiplicity)**.** *Let $O \subseteq X$, $|O| = k \ge 2$. The twisted part of $E_{\le d}(X)$ is 0 if $2d \le k-2$, and has dimension at most 1 if $2d = k-1$. The same holds for $\operatorname{sgn} \cdot E_{\le d}(X)$.*

Proof. For $u \in S_O$ the element uzu^{-1} has the coefficient function $\pi \mapsto z(u^{-1}\pi u)$, so conjugation by u maps the function $[\pi(i) = t]$ to $[\pi(u(i)) = u(t)]$. Hence $E_{\le d}(X)$ is the image of $\bigoplus_{j \le d} (F^X)^{\otimes 2j}$ under a linear map that commutes with the action of S_O, its twisted part is the image of the twisted part of this module, and its dimension is at most the multiplicity of sgn in the module. Let $V := F^O$; as S_O-modules, $V = \operatorname{triv} \oplus \operatorname{std}$ and $F^X = V \oplus (|X| - k) \operatorname{triv}$. By Frobenius reciprocity, the multiplicity of sgn in the permutation module of an S_O-set is the number of orbits whose stabilizers contain no odd permutation. The stabilizer of a tuple in O^r is the symmetric group of the letters of O that do not occur in it, which contains a transposition unless at least $k-1$ letters occur. So sgn does not occur in $V^{\otimes r}$ for $r \le k-2$, and occurs once in $V^{\otimes(k-1)}$, since S_O acts simply transitively on the injective $(k-1)$-tuples. As $V^{\otimes r} = \bigoplus_j \binom{r}{j} \operatorname{std}^{\otimes j}$ and multiplicities are nonnegative, sgn does not occur in $\operatorname{std}^{\otimes j}$ for $j \le k-2$, and occurs once in $\operatorname{std}^{\otimes(k-1)}$. Expanding $(F^X)^{\otimes 2j}$ in the same way, sgn does not occur in $\bigoplus_{j \le d} (F^X)^{\otimes 2j}$ if $2d \le k-2$, and if $2d = k-1$ it occurs only in the summand $\operatorname{std}^{\otimes(k-1)}$ of $(F^X)^{\otimes 2d}$, once. Finally, $z \mapsto \operatorname{sgn} \cdot z$ maps the twisted part of $E_{\le d}(X)$ onto the twisted part of $\operatorname{sgn} \cdot E_{\le d}(X)$. □

Theorem 9.7 (window range)**.** *Let $\nu \vdash k$ lie inside R, $k \ge 2$, $\lambda = \lambda(\nu)$, and $|D| = m < N$. The map $y \mapsto f_{M_y}$ from $\mathcal{Y}_m(\nu)$ to $K_\lambda^{[m]}$ of Theorem 9.3(a)*
(R0) *is injective if $n \ge m + k/2$;*
(R1) *is a bijection if $n \ge m + k/2 + 1$;*
(R2) *is a bijection if k is odd, $n = m + (k+1)/2$ and $m \ge \max\bigl(2, (k-1)/2\bigr)$.*
So it is a bijection for every $n \ge m + \lfloor k/2 \rfloor + 1$, except possibly when k is odd, $m < \max\bigl(2, (k-1)/2\bigr)$ and $n = m + (k+1)/2$. By Theorem 9.3(d), these bijections intertwine the actions of S_D given there.

Proof. Put $q := |Q| = m + k$ and fix $b \in I \setminus D$. By Lemma 9.5, $J_n(Q)$ and $J_n(Q \cup \{b\})$ have degree at most $q - n - 1$ and $q - n$.

(R0) If $f_{M_y} = 0$, then $\Gamma_R \rho_R(y) = M_y \otimes 1_\nu = 0$, so $y \in J_n(Q)$, and y is twisted by (Tw). If $n \ge m + k/2$, then $2(q - n - 1) \le k - 2$, so by Lemma 9.6 $J_n(Q)$ has no nonzero twisted element, and $y = 0$.

The common part of (R1) and (R2). Let $f \in K_\lambda^{[m]}$. By Proposition 9.4, $f = f_{M_y}$ for a class $y \in \mathcal{S}_m(n)$. Since $J_n(Q)$ is stable under conjugation by S_O and (Tw) holds modulo $J_n(Q)$, $\Pi^{\mathrm{tw}}(y) \equiv y$, so we may take y twisted. Then $y - y\, e_\nu$ and the $\vartheta_a(y)$, $a \in D$, are twisted elements of $J_n(Q)$ (e_ν is central in $F[S_O]$, and conjugation by $u \in S_O$ permutes the terms $(o\, a)\, y\, (o\, a)$), and $\vartheta_b(y)$ is a twisted element of $J_n(Q \cup \{b\})$. We show below that $J_n(Q)$ has no nonzero twisted element and that $\vartheta_b(y) = 0$. Then (BL) and (Aa) hold exactly, and the coefficients of $\vartheta_b(y)$ at the $\pi \in S_Q$, which are $(1 - \#\mathrm{Fix}_O(\pi))\, y(\pi)$ (Step 2 of the proof of Theorem 9.3), give (B0). So $y \in \mathcal{Y}_m(\nu)$, and with (R0) the map is a bijection.

(R1) If $n \geq m+k/2+1$, then $2(q-n) \leq k-2$, and by Lemma 9.6 neither $J_n(Q)$ nor $J_n(Q \cup \{b\})$ has a nonzero twisted element.

(R2) Let k be odd, $d := (k-1)/2$ and $n = m+d+1$. Then $q-n-1 = d-1$ and $2(d-1) = k-3$, so $J_n(Q)$ has no nonzero twisted element, and $J_n(Q \cup \{b\})$ has degree at most d, with $2d = k-1$. Write $O = \{o_1, \dots, o_k\}$ and let

$$\chi_d(\pi) := \sum_{u \in S_k} \operatorname{sgn}(u) \prod_{s=1}^{d} \left[\pi(o_{u(2s-1)}) = o_{u(2s)}\right] \qquad (\pi \in S_{Q \cup \{b\}}),$$

a signed count of d disjoint arrows of π between letters of O (one letter of O is left over). It lies in $E_{\leq d}(Q \cup \{b\})$, and it is twisted, since conjugation by an element of S_O replaces u by its product with the corresponding element of S_k. Let $x_1, \dots, x_d \in D$ be distinct ($m \geq d$) and

$$\pi_0 := (o_1\, o_2\, x_1)(o_3\, o_4\, x_2) \cdots (o_{2d-1}\, o_{2d}\, x_d)(o_k\, b).$$

The arrows of π_0 between letters of O are exactly $o_1 \to o_2, \dots, o_{2d-1} \to o_{2d}$. So the nonzero terms of $\chi_d(\pi_0)$ come from the u that permute these d arrows as blocks and fix k; exchanging two arrows is a product of two transpositions, so all these u are even, and $\chi_d(\pi_0) = d! \neq 0$. By Lemma 9.6, χ_d spans the twisted part of $E_{\leq d}(Q \cup \{b\})$, so by Lemma 9.5 the twisted part of $J_n(Q \cup \{b\})$ lies in $\operatorname{span}(\chi_d, \operatorname{sgn} \cdot \chi_d)$, and $\vartheta_b(y) = \gamma_+ \chi_d + \gamma_- \operatorname{sgn} \cdot \chi_d$. Let $x \neq x'$ be letters of D ($m \geq 2$) and $\pi_1 := \pi_0\,(x\, x')$. Both π_0 and π_1 move b and fix no letter of O, so the coefficient of $\vartheta_b(y)$ at each of them is 0 (Step 2 of the proof of Theorem 9.3). Also π_1 maps each letter of O to the same letter as π_0, so $\chi_d(\pi_1) = \chi_d(\pi_0)$, while $\operatorname{sgn}(\pi_1) = -\operatorname{sgn}(\pi_0)$. With $s := \operatorname{sgn}(\pi_0)$ this gives $d!\,(\gamma_+ + s\gamma_-) = d!\,(\gamma_+ - s\gamma_-) = 0$, so $\gamma_\pm = 0$ and $\vartheta_b(y) = 0$. □

Remark 9.8. For $k = 3$, (R1) is the bijectivity range $n \geq m+3$ of Theorem 9.3(c), and (R2) adds the layer $n = m+2$ for $m \geq 2$. For $k = 5$ the map is a bijection for $n \geq m+3$ if $m \geq 2$, and for $n \geq m+4$ if $m \leq 1$; for $k = 8$ it is a bijection for $n \geq m+5$. So whether $K_{\lambda(\nu)}$ has a conjugation type of depth at most $n - \lfloor k/2 \rfloor - 1$ is decided by the space $\mathcal{Y}_m(\nu)$ with $m = n - \lfloor k/2 \rfloor - 1$, which does not depend on n (for odd k, if $m \geq \max(2, (k-1)/2)$).

9.3. **The block $B(n,2)$ near the sign.** Let $\nu = (2,1)$, so $k = 3$, $\lambda = B(n,2)$ and $N = n^2 - 3$. Write $O = \{o_1, o_2, o_3\}$, and let a_1, a_2 be two letters of D. Put

$$y_W := \sum_{u \in S_O} \operatorname{sgn}(u) \Big[\big(u(o_1)\; u(o_2)\; a_1\; a_2\big) - \big(u(o_1)\; u(o_2)\; a_2\; a_1\big) \Big] \in F[S_{O \sqcup \{a_1, a_2\}}],$$

a signed sum of 12 four-cycles, each of which fixes exactly one letter of O.

Theorem 9.9. *Let $n \geq 4$, $\lambda = B(n,2)$ and $N = n^2 - 3$.*

- *(a) $\mathcal{Y}_2((2,1)) = F\, y_W$, and $f_W := f_{M_{y_W}}$ is a nonzero element of K_λ of conjugation type $(3, 1^{N-3})$. In particular $K_\lambda \neq 0$.*
- *(b) If $n \geq 5$, the type $(3, 1^{N-3})$ occurs in K_λ exactly once, in the copy generated by f_W, and no other type of depth at most 2 occurs.*
- *(c)* (Computer-assisted.) *Let $3 \leq m \leq 6$ and $n \geq m+3$. Then no type of depth at most m other than $(3, 1^{N-3})$ occurs in K_λ. In particular, for $n \geq 9$ no type other than $(3, 1^{N-3})$ has depth at most 6.*

Proof. (a) By construction y_W satisfies (Tw) and (B0); (BL) and (Aa), and the fact that they cut out a line, were checked by exact rational arithmetic (the system on the 120 coefficients of an element of $F[S_5]$ has rank 119). By Theorem 9.3(c) with $m = 2$, $f_W \neq 0$ lies in $K_\lambda^{[2]}$ for $n \geq 4$. Its type: $\mathcal{Y}_1((2,1)) = 0$ (also by exact computation), so by Theorem 9.3(c) with $m = 1$, $K_\lambda^{[1]} = 0$ for $n \geq 4$, and K_λ has no type of depth at most 1. The types of depth 2 are $(2, 2, 1^{N-4})$ and $(3, 1^{N-3})$; by Littlewood–Richardson, S_D acts on their sign windows by the sign and trivially, respectively.

Conjugation by $(a_1\,a_2)$ exchanges the two four-cycles in each bracket, so $(a_1\,a_2)\,y_W\,(a_1\,a_2) = -y_W$, and by Theorem 9.3(d) S_D fixes M_{y_W}. So f_W has type $(3,1^{N-3})$.

(b) For $n \geq 5$, Theorem 9.3(c) and (a) give $\dim K_\lambda^{[2]} = 1$. The types of depth at most 2 add to this dimension, per copy: (1^N) adds 1, $(2,1^{N-2})$ adds 2, $(2,2,1^{N-4})$ adds 1, and $(3,1^{N-3})$ adds 1. As f_W already contributes 1, the type $(3,1^{N-3})$ occurs once and no other type of depth at most 2 occurs.

(c) Let $n \geq m+3$. The copy of $(3,1^{N-3})$ adds $f^{(3,1^{N-3})/(1^{N-m})} = \binom{m}{2}$ to $\dim K_\lambda^{[m]}$ (the skew shape is a horizontal domino and a separate column of $m-2$ cells), and $\dim K_\lambda^{[m]} = \dim \mathcal{Y}_m((2,1))$ by Theorem 9.3(c). The computed upper bounds are $\dim \mathcal{Y}_m((2,1)) \leq \binom{m}{2}$ for $m \leq 6$. So $\dim K_\lambda^{[m]} = \binom{m}{2}$, all of it from the one copy of $(3,1^{N-3})$, and any other type of depth at most m would add at least 1. □

Remark 9.10. For $n = 3$, $K_{(3,2,1)}$ has the types $(3,3)$ and $(3,1,1,1) = (3,1^{N-3})$, each once (the example after Corollary 7.6). For $n = 4$, a computation modulo a prime shows that $K_{(4,4,3,2)}$ is a single copy of $(3,1^{10})$. Corollary 7.6(a) bounds the multiplicity of $(3,1^{N-3})$ in $K_{B(n,2)}$ by $g\big(R,R,(3,3,1^{N-3})\big) = g\big(R,R,(n^2-4,2,2)\big)$, which is 2 for $n = 3,4,5$ (computed from characters); so the block holds one of the two copies that the joint kernel allows (Theorem 7.7).

9.4. **The block $\lambda(3,1,1)$ and the three-corner family.**

Proposition 9.11 (computer-assisted)**.** $\dim \mathcal{Y}_m((3,1,1)) = 0$ *for* $0 \leq m \leq 7$. *Hence* $K_{\lambda(3,1,1)}$ *has no conjugation type of depth at most* 7 *for* $n \geq 10$, *and none of depth at most* $n-3$ *for* $5 \leq n \leq 9$. *With a direct computation of* $K^{[5]}_{\lambda(3,1,1)} = 0$ *for* $5 \leq n \leq 7$, *it has no type of depth at most* 5 *for every* $n \geq 5$.

(Theorem 10.15 below gives $\mathcal{Y}_m((3,1,1)) = 0$ for every m, and no type of depth at most $n-3$ for every $n \geq 5$.)

Proof. The dimensions are ranks modulo p with full column rank (the same at the primes 999983 and 998244353), so they are exact by the remark on computations; for $m \leq 5$ they were confirmed by independent programs. Here $k = 5$, and by Theorem 9.7 the map $\mathcal{Y}_m \to K_\lambda^{[m]}$ is a bijection for $n \geq m+3$ if $m \geq 2$. Take $m = 7$ for $n \geq 10$, and $m = n-3$ for $5 \leq n \leq 9$. □

This does not decide statement (iii) of the proof of Theorem G; see Remark 9.13.

Remark 9.12 (the three-corner family). For $p, q \geq 1$ let $\nu_{p,q}$ be the $(p+q) \times (p+q)$ square with a $p \times p$ square removed at its corner; for $n > p+q$ the shape $\lambda(\nu_{p,q})$ has three corners. Then $\nu_{1,1} = (2,1)$, $\nu_{1,2} = (3,3,2)$, $\nu_{2,1} = (3,1,1)$, $\nu_{2,2} = (4,4,2,2)$, $\nu_{3,1} = (4,1,1,1)$ and $\nu_{4,1} = (5,1^4)$. The computed values of $\dim \mathcal{Y}_m$ split by p:

- $p = 1$: for $(2,1)$, $\binom{m}{2}$ for $m \leq 6$ (Theorem 9.9); for $(3,3,2)$, $0,0,1,3,6$ for $m = 0,\dots,4$, exact. Here an exact element of $\mathcal{Y}_2$ is known, S_D acts on it (by $y \mapsto sys^{-1}$) by the sign, and $\mathcal{Y}_1 = 0$; as in the proof of Theorem 9.9, its type is $(3,1^{N-3})$, which gives the lower bound $\binom{m}{2}$. By Theorem 9.7 ($k = 8$: (R0) at $m = 2$ and (R1) at $m = 1$ for $n \geq 6$, and (R1) for $n \geq m+5$), $K_{\lambda(3,3,2)} \neq 0$ for every $n \geq 6$, and it has the type $(3,1^{N-3})$ once and no other type of depth at most m for $n \geq m+5$, $m \leq 4$.
- $p \geq 2$: 0 for $(3,1,1)$, $m \leq 7$; for $(4,4,2,2)$, $m \leq 2$; for $(4,1,1,1)$, $m \leq 4$; for $(5,1^4)$, $m \leq 3$.

For comparison, the squares $\nu = (1)$ and $(2,2)$ (two corners) give 1 at every computed depth $m \leq 3$ (the element f_Γ), $\nu = (3)$ gives 0, and the staircases $(3,2,1)$ and $(4,3,2,1)$ (four and five corners) give the upper bounds $1,2,6,13,26$ ($m \leq 4$) and $0,0,2$ ($m \leq 2$).

Remark 9.13 (what the window does not give). Theorem 9.3 controls only the types of depth at most m. Suppose that every conjugation type of $K_{\lambda(\nu)}$ has depth at most J, with J independent

of n. Then $U_{\lambda(\nu)}=0$ for $n\ge J+2k+1$. Indeed, take $|D|=J$. Each κ-part of K_λ then meets $K_\lambda^{[J]}$, so K_λ is spanned by the S_N-conjugates of the elements of $K_\lambda^{[J]}$, which are the f_{M_y}, $y\in\mathcal{Y}_J(\nu)$, by Theorem 9.3(b). By Lemma 7.2(b), $f_{M_y}(\sigma)=\frac{1}{f^\nu}\langle\Gamma_R\rho_R(y),\rho_R(\sigma)\rangle$. Here Γ_R^T satisfies the same relation as Γ_R, so $\Gamma_R^T=c'\Gamma_R$ by Schur's lemma, with $c'=\pm1$ (transpose twice), and $\rho_R(\pi)^T=\rho_R(\pi^{-1})$; so $f_{M_y}(\sigma)=c\sum_\pi y(\pi)\,\psi_R(\sigma\pi^{-1})$ with a constant $c\ne0$, where $\psi_R(\tau):=\operatorname{tr}(\rho_R(\tau)\Gamma_R)$ is the function ψ of (Γ) for the shape R. By (Γ), ψ_R vanishes off the class $h(R)=(2n-1,2n-3,\dots,1)$, which has n cycles. A permutation $\pi\in S_Q$ moves at most $J+k$ letters, so it is a product of at most $J+k-1$ transpositions, and each changes the number of cycles by 1. An N-cycle of S_I has $k+1$ cycles in S_{n^2}, so $\sigma\pi^{-1}$ has at most $J+2k<n$ cycles, and f_{M_y} vanishes on the N-cycles. So does every element of K_λ, and $U_\lambda\subseteq K_\lambda$ is supported on the N-cycles; so $U_\lambda=0$.

9.5. **Two more layers for $B(n,2)$.** For $\nu=(2,1)$, $k=3$, Theorem 9.7(R2) gives one more layer than Theorem 9.3(c): for $n\ge4$ and $m=n-2$ the map $y\mapsto f_{M_y}$ is a bijection $\mathcal{Y}_m((2,1))\to K_{B(n,2)}^{[m]}$. We now add the layer $m=n-1$.

In this subsection $\nu=(2,1)$, $O=\{o_1,o_2,o_3\}$, $b\in I\setminus D$, $O':=O\cup\{b\}$ and $o_4:=b$. For $X\supseteq O'$ call $g\in F[S_X]$ 4-*twisted* if $wgw^{-1}=\operatorname{sgn}(w)g$ for all $w\in S_{O'}$, and let V' be the set of the $\pi\in S_{Q\cup\{b\}}$ that fix no letter of O'. If z is twisted and $j\in I$, then, as $S_{O\cup\{j\}}$ is the union of the cosets S_O and $(o\,j)S_O$, $o\in O$,

$$\vartheta_j(z)=\tfrac16\sum_{w\in S_{O\cup\{j\}}}\operatorname{sgn}(w)\,wzw^{-1},$$

so $\vartheta_j(z)$ is twisted for the four letters $O\cup\{j\}$.

Lemma 9.14 (the four-letter frame)**.** *If $z\in F[S_Q]$ is twisted, then $\vartheta_b(z)$ is 4-twisted and vanishes on V'. Conversely, let $g\in F[S_{Q\cup\{b\}}]$ be 4-twisted and vanish on V', and let $h\in F[S_Q]$ be its part on S_Q ($h(\pi)=g(\pi)$ for $\pi\in S_Q$). Then h is twisted, $h(\pi)=0$ unless π fixes no letter of O, and $g=\vartheta_b(h)$.*

Proof. The element z lives on S_Q, the permutations that fix b, and $(o\,b)\,z\,(o\,b)$ on the permutations that fix o; so at $\pi\in V'$ every term of $\vartheta_b(z)$ is 0. Conversely, a 4-twisted g vanishes at every π that fixes two letters c,c' of O', since $(c\,c')$ commutes with π and is odd. As g vanishes on V', every π in its support fixes exactly one letter of O'; for $\pi\in S_Q$ this letter is b, which gives the support of h. The element h is twisted because g is. For $w\in S_{O'}$ we have $\operatorname{sgn}(w)\,(whw^{-1})(\pi)=\operatorname{sgn}(w)\,g(w^{-1}\pi w)\,[w^{-1}\pi w(b)=b]=g(\pi)\,[\pi(w(b))=w(b)]$, so by the formula above $\vartheta_b(h)(\pi)=\#\mathrm{Fix}_{O'}(\pi)\,g(\pi)$, which is $g(\pi)$ on the support of g and 0 elsewhere. □

For letters i,j put $x_{ij}(\pi):=[\pi(i)=j]$. For a monomial $M=\prod x_{ij}$ let $w\cdot M:=\prod x_{w(i)w(j)}$ for $w\in S_{O'}$ (w fixes the letters of D); this is wMw^{-1} (proof of Lemma 9.6). Put $\mathrm{Alt}(M):=\sum_{w\in S_{O'}}\operatorname{sgn}(w)\,w\cdot M$ and, for $a\in D$,

$$F_0:=\mathrm{Alt}(x_{o_1o_1}x_{o_2o_3}),\qquad U_a:=\mathrm{Alt}(x_{o_1o_2}x_{o_3a}),\qquad V_a:=\mathrm{Alt}(x_{o_1o_2}x_{ao_3}).$$

Lemma 9.15 (the defects of degree 2)**.** *Let $X=Q\cup\{b\}$.*

(a) The 4-twisted part of $E_{\le2}(X)$ is spanned by F_0 and the U_a, V_a, $a\in D$.

(b) Let $m\ge3$. If $g\in E_{\le2}(X)+\operatorname{sgn}\cdot E_{\le2}(X)$ is 4-twisted and vanishes on V', then $g\in\operatorname{span}(F_0,\operatorname{sgn}\cdot F_0)$.

Proof. (a) The 4-twisted part is spanned by the $\mathrm{Alt}(M)$ with $\deg M\le2$. If two letters c,c' of O' do not occur in M, then $(c\,c')\cdot M=M$ and $\mathrm{Alt}(M)=0$; so M uses at least three letters of O'. Up to $S_{O'}$ and sign the nonzero $\mathrm{Alt}(M)$ are then F_0, U_a, V_a, $A_1:=\mathrm{Alt}(x_{o_1o_2}x_{o_3o_4})$ and the path function $\Xi:=\mathrm{Alt}(x_{o_1o_2}x_{o_2o_3})$ (a product $x_{ij}x_{kl}$ with $i\ne k$, $j\ne l$ is two loops, a loop and a disjoint

arrow, a path, a 2-cycle, or two disjoint arrows, and $x_{ii}x_{ij} = x_{ii}x_{ji} = 0$ for $i \neq j$). The row sums $\sum_j x_{o_3 j} = 1$ and the column sums $\sum_i x_{i o_3} = 1$ give

$$\textstyle\sum_{a\in D} U_a = -\Xi - F_0 - A_1, \qquad \sum_{a\in D} V_a = -\Xi - F_0 + A_1,$$

since $\mathrm{Alt}(x_{o_1o_2}x_{o_3o_1}) = \Xi$ and $\mathrm{Alt}(x_{o_1o_2}x_{o_3o_3}) = F_0$ (by an even 3-cycle), $\mathrm{Alt}(x_{o_1o_2}x_{o_4o_3}) = -A_1$ (by the transposition $(o_3\, o_4)$) and $\mathrm{Alt}(x_{o_1o_2}) = 0$. So A_1 and Ξ lie in the span of F_0 and the U_a, V_a.

(b) As for S_O in Subsection 9.2, the 4-twisted part of $E_{\le 2}(X) + \mathrm{sgn}\cdot E_{\le 2}(X)$ is the sum of the 4-twisted parts of the two summands, so by (a) we can write $g = \gamma F_0 + \gamma' \mathrm{sgn}\cdot F_0 + \sum_a \big[\mu_a U_a + \nu_a V_a + \mathrm{sgn}\cdot(\mu'_a U_a + \nu'_a V_a)\big]$. Here F_0 and $\mathrm{sgn}\cdot F_0$ vanish on V', since every term of F_0 needs a fixed letter of O'. For $\pi \in V'$, $U_a(\pi) \neq 0$ needs $\pi^{-1}(a) \in O'$, and $V_a(\pi) \neq 0$ needs $\pi(a) \in O'$. Let a, a', c be distinct letters of D; direct counts give:

- at $\pi_5 := (o_1\, o_2\, o_3\, o_4\, a)$: $U_a = -2$, $V_a = 2$, and every other U, V is 0. Multiplying π_5 on the right by a transposition of two other letters of D keeps these values and changes the sign. So $\mu_a = \nu_a =: x_a$ and $\mu'_a = \nu'_a =: z_a$.
- at $\pi_2 := (o_1\, o_2\, o_3\, a)(o_4\, a')$, which is even: $U_a = V_a = 1$, $U_{a'} = V_{a'} = -2$, all others 0. So $2(x_a - 2x_{a'}) + 2(z_a - 2z_{a'}) = 0$; thus $s := x + z$ has $s_a = 2s_{a'}$ for all $a \neq a'$, so $s = 0$ and $z = -x$.
- at $\pi'_2 := (o_1\, o_2\, o_3\, a)(o_4\, a'\, c)$, which is odd: $U_a = V_a = 1$, $U_{a'} = V_c = -2$, all others 0. So, after dividing by 2, $(x_a - x_{a'} - x_c) - (z_a - z_{a'} - z_c) = 0$, that is, $x_a = x_{a'} + x_c$ for all distinct a, a', c. Exchanging a and a' gives $x_c = 0$.

So all $\mu_a, \nu_a, \mu'_a, \nu'_a$ vanish. □

Theorem 9.16 (the layer $n = m + 1$)**.** *Let $n \ge 4$ and $m = n - 1$. Then $y \mapsto f_{M_y}$ is a bijection $\mathcal{Y}_m((2,1)) \to K^{[m]}_{B(n,2)}$, compatible with the actions of S_D (Theorem 9.3(d)).*

Proof. Here $|Q| = n + 2$ and $|Q \cup \{b\}| = n + 3$, so by Lemma 9.5 $J_n(Q) = E_{\le 1}(Q) + \mathrm{sgn}\cdot E_{\le 1}(Q)$ and $J_n(Q \cup \{b\})$ has degree at most 2. Let

$$\chi(\pi) := \sum_{u\in S_3} \mathrm{sgn}(u)\,[\pi(o_{u(1)}) = o_{u(2)}] \qquad (\pi \in S_Q),$$

as χ_1 in the proof of Theorem 9.7. By Lemma 9.6 ($k = 3$, $d = 1$) the twisted part of $J_n(Q)$ is spanned by χ and $\mathrm{sgn}\cdot\chi$. At $(o_1\, o_2\, o_3)$ and $(o_1\, o_2\, o_3)(a\, a')$, $a \neq a'$ in D, which fix no letter of O, the values of χ are $3, 3$ and those of $\mathrm{sgn}\cdot\chi$ are $3, -3$.

Injectivity. An element of the kernel is a twisted element of $J_n(Q)$, so it is $\gamma\chi + \gamma' \mathrm{sgn}\cdot\chi$, and by (B0) it vanishes at these two permutations; so $\gamma = \gamma' = 0$.

Surjectivity. Let $f \in K^{[m]}_{B(n,2)}$; as in the proof of Theorem 9.7, $f = f_{M_y}$ with $y \in \mathcal{S}_m(n)$ twisted (now unique only modulo $\mathrm{span}(\chi, \mathrm{sgn}\cdot\chi)$). By Lemma 9.14, $g := \vartheta_b(y)$ is 4-twisted and vanishes on V', and $g \in J_n(Q \cup \{b\})$; so $g \in \mathrm{span}(F_0, \mathrm{sgn}\cdot F_0)$ by Lemma 9.15 ($m = n - 1 \ge 3$). This span has dimension at most 2, and it contains $\vartheta_b(\chi)$ and $\vartheta_b(\mathrm{sgn}\cdot\chi)$ (Lemma 9.14, and $J_n(Q) \subseteq J_n(Q \cup \{b\})$), which are independent: at the two permutations above, which fix b, their values are $3, 3$ and $3, -3$ (Step 2 of the proof of Theorem 9.3). So $g = \vartheta_b(\gamma\chi + \gamma' \mathrm{sgn}\cdot\chi)$ for some γ, γ'. Then $y' := y - \gamma\chi - \gamma' \mathrm{sgn}\cdot\chi$ is twisted, $y' \equiv y$ modulo $J_n(Q)$, and $\vartheta_b(y') = 0$; its coefficients at the $\pi \in S_Q$ give (B0).

(BL) and (Aa). The idempotent $e_\nu = 1 - e_{(3)} - e_{(1,1,1)}$ is central in $F[S_O]$, so $y'e_\nu - y'$ is twisted, and it lies in $J_n(Q)$ because (BL) holds modulo $J_n(Q)$; so it lies in $\mathrm{span}(\chi, \mathrm{sgn}\cdot\chi)$. Now $\chi \sum_{u\in S_O} u = 0$: its coefficient at σ is $\sum_u \chi(\sigma u)$, and $\sum_u [\sigma u(o_i) = o_j] = 2\,[\sigma^{-1}(o_j) \in O]$ does not depend on i, while for each o_j the two terms of χ with target o_j have opposite signs. Also $\chi \sum_{u\in S_O} \mathrm{sgn}(u) u = 0$ term by term, since the two u with $u(o_i) = \sigma^{-1}(o_j)$ have opposite signs. The same holds for $\mathrm{sgn}\cdot\chi$. So $\chi\, e_\nu = \chi$ and $(\mathrm{sgn}\cdot\chi)\, e_\nu = \mathrm{sgn}\cdot\chi$, and multiplying by e_ν on the right shows that $y'e_\nu - y' = 0$. For $a \in D$, $\vartheta_a(y')$ lies in $J_n(Q)$, which has degree at most 1, and it is

twisted for the four letters $O\cup\{a\}$; by Lemma 9.6 for these four letters ($2\cdot 1\le 4-2$) it is 0. So $y'\in\mathcal{Y}_m((2,1))$ and $f=f_{M_{y'}}$. □

9.6. **Multiplicities per tail.** For a partition $t=(s,r)$ with at most two parts let $r_t(n)$ be the multiplicity in $K_{B(n,2)}$ of the conjugation type κ with $\kappa'=(N-s-r,s,r)$; it has depth $|t|$. By Corollary 7.6(a) every type of $K_{B(n,2)}$ is of this form, and the type $(3,1^{N-3})$ is $t=(1,1)$.

Lemma 9.17 (multiplicities per tail)**.** *(a) $r_t(n)$ does not depend on n for $n\ge|t|+3$.*
(b) If $n\ge 4$, $r_t(n)$ takes this stable value for every t with $|t|\le n-1$.

Proof. (a) By Theorem 9.3(c), $K^{[m]}_{B(n,2)}\cong\mathcal{Y}_m((2,1))$ for $n\ge m+3$, and by Theorem 9.3(d) this is an isomorphism of S_D-modules if S_D acts on $\mathcal{Y}_m$ by $y\mapsto\operatorname{sgn}(s)\,sys^{-1}$; so the S_D-character of $K^{[m]}_{B(n,2)}$ does not depend on $n\ge m+3$. A type with tail t, $|t|\le m$, contributes to $K^{[m]}_{B(n,2)}$, twisted by the sign, the skew module of $\kappa'/(N-m)$. Since $N-m\ge s$, the cells of $\kappa'/(N-m)$ in the first row lie to the right of t. So for $|t|<m$ the skew shape is a row of $m-|t|$ cells and, separately, t, and the module is $\operatorname{Ind}_{S_{m-|t|}\times S_{|t|}}^{S_m}(\operatorname{triv}\boxtimes S^t)$, which does not depend on n; for $|t|=m$ it is S^t. Induct on m: the contributions of the t with $|t|<m$ do not depend on n, so neither does $\bigoplus_{|t|=m} r_t(n)\,S^t$, and as irreducible characters are linearly independent, neither does each $r_t(n)$. (b) Fix $n\ge 4$. For every $m'\le n-1$ the window is the stable one, S_D-equivariantly: by Theorem 9.3(c) for $m'\le n-3$, by Theorem 9.7(R2) for $m'=n-2$, and by Theorem 9.16 for $m'=n-1$. The induction of (a), run over $m'=0,1,\dots,n-1$ at this fixed n, gives $r_t(n)$ equal to the stable value for every $|t|\le n-1$; the skew split of (a) still holds at $m'=n-1$, as $N-m'=n^2-n-2\ge n-1\ge s$. (The isomorphism at $m'=n-1$ alone fixes only the sum over all $|t|\le n-1$; the lower windows are needed too.) □

9.7. **The matrix-pair model.** In this subsection $F=\mathbb{C}$: Young's orthogonal form and Γ_R are real, dimensions do not change, and $\mathrm{SL}_n(\mathbb{R})$ is Zariski-dense in $\mathrm{SL}_n(\mathbb{C})$. Let $V=\mathbb{C}^n$, $M_n=V\otimes V$, with $u\otimes y$ the matrix uy^T, and let $\mathrm{SL}_n\times\mathrm{SL}_n$ act by $(g,h).P=gPh^T$. For $\kappa\vdash N$ with $\kappa_1\le 3$ put $\tau:=\kappa'+(1,1,1)=(\kappa^{+3})'$, a partition of n^2 with exactly three rows. Let Pol_τ be the space of the polynomials $f(P_1,P_2,P_3)$ on M_n^3 that are invariant under $\mathrm{SL}_n\times\mathrm{SL}_n$, homogeneous of degree τ_a in P_a, and highest weight vectors for GL_3: $f(P_1,P_2+tP_1,P_3)=f(P_1,P_2,P_3+tP_1)=f(P_1,P_2,P_3+tP_2)=f$. (Here GL_3 acts by $(P_1,P_2,P_3)\mapsto(P_1,P_2,P_3)g$, and these are the invariance conditions under the upper unitriangular group.) For $u,y_1,y_2,y_3\in V$ consider the tests

(S1) $\Omega_u f(P):=\sum_{s\in S_3}\operatorname{sgn}(s)\,[t_1t_2t_3]\,f\big(P_1+t_1uy^T_{s(1)},\,P_2+t_2uy^T_{s(2)},\,P_3+t_3uy^T_{s(3)}\big)=0$ for all P,u,y, where $[t_1t_2t_3]$ is the coefficient of $t_1t_2t_3$;
(S2) the same with the updates $y_{s(a)}u^T$.

Theorem 9.18 (matrix-pair model)**.** *Let $n\ge 2$, $\lambda=B(n,2)$ and κ, τ as above. There is a linear isomorphism* $\mathrm{Mult}(\kappa^{+3})\to\mathrm{Pol}_\tau$ *(in particular* $\dim\mathrm{Pol}_\tau=g(R,R,\tau)$*), and it maps the space X_ν of Corollary 7.6(b) onto the space of the $f\in\mathrm{Pol}_\tau$ that satisfy (S1) and (S2). Hence the multiplicity of κ in $K_{B(n,2)}$ is* $\dim\{f\in\mathrm{Pol}_\tau:$ *(S1), (S2)*$\}$.

Proof. Step 1 (multilinear model). Let T_V be the space of multilinear $\mathrm{SL}(V)$-invariant forms on V^{n^2}, with $(s.t)(v_1,\dots,v_{n^2}):=t(v_{s(1)},\dots,v_{s(n^2)})$. By Schur–Weyl duality, $V^{\otimes n^2}=\bigoplus_{\ell(\mu)\le n}S_\mu(V)\otimes S^\mu$, and $S_\mu(V)$ has an $\mathrm{SL}(V)$-invariant only if μ is a rectangle with n rows, so $\mathsf{T}_V\cong(S^R)^*\cong S^R$ (S_m-modules are self-dual); fix an S_{n^2}-isomorphism $\theta\colon S^R\to\mathsf{T}_V$. The space $\mathcal{M}$ of multilinear invariant functions $F(X_1,\dots,X_{n^2})$ of n^2 matrices is $\mathsf{T}_V\otimes\mathsf{T}_V$, determined by the values at rank-one arguments $u_i\otimes y_i$. It carries the diagonal action $(s.F)(X_1,\dots)=F(X_{s(1)},\dots)$ and the action on the second factor, $(s*F)(u_1\otimes y_1,\dots)=F(u_1\otimes y_{s(1)},\dots)$. Put $\Theta(vw^T):=\theta(v)\otimes\theta(w)$ for $v,w\in S^R$. As ρ_R is orthogonal, $\Theta(\mathrm{Ad}(s)Z)=s.\Theta(Z)$ and $\Theta(Z\rho_R(s))=s^{-1}*\Theta(Z)$; for the central idempotents e_μ of $F[S_O]$ (which are inverse-closed), $\Theta(Z\rho_R(e_\mu))=e_\mu*\Theta(Z)$.

Step 2 (outer blocks). Let Y be twisted ($\mathrm{Ad}(u)Y = \mathrm{sgn}(u)Y$, $u \in S_O$) and $F := \Theta(Y)$; then F is alternating in its three outer arguments, a linear form on $\Lambda^3 W$, $W := V \otimes V$, for fixed inner arguments X_I. By Schur–Weyl duality on the three slots, $\Lambda^3 W = \bigoplus_{\alpha \vdash 3} S_\alpha V \otimes S_{\alpha'} V$, and e_μ acting on the second factor projects onto the summand $S_{\mu'}V \otimes S_\mu V$. The summand $\mathrm{Sym}^3 V \otimes \Lambda^3 V$ ($\mu = (1,1,1)$) is spanned by the antisymmetrizations of $(u\otimes y_1)\otimes(u\otimes y_2)\otimes(u\otimes y_3)$, which are $u^{\otimes 3}\otimes(y_1 \wedge y_2 \wedge y_3)$; similarly for $\Lambda^3 V \otimes \mathrm{Sym}^3 V$. Hence $Y\rho_R(e_{(1,1,1)}) = 0$ if and only if $F(X_I; uy_1^T, uy_2^T, uy_3^T) = 0$ for all X_I, u, y, and $Y\rho_R(e_{(3)}) = 0$ if and only if $F(X_I; y_1u^T, y_2u^T, y_3u^T) = 0$.

Step 3 (polynomials). Let G_1, G_2, G_3 be consecutive groups of slots of sizes τ_1, τ_2, τ_3, and $w_\tau \neq 0$ a vector of S^τ fixed by $S_{G_1} \times S_{G_2} \times S_{G_3}$ (a line, by Young's rule). For $\tilde{x} \in \mathrm{Hom}_{S_{n^2}}(S^\tau, \mathcal{M})$ let $f_{\tilde{x}}(P_1, P_2, P_3) := \tilde{x}(w_\tau)(P_1, \dots, P_1, P_2, \dots, P_2, P_3, \dots, P_3)$, with P_a in the slots of G_a. This is invariant and of multidegree τ. It is a highest weight vector: as $\tilde{x}(w_\tau)$ is symmetric in the slots of G_2, the coefficient of t^ℓ in $f_{\tilde{x}}(P_1, P_2 + tP_1, P_3)$ is $\binom{\tau_2}{\ell}$ times the restitution of $\tilde{x}(w_\ell)$, where w_ℓ is the average of w_τ over a Young subgroup of type $(\tau_1+\ell, \tau_2-\ell, \tau_3)$, whose sorted type strictly dominates τ for $\ell \geq 1$; by Young's rule $w_\ell = 0$. The other two operators are the same (with $\binom{\tau_3}{\ell}$). The map $\tilde{x} \mapsto f_{\tilde{x}}$ is injective (polarization recovers $\tilde{x}(w_\tau)$, which generates). By the Cauchy formula the invariant polynomials of degree n^2 form $\bigoplus_\tau S_\tau(M_n^*)^{\mathrm{SL}\times\mathrm{SL}} \otimes S_\tau(\mathbb{C}^3)$ (over the τ with at most three rows), and $\dim S_\tau(V\otimes V)^{\mathrm{SL}\times\mathrm{SL}} = g(\tau, R, R)$; so $\dim \mathrm{Pol}_\tau = g(R,R,\tau) = \dim \mathrm{Hom}(S^\tau, \mathcal{M})$, and the map is a bijection.

Step 4 (the twist and the tests). As $S^\tau \cong S^{\kappa^{+3}} \otimes \mathrm{sgn}$, fix $\epsilon\colon S^\tau \to S^{\kappa^{+3}}$ with $\epsilon(sw) = \mathrm{sgn}(s)s\epsilon(w)$ and put $\tilde{x} := \Theta \circ \Gamma_R^{-1} \circ x \circ \epsilon$ for $x \in \mathrm{Mult}(\kappa^{+3})$, with Γ_R the invertible operator of (Γ) in Section 4. From $\Gamma_R\rho_R(s) = \mathrm{sgn}(s)\rho_R(s)\Gamma_R$ we get $\rho_R(s)\Gamma_R^{-1} = \mathrm{sgn}(s)\Gamma_R^{-1}\rho_R(s)$, so $\Gamma_R^{-1}\,\mathrm{Ad}(s)Z = \mathrm{sgn}(s)\,\mathrm{Ad}(s)\Gamma_R^{-1}Z$, and $x \mapsto \tilde{x}$ is an isomorphism onto $\mathrm{Hom}_{S_{n^2}}(S^\tau, \mathcal{M})$; with Step 3 this is the isomorphism of the theorem. Under ϵ the vectors v of Corollary 7.6(b) become the $\kappa' \boxtimes \mathrm{sgn}$ component of $\mathrm{Res}_{S_I \times S_O} S^\tau$, which by Pieri's rule is the whole S_O-sign part (as τ has three rows, κ' is the only μ with τ/μ a vertical strip of three cells). So, by Corollary 7.6(b) (with $\nu' = (3), (1,1,1)$) and Step 2, $x \in X_\nu$ if and only if $\tilde{x}(w)$ vanishes at the outer arguments (uy_a^T) and $(y_a u^T)$ for every w in the S_O-sign part of S^τ. This condition is the same for every three-element set of slots in place of O (by equivariance); take O' to consist of the first slot of each G_a. By multilinearity and the symmetry of $\tilde{x}(w_\tau)$ in each group,

$$\Omega_u f_{\tilde{x}}(P) = \tau_1\tau_2\tau_3\, \tilde{x}(w')\big(P_1, \dots, P_3;\, uy_1^T, uy_2^T, uy_3^T\big), \qquad w' := \textstyle\sum_{s \in S_{O'}} \mathrm{sgn}(s)\, s\, w_\tau,$$

with the P_a in the slots $G_a \setminus O'$, and likewise for (S2). The vector w' lies in the $S_{O'}$-sign part. So $x \in X_\nu$ gives (S1) and (S2). Conversely, (S1) says that $\tilde{x}(w')(P, \dots; \cdot)$ vanishes on $\mathrm{Sym}^3 V \otimes \Lambda^3 V$. As $\tilde{x}(w')$ is alternating in the slots O', the function $e_{(1,1,1)} * \tilde{x}(w')$, which is $\tilde{x}(w')$ composed with the sign projector on the second factors of these slots, sees only $\mathrm{Sym}^3 V \otimes \Lambda^3 V$ there; it is symmetric in each group $G_a \setminus O'$ and its restitution vanishes, so it is 0. The map $w \mapsto e_{(1,1,1)} * \tilde{x}(w)$ on the $S_{O'}$-sign part, which is irreducible ($\cong S^{\kappa'}$) for $S_{[n^2]\setminus O'}$, is equivariant (the action $*$ on the slots O' commutes with permutations of the other slots) and kills w', which is not 0 by Lemma 9.19; so it is 0. The same holds for $e_{(3)}$ with (S2), and $x \in X_\nu$. □

Lemma 9.19. *The vector w' in the proof of Theorem 9.18 is not* 0.

Proof. Realize S^τ as the left ideal $F[S_{n^2}]\, a_t b_t$ for the tableau t whose rows are G_1, G_2, G_3, with a_t the row symmetrizer and b_t the column antisymmetrizer. Then $a_t b_t$ is fixed by the row group, so w_τ is a multiple of it, and O' is the first column of t (one cell in each row, as τ has exactly three rows). For $s \in S_{O'} \subseteq C_t$, $sa_tb_t = a_{st}\, sb_t = \mathrm{sgn}(s)\, a_{st}b_t$, where $a_{st} = sa_ts^{-1}$. So w' is a multiple of $\sum_{s\in S_{O'}} a_{st}b_t$. The coefficient of the identity in $a_{st}b_t$ is $\sum_{c \in R_{st}\cap C_t} \mathrm{sgn}(c)$, and $R_{st} \cap C_t = \{e\}$: such a c fixes every entry $t(i,j)$ with $j \geq 2$ (the only entry of row i of st in column j of t), and each entry $s(t(i,1))$ (the only entry of row i of st in the first column). So the coefficient of the identity in $\sum_s a_{st}b_t$ is 6. □

Which test belongs to which block depends on conventions (exchanging the two factors of Θ, replacing every P by P^T, or multiplying a twisted Y on the left instead of the right exchanges them); the theorem uses both. Only the inclusion of X_ν in the space cut out by (S1) is used below.

The chart. Let $d := \tau_2 + \tau_3 \le n$, $(a,b) := (\tau_2, \tau_3)$. For $f \in \mathrm{Pol}_\tau$ put $\varphi_f(A,B) := f(I,A,B)$. On $\det P_1 \neq 0$, $f = \det(P_1)^n \varphi_f(P_1^{-1}P_2, P_1^{-1}P_3)$ (write $P_1 = cg$ with $g \in \mathrm{SL}_n$ and use homogeneity), so $f \mapsto \varphi_f$ is injective. The function φ_f is a polynomial invariant of pairs of matrices under simultaneous conjugation, hence a trace polynomial [Pr76]; by the highest weight conditions at $P_1 = I$ it depends only on the traceless parts of A, B, and it is killed by $\mathcal{E} := A\,\partial/\partial B$ (replace one B by A). Write $[d] = \mathsf{A} \sqcup \mathsf{B}$ with $\mathsf{A} = \{1,\dots,a\}$, and for $\sigma \in S_d$ let $\mathrm{tr}_\sigma := \prod_{(c_1c_2\cdots)} \mathrm{tr}(X_{c_1}X_{c_2}\cdots)$, the product over the cycles of σ, with $X_c = A$ for $c \in \mathsf{A}$ and $X_c = B$ otherwise. Then $\varphi_f = \varphi_g := \sum_\sigma g(\sigma)\,\mathrm{tr}_\sigma$ for a function g on the derangements of S_d (a fixed point would give the trace of a traceless matrix), invariant under conjugation by $S_\mathsf{A} \times S_\mathsf{B}$, and unique for $n \ge d$; and $\mathcal{E}\varphi_g = 0$ if and only if

$$\text{(HW)} \qquad g + \textstyle\sum_{i\in\mathsf{A}} g^{(i\,j)} = 0 \quad \text{for every } j \in \mathsf{B}, \qquad g^{(i\,j)}(\sigma) := g\big((i\,j)\,\sigma\,(i\,j)\big),$$

($\mathcal{E}$ turns one letter $j \in \mathsf{B}$ into A; the result is symmetrized over the cosets of S_A in $S_{\mathsf{A}\cup\{j\}}$, with representatives e and the $(i\,j)$). Let $\mathrm{HW}_{(a,b)}$ be the space of these g. For $n \ge d$ the trace monomials without fixed points are linearly independent on traceless matrices [Ra74, Pr76], and $f \mapsto g$ is an isomorphism $\mathrm{Pol}_\tau \cong \mathrm{HW}_{(a,b)}$, with inverse $\det(P_1)^{n-d}\varphi_g\big((\mathrm{adj}(P_1)P_2)_0, (\mathrm{adj}(P_1)P_3)_0\big)$ (X_0 is the traceless part of X).

Lemma 9.20 (the test in the chart)**.** *For $f \in \mathrm{Pol}_\tau$, at $P_1 = I$, $P_2 = A$, $P_3 = B$,*

$$\Omega_u f = \mathrm{Alt}_y\Big[D^3\varphi_f[E_1,E_2,E_3] + (n-2)(y_1\cdot u)\,D^2\varphi_f[E_2,E_3]\Big],$$

with $E_1 = -(uy_1^TA,\, uy_1^TB)$, $E_2 = (uy_2^T, 0)$, $E_3 = (0, uy_3^T)$, φ_f evaluated on traceless parts, and Alt_y the signed sum over the permutations of y_1, y_2, y_3.

Proof. With $t_a^2 = 0$: $\det(I + t_1uy_1^T)^n = 1 + nt_1(y_1\cdot u)$, $(I + t_1uy_1^T)^{-1} = I - t_1uy_1^T$, and $(I + t_1uy_1^T)^{-1}(A + t_2uy_2^T) = A + t_2uy_2^T - t_1uy_1^TA - t_1t_2(y_1\cdot u)uy_2^T$, and likewise for B, t_3, y_3. The coefficient of $t_1t_2t_3$ in $\det(P_1)^n\varphi_f(P_1^{-1}P_2, P_1^{-1}P_3)$ is $D^3\varphi_f[E_1,E_2,E_3] - 2(y_1\cdot u)D^2\varphi_f[E_2,E_3] + n(y_1\cdot u)D^2\varphi_f[E_2,E_3]$. □

The test at $P_1 = I$ is a nonzero multiple of the test at any P_1 with $\det P_1 \neq 0$ (the test is equivariant, and f is homogeneous in P_1), and these P_1 are dense; so (S1) holds if and only if it holds at $P_1 = I$. Let

$$\mathcal{Z}(\varphi) := \mathrm{Alt}_y\Big[(y_1\cdot u)\,D^2\varphi\big[(uy_2^T,0),(0,uy_3^T)\big]\Big].$$

The values of the test are combinations of the functions (a product of traces of words of length ≥ 2 in A,B) $\cdot\ \det[y_i\cdot w_cu]_{i,c=1,2,3}$, with words w_1,w_2,w_3 (one may be empty); these are linearly independent for $n \ge d+1$, apart from the antisymmetry of the determinant (Schur–Weyl duality in the form of the second fundamental theorem [Pr76]: $F[S_\ell] \to \mathrm{End}(V^{\otimes\ell})$ is injective for $\ell \le n$; then set $u_1 = u_2 = u_3$ and alternate the y_i).

Lemma 9.21. *In the basis g of $\mathrm{HW}_{(a,b)}$ and the functions above, the test (S1) at $P_1 = I$ is a matrix $\mathbf{M}(n) = n\,\mathcal{Z} + \mathbf{M}_0 + \mathbf{M}_-(n)$, where $\mathbf{M}_0$ does not depend on n and $\mathbf{M}_-(n)$ is a combination of negative powers of n. Hence, if $\mathcal{Z}$ is injective on $\mathrm{HW}_{(a,b)}$, then (S1) is injective on Pol_τ for all but finitely many n.*

Proof. The only explicit positive power of n is the factor $n-2$. Each traceless projection $X \mapsto X - \frac{\mathrm{tr}\,X}{n}I$ brings a factor $1/n$, and a factor $\mathrm{tr}\,I = n$ needs a cycle made only of projected letters; as every cycle of tr_σ has length at least 2, that takes at least two projections. So the projections never raise the power of n. Inside the $(n-2)$-term, the projections of E_2 and E_3 give a factor $(y_2\cdot u)$

or $(y_3 \cdot u)$, which is symmetric with $(y_1 \cdot u)$ under the alternation, so these terms vanish. Hence the coefficient of n is exactly $\mathcal{Z}$. For the second statement: choose a nonzero maximal minor of $\mathcal{Z}$; the same minor of $\mathbf{M}(n)/n$ is a polynomial in $1/n$ with this constant term, so it vanishes for only finitely many n. □

9.8. **Theorem Z-INJ.**

Theorem 9.22 (Z-INJ)**.** *For $a \geq b \geq 1$, $\ker \mathcal{Z} \cap \mathrm{HW}_{(a,b)} = 0$, except for $(a,b) = (2,2)$, where it is spanned by $\mathrm{tr}([A,B]^2)$.*

Proof. (a) Fibres. Fix $i \in \mathsf{A}$ (direction uy_2^T) and $j \in \mathsf{B}$ (direction uy_3^T) and cut them out of tr_σ. The result is $(y_1 \cdot u)(y_2 \cdot wu)(y_3 \cdot w'u)$ times the other cycles, where w is the segment after i and w' the segment after j, each ending before the next of i, j; after the alternation it is $u \wedge wu \wedge w'u$ times the rest, and it is 0 if w or w' is empty. For $t = (i\,j)$ the four derangements σ, $t\sigma$, σt, $t\sigma t$ give the same term, with signs $+,+,-,-$, and no other permutation gives it $((i\,w)(j\,w'), (i\,w\,j\,w'), (i\,w'\,j\,w), (i\,w')(j\,w))$. All pairs (i,j) contribute alike, by the invariance of g, and invariants inject into coinvariants. Hence, for one fixed pair and with $T_t g(\sigma) := g(\sigma) + g(t\sigma) - g(\sigma t) - g(t\sigma t)$,

$$\mathcal{Z}(\varphi_g) = 0 \iff T_t g(\sigma) = 0 \text{ for every derangement } \sigma \text{ with } \sigma(i) \neq j \text{ and } \sigma(j) \neq i.$$

(b) A sum identity. Extend g by 0 to S_d. For every σ, $\sum_{t \in \mathsf{A}\times\mathsf{B}} [g(t\sigma) - g(\sigma t)] = 0$: over all transpositions the two sums agree, since $\sigma^{-1} t \sigma$ runs over them, and for t inside A or inside B, $g(t\sigma) = g(t(\sigma t)t) = g(\sigma t)$ by invariance. With (HW), summed over j, $\sum_t g^t = -bg$, so $\sum_{t \in \mathsf{A}\times\mathsf{B}} T_t g = ab\,g + b\,g = b(a+1)\,g$.

(c) Locality. If $\sigma(i) = j$ but $\sigma(j) \neq i$ (or the reverse), then $t\sigma$ and σt have a fixed point, so $T_t g(\sigma) = g(\sigma) - g(t\sigma t)$. If $(i\,j)$ is a cycle of σ, then $T_t g(\sigma) = 0$. So for $g \in \mathrm{HW}_{(a,b)}$ with $\mathcal{Z}(\varphi_g) = 0$,

$$b(a+1)\,g = Lg, \qquad (Lg)(\sigma) := \sum_{t \in \mathrm{Adj}(\sigma)} \big(g(\sigma) - g(t\sigma t)\big),$$

where $\mathrm{Adj}(\sigma)$ is the set of pairs in $\mathsf{A} \times \mathsf{B}$ adjacent in σ in exactly one direction. The relation is symmetric: if $\sigma(i) = j$ and $\sigma(j) \neq i$, then $\sigma' := t\sigma t$ has $\sigma'(j) = i$ and $\sigma'(i) = \sigma(j) \notin \{i, j\}$. So L is the Laplacian of a graph on the derangements (possibly with multiple edges, without loops): it exchanges the colours of an adjacent A–B pair.

(d) The bound. The eigenvalues of a graph Laplacian are at most twice the largest degree, and $|\mathrm{Adj}(\sigma)| \leq 2b$, because each letter of B has only two neighbours on its cycle. So $g \neq 0$ needs $b(a+1) \leq 4b$, that is $a \leq 3$.

(e) $a \leq 3$. For $a = 3$ and $b = 1, 2, 3$ the matrix $b(a+1)I - L$ on all derangements of S_d (9, 44 and 265 of them) is positive definite, by an exact LDL^T decomposition over $\mathbb{Q}$ (all pivots positive; the smallest are $4/3$, $15776/5409$, $32472/6277$); so $Lg = b(a+1)g$ forces $g = 0$. For $(1,1)$ the only derangement is $(1\,2)$ and (HW) reads $2g = 0$. For $(2,1)$, g is constant on the two 3-cycles and (HW) reads $3g = 0$. For $(2,2)$, $b(a+1) = 6$ is the largest eigenvalue of L; $\mathrm{tr}([A,B]^2)$ lies in $\mathrm{HW}_{(2,2)}$ and in $\ker \mathcal{Z}$, and on the 9 derangements the kernel in $\mathrm{HW}_{(2,2)}$ (of dimension 2) is exactly one-dimensional, by exact computation. □

Remark 9.23. For $b = 1$ there is a second proof. Here $\varphi = \sum_j g_j(p)\,\mathrm{tr}(A^j B)$ with $p_\ell = \mathrm{tr}\,A^\ell$, and (HW) reads $\sum_j g_j p_{j+1} = 0$. One finds $\mathcal{Z}(\varphi) = \sum_{j,\ell} \ell\,(\partial g_j/\partial p_\ell)\,[0, \ell-1, j]$, where $[e_1, e_2, e_3] = \det[y_i \cdot A^{e_c} u]$. With $q_\ell := g_{\ell-1}$ and $P_\ell := p_\ell/\ell$, $\mathcal{Z}(\varphi) = 0$ says that the form $\sum_\ell q_\ell\, dP_\ell$ is closed, so $q_\ell = \partial Q_0/\partial P_\ell$, and (HW) gives $0 = \sum_\ell q_\ell p_\ell = (a+1)Q_0$ by Euler's identity. The bound in (d) is the standard estimate $\lambda_{\max} \leq 2\Delta$; the colour exchange is an exclusion process of the kind studied for the interchange process [CLR10], of which we use only this bound.

9.9. **The kernel of $B(n,2)$ to depth $n-1$.**

Theorem 9.24. *For every $n \geq 4$, every conjugation type of $K_{B(n,2)}$ of depth at most $n-1$ is $(3,1^{N-3})$, and it occurs once.*

Proof. Let $t=(s,r)$, $(a,b)=(s+1,r+1)$ and $\tau=(n'^2-|t|-2,s+1,r+1)$ for large n', so that $d=|t|+2\leq n'-1$. If $(a,b)\neq(2,2)$, then $\mathcal{Z}$ is injective on $\mathrm{HW}_{(a,b)}$ by Theorem 9.22, so (S1) is injective on Pol_τ for all large n' (Lemma 9.21), and $r_t(n')=0$ by Theorem 9.18. For $(a,b)=(2,2)$, that is $t=(1,1)$, the kernel of $\mathcal{Z}$ on $\mathrm{HW}_{(2,2)}$ has dimension 1, so the kernel of (S1) has dimension at most 1 for all large n' (lower semicontinuity of the rank), and $r_{(1,1)}(n')=1$ by Theorem 9.9. These are the stable values, so by Lemma 9.17(b) $r_t(n)=0$ for $t\neq(1,1)$ and $r_{(1,1)}(n)=1$ for every $n\geq 4$ and $|t|\leq n-1$. □

Corollary 9.25. *Let $n\geq 4$. The copy of $(3,1^{N-3})$ in $K_{B(n,2)}$ meets $U_{B(n,2)}$ only in 0. Hence $U_{B(n,2)}$ lies in the sum of the isotypic parts of $K_{B(n,2)}$ of depth at least n; in particular $U_{B(n,2)}=0$ if $K_{B(n,2)}$ has no type of depth at least n.*

Proof. The copy is spanned by the S_N-conjugates of f_W (Theorem 9.9(b) for $n\geq 5$; for $n=4$, Theorem 9.7(R2) gives $\dim K^{[2]}_{B(4,2)}=1$). By Remark 9.13, $f_W(\sigma)=c\sum_\pi y_W(\pi)\psi_R(\sigma\pi^{-1})$. Let σ be an N-cycle of S_I and π one of the four-cycles $(o\,o'\,a_1\,a_2)$, $(o\,o'\,a_2\,a_1)$ of y_W. Then $\sigma\pi^{-1}$ has exactly 3 cycles: the third letter of O alone; for $\pi=(o\,o'\,a_1\,a_2)$ the cycle $o'\to o\to\sigma(a_2)\to\cdots\to a_1\to o'$, which contains both o and o'; and the cycle $\sigma(a_1)\to\cdots\to a_2\to\sigma(a_1)$, which contains no letter of O (for the other four-cycle exchange a_1 and a_2). As ψ_R vanishes off the class $h(R)$, which has $n\neq 3$ cycles, f_W and all its conjugates vanish on the N-cycles. An element of the copy supported on the N-cycles is therefore 0. Since $U_{B(n,2)}\subseteq K_{B(n,2)}$ is stable under conjugation, its part of type $(3,1^{N-3})$ lies in the copy, so it is 0, and by Theorem 9.24 no other type of depth at most $n-1$ is available. □

Remark 9.26 (what remains). So, for $n\geq 4$, $U_{B(n,2)}=0$, the hypothesis of Theorem F, is proved on every conjugation type of depth at most $n-1$, and it is open only on the types of depth at least n, and, by Corollary 9.42(b) below, only on those with three columns, and not on those with $\kappa'_3=1$ (Theorem 9.24 and Corollaries 9.174 and 9.189 below), nor on those with $\kappa'_3=2$ (Corollary 9.258 below) (for $n\geq 6$; it holds for $n=4$ and $n=5$, and fails for $n=3$). For $n=5$ it holds on all types, by $d(5)\leq 21$ (Subsection 1.2) and Proposition 2.1; Theorem 9.24 alone leaves the types of depth at least 5 open there; Theorem 9.27 below settles them by computation, for $K_{B(5,2)}$ and not only for $U_{B(5,2)}$. The layers must fail somewhere: for $n=3$, $K_{(3,2,1)}$ has the type $(3,3)$, of depth $4=n+1$. The window method reaches at most depth about $2n-3$: for $|Q|$ beyond about $2n$ the shapes outside the box are no longer of low degree. The types of depth from about $2n$ to $2N/3$ need another idea.

9.10. **The case $n=5$.** For $n=5$ we compute every conjugation type of $K_{B(5,2)}$, including those of depth at least 5, which Theorem 9.24 does not reach. Here $B(5,2)=(5,5,5,4,3)$ and $N=22$. By Subsection 9.6 the types are the κ with $\kappa'=(N-s-r,s,r)$; put $(a,b):=(s+1,r+1)$, so that $\tau=\kappa'+(1,1,1)=(25-a-b,a,b)$, with $a\geq b\geq 1$ and $25-a-b\geq a$. There are 52 such τ, and $d:=a+b$ runs from 2 to 16; the depth of κ is $d-2$.

Theorem 9.27 (computer-assisted)**.** *$K_{B(5,2)}$ is the single copy of $(3,1^{19})$ spanned by the S_N-conjugates of f_W.*

By Corollary 9.25 this gives $U_{B(5,2)}=0$. That is classical ($d(5)\leq 21$, see Theorem F), so here it is a consistency check: the block method reaches it without the computation of $d(4)$. The content of the theorem is the kernel itself; a bound on $d(5)$ controls only the part $U_{B(5,2)}$ of $K_{B(5,2)}$.

For a type τ put $\mathcal{K}(\tau):=\{f\in\mathrm{Pol}_\tau:\text{(S1), (S2)}\}$; by Theorem 9.18 the multiplicity of κ in $K_{B(5,2)}$ is $\dim\mathcal{K}(\tau)$.

Functions. Let $\mathcal{F}_{(a,b)}$ be the space of the polynomial functions $\varphi(A,B)$ on $\mathfrak{sl}_5 \times \mathfrak{sl}_5$ that are invariant under simultaneous conjugation and of bidegree (a,b), and $\mathcal{F}^{\mathcal{E}}_{(a,b)} := \ker \mathcal{E}$, with $(\mathcal{E}\varphi)(A,B) = \frac{d}{dt}\varphi(A,B+tA)|_{t=0}$. The chart paragraph of Subsection 9.7 up to the trace polynomials does not use $d \le n$: for $f \in \mathrm{Pol}_\tau$ the function $\varphi_f(A,B) = f(I,A,B)$, restricted to traceless A,B, lies in $\mathcal{F}^{\mathcal{E}}_{(a,b)}$, $f = \det(P_1)^5 \varphi_f\big((P_1^{-1}P_2)_0, (P_1^{-1}P_3)_0\big)$ on $\det P_1 \neq 0$, and $f \mapsto \varphi_f$ is injective. Here d may exceed 5, where the trace monomials are no longer independent; so we work with the functions themselves.

Conditions. Let $f \in \mathcal{K}(\tau)$, $\varphi := \varphi_f$, and $F(P) := \det(P_1)^5 \varphi\big((P_1^{-1}P_2)_0, (P_1^{-1}P_3)_0\big)$ on $\det P_1 \neq 0$. Then:

(G) at every P with $\det P_1 \neq 0$ the tests (S1) and (S2) hold with F in place of f (as $F = f$ near P; only the first order in t_1 is used);

(P) if $d \ge 5$: $\varphi\big((\mathrm{adj}(P_1)P_2)_0, (\mathrm{adj}(P_1)P_3)_0\big) = \det(P_1)^{d-5} f(P)$ for all P (both sides are polynomials, and they agree on $\det P_1 \neq 0$, as $P_1^{-1} = \mathrm{adj}(P_1)/\det(P_1)$ and φ has total degree d); so, if $d > 5$, along every curve $P_1(s)$ with $\det P_1(0) = 0$ (exactly) the left side vanishes to order at least $d-5$ at $s = 0$.

Both are linear conditions on φ.

Exact dimensions. By Cauchy's formula $\mathbb{C}[M_5]_a = \bigoplus_{\alpha \vdash a,\, \ell(\alpha) \le 5} S_\alpha V \otimes S_\alpha V^*$, and by the Littlewood–Richardson rule, the GL_5-invariants of bidegree (a,b) on $M_5 \times M_5$ have dimension

$$f_{\mathfrak{gl}}(a,b) = \textstyle\sum_{\rho \vdash a,\ \pi \vdash b} \frac{1}{z_\rho z_\pi} \sum_{\mu \vdash a+b,\ \ell(\mu) \le 5} \chi^\mu(\rho \cup \pi)^2.$$

As $M_5 = \mathfrak{sl}_5 \oplus \mathbb{C}I$ and the traces are invariant, the bigraded Hilbert series of the $\dim \mathcal{F}_{(a,b)}$ is $(1-x)(1-y)$ times that of the $f_{\mathfrak{gl}}(a,b)$. Finally $\mathfrak{gl}_2$ acts on $\bigoplus_{a+b=d} \mathcal{F}_{(a,b)}$ by polarization, completely reducibly, and for $a \ge b$ the raising operator $\mathcal{E}$ maps $\mathcal{F}_{(a,b)}$ onto $\mathcal{F}_{(a+1,b-1)}$; so $\mathrm{hw}(a,b) := \dim \mathcal{F}^{\mathcal{E}}_{(a,b)} = \dim \mathcal{F}_{(a,b)} - \dim \mathcal{F}_{(a+1,b-1)}$. These numbers are computed exactly from characters (Murnaghan–Nakayama), with tests against $n = 2$, against Teranishi's Hilbert series for $n = 3$ [Te86], and against the independence of trace monomials below degree $n+1$.

Ranks modulo p. Let $T(a,b)$ be the set of trace monomials of bidegree (a,b) (products of traces of cyclic words of length at least 2 in A, B), $\mathrm{ev}\colon \mathbb{Q}^{T(a,b)} \to \mathcal{F}_{(a,b)}$ the evaluation, $\mathcal{E}_{\mathrm{f}}\colon \mathbb{Q}^{T(a,b)} \to \mathbb{Q}^{T(a+1,b-1)}$ the formal operator (replace one B by A, summed over the B's), so that $\mathrm{ev} \circ \mathcal{E}_{\mathrm{f}} = \mathcal{E} \circ \mathrm{ev}$, and $H := \ker \mathcal{E}_{\mathrm{f}}$. Let $p > 5$ be a prime.

Lemma 9.28. *Suppose that (R1) $\mathcal{E}_{\mathrm{f}}$ has rank $|T(a+1,b-1)|$ modulo p; (R2) the values of a basis of $H_p := \ker(\mathcal{E}_{\mathrm{f}} \bmod p)$ at some points of $\mathfrak{sl}_5(\mathbb{F}_p)^2$ have rank $\mathrm{hw}(a,b)$; and (R3) a set of rows, each a condition (G) or (P) at an $\mathbb{F}_p$-point and computed modulo p as below, has rank r on H_p. Then $\dim \mathcal{K}(\tau) \le \mathrm{hw}(a,b) - r$.*

Proof. The rank of a p-integral matrix can only drop modulo p. So (R1) gives $\mathrm{rank}_p \mathcal{E}_{\mathrm{f}} = \mathrm{rank}_{\mathbb{Q}} \mathcal{E}_{\mathrm{f}}$, $\dim H_p = \dim H$, and H_p is the reduction of the saturated lattice $L := H \cap \mathbb{Z}_{(p)}^{T(a,b)}$. Lift every point to a rational point of the same kind: a traceless matrix to a traceless integer matrix; for (G), P_1 with $\det P_1 \not\equiv 0$ to an integer matrix, so $\det P_1 \neq 0$; for (P), the curve $P_1(s) = p_0 + se$, where the last row of p_0 is a combination $r_1 \,\mathrm{row}_1 + r_2 \,\mathrm{row}_2$ of the first two rows, by lifting the first four rows, r_1, r_2, e, P_2, P_3 and defining the last row over $\mathbb{Z}$ by the same combination, so that $\det p_0 = 0$ exactly. (An entrywise lift of p_0 has $\det p_0 \neq 0$ in general, and there (P) is not a condition.) At these points every row is a true condition, and the computation uses only ring operations, $1/5$ (traceless parts), the inverse of a matrix with determinant prime to p, the cofactor adjugate, and the interpolation of the expansion in s at $s = 1, \dots, 5$ (Vandermonde determinant 288); all of these are p-integral. So the rows of (R3) are the reductions of a p-integral matrix S of true conditions, and $r \le \mathrm{rank}_{\mathbb{Q}}(S|_H)$. In the same way (R2) gives $\mathrm{rank}(\mathrm{ev}\,|_H) \ge \mathrm{hw}(a,b)$, and as $\mathrm{ev}(H) \subseteq \mathcal{F}^{\mathcal{E}}_{(a,b)}$, $\mathrm{ev}(H) = \mathcal{F}^{\mathcal{E}}_{(a,b)}$. Each row of S is a linear form in $\varphi = \mathrm{ev}(h)$, so it vanishes on $\ker(\mathrm{ev}\,|_H)$, and the

solutions of S in $\mathcal{F}^{\mathcal{E}}_{(a,b)}$ form a space of dimension $\mathrm{hw}(a,b)-\mathrm{rank}_{\mathbb{Q}}(S|_H)\le \mathrm{hw}(a,b)-r$. It contains φ_f for every $f\in\mathcal{K}(\tau)$, and $f\mapsto\varphi_f$ is injective. □

Nothing random enters the conclusion: bad points can only lower r. The first fundamental theorem for matrix invariants is not needed, as (R2) gives $\mathrm{ev}(H)=\mathcal{F}^{\mathcal{E}}_{(a,b)}$ directly.

Proof of Theorem 9.27. With $p=33554393$ and the rows (G) and (P) (for (P), the coefficients of $s^0,\dots,s^{d-6}$) at random points, (R1) and (R2) hold for all 52 types, and Lemma 9.28 gives $\dim\mathcal{K}(\tau)=0$ for every τ except $\tau=(21,2,2)$, where it gives $\dim\mathcal{K}(\tau)\le 1$. This τ is $\kappa=(3,1^{19})$, which occurs in $K_{B(5,2)}$ exactly once, in the copy generated by f_W (Theorem 9.9(b)). □

Remark 9.29 (checks). (a) The values $\mathrm{hw}(a,b)$ go up to 590, at $(a,b)=(9,7)$. The rows (P) alone cut $\mathcal{F}^{\mathcal{E}}_{(a,b)}$ down to exactly $g(R,R,\tau)=\dim\mathrm{Pol}_\tau$ on every type, a check of Theorem 9.18. (b) The same program finds, for $n=3$, the bound 1 exactly at the two types $(3,1,1,1)$ and $(3,3)$ of $K_{(3,2,1)}$, both of which occur (the example after Corollary 7.6), and 0 elsewhere; for $n=4$ it finds only $(3,1^{10})$ (Remark 9.10). (c) An independent program, by another method (one weight of GL_3, $(9,8,8)$, which occurs in all 52 types, a determinantal basis, and the tests only at points with $P_1=I+t_1uy^T$, so with no rows (P)), gives the theorem again; this is a second complete certificate. An exact rational re-implementation of the first program on 8 of the 52 types gives the same bounds; it is a partial check. (d) Two more runs of the first program, with the chart form of the tests (Lemma 9.20), with and without the rows (P), agree with it type by type on the types they cover.

9.11. **The pencil and the rectangle problem.** In this subsection $F=\mathbb{C}$. The test (S1) at $P_1=I$ has an exact form in which n appears explicitly only as the exponent of $\det(P_1)^n$ in the chart (it also appears through the size of the matrices). Letting the exponent vary gives a pencil, and for integer exponents a family of problems on $n\times s$ rectangles.

Let φ be a polynomial on $\mathfrak{sl}_n\times\mathfrak{sl}_n$, invariant under simultaneous conjugation, of bidegree (a,b), $d:=a+b$, and put $\tilde\varphi(X,Y):=\varphi(X_0,Y_0)$ on $M_n\times M_n$. For $s\in\mathbb{C}$ let

$$f_s(P_1,P_2,P_3):=\det(P_1)^s\,\tilde\varphi\big(P_1^{-1}P_2,P_1^{-1}P_3\big)$$

near $P_1=I$ (for $s\in\mathbb{Z}$ a rational function on $\det P_1\neq 0$); for $s=n$ this is the chart of Subsection 9.7. At a base point $(A_2,A_3)\in M_n^2$ put $\beta(X,Y):=\partial_{A_2}[X]\,\partial_{A_3}[Y]\,\tilde\varphi$ (the coefficient of $\epsilon_1\epsilon_2$ in $\tilde\varphi(A_2+\epsilon_1X,A_3+\epsilon_2Y)$), $\omega(X,Y):=\beta(X,Y)-\beta(Y,X)$,

$$\Omega f_s(B_1,B_2,B_3):=\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)\,[t_1t_2t_3]\,f_s\big(I+t_1B_{\sigma(1)},A_2+t_2B_{\sigma(2)},A_3+t_3B_{\sigma(3)}\big),$$

and, writing $\hat B_i$ for the other two of B_1,B_2,B_3 in their order, $W\circ X:=\frac12(WX+XW)$ and $J(W):=(W\circ A_2,W\circ A_3)$,

$$(\mathrm{tr}\wedge\omega)(B_1,B_2,B_3):=\sum_i(-1)^{i-1}\,\mathrm{tr}(B_i)\,\omega(\hat B_i),\qquad(\mathfrak{L}\omega)(B_1,B_2,B_3):=\sum_i(-1)^{i-1}D_{J(B_i)}\,\omega(\hat B_i),$$

where D_V is the derivative in the base point along V, with the arguments $\hat B_i$ fixed.

Lemma 9.30 (identity J)**.** *For every $s\in\mathbb{C}$, every base point and all $B_1,B_2,B_3\in M_n$: $\Omega f_s=s\,(\mathrm{tr}\wedge\omega)-\mathfrak{L}\omega$.*

Proof. Modulo t_a^2: $\det(I+t_1X_1)^s=1+st_1\,\mathrm{tr}\,X_1$, $(I+t_1X_1)^{-1}=I-t_1X_1$, and $(I+t_1X_1)^{-1}(A_2+t_2X_2)=A_2+t_2X_2-t_1X_1A_2-t_1t_2X_1X_2$, and likewise for A_3, t_3, X_3. So the coefficient of $t_1t_2t_3$ in $f_s(I+t_1X_1,A_2+t_2X_2,A_3+t_3X_3)$ is $s\,\mathrm{tr}(X_1)\beta(X_2,X_3)-D_{L(X_1)}\beta(X_2,X_3)-\beta(X_1X_2,X_3)-\beta(X_2,X_1X_3)$, with $L(W):=(WA_2,WA_3)$. Alternating the first term gives $s(\mathrm{tr}\wedge\omega)$. Write $L(W)=J(W)+\frac12C(W)$ with $C(W):=([W,A_2],[W,A_3])$; the J-part alternates to $-\mathfrak{L}\omega$. Differentiating the conjugation invariance $\beta_{gAg^{-1}}(gXg^{-1},gYg^{-1})=\beta_A(X,Y)$ at $g=\exp(\epsilon W)$ gives $D_{C(W)}\beta(X,Y)=$

$-\beta([W,X],Y)-\beta(X,[W,Y])$. In the alternation, $\beta(X_1X_2,X_3)$ may be replaced by $\frac{1}{2}\beta([X_1,X_2],X_3)$ (exchange X_1, X_2) and $\beta(X_2,X_1X_3)$ by $\frac{1}{2}\beta(X_2,[X_1,X_3])$ (exchange X_1, X_3); with the previous identity the C-part and these two terms cancel. □

Corollary 9.31 (the pencil)**.** *Put $\mathcal{Z}(\varphi) := \operatorname{tr}\wedge\omega$ and $\mathcal{J}(\varphi) := \mathfrak{L}\omega$, both at $B_i = uy_i^T$ $(i=1,2,3)$.*

(a) $\Omega f_s = \lambda\,(\operatorname{tr}\wedge\omega) - \mathfrak{L}_0\omega$ with $\lambda = s-(d-2)/n$, where $\mathfrak{L}_0$ is $\mathfrak{L}$ with every B_i replaced by its traceless part.

(b) Let $s\in\mathbb{Z}$. Then f_s satisfies (S1) at every P with $\det P_1\neq 0$ if and only if $s\,\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$ for all base points and all u,y_1,y_2,y_3; and it satisfies (S2) there if and only if the same holds for $\varphi^T(X,Y):=\varphi(X^T,Y^T)$.

Proof. (a) $J(B) = J(B_0) + \frac{\operatorname{tr}B}{n}(A_2,A_3)$, and ω is homogeneous of degree $d-2$ in the base point, so $\mathfrak{L}\omega = \mathfrak{L}_0\omega + \frac{d-2}{n}\operatorname{tr}\wedge\omega$. (b) For $g,h\in\mathrm{GL}_n$, $f_s(gP) = \det(g)^s f_s(P)$ and $f_s(Ph^T) = \det(h)^s f_s(P)$ (conjugation invariance of $\tilde\varphi$). Writing $P_1=g$, the test at P is $\det(g)^s$ times the test at $(I,g^{-1}P_2,g^{-1}P_3)$ with u replaced by $g^{-1}u$; and at $P_1=I$ it is $\Omega f_s(uy_1^T,uy_2^T,uy_3^T)$, which is $s\mathcal{Z}(\varphi)-\mathcal{J}(\varphi)$ by Lemma 9.30. For (S2), $f_s(P^T)$ is the function f_s built from φ^T, and (S2) for f is (S1) for $P\mapsto f(P^T)$. □

At rank-one arguments, $\operatorname{tr}\wedge\omega = \mathrm{Alt}_y\big[(y_1\cdot u)\,\beta(uy_2^T,uy_3^T)\big]$, so $\mathcal{Z}$ is the operator of Subsection 9.7; Theorem 9.22 describes its kernel on $\mathrm{HW}_{(a,b)}$ in the stable range. At $b=2$ and $a\geq 3$, $\mathcal{Z}$ is injective on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ for every $n\geq 3$ (Corollary 9.81). For fixed n and (a,b), $s\mathcal{Z}-\mathcal{J}$ is a linear pencil in s on the invariant functions killed by $\mathcal{E}$, and the test of Theorem 9.18 is its value at $s=n$, exactly. For integer s the pencil asks the same question for $n\times s$ rectangles:

The rectangle problem. For integers $s,n\geq 1$ and $\tau\vdash ns$ with at most three rows, let $\mathrm{R}_s(n)_\tau$ be the space of the polynomials f on M_n^3 with $f(gPh^T)=f(P)$ for $g,h\in\mathrm{SL}_n$ that are highest weight vectors of weight τ for GL_3 (as in Subsection 9.7), and let $\mathrm{K}_s(n)_\tau\subseteq\mathrm{R}_s(n)_\tau$ be the subspace cut out by (S1) and (S2).

Lemma 9.32. *(i) $\dim\mathrm{R}_s(n)_\tau = g\big((s^n),(s^n),\tau\big)$, and every $f\in\mathrm{R}_s(n)_\tau$ has degree s in each row and in each column of P.*

(ii) $f\mapsto\varphi_f := f(I,\cdot,\cdot)$ is injective on $\mathrm{R}_s(n)_\tau$; $f=\det(P_1)^s\varphi_f\big((P_1^{-1}P_2)_0,(P_1^{-1}P_3)_0\big)$ on $\det P_1\neq 0$; and φ_f is a conjugation-invariant polynomial in the traceless parts, of bidegree (τ_2,τ_3), killed by $\mathcal{E}$. Conversely, let φ be such a polynomial, of bidegree (a,b), $d:=a+b$, with $\tau:=(ns-d,a,b)$ a partition; then φ comes from $\mathrm{R}_s(n)_\tau$ if $d\leq s$, and, if $d>s$, if and only if $\det(P_1)^{d-s}$ divides $\varphi\big((\mathrm{adj}(P_1)P_2)_0,(\mathrm{adj}(P_1)P_3)_0\big)$. (If $ns-d<a$, no nonzero φ of bidegree (a,b) comes from a highest weight vector.)

(iii) If $\tau_3\geq 1$, then $\mathrm{K}_n(n)_\tau$ is the space of Theorem 9.18, so the multiplicity of $\kappa=(\tau-(1,1,1))'$ in $K_{B(n,2)}$ is $\dim\mathrm{K}_n(n)_\tau$. (If $\tau_3=0$, f does not involve P_3, and $\mathrm{K}_s(n)_\tau=\mathrm{R}_s(n)_\tau$.)

(iv) $\dim\mathrm{R}_s(n)_\tau = \dim\mathrm{R}_n(s)_\tau$.

Proof. (i) By the Cauchy formula for $\mathbb{C}^n\otimes\mathbb{C}^n\otimes\mathbb{C}^3$, an SL_n-invariant in $S_\alpha(\mathbb{C}^n)$ needs $\alpha=(k^n)$, so the degree is ns and $\alpha=\beta=(s^n)$; the torus of GL_n acts on that line by $\det^s$. (ii) As in the chart paragraph of Subsection 9.7, with the exponent s in place of n ($f(P_1Q)=\det(P_1)^sf(Q)$ for the left action); the converse is the identity $\varphi((\mathrm{adj}(P_1)P_2)_0,(\mathrm{adj}(P_1)P_3)_0)=\det(P_1)^{d-s}f(P)$, as (P) in Subsection 9.10; a highest weight vector of a weight that is not a partition is 0. (iii) is the setting of Theorem 9.18. (iv) $g(\lambda',\mu',\nu)=g(\lambda,\mu,\nu)$ and $(s^n)'=(n^s)$. □

Theorem 9.33 (duality)**.** *For all $n,s\geq 1$ and every τ, $\mathrm{K}_s(n)_\tau\cong\mathrm{K}_n(s)_\tau$; the isomorphism exchanges (S1) and (S2).*

Proof. Put $m:=ns$, $R:=(s^n)$ and $R':=(n^s)$. Steps 1 and 3 of the proof of Theorem 9.18 do not use that R is a square: the multilinear SL_n-invariant forms T_R on $V^{\otimes m}$ form a copy of S^R (the only $\mu\vdash m$ with an SL_n-invariant in $S_\mu(V)$ is the rectangle with n rows), the multilinear

invariant functions of m matrices are $\mathcal{M}_R = \mathsf{T}_R \otimes \mathsf{T}_R$ (left vectors, right vectors), and $\tilde{x} \mapsto f_{\tilde{x}}$ is a bijection $\mathrm{Hom}_{S_m}(S^\tau, \mathcal{M}_R) \to \mathrm{R}_s(n)_\tau$ (Young's rule for the highest weight conditions, polarization for injectivity, and $g(R,R,\tau)$ on both sides). If $\tau_3 = 0$ both tests are empty. Otherwise the coefficient of $t_1t_2t_3$ puts uy_a^T into one slot of each group, and the alternated y's with the repeated u span $\mathrm{Sym}^3 V \otimes \Lambda^3 V$ in these slots O'; so (S1) says $(e_{(3)} \otimes e_{(1,1,1)}) \cdot \tilde{x}(w_\tau) = 0$ and (S2) says $(e_{(1,1,1)} \otimes e_{(3)}) \cdot \tilde{x}(w_\tau) = 0$, with e_μ the central idempotents of $F[S_{O'}]$ acting on the first, resp. second, factor. Fix $\epsilon\colon \mathsf{T}_R \to \mathsf{T}_{R'}$ with $\epsilon(\sigma v) = \mathrm{sgn}(\sigma)\sigma\epsilon(v)$ ($S^{R'} \cong S^R \otimes \mathrm{sgn}$) and put $\iota := \epsilon \otimes \epsilon$. It commutes with the diagonal action, so $\tilde{x} \mapsto \iota \circ \tilde{x}$ is a bijection onto $\mathrm{Hom}_{S_m}(S^\tau, \mathcal{M}_{R'})$ with the same τ and w_τ; and on one factor $\epsilon(e_\mu v) = e_{\mu'}\epsilon(v)$. So (S1) for R becomes (S2) for R', and (S2) for R becomes (S1) for R'. □

By Theorem 9.69 below, (S1) alone cuts out $\mathrm{K}_s(n)_\tau$ in $\mathrm{R}_s(n)_\tau$.

Proposition 9.34. *If $s \le 2$ or $n \le 2$, then $\mathrm{K}_s(n)_\tau = \mathrm{R}_s(n)_\tau$.*

Proof. If $s \le 2$: after moving u to e_1, f has degree $s \le 2$ in the entries of the first row (Lemma 9.32(i)), and the coefficient of $t_1t_2t_3$ is a third derivative in these entries; likewise for the columns and (S2). If $n \le 2$: an alternating trilinear form on $\mathbb{C}^n$ is 0. □

For a side 3 there is a classical invariant. Let $(K_a)_{\alpha\beta} := \varepsilon_{a\alpha\beta}$ ($a,\alpha,\beta \in \{1,2,3\}$) and $G_K(P) := \det\bigl(\sum_a K_a \otimes P_a\bigr)$, the determinant of the $3n \times 3n$ matrix with blocks $\sum_a (K_a)_{\alpha\beta} P_a$. This is the determinant of the Koszul flattening of the tensor $P \in \mathbb{C}^3 \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, that is, Strassen's equation [Str83, LO13].

Lemma 9.35. *$G_K \in \mathrm{R}_3(n)_{(n,n,n)}$, $G_K(I,A,B) = (-1)^n \det[A,B]$, and $G_K \ne 0$ for $n \ge 2$.*

Proof. $\sum_a K_a \otimes (gP_ah^T) = (I_3 \otimes g)(\sum_a K_a \otimes P_a)(I_3 \otimes h^T)$. For $x \in \mathrm{GL}_3$ acting on the index a, the cross-product rule $xu \times xv = \det(x) x^{-T}(u \times v)$ gives $\sum_a x_{ab} K_a = \det(x)\, x^{-T} K_b x^{-1}$ (the exponent below is the same for the other convention), so the determinant changes by $\det(x)^{3n-n-n} = \det(x)^n$: weight (n,n,n). At $P_1 = I$ the Schur complement of the block $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ (determinant 1) is $-[A,B]$. For $A = \mathrm{diag}(1,2,\dots,n)$ and B the cyclic shift ($Be_i = e_{i+1}$), $[A,B]$ is B times an invertible diagonal matrix, so $\det[A,B] \ne 0$. □

Theorem 9.36. *G_K satisfies (S1) and (S2) for every n. So $\mathrm{K}_3(n)_{(n,n,n)} \ne 0$ for every $n \ge 2$, and, by Theorem 9.33, $\mathrm{K}_s(3)_{(s,s,s)} \ne 0$ for every $s \ge 2$. For $n = 3$, G_K is the type $(3,3)$ of $K_{(3,2,1)}$ (the example after Corollary 7.6): Strassen's invariant of degree 9.*

Proof. At $u = e_1$, by Cauchy–Binet, (S1) says $D_{1,J}G_K = 0$ for every set $J = \{j_1 < j_2 < j_3\}$ of columns, where $D_{1,J} := \det[\partial/\partial(P_a)_{1j_b}]_{a,b}$; this suffices, as G_K is left SL_n-invariant. The first rows of P_1, P_2, P_3 enter only the three rows $(\alpha, 1)$ of the block matrix, with entry $(e_\beta \times x_j)_\alpha$ in column (β, j), where $x_j := ((P_1)_{1j}, (P_2)_{1j}, (P_3)_{1j})$. In the Laplace expansion along these three rows the complementary minors do not involve the first rows, so $D_{1,J}G_K$ is a combination of the values $F(e_{\beta_1}, e_{\beta_2}, e_{\beta_3}; e_1, e_2, e_3)$ of

$$F(a,b,c;x,y,z) := \textstyle\sum_{\pi \in S_3} \mathrm{sgn}(\pi) \det[a \times x_{\pi(1)},\, b \times x_{\pi(2)},\, c \times x_{\pi(3)}], \qquad (x_1,x_2,x_3) := (x,y,z).$$

F is invariant under SL_3 acting on all six vectors, alternating in (x,y,z) and symmetric in (a,b,c). So $F = \det(x,y,z)\,\gamma(a,b,c)$ with γ an SL_3-invariant trilinear form, hence a multiple of det, which is alternating; as γ is symmetric, $\gamma = 0$. For (S2): $K_a^T = -K_a$ gives $G_K(P^T) = (-1)^{3n} G_K(P)$, and (S2) for G_K is (S1) for $P \mapsto G_K(P^T)$. The statement for $\mathrm{K}_s(3)$ follows by Theorem 9.33 applied to G_K for $s \times s$ matrices. For $n = 3$, $\tau = (3,3,3)$ is $\kappa = (2,2,2)' = (3,3)$, and by Lemma 9.32(iii) and that example the multiplicity of $(3,3)$ in $K_{(3,2,1)}$ is 1. □

Theorem 9.37. *Let $n \ge 3$, $c := \mathrm{tr}([A,B]^2)$ and $G := \langle A,A\rangle\langle B,B\rangle - \langle A,B\rangle^2$ with $\langle X,Y\rangle := \mathrm{tr}(X_0Y_0)$. For every $\kappa \in \mathbb{C}$ the kernel of $\kappa\mathcal{Z} - \mathcal{J}$ on the invariant functions of bidegree $(2,2)$ killed*

by $\mathcal{E}$, which are $\operatorname{span}\{c, G\}$*, is the line spanned by* $(n\kappa-2)\,c + 2n\,G$*. Hence, for every* $s \geq 4$*,* $\dim \mathrm{K}_s(n)_{(ns-4,2,2)} = 1$*; at* $s = n \geq 4$ *the line is the type* $(3, 1^{N-3})$ *of* $K_{B(n,2)}$*.*

Proof. Fix a base point (A_2, A_3) and $B_i = uy_i^T$, and put $a_i := y_i \cdot u$, $p_i := y_i^T A_2 u$, $q_i := y_i^T A_3 u$, $c_i := y_i^T C u$ with $C := [A_2, A_3]$, and $D := \det(a, p, q)$ (rows a, p, q). For row vectors x, v, w, $\sum_i (-1)^{i-1} x_i (v_j w_k - v_k w_j) = \det(x, v, w)$, with (j, k) the complement of i in increasing order. For c: $\beta(X, Y) = 2\operatorname{tr}([A_2, Y][X, A_3]) + 2\operatorname{tr}(C[X, Y])$, and at $X = B_j$, $Y = B_k$ one finds $\omega(B_j, B_k) = 6(a_j c_k - a_k c_j)$ at every base point; so $\mathcal{Z}(c) = 6\det(a, a, c) = 0$. Differentiating along $J(B_i)$, with $D_{J(B)}C = \frac{1}{2}(BC + CB) + A_2 B A_3 - A_3 B A_2$, so that $y_k^T (D_{J(B_i)} C) u = \frac{1}{2}(a_k c_i + c_k a_i) + p_k q_i - q_k p_i$, and alternating: $\mathcal{J}(c) = 6[\frac{1}{2}\det(a, a, c) + \det(q, a, p) - \det(p, a, q)] = 12D$. For G: $\omega(X, Y) = 6(\langle A_2, X\rangle\langle A_3, Y\rangle - \langle A_2, Y\rangle\langle A_3, X\rangle)$; with $p_j' := \langle A_2, B_j\rangle = p_j - \frac{\operatorname{tr} A_2}{n} a_j$ and q_j' likewise, $\mathcal{Z}(G) = 6\det(a, p', q') = 6D$, and, as $\langle B_i \circ A_2, B_j\rangle = \frac{1}{2}(p_i a_j + a_i p_j) - p_i a_j / n$, alternating gives $\mathcal{J}(G) = 12D/n$. So $(\kappa\mathcal{Z} - \mathcal{J})(xc + yG) = D\big(-12x + (6\kappa - 12/n)y\big)$, which vanishes identically if and only if $x : y = (n\kappa - 2) : 2n$, as D is not identically 0 for $n \geq 3$. The functions c and G are killed by $\mathcal{E}$ and are independent ($\mathcal{Z}(c) = 0 \neq \mathcal{Z}(G)$), and the space of these functions has dimension 2 (for $n \geq 4$, four independent trace monomials of bidegree $(2, 2)$ and two of bidegree $(3, 1)$, onto which $\mathcal{E}$ maps; for $n = 3$ by an exact count). For $s \geq 4 = d$ every such φ gives $f_s \in \mathrm{R}_s(n)_{(ns-4,2,2)}$ (Lemma 9.32(ii)), and $c^T = c$, $G^T = G$, so Corollary 9.31(b) gives the line for (S1) and for (S2). At $s = n \geq 4$ it is the type $(3, 1^{N-3})$ by Lemma 9.32(iii). □

At $s = n \geq 5$ this is the multiplicity 1 of Theorem 9.9(b), by another route, and it covers $n = 4$ as well.

Remark 9.38 (computations). The following are computer-assisted (exact computations modulo primes; where they are ranks, they bound dimensions from above as in Lemma 9.28), and are not used in any proof. (a) $\mathrm{K}_4(4)$ and $\mathrm{K}_5(5)$ (Remark 9.10, Theorem 9.27) and $\mathrm{K}_5(4) \cong \mathrm{K}_4(5)$ (one weight space $(7, 7, 6)$, which occurs in every type) have, in three-row weights, only the type $(ns-4, 2, 2)$. (b) Theorem 9.33 was found from type-by-type computations for $(n, s) = (3, 4)$, $(3, 6)$ and $(3, 5)$ (at $(5, 5)$) and their duals, and agrees with later ones for $(4, 5)$ and $(5, 4)$. (c) G_K passes (S1) and (S2) in direct computations from the $3n \times 3n$ determinant for $n = 3, \dots, 6$ (by a second program for $n = 3, 4, 5$). (d) For $n = 3, \dots, 7$ (all bidegrees with $d \leq 12$ for $n \leq 5$; 15 bidegrees with $d \leq 11$ at $n = 6$; 4 with $d \leq 8$ at $n = 7$), the rational values of s at which $s\mathcal{Z} - \mathcal{J}$ has a kernel on all functions killed by $\mathcal{E}$ were computed (one-sided: a true value of small height is not missed): no integer $s \geq 4$ occurs outside $(a, b) = (2, 2)$ for $n \geq 4$, and the integers $s \geq 4$ at $n = 3$ are the side-3 values of Theorem 9.36. (e) In the stable range, at $(a, b) = (4, 2)$, $(4, 4)$ and $(6, 2)$, the kernels at $s = 2$ and $s = -2$ are lines whose coefficients, fitted exactly as rational functions of n, are exchanged by $n \mapsto -n$ together with the sign $(-1)^{\text{number of trace factors}}$ of each trace monomial (86 of 86 coefficients); at $b = 2$ this is Theorem 9.75 with Corollary 9.76(ii), for every $a \geq 2$ and $n \geq a + 6$, and $\kappa = -2$ occurs for every even $a \geq 4$ and $n \geq a + 6$ (Corollary 9.77).

9.12. **The first layer.** In this subsection $F = \mathbb{C}$ and $n \geq 3$. We show that, when $b = 1$, the pencil of Corollary 9.31 has no eigenvalue other than $(d-2)/n$, for every n and every a. Let $\mathcal{F}_{(a,b)}(n)$ be the space of the functions φ of Subsection 9.11 (polynomials on $\mathfrak{gl}_n \times \mathfrak{gl}_n$, invariant under simultaneous conjugation, of bidegree (a, b)), and $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n) := \ker \mathcal{E}$, as in Subsection 9.10 with n in place of 5. On $\mathbb{C}^n$ we use the vector $\mathbf{1} := (1, \dots, 1)$, the 1-form $\eta := \sum_l dx_l$, the Euler field $\xi := \sum_l x_l\, \partial/\partial x_l$ and the interior product ι_v with a vector field v; for a 2-form $\Theta = \frac{1}{2}\sum_{m,l} \Theta_{ml}\, dx_m \wedge dx_l$ ($\Theta_{lm} = -\Theta_{ml}$) we write $\Theta(v, w) = \sum_{m,l} \Theta_{ml} v_m w_l$, $\partial_j \Theta$ for the form whose coefficients are the $\partial\Theta_{ml}/\partial x_j$, and $(\eta \wedge \Theta)(v_1, v_2, v_3) = \eta(v_1)\Theta(v_2, v_3) - \eta(v_2)\Theta(v_1, v_3) + \eta(v_3)\Theta(v_1, v_2)$.

Lemma 9.39. *Let* $\varphi \in \mathcal{F}_{(a,1)}(n)$*. Then* $\tilde{\varphi}(X, Y) = \operatorname{tr}(h(X)Y)$ *for a unique polynomial map* $h\colon M_n \to \mathfrak{gl}_n$ *that is equivariant under conjugation, homogeneous of degree* a*, and satisfies* $h(X +$

$cI) = h(X)$. *For* $x \in \mathbb{C}^n$, $h(\mathrm{diag}(x)) = \mathrm{diag}(h_1(x), \dots, h_n(x))$ *with polynomials* h_l, *homogeneous of degree* a, *such that* $\sum_l h_l = 0$ *and* $h_l(x + c\mathbf{1}) = h_l(x)$. *Put* $\theta := \sum_l h_l\, dx_l$ *and* $\Theta := d\theta$. *Then* $\mathcal{E}\varphi = 0$ *if and only if* $\sum_l x_l h_l = 0$, *that is,* $\iota_\xi \theta = 0$.

Proof. $h(X)$ is the traceless matrix with $\mathrm{tr}(h(X)Y) = \tilde{\varphi}(X, Y)$ for all Y; it is traceless because $\tilde{\varphi}$ sees only Y_0, and equivariance, homogeneity and the invariance under $X \mapsto X + cI$ come from those of $\tilde{\varphi}$. Conjugation by an invertible diagonal matrix fixes $X = \mathrm{diag}(x)$, hence fixes $h(X)$, so $h(X)$ is diagonal; its entries h_l are polynomials in x with the stated properties. As φ is linear in Y, $\mathcal{E}\varphi = 0$ means $\varphi(X, X) = 0$, that is, $\tilde{\varphi}(X, X) = \mathrm{tr}(h(X)X) = 0$ for all X; by equivariance and density of the diagonalizable matrices this is $\sum_l x_l h_l(x) = 0$ for all x. □

Proposition 9.40 (the test in eigenvalue coordinates)**.** *Let* $\varphi \in \mathcal{F}_{(a,1)}(n)$, *take the base point* $(A_2, A_3) = (X, Y)$ *with* $X = \mathrm{diag}(x)$, $x_1, \dots, x_n$ *distinct, and* Y *arbitrary, and let* $B_k = u y_k^T$ *and* $v_k := (u_l y_{kl})_l \in \mathbb{C}^n$ $(k = 1, 2, 3)$. *Then, with the forms evaluated at* x,

$$\mathcal{Z}(\varphi)(B_1, B_2, B_3) = (\eta \wedge \Theta)(v_1, v_2, v_3), \qquad \mathcal{J}(\varphi)(B_1, B_2, B_3) = (d^x\Theta)(v_1, v_2, v_3),$$

where $d^x\Theta := \sum_j x_j\, dx_j \wedge \partial_j \Theta$. *If every* $u_l \neq 0$, *the triple* (v_1, v_2, v_3) *runs over all of* $(\mathbb{C}^n)^3$ *as the* y_k *vary.*

Proof. Step 1. Write $U = U_d + [W, X]$ with U_d the diagonal part of U and $W_{ij} = U_{ij}/(x_j - x_i)$ for $i \neq j$, $W_{ii} = 0$. Equivariance gives $Dh[[W, X]] = [W, h(X)]$, and $Dh[U_d] = \mathrm{diag}\big(\sum_m U_{mm}\partial_m h_l\big)_l$ by Lemma 9.39. So

$$\beta(U, V) = \mathrm{tr}(Dh[U]\, V) = \sum_{m,l} (\partial_m h_l)\, U_{mm} V_{ll} + \sum_{i \neq j} h[i, j]\, U_{ij} V_{ji},$$

with $h[i, j] := (h_j - h_i)/(x_j - x_i) = h[j, i]$. These are the divided differences of the Daleckiĭ–Kreĭn formula, which appear in the Hessians of spectral functions for $h = \nabla f$ [LS01]; here h need not be a gradient, and $\Theta \neq 0$ exactly when it is not. The second sum is symmetric in (U, V), so $\omega(U, V) = \Theta(\mathrm{diag}\, U, \mathrm{diag}\, V)$. As $\mathrm{diag}\, B_k = v_k$ and $\mathrm{tr}\, B_k = \eta(v_k)$, this gives $\mathcal{Z}(\varphi) = (\eta \wedge \Theta)(v_1, v_2, v_3)$.

Step 2. ω does not depend on Y, so in $J(B) = (B \circ X, B \circ Y)$ only $B \circ X$ acts. For a direction $\zeta = \zeta_d + [W', X]$ at X (W' built from ζ as W from U in Step 1), the part ζ_d moves X along diagonal matrices with distinct entries, where Step 1 applies, and the conjugation invariance of ω handles $[W', X]$:

$$D_\zeta \omega(U, V) = (\partial_{\mathrm{diag}\,\zeta}\Theta)(\mathrm{diag}\, U, \mathrm{diag}\, V) + \Theta(q(\zeta, U), \mathrm{diag}\, V) + \Theta(\mathrm{diag}\, U, q(\zeta, V)),$$

$$q_l(\zeta, U) := -[W', U]_{ll} = \sum_{j \neq l} \frac{\zeta_{lj} U_{jl} + U_{lj} \zeta_{jl}}{x_l - x_j},$$

with $\partial_w \Theta := \sum_j w_j \partial_j \Theta$.

Step 3. For $\zeta = B_k \circ X$ and $U = B_i$ we have $(B_k \circ X)_{lj} = \frac{1}{2}(x_l + x_j) u_l y_{kj}$, so

$$q_l(B_k \circ X, B_i) = \sum_{j \neq l} \tfrac{1}{2}(x_l + x_j)\, u_l u_j\, \frac{y_{kj} y_{il} + y_{ij} y_{kl}}{x_l - x_j},$$

which is symmetric in (k, i). Put $g(k, i, j) := \Theta(q(B_k \circ X, B_i), v_j)$. With (i, j) the other two indices in increasing order, the q-part of $\mathcal{J}(\varphi)$ is

$$\sum_k (-1)^{k-1} \big[g(k, i, j) - g(k, j, i)\big] = \sum_{\pi \in S_3} \mathrm{sgn}(\pi)\, g(\pi(1), \pi(2), \pi(3)),$$

and this is 0, as g is symmetric in its first two arguments.

Step 4. $\mathrm{diag}(B_k \circ X) = (x_l v_{kl})_l$, so the rest of $\mathcal{J}(\varphi)$ is $\sum_k (-1)^{k-1} (\partial_{x \cdot v_k}\Theta)(\hat{v}_k) = \sum_j x_j\, (dx_j \wedge \partial_j \Theta)(v_1, v_2, v_3)$, where $x \cdot v := (x_l v_l)_l$ and $\hat{v}_k$ is the pair of the other two vectors in their order. The last sentence of the proposition holds because $v_{kl} = u_l y_{kl}$. □

Theorem 9.41 (the first layer)**.** *Let $n \geq 3$, $a \geq 1$ and $\kappa \in \mathbb{C}$ with $\kappa \neq (a-1)/n$. If $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,1)}(n)$ satisfies $\kappa\, \mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ at every base point and for all u, y_1, y_2, y_3, then $\varphi = 0$. That is, on $\mathcal{F}^{\mathcal{E}}_{(a,1)}(n)$ the pencil $\kappa\mathcal{Z} - \mathcal{J}$ has no eigenvalue other than $\kappa = (d-2)/n$.*

Proof. Let x have distinct coordinates. By Proposition 9.40, with every $u_l \neq 0$, $\kappa\,(\eta \wedge \Theta) = d^x\Theta$ as 3-forms at x. We contract with $\mathbf{1}$. First, $\iota_{\mathbf{1}}\Theta = 0$: by Cartan's formula $\iota_{\mathbf{1}} d\theta = L_{\mathbf{1}}\theta - d\,\iota_{\mathbf{1}}\theta$, where $L_{\mathbf{1}}\theta = \sum_l (\partial_{\mathbf{1}} h_l)\, dx_l = 0$ (the h_l are invariant under $x \mapsto x + c\mathbf{1}$) and $\iota_{\mathbf{1}}\theta = \sum_l h_l = 0$. Hence $\iota_{\mathbf{1}}(\eta \wedge \Theta) = \eta(\mathbf{1})\Theta - \eta \wedge \iota_{\mathbf{1}}\Theta = n\Theta$. Second, as $\mathbf{1}$ is constant, $\iota_{\mathbf{1}}$ commutes with ∂_j, so $\iota_{\mathbf{1}}(d^x\Theta) = \sum_j x_j \big[(\iota_{\mathbf{1}} dx_j)\, \partial_j\Theta - dx_j \wedge \partial_j(\iota_{\mathbf{1}}\Theta)\big] = \sum_j x_j\, \partial_j\Theta = (a-1)\Theta$, since the coefficients of Θ are homogeneous of degree $a-1$. So $(n\kappa - (a-1))\Theta = 0$, and $\Theta = 0$ at x, hence everywhere. Now h_l is homogeneous of degree a, so $L_\xi\theta = (a+1)\theta$, while Cartan's formula and Lemma 9.39 give $L_\xi\theta = d\,\iota_\xi\theta + \iota_\xi\Theta = 0$. So $\theta = 0$: h vanishes on the diagonal matrices, hence, by equivariance, on the dense set of diagonalizable matrices, so $h = 0$ and $\varphi = 0$. Finally $d = a+1$, so $(a-1)/n = (d-2)/n$. □

Corollary 9.42. *Let $n \geq 3$.*

(a) For every integer $s \geq 1$ and every $\tau \vdash ns$ with $\tau_3 = 1$, $\mathrm{K}_s(n)_\tau = 0$.

(b) $K_{B(n,2)}$ ($N = n^2 - 3$) has no conjugation type κ with $\kappa' = (N - \sigma, \sigma)$, $\sigma \geq 0$, that is, no type with at most two columns; in the notation of Subsection 9.6, $r_{(\sigma)}(n) = 0$ for every σ. So, by Corollary 7.6(a), every conjugation type of $K_{B(n,2)}$ has exactly three columns.

Proof. (a) Let $f \in \mathrm{K}_s(n)_\tau$ and $a := \tau_2$. By Lemma 9.32(ii), $\varphi_f \in \mathcal{F}^{\mathcal{E}}_{(a,1)}(n)$ and f is the function f_s built from φ_f; by Corollary 9.31(b), $s\,\mathcal{Z}(\varphi_f) = \mathcal{J}(\varphi_f)$. If $s = (a-1)/n$, then $\tau_1 = ns - a - 1 = -2$, which is impossible. So $\varphi_f = 0$ by Theorem 9.41, and $f = 0$ by the injectivity in Lemma 9.32(ii). (b) For $\kappa' = (N - \sigma, \sigma)$ put $\tau := \kappa' + (1,1,1) = (N - \sigma + 1, \sigma + 1, 1) \vdash n^2$. By Lemma 9.32(iii) the multiplicity of κ in $K_{B(n,2)}$ is $\dim \mathrm{K}_n(n)_\tau$, which is 0 by (a) with $s = n$. □

Remark 9.43. (1) Theorem 9.24 gives (b) for $\sigma \leq n-1$ ($n \geq 4$); the new range is $\sigma \geq n$ (for $n = 3$, (b) also follows from the example after Corollary 7.6). The proof of Theorem 9.41 works with the exact test at the given n, for every s, with no expansion in $1/n$ and no condition between a and n, so the trace identities of $n \times n$ matrices never enter. For $n = s = 6$, (a) says that the whole column $\tau = (35 - a, a, 1)$, $1 \leq a \leq 17$, of $\mathrm{K}_6(6)$ is 0; Theorem 9.24 with Lemma 9.32(iii) covers $a \leq 6$.

(2) The spaces are not small. By Kostant's theorem [Ko63] the conjugation-equivariant polynomial maps $\mathfrak{sl}_n \to \mathfrak{sl}_n$ form a free module over $\mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} = \mathbb{C}[P_2, \ldots, P_n]$, $P_m := \mathrm{tr}(X^m)/m$, and a basis is given by the gradients $\nabla P_{k+1} = (X^k)_0$, $1 \leq k \leq n-1$ (Vust [Vu74, Ch. III, §2], Raïs [Rai88]; see [PY12, §2]). So $\varphi = \sum_{k=1}^{n-1} c_k\, \mathrm{tr}(X^k Y)$ with unique $c_k \in \mathbb{C}[P_2, \ldots, P_n]$ of weighted degree $a - k$, and $\mathcal{E}\varphi = 0$ if and only if $\sum_k c_k\, \mathrm{tr}(X^{k+1}) = 0$. The $\mathrm{tr}(X^{k+1})$ generate the ideal of positive weighted degree, so $(c_k) \mapsto \sum_k c_k\, \mathrm{tr}(X^{k+1})$ is onto $\mathbb{C}[P]_{a+1}$, and, with $\mathbb{C}[P]_j$ the part of weighted degree j,

$$\dim \mathcal{F}^{\mathcal{E}}_{(a,1)}(n) = \sum_{k=1}^{n-1} \dim \mathbb{C}[P]_{a-k} - \dim \mathbb{C}[P]_{a+1}.$$

For $n = 6$ this is 0 for $a \leq 3$ and $1, 1, 3, 4, 6, 8, 12, 14, 20, 25, 31, 38, 48, 56$ for $a = 4, \ldots, 17$.

(3) The contraction with $\mathbf{1}$ is the contraction with a corner $B_1 = uy^T$ for which $\mathrm{diag}(B_1)$ is a multiple of $\mathbf{1}$, that is, $B_1^0 \in \mathrm{Im}\,\mathrm{ad}_X$ at diagonal X with distinct entries. The following coordinate-free statement shows why the $\mathfrak{L}_0$-part of the test vanishes there.

Proposition I. Let $b = 1$ (φ need not be killed by $\mathcal{E}$), let $(A_2, A_3) = (X, Y)$ be any base point, $W \in M_n$, and $B_1 \in M_n$ with $B_1^0 = [W, X]$. Then $(\mathfrak{L}_0\omega)(B_1, B_2, B_3) = 0$ for all $B_2, B_3 \in M_n$.

Proof. (i) ω depends only on X, and only on the traceless parts of its arguments. (ii) For all X, W and V, $\omega_X([W,X], V) = 0$: conjugation invariance gives $\beta([W,X], V) = -\tilde{\varphi}(X, [W,V])$; and the function $X' \mapsto \tilde{\varphi}(X', [W, X'])$, which is $\mathrm{tr}([X', h(X')]W)$, is identically 0, so its derivative

along V gives $\beta(V,[W,X]) = -\tilde\varphi(X,[W,V])$. (iii) $D_{[W',X]}\omega(U,V) = -\omega([W',U],V) - \omega(U,[W',V])$ for every W' (conjugation invariance; by (i) the Y-part of the direction does not matter). (iv) $[W,X]\circ X = \frac12[W,X^2] = [W\circ X, X]$. By (iii) and (iv) the first term of $\mathfrak{L}_0\omega(B_1,B_2,B_3)$ is $-\omega([W\circ X,B_2],B_3) - \omega(B_2,[W\circ X,B_3])$. In the second, B_1 is held fixed while the base point moves to X'; by (i) and by (ii) at X', $\omega_{X'}(B_1,B_3) = \omega_{X'}([W,X-X'],B_3)$, whose derivative at $X'=X$ along V is $-\omega([W,V],B_3)$; with $V = B_2^0\circ X$ and (ii) again, $-D_{J(B_2^0)}\omega(B_1,B_3) = \omega([W,B_2\circ X],B_3)$. Likewise $D_{J(B_3^0)}\omega(B_1,B_2) = -\omega([W,B_3\circ X],B_2)$. The sum is $\omega(Q(B_2),B_3) - \omega(Q(B_3),B_2)$ with $Q(B) := [W,B\circ X] - [W\circ X,B]$, and $2Q(B) = [W,BX+XB] - [WX+XW,B] = [WB+BW,X]$, so $Q(B) = [W\circ B, X]$ and both terms vanish by (ii). □

With Corollary 9.31(a) this gives a second proof of Theorem 9.41: take $X = \mathrm{diag}(x)$ with distinct x_l, u with every $u_l \neq 0$ (so u is cyclic for X), and y_* with $y_*^T X^j u = \mathrm{tr}(X^j)/n$ $(0\le j\le n-1)$; then $\mathrm{tr}\,B_1 = 1$ and B_1^0 is orthogonal to the centralizer of X, so $B_1^0 \in \mathrm{Im}\,\mathrm{ad}_X$. By Corollary 9.31(a) and Proposition I, the test at (B_1,B_2,B_3) is then $\lambda\,(\mathrm{tr}\wedge\omega)(B_1,B_2,B_3) = \lambda\,\omega(B_2,B_3)$ by (ii), with $\lambda = \kappa - (d-2)/n$, so $\lambda\neq 0$ forces $\omega = 0$ on all corners, that is, $\Theta = 0$ by Step 1 of the proof of Proposition 9.40, and $\varphi = 0$ as in the proof of Theorem 9.41.

(4) We do not claim that $\kappa = (d-2)/n$ is an eigenvalue for every $(a,1)$. For $n=3$ its eigenspace is all of $\mathcal{F}^{\mathcal{E}}_{(a,1)}(3)$: at $\lambda = 0$ the test is $-(d^x\Theta - \frac{a-1}{n}\,\eta\wedge\Theta)$ (at diagonal X with distinct entries; elsewhere by conjugation and density), this 3-form is killed by $\iota_{\mathbf 1}$, and $\iota_{\mathbf 1}$ is injective on 3-forms on $\mathbb{C}^3$.

(5) (Computations, not used.) Proposition 9.40 agrees with the definition of Ωf_s in 12 cases ($(a,n) = (4,3)$, $(4,4)$, $(5,4)$, four values of s each); Proposition I in its λ-form holds in 16 cases ($n = 3,4,5$); and in an independent closed form of the whole test the full pencil, solved exactly for $n = 3,4,5$ and $a = 4,\dots,9$, and the pencil tool of Remark 9.38, run also on the rows with $b=1$ for $n=6$, $a\le 13$ and $n=7$, $a\le 12$ (42 nonzero rows with $b=1$ for $n=3,\dots,7$ in all), show only the eigenvalue $(d-2)/n$, with $\mathcal{Z}$ injective.

9.13. **The second layer.** In this subsection $F=\mathbb{C}$, $n\ge 3$ and $b=2$, and we keep the notation of Subsection 9.12. We write the test at diagonal base points in closed form (Propositions 9.44 and 9.45, the analogues for $b=2$ of Proposition 9.40), express the diagonal data of a kernel vector through its off-diagonal data (Lemma 9.46), and show that every eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(3,2)}(n)$ is $3/n$ or $6/n$, for every $n\ge 4$ (Theorem 9.48). We write $\mu := n\kappa - a$; so $\mu = 0$ is $\kappa = (d-2)/n$, the only possible eigenvalue when $b=1$ (Theorem 9.41).

The torus form. Let $\varphi\in\mathcal{F}_{(a,2)}(n)$. Then $\tilde\varphi(X,Y) = Q_X(Y,Y)$, where Q_X is a symmetric bilinear form on M_n that is polynomial and homogeneous of degree a in X, with $Q_{gXg^{-1}}(gVg^{-1},gV'g^{-1}) = Q_X(V,V')$, $Q_{X+cI} = Q_X$ and $Q_X(I,\cdot) = 0$ (as $\tilde\varphi$ sees only X_0 and Y_0). Let $X = \mathrm{diag}(x)$ with $x_1,\dots,x_n$ distinct, and $x_{ij} := x_i - x_j$. Conjugation by invertible diagonal matrices fixes X, so only the products $V_{rt}V'_{tr}$ $(r\neq t)$ and $V_{rr}V'_{tt}$ occur in Q_X:

$$Q_X(V,V') = \sum_{r\neq t} \mathsf{F}_{rt}V_{rt}V'_{tr} + \sum_{r,t}\mathsf{G}_{rt}V_{rr}V'_{tt}, \tag{1}$$

with $\mathsf{F}_{rt} = \mathsf{F}_{tr}$ and $\mathsf{G}_{rt} = \mathsf{G}_{tr}$ polynomials in x, homogeneous of degree a and invariant under $x\mapsto x+c\mathbf 1$. $Q_X(I,\cdot) = 0$ gives $\mathsf{G}\mathbf 1 = 0$. Since $(\mathcal{E}\varphi)(A,B) = \frac{d}{dt}\varphi(A,B+tA)|_{t=0}$ gives $\widetilde{\mathcal{E}\varphi}(X,Y) = 2Q_X(X,Y)$, and $Q_X(X,V) = \sum_t(\mathsf{G}x)_tV_{tt}$ at $X=\mathrm{diag}(x)$, we have $\mathcal{E}\varphi = 0$ if and only if $\mathsf{G}x = 0$ for all x (by equivariance and density, as in Lemma 9.39). For $U\in M_n$ write $U = U_d + [W_U,X]$ as in Step 1 of the proof of Proposition 9.40 (U_d diagonal, $(W_U)_{rt} = U_{rt}/x_{tr}$ for $r\neq t$). Differentiating the equivariance at $g = \exp(\epsilon W_U)$ gives, for the derivative D_U of $X\mapsto Q_X$ along U,

$$(D_UQ)(V,V') = (\partial_{\mathrm{diag}\,U}Q)(V,V') - Q([W_U,V],V') - Q(V,[W_U,V']), \tag{2}$$

where $\partial_w := \sum_m w_m\partial_m$ acts on the coefficients F_{rt}, G_{rt} in (1). (The torus form and (2) are standard; compare the Daleckiĭ–Kreĭn formula in Step 1 of the proof of Proposition 9.40 and [LS01].) Here

$\beta(U,V) = 2(D_U Q)(Y,V)$, so the test is linear in Y, and it suffices to take $Y = E_{pq}$ $(p \neq q)$ and Y diagonal. We take the corners with $u = \mathbf{1}$, $B_k = \mathbf{1}v_k^T$ $(v_k \in \mathbb{C}^n)$, so that $\operatorname{diag} B_k = v_k$ and $\operatorname{tr} B_k = \eta(v_k)$; we use only these corners, which is enough, as the test is used here only as a necessary condition.

Notation. For $r \neq s$ put $\mathsf{H}_{rs} := x_{rs}\mathsf{F}_{rs}$ (so $\mathsf{H}_{sr} = -\mathsf{H}_{rs}$; $\mathsf{H}_{rr} := 0$) and $\mathsf{H}_{rwt} := \mathsf{H}_{rw} + \mathsf{H}_{wt} + \mathsf{H}_{tr}$. For $p \neq q$ let $\psi^{(pq)} := \sum_{w \neq p} \psi_w^{(pq)}\, dx_w$, where

$$\psi_w^{(pq)} := \partial_w \mathsf{F}_{pq} + \frac{\mathsf{G}_{qw} - \mathsf{G}_{pw}}{x_{pq}} + \mathsf{c}_w^{(pq)} \quad (w \notin \{p,q\}), \qquad \psi_q^{(pq)} := \partial_q \mathsf{F}_{pq} + \frac{\mathsf{G}_{qq} - \mathsf{G}_{pq} - \mathsf{F}_{pq}}{x_{pq}},$$
$$\mathsf{c}_w^{(pq)} := \frac{\mathsf{F}_{qw} - \mathsf{F}_{pq}}{x_{pw}} + \frac{\mathsf{F}_{pw} - \mathsf{F}_{pq}}{x_{wq}} = \frac{\mathsf{H}_{pwq}}{x_{pw}x_{wq}},$$

and $\mathsf{c}^{(pq)} := \sum_{w \notin \{p,q\}} \mathsf{c}_w^{(pq)}\, dx_w$. For $1 \le j \le n$ let $\gamma_j := \sum_t \mathsf{G}_{jt}\, dx_t$ and $\Theta_j := d\gamma_j$, and for a constant $y \in \mathbb{C}^n$ let $\Theta_y := \sum_j y_j \Theta_j$. As in Subsection 9.12, ∂_m acts on forms coefficientwise and $d^x := \sum_m x_m\, dx_m \wedge \partial_m$. Finally $\mathsf{a}_r := \frac{1}{2}\sum_{m \neq r} \frac{x_r + x_m}{x_{mr}}\, dx_m$, a 1-form.

Proposition 9.44 (the test at $Y = E_{pq}$)**.** *Let $\varphi \in \mathcal{F}_{(a,2)}(n)$, $X = \operatorname{diag}(x)$ with distinct x_l, $Y = E_{pq}$ with $p \neq q$, and $B_k = \mathbf{1}v_k^T$ $(k = 1,2,3)$. Then*

$$(\kappa\mathcal{Z} - \mathcal{J})(\varphi)(B_1, B_2, B_3) = 2\,(\Psi_{pq} \wedge dx_p)(v_1, v_2, v_3),$$

where, with $\psi := \psi^{(pq)}$ and $\delta_m\psi := (\psi^{(pq)} - \psi^{(pm)})/x_{mq} - (\psi^{(pq)} - \psi^{(mq)})/x_{mp}$,

$$\Psi_{pq} := \kappa\,\eta \wedge \psi - x_q\, dx_q \wedge \Big(\partial_q\psi - \frac{\psi}{x_{pq}}\Big) - \sum_{m \notin \{p,q\}} x_m\, dx_m \wedge (\partial_m\psi - \delta_m\psi) - \frac{x_p\Theta_p - x_q\Theta_q}{x_{pq}}.$$

The same holds with every explicit coefficient x_l in Ψ_{pq} (not the arguments of F, G, ψ, Θ) replaced by $x_l + c$, for any $c \in \mathbb{C}$. (The dx_p-components of the 1-forms $\psi^{(\cdot,\cdot)}$ never matter, as every term is wedged with dx_p.)

Proof. Step 1 (ω at $Y = E_{pq}$ and at $Y = E_{jj}$). We claim that for all $U, V \in M_n$, with $u := \operatorname{diag} U$ and $v := \operatorname{diag} V$,

$$\tfrac{1}{2}\,\omega_{E_{pq}}(U,V) = V_{qp}\,\alpha(u) - U_{qp}\,\alpha(v) + \sum_{r \notin \{p,q\}} \mathsf{c}_r^{(pq)}\,(U_{qr}V_{rp} - V_{qr}U_{rp}),$$
$$\alpha(w) := \partial_w \mathsf{F}_{pq} + \big[(\mathsf{G}w)_q - (\mathsf{G}w)_p + \mathsf{F}_{pq}(w_p - w_q)\big]/x_{pq},$$

and $\omega_{E_{jj}}(U,V) = 2\Theta_j(u,v)$, where ω_Y is ω at the base point (X,Y). By (1), $Q(E_{pq},V) = \mathsf{F}_{pq}V_{qp}$, $Q(E_{rq},V) = \mathsf{F}_{rq}V_{qr}$ $(r \neq q)$, $Q(E_{pt},V) = \mathsf{F}_{pt}V_{tp}$ $(t \neq p)$ and $Q(E_{jj},V) = (\mathsf{G}v)_j$. With $W := W_U$ we have $[W,E_{pq}] = \sum_r W_{rp}E_{rq} - \sum_t W_{qt}E_{pt}$, so

$$Q([W,E_{pq}],V) = \sum_{r \notin \{p,q\}} \frac{U_{rp}\mathsf{F}_{rq}V_{qr}}{x_{pr}} - \sum_{t \notin \{p,q\}} \frac{U_{qt}\mathsf{F}_{pt}V_{tp}}{x_{tq}} + \frac{U_{qp}\big((\mathsf{G}v)_q - (\mathsf{G}v)_p\big)}{x_{pq}},$$
$$Q(E_{pq},[W,V]) = \mathsf{F}_{pq}[W,V]_{qp} = \mathsf{F}_{pq}\Big(\sum_{m \neq q} \frac{U_{qm}V_{mp}}{x_{mq}} - \sum_{m \neq p} \frac{V_{qm}U_{mp}}{x_{pm}}\Big),$$

and $(\partial_u Q)(E_{pq},V) = (\partial_u \mathsf{F}_{pq})V_{qp}$. Insert these in (2) and antisymmetrize in (U,V): the F_{rq}- and F_{pt}-terms give the parts F_{qr}/x_{pr} and F_{pr}/x_{rq} of $\mathsf{c}_r^{(pq)}(U_{qr}V_{rp} - V_{qr}U_{rp})$; the F_{pq}-terms with $m \notin \{p,q\}$ give the rest, $-\mathsf{F}_{pq}(1/x_{mq} + 1/x_{pm})$; and the terms $m = p$ (first sum) and $m = q$ (second sum) give $\mathsf{F}_{pq}[V_{qp}(u_p - u_q) - U_{qp}(v_p - v_q)]/x_{pq}$. At $Y = E_{jj}$: $[W,E_{jj}]$ is off-diagonal, and

$$Q([W,E_{jj}],V) = \sum_{r \neq j} \frac{\mathsf{F}_{rj}(U_{rj}V_{jr} + U_{jr}V_{rj})}{x_{jr}}, \qquad Q(E_{jj},[W,V]) = \sum_{t,\, m \neq t} \frac{\mathsf{G}_{jt}(U_{tm}V_{mt} + V_{tm}U_{mt})}{x_{mt}}$$

are symmetric in (U,V); so they cancel in ω, and what is left is $2\sum_{m,t}\partial_m \mathsf{G}_{jt}\,(u_m v_t - v_m u_t) = 2\Theta_j(u,v)$.

On corners, $U=\mathbf{1}v^T$ and $V=\mathbf{1}v'^T$ give $U_{rt}=v_t$, $V_{rt}=v'_t$, and Step 1 becomes $\omega_{E_{pq}}(\mathbf{1}v^T,\mathbf{1}v'^T) = 2(\psi^{(pq)}\wedge dx_p)(v,v')$ (the dx_p-component of the 1-form $\alpha+\mathsf{c}^{(pq)}$ cancels). So $\operatorname{tr}\wedge\omega = 2\,\eta\wedge\psi\wedge dx_p$.

Step 2 (the motion). Let $B=\mathbf{1}w^T$. Then $(B\circ X)_{rt}=w_t(x_r+x_t)/2$, so $B\circ X=\operatorname{diag}(x\cdot w)+[W'_w,X]$ with $(W'_w)_{rt}=w_t(x_r+x_t)/(2x_{tr})$ for $r\neq t$. The conjugation invariance of ω, in the base point and the arguments jointly, and the linearity of ω in Y give, with $Y'(w):=B\circ Y-[W'_w,Y]$,

$$D_{J(B)}\omega(U,V)=(\partial_{x\cdot w}\omega)(U,V)+\omega_{Y'(w)}(U,V)-\omega([W'_w,U],V)-\omega(U,[W'_w,V]),$$

where $\partial_{x\cdot w}$ moves X along diagonal matrices. At $Y=E_{pq}$ a direct computation of the entries gives

$$Y'(w)=\sum_{r\notin\{p,q\}} w_p\frac{x_r}{x_{rp}}E_{rq}+\sum_{t\notin\{p,q\}} w_t\frac{x_t}{x_{tq}}E_{pt}+\frac{w_p+w_q}{2}E_{pq}+\frac{w_p}{x_{pq}}(x_pE_{pp}-x_qE_{qq}).$$

Step 3 (the conjugation terms). Put $\mathsf{a}(w):=(\mathsf{a}_r(w))_r$. Then $W'_w\mathbf{1}=\mathsf{a}(w)$ and $v^TW'_w=-(w\cdot\mathsf{a}(v))^T$ with $(w\cdot\mathsf{a}(v))_t:=w_t\mathsf{a}_t(v)$, so $[W'_w,\mathbf{1}v^T]=\mathsf{a}(w)v^T+\mathbf{1}(w\cdot\mathsf{a}(v))^T$. By Step 1 for general U, V (the dx_p-components again cancel),

$$\begin{aligned}
&\tfrac12\big[\omega(\mathsf{a}(w)v^T,\mathbf{1}v'^T)+\omega(\mathbf{1}v^T,\mathsf{a}(w)v'^T)\big]\\
&\qquad=v'_p\big[\psi(\mathsf{a}(w)\cdot v)+\mathsf{a}_q(w)\psi(v)\big]-v_p\big[\psi(\mathsf{a}(w)\cdot v')+\mathsf{a}_q(w)\psi(v')\big],\\
&\tfrac12\big[\omega(\mathbf{1}(w\cdot\mathsf{a}(v))^T,\mathbf{1}v'^T)+\omega(\mathbf{1}v^T,\mathbf{1}(w\cdot\mathsf{a}(v'))^T)\big]\\
&\qquad=v'_p\,\psi(w\cdot\mathsf{a}(v))-v_p\,\psi(w\cdot\mathsf{a}(v'))-w_p\big[\mathsf{a}_p(v)\psi(v')-\mathsf{a}_p(v')\psi(v)\big].
\end{aligned}$$

Here $\psi(\mathsf{a}(w)\cdot v)+\psi(w\cdot\mathsf{a}(v))=\sum_m\psi_m S_m(w,v)$ with $S_m(w,v):=\mathsf{a}_m(w)v_m+w_m\mathsf{a}_m(v)$, which is symmetric in (w,v). With (i,j) the other two indices in increasing order, put $g(k,i,j):=(v_j)_p\sum_m\psi_m S_m(v_k,v_i)$; the corresponding part of $\sum_k(-1)^{k-1}\big[\omega([W'_{v_k},B_i],B_j)+\omega(B_i,[W'_{v_k},B_j])\big]$ is twice the full alternation of g over S_3, which is 0 as g is symmetric in its first two arguments (as in Step 3 of the proof of Proposition 9.40). What is left alternates to $2(\mathsf{a}_q\wedge\psi\wedge dx_p-dx_p\wedge\mathsf{a}_p\wedge\psi)=2(\mathsf{a}_q-\mathsf{a}_p)\wedge\psi\wedge dx_p$.

Step 4 (assembly). By Lemma 9.30 with $s=\kappa$, $(\kappa\mathcal{Z}-\mathcal{J})(\varphi)=\kappa(\operatorname{tr}\wedge\omega)-\mathfrak{L}\omega$, and by Step 2, $\mathfrak{L}\omega=\text{(I)}+\text{(II)}-\text{(III)}$ with: $\text{(I)}=\sum_k(-1)^{k-1}(\partial_{x\cdot v_k}\omega)(\hat v_k)=2\sum_m x_m\,dx_m\wedge\partial_m\psi\wedge dx_p$, since the corner formula of Step 1 holds at every diagonal X with distinct entries and $x\cdot w$ moves X along such matrices; $\text{(II)}=\sum_k(-1)^{k-1}\omega_{Y'(v_k)}(\hat v_k)=\sum_E l_E\wedge\omega_E$, where $Y'(w)=\sum_E l_E(w)E$; with $\omega_{E_{rq}}=2\psi^{(rq)}\wedge dx_r$, $\omega_{E_{pt}}=2\psi^{(pt)}\wedge dx_p$ (Step 1 for these pairs) and $\omega_{E_{jj}}=2\Theta_j$ on corners,

$$\begin{aligned}
\tfrac12\text{(II)}=&\sum_{r\notin\{p,q\}}\frac{x_r}{x_{rp}}\,dx_p\wedge\psi^{(rq)}\wedge dx_r+\sum_{t\notin\{p,q\}}\frac{x_t}{x_{tq}}\,dx_t\wedge\psi^{(pt)}\wedge dx_p\\
&+\frac12\,dx_q\wedge\psi\wedge dx_p+\frac{1}{x_{pq}}\,dx_p\wedge(x_p\Theta_p-x_q\Theta_q);
\end{aligned}$$

$\text{(III)}=2(\mathsf{a}_q-\mathsf{a}_p)\wedge\psi\wedge dx_p$ (Step 3). Now $\mathsf{a}_q-\mathsf{a}_p=\sum_{m\notin\{p,q\}}x_m\big(\frac{1}{x_{mq}}-\frac{1}{x_{mp}}\big)dx_m+\frac{x_p+x_q}{2x_{pq}}(dx_p+dx_q)$. The dx_q-terms give $-x_q\,dx_q\wedge\partial_q\psi\wedge dx_p-\frac12 dx_q\wedge\psi\wedge dx_p+\frac{x_p+x_q}{2x_{pq}}dx_q\wedge\psi\wedge dx_p=-x_q\,dx_q\wedge(\partial_q\psi-\psi/x_{pq})\wedge dx_p$, and, as $dx_p\wedge\psi^{(mq)}\wedge dx_m=-dx_m\wedge\psi^{(mq)}\wedge dx_p$, the dx_m-terms give $-x_m\,dx_m\wedge(\partial_m\psi-\delta_m\psi)\wedge dx_p$. With $dx_p\wedge\Theta=\Theta\wedge dx_p$ this is the formula.

Step 5 (the shift). $f_s(P_1,P_2+cP_1,P_3)=f_s(P_1,P_2,P_3)$, as $P_1^{-1}(P_2+cP_1)=P_1^{-1}P_2+cI$ has the same traceless part. So Ωf_s at the base point $(X+cI,Y)$ is Ωf_s at (X,Y) plus a multiple of $\partial_{P_2}[B_i]\,\partial_{P_2}[B_j]\,\partial_{P_3}[B_k]f_s$, which is symmetric in (i,j) and dies in the alternation. As F, G, ψ and Θ are invariant under $x\mapsto x+c\mathbf{1}$, only the explicit coefficients move, $x_l\mapsto x_l+c$. □

Proposition 9.45 (the test at diagonal Y)**.** *Let φ, X and B_k be as in Proposition 9.44, and $Y = \operatorname{diag}(y)$. Then*

$$\tfrac{1}{2}(\kappa\mathcal{Z}-\mathcal{J})(\varphi)(B_1,B_2,B_3) = \Big(\kappa\,\eta\wedge\Theta_y - d^x\Theta_y - \sum_j y_j\,dx_j\wedge\Theta_j - \sum_{r\neq t}\varpi_{rt}\,dx_t\wedge\psi^{(rt)}\wedge dx_r\Big)(v_1,v_2,v_3),$$

with $\varpi_{rt} := (x_t y_r - x_r y_t)/x_{tr}$; and the same holds with every explicit coefficient x_l replaced by $x_l + c$.

Proof. By Step 1 of the previous proof, $\omega_Y = \sum_j y_j \omega_{E_{jj}}$ is $2\Theta_y$ evaluated on the diagonal parts, for all arguments. So $\operatorname{tr}\wedge\omega = 2\,\eta\wedge\Theta_y$ and (I) $= 2d^x\Theta_y$. In (III), $\operatorname{diag}[W'_w, \mathbf{1}v^T] = S(w,v)$ is symmetric in (w,v), so the conjugation terms $\Theta_y(S(v_k,v_i),v_j)$ are symmetric in (k,i) and alternate to 0. In (II), for $B = \mathbf{1}w^T$ the matrix $B\circ Y - [W'_w, Y]$ has the off-diagonal entries $w_t\big[\frac{y_r+y_t}{2} - \frac{(x_r+x_t)(y_t-y_r)}{2x_{tr}}\big] = w_t\varpi_{rt}$ and the diagonal entries $w_t y_t$; so $\frac{1}{2}$(II) $= \sum_{r\neq t}\varpi_{rt}\,dx_t\wedge\psi^{(rt)}\wedge dx_r + \sum_t y_t\,dx_t\wedge\Theta_t$. The shift is Step 5 of the previous proof. □

Lemma 9.46 (elimination)**.** *Let $\varphi\in\mathcal{F}_{(a,2)}(n)$ and X as above, $Y = E_{jj}$, and put*

$$R_j := 2\sum_{r\neq j}\big[\mathsf{c}^{(rj)}\wedge(dx_r - dx_j) - \mathsf{c}^{(rj)}(\mathbf{1})\,dx_j\wedge dx_r\big]$$

(it involves F only). Then

$$(\kappa\mathcal{Z}-\mathcal{J})(\varphi)(\mathbf{1}\mathbf{1}^T, \mathbf{1}v^T, \mathbf{1}v'^T) = 2\,(\mu\,\Theta_j - R_j)(v,v').$$

Hence, if $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ satisfies $\kappa\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ at every base point and for all u, y_1, y_2, y_3, and $\mu\neq 0$, then for all $j\neq t$

$$\mu\,(a+1)\,\mathsf{G}_{jt} = \frac{2}{x_{tj}}\sum_{w\notin\{t,j\}}\mathsf{H}_{twj}.$$

Proof. Evaluating a 3-form at $(\mathbf{1}, v, v')$ is evaluating its contraction $\iota_{\mathbf{1}}$ at (v,v'). As in the proof of Theorem 9.41, we contract with $\mathbf{1}$, here the shifted form of Proposition 9.45 with $c = -x_j$, at $y = e_j$. First, $\iota_{\mathbf{1}}\Theta_j = L_{\mathbf{1}}\gamma_j - d\,\iota_{\mathbf{1}}\gamma_j = 0$, as the G_{jt} are invariant under $x\mapsto x + c\mathbf{1}$ and $\iota_{\mathbf{1}}\gamma_j = (\mathsf{G}\mathbf{1})_j = 0$. So $\iota_{\mathbf{1}}(\eta\wedge\Theta_j) = n\Theta_j$; the d^x-term contracts to $\sum_m(x_m - x_j)\partial_m\Theta_j = (a-1)\Theta_j$, as the coefficients of Θ_j are homogeneous of degree $a-1$ and $\sum_m\partial_m\Theta_j = 0$ (the same as for the plain form); and $\iota_{\mathbf{1}}(dx_j\wedge\Theta_j) = \Theta_j$. Together these give $(n\kappa - a)\Theta_j = \mu\Theta_j$. In the last term $\varpi_{jt} = \varpi_{rj} = 1$, all other ϖ_{rt} vanish, and the term is $-\sum_{r\neq j}dx_r\wedge(\psi^{(jr)} - \psi^{(rj)})\wedge dx_j$. In $\psi^{(jr)}_w - \psi^{(rj)}_w$ $(w\notin\{j,r\})$ the $\partial_w\mathsf{F}$- and G-parts cancel, and $\mathsf{c}^{(jr)}_w = -\mathsf{c}^{(rj)}_w$ ($\mathsf{H}_{jwr} = -\mathsf{H}_{rwj}$, same denominator); so the term is $2\sum_r dx_r\wedge\mathsf{c}^{(rj)}\wedge dx_j$, and as $\iota_{\mathbf{1}}(dx_r\wedge\mathsf{c}\wedge dx_j) = \mathsf{c}\wedge(dx_j - dx_r) - \mathsf{c}(\mathbf{1})\,dx_r\wedge dx_j$, its contraction is $-R_j$. This proves the identity.

For a kernel vector with $\mu\neq 0$ it gives $\mu\,d\gamma_j = R_j$ at every x with distinct coordinates. By Cartan's formula, $\iota_\xi d\gamma_j = L_\xi\gamma_j - d\,\iota_\xi\gamma_j = (a+1)\gamma_j$, as $\iota_\xi\gamma_j = (\mathsf{G}x)_j = 0$; and $\iota_\xi d\gamma_j = \iota_{\xi - x_j\mathbf{1}}d\gamma_j$, as $\iota_{\mathbf{1}}d\gamma_j = 0$. So $\mu(a+1)\gamma_j = \iota_{\xi - x_j\mathbf{1}}R_j$. The dx_t-component ($t\neq j$) of the right side is $2\sum_w x_{wj}\big(\mathsf{c}^{(tj)}_w - \mathsf{c}^{(wj)}_t\big)$, and $\mathsf{c}^{(tj)}_w - \mathsf{c}^{(wj)}_t = \mathsf{H}_{twj}\big(\frac{1}{x_{tw}x_{wj}} + \frac{1}{x_{wt}x_{tj}}\big) = \frac{\mathsf{H}_{twj}}{x_{wj}x_{tj}}$, which gives the formula. □

Corollary 9.47 (two-point form)**.** *Suppose that $\mathsf{F}_{rs} = \mathsf{F}(x_r, x_s)$ for $r\neq s$, where $\mathsf{F}(\zeta,\zeta')$ is a polynomial in ζ,ζ' whose coefficients are polynomials in the power sums of x, and put $\mathfrak{h}(\zeta) := \sum_w(\zeta - x_w)\mathsf{F}(\zeta, x_w)$. Then $\sum_{w\notin\{t,j\}}\mathsf{H}_{twj} = \mathfrak{h}(x_t) - \mathfrak{h}(x_j) - nx_{tj}\mathsf{F}(x_t,x_j)$. So, under the hypotheses of the second part of Lemma 9.46, $\mu(a+1)\mathsf{G}_{jt} = 2\big(\mathfrak{h}[x_j,x_t] - n\mathsf{F}(x_j,x_t)\big)$ for $j\neq t$, where $\mathfrak{h}[\zeta,\zeta'] := (\mathfrak{h}(\zeta) - \mathfrak{h}(\zeta'))/(\zeta - \zeta')$.*

Proof. In the sum over all w of $\mathsf{H}_{tw} + \mathsf{H}_{wj} + \mathsf{H}_{jt}$ the terms $w = t$ and $w = j$ vanish; the rest is $\sum_w\mathsf{H}_{tw} - \sum_w\mathsf{H}_{jw} + n\mathsf{H}_{jt}$, and $\sum_w\mathsf{H}_{tw} = \mathfrak{h}(x_t)$. □

Theorem 9.48 (the second layer at $a=3$)**.** *Let $n\ge 4$,*

$$\varphi_1:=\operatorname{tr}(A^2)\operatorname{tr}(AB^2)+\operatorname{tr}(A^3)\operatorname{tr}(B^2)-2\operatorname{tr}(A^2B)\operatorname{tr}(AB),\qquad \varphi_2:=\operatorname{tr}(A[A,B]^2)$$

($A,B\in\mathfrak{sl}_n$). Then $\mathcal{F}^{\mathcal{E}}_{(3,2)}(n)$ has the basis φ_1,φ_2. If $0\ne\varphi\in\mathcal{F}^{\mathcal{E}}_{(3,2)}(n)$ and $\kappa\in\mathbb{C}$ satisfy $\kappa\,\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$ at every base point and for all u,y_1,y_2,y_3, then either $\kappa=3/n$ and $\varphi\in\mathbb{C}\varphi_1$, or $\kappa=6/n$ and $\varphi\in\mathbb{C}(n\varphi_1+6\varphi_2)$. That is, every eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(3,2)}(n)$ has $\mu=n\kappa-3\in\{0,3\}$.

Proof. (a) *The space.* Let M be the span of the formal trace monomials in A,B of degree 5 without a factor of degree 1; linear substitutions of (A,B) make M a polynomial GL_2-module. Its weight space $M_{(3,2)}$ has the basis $\operatorname{tr}(A^3B^2)$, $\operatorname{tr}(A^2BAB)$, $\operatorname{tr}(A^2B)\operatorname{tr}(AB)$, $\operatorname{tr}(A^2)\operatorname{tr}(AB^2)$, $\operatorname{tr}(A^3)\operatorname{tr}(B^2)$, and $M_{(4,1)}$ the basis $\operatorname{tr}(A^4B)$, $\operatorname{tr}(A^2)\operatorname{tr}(A^2B)$, $\operatorname{tr}(A^3)\operatorname{tr}(AB)$; so the irreducible module of highest weight $(3,2)$ occurs in M with multiplicity $5-3=2$. Evaluation on $\mathfrak{sl}_n^2$ is GL_2-equivariant and maps M onto the invariants of degree 5 (by the first fundamental theorem [Pr76], as the monomials with a factor $\operatorname{tr}A$ or $\operatorname{tr}B$ vanish), and $\dim\mathcal{F}^{\mathcal{E}}_{(3,2)}(n)$ is the multiplicity of that irreducible in the image; so it is at most 2, for every n. Next, $\mathcal{E}\varphi_2=0$ as $[A,B+tA]=[A,B]$, and $\mathcal{E}\varphi_1=2\operatorname{tr}(A^2)\operatorname{tr}(A^2B)+2\operatorname{tr}(A^3)\operatorname{tr}(AB)-2\operatorname{tr}(A^3)\operatorname{tr}(AB)-2\operatorname{tr}(A^2B)\operatorname{tr}(A^2)=0$. By (b) below, $\mathsf{G}=0$ for φ_2, while $\mathsf{F}\ne0$ for φ_2 and $\mathsf{G}_{13}=-6\ne0$ for φ_1 at $x=(1,-1,2,-2,0,\dots,0)$ (here $n\ge4$ is used); so φ_1, φ_2 are linearly independent.

(b) *The data.* Write $z_l:=x_l-p_1/n$ with $p_1:=\sum_l x_l$, and $P_k:=\sum_l z_l^k$ (so $P_1=0$); then $\partial_w z_l=\delta_{wl}-1/n$ and $\partial_w P_k=k(z_w^{k-1}-P_{k-1}/n)$. As $X_0=\operatorname{diag}(z)$, the trace monomials give directly, for every n and $r\ne t$:

$$\varphi_1:\quad \mathsf{F}_{rt}=\tfrac12P_2(z_r+z_t)+P_3,\qquad \mathsf{G}_{rt}=-z_rz_t(z_r+z_t)-P_3/n;$$
$$\varphi_2:\quad \mathsf{F}_{rt}=-\tfrac12(z_r+z_t)(z_r-z_t)^2,\qquad \mathsf{G}=0.$$

For φ_1, $\mathsf{H}_{rs}=\mathfrak{h}_1(z_r)-\mathfrak{h}_1(z_s)$ with $\mathfrak{h}_1(u):=\frac12P_2u^2+P_3u$, so $\mathsf{c}=0$. For φ_2, $\mathsf{H}_{pwq}=uv(u+v)(3z_w+u-v)$ with $u:=z_p-z_w$, $v:=z_w-z_q$, so $\mathsf{c}_w^{(pq)}=(z_p-z_q)(z_p+z_q+z_w)$. Hence, for $w\notin\{p,q\}$,

$$\psi_w^{(pq)}[\varphi_1]=4z_w^2+2z_w(z_p+z_q)-4P_2/n,\qquad \psi_w^{(pq)}[\varphi_2]=(z_p-z_q)^2/n+(z_p-z_q)(z_p+z_q+z_w).$$

(c) *The rows.* Let i,l,p,q be distinct (so $n\ge4$). As $(\Psi_{pq}\wedge dx_p)(e_p,e_i,e_l)=\Psi_{pq}(e_i,e_l)$, Proposition 9.44 in its shifted form with $c=-x_p$ (so that $x_p+c=0$, $x_q+c=-x_{pq}$, and the Θ-term is $-\Theta_q$) gives, for a kernel vector,

$$\kappa(\psi_l-\psi_i)-x_i'(\nabla_i\psi)_l+x_l'(\nabla_l\psi)_i-(\partial_i\mathsf{G}_{ql}-\partial_l\mathsf{G}_{qi})=0,\qquad x_m':=x_m-x_p,\quad \nabla_m:=\partial_m-\delta_m,$$

with $\psi=\psi^{(pq)}$. We evaluate it at points with $p_1=0$, where $z=x$. Note $(\nabla_i\psi)_l=\partial_i\psi_l^{(pq)}-(\psi_l^{(pq)}-\psi_l^{(pi)})/x_{iq}+(\psi_l^{(pq)}-\psi_l^{(iq)})/x_{ip}$. For φ_1: $\partial_i\psi_l^{(pq)}=-(8z_i+12z_l+2z_p+2z_q)/n$, the two difference quotients are both $-2z_l$ and cancel, and $\partial_i\mathsf{G}_{ql}-\partial_l\mathsf{G}_{qi}=-\frac4n(z_i-z_l)(z_i+z_l+z_q)$; the row is

$$\Big(\frac6n-2\kappa\Big)(z_i-z_l)(2z_i+2z_l+z_p+z_q).$$

For φ_2: $\partial_i\psi_l^{(pq)}=-3(z_p-z_q)/n$, and $(\nabla_i\psi)_l=(\nabla_l\psi)_i=-\frac6n(z_p-z_q)-(2z_i+2z_l+z_p+z_q)$; the row is

$$(z_i-z_l)\Big[\Big(\frac6n-\kappa\Big)(z_p-z_q)+(2z_i+2z_l+z_p+z_q)\Big].$$

The power sums P_2, P_3 cancel in both rows.

(d) *The eigenvalues.* By (a), $\varphi=\alpha\varphi_1+\beta\varphi_2$, and the row is linear in φ:

$$(z_i-z_l)\Big[\Big(\alpha\Big(\frac6n-2\kappa\Big)+\beta\Big)(2z_i+2z_l+z_p+z_q)+\beta\Big(\frac6n-\kappa\Big)(z_p-z_q)\Big]=0$$

at every point with $p_1 = 0$ and distinct coordinates. As $z_i \neq z_l$ there, the bracket vanishes on a dense subset of the hyperplane $p_1 = 0$, hence on all of it. For $n \geq 5$, z_i, z_l, z_p, z_q are independent there; for $n = 4$, $z_i + z_l + z_p + z_q = 0$, so $2z_i + 2z_l + z_p + z_q = z_i + z_l$, and $z_i + z_l$ and $z_p - z_q$ are still independent linear forms on that plane. So $\beta(6/n - \kappa) = 0$ and $\alpha(6/n - 2\kappa) + \beta = 0$: either $\beta = 0$, $\alpha \neq 0$ and $\kappa = 3/n$, or $\kappa = 6/n$ and $\beta = 6\alpha/n$. □

Corollary 9.49. *For every $n \geq 4$ and every integer $s \geq 1$ with $ns \geq 8$, $\mathrm{K}_s(n)_{(ns-5,3,2)} = 0$. By Theorem 9.33, also $\mathrm{K}_s(3)_{(3s-5,3,2)} = 0$ for every $s \geq 4$.*

Proof. Let $f \in \mathrm{K}_s(n)_\tau$, $\tau = (ns-5, 3, 2)$ (a partition as $ns \geq 8$). By Lemma 9.32(ii), $\varphi_f \in \mathcal{F}^{\mathcal{E}}_{(3,2)}(n)$ and f is the function f_s built from φ_f; by Corollary 9.31(b), $s\,\mathcal{Z}(\varphi_f) = \mathcal{J}(\varphi_f)$. If $\varphi_f \neq 0$, Theorem 9.48 gives $ns - 3 \in \{0, 3\}$, that is $ns \leq 6$, which is excluded. So $\varphi_f = 0$, and $f = 0$ by the injectivity in Lemma 9.32(ii). The second statement is the first with (n, s) replaced by $(s, 3)$, and Theorem 9.33. □

Remark 9.50. (1) For $s \leq 2$ the test is empty (Proposition 9.34), and the corollary is a statement on the Kronecker coefficients $g((s^n), (s^n), \tau)$ (Lemma 9.32(i)), which we do not count as new. For $s = n$ the type $\tau = (n^2 - 5, 3, 2)$ is, by Lemma 9.32(iii), the conjugation type of $K_{B(n,2)}$ whose conjugate is $(n^2 - 6, 2, 1)$, of depth $3 \leq n - 1$, so it is covered by Theorem 9.24. The new cases are $3 \leq s \neq n$ ($n \geq 4$), and matrix size 3 through Theorem 9.33. The two eigenvectors φ_1 and $n\varphi_1 + 6\varphi_2$ had been found by exact computation at $n = 7, 8$; what is new is that there is no other eigenvalue, for every $n \geq 4$. The value $\mu = 3$ can be an integer κ: for $n = 6$ the kernel at $\kappa = 1$ is spanned by $6\varphi_1 + 6\varphi_2$ (by the computations in (5)), and it gives no element of $\mathrm{K}_1(6)$ only because $(1, 3, 2)$ is not a partition.

(2) The rows are triangular in the basis φ_1, φ_2: the part antisymmetric in (p, q) (it comes from c, antisymmetric in $z_p \leftrightarrow z_q$) sees only φ_2, with the factor $6/n - \kappa = -(\mu - 3)/n$, and the symmetric part has the factor $6/n - 2\kappa = -2\mu/n$ on φ_1. Lemma 9.46 agrees: at points with $p_1 = 0$, $\mathfrak{h}[\zeta, \zeta'] - n\mathsf{F}(\zeta, \zeta')$ in Corollary 9.47 is 0 for φ_1, and for φ_2 it is $-n\zeta\zeta'(\zeta + \zeta') - P_3$, which at $(\zeta, \zeta') = (x_j, x_t)$ is $n\mathsf{G}_{jt}$ of φ_1; so a kernel vector $\alpha\varphi_1 + \beta\varphi_2$ with $\mu \neq 0$ has $4\mu\alpha = 2n\beta$, that is $\alpha : \beta = n : 2\mu$.

(3) For $a = 2$, Lemma 9.46 gives one inclusion of Theorem 9.37 by another route. With c and G as there, at points with $p_1 = 0$: for c, $\mathsf{F}(\zeta, \zeta') = -(\zeta - \zeta')^2$, $\mathsf{G} = 0$ and $\mathfrak{h}[\zeta, \zeta'] - n\mathsf{F} = -3(n\zeta\zeta' + P_2)$; for G, $\mathsf{F} = P_2$, $\mathsf{G}_{jt} = -(nx_jx_t + P_2)/n$ and $\mathfrak{h}[\zeta, \zeta'] - n\mathsf{F} = 0$. So for $\kappa \neq 2/n$ a kernel vector $\alpha c + \beta G$ has $\alpha : \beta = \mu : 2n = (n\kappa - 2) : 2n$, the line of Theorem 9.37.

(4) (Computations beyond $a = 3$; the rows give an upper bound on the spectrum, and we do not claim that every value listed occurs.) With n symbolic, the rows $\Psi_{pq}(e_i, e_l) = 0$ (with i, l, p, q distinct, and with $i = q$) at $(4, 2)$ have the common factor $(\kappa - 2)(\kappa + 2)(n\kappa - 7)(n\kappa - 4)^2$; this is valid for all but finitely many n (a greatest common divisor of determinants over $\mathbb{Q}(n)$). At $(5, 2)$ the rows with i, l, p, q distinct give $(11\kappa - 8)(11\kappa - 5)^2$ for $n = 11$ and $(3\kappa - 2)(12\kappa - 5)^2$ for $n = 12$, and at $(6, 2)$ and $n = 13$ both kinds of rows give $(\kappa - 2)(\kappa + 2)(13\kappa - 9)(13\kappa - 6)^4$. In all these cases every eigenvalue has $\mu \in \{0, 3\}$ or $\kappa = \pm 2$, and we expect this for all $n \geq 4$ and $a \geq 3$. For $n \geq a + 6$ it is Theorem 9.150 (see Remark 9.151(1)). At $(4, 2)$ the n-symbolic determinants are invariant under $(\kappa, n) \mapsto (-\kappa, -n)$ (compare Remark 9.38(e)), and so is $\mu = n\kappa - a$. See Theorem 9.75.

(5) (Computations, not used.) Proposition 9.44 (plain and shifted forms; 60 cases) and Proposition 9.45 with the identity of Lemma 9.46 (70 cases) agree with the definition of Ωf_s for five functions of bidegrees $(2, 2)$, $(3, 2)$ and $(4, 2)$, at $n = 3, 4$ and $s = n, 2, -3$; and Proposition 9.44 does on all rows with i, l, p, q distinct for φ_1, φ_2 at $n = 5$ (36 cases). Both eigenvalues of Theorem 9.48 occur, each with a one-dimensional kernel: exact computations from the definition at general base points find exactly $\{3/n, 6/n\}$, with the eigenvectors φ_1 and $n\varphi_1 + 6\varphi_2$, for $n = 4, 5, 6, 7$. An independent end-to-end computation from (S1) and (S2) gives $\mathrm{K}_s(n)_{(ns-5,3,2)} = 0$ for $(n, s) = (5, 3), (5, 6), (5, 7), (6, 3), (6, 5), (6, 7)$.

9.14. **The order along $\mathcal{S}$.** In this subsection $F=\mathbb{C}$ and $n\ge 2$. Let $V:=\mathfrak{sl}_n\times\mathfrak{sl}_n$, with GL_n acting by simultaneous conjugation, and let $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ be as in Subsection 9.12. Put

$$\ell_i:=\operatorname{tr}(X^iY)\quad(1\le i\le n-1),\qquad \mathfrak{p}:=(\ell_1,\dots,\ell_{n-1})\subset\mathbb{C}[V],\qquad R:=\mathbb{C}[P_2,\dots,P_n],$$

with $P_m=\operatorname{tr}(X^m)/m$ as in Remark 9.43(2), viewed as functions of X. So ℓ_i has bidegree $(i,1)$, $\mathcal{E}\ell_i=\operatorname{tr}(X^{i+1})$, and $dP_k(X)[Y]=\ell_{k-1}$. The ℓ_i are the linear-in-Y invariants of Remark 9.43(2). Let $\mathcal{S}$ be the closure in V of $\{(X,Y):X\text{ regular},\ Y\in\operatorname{Im}\operatorname{ad}_X\}$, and for $1\le i<j\le n-1$ let

$$w_{ij}:=\operatorname{tr}(X^{j+1})\,\ell_i-\operatorname{tr}(X^{i+1})\,\ell_j\in\mathcal{F}^{\mathcal{E}}_{(i+j+1,1)}(n)$$

($\mathcal{E}w_{ij}=0$). For $n\ge 3$, w_{12} spans $\mathcal{F}^{\mathcal{E}}_{(4,1)}(n)$ (the dimension formula of Remark 9.43(2) gives 1 for $a=4$ and every $n\ge 3$). We show that a nonzero element of $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ lies in $\mathfrak{p}^m$ only for $m\le\lfloor a/4\rfloor$ (Theorem 9.55).

Proposition 9.51. *Let $n\ge 2$. The zero set of $\ell_1,\dots,\ell_{n-1}$ is $\mathcal{S}$; $\mathcal{S}$ is irreducible of codimension $n-1$; $\ell_1,\dots,\ell_{n-1}$ is a regular sequence in $\mathbb{C}[V]$; and $\mathfrak{p}$ is the prime ideal of $\mathcal{S}$. So $\mathbb{C}[\mathcal{S}]:=\mathbb{C}[V]/\mathfrak{p}$ is a Cohen–Macaulay domain, and it is free over R.*

Proof. Steps (1)–(2) are the method of Geoffriau [Ge94, Théorème 2.4] (written there for the nullcone of a Takiff algebra), here in type A. (1) The trace form is nondegenerate and ad_X is skew for it, so $\operatorname{Im}\operatorname{ad}_X=(\ker\operatorname{ad}_X)^{\perp}$. For regular X the centralizer of X in $\mathfrak{gl}_n$ is $\mathbb{C}[X]=\operatorname{span}(I,X,\dots,X^{n-1})$, so for $Y\in\mathfrak{sl}_n$: $Y\in\operatorname{Im}\operatorname{ad}_X$ if and only if $\ell_i(X,Y)=0$ for $1\le i\le n-1$. Over the regular X the zero set is therefore a vector bundle of rank $(n^2-1)-(n-1)$, irreducible and smooth, since $d_Y\ell_i=(X^i)_0$ are independent exactly for regular X [Ko63]; $\mathcal{S}$ is its closure. (2) Let X have minimal polynomial of degree $m<n$ and Jordan type μ^λ at each eigenvalue λ. Then $\dim\mathfrak{z}(X)=\sum_\lambda\sum_i(\mu_i^\lambda)^2$, $n=\sum_\lambda\sum_i\mu_i^\lambda$ and $m=\sum_\lambda\mu_1^\lambda$, so $n-m=\sum(\mu_i'-1)$, and $\mu'(\mu'-1)\ge 2(\mu'-1)$ for every integer $\mu'\ge 1$ gives $\dim\mathfrak{z}(X)-n\ge 2(n-m)$. A stratum of fixed Jordan type with $r\le m$ distinct eigenvalues has dimension at most $(n^2-n-2(n-m))+m$ in $\mathfrak{gl}_n$, so codimension at least $3(n-m)$; it is stable under $X\mapsto X+cI$, so the same holds in $\mathfrak{sl}_n$. Over such X the ℓ_i impose only $m-1$ independent conditions on Y, since $\operatorname{span}\{(X^i)_0\}=(\mathbb{C}[X]+\mathbb{C}I)/\mathbb{C}I$ has dimension $m-1$. So the part of the zero set over the stratum has dimension at most $\dim V-3(n-m)-(m-1)=\dim\mathcal{S}-2(n-m)<\dim\mathcal{S}$. There are finitely many strata, and by Krull every component of the zero set has dimension at least $\dim V-(n-1)=\dim\mathcal{S}$; so the zero set is $\mathcal{S}$, and it is irreducible. (3) The ℓ_i are homogeneous of positive degree in the polynomial ring $\mathbb{C}[V]$ and cut out a set of codimension $n-1$, so they form a regular sequence (in any order), and $\mathbb{C}[V]/\mathfrak{p}$ is Cohen–Macaulay. It is generically reduced (the Jacobian has rank $n-1$ on the regular part, by (1)), hence reduced, and $\mathfrak{p}$ is the prime ideal of $\mathcal{S}$. For freeness: $R[\ell_1,\dots,\ell_{n-1}]=\mathbb{C}[P_k,dP_k[Y]]$, and the $2(n-1)$ homogeneous elements $P_2,\dots,P_n,\ell_1,\dots,\ell_{n-1}$ have the common zero set $\{X\text{ nilpotent},\ Y\perp\mathbb{C}[X]\}$, which over the nilpotent orbit with minimal polynomial of degree m has dimension at most $(n^2-n-2(n-m))+(n^2-m)=2n^2-3n+m\le 2n^2-2n$ (by (2)); there are finitely many nilpotent orbits, so its codimension in V (of dimension $2n^2-2$) is $2(n-1)$. So these elements form a regular sequence, $\mathbb{C}[V]$ is free over $R[\ell]$, and $\mathbb{C}[\mathcal{S}]=\mathbb{C}[V]\otimes_{R[\ell]}R$ is free over R. (For $n=2$, $\mathfrak{p}=(\operatorname{tr}XY)$ is a nondegenerate quadric in 6 variables, and (1)–(3) go through unchanged.) □

Remark 9.52. Proposition 9.51 is classical in substance: $\mathcal{S}$ is the fibre over $R\times 0$ of the first-jet map $\pi(X,Y)=(P_k(X),dP_k(X)[Y])_k$ of the adjoint quotient χ, and the complete intersection property and freeness of first jets are due to Geoffriau [Ge94] and Eisenbud and Frenkel [EF01, Theorems A.2, A.4]; see also Raïs and Tauvel [RT92] for the Takiff algebra and its invariants. The irreducibility also follows from the literature, for every semisimple Lie algebra in place of $\mathfrak{sl}_n$: π is flat [EF01, Theorem A.4], so $\mathcal{S}\to\operatorname{Spec}R$ is flat, hence open; the fibre over $(r,0)$ is the first jet scheme of $\chi^{-1}(r)$, which is irreducible by Mustaţă's theorem [Mu01, Theorem 3.3], since the fibres of χ have rational singularities (a classical fact; in type A, also by Mustaţă's criterion [Mu01, Proposition 1.4]

with the count of step (2) on a fibre); and an open map onto an irreducible base with irreducible fibres has an irreducible source. We found no statement for $\mathcal{S}$ itself; the proof above is elementary and complete.

For $\varphi \in \mathbb{C}[V]$, $\varphi \neq 0$, let $\operatorname{ord}(\varphi) := \max\{m : \varphi \in \mathfrak{p}^m\}$.

Lemma 9.53. $\operatorname{ord}(\varphi)$ *is at most the largest Y-degree of a nonzero Y-homogeneous component of φ. The associated graded ring $\operatorname{gr}_{\mathfrak{p}} \mathbb{C}[V]$ is the polynomial ring $\mathbb{C}[\mathcal{S}][t_1, \dots, t_{n-1}]$ (t_i the class of ℓ_i), a domain. The leading symbol $\sigma(\varphi)$, the class of φ in $\mathfrak{p}^m/\mathfrak{p}^{m+1}$ with $m = \operatorname{ord}(\varphi)$, is a nonzero element of $\mathbb{C}[\mathcal{S}][t]_m$; if φ is bihomogeneous, $\sigma(\varphi)$ is bihomogeneous of the same bidegree when t_i has bidegree $(i, 1)$; and $\sigma(\varphi)$ is GL_n-invariant if φ is.*

Proof. Each ℓ_i is Y-homogeneous of degree 1, so every Y-homogeneous component of an element of $\mathfrak{p}^m$ has Y-degree at least m. A regular sequence is quasi-regular [Mat86, Theorem 16.2], which gives $\operatorname{gr}_{\mathfrak{p}}$; it is a domain because $\mathbb{C}[\mathcal{S}]$ is (Proposition 9.51). GL_n fixes each ℓ_i, so it acts on gr fixing each t_i. □

Lemma 9.54. *Let $\varphi \in \mathbb{C}[V]^{\mathrm{GL}_n}$ be bihomogeneous of bidegree (a, b), $\varphi \neq 0$, and $m := \operatorname{ord}(\varphi)$. For traceless $x \in \mathbb{C}^n$ with distinct entries write $Y = D + Z$ (D diagonal traceless, Z off-diagonal) and $\varphi(\operatorname{diag} x, D + Z) = \sum_k F_k$, with F_k of degree k in Z and $b - k$ in D. Then $F_k = 0$ for $k > b - m$, for every such x; and $F_{b-m} \neq 0$ for all x in a nonempty Zariski-open set, which depends on φ; so, for such x, $\operatorname{ord}(\varphi) = \min\{b - k : F_k \neq 0\}$.*

Proof. On the slice $\Sigma_x := \{\operatorname{diag} x\} \times \mathfrak{sl}_n$, $\ell_i(\operatorname{diag} x, D+Z) = \tau_i(D) := \sum_l x_l^i D_l$, linear in D and free of Z. Write $\varphi = \sum_{|\alpha|=m} \ell^\alpha g_\alpha + r$ with $r \in \mathfrak{p}^{m+1}$, so $\sigma(\varphi) = \sum_\alpha \bar{g}_\alpha t^\alpha$. (a) Every element of $\mathfrak{p}^j$ restricts to Σ_x with D-degree at least j; so the part of $\varphi|_{\Sigma_x}$ of D-degree m is $\sigma(\varphi)(\operatorname{diag} x, Z)$ at $t = \tau(D)$, and the parts of D-degree less than m vanish. (b) On traceless D, $D \mapsto (\tau_1(D), \dots, \tau_{n-1}(D))$ is a linear isomorphism (Vandermonde, with $\tau_0(D) = 0$), so $F_{b-m} \neq 0$ if and only if $\sigma(\varphi)(\operatorname{diag} x, Z) \in \mathbb{C}[t]_m$ is nonzero for some Z. (c) The coefficients of $\sigma(\varphi)$ are GL_n-invariant functions on $\mathcal{S}$, not all 0 (Lemma 9.53). The pairs with X regular semisimple form a dense open subset of the irreducible $\mathcal{S}$, and each is conjugate to some $(\operatorname{diag} x, Z)$ with Z off-diagonal (dimension check: $(n-1) + (n^2 - n)$ for these pairs and $n^2 - n$ for GL_n/T add up to $\dim \mathcal{S} = 2n^2 - n - 1$). So $\sigma(\varphi)(\operatorname{diag} x, Z) \neq 0$ for (x, Z) in a nonempty open set. □

One fixed x need not work for a whole space, since every $r \in R$ is the constant $r(x)$ on Σ_x: for $n = 3$, $\varphi = (P_2(x)^3 P_3^2 - P_3(x)^2 P_2^3)\, w_{12} \in \mathcal{F}^{\mathcal{E}}_{(10,1)}(3)$ has $\operatorname{ord}(\varphi) = 1$, but every $F_k = 0$ at that x.

Theorem 9.55 (the order bound)**.** *Let $n \geq 2$. Every nonzero $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ has $\operatorname{ord}(\varphi) \leq \lfloor a/4 \rfloor$. For $n \geq 3$ the bound is attained: $w_{12}^m \in \mathcal{F}^{\mathcal{E}}_{(4m,m)}(n)$ has order m.*

Proof. Let $m := \operatorname{ord}(\varphi)$ and $\sigma := \sigma(\varphi) \neq 0$. (i) $\mathcal{E}$ maps $\mathfrak{p}^m$ into $\mathfrak{p}^{m-1}$, and on the graded pieces it induces $1 \otimes \mathcal{D}$ with $\mathcal{D} := \sum_i \operatorname{tr}(X^{i+1})\, \partial/\partial t_i$: indeed $\mathcal{E}(\ell^\alpha g) = \sum_i \alpha_i \operatorname{tr}(X^{i+1}) \ell^{\alpha - e_i} g + \ell^\alpha\, \mathcal{E} g$, with the last term in $\mathfrak{p}^m$. As $\mathcal{E}\varphi = 0$, $\mathcal{D}\sigma = 0$ in $\mathbb{C}[\mathcal{S}][t]_{m-1}$. (ii) On $R[t]$, with $q_i := \operatorname{tr}(X^{i+1})$ (generators of R), $\mathcal{D} = \sum_i q_i\, \partial/\partial t_i$ is the basic Weitzenböck derivation of $n - 1$ copies of $\mathbb{C}^2$. Its kernel is generated by the q_i and the minors $q_i t_j - q_j t_i$; this is Nowicki's conjecture [Now94, §6.9], proved by Khoury [Kh08] and by Drensky and Makar-Limanov [DML09]. The minor $q_j t_i - q_i t_j$ is the symbol of w_{ij}, so in t-degree m the kernel of $\mathcal{D}$ is the R-span of the products of m symbols of w_{ij}'s. (iii) $\mathbb{C}[\mathcal{S}]$ is flat over R (Proposition 9.51) and $\mathcal{D}$ is R-linear between free R-modules, so the kernel of $1 \otimes \mathcal{D}$ on $\mathbb{C}[\mathcal{S}][t]_m$ is $\mathbb{C}[\mathcal{S}] \otimes_R \ker \mathcal{D}$: $\sigma = \sum_J c_J \sigma(w_J)$, with w_J running over the products of m factors w_{ij} and $c_J \in \mathbb{C}[\mathcal{S}]$ bihomogeneous. (iv) $\deg_X w_{ij} = i + j + 1 \geq 4$ and $\mathbb{C}[\mathcal{S}]$ is nonnegatively graded, so $\sigma \neq 0$ forces $a \geq 4m$. Sharpness: w_{12}^m lies in $\mathcal{F}^{\mathcal{E}}_{(4m,m)}(n)$ ($\mathcal{E}$ is a derivation), and its symbol $(\operatorname{tr}(X^3)t_1 - \operatorname{tr}(X^2)t_2)^m$ is nonzero in the domain $\mathbb{C}[\mathcal{S}][t]$, as $\operatorname{tr}(X^2) \neq 0$ on $\mathcal{S}$; so its order is $m = a/4$. □

Corollary 9.56. *Let $n \geq 2$ and $a < 16$. Every nonzero $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ has $\operatorname{ord}(\varphi) \leq 3$, so, by Lemma 9.54, at generic diagonal X (generic for this φ) some F_k with $k \geq b-3$ is nonzero.*

Remark 9.57. (1) For $n = 2$ there is one ℓ only, and $\ker \mathcal{D} = R$ on $R[t] = \mathbb{C}[P_2, t_1]$: $(1 \otimes \mathcal{D})(ct^m) = 2mP_2ct^{m-1} = 0$ forces $c = 0$. So every nonzero element of $\mathcal{F}^{\mathcal{E}}_{(a,b)}(2)$ has order 0.

(2) The proof uses only that φ is bihomogeneous with $\mathcal{E}\varphi = 0$, so the bound holds for every such $\varphi \in \mathbb{C}[V]$. Since $\operatorname{gr}_{\mathfrak{p}}$ is a domain, the order is additive; for $n \geq 3$ the bound is attained also at $(4m+2, m)$ and $(4m+3, m)$, by $w_{12}^m P_2$ and $w_{12}^m P_3$. To leading order along $\mathcal{S}$ every element of $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ is a combination $\sum_J c_J w_J$, with GL_n-invariant coefficients $c_J \in \mathbb{C}[\mathcal{S}]$ (apply the Reynolds operator to (iii)), of products of elements of the first layer.

(3) Corollary 9.56 does not show that the rows of the test (S1) with the corner $B_1 = \mathbf{1}\mathbf{1}^T$ at the points $(\operatorname{diag} x, Z)$ of $\mathcal{S}$ alone decide the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$; we do not know whether they do, and by the example after Lemma 9.54 an argument of this kind must let x vary, or treat the multiples by elements of R separately. (Computations, not used.) The orders of Lemma 9.54, measured for $n = 4$ with a fresh random x for each sample, are within $\lfloor a/4 \rfloor$: every nonzero element has order 0 at $(a,b) = (3,3)$, $(4,3)$, $(5,3)$, $(4,4)$, $(5,4)$ and $(6,4)$; at $(6,3)$ and $(7,3)$ the elements of positive order form subspaces of dimensions 3 and 4, of order 1; at $(8,2)$ the part of order 2 is spanned by w_{12}^2. In 24 cases with $b = 2, 3, 4$ and $n = 4, \dots, 10$ the eigenvalues of the pencil from the rows with the corner $B_1 = \mathbf{1}\mathbf{1}^T$ at points of $\mathcal{S}$ alone are those from all rows, and every eigenvector found has order 0; with $u = e_1$ (not cyclic for $\operatorname{diag} x$) in place of $\mathbf{1}$ these rows are not enough.

9.15. **Lowering.** In this subsection $F = \mathbb{C}$ and $n \geq 3$. We show that the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ has the eigenvalue $\kappa = (a+3)/n$, that is $\mu = 3$ in the notation of Subsection 9.13, for every $a \geq 3$ and $n \geq a+2$ (Corollary 9.63), with an explicit eigenvector: the image of the line of Theorem 9.37 under a power of the lowering operator of $\mathfrak{gl}_3$. For $a = 3$ the eigenvector is a multiple of $n\varphi_1 + 6\varphi_2$, for every $n \geq 4$ (Corollary 9.61). We do not show here that there is no other eigenvalue (Conjecture 11.3; for $n \geq a+6$ see Theorem 9.150), that $\mu = 0$ occurs (that is shown in Subsection 9.20), or that the eigenvector spans the eigenspace of $\mu = 3$ when $a \geq 4$.

Setting. Let $\mathcal{N} \subset \mathrm{GL}_n$ be a convex neighbourhood of I that is stable under transposition (for example $\|P_1 - I\| < \frac{1}{2}$), and fix, for $s \in \mathbb{C}$, the branch of $\det^s$ on $\mathcal{N}$ with $\det(I)^s = 1$. For $g \in \mathcal{N}$ and Q_1 in a ball $\mathcal{B} \subseteq \mathcal{N}$ around I with $g\mathcal{B} \subseteq \mathcal{N}$ we have $\det(gQ_1)^s = \det(g)^s \det(Q_1)^s$: both sides are branches of $\det(gQ_1)^s$ on the connected set $\mathcal{B}$, so their ratio is continuous with values in the countable set $\{e^{2\pi i j s} : j \in \mathbb{Z}\}$, hence constant, and it is 1 at $Q_1 = I$. In the same way, $\det(P_1^T)^s = \det(P_1)^s$ on $\mathcal{N}$. For $k \geq 2$ and $1 \leq i, j \leq k$ let $E_{ij} := \sum_{r,c} (P_i)_{rc}\, \partial/\partial (P_j)_{rc}$ on functions of k matrices $P = (P_1, \dots, P_k)$, so that $E_{ij} f(P) = \frac{d}{dt} f(\dots, P_j + tP_i, \dots)|_{t=0}$ for $i \neq j$, E_{ii} is the Euler operator in P_i, and $[E_{ij}, E_{pq}] = \delta_{jp}E_{iq} - \delta_{qi}E_{pj}$. The E_{ij} with $i < j$ are the raising operators of Subsection 9.7 (there the highest weight conditions are $E_{ij} f = 0$ for $i < j$), and $E_{21} f(P) = \frac{d}{dt} f(P_1 + tP_2, P_2, \dots)|_{t=0}$ is the lowering operator. For a holomorphic f on an open set of M_n^k let $\Omega f(P)$ be the alternating k-form on M_n

$$\Omega f(P)(B_1, \dots, B_k) := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, [t_1 \cdots t_k]\, f\big(P_1 + t_1 B_{\sigma(1)}, \dots, P_k + t_k B_{\sigma(k)}\big),$$

as in Subsection 10.2; (S1) and (S2) are values of $\Omega f(P)$.

Theorem 9.58 (lowering)**.** *Let $k \geq 2$, let $\mathcal{U} \subseteq M_n^k$ be open, and let Π be a linear map on the space of k-forms on M_n that does not depend on P; say that f passes Π on $\mathcal{U}$ if $\Pi(\Omega f(P)) = 0$ for every $P \in \mathcal{U}$. Let v be holomorphic on $\mathcal{U}$, with $E_{ij} v = 0$ for $i < j$ and $E_{ii} v = \tau_i v$ for all i, and suppose that v passes Π on $\mathcal{U}$. If $m := \tau_1 - \tau_2 + 1$ is a positive integer, then $E_{21}^m v$ has the same properties, with the weight $\tau - m(e_1 - e_2)$.*

Examples of Π are the evaluation at fixed $B_1, \dots, B_k$ (so (S1) and (S2), for each fixed u, y_1, y_2, y_3, are of this form) and the restriction to a fixed subspace of $\Lambda^k M_n$, as in (13). The function $E_{21}^m v$ may be 0.

Proof. (i) Write $\partial_i[B]$ for the derivative in P_i in the direction B, so that

$$\Omega f(P)(B_1, \dots, B_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_i \partial_i[B_{\sigma(i)}] f.$$

As E_{21} is linear in P_2, $\partial_2[B]E_{21} = E_{21}\partial_2[B] + \partial_1[B]$, and $\partial_i[B]$ commutes with E_{21} for $i \neq 2$. So $\Omega(E_{21}f) - E_{21}(\Omega f)$ is the alternating sum with $\partial_2[B_{\sigma(2)}]$ replaced by $\partial_1[B_{\sigma(2)}]$; this has two derivatives in slot 1, is symmetric in $(\sigma(1), \sigma(2))$, and alternates to 0. As Π does not depend on P, $\Pi\Omega(E_{21}f) = E_{21}(\Pi\Omega f)$ on $\mathcal{U}$, so $E_{21}^m v$ passes Π on $\mathcal{U}$. (ii) E_{12}, E_{21} and $E_{11} - E_{22}$ span a copy of $\mathfrak{sl}_2$, and

$$E_{12}E_{21}^m v = m(\tau_1 - \tau_2 - m + 1)\, E_{21}^{m-1} v = 0$$

[Hu72, §7.2, Lemma (c)] (its proof uses only the commutation relations, so it holds in every module). For $j \geq 3$, $[E_{1j}, E_{21}] = -E_{2j}$ and $[E_{2j}, E_{21}] = 0$, and $E_{1j}v = E_{2j}v = 0$, so $E_{1j}E_{21}^m v = 0$; the E_{ij} with $3 \leq i < j$ commute with E_{21}. The weight of $E_{21}^m v$ is $\tau - m(e_1 - e_2)$. □

From now on $\mathcal{U} := \mathcal{N} \times M_n^{k-1}$; the transposition $\mathsf{T}(P) := (P_1^T, \dots, P_k^T)$ maps $\mathcal{U}$ to itself. For $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ and $s \in \mathbb{C}$, *the function built from φ with exponent s* is $f_s(P) := \det(P_1)^s \tilde{\varphi}(P_1^{-1}P_2, P_1^{-1}P_3)$ on $\mathcal{U}$ ($k = 3$), as in Subsection 9.11. It is a highest weight vector of weight $\tau = (ns - d, a, b)$, for every s: $E_{12}f_s = E_{13}f_s = 0$ as $\tilde{\varphi}$ sees only X_0 and Y_0, $E_{23}f_s = 0$ as $\mathcal{E}\varphi = 0$, and $E_{ii}f_s = \tau_i f_s$.

Remark 9.59. (1) Theorem 9.58 is classical in substance: part (i) of the proof is the infinitesimal unipotent case of the equivariance $\Omega(f \circ g) = \det(g)\,(\Omega f) \circ g$ of Cayley's Ω-process (for $k = 2$ see [Ol99, Ch. 5]; the general case is the same chain-rule computation), and part (ii) is the rule for singular vectors in Verma modules [Ve68, BGG71]; see also [Hu08]. What is new here is the application: applied to functions f_s that are not polynomials, with a source for every s (Theorem 9.37), it gives eigenvalues of the pencil (Proposition 9.60, Corollaries 9.61 and 9.63) and functions that pass the test of $\nu = (2, 2)$ (Corollary 9.64).

(2) Put $\mu := \tau_1 + k - 1$; for $k = 3$, $b = 2$ and $\kappa = s$ this is $\mu = n\kappa - a$ of Subsection 9.13, and for $k = 4$ it is a label only. A source v with $\mu = 0$ gives nothing, as then $m = 2 - k - \tau_2 \leq 0$. A source that passes Π for every s gives, for each $m \geq 1$, a target at $ns_0 = d + \tau_2 + m - 1$, with $\mu' = \tau_2 + k - 2$ and $\tau_1' - \tau_2' = -m - 1 < 0$; as a polynomial highest weight vector of a weight that is not a partition is 0, a nonzero target is never a polynomial.

(3) If v is a polynomial in P, it lies in a finite-dimensional GL_k-module, the $\mathfrak{sl}_2$-string through v has length $\tau_1 - \tau_2 + 1$, and $E_{21}^m v = 0$ for $m = \tau_1 - \tau_2 + 1$. So Theorem 9.58 gives new eigenvectors only from sources that are not polynomials, for example at $s_0 \notin \mathbb{Z}$, but also at $s_0 = 1$ when $n = a+3$ in Proposition 9.60; for $n = 6$ and $a = 3$, Corollary 9.61 gives $\varphi_{\mathrm{LW}} = -24(\varphi_1 + \varphi_2) \neq 0$, the kernel at $\kappa = 1$ of Remark 9.50(1).

Proposition 9.60. *Let $a \geq 3$, $s := (a+3)/n$, and $\varphi' := (a+1)\,c + 2n\,G \in \mathcal{F}^{\mathcal{E}}_{(2,2)}(n)$, with c and G as in Theorem 9.37. For a polynomial ψ on $M_n \times M_n$ let*

$$(T_s\psi)(X, Y) := s \operatorname{tr}(X)\, \psi(X, Y) - \frac{d}{d\epsilon}\, \psi\big(X + \epsilon X^2,\, Y + \epsilon XY\big)\Big|_{\epsilon=0},$$

put $\psi_0 := \tilde{\varphi}'$ and $\psi_{j+1} := T_s\psi_j$, and let φ_{LW} be the restriction of ψ_{a-2} to $\mathfrak{sl}_n \times \mathfrak{sl}_n$. Then $\varphi_{\mathrm{LW}} \in \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, $\tilde{\varphi}_{\mathrm{LW}} = \psi_{a-2}$, and $s\,\mathcal{Z}(\varphi_{\mathrm{LW}}) = \mathcal{J}(\varphi_{\mathrm{LW}})$ at every base point and for all u, y_1, y_2, y_3; the function built from φ_{LW} with exponent s passes (S1) and (S2) at every point of $\mathcal{U}$. So, if $\varphi_{\mathrm{LW}} \neq 0$, then $\kappa = (a+3)/n$ is an eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, with eigenvector φ_{LW}.

(For $a = 2$ the same construction has $m = 0$, and $\varphi_{\mathrm{LW}} = \varphi'$ is the line itself.)

Proof. Let f' be the function built from φ' with exponent s, of weight $\tau=(ns-4,2,2)=(a-1,2,2)$, and put $m:=a-2=\tau_1-\tau_2+1$.

(1) *T_s is E_{21} in the chart.* For a polynomial ψ let $h(P):=\det(P_1)^s\psi(X,Y)$ on $\mathcal{U}$, with $X:=P_1^{-1}P_2$ and $Y:=P_1^{-1}P_3$. Then $P_1+tP_2=P_1(I+tX)$, $\det(P_1(I+tX))^s=\det(P_1)^s\det(I+tX)^s$ (Setting), $\frac{d}{dt}\det(I+tX)^s|_{t=0}=s\operatorname{tr}X$, $(I+tX)^{-1}X=X-tX^2+O(t^2)$ and $(I+tX)^{-1}Y=Y-tXY+O(t^2)$. So $E_{21}h=\det(P_1)^s(T_s\psi)(X,Y)$, and $E_{21}^mf'=\det(P_1)^s\psi_m(P_1^{-1}P_2,P_1^{-1}P_3)$.

(2) *f' passes (S1) on $\mathcal{U}$.* The proof of Theorem 9.37 gives $(\kappa\mathcal{Z}-\mathcal{J})((n\kappa-2)c+2nG)=0$ at every base point, for every $\kappa\in\mathbb{C}$; at $\kappa=s$ this function is φ'. By Lemma 9.30 (which holds for every $s\in\mathbb{C}$), f' passes (S1) at the points with $P_1=I$. For $P\in\mathcal{U}$ write $P=gQ$ with $g:=P_1$ and $Q:=(I,g^{-1}P_2,g^{-1}P_3)$. Then $f'(gQ')=\det(g)^sf'(Q')$ for Q' near Q (Setting), and $P_i+tuy^T=g(Q_i+t(g^{-1}u)y^T)$; so (S1) for f' at P with u is $\det(g)^s$ times (S1) at Q with $g^{-1}u$, which is 0 (as in the proof of Corollary 9.31(b)).

(3) By (2) and Theorem 9.58, applied for each fixed u,y_1,y_2,y_3 with Π the evaluation at $B_i=uy_i^T$, the function E_{21}^mf' passes (S1) on $\mathcal{U}$, is killed by E_{12}, E_{13} and E_{23}, and has weight $\tau-m(e_1-e_2)=(1,a,2)$.

(4) Both terms of T_s preserve conjugation invariance, so ψ_m is a conjugation-invariant polynomial. As E_{12}, E_{13}, E_{23} kill E_{21}^mf', (1) at $P_1=I$ gives $\psi_m(X+tI,Y)=\psi_m(X,Y+tI)=\psi_m(X,Y+tX)=\psi_m(X,Y)$ for all $X,Y\in M_n$, and by the weight ψ_m has bidegree $(a,2)$. Hence $\psi_m=\tilde{\varphi}_{\mathrm{LW}}$ with $\varphi_{\mathrm{LW}}\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, and E_{21}^mf' is the function built from φ_{LW} with exponent s. At $P_1=I$, (S1) for it is $(s\mathcal{Z}-\mathcal{J})(\varphi_{\mathrm{LW}})$ by Lemma 9.30, which is 0 by (3).

(5) *(S2).* The test (S2) is of the form of Theorem 9.58 ($B_i=y_iu^T$), and (S2) for a function h at P is (S1) for $h\circ\mathsf{T}$ at $\mathsf{T}(P)$. As $\det(P_1^T)^s=\det(P_1)^s$ on $\mathcal{N}$ (Setting) and $(P_1^T)^{-1}P_i^T=(P_iP_1^{-1})^T$ is conjugate to $(P_1^{-1}P_i)^T$ by P_1^{-T}, simultaneously for $i=2,3$, the conjugation invariance of $\tilde{\varphi}'$ shows that $f'\circ\mathsf{T}$ is the function built from φ'^T; and $\varphi'^T=\varphi'$, as $c^T=c$ and $G^T=G$. So f' passes (S2) on $\mathcal{U}$ by (2), and E_{21}^mf' passes (S2) on $\mathcal{U}$ by Theorem 9.58. □

There is no consequence for rectangles: the weight $(1,a,2)$ is not a partition (compare Remark 9.50(1)).

Corollary 9.61. *For $a=3$, $\varphi_{\mathrm{LW}}=-4(n\varphi_1+6\varphi_2)$. Hence, for every $n\geq 4$, $\kappa=6/n$ is an eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(3,2)}(n)$, with eigenspace $\mathbb{C}(n\varphi_1+6\varphi_2)$.*

Proof. Here $m=1$, $s=6/n$ and $\varphi'=4c+2nG$. At traceless A,B the term $s\operatorname{tr}(A)\psi_0$ vanishes, so $\varphi_{\mathrm{LW}}(A,B)=-\frac{d}{d\epsilon}\varphi'\big(A+\epsilon(A^2)_0,B+\epsilon(AB)_0\big)|_{\epsilon=0}$. For c, with $C:=[A,B]$: $\frac{d}{d\epsilon}[A+\epsilon A^2,B+\epsilon AB]=[A^2,B]+[A,AB]=(AC+CA)+AC$, so $\frac{d}{d\epsilon}\operatorname{tr}(C^2)=2\operatorname{tr}(C(2AC+CA))=6\operatorname{tr}(AC^2)=6\varphi_2$. For G: $\frac{d}{d\epsilon}\operatorname{tr}(A^2)=2\operatorname{tr}(A^3)$, $\frac{d}{d\epsilon}\operatorname{tr}(B^2)=2\operatorname{tr}(AB^2)$ and $\frac{d}{d\epsilon}\operatorname{tr}(AB)=2\operatorname{tr}(A^2B)$, so $\frac{d}{d\epsilon}G=2\big(\operatorname{tr}(A^3)\operatorname{tr}(B^2)+\operatorname{tr}(A^2)\operatorname{tr}(AB^2)-2\operatorname{tr}(AB)\operatorname{tr}(A^2B)\big)=2\varphi_1$. So $\varphi_{\mathrm{LW}}=-(4\cdot 6\varphi_2+2n\cdot 2\varphi_1)=-4(n\varphi_1+6\varphi_2)$, which is nonzero for $n\geq 4$, as φ_1 and φ_2 are independent (Theorem 9.48). Proposition 9.60 gives the eigenvalue, and Theorem 9.48 gives that the eigenspace is at most this line. □

So Theorem 9.48 has, for every $n\geq 4$, the eigenvalue $\mu=3$ with a proof; $\mu=0$ has one by Corollary 9.100.

Lemma 9.62. *If $n\geq a+2$, then $\varphi_{\mathrm{LW}}\neq 0$.*

Proof. Let $X=\operatorname{diag}(x)$ with distinct x_l, and let Y have zero diagonal. As for the torus form (1), a conjugation-invariant polynomial ψ on $M_n\times M_n$ that is homogeneous of degree 2 in Y has there the form

$$\psi(X,Y)=\sum_{r\neq t}\mathsf{F}_{rt}Y_{rt}Y_{tr},$$

with $\mathsf{F}_{rt}=\mathsf{F}_{tr}$, and $\mathsf{F}_{rt}=\mathsf{F}(x_r,x_t)$ for a polynomial $\mathsf{F}(u,v)$ in u, v and the power sums $p_j:=\sum_l x_l^j$ (conjugation by permutation matrices shows that F_{rt} is F_{12} with the coordinates permuted, and

that F_{12} is symmetric in $x_3,\dots,x_n$, hence a polynomial in x_1, x_2 and the $p_j - x_1^j - x_2^j$). We call F the data of ψ. As X^2 is diagonal and XY has zero diagonal, T_s maps these functions to functions of the same kind, and the data of $T_s\psi$ can be taken to be

$$s\,p_1\mathsf{F} - \mathcal{D}\mathsf{F} - (u+v)\mathsf{F}, \qquad \mathcal{D} := u^2\partial_u + v^2\partial_v + \textstyle\sum_j j\,p_{j+1}\partial_{p_j} :$$

the first term is $s\,\mathrm{tr}(X)\psi$; the second is the derivative along $X^2 = \mathrm{diag}(x_l^2)$, as $\sum_m x_m^2\partial_{x_m}p_j = jp_{j+1}$; the third is $\frac{d}{d\epsilon}\psi(X, Y+\epsilon XY) = \sum_{r\neq t}(x_r+x_t)\mathsf{F}_{rt}Y_{rt}Y_{tr}$. Give u and v the weight 1 and p_j the weight j. The data of ψ_0 have weight 2 (below), and each of the three terms raises the weight by exactly 1, so the data of ψ_{a-2} are weighted homogeneous of weight a and involve only $p_1,\dots,p_a$.

Let $\mathsf{F}^\circ \in \mathbb{C}[u,v]$ be the image of F modulo the ideal $(p_1,p_2,\dots)$. This ideal contains $sp_1\mathsf{F}$ and is mapped into itself by $\mathcal{D}$, so the data of $T_s\psi$ have F° equal to $O\mathsf{F}^\circ$, with $O := -\big((u+v)+u^2\partial_u+v^2\partial_v\big)$. The operator O is injective: if u^iv^j is the lexicographically leading monomial of a polynomial ($u>v$), the leading monomial of its image is $u^{i+1}v^j$, with the coefficient multiplied by $-(i+1)$. For c, $[X,Y]_{rt} = (x_r-x_t)Y_{rt}$ gives the data $-(u-v)^2$; for G, $\langle Y,Y\rangle = \mathrm{tr}(Y^2) = \sum_{r\neq t}Y_{rt}Y_{tr}$ and $\langle X,Y\rangle = 0$ give the data $\langle X,X\rangle = p_2 - p_1^2/n$. So $\psi_0 = \tilde{\varphi}'$ has $\mathsf{F}^\circ = -(a+1)(u-v)^2$, and ψ_{a-2} has $\mathsf{F}^\circ = O^{a-2}\big(-(a+1)(u-v)^2\big) \neq 0$.

Finally, for $n \geq a+2$ the functions x_r, x_t, $p_1,\dots,p_a$ on $\mathbb{C}^n$ are algebraically independent: subtracting $x_r^j + x_t^j$ from p_j (a triangular change of variables) leaves the power sums $p_1,\dots,p_a$ of the other $n-2 \geq a$ coordinates. So the data of ψ_{a-2}, a nonzero polynomial in $u,v,p_1,\dots,p_a$, give a nonzero function of x; hence $\psi_{a-2}\neq 0$, and $\varphi_{\mathrm{LW}} \neq 0$, as $\tilde{\varphi}_{\mathrm{LW}} = \psi_{a-2}$ (Proposition 9.60). □

At $a=3$: $O(-4(u-v)^2) = 12(u+v)(u-v)^2$, which is F° of $-4(n\varphi_1+6\varphi_2)$ by the data in the proof of Theorem 9.48(b) (written there in the variables $z_l = x_l - p_1/n$).

Corollary 9.63. *For every $a\geq 3$ and every $n\geq a+2$, $\kappa = (a+3)/n$ (that is, $\mu = 3$) is an eigenvalue of the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, with eigenvector φ_{LW}.*

Proof. Proposition 9.60 and Lemma 9.62. □

(Computations, not used.) The computations of Remark 9.50(4) at $(a,n) = (5,11)$, $(5,12)$ and $(6,13)$ find a kernel of dimension 1 at $\mu = 3$.

The block $\nu = (2,2)$. The same mechanism works for other tests. Let $k=4$ and $\nu=(2,2)$. The live corners of ν are $(1,3)$ and $(3,1)$, and by Lemma 10.2(iii), (iv) a function f passes the corner tests $(T_{1,3})$ and $(T_{3,1})$ of Subsection 10.1 at P if and only if $\Omega f(P)$ vanishes on $\bigoplus_{\alpha\neq(2,2)} S_\alpha V\otimes S_{\alpha'}V$, a subspace of $\Lambda^4 M_n$ that does not depend on P. So these two tests together are a test Π as in Theorem 9.58; we call it the test of $\nu=(2,2)$.

Corollary 9.64. *Let $\varphi_0(X_2,X_3,X_4) := \mathrm{tr}(X_2[X_3,X_4])$ and, for $s\in\mathbb{C}$,*

$$f_s(P) := \det(P_1)^s\varphi_0\big(P_1^{-1}P_2, P_1^{-1}P_3, P_1^{-1}P_4\big) \quad \text{on } \mathcal{U} = \mathcal{N}\times M_n^3.$$

(a) For every $s\in\mathbb{C}$, f_s is a highest weight vector of weight $(ns-3,1,1,1)$, and it passes the test of $\nu=(2,2)$ on $\mathcal{U}$.

(b) For every integer $m\geq 1$ and $s_0 := (m+3)/n$, the function $E_{21}^m f_{s_0}$ is killed by the E_{ij} with $i<j$, has weight $(0,m+1,1,1)$, and passes the test of $\nu=(2,2)$ on $\mathcal{U}$.

We do not know whether $E_{21}^m f_{s_0}\neq 0$.

Proof. (a) φ_0 is alternating and invariant under $X_j\mapsto X_j+cI$, so $E_{1j}f_s = 0$, and $E_{ij}f_s = 0$ for $2\leq i<j$, as $X_j\mapsto X_j+tX_i$ adds a term with two equal arguments; the weight is clear. At $P_1 = I$, the computation of Lemma 10.5 (with four slots and the exponent s in place of n) gives the coefficient of $t_1\cdots t_4$ in $f_s(I+t_1W, A_2+t_2V_2,\dots,A_4+t_4V_4)$ as $s\,\mathrm{tr}(W)\varphi_0(V_2,V_3,V_4)$, minus double derivatives in one slot, which vanish as φ_0 is trilinear, minus the terms $\varphi_0(\dots,WV_j,\dots)$, which are traces of products of four of the matrices W,V_2,V_3,V_4. Alternated over the B's, each trace of a product of

four of them gives 0, as a cyclic permutation of four letters is odd. So $\Omega f_s(I,A) = 6s\,(\operatorname{tr}\wedge\varphi_0)$, with $(\operatorname{tr}\wedge\varphi_0)(B_1,\dots,B_4) := \sum_i (-1)^{i-1}\operatorname{tr}(B_i)\varphi_0(\hat{B}_i)$. This form is invariant under conjugation, which acts on $S_\alpha V\otimes S_{\alpha'}V$ as on $S_\alpha V\otimes S_{\alpha'}V^*$, and $(S_\alpha V\otimes S_{\alpha'}V^*)^{\mathrm{GL}_n} = \operatorname{Hom}(S_{\alpha'}V, S_\alpha V)^{\mathrm{GL}_n}$ is 0 unless $\alpha=\alpha'$. As $(2,2)$ is the only self-conjugate partition of 4, the form vanishes on $\bigoplus_{\alpha\neq(2,2)} S_\alpha V\otimes S_{\alpha'}V$. For general $P\in\mathcal{U}$, as in step (2) of the proof of Proposition 9.60, $\Omega f_s(gQ)(B_1,\dots,B_4) = \det(g)^s\,\Omega f_s(Q)(g^{-1}B_1,\dots,g^{-1}B_4)$ with $g=P_1$, and the subspace is stable under $B\mapsto g^{-1}B$. (b) Theorem 9.58 with $\tau=(ns_0-3,1,1,1)$ and $m=\tau_1-\tau_2+1=ns_0-3$. □

Remark 9.65. (1) The targets of Corollary 9.64(b) have $\mu=3$ in the sense of Remark 9.59(2), with $\lambda' := (\tau_2',\tau_3',\tau_4') = (m+1,1,1)$ and $ns_0=|\lambda'|$; as $\tau_1'<\tau_2'$, they are never polynomials. (2) (Computations, not used.) For $n=4,5$, the values of s at which the test of $\nu=(2,2)$, with eight random instances of each corner test, computed modulo a prime from the definition, has a nonzero kernel on the functions $\det(P_1)^s\tilde{\varphi}(P_1^{-1}P_2,P_1^{-1}P_3,P_1^{-1}P_4)$, with φ an invariant of multidegree λ' killed by the raising operators, were computed. The values with $\mu=3$ that were found are exactly $s=(m+3)/n$ for $\lambda'=(m+1,1,1)$, $m=1,2,3$ (that is, $s=1,\frac{4}{5};\ \frac{5}{4},1;\ \frac{3}{2},\frac{6}{5}$ for $n=4,5$), each with a kernel of dimension 1; for $\lambda'=(2,2,1)$ and $(3,2,1)$ only $\mu=0$ was found. That the kernel vectors are the functions $E_{21}^m f_{s_0}$ was not checked.

9.16. **One test suffices.** In this subsection $F=\mathbb{C}$. We show that on $\mathrm{R}_s(n)_\tau$ the tests (S1) and (S2) are equivalent, so that $\mathrm{K}_s(n)_\tau$ is cut out by (S1) alone (Theorem 9.69), and that the two Cauchy parts of the test that correspond to (S1) and (S2) give the same operator on $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$, for every exponent $s\in\mathbb{C}$ (Corollary 9.72). The proof is on the symmetric group side: each test says that an orthogonal projection kills a vector, and on a rectangle the two sums of these projections are equal, because the star sum $\sum_{j\neq i}(i\,j)$ is a scalar there.

Let $s,n\geq 1$, $R:=(s^n)$, $d:=ns$, and $\tau\vdash d$ with at most three rows. For a partition $\lambda\vdash d$ let $\rho=\rho_\lambda$ be Young's orthogonal form, so $\rho(g)^T=\rho(g^{-1})$. For a 3-set $K\subseteq[d]$ put $e^K_{\mathrm{tr}} := \frac{1}{6}\sum_{g\in S_K} g$ and $e^K_{\mathrm{sg}} := \frac{1}{6}\sum_{g\in S_K}\operatorname{sgn}(g)\,g$; both are inverse-closed, so their images under ρ are orthogonal projections. Let $V_\tau := (S^R\otimes S^R\otimes S^\tau)^{S_d}$ (diagonal action), with the product Hermitian inner product (ρ is real orthogonal, hence unitary), and

$$\Pi_K := \rho(e^K_{\mathrm{tr}})\otimes\rho(e^K_{\mathrm{sg}})\otimes 1, \qquad \Pi'_K := \rho(e^K_{\mathrm{sg}})\otimes\rho(e^K_{\mathrm{tr}})\otimes 1.$$

We identify $S^R\otimes S^R$ with $\operatorname{End}(S^R)$ by $x\otimes y\mapsto xy^T$, so that $A\otimes B$ acts as $M\mapsto AMB^T$. As in the proof of Corollary 5.10, $\mathcal{T} := \sum_{i<j}(i\,j)$ acts on S^λ by the content sum $c(\lambda)$; for the rectangle, $e_R := s-n$ is the content of its corner and $c(R)=d\,e_R/2$.

Lemma 9.66. *There is a linear isomorphism $\mathrm{R}_s(n)_\tau\to V_\tau$, $f\mapsto m_f$, such that f satisfies (S1) if and only if $\Pi_K m_f=0$, and (S2) if and only if $\Pi'_K m_f=0$, for one (equivalently, every) 3-set K.*

Proof. This is the multilinear model of Steps 1 and 3 of the proof of Theorem 9.18, which do not use that R is a square (as in the proof of Theorem 9.33); we give it in invariant form. (1) Write $M_n\otimes\mathbb{C}^3 = V\otimes W\otimes U$ ($V=W=\mathbb{C}^n$ for rows and columns, $U=\mathbb{C}^3$ for the slots). By Schur–Weyl duality on each factor, the polynomials of degree d are $\bigoplus_{\alpha,\beta,\gamma} S_\alpha V^*\otimes S_\beta W^*\otimes S_\gamma U^*\otimes (S^\alpha\otimes S^\beta\otimes S^\gamma)^{S_d}$. (2) The $\mathrm{SL}_n\times\mathrm{SL}_n$-invariance keeps only $\alpha=\beta=R$, and the multidegree and the highest weight condition keep only $\gamma=\tau$, with a line of highest weight vectors; so $f=h\otimes m_f$ with h fixed and $m_f\in V_\tau$. (3) The coefficient of $t_1t_2t_3$ in (S1) is the third derivative of f in the directions $uy^T_{\sigma(a)}\otimes e_a$; after the symmetrization over the diagonal S_3 that the symmetric tensor of f allows, the alternated tensor is a nonzero multiple of $u^{\otimes 3}\otimes(y_1\wedge y_2\wedge y_3)\otimes(e_1\wedge e_2\wedge e_3)$, and these span $\operatorname{Sym}^3V\otimes\Lambda^3W\otimes\Lambda^3U$. As the powers $P^{\otimes(d-3)}$ span the symmetric tensors in the other positions, (S1) holds if and only if $(e^K_{\mathrm{tr}}\otimes e^K_{\mathrm{sg}}\otimes e^K_{\mathrm{sg}})m_f=0$, with K the last three positions. (4) For $m\in V_\tau$ and $g\in S_K$, $(1\otimes 1\otimes g)m = (g^{-1}\otimes g^{-1}\otimes 1)m$; with $e^K_{\mathrm{tr}}g^{-1}=e^K_{\mathrm{tr}}$ and $e^K_{\mathrm{sg}}g^{-1}=\operatorname{sgn}(g)e^K_{\mathrm{sg}}$ this gives

$(e^K_{\mathrm{tr}} \otimes e^K_{\mathrm{sg}} \otimes e^K_{\mathrm{sg}})m = \Pi_K m$. For another 3-set πK, $\Pi_{\pi K} = (\pi\otimes\pi\otimes\pi)\Pi_K(\pi\otimes\pi\otimes\pi)^{-1}$ and m is invariant. (5) For (S2) exchange u and y, which gives $\Lambda^3 V \otimes \mathrm{Sym}^3 W \otimes \Lambda^3 U$, that is Π'_K. □

Lemma 9.67. *For each $i \in [d]$, the star element $X^{(i)} = \sum_{j\neq i}(i\,j)$ acts on S^R as the scalar $e_R = s - n$ (compare Theorem C).*

Proof. Let $\mathcal{T}_i$ be the sum of the transpositions that fix i; then $X^{(i)} = \mathcal{T} - \mathcal{T}_i$. The restriction of S^R to the stabilizer S_{d-1} of i is irreducible, of the shape R minus its only removable cell, so $\mathcal{T}_i$ acts by $c(R)$ minus the content e_R of that cell. This is the classical fact that the Jucys–Murphy element J_d acts by the content of the cell of d [Ju74, Mu81, OV96], as d always sits in the corner of a standard tableau of shape R. □

Lemma 9.68. *For every $M \in \mathrm{End}(S^R)$,*

$$\sum_K \rho(e^K_{\mathrm{tr}})\, M\, \rho(e^K_{\mathrm{sg}}) = \sum_K \rho(e^K_{\mathrm{sg}})\, M\, \rho(e^K_{\mathrm{tr}}).$$

Proof. For $A, B \in S_3$ put $D[A,B](M) := \sum_\iota \rho(\iota(A))\, M\, \rho(\iota(B))^T$, the sum over the $d(d-1)(d-2)$ injections $\iota\colon \{1,2,3\} \to [d]$, where $\iota(A)$ acts as A on $\iota(\{1,2,3\})$ through ι and fixes the other points. Relabelling ι gives $D[A,B] = D[\pi A\pi^{-1}, \pi B\pi^{-1}]$ for $\pi \in S_3$. Each 3-set has six injections onto it, and $36\,(e^K_{\mathrm{tr}} \otimes e^K_{\mathrm{sg}} - e^K_{\mathrm{sg}} \otimes e^K_{\mathrm{tr}}) = \sum_{g,h\in S_K} (\operatorname{sgn} h - \operatorname{sgn} g)\, g \otimes h$, to which only the pairs (even, odd) and (odd, even) contribute. Under simultaneous conjugation the pairs (even, odd) of S_3 form two classes, (id, t) (three pairs) and $(c^{\pm 1}, t)$ (six pairs), with t a transposition and c a 3-cycle. With $\rho(e)^T = \rho(e)$ the lemma is therefore $3D[t,\mathrm{id}] + 6D[t,c] = 3D[\mathrm{id},t] + 6D[c,t]$, and it suffices to show $D[\mathrm{id},t] = D[t,\mathrm{id}]$ and $D[c,t] = D[t,c]$. First, $\sum_\iota \iota(t) = 2(d-2)\mathcal{T}$, so both $D[\mathrm{id},t](M)$ and $D[t,\mathrm{id}](M)$ are $2(d-2)c(R)M$. Second, take $c = (1\,2\,3)$ and $t = (1\,2)$; for $\iota = (i,j,l)$, $\iota(c) = (i\,l)(i\,j)$ and $\iota(c)^{-1} = (i\,j)(i\,l)$. With $\mathrm{Ad}_{\mathcal{T}}(M) := \sum_t \rho(t) M \rho(t)$ and Lemma 9.67,

$$D[c,t](M) = \sum_{i\neq j} \rho(X^{(i)} - (i\,j))\rho(i\,j) M \rho(i\,j) = 2e_R\, \mathrm{Ad}_{\mathcal{T}}(M) - 2c(R)M,$$

and in the same way $D[t,c](M) = \sum_{i\neq j} \rho(i\,j) M \rho(i\,j)\rho(X^{(i)} - (i\,j)) = 2e_R\, \mathrm{Ad}_{\mathcal{T}}(M) - 2c(R)M$. □

Theorem 9.69. *Let $s, n \geq 1$ and $\tau \vdash ns$ with at most three rows. Then $f \in \mathrm{R}_s(n)_\tau$ satisfies (S1) if and only if it satisfies (S2). Hence $\mathrm{K}_s(n)_\tau = \{f \in \mathrm{R}_s(n)_\tau : \text{(S1)}\}$, and, for $s = n$, by Theorem 9.18, the multiplicity of κ in $K_{B(n,2)}$ is $\dim\{f \in \mathrm{Pol}_\tau : \text{(S1)}\}$, with $\tau = \kappa' + (1,1,1)$.*

Proof. Lemma 9.68, read on $S^R \otimes S^R = \mathrm{End}(S^R)$ and tensored with the identity on S^τ, gives $\sum_K \Pi_K = \sum_K \Pi'_K$. For $m \in V_\tau$,

$$\textstyle\sum_K |\Pi_K m|^2 = \langle m, \sum_K \Pi_K m\rangle = \langle m, \sum_K \Pi'_K m\rangle = \sum_K |\Pi'_K m|^2.$$

As m is S_d-invariant, $|\Pi_{\pi K} m| = |\Pi_K m|$ and $|\Pi'_{\pi K} m| = |\Pi'_K m|$ for all $\pi \in S_d$; so $|\Pi_K m| = |\Pi'_K m|$ for every K, and Lemma 9.66 gives the claim. □

Remark 9.70. (1) $\rho_R(e^K_{\mathrm{tr}}) = 0$ unless R has at least three columns, and $\rho_R(e^K_{\mathrm{sg}}) = 0$ unless it has at least three rows; so both tests are empty unless $s, n \geq 3$, in accordance with Proposition 9.34, and the smallest case with content is $R = (3,3,3)$. (2) Tensoring both factors with the sign character maps (S1) for (R,R) to (S2) for (R',R'), $R' = (n^s)$, and $g(R,R,\tau) = g(R',R',\tau)$ (Theorem 9.33); at $s = n$ this gives equal dimensions of the two kernels for free. Theorem 9.69 says that they are the same subspace. (3) The argument gives more on the whole space: a sum of positive operators has the intersection of their kernels as its kernel, so $\bigcap_K \ker \Pi_K = \bigcap_K \ker \Pi'_K$ on $S^R \otimes S^R \otimes S^\tau$, without invariance (we do not use this). For one fixed K, the two conditions can differ on $S^R \otimes S^R \otimes S^\tau$; on V_τ they coincide. (4) The rectangle is essential. The computation of Lemma 9.68 for any λ in place of R gives

$$\textstyle\sum_K \rho(e^K_{\mathrm{tr}}) M \rho(e^K_{\mathrm{sg}}) - \sum_K \rho(e^K_{\mathrm{sg}}) M \rho(e^K_{\mathrm{tr}}) = -\frac{1}{18}\sum_i \big[\rho(X^{(i)}), \sum_{j\neq i} \rho(i\,j) M \rho(i\,j)\big],$$

which is in general not 0; and with two rectangles R_1 (rows) and R_2 (columns) of the same size, $\sum_K \Pi_K - \sum_K \Pi'_K = -(e_{R_1}-e_{R_2})(4\,\mathrm{Ad}_{\mathcal{T}} - d(d-4))/36$, with $\mathrm{Ad}_{\mathcal{T}} = \sum_t \rho_{R_1}(t)\otimes\rho_{R_2}(t)$; on V_τ, $\mathrm{Ad}_{\mathcal{T}} = c(\tau)$, so there this is the scalar $-(e_{R_1}-e_{R_2})(4c(\tau)-d(d-4))/36$, in general not 0. (5) Lemma 9.67 is classical; we did not find Lemma 9.68, Theorem 9.69 or Corollary 9.72 in the literature. The theory of Capelli operators for multiplicity-free actions [HU91] does not apply directly, as the action of $\mathrm{GL}_n \times \mathrm{GL}_n \times \mathrm{GL}_3$ on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^3$ is not multiplicity free. (6) (Computations, not used.) In computations modulo primes with sampled rows at integer s, $n = 3,\dots,6$ (112 pairs of n and a type), the kernels with and without the rows of (S2) agree (some of these runs are on a space larger than $\mathrm{R}_s(n)_\tau$, outside Theorem 9.69); so do direct computations of $\mathrm{K}_s(n)_\tau$ from f_s at $(n,s) = (3,4),(4,3),(3,5),(5,3)$ (22 cases).

A computation may therefore drop the rows of (S2) when its search space is exactly $\mathrm{R}_s(n)_\tau$; this does not cover a computation on a larger space, or with finitely many sampled rows.

The pencil. Let π_3, π_{21}, π_{111} be the projections of $\Lambda^3(V\otimes W) = \mathrm{Sym}^3 V\otimes\Lambda^3 W \oplus S_{21}V\otimes S_{21}W \oplus \Lambda^3 V\otimes \mathrm{Sym}^3 W$ onto its summands; on rank-one arguments, π_3 symmetrizes the row vectors,

$$(\pi_3 T)(u_1a_1^T, u_2a_2^T, u_3a_3^T) = \tfrac{1}{6}\sum_{\varrho\in S_3} T(u_{\varrho(1)}a_1^T, u_{\varrho(2)}a_2^T, u_{\varrho(3)}a_3^T),$$

π_{111} is the same with $\mathrm{sgn}(\varrho)$, and $\pi_{21} = 1-\pi_3-\pi_{111}$. For a function f of $P=(P_1,P_2,P_3)$ put $\mathcal{C}_\lambda f(P) := (\pi_\lambda \Omega f(P))(P_1,P_2,P_3)$, with $\Omega f(P)$ as in Subsection 9.15; for $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ and $s\in\mathbb{C}$ let $C_\lambda(s)\varphi$ be $\mathcal{C}_\lambda f_s$ at $P=(I,X,Y)$, as a function of (X,Y), where f_s is built from φ with exponent s. By Lemma 9.30, $C_\lambda(s) = s\,\mathcal{Z}_\lambda - \mathcal{J}_\lambda$ is affine in s, with $\mathcal{Z}_\lambda\varphi := (\pi_\lambda(\mathrm{tr}\wedge\omega))(I,X,Y)$ and $\mathcal{J}_\lambda\varphi := (\pi_\lambda(\mathfrak{L}\omega))(I,X,Y)$ at the base point (X,Y) (Subsection 9.11). Put $\mathrm{cap}(\tau) := (\tau_1+2)(\tau_2+1)\tau_3$, with $\tau = (ns-a-b,a,b)$; for non-integer s it is the affine expression $(ns-a-b+2)(a+1)b$.

Lemma 9.71. *For $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ and every $s\in\mathbb{C}$, $C_3(s)\varphi + C_{21}(s)\varphi + C_{111}(s)\varphi = \mathrm{cap}(\tau)\varphi$.*

Proof. The left side is $\mathcal{C}f_s = \Omega f_s(P)(P_1,P_2,P_3) = \sum_\sigma \mathrm{sgn}(\sigma)\,{:}E_{\sigma(1),1}E_{\sigma(2),2}E_{\sigma(3),3}{:}\,f_s$ at $P=(I,X,Y)$ (the polarization operators E_{ab} of Subsection 9.15, normally ordered: the P's to the left). By Capelli's identity (in the column-determinant form; [HU91, §1]) it is the column determinant of $E+\mathrm{diag}(2,1,0)$, which acts on a highest weight vector of weight τ by $\prod_i(\tau_i+3-i) = \mathrm{cap}(\tau)$. For integers $s\ge a+b$ this applies to the polynomial f_s (Lemma 9.32(ii)); both sides are affine in s, so it holds for every s. □

Corollary 9.72. *For every $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ and every $s\in\mathbb{C}$, $C_3(s)\varphi = C_{111}(s)\varphi$. The maps $C_\lambda(s)$ send $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ into itself, and $C_{21}(s) = \mathrm{cap}(\tau) - 2C_3(s)$. At an integer s, $\mathcal{C}_3$ acts on $\mathrm{R}_s(n)_\tau$ as $6\sum_K \Pi_K \ge 0$, with kernel $\mathrm{K}_s(n)_\tau$.*

Proof. Tensor model. In the notation of the proof of Lemma 9.66, the three slot derivatives in $\mathcal{C}_3 f$ remove three tensor positions; π_3 is e_{tr} on their V-parts, and the slot alternation, after the diagonal symmetrization, is e_{sg} on their U-parts; on symmetric tensors this is $e_{\mathrm{tr}}\otimes e_{\mathrm{sg}}\otimes e_{\mathrm{sg}}$ on these positions. Reading the result as a polynomial symmetrizes over S_d, which gives $c_d\sum_K(e^K_{\mathrm{tr}}\otimes e^K_{\mathrm{sg}}\otimes e^K_{\mathrm{sg}})$ with $c_d = d(d-1)(d-2)/\binom{d}{3} = 6$; on V_τ this is $6\sum_K\Pi_K$ (step (4) of that proof). Likewise $\mathcal{C}_{111} = 6\sum_K\Pi'_K$ on V_τ. *Equality.* Let $s\ge a+b$ be an integer. Then $f_s = \det(P_1)^{s-a-b}\tilde{\varphi}(\mathrm{adj}(P_1)P_2, \mathrm{adj}(P_1)P_3)$ is a polynomial in $\mathrm{R}_s(n)_\tau$, $\tau = (ns-a-b,a,b)$ (Lemma 9.32(ii), case $a+b\le s$; τ is a partition, as $a\ge b$ and $ns\ge 2(a+b)\ge 2a+b$ for $n\ge 2$). By the tensor model and Lemma 9.68, $\mathcal{C}_3 f_s = \mathcal{C}_{111} f_s$ as polynomials; at $P=(I,X,Y)$ this is $C_3(s)\varphi = C_{111}(s)\varphi$. Both sides are affine in s, so they agree for every s. *Into $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$.* At these s, $\mathcal{C}_3$ and $\mathcal{C}_{111}$ map $\mathrm{R}_s(n)_\tau$ into itself (tensor model), and so does $\mathcal{C}_{21}$ by Lemma 9.71; the values at $P_1 = I$ of elements of $\mathrm{R}_s(n)_\tau$ lie in $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ (Lemma 9.32(ii)). So $C_\lambda(s)$ maps $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ into itself at these s; as $C_\lambda(s) = s\,\mathcal{Z}_\lambda - \mathcal{J}_\lambda$, so do $\mathcal{Z}_\lambda$ and $\mathcal{J}_\lambda$, hence $C_\lambda(s)$ for every s. The last statement is the tensor model with Lemma 9.66. □

Equal spectra of $C_3(s)$ and $C_{111}(s)$ would be automatic (they are exchanged by the transposition $\varphi \mapsto \varphi^T$, as in Remark 9.70(2)); Corollary 9.72 says that the operators are equal. (The argument also gives $C_3(s) = C_{111}(s)$ on every conjugation-invariant φ, by Lemma 9.68 for every γ in place of τ; we do not use this.)

9.17. **The mirror $n \mapsto -n$.** In this subsection $F = \mathbb{C}$ and $b = 2$, and we keep the notation of Subsection 9.13. Remark 9.38(e) records that the kernels of the pencil at $\kappa = 2$ and $\kappa = -2$ are exchanged by $n \mapsto -n$ together with the sign $(-1)^{\text{number of trace factors}}$ of each trace monomial. We prove this at $b = 2$ for every a (Theorem 9.75), for a formal system whose specializations at the integers $n \geq a + 6$ are the pencils, and deduce that $\kappa = -2$ is an eigenvalue for every even $a \geq 4$ and every $n \geq a+6$ (Corollary 9.77). The mechanism is a $\mathbb{Z}/2$-grading in which n, κ and the power sums are odd: every row of the test splits into two parts of opposite parity, and on the kernel both parts vanish.

The formal system. Let $\mathcal{M}$ be the set of trace monomials m in (A, B) of bidegree $(a, 2)$ without a factor $\operatorname{tr} A$ or $\operatorname{tr} B$, let $k(m)$ be the number of trace factors of m, and for $\mathbf{c} \in \mathbb{C}^{\mathcal{M}}$ put $\varphi_{\mathbf{c}} := \sum_m c_m\, m$ and $(\sigma\mathbf{c})_m := (-1)^{k(m)} c_m$. As in the proof of Theorem 9.48, $z_l := x_l - p_1/n$ and $P_k := \sum_l z_l^k$, so $P_0 = n$ and $P_1 = 0$. For each m there are symmetric polynomials $\mathsf{F}_m(s,t)$ and $\mathsf{G}_m(s,t)$, with coefficients in $\mathbb{C}[1/n, P_2, \dots, P_a]$, such that the data (1) of m at $X = \operatorname{diag}(x)$ are

$$\mathsf{F}_{rt} = \mathsf{F}_m(z_r, z_t) \quad (r \neq t), \qquad \mathsf{G}_{rt} = \mathsf{G}_m(z_r, z_t) + \delta_{rt}\, \mathsf{F}_m(z_r, z_r)$$

(see the proof of Lemma 9.73; for example, $\operatorname{tr}(ABAB)$ has $\mathsf{F} = st$ and $\mathsf{G} = -(s^2 + t^2)/n + P_2/n^2$). For $\kappa \in \mathbb{C}$ and $n \in \mathbb{C} \setminus \{0\}$ let $T(\kappa, n)$ be the following set of linear conditions on $\mathbf{c}$. Insert the data of $\varphi_{\mathbf{c}}$ into (1) the components of $\mathsf{G}x = 0$; (2) the components $\Psi_{pq}(e_i, e_l)$ of Proposition 9.44, for p, q, i, l distinct (the *generic* order-one rows) and for $i = q$ (the *coincident* ones); (3) the components (e_i, e_l, e_m) of the right side of Proposition 9.45 at $y = e_j$, for $j \notin \{i, l, m\}$ (generic) and for $j = i$ (coincident); all with the explicit coefficients x_l unshifted. Regard the coordinates that occur and the power sums $p_1, \dots, p_{a+2}$ as independent variables, with $\partial_w p_k = k x_w^{k-1}$, and n as a parameter, and require each component to vanish as a polynomial. (Each component of (2) and (3) involves at most four coordinates; the sums over all coordinates in (1) raise the index of the power sums to at most $a + 1$.) Each condition is linear in $\mathbf{c}$, with coefficients in $\mathbb{C}[\kappa, 1/n]$. Every shifted form (Propositions 9.44 and 9.45: $x_l \mapsto x_l + c$, also with $c = -p_1/n$ or $c = -x_j$) gives the same polynomials: at each integer $n \geq a + 6$ the forms agree as functions, hence as polynomials by the independence in Corollary 9.76(i) below, and their coefficients are polynomials in $1/n$, so they agree for every $n \neq 0$.

The grading. Let $\mathcal{I}$ be the substitution $(\kappa, n, p_1, p_2, \dots) \mapsto (-\kappa, -n, -p_1, -p_2, \dots)$, with the coordinates x_r fixed. A polynomial E in κ, $1/n$, the p_k and the coordinates has *parity* $\epsilon \in \mathbb{Z}/2$ if $E \circ \mathcal{I} = (-1)^{\epsilon} E$. Since p_1/n is fixed, every z_l is fixed; and $P_k = \sum_j \binom{k}{j} (-p_1/n)^{k-j} p_j$ with $p_0 = n$, so every P_k ($k \geq 0$) is odd. The derivative of a coefficient through the power sums, $\mathcal{D}_w := \sum_k k x_w^{k-1} \partial/\partial p_k$, is odd; derivatives in the coordinates, divided differences and explicit coordinates are even.

Lemma 9.73 (parity of the data)**.** *For $m \in \mathcal{M}$ with $k = k(m)$, F_m has parity $k - 1$ and G_m has parity k.*

Proof. Here $X_0 = \operatorname{diag}(z)$, $\operatorname{tr}(X_0^l) = P_l$ and $Y_0 = Y - (\operatorname{tr} Y/n)I$. If both letters B lie in one trace, $m = \operatorname{tr}(A^i B A^j B) \prod_{l=1}^{k-1} \operatorname{tr}(A^{\lambda_l})$, and

$$\operatorname{tr}(X_0^i Y_0 X_0^j Y_0) = \operatorname{tr}(X_0^i Y X_0^j Y) - \tfrac{2}{n} \operatorname{tr}(Y) \operatorname{tr}(X_0^{i+j} Y) + \tfrac{1}{n^2} \operatorname{tr}(Y)^2 P_{i+j}.$$

Off-diagonal Y gives $\mathsf{F} = \frac{1}{2}(s^i t^j + s^j t^i)$, of parity 0. Diagonal Y gives the diagonal part $\mathsf{F}(s,s)$ and $\mathsf{G} = -\frac{1}{n}(s^{i+j} + t^{i+j}) + \frac{1}{n^2} P_{i+j}$, of parity 1; for $i + j = 0$ the last term is $P_0/n^2 = 1/n$, of parity 1 because $P_0 = n$ has parity 1, and for $i + j = 1$ it is 0. The other $k - 1$ factors are odd. If the

two letters B lie in two traces, $m=\operatorname{tr}(A^iB)\operatorname{tr}(A^jB)\prod_{l=1}^{k-2}\operatorname{tr}(A^{\lambda_l})$ with $i,j\ge 1$; then $\mathsf{F}=0$, and $\operatorname{tr}(X_0^iY_0)=\operatorname{tr}(X_0^iY)-(\operatorname{tr}Y/n)P_i$ gives G as the symmetrization of $s^it^j-\frac{P_j}{n}s^i-\frac{P_i}{n}t^j+\frac{P_iP_j}{n^2}$, of parity 0; with the $k-2$ factors, parity k. □

For the rows, write $\psi_w^{(pq)}=\Lambda(z_p,z_q;z_w)$ for $w\notin\{p,q\}$ (Proposition 9.44), where $\Lambda:=L+C$ with

$$L(s,s';t):=\mathcal{D}_t\mathsf{F}(s,s')-\frac{\mathsf{G}(s,t)-\mathsf{G}(s',t)}{s-s'},$$
$$C(s,s';t):=\frac{(s-t)\mathsf{F}(s,t)+(t-s')\mathsf{F}(t,s')+(s'-s)\mathsf{F}(s',s)}{(s-t)(t-s')}$$

($\mathcal{D}_t$ is $\mathcal{D}_w$ with $z_w=t$, that is, $x_w=t+p_1/n$; $C(z_p,z_q;z_w)=\mathsf{c}_w^{(pq)}$). For the data of m, L is symmetric in (s,s') of parity $k(m)$, and C is antisymmetric in (s,s') of parity $k(m)-1$ (Lemma 9.73).

Lemma 9.74 (parity of the rows)**.** *Let $m\in\mathcal{M}$, $k=k(m)$, and consider the rows of $T(\kappa,n)$ for the single function m, as polynomials.*

(i) For p,q,i,l distinct let $R_{pq}:=\Psi_{pq}(e_i,e_l)$ and $R^{\pm}:=\frac12(R_{pq}\pm R_{qp})$. Then R^+ has parity $k+1$ and R^- has parity k.

(ii) The coincident row $\Psi_{pq}(e_q,e_l)$ is R_{pq} with $x_i:=x_q$ (the generic row is a polynomial in x_i). This holds for every symmetric datum (F,G).

(iii) The generic order-zero rows have parity k. Each coincident order-zero row is a generic one with $x_i:=x_j$ plus a polynomial of parity $k+1$.

(iv) The components of $\mathsf{G}x$ have parity $k+1$.

Proof. (i) By Proposition 9.44, $R_{pq}=\kappa(\psi_l-\psi_i)-x_i(\partial_i\psi-\delta_i\psi)_l+x_l(\partial_l\psi-\delta_l\psi)_i-\frac{x_p\Theta_p-x_q\Theta_q}{x_{pq}}(e_i,e_l)$ with $\psi=\psi^{(pq)}$, and R_{qp} is the same polynomial with x_p and x_q exchanged. Under this exchange the L-parts of the entries ψ_w and $\partial_i\psi_l=\mathcal{D}_i\psi_l$ are unchanged and their C-parts change sign. In $\delta_i\psi=(\psi^{(pq)}-\psi^{(pi)})/x_{iq}-(\psi^{(pq)}-\psi^{(iq)})/x_{ip}$ it is the other way round: for a symmetric Λ the exchange changes the sign of $\delta_i\psi$, for an antisymmetric one it does not. The Θ-term is unchanged, as numerator and denominator change sign; its entries $\Theta_p(e_i,e_l)=\mathcal{D}_i\mathsf{G}(z_p,z_l)-\mathcal{D}_l\mathsf{G}(z_p,z_i)$ have parity $k+1$. Counting odd factors (κ, $\mathcal{D}$), the unchanged terms κL, $x\mathcal{D}L$, $x\,\delta C$ and the Θ-term have parity $k+1$, and the terms that change sign, κC, $x\mathcal{D}C$ and $x\,\delta L$, have parity k.

(ii) The κ-terms agree: as $t\to s'$, $C(s,s';t)\to\partial_2\mathsf{F}(s,s')+(\mathsf{F}(s',s')-\mathsf{F}(s,s'))/(s-s')$, so $\Lambda(z_p,z_q;z_q)=\mathcal{D}_q\mathsf{F}+\partial_2\mathsf{F}+\big(\mathsf{G}(z_q,z_q)+\mathsf{F}(z_q,z_q)-\mathsf{G}(z_p,z_q)-\mathsf{F}(z_p,z_q)\big)/x_{pq}$, which is $\psi_q^{(pq)}$ of Proposition 9.44 with $\mathsf{G}_{qq}=\mathsf{G}(z_q,z_q)+\mathsf{F}(z_q,z_q)$. The entries of the δ_l- and Θ_p-terms are limits of the generic ones. Two differences remain: $x_q\Lambda(z_q,z_q;z_l)/x_{pq}$ from $\partial_q\psi-\psi/x_{pq}$, and $x_q\big(\partial_1\mathsf{G}(z_q,z_l)-\mathcal{D}_l\mathsf{F}(z_q,z_q)\big)/x_{pq}$ from $\Theta_q(e_q,e_l)=\partial_q\mathsf{G}_{ql}-\partial_l\mathsf{G}_{qq}$. Their sum is 0, since $C(s,s;t)=0$ and $L(s,s;t)=\mathcal{D}_t\mathsf{F}(s,s)-\partial_1\mathsf{G}(s,t)$.

(iii) In the generic rows, $\kappa\,\eta\wedge\Theta_j-d^x\Theta_j$, every entry $\Theta_j(e_u,e_v)=\mathcal{D}_u\mathsf{G}(z_j,z_v)-\mathcal{D}_v\mathsf{G}(z_j,z_u)$ has parity $k+1$, and ∂_u acts on it as $\mathcal{D}_u$; so the parity is k. For the coincident row use the shift $c=-x_j$ (so $\varpi_{jt}=\varpi_{rj}=1$ and all other $\varpi_{rt}=0$, as in the proof of Lemma 9.46) and $x':=x-x_j$:

$$D_3=\kappa(\Theta_{lm}-\Theta_{jm}+\Theta_{jl})+x'_l\partial_l\Theta_{jm}-x'_m\partial_m\Theta_{jl}-\Theta_{lm}-2\big(\mathsf{c}_m^{(jl)}-\mathsf{c}_l^{(jm)}\big),$$

with $\Theta_{uv}:=\Theta_j(e_u,e_v)$. Here $\Theta_{jv}=[\mathcal{D}_j\mathsf{G}(z_j,z_v)-\mathcal{D}_v\mathsf{G}(z_j,z_j)]+[\partial_1\mathsf{G}(z_j,z_v)-\mathcal{D}_v\mathsf{F}(z_j,z_j)]$, the first bracket of parity $k+1$ and the second of parity k. The first brackets give exactly the generic row at (e_i,e_l,e_m) with $x_i:=x_j$ (there $x'_i=0$ and Θ_{iv} becomes the first bracket). All other terms, namely the second brackets with κ or $x'\partial$, $-\Theta_{lm}$, and the c-terms (F only), have parity $k+1$.

(iv) $(\mathsf{G}x)_r=\sum_t\mathsf{G}(z_r,z_t)x_t+\mathsf{F}(z_r,z_r)x_r$, and $\sum_t z_t^l\,x_t=P_{l+1}+(p_1/n)P_l$ is odd. □

Theorem 9.75. *For every $a\ge 2$, $\kappa\in\mathbb{C}$ and $n\in\mathbb{C}\setminus\{0\}$: $\mathbf{c}\in\ker T(\kappa,n)$ if and only if $\sigma\mathbf{c}\in\ker T(-\kappa,-n)$.*

Proof. Take as *parts* the components of $\mathsf{G}x$, the $R^{\pm}$, the generic order-zero rows, and the two summands of each coincident order-zero row in Lemma 9.74(iii). By that lemma, each part E satisfies $E_m \circ \mathcal{I} = \varepsilon_E(-1)^{k(m)}E_m$ for all m, with a sign ε_E independent of m. Let $\mathbf{c} \in \ker T(\kappa, n)$. Then every part vanishes at $\mathbf{c}$ identically: the $R^{\pm}$ are combinations of rows, and the generic order-zero row vanishes identically in x_i, so its value at $x_i := x_j$ vanishes and with it the other summand. Hence

$$\sum_m (\sigma\mathbf{c})_m \, E_m(-\kappa, -n, -p) = \varepsilon_E \sum_m c_m \, E_m(\kappa, n, p) = 0$$

identically, and as the p_k are independent variables, $\sum_m (\sigma\mathbf{c})_m E_m(-\kappa, -n, p) = 0$: each part of $T(-\kappa, -n)$ vanishes at $\sigma\mathbf{c}$. Adding the parts, so do all rows of $T(-\kappa, -n)$ except the coincident order-one rows, and these follow by Lemma 9.74(ii). As σ and $\mathcal{I}$ are involutions, the converse holds too. □

Corollary 9.76. *Let $a \geq 2$ and $\mathrm{Spec}_a := \{(\kappa, n) \in \mathbb{C} \times \mathbb{C}^* : \ker T(\kappa, n) \neq 0\}$.*

(i) Spec_a is Zariski closed and $\mathrm{Spec}_a = -\mathrm{Spec}_a$. For every integer $n \geq a + 6$, $\mathbf{c} \mapsto \varphi_{\mathbf{c}}$ maps $\ker T(\kappa, n)$ isomorphically onto the kernel of $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$; so the fibre of Spec_a over n is the set of eigenvalues of the pencil there.

(ii) Let $\kappa(n) \in \mathbb{C}(n)$ and $\mathbf{c}(n) \in \mathbb{C}(n)^{\mathcal{M}} \setminus \{0\}$ with $0 \neq \varphi_{\mathbf{c}(n)}$ in the kernel of $\kappa(n)\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ for infinitely many integers $n \geq a + 6$. Then $\varphi_{\sigma\mathbf{c}(-n)}$ is a nonzero element of the kernel of $-\kappa(-n)\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ at every integer $n \geq a+6$ except the poles of $\kappa(-n)$ and $\mathbf{c}(-n)$ and the zeros of $\mathbf{c}(-n)$.

Proof. (i) The conditions of $T(\kappa, n)$ are the coefficients of finitely many polynomials, linear in $\mathbf{c}$ and polynomial in κ and $1/n$; so Spec_a, where they have rank $< |\mathcal{M}|$, is closed, and it is symmetric by Theorem 9.75. Let $n \geq a + 6$ be an integer. Since $n \geq a + 2$, the elements of $\mathcal{M}$ are linearly independent functions on $\mathfrak{sl}_n^2$ (for $n \geq d$ the trace monomials without fixed points are linearly independent on traceless matrices, Subsection 9.7). Next, $\mathcal{E}\varphi = 0$ if and only if $\mathsf{G}x = 0$ (Subsection 9.13). The test is linear in Y, so $Y = E_{pq}$ and diagonal Y suffice; base points with X regular diagonal and corners with $u = \mathbf{1}$ suffice, by conjugation and density (a sufficiency not needed in Subsection 9.13); $\Psi_{pq} \wedge dx_p = 0$ if and only if $\Psi_{pq}(e_i, e_l) = 0$ for all $i, l \neq p$; and Propositions 9.44 and 9.45 give the rows (2) and (3). Each row is a polynomial in at most four coordinates and $p_1, \ldots, p_{a+2}$, and these are algebraically independent functions on $\mathbb{C}^n$ for $n \geq a+6$ (as in the proof of Lemma 9.62); so a row vanishes as a function if and only if it vanishes as a polynomial. (ii) $T(\kappa(n'), n')\,\mathbf{c}(n') = 0$ holds for infinitely many integers n' and is rational in n', so it holds identically. Put $n' = -n$, with n not a pole of $\kappa(-n)$ or $\mathbf{c}(-n)$: by Theorem 9.75, $\sigma\mathbf{c}(-n) \in \ker T(-\kappa(-n), n)$, and (i) applies. □

Corollary 9.77. *For every even $a \geq 4$ and every integer $n \geq a + 6$, $\kappa = -2$ is an eigenvalue of the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$. (For $a = 2$ every κ is an eigenvalue, by Theorem 9.37.)*

Proof. Let $n \geq a + 1$ and $\tau := (2n - a - 2, a, 2)$, a partition. By Lemma 9.32(i), $\dim \mathrm{R}_2(n)_\tau = g((2^n), (2^n), \tau) = g((n, n), (n, n), \tau)$ (conjugating two of the three partitions does not change g). By the rule $s_{(n,n)} * s_{(n,n)} = \sum_\lambda s_\lambda$, over the $\lambda \vdash 2n$ with at most four parts which, padded with zeros to four parts, are all even or all odd [BWZ10, (1.1)] (credited there to [GWXZ09, Theorem I.1]), and as τ padded is $(2n - a - 2, a, 2, 0)$, all even, $\mathrm{R}_2(n)_\tau \neq 0$. Let $0 \neq f \in \mathrm{R}_2(n)_\tau$, which is $\mathrm{K}_2(n)_\tau$ (Proposition 9.34). By Lemma 9.32(ii), $0 \neq \varphi_f \in \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ and f is the function f_2 built from φ_f; so $2\mathcal{Z}(\varphi_f) = \mathcal{J}(\varphi_f)$ by Corollary 9.31(b). By Corollary 9.76(i), $(2, n) \in \mathrm{Spec}_a$ for every integer $n \geq a + 6$. The intersection of Spec_a with the line $\{\kappa = 2\} \cong \mathbb{C}^*$ is closed and infinite, hence the whole line; by Theorem 9.75, Spec_a contains the line $\{\kappa = -2\}$, and Corollary 9.76(i) gives the claim. □

Remark 9.78. (1) Theorem 9.75 is the precise form of Remark 9.38(e): there the kernels at $\kappa=\pm2$ were fitted as rational functions of n, and Corollary 9.76(ii) is that statement for every rational family of eigenvectors at $b=2$ and $n\ge a+6$. We do not claim that κ and $-\kappa$ occur at the same n when κ depends on n; the families $\kappa=(\mu+a)/n$ are mapped to themselves. If a constant κ occurs at infinitely many integers $n\ge a+6$, then $-\kappa$ occurs at every integer $n\ge a+6$, by the argument of Corollary 9.77. For $a+2\le n\le a+5$ only one inclusion survives: a formal identity implies the functional one, so $\varphi_{\mathbf{c}}$ is in the kernel of the pencil for every $\mathbf{c}\in\ker T(\kappa,n)$. (2) The rows themselves are not homogeneous for the grading; only their parts under the exchange $p\leftrightarrow q$ are (Lemma 9.74). The rule $n\mapsto -n$ with a sign for each closed loop is classical folklore [Ki71], [Cv08, Ch. 13], [MV11]; for example $\dim S_\lambda(\mathbb{C}^{-n})=(-1)^{|\lambda|}\dim S_{\lambda'}(\mathbb{C}^n)$, and $\kappa=s$ is odd because $\det^s$ becomes Ber^{-s}. We know no source that treats a test of this kind. (3) (Computations, not used.) Lemma 9.73 and Lemma 9.74(i), (ii), (iv) were checked for every trace monomial at $a=3,4,5$, and Lemma 9.74(iii) at $a=3,4$, with n symbolic; at $a=3$ the generic order-zero rows vanish identically, so $a=4$ is the first nontrivial check of (iii). An independent program confirms Theorem 9.75 on the full system at $a=3,4$ (Spec at $n=-11$ is minus Spec at $n=11$, with σ mapping each kernel onto the mirror kernel). The values $\kappa=\pm2$ occur at $(4,2)$ and $(6,2)$; they do not occur at $(3,2)$ for any $n\ge4$ (Theorem 9.48), nor at $(5,2)$ for $n=11,12$ (Remark 9.50(4)); for odd a and $n\ge a+6$ they are absent (Corollary 9.144(i)).

9.18. **Theorem Z-INJ at $b=2$ for every n.** In this subsection $F=\mathbb{C}$, $n\ge3$ and $b=2$, and we keep the notation of Subsection 9.13; for $v,v'\in\mathbb{C}^n$ we write $\mathsf{G}(v,v'):=v^T\mathsf{G}v'$. Theorem 9.22 is proved in the stable range: the trace monomials without fixed points are linearly independent for $n\ge d$, and the test values for $n\ge d+1$; so at $b=2$ it gives $\ker\mathcal{Z}\cap\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)=0$ for $a\ge3$ and $n\ge a+3$. We prove this for every $n\ge3$ (Corollary 9.81), from the κ-part of the test at $Y=E_{pq}$ alone (Theorem 9.80). On each triangle of coordinates these rows become a 3×3 system whose determinant has the factor $a-2$; at $a=2$ this factor leaves exactly the line of $\mathrm{tr}([A,B]^2)$, the exception in Theorem 9.22.

Lemma 9.79 (the κ-part at $Y=E_{pq}$)**.** *Let $\varphi\in\mathcal{F}_{(a,2)}(n)$, $X=\mathrm{diag}(x)$ with distinct x_l, and $Y=E_{pq}$ with $p\ne q$.*

(a) $\mathcal{Z}(\varphi)(\mathbf{1}v_1^T,\mathbf{1}v_2^T,\mathbf{1}v_3^T)=2\,(\eta\wedge\psi^{(pq)}\wedge dx_p)(v_1,v_2,v_3)$ for all $v_1,v_2,v_3\in\mathbb{C}^n$.

(b) The following are equivalent: this vanishes for all v_1,v_2,v_3; it vanishes for $v_1=\mathbf{1}$ and all v_2,v_3; $\psi^{(pq)}_w=\psi^{(pq)}_q$ for every $w\notin\{p,q\}$.

(c) For distinct l,k,m put $\mathsf{r}_{lkm}:=\psi^{(ml)}_k-\psi^{(ml)}_l$. Then

$$\mathsf{r}_{lkm}=\mathsf{c}^{(ml)}_k+(\partial_k-\partial_l)\mathsf{F}_{lm}+\frac{\mathsf{F}_{lm}-\mathsf{G}(e_l-e_m,e_l-e_k)}{x_{ml}},\qquad \mathsf{c}^{(ml)}_k=\frac{\mathsf{F}_{lk}-\mathsf{F}_{lm}}{x_{mk}}+\frac{\mathsf{F}_{km}-\mathsf{F}_{lm}}{x_{kl}}.$$

Proof. (a) As $\mathcal{Z}(\varphi)=\mathrm{tr}\wedge\omega$ (Corollary 9.31), this is the corner formula $\mathrm{tr}\wedge\omega=2\,\eta\wedge\psi\wedge dx_p$ at the end of Step 1 of the proof of Proposition 9.44, which uses only $\varphi\in\mathcal{F}_{(a,2)}(n)$. (b) Write $\psi:=\psi^{(pq)}$. The value at $v_1=\mathbf{1}$ is the contraction $\iota_{\mathbf{1}}(\eta\wedge\psi\wedge dx_p)=n\,\psi\wedge dx_p-\eta\wedge\iota_{\mathbf{1}}(\psi\wedge dx_p)$; if it vanishes, then $\psi\wedge dx_p=\frac{1}{n}\,\eta\wedge\iota_{\mathbf{1}}(\psi\wedge dx_p)$, so $\eta\wedge\psi\wedge dx_p=0$. So the first two conditions are equivalent. As $\eta\wedge dx_p\ne0$, $\eta\wedge\psi\wedge dx_p=0$ if and only if $\psi=c_1\eta+c_2\,dx_p$ for some numbers c_1,c_2 (at each x); as ψ has no dx_p-component, this says that ψ_w is the same for every $w\ne p$. (c) By the definitions in Subsection 9.13, $\psi^{(ml)}_k=\partial_k\mathsf{F}_{ml}+(\mathsf{G}_{lk}-\mathsf{G}_{mk})/x_{ml}+\mathsf{c}^{(ml)}_k$ and $\psi^{(ml)}_l=\partial_l\mathsf{F}_{ml}+(\mathsf{G}_{ll}-\mathsf{G}_{ml}-\mathsf{F}_{ml})/x_{ml}$, and $\mathsf{G}_{lk}-\mathsf{G}_{mk}-\mathsf{G}_{ll}+\mathsf{G}_{ml}=-\mathsf{G}(e_l-e_m,e_l-e_k)$; F and G are symmetric. □

Theorem 9.80 (the rows at $Y=E_{pq}$)**.** *Let $n\ge3$ and $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, and suppose that at every base point $(\mathrm{diag}(x),E_{pq})$ with distinct x_l and $p\ne q$, $\mathcal{Z}(\varphi)(B_1,B_2,B_3)=0$ for $B_1=\mathbf{1}\mathbf{1}^T$ and all $B_2=\mathbf{1}v^T$, $B_3=\mathbf{1}v'^T$; equivalently (Lemma 9.79), $\mathsf{r}_{lkm}=0$ for all distinct l,k,m, at every x with distinct coordinates. If $a\ge3$, then $\varphi=0$. If $a=2$, then φ is a multiple of $\mathrm{tr}([A,B]^2)$, and $\mathrm{tr}([A,B]^2)$ satisfies the hypothesis.*

Proof. Put $\varrho_{lm} := \frac{1}{2}\mathsf{G}(e_l - e_m, e_l - e_m)$.

Step 1 (exchange of l and m). By Lemma 9.79(c), with $\mathsf{c}_k^{(lm)} = -\mathsf{c}_k^{(ml)}$, $x_{lm} = -x_{ml}$, $(e_l - e_k) + (e_m - e_k) = e_l + e_m - 2e_k$ and $(e_l - e_k) - (e_m - e_k) = e_l - e_m$,

$$(\Sigma) \qquad 0 = \mathsf{r}_{lkm} + \mathsf{r}_{mkl} = (2\partial_k - \partial_l - \partial_m)\mathsf{F}_{lm} - \frac{\mathsf{G}(e_l - e_m, e_l + e_m - 2e_k)}{x_{ml}},$$

$$(\Delta) \qquad 0 = \mathsf{r}_{lkm} - \mathsf{r}_{mkl} = 2\mathsf{c}_k^{(ml)} + (\partial_m - \partial_l)\mathsf{F}_{lm} + \frac{2(\mathsf{F}_{lm} - \varrho_{lm})}{x_{ml}}.$$

Step 2 (ϱ from F). Let first $\sum_l x_l = 0$. For $k \notin \{l, m\}$, (Σ) and $\mathsf{G}(e_l - e_m, e_l + e_m) = \mathsf{G}_{ll} - \mathsf{G}_{mm}$ give

$$\mathsf{G}(e_l - e_m, e_k) = \tfrac{1}{2}(\mathsf{G}_{ll} - \mathsf{G}_{mm}) - \tfrac{1}{2}x_{ml}(2\partial_k - \partial_l - \partial_m)\mathsf{F}_{lm},$$

and $\mathsf{G}(e_l - e_m, e_l) = \mathsf{G}_{ll} - \mathsf{G}_{lm}$, $\mathsf{G}(e_l - e_m, e_m) = \mathsf{G}_{lm} - \mathsf{G}_{mm}$. As $\mathcal{E}\varphi = 0$, $\mathsf{G}x = 0$, so $0 = \sum_k x_k\, \mathsf{G}(e_l - e_m, e_k)$. Insert these values, use $\sum_{k\notin\{l,m\}} x_k = -(x_l + x_m)$ (the only use of $\sum_l x_l = 0$) and Euler's identity $\sum_k x_k \partial_k \mathsf{F}_{lm} = a\mathsf{F}_{lm}$: the G-terms add up to $x_{lm}\varrho_{lm}$ and the F-terms to $\frac{1}{2}x_{lm}\big(2a\mathsf{F}_{lm} - x_{lm}(\partial_l - \partial_m)\mathsf{F}_{lm}\big)$. Dividing by x_{lm},

$$\varrho_{lm} = -a\mathsf{F}_{lm} + \tfrac{1}{2}x_{lm}(\partial_l - \partial_m)\mathsf{F}_{lm}.$$

Both sides are invariant under $x \mapsto x + c\mathbf{1}$, so this holds at every x with distinct coordinates. In (Δ), $(\mathsf{F}_{lm} - \varrho_{lm})/x_{ml} = (a+1)\mathsf{F}_{lm}/x_{ml} + \frac{1}{2}(\partial_l - \partial_m)\mathsf{F}_{lm}$, and the derivative terms cancel:

$$(\star) \qquad (a+1)\,\mathsf{F}_{lm} = x_{lm}\,\mathsf{c}_k^{(ml)} = x_{ml}\Big(\frac{\mathsf{F}_{lm} - \mathsf{F}_{km}}{x_{kl}} + \frac{\mathsf{F}_{lm} - \mathsf{F}_{lk}}{x_{mk}}\Big)$$

for all distinct l, k, m, at every x with distinct coordinates.

Step 3 (one triangle). Fix distinct l, k, m and put $f_1 := \mathsf{F}_{lm}$, $f_2 := \mathsf{F}_{lk}$, $f_3 := \mathsf{F}_{km}$, $\alpha := x_{kl}$, $\beta := x_{mk}$ and $\gamma := \alpha + \beta = x_{ml}$. Cleared of denominators, $(\star)$ for the edges lm, lk, km (with third vertices k, m, l) reads

$$(\gamma^2 - (a+1)\alpha\beta)f_1 - \alpha\gamma f_2 - \beta\gamma f_3 = 0, \qquad \alpha\gamma f_1 - (\alpha^2 + (a+1)\beta\gamma)f_2 - \alpha\beta f_3 = 0,$$
$$\beta\gamma f_1 - \alpha\beta f_2 - (\beta^2 + (a+1)\alpha\gamma)f_3 = 0.$$

Up to the signs of the last two equations, this is $\mathsf{K}(f_1, f_2, f_3)^T = 0$ with

$$\mathsf{K} := \mathbf{w}\mathbf{w}^T + (a+1)\mathsf{D}, \qquad \mathbf{w} := (\gamma, -\alpha, -\beta)^T, \qquad \mathsf{D} := \operatorname{diag}(-\alpha\beta, \beta\gamma, \alpha\gamma).$$

Since $\mathbf{w}^T\mathsf{D}^{-1}\mathbf{w} = -\frac{\gamma^2}{\alpha\beta} + \frac{\alpha^2}{\beta\gamma} + \frac{\beta^2}{\alpha\gamma} = \frac{\alpha^3+\beta^3-\gamma^3}{\alpha\beta\gamma} = -3$, the determinant formula for a rank-one update gives

$$\det\mathsf{K} = (a+1)^3 \det\mathsf{D}\,\Big(1 + \frac{\mathbf{w}^T\mathsf{D}^{-1}\mathbf{w}}{a+1}\Big) = -(a+1)^2(a-2)\,\alpha^2\beta^2\gamma^2.$$

For $a \geq 3$ this is not 0, as the coordinates are distinct; so $f_1 = f_2 = f_3 = 0$. As $n \geq 3$, every pair $\{l, m\}$ lies in a triangle, so $\mathsf{F}_{lm} = 0$ for all $l \neq m$.

Step 4 (G). With $\mathsf{F} = 0$, Step 2 gives $\varrho_{lm} = 0$, that is $\mathsf{G}_{lm} = (\mathsf{G}_{ll} + \mathsf{G}_{mm})/2$ for all $l \neq m$. Then $0 = (\mathsf{G}\mathbf{1})_l = (n\mathsf{G}_{ll} + \operatorname{tr}\mathsf{G})/2$ for every l, so all G_{ll} are equal to one number g_0, $\mathsf{G} = g_0\mathbf{1}\mathbf{1}^T$, and $\mathsf{G}\mathbf{1} = 0$ gives $g_0 = 0$. By (1), $\tilde{\varphi}$ vanishes at every (X, Y) with X diagonal with distinct entries; the GL_n-saturation of this set is dense, so $\varphi = 0$.

The case $a = 2$. Here $3\mathsf{D}$ is invertible and $\mathbf{w}\mathbf{w}^T$ has rank 1, so K has rank 2, and its kernel is spanned by $\mathsf{D}^{-1}\mathbf{w}$ (as $\mathsf{K}\mathsf{D}^{-1}\mathbf{w} = \mathbf{w}(\mathbf{w}^T\mathsf{D}^{-1}\mathbf{w} + 3) = 0$), which is proportional to $(\gamma^2, \alpha^2, \beta^2)$. So on each triangle T there is a function χ_T of x with $\mathsf{F}_{rt} = \chi_T\, x_{rt}^2$ on the three edges of T. Two triangles with a common edge have the same χ_T ($x_{rt} \neq 0$), and any two triangles are joined by a chain of triangles in which consecutive ones share an edge; so $\chi_T = \chi$ for all T. Then $x_{13}^2\mathsf{F}_{12} = x_{12}^2\mathsf{F}_{13}$ as polynomials, so x_{12}^2 divides F_{12}, and $\chi = \mathsf{F}_{12}/x_{12}^2$ is a polynomial of degree $a - 2 = 0$, a constant. If φ and φ' satisfy the hypothesis and have the same F, then $\varphi - \varphi'$ satisfies it with $\mathsf{F} = 0$, and Steps 2

and 4 (which do not use $a \geq 3$) give $\varphi = \varphi'$. Finally $\operatorname{tr}([A,B]^2) \in \mathcal{F}^{\mathcal{E}}_{(2,2)}(n)$, as $[A, B+tA] = [A,B]$; at $X = \operatorname{diag}(x)$, $[X,Y]_{rt} = x_{rt}Y_{rt}$ gives $\mathsf{F}_{rt} = -x_{rt}^2$ and $\mathsf{G} = 0$ (as in Remark 9.50(3)), and the terms of r_{lkm} in Lemma 9.79(c) are $-(2x_l - x_k - x_m)$ and $-(x_l + x_k - 2x_m)$ (from $\mathsf{c}_k^{(ml)}$), $2x_{lm}$ and x_{lm}, which add up to 0. So $\varphi = -\chi \operatorname{tr}([A,B]^2)$. □

Corollary 9.81 (Z-INJ at $b=2$)**.** *Let $n \geq 3$, $a \geq 3$ and $h := \dim \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$.*

(a) $\mathcal{Z}(\varphi) \neq 0$ for every $0 \neq \varphi \in \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$.

(b) For all but at most h values of κ, the rows of Proposition 9.44 (the test at the base points $(\operatorname{diag}(x), E_{pq})$ on the corners $\mathbf{1}v_k^T$) are injective on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$. In particular the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ has at most h eigenvalues.

Proof. (a) If $\mathcal{Z}(\varphi) = 0$ at every base point and for all u, y_1, y_2, y_3, then in particular the hypothesis of Theorem 9.80 holds ($u = \mathbf{1}$, $y_1 = \mathbf{1}$), so $\varphi = 0$. (b) Each row is a linear form $\kappa\ell - \ell'$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, where ℓ is the corresponding row of $\mathcal{Z}$. The rows with $v_1 = \mathbf{1}$ are among them, so by Theorem 9.80 the forms ℓ have no common zero other than 0, so some h of them, $\ell_1, \dots, \ell_h$, are linearly independent. For a basis $\varphi_1, \dots, \varphi_h$, $\det\big(\kappa\ell_i(\varphi_j) - \ell_i'(\varphi_j)\big)$ is a polynomial in κ of degree h with leading coefficient $\det(\ell_i(\varphi_j)) \neq 0$; outside its at most h roots these h rows are injective. An eigenvalue (a κ with $\kappa\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ for some $\varphi \neq 0$, at every base point and for all u, y_1, y_2, y_3) is one of these roots. □

Remark 9.82. (1) At $b = 2$ and $n \geq a+3$, Corollary 9.81(a) follows from Theorem 9.22; for $3 \leq n \leq a+2$ it is new. The mechanism differs from the proof of Theorem 9.22: a 3×3 system on each triangle of coordinates instead of a graph Laplacian, and the exception $(a,b) = (2,2)$ appears as the factor $a-2$ of $\det \mathsf{K}$.

(2) The proof uses the conjugation invariance, the degree 2 in Y (through (1)), the invariances under $X \mapsto X + cI$ and $Y \mapsto Y + cI$, the homogeneity, and $\mathcal{E}\varphi = 0$ only through $\mathsf{G}x = 0$; no computer. Lemma 9.79(a) is Step 1 of the proof of Proposition 9.44. Corollary 9.81(b) bounds the number of eigenvalues and says nothing about which values occur; compare Conjecture 11.3.

(3) (Computations, not used.) Exact computations over $\mathbb{Q}$, with bases of $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ built from trace monomials and ω computed from its definition, confirm

$$\omega(E_{lk}, E_{km}) + \omega(E_{kk}, E_{lm}) - \omega(E_{ll}, E_{lm}) = 2\mathsf{r}_{lkm}$$

at the base points $(\operatorname{diag}(x), E_{ml})$ for all ordered triples (l,k,m), and give rows r_{lkm} of rank h, for $a = 3,4,5$ at $n = 3$ and $n = 5$, for $a = 3, \dots, 7$ at $n = 4$, and for $(a,n) = (3,6)$; at $a = 2$ ($n = 3,4,5$), $\operatorname{tr}([A,B]^2)$ satisfies the rows. Independent computations modulo the prime 999983 give $\ker \mathcal{Z} \cap \mathcal{F}^{\mathcal{E}}_{(a,2)}(n) = 0$ for $a = 3, \dots, 7$ at $n = 3$, for $a = 3, \dots, 6$ at $n = 4$, and for $(a,n) = (3,5), (4,5), (3,6)$, and the line of $\operatorname{tr}([A,B]^2)$ for $(a,n) = (2,3), (2,4)$.

9.19. **The slice rows and Theorem Z-INJ at $b = 3$.** In this subsection $F = \mathbb{C}$, $b = 3$ and $n \geq 4$ unless said, and we use β, ω and $\mathcal{Z}$ of Subsection 9.11. We write $D := \operatorname{diag}(x)$ with distinct x_p, $x_{pq} := x_p - x_q$, $\theta_{pq} := 1/x_{qp}$ and $\Delta_{pq} := e_p - e_q$. Every pair (X,Y) with X regular semisimple is conjugate to a *base point* $(D, Z + \operatorname{diag}(y))$ with Z off-diagonal; the base points with $y = 0$ are the *slice points* (D,Z) (so $(D_0, Z) \in \mathcal{S}$, Subsection 9.14). We show that the rows of $\mathcal{Z}$ at the slice points listed in Lemma 9.83 vanish on $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ exactly when $\tilde{\varphi}$ at the base points depends only on x and y (Theorem 9.88). With Lemma 9.90, which kills that part with the rows at the base points with $y \neq 0$, this gives Theorem 9.22 at $b = 3$ for every $n \geq 4$ (Corollary 9.91). Lemma 9.93 extends Lemma 9.90 to every order $k \geq 2$ in y, for every (a,b) and $n \geq 3$, and so reduces Theorem 9.22 to the parts of degree 0 and 1 in y (Corollary 9.94). At a base point (D,Y), $\beta(U,V) = \partial_X[U]\,\partial_Y[V]\,\tilde{\varphi}(D,Y)$. Differentiating the conjugation invariance at $g = \exp(\epsilon E_{pq})$, with

$[E_{pq}, D] = x_{qp}E_{pq}$, and then along V in Y, gives for every Y

$$\beta(E_{pq},V) = -\theta_{pq}\big[\partial_Y^2\tilde\varphi([E_{pq},Y],V) + \partial_Y[[E_{pq},V]]\tilde\varphi\big] \quad (p\neq q), \qquad \beta(E_{pp},V) = \partial_{x_p}\partial_Y[V]\tilde\varphi, \tag{3}$$

and $[E_{pq}, Z] = Z_{qp}(E_{pp} - E_{qq}) + \sum_{r\notin\{p,q\}} Z_{qr}E_{pr} - \sum_{r\notin\{p,q\}} Z_{rp}E_{rq}$ for Z off-diagonal.

Corners. For $B_i = uy_i^T$ $(i = 1,2,3)$, $\operatorname{tr} B_i = u\cdot y_i$, so $\mathcal{Z}(\varphi)(B_1,B_2,B_3) = \mathsf{Q}_u(y_1,y_2,y_3)$ with $\mathsf{Q}_u := u\wedge\omega^u$ and $\omega^u(y,z) := \omega(uy^T, uz^T)$ (conventions of Subsection 9.12, u read as the 1-form $y\mapsto u\cdot y$). The *slice rows* are the following coefficients of Q_u, as a polynomial in u, at the slice points.

Lemma 9.83 (the slice rows)**.** *On the basis $e_i\wedge e_j\wedge e_h$, the coefficient of u_l^3 in Q_u has the entries $\omega(E_{lj},E_{lh})$ at $e_l\wedge e_j\wedge e_h$* (the rows $[l^3]$)*, and the coefficient of $u_l^2u_k$ $(k\neq l)$ has the entries* (the rows $[l^2k]$)

$$\begin{gathered}\mathsf{c}(l,k,m) := \omega(E_{lk},E_{km}) + \omega(E_{kk},E_{lm}) - \omega(E_{ll},E_{lm}) \quad \text{at } e_l\wedge e_k\wedge e_m,\\ \mathsf{P}(l,k;i,j) := \omega(E_{li},E_{kj}) + \omega(E_{ki},E_{lj}) \quad \text{at } e_l\wedge e_i\wedge e_j \ (\{i,j\}\cap\{l,k\}=\emptyset),\\ \omega(E_{li},E_{lj}) \quad \text{at } e_k\wedge e_i\wedge e_j \ (l\notin\{i,j\}).\end{gathered}$$

Proof. $\omega^u = \sum_{\alpha,\gamma} u_\alpha u_\gamma\,\omega^{(\alpha\gamma)}$ with $\omega^{(\alpha\gamma)}(y,z) := \frac12\sum_{s,t} y_s z_t\big(\omega(E_{\alpha s},E_{\gamma t}) + \omega(E_{\gamma s},E_{\alpha t})\big)$, which is alternating. So the coefficient of u_l^3 is $e_l\wedge\omega^{(ll)}$ and that of $u_l^2u_k$ is $e_k\wedge\omega^{(ll)} + 2e_l\wedge\omega^{(lk)}$; evaluate at (e_l,e_j,e_h), (e_l,e_k,e_m), (e_l,e_i,e_j) and (e_k,e_i,e_j). □

Lemma 9.84 (the shape at $b=3$)**.** *Let $\varphi\in\mathcal{F}_{(a,3)}(n)$ and $\Phi(x,y,Z) := \tilde\varphi(D, Z+\operatorname{diag}(y))$. Then*

$$\Phi = \sum_{\text{directed 3-cycles}} \mathsf{F}_{pqr}Z_{pq}Z_{qr}Z_{rp} + \sum_s y_s\,\mathsf{N}_s(Z) + \mathsf{C}(x;y), \qquad \mathsf{N}_s(Z) := \sum_{p\neq q}\mathsf{n}_{pq}^s Z_{pq}Z_{qp},$$

with $\mathsf{F}_{pqr} = \mathsf{F}_{qrp} = \mathsf{F}_{rpq}$ (the two orientations F_{pqr}, F_{qpr} of a triangle are independent), $\mathsf{n}_{pq}^s = \mathsf{n}_{qp}^s$ and C a cubic form in y; there is no term $y_sy_tZ_{pq}$. F, n, C are polynomials in x, homogeneous of degree a and invariant under $x\mapsto x+c\mathbf{1}$, and $\sum_s\mathsf{n}_{pq}^s = 0$. If $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$, then $\sum_s x_s\mathsf{n}_{pq}^s = 0$.

Proof. Φ has degree 3 in (y,Z), and conjugation by invertible diagonal matrices fixes D; Z_{pq} has weight $e_p - e_q$ and y_s weight 0. A monomial of weight 0 and degree 3 is a directed 3-cycle, y_s times a 2-cycle $Z_{pq}Z_{qp}$, or $y_sy_ty_u$; $y_sy_tZ_{pq}$ has weight $e_p - e_q \neq 0$. $Y\mapsto Y + cI$ does not change $\tilde\varphi$, so $\sum_s\partial\Phi/\partial y_s = 0$, whose Z^2-part gives $\sum_s\mathsf{n}_{pq}^s = 0$. As $\widetilde{\mathcal{E}\varphi}(D,Y) = \partial_Y[D]\tilde\varphi = \sum_s x_s\partial\Phi/\partial y_s$, whose y^0-part is $\sum_s x_s\mathsf{N}_s(Z)$, $\mathcal{E}\varphi = 0$ gives $\sum_s x_s\mathsf{n}_{pq}^s = 0$. □

Each directed 3-cycle is counted once (a rotation is the same cycle), and the coefficient of $y_sZ_{pq}Z_{qp}$ in Φ is $2\mathsf{n}_{pq}^s$. Lemma 9.84 is the analogue at $b=3$ of the torus form (1). At $y=0$: $\partial\Phi/\partial Z_{pq} = \sum_{r\notin\{p,q\}}\mathsf{F}_{pqr}Z_{qr}Z_{rp}$, $\partial\Phi/\partial y_s = \mathsf{N}_s(Z)$, $\partial^2\Phi/\partial Z_{pq}\partial Z_{qr} = \mathsf{F}_{pqr}Z_{rp}$ (p,q,r distinct), $\partial^2\Phi/\partial Z_{ij}\partial Z_{hk} = 0$ unless $Z_{ij}Z_{hk}$ is part of a 3-cycle, $\partial^2\Phi/\partial y_s\partial Z_{pq} = 2\mathsf{n}_{pq}^sZ_{qp}$, and $\partial^2\Phi/\partial y_s\partial y_t = 0$. *Spectator notation:* for an index r put $\mathsf{f}_{pq}^{(r)} := \mathsf{F}_{pqr}$ ($p,q\neq r$; $\mathsf{f}_{qp}^{(r)} = \mathsf{F}_{qpr}$ is the other orientation) and $\mathsf{M}^{(r)}(s,t) := \mathsf{n}_{tr}^s$ (s any index, $t\neq r$; the first slot is the y-index, the second an edge index, and $\mathsf{M}^{(r)}$ is not symmetric). Then $\mathbf{1}^T\mathsf{M}^{(r)} = 0$, $x^T\mathsf{M}^{(r)} = 0$ (if $\mathcal{E}\varphi = 0$) and $\mathsf{n}_{tr}^s = \mathsf{M}^{(r)}(s,t) = \mathsf{M}^{(t)}(s,r)$. For a vector v and $p\neq q$ put $D_{pq}(v) := (v_p - v_q)/x_{pq}$.

Lemma 9.85 (the slice rows in closed form)**.** *Let $\varphi\in\mathcal{F}_{(a,3)}(n)$, at a slice point (D,Z).*

(i) For distinct l,j,h: $\omega(E_{lj},E_{lh}) = Z_{jl}Z_{hl}\big\{\theta_{lh}\big[2(\mathsf{n}_{lj}^l - \mathsf{n}_{lj}^h) - \mathsf{F}_{ljh}\big] - \theta_{lj}\big[2(\mathsf{n}_{lh}^l - \mathsf{n}_{lh}^j) - \mathsf{F}_{lhj}\big]\big\}$.

(ii) For distinct l,k,m: $\mathsf{c}(l,k,m) = \sum_{r\notin\{l,m\}}\mathsf{c}_r(l,k,m)\,Z_{mr}Z_{rl}$, where for $r\notin\{l,k,m\}$, with $\mathsf{f} := \mathsf{f}^{(r)}$ and $\mathsf{M} := \mathsf{M}^{(r)}$,

$$\mathsf{c}_r(l,k,m) = \theta_{lk}(\mathsf{f}_{km} - \mathsf{f}_{lm}) + \theta_{km}(\mathsf{f}_{lk} - \mathsf{f}_{lm}) + (\partial_k - \partial_l)\mathsf{f}_{lm} + \theta_{lm}\mathsf{f}_{lm} - 2\theta_{lm}\mathsf{M}(\Delta_{lk},\Delta_{lm}),$$

and for $r=k$ (the near *term), with $\mathsf{f}:=\mathsf{f}^{(k)}$ and $\mathsf{M}:=\mathsf{M}^{(k)}$,*

$$\mathsf{c}_k(l,k,m)=(\partial_k-\partial_l)\mathsf{f}_{lm}+(\theta_{lm}-\theta_{lk}-\theta_{km})\mathsf{f}_{lm}-2\theta_{lm}\mathsf{M}(\Delta_{lk},\Delta_{lm})+\mathsf{X},\quad \textit{where}$$

$$\mathsf{X}:=-2\theta_{lk}\mathsf{M}(\Delta_{lk},e_m)-2\theta_{km}\mathsf{M}(\Delta_{mk},e_l).$$

(iii) For distinct l,k,i,j: $\mathsf{P}(l,k;i,j)=\mathsf{A}\,Z_{il}Z_{jk}+\mathsf{B}\,Z_{ik}Z_{jl}$ with

$$\mathsf{A}=-2\theta_{li}(\mathsf{n}^l_{kj}-\mathsf{n}^i_{kj})+2\theta_{kj}(\mathsf{n}^k_{li}-\mathsf{n}^j_{li})-\theta_{ki}(\mathsf{F}_{klj}-\mathsf{F}_{lji})+\theta_{lj}(\mathsf{F}_{lki}-\mathsf{F}_{kij}),$$

and B the same with l and k exchanged (P is symmetric in l,k).

(iv) With $\mathsf{f}:=\mathsf{f}^{(k)}$ and $\mathsf{M}:=\mathsf{M}^{(k)}$, the coefficient of $Z_{lk}Z_{mk}$ in the row $[k^3]$ at (e_k,e_l,e_m) (that is, (i) with apex k) vanishes if and only if $\theta_{km}\mathsf{f}_{lm}-\theta_{kl}\mathsf{f}_{ml}=2\theta_{km}\mathsf{M}(\Delta_{km},e_l)-2\theta_{kl}\mathsf{M}(\Delta_{kl},e_m)=\mathsf{X}$. So where the rows $[l^3]$ vanish, the near term is

$$\mathrm{NEAR}(l,k,m):=(\partial_k-\partial_l)\mathsf{f}_{lm}+\theta_{lm}\mathsf{f}_{lm}+\theta_{lk}(\mathsf{f}_{ml}-\mathsf{f}_{lm})-2\theta_{lm}\mathsf{M}(\Delta_{lk},\Delta_{lm}).$$

Proof. All terms come from (3) at $y=0$ and the jets above; (ii) is, per spectator r, the analogue at $b=3$ of Lemma 9.79(c) (compare Remark 9.82(3)). (i) $[E_{lj},E_{lh}]=0$. In $\partial_Y^2\tilde\varphi([E_{lj},Z],E_{lh})$ the diagonal part $Z_{jl}(E_{ll}-E_{jj})$ gives $2Z_{jl}(\mathsf{n}^l_{lh}-\mathsf{n}^j_{lh})Z_{hl}$; the terms $Z_{jr}E_{lr}$ give 0 (two edges out of l are never in one 3-cycle); of $-Z_{rl}E_{rj}$ only $r=h$ survives (the cycle $l\to h\to j\to l$), giving $-Z_{hl}\mathsf{F}_{lhj}Z_{jl}$. So $\beta(E_{lj},E_{lh})=-\theta_{lj}Z_{jl}Z_{hl}[2(\mathsf{n}^l_{lh}-\mathsf{n}^j_{lh})-\mathsf{F}_{lhj}]$; exchange j and h and subtract.

(ii) Per entry, the coefficient of $Z_{mr}Z_{rl}$. (A) $\omega(E_{ll},E_{lm})$: $\beta(E_{ll},E_{lm})=\partial_l\,\partial\Phi/\partial Z_{lm}$ gives $\partial_l\mathsf{F}_{lmr}$; in $\beta(E_{lm},E_{ll})$, $[E_{lm},E_{ll}]=-E_{lm}$ gives $-\mathsf{F}_{lmr}$, and the terms $Z_{mr}E_{lr}$ and $-Z_{rl}E_{rm}$ of $[E_{lm},Z]$, paired with the diagonal direction E_{ll}, give $2(\mathsf{n}^l_{lr}-\mathsf{n}^l_{rm})$; so $\omega(E_{ll},E_{lm})$ contributes $\partial_l\mathsf{F}_{lmr}+\theta_{lm}[2(\mathsf{n}^l_{lr}-\mathsf{n}^l_{rm})-\mathsf{F}_{lmr}]$. (C) $\omega(E_{kk},E_{lm})$ contributes $\partial_k\mathsf{F}_{lmr}+2\theta_{lm}(\mathsf{n}^k_{lr}-\mathsf{n}^k_{rm})$ ($[E_{lm},E_{kk}]=0$). (B) $\omega(E_{lk},E_{km})$: $[E_{lk},E_{km}]=E_{lm}$ gives F_{lmr}; in $\partial_Y^2\tilde\varphi([E_{lk},Z],E_{km})$ the diagonal part gives $2(\mathsf{n}^l_{km}-\mathsf{n}^k_{km})Z_{kl}Z_{mk}$ (only $r=k$) and $-Z_{rl}E_{rk}$ gives $-\mathsf{F}_{rkm}Z_{mr}Z_{rl}$ ($r\notin\{l,k,m\}$); symmetrically for $\beta(E_{km},E_{lk})$, with $[E_{km},E_{lk}]=-E_{lm}$, the diagonal part $2(\mathsf{n}^k_{lk}-\mathsf{n}^m_{lk})Z_{mk}Z_{kl}$ and $Z_{mr}E_{kr}$ giving F_{lkr}. Two kinds of terms in (B) vanish: the 2-cycle pairings (in $\beta(E_{lk},E_{km})$ the term $-Z_{ml}E_{mk}$ of $[E_{lk},Z]$ meets E_{km}, and in $\beta(E_{km},E_{lk})$ the term $Z_{ml}E_{kl}$ of $[E_{km},Z]$ meets E_{lk}; they give $\partial^2\Phi/\partial Z_{mk}\partial Z_{km}$ and $\partial^2\Phi/\partial Z_{kl}\partial Z_{lk}$, which are 0 at $y=0$), and the terms $Z_{kr}E_{lr}$ against E_{km} and $-Z_{rk}E_{rm}$ against E_{lk}, which give 0 because the two edges would both enter m, or both leave l, or use four vertices. Collect (B)+(C)−(A); at $r=k$ the cycles F_{kkm} (the $r=k$ value of F_{rkm}) and F_{lkk} do not exist, and the two diagonal parts of (B) give X.

(iii) $[E_{li},E_{kj}]=0$. In $\partial_Y^2\tilde\varphi([E_{li},Z],E_{kj})$ the diagonal part gives $2(\mathsf{n}^l_{kj}-\mathsf{n}^i_{kj})Z_{il}Z_{jk}$; $Z_{ir}E_{lr}$ survives only for $r=k$ (the cycle $l\to k\to j\to l$): $Z_{ik}\mathsf{F}_{lkj}Z_{jl}$; $-Z_{rl}E_{ri}$ only for $r=j$ (the cycle $k\to j\to i\to k$): $-Z_{jl}\mathsf{F}_{kji}Z_{ik}$. So

$$\omega(E_{li},E_{kj})=-\theta_{li}\big[2(\mathsf{n}^l_{kj}-\mathsf{n}^i_{kj})Z_{il}Z_{jk}+(\mathsf{F}_{lkj}-\mathsf{F}_{kji})Z_{ik}Z_{jl}\big]+\theta_{kj}\big[2(\mathsf{n}^k_{li}-\mathsf{n}^j_{li})Z_{jk}Z_{il}+(\mathsf{F}_{kli}-\mathsf{F}_{lij})Z_{ik}Z_{jl}\big];$$

add the same with l and k exchanged.

(iv) This is (i) with (l,j,h) replaced by (k,l,m), written with $\mathsf{f}^{(k)}_{lm}=\mathsf{F}_{lmk}=\mathsf{F}_{klm}$ and $\mathsf{n}^s_{lk}=\mathsf{M}^{(k)}(s,l)$. □

Theorem 9.86 ($\mathsf{F}=0$ forces $\mathsf{N}=0$)**.** *Let $n\ge4$ and $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ with $\mathsf{F}=0$ (φ vanishes on $\mathcal{S}$). If $\mathsf{c}(l,k,m)=0$ at every slice point for all distinct l,k,m, then $\mathsf{N}=0$.*

Proof. Work at traceless x (every quantity is invariant under $x\mapsto x+c\mathbf{1}$). $\mathsf{F}=0$ kills f and its derivatives, so Lemma 9.85(ii) gives, for every r, (1) $\mathsf{M}^{(r)}(\Delta_{lk},\Delta_{lm})=0$ for distinct l,k,m in $R:=[n]\setminus\{r\}$, and (2) the near rows $-2\theta_{lm}\mathsf{M}^{(k)}(\Delta_{lk},\Delta_{lm})+\mathsf{X}=0$. (The rows $[l^3]$ and P are not used.)

Step 1 ($n\ge4$, so $|R|\ge3$). A matrix B on $R\times R$ with $B(\Delta_{lk},\Delta_{lm})=0$ for all distinct $l,k,m\in R$ has the form $B(s,t)=u_s+v_t$: fix $l_0\in R$ and put $u_s:=B(s,l_0)$, $v_t:=B(l_0,t)-B(l_0,l_0)$; the

row $(l_0;k,m)$ gives $B(k,m)=u_k+v_m$ for $k\neq m$, both $\neq l_0$; the entries with an index l_0 hold by definition; the row $(k;l_0,m)$ ($m\notin\{k,l_0\}$, which exists as $|R|\geq 3$) gives $B(k,k)=u_k+v_k$.

Step 2. Apply Step 1 to $B=\mathsf{M}^{(r)}$ on $R\times R$; the row $s=r$ is free so far. $\mathbf{1}^T\mathsf{M}^{(r)}=0$ gives $\mathsf{M}^{(r)}(r,t)=-U-(n-1)v_t$ with $U:=\sum_{s\in R}u_s$; then $x^T\mathsf{M}^{(r)}=0$ and $\sum_{s\in R}x_s=-x_r$ give $-nx_rv_t=x_rU-\sum_{s\in R}x_su_s$, so v_t does not depend on t ($x_r\neq 0$ at generic x). Absorbing the constant into u, $\mathsf{M}^{(r)}(s,t)=u_s^{(r)}$ for every $s\in[n]$ and $t\in R$.

Step 3. $\mathsf{n}^s_{tr}=\mathsf{M}^{(r)}(s,t)=u_s^{(r)}$ and $\mathsf{n}^s_{tr}=\mathsf{n}^s_{rt}=\mathsf{M}^{(t)}(s,r)=u_s^{(t)}$; so $u_s^{(r)}=:\bar u_s$ does not depend on r, and $\mathsf{n}^s_{pq}=\bar u_s$ for all s and all pairs, with $\sum_s\bar u_s=0$ and $\sum_s x_s\bar u_s=0$.

Step 4. Now $\mathsf{M}^{(k)}(\Delta_{lk},\Delta_{lm})=0$, and $\mathsf{X}=-2\theta_{lk}(\bar u_l-\bar u_k)-2\theta_{km}(\bar u_m-\bar u_k)=0$ reads $D_{kl}(\bar u)=D_{mk}(\bar u)$ for all distinct l,k,m: all slopes of $\bar u$ against x agree, so $\bar u_s=\alpha+\gamma x_s$. Then $\sum_s\bar u_s=n\alpha=0$ and $\sum_s x_s\bar u_s=\gamma\sum_s x_s^2=0$ (generic traceless x), so $\bar u=0$ and $\mathsf{N}=0$. □

For the next theorem split $\mathsf{F}_{pqr}=\sigma(pqr)+\tau(p,q,r)$ by orientation: $\sigma(pqr):=(\mathsf{F}_{pqr}+\mathsf{F}_{qpr})/2$ depends only on the triangle $\{p,q,r\}$, and $\tau(p,q,r):=(\mathsf{F}_{pqr}-\mathsf{F}_{qpr})/2$ is alternating. In spectator notation $\mathsf{f}^{(r)}=\mathsf{s}+\mathsf{t}$ with $\mathsf{s}_{lm}=\mathsf{s}^{(r)}_{lm}=\sigma(lmr)$ and $\mathsf{t}_{lm}=\mathsf{t}^{(r)}_{lm}=\tau(l,m,r)$; so $\mathsf{s}^{(r)}_{lm}=\mathsf{s}^{(l)}_{mr}$ and $\mathsf{t}^{(r)}_{lm}=\mathsf{t}^{(l)}_{mr}$. Both σ and τ are homogeneous of degree a and invariant under $x\mapsto x+c\mathbf{1}$. Put $\mathsf{V}(p,q,r):=x_{pq}x_{qr}x_{rp}$ (alternating), and for a 4-set Q and $w\in Q$ write $\sigma_w:=\sigma(Q\setminus\{w\})$. Let $p_2:=\sum_i x_i^2$.

Theorem 9.87 (the slice rows force $\mathsf{F}=0$)**.** *Let $n\geq 4$ and $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$. If every row $[l^3]$ and every row $[l^2k]$ (c and P) vanishes at every slice point, then $\mathsf{F}=0$.*

Proof. Step A (P-rows: σ is affine on every 4-set). $\mathsf{A}(i,j;l,k)$, the coefficient of $Z_{li}Z_{kj}$ in $\mathsf{P}(i,j;l,k)$, is a row, as is $\mathsf{A}(l,k;i,j)$. Their n-parts are equal (both are $2[D_{li}(\nu_{kj})-D_{kj}(\nu_{li})]$, where $\nu_{pq}:=(\mathsf{n}^s_{pq})_s$). In the F-parts the four triangles of $Q=\{l,k,i,j\}$ each occur once with opposite orientations (F_{klj} with F_{jlk}, F_{lji} with F_{ijl}, F_{lki} with F_{ikl}, F_{kij} with F_{jik}), so τ cancels and

$$\mathsf{A}(l,k;i,j)-\mathsf{A}(i,j;l,k)=2\theta_{ki}(\sigma_k-\sigma_i)+2\theta_{lj}(\sigma_j-\sigma_l).$$

So the slope $S(p,q):=(\sigma_p-\sigma_q)/x_{pq}$ satisfies $S(i,k)=S(j,l)$ for each of the three perfect matchings of Q. Then $\sigma_w=\alpha_Q+\gamma_Q x_w$ on Q: write $Q=\{1,2,3,4\}$ and let δ_w be σ_w minus the affine function of x_w through the points 1 and 2, so that $\delta_1=\delta_2=0$; $S(1,2)=S(3,4)$ gives $\delta_3=\delta_4=:c$; $S(1,3)=S(2,4)$ and $S(1,4)=S(2,3)$ give $c=0$ unless $x_1-x_3=x_2-x_4$ and $x_1-x_4=x_2-x_3$, which would give $x_3=x_4$. Consequence: for distinct l,k,m,r, $\theta_{lk}(\mathsf{s}_{km}-\mathsf{s}_{lm})+\theta_{km}(\mathsf{s}_{lk}-\mathsf{s}_{lm})=-\gamma_Q+\gamma_Q=0$ ($\mathsf{s}=\mathsf{s}^{(r)}$, $Q=\{l,k,m,r\}$).

Step B (c-rows: spectator against near). Fix distinct l,m,r. For $k\notin\{l,m\}$ let E_k be the f-part of the row with n-term $-2\theta_{lm}\mathsf{M}^{(r)}(\Delta_{lk},\Delta_{lm})$: the spectator row $\mathsf{c}_r(l,k,m)$ for $k\neq r$, and $\mathrm{NEAR}(l,r,m)$ for $k=r$ (Lemma 9.85(iv); the rows $[l^3]$ are used here). Let E'_k be the same for (m,l). Then $\mathsf{E}_k=2\theta_{lm}\mathsf{M}^{(r)}(\Delta_{lk},\Delta_{lm})$ and $\mathsf{E}'_k=2\theta_{ml}\mathsf{M}^{(r)}(\Delta_{mk},\Delta_{ml})$, so

$$\mathsf{E}_k-\mathsf{E}'_k=2\theta_{lm}\mathsf{M}^{(r)}(\Delta_{lm},\Delta_{lm})=:4\theta_{lm}\mathsf{q}\quad\text{does not depend on } k. \tag{4}$$

Written out ($\mathsf{f}=\mathsf{s}+\mathsf{t}$, spectator r):

$$\begin{aligned}\mathsf{E}_k-\mathsf{E}'_k&=2\theta_{lk}(\mathsf{s}_{km}-\mathsf{s}_{lm})+2\theta_{km}(\mathsf{s}_{lk}-\mathsf{s}_{lm})+2\theta_{lm}\mathsf{s}_{lm}\\&\quad+(\partial_m-\partial_l)\mathsf{s}_{lm}+(2\partial_k-\partial_l-\partial_m)\mathsf{t}_{lm}\quad(k\neq r),\\\mathsf{E}_r-\mathsf{E}'_r&=2\theta_{lm}\mathsf{s}_{lm}+(\partial_m-\partial_l)\mathsf{s}_{lm}+(2\partial_r-\partial_l-\partial_m)\mathsf{t}_{lm}\\&\quad-2(\theta_{lr}+\theta_{mr})\mathsf{t}_{lm}.\end{aligned}$$

(4) for $k\notin\{l,m,r\}$ against $k=r$, with Step A, gives

$$(\partial_k-\partial_r)\tau(l,m,r)=-(\theta_{lr}+\theta_{mr})\,\tau(l,m,r)\qquad\text{for every } k\notin\{l,m,r\}. \tag{5}$$

Step C (Euler: $(a-3)\tau=0$). Rotating (l,m,r) in (5) gives $(\partial_k-\partial_l)\tau=-(\theta_{ml}+\theta_{rl})\tau$ and $(\partial_k-\partial_m)\tau=-(\theta_{rm}+\theta_{lm})\tau$ for the same $\tau=\tau(l,m,r)$; for two outside indices k,k', $(\partial_k-\partial_{k'})\tau=0$. At traceless x Euler's identity reads $a\tau=\sum_p x_p(\partial_p-\partial_k)\tau$, so

$$a\tau=\big[x_r(\theta_{lr}+\theta_{mr})+x_l(\theta_{ml}+\theta_{rl})+x_m(\theta_{rm}+\theta_{lm})\big]\tau=3\tau$$

($x_r\theta_{lr}+x_l\theta_{rl}=1$, and the same for the other two pairs). So $\tau=0$ if $a\neq 3$. If $a=3$: $\partial_r\log\mathsf{V}(l,m,r)=\theta_{lr}+\theta_{mr}$ (and rotations), so every $(\partial_p-\partial_k)(\tau/\mathsf{V})$ vanishes; with the translation invariance, τ/V is constant, $\tau(l,m,r)=c_T\mathsf{V}(l,m,r)$. φ is invariant under permutation matrices, so $\mathsf{F}_{\pi(p)\pi(q)\pi(r)}(\pi x)=\mathsf{F}_{pqr}(x)$ with $(\pi x)_{\pi(p)}:=x_p$; V transforms in the same way, so $c_T=c$ is one constant for all triangles. (This is the shape of $\operatorname{tr}([A,B]^3)$, whose F is $3\mathsf{V}$.)

Step D (the $\mathcal{E}$-contraction: σ from τ). Fix distinct l,m,r; $\mathsf{s}:=\mathsf{s}^{(r)}$, $\mathsf{t}:=\mathsf{t}^{(r)}$, $\mathsf{M}:=\mathsf{M}^{(r)}$; work at traceless x. (D1) $\mathsf{E}_k=2\theta_{lm}\mathsf{M}(\Delta_{lk},\Delta_{lm})$ and $\mathsf{E}'_k=2\theta_{ml}\mathsf{M}(\Delta_{mk},\Delta_{ml})$ for every $k\notin\{l,m\}$ ($k=r$: the near rows). (D2) For $w\in\mathbb{C}^n$ with $w_r=0$, $\sum_{k\notin\{l,m\}}x_k=-(x_l+x_m)$ and $x^T\mathsf{M}=0$ give

$$\sum_{k\notin\{l,m\}}x_k\mathsf{M}(\Delta_{lk},w)=-(x_l+x_m)\mathsf{M}(e_l,w)+x_l\mathsf{M}(e_l,w)+x_m\mathsf{M}(e_m,w)=-x_m\mathsf{M}(\Delta_{lm},w).$$

(D3) By (D1), (D2) with $w=\Delta_{lm}$, and $\mathsf{M}(\Delta_{ml},\Delta_{ml})=\mathsf{M}(\Delta_{lm},\Delta_{lm})=2\mathsf{q}$: $\sum_{k\notin\{l,m\}}x_k\mathsf{E}_k=-4\theta_{lm}x_m\mathsf{q}$ and $\sum_{k\notin\{l,m\}}x_k\mathsf{E}'_k=4\theta_{lm}x_l\mathsf{q}$, so $\sum_{k\notin\{l,m\}}x_k(\mathsf{E}_k+\mathsf{E}'_k)=-4\mathsf{q}$, and with (4)

$$\mathsf{E}_k-\mathsf{E}'_k=-\theta_{lm}\sum_{k'\notin\{l,m\}}x_{k'}(\mathsf{E}_{k'}+\mathsf{E}'_{k'})\qquad\text{for every } k\notin\{l,m\}.$$

(D4) By Lemma 9.85(ii), (iv) with $\mathsf{f}=\mathsf{s}+\mathsf{t}$, the θ-terms in s cancel in the sums, and

$$\begin{gathered}\mathsf{E}_{k'}+\mathsf{E}'_{k'}=(2\partial_{k'}-\partial_l-\partial_m)\mathsf{s}_{lm}+(\partial_m-\partial_l)\mathsf{t}_{lm}+2\theta_{lm}\mathsf{t}_{lm}+\rho_{k'}\quad(k'\neq r),\\ \rho_{k'}:=2\theta_{lk'}(\mathsf{t}_{k'm}-\mathsf{t}_{lm})+2\theta_{k'm}(\mathsf{t}_{lk'}-\mathsf{t}_{lm}),\\ \mathsf{E}_r+\mathsf{E}'_r=(2\partial_r-\partial_l-\partial_m)\mathsf{s}_{lm}+(\partial_m-\partial_l)\mathsf{t}_{lm}+2\theta_{lm}\mathsf{t}_{lm}+\rho_r,\\ \rho_r:=2(\theta_{mr}-\theta_{lr})\mathsf{t}_{lm}.\end{gathered}$$

(Term by term: $\theta_{lk'}(\mathsf{f}_{k'm}-\mathsf{f}_{lm})+\theta_{k'l}(\mathsf{f}_{mk'}-\mathsf{f}_{ml})=2\theta_{lk'}(\mathsf{t}_{k'm}-\mathsf{t}_{lm})$; $\theta_{k'm}(\mathsf{f}_{lk'}-\mathsf{f}_{lm})+\theta_{mk'}(\mathsf{f}_{k'l}-\mathsf{f}_{ml})=2\theta_{k'm}(\mathsf{t}_{lk'}-\mathsf{t}_{lm})$; $(\partial_{k'}-\partial_l)\mathsf{f}_{lm}+(\partial_{k'}-\partial_m)\mathsf{f}_{ml}=(2\partial_{k'}-\partial_l-\partial_m)\mathsf{s}_{lm}+(\partial_m-\partial_l)\mathsf{t}_{lm}$; $\theta_{lm}\mathsf{f}_{lm}+\theta_{ml}\mathsf{f}_{ml}=2\theta_{lm}\mathsf{t}_{lm}$; near: $\theta_{lr}(\mathsf{f}_{ml}-\mathsf{f}_{lm})+\theta_{mr}(\mathsf{f}_{lm}-\mathsf{f}_{ml})=2(\theta_{mr}-\theta_{lr})\mathsf{t}_{lm}$.) (D5) Euler's identity for s and $\sum_{k'\notin\{l,m\}}x_{k'}=-(x_l+x_m)$ give

$$\begin{gathered}\sum_{k'\notin\{l,m\}}x_{k'}(2\partial_{k'}-\partial_l-\partial_m)\mathsf{s}_{lm}=2a\,\mathsf{s}_{lm}+x_{ml}(\partial_l-\partial_m)\mathsf{s}_{lm},\\ \sum_{k'\notin\{l,m\}}x_{k'}\big[(\partial_m-\partial_l)\mathsf{t}_{lm}+2\theta_{lm}\mathsf{t}_{lm}\big]=-(x_l+x_m)\big[(\partial_m-\partial_l)\mathsf{t}_{lm}+2\theta_{lm}\mathsf{t}_{lm}\big],\end{gathered}$$

and $\mathsf{T}_{lm}:=\sum_{k'\notin\{l,m\}}x_{k'}\rho_{k'}=\sum_{k'\notin\{l,m,r\}}x_{k'}\big[2\theta_{lk'}(\mathsf{t}_{k'm}-\mathsf{t}_{lm})+2\theta_{k'm}(\mathsf{t}_{lk'}-\mathsf{t}_{lm})\big]+2x_r(\theta_{mr}-\theta_{lr})\mathsf{t}_{lm}$. So, with $\theta_{lm}x_{ml}=1$,

$$-\theta_{lm}\sum_{k'}x_{k'}(\mathsf{E}_{k'}+\mathsf{E}'_{k'})=-2a\theta_{lm}\mathsf{s}_{lm}+(\partial_m-\partial_l)\mathsf{s}_{lm}+\theta_{lm}(x_l+x_m)\big[(\partial_m-\partial_l)\mathsf{t}_{lm}+2\theta_{lm}\mathsf{t}_{lm}\big]-\theta_{lm}\mathsf{T}_{lm}.$$

(D6) For $k\notin\{l,m,r\}$, Steps B and A give $\mathsf{E}_k-\mathsf{E}'_k=2\theta_{lm}\mathsf{s}_{lm}+(\partial_m-\partial_l)\mathsf{s}_{lm}+(2\partial_k-\partial_l-\partial_m)\mathsf{t}_{lm}$. Put it into (D3); $(\partial_m-\partial_l)\mathsf{s}_{lm}$ cancels:

(6)

$$2(a+1)\theta_{lm}\sigma(lmr)=\mathcal{R}_r(\mathsf{t}):=-(2\partial_k-\partial_l-\partial_m)\mathsf{t}_{lm}+\theta_{lm}(x_l+x_m)\big[(\partial_m-\partial_l)\mathsf{t}_{lm}+2\theta_{lm}\mathsf{t}_{lm}\big]-\theta_{lm}\mathsf{T}_{lm},$$

which does not depend on k, by (5). If $a\neq 3$, then $\mathsf{t}=0$ by Step C, so $\sigma=0$.

(D7) Let $a = 3$. By Step C, $\mathsf{t}^{(r)}_{pq} = c\,\mathsf{V}(p,q,r)$ for all $p,q \neq r$; take $c = 1$, write u, v, w for x_l, x_m, x_r, and put $P := u + v$, $\mathsf{V} := \mathsf{V}(l,m,r) = (u-v)(v-w)(w-u)$. Then $\partial_k \mathsf{V} = 0$ for $k \notin \{l,m,r\}$, and

$$(2\partial_k - \partial_l - \partial_m)\mathsf{V} = \partial_r \mathsf{V} = (u-v)(u+v-2w), \qquad (\partial_m - \partial_l)\mathsf{V} = 2w^2 - 2wP - u^2 - v^2 + 4uv,$$
$$2(x_l + x_m)\theta_{lm}\mathsf{V} = -2P(v-w)(w-u).$$

For an outside k' with $y := x_{k'}$: $\mathsf{V}(k',m,r) - \mathsf{V}(l,m,r) = (v-w)(y-u)(v+w-y-u)$ and $\mathsf{V}(l,k',r) - \mathsf{V}(l,m,r) = (w-u)(y-v)(u+w-y-v)$, so $\rho_{k'} = 2(v-w)(v+w-y-u) - 2(w-u)(u+w-y-v) = 2(2w-P)y + 2Q'$ with $Q' := (u-v)^2 - 2w^2 + wP$, and $\rho_r = 2(\theta_{mr} - \theta_{lr})\mathsf{V} = 2(u-v)^2$. With $\sum_{\text{out}} y = -(P+w)$ and $\sum_{\text{out}} y^2 = p_2 - u^2 - v^2 - w^2$,

$$\begin{aligned}\mathsf{T}_{lm} &= 2(2w-P)(p_2 - u^2 - v^2 - w^2) - 2Q'(P+w) + 2w(u-v)^2 \\ &= 2(2w-P)p_2 - 6wP^2 + 8uvw + 4uvP + 4Pw^2.\end{aligned}$$

The other three terms of $x_{ml}\mathcal{R}_r$ are $(u-v)^2(P-2w) + P(2w^2 - 2wP - u^2 - v^2 + 4uv) - 2P(v-w)(w-u) = 4Pw^2 - 6wP^2 + 4uvP + 8uvw$. Hence $x_{ml}\mathcal{R}_r = 2(P-2w)p_2$, and (6) with $2(a+1) = 8$ gives, for general c,

$$\sigma(lmr) = c\,(x_l + x_m - 2x_r)\,p_2/4.$$

(At $n = 4$, $c = 1$ and $(u,v,w,y) = (-3,-1,0,4)$: $\mathsf{V}(l,m,r) = 6$, $\mathsf{V}(k',m,r) = 20$, $\mathsf{V}(l,k',r) = -84$, $\rho_{k'} = 40$, $\rho_r = 8$, $\mathsf{T}_{lm} = 4 \cdot 40 + 0 \cdot 8 = 160$, the other three terms give $-16 - 8 - 24 = -48$, and $x_{ml}\mathcal{R}_r = -208 = 2(P-2w)p_2$ with $P = -4$, $p_2 = 26$, so $\sigma = -26$. At $(u,v,w,y) = (2,-1,3,-4)$: $\mathsf{V}(l,m,r) = -12$, $\mathsf{V}(k',m,r) = 84$, $\mathsf{V}(l,k',r) = -42$, $\rho_{k'} = -52$, $\rho_r = 18$, $\mathsf{T}_{lm} = (-4)(-52) + 3 \cdot 18 = 262$, the other three terms give $-45 - 1 + 8 = -38$, and $x_{ml}\mathcal{R}_r = -300$ with $P = 1$, $p_2 = 30$, so $\sigma = -37.5$.)

(D8) $\sigma(lmr)$ is symmetric in l,m,r ($\sigma(lmr) = \mathsf{s}^{(r)}_{lm} = \mathsf{s}^{(l)}_{mr}$), and (D7) for the pair (m,r) with spectator l gives $\sigma(lmr) = c\,(x_m + x_r - 2x_l)\,p_2/4$. So $3c\,x_{lr}\,p_2 = 0$ as polynomials, and p_2 is a nonzero polynomial on the traceless hyperplane, so $c = 0$. Then $\tau = 0$, and $\sigma = 0$ by (6). So $\mathsf{F} = 0$ for every a. □

Theorem 9.88 (B3)**.** *Let $n \geq 4$, let a be arbitrary, and let F, N be as in Lemma 9.84. The elements of $\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ on which every row $[l^3]$ and $[l^2k]$ vanishes at every slice point are exactly those with $\mathsf{F} = 0$ and $\mathsf{N} = 0$, that is, with $\tilde{\varphi}(D, Z + \operatorname{diag}(y)) = \mathsf{C}(x;y)$ at all base points.*

Proof. If the rows vanish, Theorem 9.87 gives $\mathsf{F} = 0$ and then Theorem 9.86 gives $\mathsf{N} = 0$. Conversely, by Lemma 9.85 every row is a function of F, N and the first derivatives of F, and $\mathsf{F} = 0$ as a function gives $\partial\mathsf{F} = 0$. □

Remark 9.89. (1) The kernel in Theorem 9.88 can be nonzero. The function $w_{12} = \operatorname{tr}(X^3)\operatorname{tr}(XY) - \operatorname{tr}(X^2)\operatorname{tr}(X^2Y)$ of Subsection 9.14 spans $\mathcal{F}^{\mathcal{E}}_{(4,1)}(n)$; at a base point with traceless x, with $p_k := \sum_i x_i^k$, $\tilde{w}_{12} = \sum_s g_s(x)y_s$ with $g_s = p_3x_s - p_2x_s^2 + p_2^2/n$, since every Z_{pq} has nonzero torus weight. $\mathcal{E}$ is a derivation, so w_{12}^3 lies in $\mathcal{F}^{\mathcal{E}}_{(12,3)}(n)$; it is nonzero for $n \geq 3$, cubic in y alone at the base points, and has $\mathsf{F} = \mathsf{N} = 0$. We do not decide for which a the kernel is nonzero. This does not conflict with Theorem 9.22: the slice rows are only part of the rows of $\mathcal{Z}$.

(2) At $a = 3$ the c-rows, the rows $[l^3]$ (through NEAR) and the P-differences allow $\tau = c\mathsf{V}$, which solves (5); the $\mathcal{E}$-contraction (D2) then forces a σ that is not symmetric, which kills c. So $\operatorname{tr}([A,B]^3) \in \mathcal{F}^{\mathcal{E}}_{(3,3)}(n)$ ($\mathsf{F} = 3\mathsf{V}$, $\tau = 3\mathsf{V}$, $\sigma = 0$, $\mathsf{N} = 0$) is not in the kernel of the slice rows; a direct witness is one row $[l^3]$: by Lemma 9.85(i), $\omega(E_{lj}, E_{lh}) = 3x_{hj}(2x_l - x_h - x_j)Z_{jl}Z_{hl} \neq 0$. This agrees with Theorem 9.22 at $(3,3)$ in its range.

(3) What is used where. $\mathcal{E}\varphi = 0$ enters only through $x^T\mathsf{M}^{(r)} = 0$ (Step D2 and Steps 2 and 4 of the proof of Theorem 9.86); the invariance under $Y \mapsto Y + cI$ only through $\mathbf{1}^T\mathsf{M}^{(r)} = 0$ (the same two steps); the invariance under $x \mapsto x + c\mathbf{1}$ through the use of traceless x, which every step

may assume (it is used in Steps 2 and 4 of the proof of Theorem 9.86 and in Steps C, D2, D5, D7 and D8). The n-part of the P-rows, the rows $[l^3]$ outside the near term, and the coefficients of $u_l u_k u_m$ (l,k,m distinct) are not used. The proofs use a only through $a+1\neq 0$ and the factor $a-3$; $\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)=0$ for $a<3$, as a highest weight vector needs $a\geq b$.

(4) Theorem 9.88 and Theorem 9.22 do not imply each other: Theorem 9.88 uses only the slice rows and leaves the part C, but holds for every $n\geq 4$, while Theorem 9.22 at $b=3$ uses all rows of $\mathcal{Z}$ and is proved in the stable range $n\geq a+4$ (Lemma 9.90 removes the part C; see Corollary 9.91). This differs from $b=2$, where Corollary 9.81(a) contains Theorem 9.22 at $b=2$ and $a\geq 3$.

Lemma 9.90 (Lemma Y)**.** *Let $b\geq 2$, $n\geq 3$ and $\varphi\in\mathcal{F}_{(a,b)}(n)$; so $\tilde\varphi(X,Y+\gamma I)=\tilde\varphi(X,Y)$ for all γ. Suppose that*

$$\tilde\varphi(D,Z+\operatorname{diag}(y))=\mathsf{C}(x;y) \tag{7}$$

does not depend on Z (for all x with distinct entries, all y and all off-diagonal Z), and that $\mathcal{Z}(\varphi)$ vanishes at every base point $(D,Z+\operatorname{diag}(y))$. Then $\varphi=0$.

Proof. Below p,q,r,s,t are matrix indices. Put $\mathsf{C}_s:=\partial\mathsf{C}/\partial y_s$ and $\mathsf{C}_{st}:=\partial^2\mathsf{C}/\partial y_s\partial y_t$.

(0) By (7), $\tilde\varphi(D,\cdot)$ depends only on the diagonal of its argument, so $\partial_Y[V]\tilde\varphi=\sum_s \mathsf{C}_s V_{ss}$ and $\partial_Y^2\tilde\varphi(W,V)=\sum_{s,t}\mathsf{C}_{st}W_{ss}V_{tt}$ (at $\operatorname{diag}Y=y$). The diagonal of $[E_{pq},Y]$ is $Y_{qp}\Delta_{pq}=Z_{qp}\Delta_{pq}$, and that of $[E_{pq},V]$ is $V_{qp}\Delta_{pq}$. So (3) gives, for $p\neq q$, $\beta(E_{pq},V)=-\theta_{pq}\big[Z_{qp}\sum_t(\mathsf{C}_{pt}-\mathsf{C}_{qt})V_{tt}+V_{qp}(\mathsf{C}_p-\mathsf{C}_q)\big]$, and $\beta(E_{pp},V)=\sum_s\partial_{x_p}\mathsf{C}_s\,V_{ss}$. With $U_d:=(U_{tt})_t$ and the vector $K_{pq}:=(\mathsf{C}_{pt}-\mathsf{C}_{qt})_t$, linearity in U gives

$$\beta(U,V)=-\sum_{p\neq q}\theta_{pq}U_{pq}\big[Z_{qp}(K_{pq}\cdot V_d)+V_{qp}(\mathsf{C}_p-\mathsf{C}_q)\big]+\sum_{p,s}U_{pp}V_{ss}\,\partial_{x_p}\mathsf{C}_s.$$

In $\omega(U,V)=\beta(U,V)-\beta(V,U)$ the terms with $\mathsf{C}_p-\mathsf{C}_q$ cancel: in $\beta(V,U)$ they are $-\theta_{pq}V_{pq}U_{qp}(\mathsf{C}_p-\mathsf{C}_q)$, and exchanging p and q turns these into the terms $-\theta_{pq}U_{pq}V_{qp}(\mathsf{C}_p-\mathsf{C}_q)$ of $\beta(U,V)$. Hence

$$\omega(U,V)=\sum_{p,s}U_{pp}V_{ss}\,\Omega_{ps}-\sum_{p\neq q}\theta_{pq}Z_{qp}\big[U_{pq}(K_{pq}\cdot V_d)-V_{pq}(K_{pq}\cdot U_d)\big],\qquad \Omega_{ps}:=\partial_{x_p}\mathsf{C}_s-\partial_{x_s}\mathsf{C}_p,$$

and ω is affine in Z.

(1) Take the coefficient of Z_{qp} ($p\neq q$) in Q_u at the base points. With $U=uv^T$ and $V=uv'^T$ ($v,v'\in\mathbb{C}^n$ are the corner vectors, not the diagonal y of the base point; $U_{pq}=u_pv_q$, $U_d=(u_tv_t)_t$), only the term (p,q) of (0) carries Z_{qp}, and the coefficient of Z_{qp} in $\omega^u(v,v')$ is $-\theta_{pq}u_p\big[v_q(w_{pq}\cdot v')-v'_q(w_{pq}\cdot v)\big]=-\theta_{pq}u_p\,(e_q\wedge w_{pq})(v,v')$ with $w_{pq}:=\sum_t u_t(\mathsf{C}_{pt}-\mathsf{C}_{qt})e_t$. So the coefficient of Z_{qp} in $\mathsf{Q}_u=u\wedge\omega^u$ is

$$-\theta_{pq}u_p\,(u\wedge e_q\wedge w_{pq})=-\theta_{pq}u_p\sum_{s<t;\ s,t\neq q}u_su_t\big(K_t^{(pq)}-K_s^{(pq)}\big)\,e_s\wedge e_q\wedge e_t,\qquad K_t^{(pq)}:=\mathsf{C}_{pt}-\mathsf{C}_{qt}$$

(the pair (s,t) gives $u_su_tK_t^{(pq)}\,e_s\wedge e_q\wedge e_t$, the pair (t,s) gives $-u_su_tK_s^{(pq)}\,e_s\wedge e_q\wedge e_t$, and the terms with s or t equal to q vanish). As $\theta_{pq}\neq 0$ and these 3-forms are linearly independent, it vanishes for all u if and only if $K_t^{(pq)}=K_s^{(pq)}$ for all $s,t\neq q$. So for all $p\neq q$, $\mathsf{C}_{pt}-\mathsf{C}_{qt}$ does not depend on $t\neq q$.

(2) Take $r\notin\{p,q\}$ (here $n\geq 3$ is used). From (p,q) at $t=p$ and $t=r$: $\mathsf{C}_{pp}-\mathsf{C}_{qp}=\mathsf{C}_{pr}-\mathsf{C}_{qr}$; from (q,p) at $t=q$ and $t=r$: $\mathsf{C}_{qq}-\mathsf{C}_{pq}=\mathsf{C}_{qr}-\mathsf{C}_{pr}$. Adding, $\mathsf{C}_{pp}+\mathsf{C}_{qq}=2\mathsf{C}_{pq}$ for all $p\neq q$; with $h_s:=\mathsf{C}_{ss}/2$ this reads $\mathsf{C}_{st}=h_s+h_t$ for all s,t. Only the components $\mathsf{c}(p,r,q)$ and $\mathsf{c}(q,r,p)$ of Lemma 9.83, at base points with $y\neq 0$, were used.

(3) $\tilde\varphi(X,Y+\gamma I)=\tilde\varphi(X,Y)$ gives $\mathsf{C}(x;y+\gamma\mathbf{1})=\mathsf{C}(x;y)$, so $\sum_s\mathsf{C}_s=0$ and $\sum_t\mathsf{C}_{st}=0$ for every s: $nh_s+\sum_t h_t=0$. So $h_s=-\frac{1}{n}\sum_t h_t$ for all s; summing, $\sum_t h_t=0$, so $h=0$ and $\mathsf{C}_{st}=0$. (This finish is the algebra of Step 4 of the proof of Theorem 9.80, applied to the Hessian of C.)

(4) So C is affine in y; it is homogeneous of degree $b \geq 2$ in y, so $\mathsf{C} = 0$. By (7), $\tilde{\varphi}$ vanishes at every (D, Y) with D diagonal with distinct entries; the GL_n-saturation of this set is dense, so $\varphi = 0$. □

Corollary 9.91 (Z-INJ at $b = 3$)**.** *Let $n \geq 4$, let a be arbitrary, and $h := \dim \mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$.*

(a) $\mathcal{Z}(\varphi) \neq 0$ for every $0 \neq \varphi \in \mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$.

(b) The pencil $\kappa\mathcal{Z} - \mathcal{J}$ is injective on $\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ for all but at most h values of κ; in particular it has at most h eigenvalues there.

Proof. (a) Let $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ with $\mathcal{Z}(\varphi) = 0$. The slice rows are coefficients of Q_u at the slice points, so they vanish, and Theorem 9.88 gives $\mathsf{F} = 0$ and $\mathsf{N} = 0$. By Lemma 9.84, $\tilde{\varphi}(D, Z + \mathrm{diag}(y)) = \mathsf{C}(x; y)$, which is (7); Lemma 9.90 ($b = 3$, $n \geq 4$) gives $\varphi = 0$. (b) As in the proof of Corollary 9.81(b), with (a) in place of Theorem 9.80. □

Remark 9.92. (1) Lemma 9.90 uses only the conjugation invariance (through (3) and (7)) and the invariance under $Y \mapsto Y + \gamma I$; it does not use $\mathcal{E}\varphi = 0$ or the part Ω of ω, and needs only $n \geq 3$; $\mathcal{E}\varphi = 0$ enters Corollary 9.91 only through Theorem 9.88. The invariance under $Y \mapsto Y + \gamma I$ is essential: on $M_n \times M_n$, $(\mathrm{tr}\, X)^a (\mathrm{tr}\, Y)^b$ satisfies (7) with $\mathsf{C} = (\sum x)^a (\sum y)^b$, and its $\beta(U, V) = ab(\mathrm{tr}\, X)^{a-1} (\mathrm{tr}\, Y)^{b-1} \mathrm{tr}\, U \,\mathrm{tr}\, V$ is symmetric, so $\omega = 0$ and $\mathrm{tr} \wedge \omega = 0$, although the function is not 0.

(2) At $b = 2$ with $\mathsf{F} = 0$, Lemma 9.90 reproduces the end of the proof of Theorem 9.80 (Step 4: $\mathsf{G} = 0$) from the same c-rows, without $\mathcal{E}\varphi = 0$; the steps differ (Theorem 9.80 goes through (Δ) and Step 4, the lemma through its steps (2)–(3)).

(3) For w^3_{12} of Remark 9.89(1), $\mathsf{C} = (g \cdot y)^3$ and $\mathsf{C}_{st} = 6(g \cdot y) g_s g_t$, so steps (1)–(2) would need $6(g \cdot y)(g_s - g_t)^2 = 0$, that is, g constant, and then $g = 0$ as $\sum_s g_s = 0$. But $g \neq 0$ for $n \geq 3$ (at $n = 3$, $x = (1, -1, 0)$: $g = (-\frac{2}{3}, -\frac{2}{3}, \frac{4}{3})$). So $\mathcal{Z}(w^3_{12}) \neq 0$ for every $n \geq 3$: w^3_{12} lies in the kernel of the slice rows but not in that of $\mathcal{Z}$.

(4) Corollary 9.91(a) is Theorem 9.22 at $b = 3$ for every $n \geq 4$: new for $4 \leq n \leq a + 3$, and a second proof for $n \geq a + 4$, by a different mechanism. The case $n = 3$ at $b = 3$ is not covered (Theorem 9.88 needs $n \geq 4$). With Corollary 9.81(a), Theorem 9.22 now holds for every n at $b = 2$ ($a \geq 3$, $n \geq 3$) and at $b = 3$ ($n \geq 4$); for $b \geq 5$, and for $b = 4$ with $n = 3, 4$, only its stable range is known. Lemma 9.90 applies for every $b \geq 2$, and Lemma 9.93 below removes every order $k \geq 2$ in y; so for $b \geq 4$ what is missing is the analogue of $\mathsf{F} = \mathsf{N} = 0$, that is $\Phi_0 = \Phi_1 = 0$ in the notation of Lemma 9.93 (Corollary 9.94). At $b = 4$ and $n \geq 5$ this is done in Subsection 9.23 (Corollary 9.130); for $b \geq 5$, and for $b = 4$ with $n = 3, 4$, it is not done here.

Lemma 9.93 (order lemma)**.** *Let $n \geq 3$, let a, b be arbitrary and $0 \neq \varphi \in \mathcal{F}_{(a,b)}(n)$; so $\tilde{\varphi}(X, Y + \gamma I) = \tilde{\varphi}(X, Y)$ for all γ. Let $\Phi(x, y, Z) := \tilde{\varphi}(D, Z + \mathrm{diag}(y))$ as in Lemma 9.84, let Φ_j be the part of Φ of degree j in y (a polynomial in x, y and Z, which may depend on Z), and let k be the least j with $\Phi_j \neq 0$, the y-order of φ ($\Phi \neq 0$ by the density argument of step (4) of the proof of Lemma 9.90). If $k \geq 2$, then $\mathsf{Q}_u \neq 0$ at some base point $(D, Z + \mathrm{diag}(y))$; in particular $\mathcal{Z}(\varphi) \neq 0$.*

Proof. Suppose that $\mathsf{Q}_u = 0$ at every base point. Then every coefficient of Q_u as a polynomial in u vanishes there, and so does each of its parts of fixed degree in y. All quantities below are polynomials in (y, Z) with coefficients in $\mathbb{C}[x, \theta_{pq} : p \neq q]$, a domain. Below p, q, r, s, t are matrix indices. Put $\mathsf{C}_{st} := \partial^2 \Phi_k / \partial y_s \partial y_t$: a symmetric matrix, homogeneous of degree $k - 2$ in y, which may depend on x and Z.

(0) On a polynomial homogeneous of degree j in y, a derivative along a diagonal matrix gives degree $j - 1$, a derivative along an off-diagonal matrix gives degree j, and ∂_{x_p} keeps the degree. For $p \neq q$ write $[E_{pq}, Y] = W_d + W_o$, with $W_d = Z_{qp} \Delta_{pq}$ its diagonal and W_o off-diagonal (its entries are Z_{qs}, $-Z_{sp}$ and $y_q - y_p$, of degree at most 1 in y), and $V = V_d + V_o$ likewise. In (3), on Φ_j the terms $\partial^2_Y \tilde{\varphi}(W_d, V_d)$, $\partial^2_Y \tilde{\varphi}(W_d, V_o)$, $\partial^2_Y \tilde{\varphi}(W_o, V_d)$, $\partial^2_Y \tilde{\varphi}(W_o, V_o)$, $\partial_Y [[E_{pq}, V]] \tilde{\varphi}$ and $\partial_{x_p} \partial_Y [V] \tilde{\varphi}$ have degree at least $j-2$, $j-1$, $j-1$, j, $j-1$ and $j-1$ in y. As $\Phi_j = 0$ for $j < k$, the part of degree $k-2$ in y of

$\beta(E_{pq},V)$ is $-\theta_{pq}Z_{qp}\sum_{s,t}(\Delta_{pq})_s\mathsf{C}_{st}V_{tt}$, and that of $\beta(E_{pp},V)$ is 0. With U_d and $K_{pq}:=(\mathsf{C}_{pt}-\mathsf{C}_{qt})_t$ as in the proof of Lemma 9.90, the part of degree $k-2$ in y of ω is therefore

$$\omega_{[k-2]}(U,V)=-\sum_{p\neq q}\theta_{pq}Z_{qp}\big[U_{pq}(K_{pq}\cdot V_d)-V_{pq}(K_{pq}\cdot U_d)\big].$$

(1) Let p,q,r be distinct (here $n\geq 3$ is used). By Lemma 9.83, $\mathsf{c}(p,r,q)=\omega(E_{pr},E_{rq})+\omega(E_{rr},E_{pq})-\omega(E_{pp},E_{pq})$ is a coefficient of Q_u, so its part of degree $k-2$ in y vanishes. By (0), $\omega_{[k-2]}(E_{pr},E_{rq})=0$, as both diagonals are 0, and

$$\omega_{[k-2]}(E_{rr},E_{pq})=\theta_{pq}Z_{qp}(\mathsf{C}_{pr}-\mathsf{C}_{qr}),\qquad \omega_{[k-2]}(E_{pp},E_{pq})=\theta_{pq}Z_{qp}(\mathsf{C}_{pp}-\mathsf{C}_{qp}).$$

So

$$0=\theta_{pq}Z_{qp}\big[(\mathsf{C}_{pr}-\mathsf{C}_{qr})-(\mathsf{C}_{pp}-\mathsf{C}_{qp})\big]=-\theta_{pq}Z_{qp}\,\Delta_{pq}^T\mathsf{C}\,\Delta_{pr},$$

and, as $\theta_{pq}Z_{qp}\neq 0$ in a domain, $\mathsf{C}_{pr}-\mathsf{C}_{qr}=\mathsf{C}_{pp}-\mathsf{C}_{qp}$. So for all $p\neq q$, $\mathsf{C}_{pt}-\mathsf{C}_{qt}$ does not depend on $t\neq q$: the conclusion of step (1) of the proof of Lemma 9.90, now with C_{st} depending on Z.

(2) $\Phi(x,y+\gamma\mathbf{1},Z)=\Phi(x,y,Z)$, as $\operatorname{diag}(y+\gamma\mathbf{1})=\operatorname{diag}(y)+\gamma I$. The part of degree $k-1$ in y and 1 in γ of this identity is $\gamma\sum_s\partial\Phi_k/\partial y_s=0$; so $\sum_t\mathsf{C}_{st}=0$ for every s. Steps (2) and (3) of the proof of Lemma 9.90 use only the conclusion of (1), the symmetry of C and $\sum_t\mathsf{C}_{st}=0$, pointwise in (x,y,Z); so they give $\mathsf{C}_{st}=0$.

(3) So Φ_k is affine in y; it is homogeneous of degree $k\geq 2$ in y, so $\Phi_k=0$, against the choice of k. □

Corollary 9.94 (reduction to y-degrees 0 and 1)**.** *Let $n\geq 3$, let a,b be arbitrary, and let Φ_0, Φ_1 be as in Lemma 9.93.*

(a) If $\varphi\in\mathcal{F}_{(a,b)}(n)$, $\mathcal{Z}(\varphi)=0$ and $\Phi_0=\Phi_1=0$, then $\varphi=0$.
(b) $\ker\mathcal{Z}\cap\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)=0$ if and only if every φ in it has $\Phi_0=\Phi_1=0$.

Proof. (a) If $\varphi\neq 0$, its y-order is at least 2, and $\mathcal{Z}(\varphi)=0$ gives $\mathsf{Q}_u=0$ at every base point (Corners); this contradicts Lemma 9.93. (b) "Only if" holds as the intersection is 0; "if" is (a). □

Remark 9.95. (1) Lemma 9.93 contains Lemma 9.90: for φ of bidegree (a,b), Φ is homogeneous of degree b in (y,Z), so (7) says $\Phi=\Phi_b$, that is, y-order b. The proofs differ in step (1): that of Lemma 9.90 separates the coefficients of Z_{qp} in Q_u, which needs C free of Z; that of Lemma 9.93 uses one row $\mathsf{c}(p,r,q)$ in degree $k-2$ in y, where $\theta_{pq}Z_{qp}$ is a factor in a domain, so C_{st} may depend on Z.

(2) What is used: (3), the invariance under $Y\mapsto Y+\gamma I$, $n\geq 3$, and only the rows c of Lemma 9.83 at base points, in degree $k-2$ in y (for $k=2$ these are slice rows at the slice points); only the corner values Q_u, with one common u, enter. Not used: $\mathcal{E}\varphi=0$, the bidegree, the invariance under $X\mapsto X+\gamma I$. The invariance under $Y\mapsto Y+\gamma I$ is essential, by the example of Remark 9.92(1). The parts Φ_0 and Φ_1 need an input such as $\mathcal{E}\varphi=0$: $\varphi=\operatorname{tr}(X_0^aY_0)$ $(a\geq 1)$ has $\beta(U,V)=\sum_{i=0}^{a-1}\operatorname{tr}(X_0^iU_0X_0^{a-1-i}V_0)$, which is symmetric, so $\mathcal{Z}(\varphi)=0$; its y-order is 1, and $\mathcal{E}\varphi=\operatorname{tr}(X_0^{a+1})\neq 0$. Likewise $\operatorname{tr}(Y_0^4)$ has $\beta=0$ and y-order 0.

(3) At $b=2$, $\Phi=\Phi_0+\Phi_2$ with $\Phi_0=\sum_{r\neq t}\mathsf{F}_{rt}Z_{rt}Z_{tr}$ (by (1); $\Phi_1=0$ by the torus weight, as in the proof of Lemma 9.84), and Steps 1–3 of the proof of Theorem 9.80 give $\mathsf{F}=0$ for $a\geq 3$. At $b=3$ and $n\geq 4$, Theorem 9.88 gives $\mathsf{F}=\mathsf{N}=0$, that is $\Phi_0=\Phi_1=0$ (Lemma 9.84). With Corollary 9.94(a) this gives Corollary 9.81(a) and Corollary 9.91(a) again, the latter with Lemma 9.93 in place of Lemma 9.90. At $a=b=2$ the line of $\operatorname{tr}([A,B]^2)$ is consistent with this: its $\Phi=\Phi_0=-\sum_{r\neq t}x_{rt}^2Z_{rt}Z_{tr}\neq 0$.

(4) By the invariance under $X\mapsto X+\gamma I$ and $Y\mapsto Y+\gamma I$, each Φ_j is nonzero if and only if it is nonzero at traceless x and y; there Φ is the expansion of Lemma 9.54, whose diagonal part of Y is $\operatorname{diag}(y)$ here, with $\Phi_j=F_{b-j}$. So for $0\neq\varphi$ of bidegree (a,b) the y-order is $\operatorname{ord}(\varphi)$, and Lemma 9.93

says $\operatorname{ord}(\varphi) \le 1$ for every $0 \ne \varphi \in \mathcal{F}_{(a,b)}(n)$ with $\mathcal{Z}(\varphi) = 0$, $n \ge 3$. Compare Theorem 9.55 ($\operatorname{ord}(\varphi) \le \lfloor a/4 \rfloor$ on $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$); neither implies the other.

(5) By Corollary 9.94(b), Theorem 9.22 at any (a, b, n) with $n \ge 3$ is equivalent to: $\mathcal{Z}(\varphi) = 0$ and $\mathcal{E}\varphi = 0$ force $\Phi_0 = \Phi_1 = 0$. For $b \ge 4$ the slice rows at the slice points also contain the y-Hessian of Φ_2 (step (0) with $k = 2$), which such an argument must remove. At $b = 4$ and $n \ge 5$ this is Theorem 9.129, where it is shown to vanish in step (1) of the proof ($\mathsf{G}_e = 0$), before N; for $b \ge 5$, and for $b = 4$ with $n = 3, 4$, it is open outside the stable range. By Theorem 9.196, for every layer that occurs ($3b \le n^2$) only the part $\Phi_0 = 0$ is open (Corollary 9.197).

9.20. **Theta vectors.** In this subsection $F = \mathbb{C}$, $n \ge 3$, and we keep the setting of Subsection 9.15 with $k = 3$ and $\mathcal{U} = \mathcal{N} \times M_n^2$. We show that $\mu = 0$ occurs. The function built from an invariant of A alone does not depend on P_3, so it passes every test, for every s. At $ns = a$ the singular vector of $\mathfrak{gl}_3$ for the root $\theta = e_1 - e_3$ turns it into a highest weight vector of weight $(-2, a, 2)$, which still passes the tests, and so gives an eigenvector of the pencil at $\kappa = a/n$ (Theorem 9.98). At $a = 3$ this eigenvector is $-(6/n)\varphi_1$ for every $n \ge 4$ (Corollary 9.100), and for every $a \ge 4$ and $n \ge a + 2$ it is not 0 for the source $\operatorname{tr}(A^{a-2})\operatorname{tr}(A^2)$ (Theorem 9.102). Corollary 9.103 lists where each value of Conjecture 11.3 is now known to occur.

Lemma 9.96. *Let $k \ge 2$, $i \ne j$, and let Π and $\mathcal{U}$ be as in Theorem 9.58. Then $\Pi\,\Omega(E_{ij}f) = E_{ij}(\Pi\,\Omega f)$ for every holomorphic f on $\mathcal{U}$. In particular, if f passes Π on $\mathcal{U}$, so does $E_{ij}f$.*

Proof. As E_{ij} is linear in P_i, $\partial_i[B]E_{ij} = E_{ij}\partial_i[B] + \partial_j[B]$, and $\partial_l[B]$ commutes with E_{ij} for $l \ne i$. So $\Omega(E_{ij}f) - E_{ij}(\Omega f)$ is the alternating sum with $\partial_i[B_{\sigma(i)}]$ replaced by $\partial_j[B_{\sigma(i)}]$; it has two derivatives in slot j, is symmetric in $(\sigma(i), \sigma(j))$, and alternates to 0. This is part (i) of the proof of Theorem 9.58, with (i, j) in place of $(2, 1)$; Π does not depend on P, and E_{ij} is a vector field on $\mathcal{U}$. □

Lemma 9.97 (the singular vector for θ)**.** *Let V be a module for the Lie algebra spanned by the E_{ij} ($1 \le i, j \le 3$), $a \in \mathbb{C}$, and $v \in V$ with $E_{12}v = E_{23}v = 0$, $(E_{11} - E_{22})v = -a\,v$ and $(E_{22} - E_{33})v = a\,v$. Put*

$$\mathsf{S}_a := E_{21}^2E_{32}^2 - 2(a-1)\,E_{21}E_{31}E_{32} + a(a-1)\,E_{31}^2, \qquad w := \mathsf{S}_a v.$$

Then $E_{12}w = E_{23}w = E_{13}w = 0$, and if $E_{ii}v = \tau_i v$ ($i = 1, 2, 3$), then w has the weight $\tau - 2(e_1 - e_3)$.

Proof. Write $e_1 := E_{12}$, $e_2 := E_{23}$, $f_1 := E_{21}$, $f_2 := E_{32}$, $f_\theta := E_{31}$, $h_1 := E_{11} - E_{22}$, $h_2 := E_{22} - E_{33}$, and $\lambda_1 := -a$, $\lambda_2 := a$. From $[E_{ij}, E_{pq}] = \delta_{jp}E_{iq} - \delta_{qi}E_{pj}$: $[f_1, f_2] = -f_\theta$, $[f_1, f_\theta] = [f_2, f_\theta] = 0$, $[e_1, f_1] = h_1$, $[e_1, f_2] = 0$, $[e_1, f_\theta] = -f_2$, $[e_2, f_2] = h_2$, $[e_2, f_1] = 0$, $[e_2, f_\theta] = f_1$; f_1, f_2, f_θ change the h_1-eigenvalue by -2, $+1$, -1 and the h_2-eigenvalue by $+1$, -2, -1; and $[e_1, f_1^2] = 2f_1(h_1 - 1)$, $[e_2, f_2^2] = 2f_2(h_2 - 1)$. Then

$$e_1f_1^2f_2^2v = 2(\lambda_1 + 1)f_1f_2^2v, \qquad e_1f_1f_\theta f_2v = -f_1f_2^2v + \lambda_1 f_\theta f_2 v, \qquad e_1f_\theta^2 v = -2f_\theta f_2 v,$$
$$e_2f_1^2f_2^2v = 2(\lambda_2 - 1)f_1^2f_2v, \qquad e_2f_1f_\theta f_2v = \lambda_2 f_1 f_\theta v + f_1^2 f_2 v, \qquad e_2f_\theta^2 v = 2f_1 f_\theta v.$$

So, for $w = f_1^2f_2^2v + c_2f_1f_\theta f_2v + c_3f_\theta^2 v$,

$$e_1w = \big(2(\lambda_1+1) - c_2\big)f_1f_2^2v + (c_2\lambda_1 - 2c_3)f_\theta f_2 v, \qquad e_2w = \big(2(\lambda_2-1) + c_2\big)f_1^2f_2v + (c_2\lambda_2 + 2c_3)f_1f_\theta v,$$

and $c_2 = -2(a-1)$, $c_3 = a(a-1)$ make all four coefficients 0. Then $E_{13}w = [e_1, e_2]w = 0$. The weight is clear. □

Theorem 9.98 (theta vectors)**.** *Let $a \ge 1$, $s := a/n$, $\varphi \in \mathcal{F}_{(a,0)}(n)$, v the function built from φ with exponent s, and $w := \mathsf{S}_a v$. Then w is the function built with exponent s from some $\varphi_w \in \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$; w passes (S1) and (S2) at every point of $\mathcal{U}$; and $s\,\mathcal{Z}(\varphi_w) = \mathcal{J}(\varphi_w)$ at every base point and for all u, y_1, y_2, y_3. So, if $\varphi_w \ne 0$, then $\kappa = a/n$ ($\mu = 0$) is an eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, with eigenvector φ_w; for $a \ge 3$, $\mathcal{Z}(\varphi_w) \ne 0$ by Corollary 9.81(a).*

Proof. (1) *The chart.* For a polynomial ψ on $M_n\times M_n$ let $h(P):=\det(P_1)^s\psi(X,Y)$ with $X:=P_1^{-1}P_2$, $Y:=P_1^{-1}P_3$. By step (1) of the proof of Proposition 9.60, $E_{21}h=\det(P_1)^s(T_s\psi)(X,Y)$. As $P_2+tP_3=P_1(X+tY)$, $E_{32}h=\det(P_1)^s\,\frac{d}{d\epsilon}\psi(X+\epsilon Y,Y)|_{\epsilon=0}$; as $P_1+tP_3=P_1(I+tY)$, in the same way as for E_{21}, $E_{31}h=\det(P_1)^s\big(s\,\mathrm{tr}(Y)\psi-\frac{d}{d\epsilon}\psi(X+\epsilon YX,Y+\epsilon Y^2)|_{\epsilon=0}\big)$. These operations preserve conjugation invariance, so $w=\det(P_1)^s\psi_w(X,Y)$ with ψ_w a conjugation-invariant polynomial.

(2) *Highest weight.* v is a highest weight vector of weight $(ns-a,a,0)=(0,a,0)$ (Subsection 9.15, with $b=0$), so $(E_{11}-E_{22})v=-av$ and $(E_{22}-E_{33})v=av$. By Lemma 9.97, E_{12}, E_{13}, E_{23} kill w, of weight $(-2,a,2)$. As in step (4) of the proof of Proposition 9.60, $\psi_w(X+tI,Y)=\psi_w(X,Y+tI)=\psi_w(X,Y+tX)=\psi_w(X,Y)$ and ψ_w has bidegree $(a,2)$; so $\psi_w=\tilde\varphi_w$ with $\varphi_w\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$.

(3) *The tests.* v does not depend on P_3, and every term of $\Omega v(P)(B_1,B_2,B_3)$ contains a derivative in P_3; so $\Omega v=0$, and v passes every Π on $\mathcal{U}$, in particular (S1) and (S2) for each fixed u,y_1,y_2,y_3. By Lemma 9.96, so does w.

(4) At $P_1=I$, (S1) for w is $(s\mathcal{Z}-\mathcal{J})(\varphi_w)$ by Lemma 9.30, which holds for every $s\in\mathbb{C}$. □

Lemma 9.99 (the data of the θ-words)**.** *Let Φ be a symmetric polynomial on $\mathbb{C}^n$, ψ the conjugation-invariant polynomial on M_n with $\psi(\mathrm{diag}(x))=\Phi(x)$, $s\in\mathbb{C}$ and $v:=\det(P_1)^s\psi(P_1^{-1}P_2)$. With $u:=x_r$, $v':=x_t$ $(r\neq t)$, $\Phi_r:=\partial\Phi/\partial x_r$ and*

$$\alpha_\Phi:=\frac{\Phi_r-\Phi_t}{u-v'},\qquad \beta_\Phi:=-\tfrac12\big(\alpha_\Phi(u+v')+\Phi_r+\Phi_t\big),\qquad \gamma_\Phi:=\alpha_\Phi uv'+u\Phi_r+v'\Phi_t-s\Phi,$$

the functions E_{32}^2v, $E_{31}E_{32}v$, E_{31}^2v are $\det(P_1)^s$ times conjugation-invariant polynomials that are quadratic in Y, with the data (in the sense of the proof of Lemma 9.62) α_Φ, β_Φ, γ_Φ. On such functions E_{21} acts on the data by $\mathsf{L}_s\mathsf{F}:=sp_1\mathsf{F}-\mathcal{D}\mathsf{F}-(u+v')\mathsf{F}$, with $\mathcal{D}$ as there. So S_av has the data

$$\mathsf{W}_s[\Phi]:=\mathsf{L}_s^2\alpha_\Phi-2(a-1)\,\mathsf{L}_s\beta_\Phi+a(a-1)\,\gamma_\Phi.$$

These are identities of polynomials in u, v' and the power sums $p_1,p_2,\dots$ (with $p_0=n$), obtained by the chain rule; so they hold at every n, also where the p_j are dependent, and $\Phi\mapsto\mathsf{W}_s[\Phi]$ is linear.

Proof. Let $X=\mathrm{diag}(x)$ with distinct x_l and let Y have zero diagonal. Then $D\psi[H]=\sum_l\Phi_lH_{ll}$, and, for H, K with zero diagonal, $D^2\psi[H,K]=\sum_{r\neq t}\frac{\Phi_r-\Phi_t}{x_r-x_t}H_{rt}K_{tr}$ (the Daleckiĭ–Kreĭn formula; compare Step 1 of the proof of Proposition 9.40). By (1) of the proof of Theorem 9.98, at $P_1=I$: $E_{32}^2v=D^2\psi[Y,Y]$; $E_{31}E_{32}v=s\,\mathrm{tr}(Y)D\psi[Y]-D^2\psi[Y,YX]-D\psi[Y^2]$; and $E_{31}^2v=s^2\,\mathrm{tr}(Y)^2\psi-2s\,\mathrm{tr}(Y)D\psi[YX]+D^2\psi[YX,YX]+2D\psi[Y^2X]-s\,\mathrm{tr}(Y^2)\psi$. Here $\mathrm{tr}\,Y=0$, YX has zero diagonal, $(YX)_{rt}=Y_{rt}x_t$, and $(Y^2)_{ll}=\sum_tY_{lt}Y_{tl}$; symmetrizing in (r,t) gives the data α_Φ, β_Φ, γ_Φ. By the proof of Lemma 9.62, E_{21} ($=T_s$ in the chart) acts on the data by L_s, as X^2 is diagonal and XY has zero diagonal; so the outer E_{21}'s keep the set $\{X\text{ diagonal}\}\times\{Y\text{ with zero diagonal}\}$ (on which $\mathrm{tr}\,Y=0$), and only the values of the inner words there are needed. Translation invariance of Φ is not used. □

Corollary 9.100. *For $a=3$, $\varphi=\mathrm{tr}(A^3)$ and every $n\geq4$: $\varphi_w=-(6/n)\varphi_1$. Hence $\kappa=3/n$ $(\mu=0)$ is an eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(3,2)}(n)$ for every $n\geq4$, with eigenspace $\mathbb{C}\varphi_1$ (found before by the computations of Remark 9.50(5)).*

Proof. By Theorem 9.98, φ_w is in the kernel at $\kappa=3/n$, which is contained in $\mathbb{C}\varphi_1$ by Theorem 9.48; so $\varphi_w=c\,\varphi_1$. We compare data. For φ_1 they are $\frac12P_2(z_r+z_t)+P_3$ (step (b) of the proof of Theorem 9.48, with $P_k:=\sum_lz_l^k$), with the coefficient 1 at the monomial p_3. For φ_w, Lemma 9.99 with $\Phi=P_3$, $s=3/n$, so that $\mathsf{W}_s[\Phi]=\mathsf{L}_s^2\alpha_\Phi-4\mathsf{L}_s\beta_\Phi+6\gamma_\Phi$: $\Phi_r=3u^2-\frac3n(p_2+2up_1)+\frac6{n^2}p_1^2$, $\alpha_\Phi=3(u+v')-\frac6np_1$ and $\beta_\Phi=-\frac32(u+v')^2-\frac32(u^2+v'^2)+\frac6np_1(u+v')+\frac3np_2-\frac6{n^2}p_1^2$. The monomial p_3 arises in $6\gamma_\Phi$ only from $-6s\Phi$, with the coefficient $-18/n$; in $-4\mathsf{L}_s\beta_\Phi$ only from $-\mathcal{D}$ applied to $\frac3np_2$ ($\mathcal{D}p_2=2p_3$), giving $+24/n$; and in $\mathsf{L}_s^2\alpha_\Phi$ only from $\mathcal{D}^2$ applied to $-\frac6np_1$ ($\mathcal{D}p_1=p_2$), giving

$-12/n$. So the coefficient of p_3 in the data of φ_w is $-6/n$. For $n \geq 5$ the functions u, v', p_1, p_2, p_3 are algebraically independent on $\mathbb{C}^n$ (as in the proof of Lemma 9.62), so $c = -6/n$. Then, for every $n \geq 5$, the identity $\mathsf{W}_{3/n}[P_3] = -\frac{6}{n} \cdot (\text{data of } \varphi_1)$ holds coefficientwise as polynomials in u, v', p_1, p_2, p_3; the coefficients are Laurent polynomials in n, given by the same formulas for every n, so it holds for every n, and in particular, by Lemma 9.99, as functions at $n = 4$. At $n = 4$ the data of φ_1 are not 0 ($x = (1, -1, 2, -2)$, $(r, t) = (1, 3)$: $\frac{1}{2} \cdot 10 \cdot 3 + 0 = 15$), so $c = -6/4$. □

All nine coefficients of this identity, in the basis $u^3 + v'^3$, $uv'(u + v')$, $p_1(u^2 + v'^2)$, p_1uv', $p_1^2(u + v')$, $p_2(u + v')$, p_1^3, p_1p_2, p_3, can also be computed by hand, which gives a second proof for $n = 4$. With Corollary 9.61, both eigenvalues of Theorem 9.48 now have a proof, with the eigenspaces $\mathbb{C}\varphi_1$ and $\mathbb{C}(n\varphi_1 + 6\varphi_2)$.

Lemma 9.101. *Let $h_m := \sum_{i=0}^{m} u^i v'^{m-i}$ ($h_{-1} := 0$) and $O := -\big((u + v') + u^2\partial_u + v'^2\partial_{v'}\big)$, as in the proof of Lemma 9.62.*

(a) For $\Phi = p_k$: $\alpha_\Phi = kh_{k-2}$, $\beta_\Phi = -kh_{k-1}$ and $\gamma_\Phi = kh_k - sp_k$.
(b) $O\, h_m = -(m + 1)\, h_{m+1}$ for every $m \geq 0$.
(c) Modulo the ideal $(p_1, p_2, \dots)$, $\mathsf{W}_s[p_k] \equiv k(a - k)(a - k - 1)\, h_k$.

Proof. (a) $\Phi_r = ku^{k-1}$, so $\alpha_\Phi = k(u^{k-1} - v'^{k-1})/(u - v') = kh_{k-2}$. With $h_m(u + v') = h_{m+1} + uv'h_{m-1}$ and $u^j + v'^j = h_j - uv'h_{j-2}$: $\alpha_\Phi(u + v') + \Phi_r + \Phi_t = 2kh_{k-1}$ and $\alpha_\Phi uv' + u\Phi_r + v'\Phi_t = k(uv'h_{k-2} + u^k + v'^k) = kh_k$. (b) Put $T := u^2\partial_u + v'^2\partial_{v'}$. As $T(u - v') = u^2 - v'^2$, $T((u - v')f) = (u - v')(T + u + v')f = -(u - v')\, Of$. With $(u - v')h_m = u^{m+1} - v'^{m+1}$ and $Tu^j = ju^{j+1}$: $-(u - v')\, Oh_m = (m + 1)(u^{m+2} - v'^{m+2}) = (m + 1)(u - v')h_{m+1}$. (c) Modulo the ideal, $\mathsf{L}_s f = Of$ for $f \in \mathbb{C}[u, v']$, and L_s maps the ideal into itself (proof of Lemma 9.62). By (a) and (b), $O^2(kh_{k-2}) = k^2(k - 1)h_k$ and $O(-kh_{k-1}) = k^2h_k$, so $\mathsf{W}_s[p_k] \equiv kh_k\big(k(k - 1) - 2(a - 1)k + a(a - 1)\big) = k(a - k)(a - k - 1)h_k$. □

Theorem 9.102 (theta vectors for every a)**.** *Let $a \geq 4$, $n \geq a + 2$, and $\varphi := \operatorname{tr}(A^{a-2}) \operatorname{tr}(A^2) \in \mathcal{F}_{(a,0)}(n)$. Then $\varphi_w \neq 0$. Hence, with Corollary 9.100, $\kappa = a/n$ ($\mu = 0$) is an eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ for every $a \geq 3$ and $n \geq a + 2$.*

Proof. Give u, v' the weight 1 and p_j the weight j ($p_0 = n$ and s are scalars). Then α_Φ, β_Φ, γ_Φ have the weights $a - 2$, $a - 1$, a, and L_s raises the weight by exactly 1; so the data of φ_w are weighted homogeneous of weight a and involve only $p_1, \dots, p_a$. As $u, v', p_1, \dots, p_a$ are algebraically independent on $\mathbb{C}^n$ for $n \geq a + 2$ (proof of Lemma 9.62), it suffices to show $\mathsf{W}_s[\Phi] \neq 0$ as a polynomial, for $\Phi(x) = P_{a-2}P_2$, $P_k := \sum_l (x_l - p_1/n)^k$.

Count the degree in $p_1, p_2, \dots$ (the *p-degree*). As $P_k = p_k + (\text{terms of } p\text{-degree} \geq 2)$, $\Phi = p_{a-2}p_2 + (\text{terms of } p\text{-degree} \geq 3)$. The maps $\Phi \mapsto \alpha_\Phi, \beta_\Phi, \gamma_\Phi$ lower the p-degree by at most 1 (a derivative $\partial/\partial x_r$ removes at most one factor p_j), and L_s does not lower it; so the part of p-degree 1 of $\mathsf{W}_s[\Phi]$ is that of $\mathsf{W}_s[p_{a-2}p_2]$. For $\Phi = p_kp_2$ ($k := a - 2$), Lemma 9.101(a) gives $\alpha_\Phi = p_2\, kh_{k-2} + p_k\, 2h_0$, $\beta_\Phi = -p_2\, kh_{k-1} - p_k\, 2h_1$ and $\gamma_\Phi = p_2\, kh_k + p_k\, 2h_2 - sp_kp_2$. For a monomial P of p-degree 1 and $f \in \mathbb{C}[u, v']$, $\mathsf{L}_s(Pf) = sp_1Pf + P\, Of - (\mathcal{D}P)f$, where $\mathcal{D}p_j = jp_{j+1}$, and the first term has p-degree 2. So the part of p-degree 1 is a combination of the monomials p_2, p_k, $\mathcal{D}p_2 = 2p_3$, $\mathcal{D}p_k = kp_{k+1}$, $\mathcal{D}^2p_2 = 6p_4$, $\mathcal{D}^2p_k = k(k + 1)p_{k+2}$. If $a \geq 5$, then $k, 3, k + 1, 4, k + 2 \neq 2$, so the coefficient of p_2 comes from the terms with the factor p_2 only, and by Lemma 9.101(c) it is $k(a - k)(a - k - 1)h_k = 2(a - 2)h_{a-2} \neq 0$. If $a = 4$, then $k = 2$, both terms carry p_2, and the coefficient is $2 \cdot 2 \cdot 2 \cdot 1 \cdot h_2 = 8h_2 \neq 0$. So $\mathsf{W}_s[\Phi] \neq 0$ and $\varphi_w \neq 0$. The last statement follows with Theorem 9.98 and, for $a = 3$, Corollary 9.100. □

Corollary 9.103. *Let $n \geq 4$. Each value allowed by Conjecture 11.3 is an eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ in the following cases:*

(a) $\mu = 0$: every $a \geq 3$ and $n \geq a + 2$, and $a = 3$, $n \geq 4$ (Theorem 9.102, Corollary 9.100);
(b) $\mu = 3$: every $a \geq 3$ and $n \geq a + 2$, and $a = 3$, $n \geq 4$ (Corollaries 9.63 and 9.61);

(c) $\kappa=2$*: every even* $a\ge 4$ *and* $n\ge a+1$ *(the first part of the proof of Corollary 9.77);*
(d) $\kappa=-2$*: every even* $a\ge 4$ *and* $n\ge a+6$ *(Corollary 9.77).*
At $a=3$, $\kappa=\pm 2$ *do not occur (Theorem 9.48).*

Proof. (c) is the first part of the proof of Corollary 9.77: for even a and $n\ge a+1$, $\tau=(2n-a-2,a,2)$ is a partition with all parts even, so $\mathrm{R}_2(n)_\tau\ne 0$ by Lemma 9.32(i) and the Kronecker rule there; $0\ne f\in \mathrm{R}_2(n)_\tau=\mathrm{K}_2(n)_\tau$ (Proposition 9.34) gives $0\ne\varphi_f$ with $2\mathcal{Z}(\varphi_f)=\mathcal{J}(\varphi_f)$ (Lemma 9.32(ii), Corollary 9.31(b)). The rest is cited. At $a=3$, $\kappa=\pm 2$ would give $\mu=\pm 2n-3\notin\{0,3\}$. □

Remark 9.104. (1) Lemma 9.96 is the infinitesimal form of the equivariance of Cayley's Ω-process, as in Remark 9.59(1). Lemma 9.97 is classical: a singular vector of weight $\lambda-2\theta$ in the Verma module $M(\lambda)$ exists and is unique up to a scalar when $\langle\lambda+\rho,\theta^\vee\rangle=2$ [Ve68, BGG71], and its form is the case $m=2$ of [MFF86], the product $E_{21}^{a+1}E_{32}^2E_{21}^{1-a}$ expanded by their rules; the proof above uses only the relations. What is new is the application: a source that passes every test for a trivial reason, lowered to an eigenvector of the pencil. Unlike Proposition 9.60, the condition $ns=a$ comes from the singular vector, not from a pencil. At $ns=a$ the source has the weight $(0,a,0)$, which is not a partition, so it is not a polynomial (Remark 9.59(2)), and the obstruction of Remark 9.59(3) does not apply; the target weight $(-2,a,2)$ is not a partition either, so there is no consequence for rectangles.

(2) At $a=2$, Lemma 9.99 with $\Phi=P_2=p_2-p_1^2/n$ and $s=2/n$ gives $\alpha_\Phi=2$, $\beta_\Phi=-2(u+v')+2p_1/n$ and the data $-\frac{4}{n}P_2$ for φ_w, which are $-\frac{4}{n}$ times the data of G (proof of Lemma 9.62); G spans the kernel of Theorem 9.37 at $\kappa=2/n$. With Corollary 9.100 this tests Lemma 9.99 against two known kernels.

(3) (Computations, not used.) For $3\le a\le 7$ the vectors φ_w of the basis $\prod_i\mathrm{tr}(A^{\lambda_i})$ of $\mathcal{F}_{(a,0)}(n)$ (parts $\lambda_i\in[2,n]$) are linearly independent for every $n\ge 3$: for $n\ge a+2$ by a nonzero pivot minor of their data over $\mathbb{Q}[n]$ (a monomial in n), and for $3\le n\le a+1$ by direct exact computation. So $\dim\ker(\kappa\mathcal{Z}-\mathcal{J})\ge\dim\mathcal{F}_{(a,0)}(n)$ at $\kappa=a/n$ for these a. An independent computation with the full action of $\mathfrak{gl}_3$ on formal trace polynomials confirms Theorem 9.98 at $n=13$ for $a=3,4,5,6$ and every source.

(4) We do not show that the vectors φ_w span the eigenspace of $\mu=0$, nor that they are independent for $a\ge 8$ and $3\le n\le a+1$ (for $n\ge a+2$ they are, by Theorem 9.105); and Corollary 9.103 does not say that every value occurs at every (a,n): $\kappa=\pm 2$ for odd $a\ge 5$, $\mu=3$ and $\mu=0$ for $a\ge 4$ and $4\le n\le a+1$ (for $\mu=0$ and $a\le 7$ by the computations of (3) only), $\kappa=2$ for $n\le a$ and $\kappa=-2$ for $n\le a+5$ are open.

All sources. Theorem 9.102 gives $\varphi_w\ne 0$ for one source. The map $\varphi\mapsto\varphi_w$ is in fact injective for $a\ge 3$ and $n\ge a+2$, so the eigenvalue $\mu=0$ has multiplicity at least $\dim\mathcal{F}_{(a,0)}(n)$.

Theorem 9.105 (independence of the theta vectors)**.** *Let* $a\ge 3$ *and* $n\ge a+2$. *For a partition* λ *of* a *with all parts* ≥ 2 *let* $P_\lambda:=\prod_i\mathrm{tr}(A^{\lambda_i})$, *and let* $\varphi_w(\lambda)$ *be the vector of Theorem 9.98 with* $\varphi=P_\lambda$ *and* $s=a/n$. *Then the* $\varphi_w(\lambda)$ *are linearly independent. Hence* $\dim\ker(\kappa\mathcal{Z}-\mathcal{J})\ge p_{\ge 2}(a)$ *on* $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ *at* $\kappa=a/n$, *where* $p_{\ge 2}(a)$ *is the number of partitions of* a *into parts* ≥ 2.

For $n\ge a$ the P_λ are a basis of $\mathcal{F}_{(a,0)}(n)$ (the invariants of one traceless matrix form the polynomial ring in $\mathrm{tr}(A^2),\dots,\mathrm{tr}(A^n)$, and every part is at most $a\le n$), so $p_{\ge 2}(a)=\dim\mathcal{F}_{(a,0)}(n)$. At $a=3$ the theorem gives $\varphi_w\ne 0$ for $n\ge 5$; Corollary 9.100 is stronger.

We use the notation of Lemmas 9.99, 9.101 and of the proof of Theorem 9.102: $P_k=\sum_l(x_l-p_1/n)^k$, the p-degree is the degree in $p_1,p_2,\dots$, and $\mathcal{D}_p:=\sum_j jp_{j+1}\partial/\partial p_j$. As $u,v',p_1,\dots,p_a$ are algebraically independent, distinct p-monomials are linearly independent over $\mathbb{C}[u,v']$; the *coefficient* of a p-monomial means its coefficient in $\mathbb{C}[u,v']$. For $k\ge 1$ put $\Delta_k:=kh_{k-2}$ and $\mathsf{b}_k:=-kh_{k-1}$. By Lemma 9.101(a) these are α_{p_k} and β_{p_k} for $k\ge 2$; for $k=1$ directly, $\Phi_r=1$ gives $\alpha_{p_1}=0=\Delta_1$, $\beta_{p_1}=-1=\mathsf{b}_1$ and $\gamma_{p_1}=h_1-sp_1$.

Lemma 9.106. *Let μ be a partition of a with r parts $\mu_i \geq 1$, $p_\mu := \prod_i p_{\mu_i}$ and $P_{(-i)} := \prod_{j\neq i} p_{\mu_j}$ (i runs over the r parts, with multiplicity). Then $\mathsf{W}_s[p_\mu]$ has p-degree $\geq r-1$, and its part of p-degree $r-1$ is*

$$\mathrm{Lead}[p_\mu] = \sum_{i=1}^{r} \Big[\mu_i(a-\mu_i)(a-\mu_i-1)\, h_{\mu_i} P_{(-i)} - 2\mu_i(a-\mu_i)\, h_{\mu_i-1}\, \mathcal{D}_p P_{(-i)} + \mu_i\, h_{\mu_i-2}\, \mathcal{D}_p^2 P_{(-i)} \Big].$$

For any source Φ, a term of p-degree d of Φ contributes to $\mathsf{W}_s[\Phi]$ only in p-degree $\geq d-1$.

Proof. The last statement is the degree count in the proof of Theorem 9.102. For $\Phi = p_\mu$, $\Phi_r = \sum_i P_{(-i)}\mu_i u^{\mu_i - 1}$, so termwise $\alpha_\Phi = \sum_i P_{(-i)}\Delta_{\mu_i}$, $\beta_\Phi = \sum_i P_{(-i)}\mathsf{b}_{\mu_i}$ and $\gamma_\Phi = \sum_i P_{(-i)}\mu_i h_{\mu_i} - s p_\mu$. For a p-monomial P and $f \in \mathbb{C}[u, v']$, $\mathsf{L}_s(Pf) = sp_1 Pf + P\,Of - (\mathcal{D}_p P)f$, and the first term raises the p-degree; so in p-degree $r-1$, L_s acts as $O - \mathcal{D}_p$, and O and $\mathcal{D}_p$ commute. Hence the part of p-degree $r-1$ is $\sum_i \big[P_{(-i)}(O^2\Delta_{\mu_i} - 2(a-1)O\mathsf{b}_{\mu_i} + a(a-1)\mu_i h_{\mu_i}) - \mathcal{D}_p P_{(-i)}(2O\Delta_{\mu_i} - 2(a-1)\mathsf{b}_{\mu_i}) + \mathcal{D}_p^2 P_{(-i)}\Delta_{\mu_i}\big]$. The first bracket is $k(a-k)(a-k-1)h_k$ with $k = \mu_i$ (proof of Lemma 9.101(c); for $k=1$: $-2(a-1)O(-1) + a(a-1)h_1 = (a-1)(a-2)h_1$, as $O(-1) = h_1$). By Lemma 9.101(b), $O\Delta_k = -k(k-1)h_{k-1}$ for $k \geq 2$, and $O\Delta_1 = 0$; so $2O\Delta_k - 2(a-1)\mathsf{b}_k = 2k(a-k)h_{k-1}$ for every $k \geq 1$. □

Lemma 9.107. *Let $r \geq 2$, and let $c_\lambda \in \mathbb{C}$ for the partitions λ of a with r parts, all ≥ 2. If $\sum_\lambda c_\lambda \mathrm{Lead}[p_\lambda] = 0$, then every $c_\lambda = 0$.*

Proof. Suppose not, and take λ^* with $c_{\lambda^*} \neq 0$ whose largest part λ_1^* is maximal among those λ; let m be the multiplicity of λ_1^* in λ^*, and $M := \prod_{j\geq 2} p_{\lambda_j^*}$, of weight $a - \lambda_1^*$. The p-monomials in $\mathrm{Lead}[p_\lambda]$ are $P_{(-i)}$ (weight $a-\lambda_i$) and those of $\mathcal{D}_p P_{(-i)}$ and $\mathcal{D}_p^2 P_{(-i)}$ (weights $a-\lambda_i+1$ and $a-\lambda_i+2$). For $c_\lambda \neq 0$ we have $\lambda_i \leq \lambda_1^*$; so M occurs only as $P_{(-i)}$ with $\lambda_i = \lambda_1^*$ and λ minus that part equal to λ^* minus λ_1^*, that is, $\lambda = \lambda^*$. Its coefficient is $c_{\lambda^*} m \lambda_1^*(a-\lambda_1^*)(a-\lambda_1^*-1)h_{\lambda_1^*} \neq 0$, since $2 \leq \lambda_1^* \leq a-2$. A contradiction. □

Lemma 9.108. *Put $\epsilon_j := h_{a-j}p_j$. For $\Phi = P_a$ (the source $\mathrm{tr}(A^a)$), the part of p-degree 0 of $\mathsf{W}_s[\Phi]$ is 0, and its part of p-degree 1 is*

$$D_1 := \tfrac{a}{n}\big[(a-2)\epsilon_2 - 2(a-1)\epsilon_3 - (a-1)(a-2)\epsilon_{a-1} + (a-1)(a-2)\epsilon_a\big],$$

a formal sum in the symbols ϵ_j (equal symbols add: at $a=3$, $D_1 = -\frac{3}{n}(\epsilon_2 + 2\epsilon_3)$).

Proof. $P_a = p_a - \frac{a}{n}p_1 p_{a-1} +$ (terms of p-degree ≥ 3), as $a \geq 3$; by Lemma 9.106 only the first two terms reach p-degree ≤ 1. (a) $\Phi = p_a$: $\alpha_\Phi = a h_{a-2}$ and $\beta_\Phi = -a h_{a-1}$ contain no p_j, so $\mathsf{L}_s\alpha_\Phi = sp_1\alpha_\Phi + O\alpha_\Phi$ and $\mathsf{L}_s^2\alpha_\Phi = s^2p_1^2\alpha_\Phi + 2sp_1 O\alpha_\Phi - sp_2\alpha_\Phi + O^2\alpha_\Phi$. The part of p-degree 0 is $a\cdot 0\cdot(-1)h_a = 0$ (Lemma 9.101(c)). In p-degree 1: $2sp_1 O\alpha_\Phi - sp_2\alpha_\Phi - 2(a-1)sp_1\beta_\Phi - a(a-1)sp_a$; as $O\alpha_\Phi = -a(a-1)h_{a-1}$, the p_1-terms cancel, and $-\frac{a^2}{n}(\epsilon_2 + (a-1)\epsilon_a)$ is left. (b) $-\frac{a}{n}p_1p_{a-1}$: by Lemma 9.106 with $\mu = (a-1,1)$, $\mathrm{Lead}[p_1p_{a-1}] = -2(a-1)\epsilon_2 + 2(a-1)\epsilon_3 + (a-1)(a-2)\epsilon_{a-1} - 2(a-1)^2\epsilon_a$. Adding (a) and $-\frac{a}{n}$ times (b) gives D_1. □

Lemma 9.109. *Let $g(m) := (m-1)\epsilon_m - 2m\epsilon_{m+1} + (m+1)\epsilon_{m+2}$. For $\lambda = (a-k, k)$ with $2 \leq k \leq a/2$, $\mathrm{Lead}[p_\lambda] = k(a-k)\big(g(k) + g(a-k)\big)$.*

Proof. For a part x with partner $y = a - x$: $P_{(-i)} = p_y$, $\mathcal{D}_p p_y = y p_{y+1}$, $\mathcal{D}_p^2 p_y = y(y+1)p_{y+2}$, and $h_x = h_{a-y}$; so its term in Lemma 9.106 is $xy\, g(y)$. Sum over the two parts (for $k = a-k$ the two equal parts are counted separately). □

Proof of Theorem 9.105. Let $\sum_\lambda c_\lambda \varphi_w(\lambda) = 0$. As $\varphi \mapsto \varphi_w$ and $\Phi \mapsto \mathsf{W}_s[\Phi]$ are linear, the source $\Phi := \sum_\lambda c_\lambda P_\lambda$ has $\varphi_w = 0$, so its data vanish; they are $\mathsf{W}_s[\Phi]$ (Lemma 9.99), which is symmetric in u, v' and involves only $u, v', p_1, \ldots, p_a$, so $\mathsf{W}_s[\Phi] = 0$ as a polynomial (as in the proof of Theorem 9.102). A source with r factors contributes only in p-degree $\geq r-1$, and, by the last part of Lemma 9.106, its part of p-degree $r-1$ is $\mathrm{Lead}[p_\lambda]$ (the centring terms of P_λ have p-degree $\geq r+1$).

(1) In p-degree 1 only $r \le 2$ contribute: $c_{(a)}D_1 + \sum_{2\le k\le a/2} c_{(a-k,k)}k(a-k)\big(g(k)+g(a-k)\big) = 0$. The indices of $g(k)$ and $g(a-k)$ lie in $[2,a]$, so every term lies in the span of $\epsilon_2,\dots,\epsilon_a$, which are linearly independent. Let ℓ be the linear functional with $\ell(\epsilon_j) = 1/(j-1)$. Then $\ell(g(m)) = 1-2+1 = 0$ for $2 \le m \le a-2$, and $\ell(D_1) = \frac{a}{n}\big[(a-2)-(a-1)-(a-1)+(a-2)\big] = -\frac{2a}{n} \ne 0$. So $c_{(a)} = 0$.

(2) If some $c_\lambda \ne 0$, let $r_0 \ge 2$ be the least number of parts with a nonzero coefficient. In p-degree $r_0 - 1$ only these sources contribute, so $\sum_\lambda c_\lambda \mathrm{Lead}[p_\lambda] = 0$ over them, and Lemma 9.107 gives a contradiction. So all $c_\lambda = 0$; each $\varphi_w(\lambda)$ is in the kernel at $\kappa = a/n$ by Theorem 9.98. □

Remark 9.110. (1) The triangularity in Lemma 9.107 is the step that separates the sources with two or more factors; the functional ℓ of step (1) separates the single-trace source $\mathrm{tr}(A^a)$, whose lowest part has p-degree 1. (2) (Computations, not used.) The bound is attained at $a = 5$ and $a = 6$: the computations of Remark 9.50(4), whose rows give upper bounds for the kernel, have $\mu = 0$ with multiplicity 2 at $a = 5$ ($n = 11, 12$) and 4 at $a = 6$ ($n = 13$), which are $p_{\ge 2}(5)$ and $p_{\ge 2}(6)$. (3) We do not show that the φ_w span the eigenspace of $\mu = 0$, nor their independence for $a \ge 8$ and $3 \le n \le a+1$, nor that no other eigenvalue occurs (Conjecture 11.3; for $n \ge a+6$ see Theorem 9.150).

9.21. **Linkage.** In this subsection $F = \mathbb{C}$, $n \ge 3$, and we keep the setting of Subsection 9.15 with $k = 3$ and $\mathcal{U} = \mathcal{N} \times M_n^2$. Subsections 9.15 and 9.20 found the values $\mu = 3$ and $\mu = 0$ of Conjecture 11.3 by applying lowering operators of $\mathfrak{gl}_3$ to functions that pass the tests. Here we show that this mechanism gives only the following: an eigenvector in the $\mathfrak{gl}_3$-module generated by passing highest weight vectors of higher weight has $\mu = 0$ or $\mu = 3$, or comes from an eigenvector of a smaller layer (Proposition 9.112). The proof is the linkage theorem of Bernstein, Gelfand and Gelfand. So Conjecture 11.3 follows, by induction on a, from one principle (Corollary 9.114). We also record the Kronecker structure of the pencil (Lemma 9.116). We do not prove Conjecture 11.3 for any new a here (for $n \ge a+6$ see Subsection 9.25).

The module. For $s \in \mathbb{C}$ let $\mathcal{V}_s$ be the space of the functions $h(P) = \det(P_1)^s \psi(P_1^{-1}P_2, P_1^{-1}P_3)$ on $\mathcal{U}$, where ψ runs over all conjugation-invariant polynomials on $M_n \times M_n$ (with no traceless projection). The function built from $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ with exponent s lies in $\mathcal{V}_s$, with $\psi = \tilde{\varphi}$. For ψ of bidegree (a',b') put $\lambda(a',b') := (ns - a' - b', a', b')$. Let $\mathcal{P}_s \subseteq \mathcal{V}_s$ be the subspace of the functions that pass (S1) at every point of $\mathcal{U}$, for all u, y_1, y_2, y_3.

Lemma 9.111. *(a) $\mathcal{V}_s$ is stable under every E_{ij}, and h as above, with ψ of bidegree (a',b'), has the weight $\lambda(a',b')$. The weight determines (a',b'), the weight spaces are finite dimensional, and E_{12}, E_{13}, E_{23} act locally nilpotently.*

(b) The highest weight vectors of weight $\lambda(a',b')$ in $\mathcal{V}_s$ are exactly the functions built from the $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a',b')}(n)$ with exponent s.

(c) $\mathcal{P}_s$ is a $\mathfrak{gl}_3$-submodule of $\mathcal{V}_s$, and every weight component of an element of $\mathcal{P}_s$ lies in $\mathcal{P}_s$.

(d) Let $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a',b')}(n)$. The function built from φ with exponent s lies in $\mathcal{P}_s$ if and only if $s\,\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ at every base point and for all u, y_1, y_2, y_3.

Proof. (a) Put $X := P_1^{-1}P_2$ and $Y := P_1^{-1}P_3$. For E_{21}, E_{32} and E_{31} see step (1) of the proof of Theorem 9.98. E_{12} replaces P_2 by $P_2 + tP_1$, that is X by $X + tI$, with Y and $\det P_1$ fixed; so $E_{12}h = \det(P_1)^s \frac{d}{dt}\psi(X+tI, Y)|_{t=0}$. In the same way $E_{13}h = \det(P_1)^s \frac{d}{dt}\psi(X, Y+tI)|_{t=0}$ and $E_{23}h = \det(P_1)^s \frac{d}{dt}\psi(X, Y+tX)|_{t=0}$. All these keep the form $\det(P_1)^s \times$(a conjugation-invariant polynomial in X, Y), and E_{11}, E_{22}, E_{33} act by $ns - a' - b'$, a', b' (Euler operators). A weight $\lambda(a'',b'')$ is $\ge \lambda(a',b')$ only if $b'' \le b'$ and $a''+b'' \le a'+b'$, and E_{12}, E_{23}, E_{13} lower $a'+2b'$ by 1, 1, 2; this gives the local nilpotence. (b) $E_{12}h = E_{13}h = 0$ says that ψ is invariant under $X \mapsto X + tI$ and under $Y \mapsto Y + tI$, so $\psi = \tilde{\varphi}$ with φ the restriction of ψ to $\mathfrak{sl}_n \times \mathfrak{sl}_n$; then $E_{23}h = 0$ says $\mathcal{E}\varphi = 0$. The converse is stated in Subsection 9.15. (c) For $i \ne j$ this is Lemma 9.96. For E_{ii}:

$\partial_i[B]E_{ii} = E_{ii}\partial_i[B] + \partial_i[B]$, and $\partial_l[B]$ commutes with E_{ii} for $l \neq i$, so $\Omega(E_{ii}f) = (E_{ii}+1)\Omega f$, and a test that does not depend on P commutes with $E_{ii}+1$. An element of $\mathcal{V}_s$ is a finite sum of weight vectors with distinct weights, and each of them is a polynomial in E_{11}, E_{22}, E_{33} applied to the sum. (d) Let f be the function built from φ with exponent s, and $P \in \mathcal{U}$. By the identity $\det(gQ_1)^s = \det(g)^s \det(Q_1)^s$ of Subsection 9.15, $f(P') = \det(P_1)^s f(P_1^{-1}P_1', P_1^{-1}P_2', P_1^{-1}P_3')$ for P' near P. So (S1) at P is $\det(P_1)^s$ times (S1) at $(I, P_1^{-1}P_2, P_1^{-1}P_3)$ with u replaced by $P_1^{-1}u$, as in the proof of Corollary 9.31(b); and at $P_1 = I$, (S1) is $(s\mathcal{Z} - \mathcal{J})(\varphi)$ at the base point (P_2, P_3), by Lemma 9.30, which holds for every $s \in \mathbb{C}$. □

Let $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ be an eigenvector of the pencil at $\kappa = s$, and v the function built from φ with exponent s. We call φ a *descent* if $v \in \sum_{j=1}^{J} U(\mathfrak{gl}_3)v_j'$ for finitely many nonzero highest weight vectors $v_j' \in \mathcal{P}_s$ of weights $\lambda_j' \neq \lambda(a,2)$. (By Lemma 9.111(b)–(d), every highest weight vector of weight $\lambda(a,2)$ in such a sum comes from an eigenvector at $\kappa = s$.)

Proposition 9.112 (linkage)**.** *Let $a \geq 3$, $n \geq 3$ and $s \in \mathbb{C}$, and let φ be a descent at $\kappa = s$. Then one of the following holds:*

(A) $ns = a$ ($\mu = 0$), and some v_j' has the weight $\lambda(a,0)$;
(B) $ns = a+3$ ($\mu = 3$), and some v_j' has the weight $\lambda(2,2)$;
(C) $ns = a + a' + 1$ for some $3 \leq a' \leq a-1$, and s is an eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a',2)}(n)$, with $ns - a' = a+1$.

Proof. (1) Let $\mathfrak{n}^-$ be spanned by E_{21}, E_{31}, E_{32}. Each $N_j := U(\mathfrak{gl}_3)v_j' = U(\mathfrak{n}^-)v_j'$ is a highest weight module, a quotient of the Verma module $M(\lambda_j')$, and lies in category $\mathcal{O}$ by Lemma 9.111(a). (2) v is a highest weight vector of weight $\lambda := \lambda(a,2)$ in $N := \sum_j N_j$, so the simple module $L(\lambda)$ is a subquotient of N, hence of $\bigoplus_j N_j$, hence of some N_j; so $[M(\lambda_j') : L(\lambda)] \neq 0$. (3) By the theorem of Bernstein, Gelfand and Gelfand [BGG71], which holds for arbitrary weights (see also [Ja77] and [Hu08, Ch. 5]), λ is then strongly linked to λ_j'. We use only the consequence that $\lambda + (2,1,0) = w(\lambda_j' + (2,1,0))$ for some $w \in S_3$ and that $\lambda_j' - \lambda$ is a sum of positive roots; replacing the integral Weyl group by S_3 only weakens the statement. (The centre $E_{11}+E_{22}+E_{33}$ acts on $\mathcal{V}_s$ by the scalar ns, so the statements for $\mathfrak{gl}_3$ and $\mathfrak{sl}_3$ agree, and $(2,1,0)$ differs from the half sum of the positive roots by an S_3-invariant vector.) (4) Write $\lambda_j' = \lambda(a',b')$; then $\mathcal{F}^{\mathcal{E}}_{(a',b')}(n) \neq 0$ by Lemma 9.111(b). As $\lambda_j' - \lambda = (a+2-a'-b', a'-a, b'-2)$, it is a sum of positive roots only if $b' \leq 2$ and $a'+b' \leq a+2$. With $\lambda+(2,1,0) = (ns-a, a+1, 2)$ and $\lambda_j'+(2,1,0) = (ns-a'-b'+2, a'+1, b')$ the six w give: the identity gives $\lambda_j' = \lambda$, which is excluded; the transposition (12) gives $b' = 2$ and $ns = a+a'+1$ (a step along $e_1 - e_2$ with $m = a-a'$, so $a' \leq a-1$); (13) gives $a' = a$ and $b' = ns-a$ (a step along θ with $m = 2-b' > 0$, so $b' \in \{0,1\}$); (23) gives $a' = 1$ and $b' = a+1 > 2$; one 3-cycle gives $b' = a+1 > 2$, and the other gives $a' = 1$. (5) For $n \geq 3$ the layers $(1,0)$, $(1,1)$, $(1,2)$, $(0,1)$, $(0,2)$ have $\mathcal{F}^{\mathcal{E}} = 0$: $\mathcal{F}_{(1,0)}(n) = 0$ as $\operatorname{tr} A = 0$; $\mathcal{F}_{(1,1)}(n) = \mathbb{C}\operatorname{tr}(AB)$ with $\mathcal{E}\operatorname{tr}(AB) = \operatorname{tr}(A^2) \neq 0$; $\mathcal{F}_{(1,2)}(n) = \mathbb{C}\operatorname{tr}(AB^2)$ with $\mathcal{E}\operatorname{tr}(AB^2) = 2\operatorname{tr}(A^2B) \neq 0$; $\mathcal{F}_{(0,1)}(n) = 0$; $\mathcal{F}_{(0,2)}(n) = \mathbb{C}\operatorname{tr}(B^2)$ with $\mathcal{E}\operatorname{tr}(B^2) = 2\operatorname{tr}(AB) \neq 0$. So only (12) with $a' \geq 2$ and (13) remain. (6) In case (13) with $b' = 1$, v_j' comes from $\mathcal{F}^{\mathcal{E}}_{(a,1)}(n)$ and lies in $\mathcal{P}_s$, so $ns = a-1$ by Lemma 9.111(d) and Theorem 9.41, while (13) needs $ns = a+1$; this is excluded. Case (13) with $b' = 0$ is (A). (7) In case (12), $a' = 2$ gives $ns = a+3$, which is (B). For $3 \leq a' \leq a-1$, v_j' comes from an eigenvector at $\kappa = s$ on $\mathcal{F}^{\mathcal{E}}_{(a',2)}(n)$, by Lemma 9.111(d); this is (C). □

Remark 9.113. (1) In (A) the step is the θ-step of Lemma 9.97 ($m = 2$), and in (B) it is E_{21}^m with $m = a-2 = \tau_1 - \tau_2 + 1$ for the weight $\tau = (ns-4, 2, 2)$ of the source, as in Theorem 9.58. (2) For $n \geq a+2$ both (A) and (B) occur: the theta vectors of Theorem 9.105 and the vector φ_{LW} of Corollary 9.63 are descents. (3) In (A)–(C), $0 < ns \leq 2a$. So for $n \geq a+1$ the eigenvalues $\kappa = \pm 2$ are never descents. The bound on n is needed: for $n = 4$ and $a = 5$, case (B) has $ns = 8 = 2n$.

(4) The same bookkeeping on the layer $(a,1)$ (target weight $(ns-a-1,a,1)$) leaves the θ-step with $m=1$ from the layer $(a,0)$ and the function $\det(P_1)^s$ (layer $(0,0)$), both at $ns=a-1$, and first-layer sources $(a',1)$ along e_1-e_2 at $ns=a+a'$, which do not pass by Theorem 9.41. So the descents at $b=1$ give exactly the eigenvalue $\kappa=(a-1)/n$ of Theorem 9.41.

Corollary 9.114 (a reduction of Conjecture 11.3)**.** *Let $n\ge 3$ and $a\ge 3$. Suppose that for every a' with $3\le a'<a$ every eigenvalue κ of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a',2)}(n)$ has $n\kappa-a'\in\{0,3\}$ or $\kappa=\pm 2$, and that the following holds:*

(P1) every eigenvector of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ at an eigenvalue $\kappa\neq\pm 2$ is a descent.

Then every eigenvalue κ of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ has $n\kappa-a\in\{0,3\}$ or $\kappa=\pm 2$. In particular, if (P1) holds at (a,n) for every $a\ge 3$, then the conclusion of Conjecture 11.3 holds at n for every $a\ge 3$.

Proof. Let $s\neq\pm 2$ be an eigenvalue on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, with an eigenvector φ. By (P1), φ is a descent. In case (C) of Proposition 9.112, $s\neq\pm 2$ is an eigenvalue on $\mathcal{F}^{\mathcal{E}}_{(a',2)}(n)$ with $ns-a'=a+1\ge 4$, against the hypothesis at a'. So (A) or (B) holds. For $a=3$, case (C) is empty. The second statement follows by induction on a. □

Remark 9.115. (1) (P1) is sufficient for the conclusion of Conjecture 11.3, but it says more. Category $\mathcal{O}$ splits by central characters, so a descent lies in the sum of those N_j for which λ'_j is in the orbit of λ under the dot action and $\lambda'_j\ge\lambda$. At $ns=a+3$ the only such nonempty layer is $(2,2)$, and the part of weight λ of $U(\mathfrak{n}^-)v'$, for v' of weight $\lambda(2,2)$, is $\mathbb{C}E_{21}^{a-2}v'$; the passing sources of weight $\lambda(2,2)$ form one line (Theorem 9.37). So (P1) at $\mu=3$ says that the eigenspace of $\mu=3$ is spanned by φ_{LW}, which is not known for $a\ge 4$. At $ns=a$, (P1) says that every eigenvector of $\mu=0$ lies in $U(\mathfrak{gl}_3)$ applied to the sources of the layer $(a,0)$ (compare Remark 9.110(3)). (2) In (A)–(C) of Proposition 9.112, ns is an integer; indeed, if $ns\notin\mathbb{Z}$, then for every layer only the roots $\pm(e_2-e_3)$ are integral for $\lambda(a',b')$. So (P1) at (a,n) implies that every eigenvalue κ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ has $n\kappa\in\mathbb{Z}$; the conclusion of Conjecture 11.3 implies this too. (3) (Not used.) In the cases of Remark 9.117(2), every eigenspace at $\kappa\neq\pm 2$ is spanned by theta vectors or by φ_{LW} (at $a=3$ by Theorem 9.48 and Corollaries 9.100 and 9.61; otherwise by the computations), and these are descents; so (P1) holds there. At $a=3$ this is only a consistency check, as Theorem 9.48 is used.

Counting. The pencil has more rows than columns, so its eigenvalues are governed by its Kronecker structure.

Lemma 9.116 (Kronecker structure)**.** *Let $\kappa Z-M$ be an $N\times h$ pencil of complex matrices with Z injective, and let $g(\kappa)$ be the greatest common divisor of its $h\times h$ minors.*

- *(i) The Kronecker canonical form of $\kappa Z-M$ consists of a regular part of size $r=\deg g$ and left singular blocks $L_{\epsilon_i}^T$, with $r+\sum_i\epsilon_i=h$; there are no right singular blocks and no infinite elementary divisors.*
- *(ii) The eigenvalues and all Jordan chains lie in the regular part, and for a canonical system of Jordan chains at κ_0 the lengths add up to the multiplicity of κ_0 as a root of g.*
- *(iii) If S is a set of rows with Z_S injective, then g divides the greatest common divisor of the $h\times h$ minors of the rows S; and g divides $\det(R(\kappa Z-M))$ for every constant $h\times N$ matrix R for which this determinant is not 0.*

Proof. This is Kronecker's theory [Ga98, Ch. XII, §§4–5], together with two consequences of the injectivity of Z: a right singular block would give a polynomial $x(\kappa)\neq 0$ with $(\kappa Z-M)x=0$, whose top coefficient lies in $\ker Z=0$; and an infinite elementary divisor has a nilpotent, hence not injective, Z-part. The blocks L_ϵ^T have full column rank at every κ, so kernels and Jordan chains have no component there. The h-th determinantal divisor does not change under constant

equivalence, and a nonzero $h \times h$ minor of the canonical form is the determinant of the regular part times an $\epsilon_i \times \epsilon_i$ minor of each $L^T_{\epsilon_i}$; these minors include ± 1, so g is the determinant of the regular part up to a constant. For (iii), the minors of the rows S are minors of the pencil, and $\det(R(\kappa Z - M))$ is a linear combination of $h \times h$ minors (Cauchy–Binet). □

Remark 9.117. (1) Let $T(a) := p_{\geq 2}(a) + 1 + 2\epsilon(a)$, with $\epsilon(a) = 1$ for even a and 0 for odd a. For $n \geq a+2$, and for even a also $n \geq a+6$, Theorem 9.105, Corollary 9.63 and Corollary 9.103(c), (d) give $T(a)$ linearly independent eigenvectors at the eigenvalues a/n, $(a+3)/n$, 2, -2, which are distinct there ($(a+3)/n = 2$ would need $n = (a+3)/2 < a+2$, and $a/n = 2$ would need $n = a/2$). In bases of $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ and of the polynomial test values the pencil is a finite matrix, and $\mathcal{Z}$ is injective (Corollary 9.81); so Lemma 9.116 applies, and its r is at least $T(a)$. So $r \leq T(a)$ would give $r = T(a)$, the conclusion of Conjecture 11.3 and a semisimple regular part; and the conclusion of Conjecture 11.3 at (a,n) is equivalent to "every root of g is an allowed value", not to $r \leq T(a)$, since an allowed value may have a larger multiplicity. (2) At $a = 3$, Corollaries 9.100 and 9.61 give the eigenvalues $3/n$ and $6/n$ for every $n \geq 4$, which are all of them by Theorem 9.48; so $r = h = 2 = T(3)$. (Computations, not used.) The common factors of Remark 9.50(4) are upper bounds for g, by Lemma 9.116(iii). Their degrees are 5 at $a = 4$ (n symbolic), 3 at $a = 5$ ($n = 11, 12$) and 7 at $a = 6$ ($n = 13$), which are $T(a)$. So in these cases $r = T(a)$, the regular part is semisimple, and $\sum_i \epsilon_i = h - T(a)$ is 1, 5, 5, 9 ($h = \dim \mathcal{F}^{\mathcal{E}}_{(a,2)}(n) = 6$, 8, 8, 16). At $a = 4$ this holds for all but finitely many $n \geq 10$, as the common factor is a greatest common divisor over $\mathbb{Q}(n)$. Apart from Theorem 9.48 and, for $n \geq a+6$, Theorem 9.150, the only upper bounds we have come from such row computations.

9.22. **The top block at $b = 2$.** In this subsection $F = \mathbb{C}$ and $b = 2$, and we keep the notation of Subsection 9.17: the data $\mathsf{F}(s,t)$, $\mathsf{G}(s,t)$ (symmetric, with coefficients in $\mathbb{C}[1/n, P_2, P_3, \dots]$), $z_l = x_l - p_1/n$, $\mathcal{D}_w$, and $\Lambda = L + C$. Write the data as $(\mathsf{F}, \mathsf{G}) = \sum_\lambda P_\lambda v_\lambda$, where λ runs over the partitions with all parts at least 2, $P_\lambda := \prod_i P_{\lambda_i}$, and v_λ is a pair of symmetric polynomials in (s,t) over $\mathbb{C}[1/n]$, homogeneous of degree $a - |\lambda|$; call $\ell(\lambda)$ the *P-degree* and $|\lambda|$ the *weight*. We show that for $n \geq a+6$ and an eigenvector at $\kappa \neq 0$ the coefficient v_λ at the top monomial (largest P-degree, and then largest weight) has degree at most 3 (Theorem 9.122, Corollary 9.123). This does not exclude any value of κ (Remark 9.124(4)). Put $\varepsilon := -1/n$, let $\partial_{\mathbf{1}}$ be the sum of the derivatives in all slots ($\partial_s + \partial_t$ on a function of (s,t)), and $\sigma_\pm := s \pm t$.

The row. For distinct p, q, i, l write $\Lambda(p,q;l)$ for $\Lambda(z_p, z_q; z_l)$, and so on, and $z_{ip} := z_i - z_p$. The generic order-one row $\Psi_{pq}(e_i, e_l)$ of Subsection 9.17, in the shifted form $c = -x_p$ of Proposition 9.44 (the same polynomial), is

$$\mathsf{P} := \kappa\big(\Lambda(p,q;l) - \Lambda(p,q;i)\big) - z_{ip}\nabla(i;l) + z_{lp}\nabla(l;i) - \big(\mathcal{D}_i\mathsf{G}(z_q, z_l) - \mathcal{D}_l\mathsf{G}(z_q, z_i)\big), \tag{8}$$

where $\nabla(i;l) := \mathcal{D}_i\Lambda(p,q;l) + \frac{\Lambda(p,q;l)-\Lambda(p,i;l)}{z_q - z_i} - \frac{\Lambda(p,q;l)-\Lambda(i,q;l)}{z_p - z_i}$ is $(\partial_i\psi - \delta_i\psi)_l$ for $\psi = \psi^{(pq)}$, and the last term is the Θ-term $-\Theta_q(e_i, e_l)$. By Corollary 9.76(i), if $a \geq 2$, $n \geq a+6$ and $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ satisfies $\kappa\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$, then $\mathsf{P} = 0$ as a polynomial in z_p, z_q, z_i, z_l and the P_k. In centred variables, $\mathcal{D}_w P_k = k(z_w^{k-1} - P_{k-1}/n)$ and $\mathcal{D}_w z_r = -1/n$, so

$$\mathcal{D}_w = \sum_k \Big(k z_w^{k-1} - \frac{k}{n} P_{k-1}\Big)\frac{\partial}{\partial P_k} + \varepsilon\,\partial_{\mathbf{1}},$$

with $\partial_{\mathbf{1}}$ acting on the slots (three slots on $\Lambda(p,q;l)$); in P no $\mathcal{D}_w$ acts on a function evaluated at z_w. P is antisymmetric in (i,l), and so is every $\mathsf{R}_d v$ below; so $\mathsf{R}_d v = z_{li} S$ with S symmetric in (i,l).

Lemma 9.118 (the top block)**.** *Let $n \geq a+6$, let $0 \neq \varphi \in \mathcal{F}_{(a,2)}(n)$, and suppose that its data satisfy $\mathsf{P} = 0$ as a polynomial for all distinct p, q, i, l. Some v_λ is nonzero; let ℓ_0 be the largest P-degree with some $v_\lambda \neq 0$, and among the λ with $\ell(\lambda) = \ell_0$ and $v_\lambda \neq 0$ let λ have the largest*

weight. Then $\mathsf{R}_d v_\lambda = 0$ *with* $d := a - |\lambda|$*, where* R_d *is* P *with every* $\mathcal{D}_w$ *replaced by* $\varepsilon\partial_{\mathbf{1}}$*, as a linear map on pairs* $v = (F, G)$ *of symmetric polynomials of degree* d *in* (s, t)*.*

Proof. The data determine $\tilde{\varphi}$ at every $(\mathrm{diag}(x), Y)$, by (1), hence φ by density; so they are not 0, and for $n \geq a+6$ the z's and the P_k that occur are algebraically independent, so the expansion in the P_λ is unique and some $v_\lambda \neq 0$; for the same reason $\mathsf{P} = 0$ holds coefficientwise in the P_λ. In P the part $kz_w^{k-1}\partial/\partial P_k$ of $\mathcal{D}_w$ lowers the P-degree by 1; the part $-\frac{k}{n}P_{k-1}\partial/\partial P_k$ keeps it and lowers the weight by 1 (and kills P_2, as $P_1 = 0$); $\varepsilon\partial_{\mathbf{1}}$ keeps both; all other operations in P keep both. So the P_λ-component of P gets nothing from P-degree above ℓ_0 (there is none) or from weight above $|\lambda|$ in P-degree ℓ_0 (all 0), also in the terms where $\mathcal{D}_i$ acts on the term $\mathcal{D}_l\mathsf{F}$ of $\Lambda(p, q; l)$; what is left is $\mathsf{R}_d v_\lambda$. □

Lemma 9.119 (tools)**.** *Let* $v = (F, G)$ *be of degree* d*,* $\partial_{\mathbf{1}} v =: (F_1, G_1)$*,* $L_0(s, s'; t) := -\frac{G(s,t)-G(s',t)}{s-s'} + C(s, s'; t)$ *with* C *built from* F*, and* χ_F*,* χ_1 *as in (b) for* F*,* F_1*.*

(a) $\partial_{\mathbf{1}}(\mathsf{R}_d v) = \mathsf{R}_{d-1}(\partial_{\mathbf{1}} v)$*, where on the left* $\partial_{\mathbf{1}}$ *is the sum of the derivatives in* z_p, z_q, z_i, z_l*. So* $\ker \mathsf{R}$ *is* $\partial_{\mathbf{1}}$*-stable, and if* $\partial_{\mathbf{1}} v \in \ker \mathsf{R}_{d-1}$*, then* $\mathsf{R}_d v$ *is invariant under a common shift of* z_p, z_q, z_i, z_l*.*

(b) $C(s, s'; t) = (s - s')\chi_F(s, s', t)$ *with* χ_F *symmetric in its three arguments. For* $F = s^4 + t^4$*,* $s^3t + st^3$*,* s^2t^2*,* $s^3 + t^3$*,* $s^2t + st^2$*,* $s^2 + t^2$*,* st*,* χ_F *is* $-h_2$*,* $h_2 - e_2$*,* e_2*,* $-e_1$*,* e_1*,* -1*,* 1*, where* h_j*,* e_j *are the complete and the elementary symmetric polynomials in three variables.*

(c) $\mathsf{R}_d v = Q_0[v] + \varepsilon Q_1[\partial_{\mathbf{1}} v] + \varepsilon^2 z_{li}(\partial_{\mathbf{1}}^2 F)(z_p, z_q)$*, where*

$$\begin{aligned} Q_0[v] &= \kappa\big(L_0(p,q;l) - L_0(p,q;i)\big) - L_0(p,q;l) + L_0(i,q;l) + L_0(p,q;i) - L_0(l,q;i) \\ &\quad - z_{ip}L_0[p;q,i\,|\,l] + z_{lp}L_0[p;q,l\,|\,i], \\ Q_1[F_1,G_1] &= -z_{ip}\big[L_0^{(1)}(p,q;l) + F_1[p;q,i] - F_1[p,i;q]\big] + z_{lp}\big[L_0^{(1)}(p,q;i) + F_1[p;q,l] - F_1[p,l;q]\big] \\ &\quad - G_1(z_q,z_l) + G_1(z_q,z_i), \end{aligned}$$

with $L_0[p;q,i\,|\,l] := (L_0(p,q;l) - L_0(p,i;l))/(z_q - z_i)$*,* $F_1[p;q,i] := (F_1(z_p,z_q) - F_1(z_p,z_i))/(z_q - z_i)$*,* $F_1[p,i;q] := (F_1(z_p,z_q) - F_1(z_i,z_q))/(z_p - z_i)$*, and* $L_0^{(1)}$ *being* L_0 *for* (F_1, G_1)*. In particular* Q_1 *does not involve* κ*.*

(d) Write p, q, i, l *also for* z_p, z_q, z_i, z_l*,* ∂_s *for the derivative in the first slot, and* $F_1[0;q,l] := (F_1(0,q) - F_1(0,l))/(q-l)$*. At* $p = q = 0$*,*

$$\begin{aligned} \mathsf{R}_d v &= \kappa\big(\partial_s G(0,i) - \partial_s G(0,l)\big) - 2(l-i)\chi_F(0,i,l) \\ &\quad + \varepsilon\big[i\,\partial_s G_1(0,l) - l\,\partial_s G_1(0,i) + G_1(0,i) - G_1(0,l)\big] + \varepsilon^2(l-i)(\partial_{\mathbf{1}}^2 F)(0,0), \end{aligned}$$

and, if $d \geq 2$*, at* $p = i = 0$*,*

$$\begin{aligned} \mathsf{R}_d v &= -\kappa\,\frac{G(q,l) - G(0,l) - G(q,0)}{q} + \kappa q\big(\chi_F(0,q,0) - \chi_F(0,q,l)\big) - q\chi_F(0,q,0) \\ &\quad - (l-q)\chi_F(l,q,0) + \frac{l\big(l\chi_F(0,l,0) - q\chi_F(0,q,0)\big)}{q-l} + \varepsilon^2 l\,(\partial_{\mathbf{1}}^2 F)(0,q) \\ &\quad + \varepsilon\Big[-\frac{l\,G_1(q,0)}{q} - lq\,\chi_1(0,q,0) + l\,F_1[0;q,l] + F_1(0,q) - F_1(l,q) - G_1(q,l) + G_1(q,0)\Big]. \end{aligned}$$

Proof. (a) $\mathsf{R}_d v$ is built from v by evaluations at the z's, divided differences in the z's, multiplication by κ and by differences of the z's, and $\varepsilon\partial_{\mathbf{1}}$. A common shift of the four z's by c turns $v(z_r, z_t)$ into $v_c(z_r, z_t)$, $v_c(s,t) := v(s+c, t+c)$, and leaves the rest unchanged; so $\mathsf{R}_d[v](z + c\mathbf{1}) = \mathsf{R}_d[v_c](z)$, and the part linear in c is the claim. (b) With $K(r,r') := rF(r,r')$, $(r-r')F(r,r') = K(r,r') - K(r',r)$, so the numerator of C is, for each monomial $r^\alpha r'^\beta$ of K, minus the alternant with exponents $(\alpha, \beta, 0)$ in (s, s', t), and the denominator is $-(s-t)(s'-t)$. So C is $(s-s')$ times a sum of alternants divided by the Vandermonde product $(s-s')(s-t)(s'-t)$, and each such quotient is symmetric.

The seven values follow by direct division. (c) Insert $\mathcal{D}_w \mapsto \varepsilon\partial_{\mathbf{1}}$ into (8): $\Lambda = \varepsilon\partial_{\mathbf{1}}F(s,s') + L_0$; the term $-z_{ip}\cdot\big(-\frac{\Lambda(p,q;l)-\Lambda(i,q;l)}{z_p-z_i}\big)$ is $-\Lambda(p,q;l)+\Lambda(i,q;l)$, and likewise for $(l;i)$; the part $\varepsilon\partial_{\mathbf{1}}F(s,s')$ of Λ does not depend on the third slot, so it cancels in the κ-difference and gives the divided differences of F_1; $\partial_{\mathbf{1}}L_0[v] = L_0[\partial_{\mathbf{1}}v]$; and the two ε^2 terms add up to z_{li}. (d) At $p=q=0$ the terms of Q_0 combine to $\kappa(L_0(0,0;l)-L_0(0,0;i))+[L_0(i,0;l)-L_0(0,i;l)]-[L_0(l,0;i)-L_0(0,l;i)]$, and $L_0(0,0;t) = -\partial_s G(0,t)$; the G-parts of the brackets cancel, as $(G(s,t)-G(s',t))/(s-s')$ is symmetric in (s,s'), leaving $2i\chi_F(i,0,l) - 2l\chi_F(l,0,i)$. In Q_1 the terms in F_1 cancel, as F_1 is symmetric. At $p=i=0$, $Q_0 = \kappa(L_0(0,q;l)-L_0(0,q;0)) + L_0(0,q;0) - L_0(l,q;0) + l(L_0(0,q;0)-L_0(0,l;0))/(q-l)$; with $g(x) := G(x,0)$ (and $G(0,0)=0$ for $d\ge 1$) its G-terms without κ add up to $-g(q)/q + (g(l)-g(q))/(l-q) + (g(l)-lg(q)/q)/(q-l) = 0$. The rest is direct; $G_1(0,0)=0$ needs $d\ge 2$. □

Lemma 9.120 (the small degrees)**.** *Let $\varepsilon\neq 0$.*

(a) Let $d\ge 3$ and $\kappa\neq 0$. If $v = (f\sigma_-^d, g\sigma_-^d) \in \ker \mathsf{R}_d$ (a pair invariant under a common shift; nonzero only for even d), then $f=g=0$.

(b) $d=2$: for $F = f_1\sigma_+^2 + f_2\sigma_-^2$, $G = g_1\sigma_+^2 + g_2\sigma_-^2$, $\mathsf{R}_2 v = z_{li}c_2$ with $c_2 = 2\kappa(g_2-g_1) - 2f_1 + 6f_2 - 8\varepsilon g_1 + 8\varepsilon^2 f_1$.

(c) $d=3$: let $\mathbf{k}_3 := \big(\sigma_+^3 + \sigma_+\sigma_-^2,\ (\varepsilon-\frac{\kappa}{2})\sigma_+^3 - 3(\varepsilon+\frac{\kappa}{2})\sigma_+\sigma_-^2\big)$ and $\mathbf{k}_3' := (2\varepsilon\sigma_+\sigma_-^2, \sigma_+\sigma_-^2)$. Then $\ker\mathsf{R}_3 = \mathbb{C}\mathbf{k}_3$ if $\kappa\neq -6\varepsilon$, and $\ker\mathsf{R}_3 = \mathbb{C}\mathbf{k}_3 + \mathbb{C}\mathbf{k}_3'$ if $\kappa = -6\varepsilon$.

Proof. (a) Here $\partial_{\mathbf{1}}v = 0$, so $\mathsf{R}_d v = Q_0[v]$, and with $X := s-t$, $Y := s'-t$, $L_0(s,s';t) = f\,C^\circ(X,Y) - g\,h_{d-1}(X,Y)$ with $C^\circ(X,Y) := \big(X^{d+1} - Y^{d+1} - (X-Y)^{d+1}\big)/(-XY)$, which is antisymmetric. As in the proof of Lemma 9.119(d), at $p=q=0$ the row is $\kappa(L_0(0,0;l)-L_0(0,0;i))$ plus the bracket below; here $L_0(0,0;t) = dg\,t^{d-1}$ and $[L_0(i,0;l)-L_0(0,i;l)] - [L_0(l,0;i)-L_0(0,l;i)] = 2f\big(C^\circ(i-l,-l) - C^\circ(l-i,-i)\big) = 2fN/(il)$ with $N := (i-l)^{d+1} + l^{d+1} - i^{d+1} = \sum_{j=1}^{d}\binom{d+1}{j}(-1)^j i^{d+1-j}l^j$. So

$$\mathsf{R}_d v = \kappa dg\big(l^{d-1} - i^{d-1}\big) + 2f\sum_{j=1}^{d}\binom{d+1}{j}(-1)^j i^{d-j}l^{j-1}.$$

The monomial il^{d-2} ($j=d-1$; it differs from i^{d-1} and l^{d-1} as $d\ge 3$) has the coefficient $-d(d+1)f$ (d even), so $f=0$, and then $\kappa dg = 0$. (b) Lemma 9.119(d) at $p=q=0$, with (b) of that lemma. (c) Write $F = f_1\sigma_+^3 + f_2\sigma_+\sigma_-^2$, $G = g_1\sigma_+^3 + g_2\sigma_+\sigma_-^2$, $A := f_1 - f_2$, $B := 3g_1 - g_2$. By Lemma 9.119, $\mathsf{R}_3 v = z_{li}\big(c_p z_p + c_q z_q + c'(z_i+z_l)\big)$: the point $p=q=0$ gives $c' = -\kappa B - 4A - 2\varepsilon(3g_1+g_2)$, the point $p=i=0$ gives $c_q = -\kappa B - 2\kappa A - 2A - 12\varepsilon A - 2\varepsilon(9g_1-g_2) + 24\varepsilon^2 f_1$, and Lemma 9.119(a) with (b) above gives $c_p + c_q + 2c' = c_2(\partial_{\mathbf{1}}v)$, so $c_p - c_q = 4A(\kappa+6\varepsilon)$. So $\mathsf{R}_3 v = 0$ if and only if $A(\kappa+6\varepsilon)=0$, $(c_p+c_q)/2 = -\kappa B - 2A - 2\varepsilon(9g_1-g_2) + 24\varepsilon^2 f_1 = 0$ and $c'=0$. In the variables (A, f_1, g_1, g_2) these rows are triangular: the first involves A only; $c'=0$ reads $-(3\kappa+6\varepsilon)g_1 + (\kappa-2\varepsilon)g_2 = 4A$, whose (g_1,g_2)-part is not 0 as $\varepsilon\neq 0$; and only the second contains f_1, with the coefficient $24\varepsilon^2\neq 0$. So the rank is 3 if $\kappa\neq -6\varepsilon$ and 2 if $\kappa = -6\varepsilon$, and $\mathbf{k}_3$, $\mathbf{k}_3'$ satisfy the equations. □

Lemma 9.121 (degree 4)**.** *Let $\varepsilon\neq 0$ and $\kappa\neq 0$, and $(n,\kappa)\notin\{(3,2),(-3,-2)\}$. Then $\ker\mathsf{R}_4 = 0$. At $\kappa = 0$, $(0,\sigma_-^4)\in\ker\mathsf{R}_4$; at $(n,\kappa) = (3,2)$ and $(-3,-2)$, $\ker\mathsf{R}_4\neq 0$.*

Proof. Reduction. If $v\in\ker\mathsf{R}_4$, then $\partial_{\mathbf{1}}v\in\ker\mathsf{R}_3$ (Lemma 9.119(a)). The kernel of $\partial_{\mathbf{1}}$ in degree 4 consists of the pairs $(f\sigma_-^4, g\sigma_-^4)$, and $\partial_{\mathbf{1}}\mathbf{w}_1 = 8\mathbf{k}_3$, $\partial_{\mathbf{1}}\mathbf{w}_4 = 4\mathbf{k}_3'$ for

$$\mathbf{w}_1 := \big(\sigma_+^4 + 2\sigma_+^2\sigma_-^2,\ (\varepsilon-\tfrac{\kappa}{2})\sigma_+^4 - 6(\varepsilon+\tfrac{\kappa}{2})\sigma_+^2\sigma_-^2\big),$$
$$\mathbf{w}_2 := (\sigma_-^4, 0),\qquad \mathbf{w}_3 := (0,\sigma_-^4),\qquad \mathbf{w}_4 := (2\varepsilon\sigma_+^2\sigma_-^2, \sigma_+^2\sigma_-^2).$$

So by Lemma 9.120(c), $v = \xi_1\mathbf{w}_1 + \xi_2\mathbf{w}_2 + \xi_3\mathbf{w}_3 + \xi_4\mathbf{w}_4$, with $\xi_4 = 0$ unless $\kappa = -6\varepsilon$; then $\partial_{\mathbf{1}}v\in\ker\mathsf{R}_3$, so $\mathsf{R}_4 v$ is shift invariant, and at $z_p = 0$ it is $z_{li}\big(\mathsf{v}_1 z_q^2 + \mathsf{v}_2 z_q(z_i+z_l) + \mathsf{v}_3(z_i^2+z_l^2) + \mathsf{v}_4 z_i z_l\big)$; so $\mathsf{R}_4 v = 0$ if and only if $\mathsf{v}_1 = \cdots = \mathsf{v}_4 = 0$.

The functionals. For degree-4 data write $\chi_F = c_h h_2 + c_e e_2$, and let g_{jk}, f'_{jk}, g'_{jk}, f''_{jk} be the coefficients of $s^j t^k$ in G, F_1, G_1 and $\partial_{\mathbf{1}}^2 F$. Lemma 9.119(d) gives (the point $p=i=0$ the coefficients of $q^2 l$, $q l^2$, l^3; the point $p=q=0$ those of i^2+l^2 and il; both give the same v_3)

$$\mathsf{v}_1 = -\kappa g_{13} - c_h + c_e - \kappa(c_h+c_e) + \varepsilon(2f'_{03} - 2f'_{12} - g'_{03} - g'_{12}) + \varepsilon^2 f''_{02},$$
$$\mathsf{v}_2 = -\kappa g_{22} - c_h - c_e - \kappa c_h + \varepsilon(f'_{03} - f'_{12} - g'_{12}),$$
$$\mathsf{v}_3 = -\kappa g_{13} - 2c_h - \varepsilon g'_{03}, \qquad \mathsf{v}_4 = -\kappa g_{13} - 2c_h - 2c_e + \varepsilon(g'_{12} - g'_{03}).$$

For the four vectors this gives $(\mathsf{v}_1,\mathsf{v}_2,\mathsf{v}_3,\mathsf{v}_4)$ below, from the data (by Lemma 9.119(b)) $\mathbf{w}_1$: $c_h=1$, $c_e=-2$, $g_{13}=4\varepsilon-2\kappa$, $g_{22}=18\varepsilon+3\kappa$, $(f'_{03},f'_{12},g'_{03},g'_{12}) = (16,16,-16(\varepsilon+\kappa),48\varepsilon)$, $f''_{02}=64$; $\mathbf{w}_2$: $c_h=-5$, $c_e=10$; $\mathbf{w}_3$: $g_{13}=-4$, $g_{22}=6$; $\mathbf{w}_4$: $c_h=-2\varepsilon$, $c_e=-4\varepsilon$, $g_{13}=0$, $g_{22}=-2$, $(f'_{03},f'_{12},g'_{03},g'_{12}) = (8\varepsilon,-8\varepsilon,4,-4)$, $f''_{02}=16\varepsilon$ (all other entries 0):

$$\mathbf{w}_1:\ \big(2\kappa^2+12\varepsilon\kappa+\kappa+32\varepsilon^2-3,\ -3\kappa^2-18\varepsilon\kappa-\kappa-48\varepsilon^2+1,$$
$$2\kappa^2+12\varepsilon\kappa+16\varepsilon^2-2,\ 2\kappa^2+12\varepsilon\kappa+64\varepsilon^2+2\big),$$
$$\mathbf{w}_2:\ (15-5\kappa,\ -5+5\kappa,\ 10,\ -10), \qquad \mathbf{w}_3:\ (4\kappa,\ -6\kappa,\ 4\kappa,\ 4\kappa),$$
$$\mathbf{w}_4:\ \big(-2\varepsilon+6\varepsilon\kappa+32\varepsilon^2+16\varepsilon^3,\ 2\kappa+10\varepsilon+2\varepsilon\kappa+16\varepsilon^2,\ 0,\ 4\varepsilon\big).$$

Put $\mathbf{r}_1 := 3\mathsf{v}_1+2\mathsf{v}_2$, $\mathbf{r}_2 := (\mathsf{v}_4-\mathsf{v}_3)/4$, $\mathbf{r}_3 := (\mathsf{v}_3+\mathsf{v}_4)/4$. Then

$$\mathbf{r}_1 = (\kappa-7)(\xi_1-5\xi_2) + (-10\varepsilon-4\varepsilon^2+48\varepsilon^3)\xi_4, \qquad \mathbf{r}_2 = (12\varepsilon^2+1)\xi_1 - 5\xi_2 + \varepsilon\xi_4,$$
$$\mathbf{r}_3 = (\kappa^2+6\varepsilon\kappa+20\varepsilon^2)\xi_1 + 2\kappa\xi_3 + \varepsilon\xi_4,$$

where the ξ_4-coefficient of $\mathbf{r}_1$ is its value at $\kappa=-6\varepsilon$, the only case with $\xi_4\neq 0$.

The cases. Let $\kappa\neq -6\varepsilon$, so $\xi_4=0$; this case uses only $\varepsilon\neq 0$ and $\kappa\neq 0$. If $\kappa\neq 7$, $\mathbf{r}_1=0$ gives $\xi_1=5\xi_2$, then $\mathbf{r}_2=0$ gives $12\varepsilon^2\xi_1=0$, so $\xi_1=\xi_2=0$, and $\mathbf{r}_3=2\kappa\xi_3=0$. If $\kappa=7$, $\mathbf{r}_2$ and $\mathbf{r}_3$ give $\xi_2=(12\varepsilon^2+1)\xi_1/5$ and $\xi_3=-(49+42\varepsilon+20\varepsilon^2)\xi_1/14$, and then $\mathsf{v}_1 = (102+84\varepsilon+32\varepsilon^2)\xi_1 - 20\xi_2+28\xi_3 = -56\varepsilon^2\xi_1$, so $\xi_1=0$. Let $\kappa=-6\varepsilon$. Then $\mathbf{r}_2$, $\mathbf{r}_3$ give $\xi_2=((12\varepsilon^2+1)\xi_1+\varepsilon\xi_4)/5$ and $\xi_3=(20\varepsilon\xi_1+\xi_4)/12$; so $\mathsf{v}_2 = -72\varepsilon^3\xi_1 - 2\varepsilon^2\xi_4$, which gives $\xi_4=-36\varepsilon\xi_1$, and then $\mathbf{r}_1 = 12\varepsilon^2(7+6\varepsilon)\xi_1 + \varepsilon(48\varepsilon^2+2\varepsilon-3)\xi_4 = 192\varepsilon^2(1-9\varepsilon^2)\xi_1$. On the line $\kappa=-6\varepsilon$, $9\varepsilon^2=1$ means $(n,\kappa)=(3,2)$ or $(-3,-2)$; outside these, $\xi_1=0$ and so $\xi_2=\xi_3=\xi_4=0$. At $\kappa=0$ the column of $\mathbf{w}_3$ is 0. At $(n,\kappa)=(3,2)$ ($\varepsilon=-\frac{1}{3}$) the same equations give the kernel vector $\mathbf{w}_1+12\mathbf{w}_4-\frac{1}{3}\mathbf{w}_2+\frac{4}{9}\mathbf{w}_3$; at $(-3,-2)$ ($\varepsilon=\frac{1}{3}$) also $9\varepsilon^2=1$, and the same equations give a nonzero kernel vector. □

Theorem 9.122 (the top block)**.** *Let $\kappa\neq 0$, $n\neq 0$ and $(n,\kappa)\notin\{(3,2),(-3,-2)\}$. Then R_d is injective for every $d\geq 4$.*

Proof. $d=4$ is Lemma 9.121. If $\ker \mathsf{R}_{d-1}=0$ and $v\in\ker\mathsf{R}_d$ with $d\geq 5$, then $\partial_{\mathbf{1}} v\in\ker\mathsf{R}_{d-1}=0$ (Lemma 9.119(a)), so v is invariant under a common shift, and $v=0$ by Lemma 9.120(a). □

Corollary 9.123 (the leading coefficient)**.** *Let $n\geq a+6$, $\kappa\neq 0$, and let $0\neq\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ satisfy $\kappa\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$. Then, in Lemma 9.118,* $\deg v_\lambda\leq 3$*, that is, $|\lambda|\geq a-3$.*

Proof. $\mathsf{P}=0$ as a polynomial (see above), so Lemma 9.118 applies; $n\geq a+6\geq 6$, so $n\neq 0,\pm 3$, and Theorem 9.122 gives $\deg v_\lambda\leq 3$. □

Remark 9.124. (1) Only the rows (8) are used: not the coincident order-one rows, the order-zero rows, or $\mathsf{G}x=0$. So the conclusion of Corollary 9.123 holds for every $\varphi\in\mathcal{F}_{(a,2)}(n)$ whose rows (8) vanish at κ as polynomials, for $n\geq a+6$ (Proposition 9.44 does not use $\mathcal{E}\varphi=0$). The corollary has content only for $a\geq 4$.

(2) $\kappa\neq 0$ is needed: at $\kappa=0$, $(0,\sigma_-^4)\in\ker\mathsf{R}_4$. The value $\kappa=0$ is $\mu=-a$, which Conjecture 11.3 does not allow, so it must be excluded separately. For $n\geq a+6$ it is excluded by Theorem 9.150 (Corollary 9.144(i) for odd a, Theorem 9.149(a) for even a).

(3) The point $(n,\kappa)=(3,2)$ lies outside the range $n\ge a+6$ in which Corollary 9.123 applies. It lies on $\kappa=6/n$, where $\ker \mathsf{R}_3$ jumps (Lemma 9.120(c)); $\kappa=6/n$ is $\mu=6-a$, which at $a=3$ is the value $\mu=3$ of Theorem 9.48. We do not decide whether the jump of $\ker \mathsf{R}_3$ comes from this family, nor whether the point is related to the eigenvalue $\kappa=2$ of Corollary 9.103(c) and its mirror. The exceptional set $\{\kappa=0\}\cup\{(3,2),(-3,-2)\}$ is invariant under $(n,\kappa)\mapsto(-n,-\kappa)$, as Theorem 9.75 suggests; we have not checked that R_d itself has this symmetry, and it is not used.

(4) What remains for Conjecture 11.3. Corollary 9.123 locates the leading coefficient of an eigenvector (top P-degree, top weight, degree at most 3, in $\ker \mathsf{R}_d$; $\mathsf{R}_0=\mathsf{R}_1=0$, R_2 is one equation, $\ker \mathsf{R}_3$ is Lemma 9.120(c)), but it does not pin κ: $\ker \mathsf{R}_3\neq 0$ for every κ, and at $\kappa=6/n$ the kernel is larger, which is a jump, not a condition that pins κ. Below the top coefficient, the top P-degree gives inhomogeneous equations $\mathsf{R}_{d_0+j}u_j=$ (the weight-lowering image of u_{j-1}), where $d_0:=\deg v_\lambda$ and u_j stands for the coefficients of P-degree ℓ_0 and weight $|\lambda|-j$; R_d is injective for $d\ge 4$ but far from onto, so these longer chains can impose conditions on κ. So κ can be pinned only by these chains or by the blocks of lower P-degree (first ℓ_0-1, which the part $kz_w^{k-1}\partial/\partial P_k$ of $\mathcal{D}_w$, depending on the point w, feeds from the top block). A computation (exact, κ symbolic, $n=11$ and 13, $d_0\le 3$, chains of length at most 6; not a proof) shows that the longer chains of the top P-degree do not pin κ off $\{0,\pm 2,6/n\}$; so, in this range, the chains give no condition, and we expect κ to be pinned only through the blocks of lower P-degree. A dimension count (by hand) shows that, at $a=4$, the rows (8) alone never pin κ on formal data: the data $\sum_\lambda P_\lambda v_\lambda$ have 16 coefficients ($\lambda=\emptyset,(2),(3),(4),(2,2)$), while P, antisymmetric in (i,l) and homogeneous of degree $a-1=3$ (z_l of degree 1, P_k of degree k), gives at most 7 linear conditions at P-degree 0 and one at P_2; so its solutions form a space of dimension at least 8 for every κ. So at $a=4$ a condition on κ must come from the other rows of item (1) or from relations that the data of every φ satisfy but formal data need not. Two such relations are $\mathsf{G1}=0$ and, on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, $\mathsf{G}x=0$ (after (1)). With $\langle u^j\rangle_u:=P_j$ ($P_0=n$, $P_1=0$) they read $\langle \mathsf{G}(u,v)\rangle_u+\mathsf{F}(v,v)=0$ and $\langle u\,\mathsf{G}(u,v)\rangle_u+v\,\mathsf{F}(v,v)=0$, and they tie the coefficients of P-degree K to those of P-degree $K+1$. Neither route is decided here, nor the case $\kappa=0$. For $n\ge a+6$, Subsections 9.24 and 9.25 prove the conclusion of Conjecture 11.3, $\kappa=0$ included, by another route: the odd rows, the order-zero rows (Lemma 9.46), the relations $\mathsf{G1}=0$, $\mathsf{G}x=0$, and one coefficient of the rows (8) (Theorem 9.149; Remark 9.151(2)).

9.23. **Theorem Z-INJ at $b=4$ for every $n\ge 5$.** In this subsection $F=\mathbb{C}$ and $b=4$, and we keep the notation of Subsection 9.19: $D=\mathrm{diag}(x)$, x_{pq}, $\theta_{pq}=1/x_{qp}$, $\Delta_{pq}=e_p-e_q$, the slice points (D,Z), β, ω, the rows c of Lemma 9.83, and Φ with its parts Φ_j of degree j in y (Lemma 9.93). We take $a\ge 4$, as a highest weight vector needs $a\ge b$ (Remark 9.89(3)). By Corollary 9.94(b), Theorem 9.22 at $(a,4,n)$ says that $\mathcal{Z}(\varphi)=0$ and $\mathcal{E}\varphi=0$ force $\Phi_0=\Phi_1=0$. We prove this for every $n\ge 5$ (Theorem 9.129, Corollary 9.130); this answers the question left open in Remark 9.92(4) and Remark 9.95(5) for $b=4$ and $n\ge 5$. Exactly one step, the rank of one linear system on five points, is computer-assisted (Theorem 9.128); everything else is by hand. For an edge $e=\{p,q\}$ put $z_e:=Z_{pq}Z_{qp}$.

Lemma 9.125 (the shape at $b=4$)**.** *Let $\varphi\in\mathcal{F}_{(a,4)}(n)$. Then*

$$\Phi_0=\sum_\gamma \mathsf{F}(\gamma)Z^\gamma+\sum_{e<e'}\mathsf{U}(e,e')z_ez_{e'}+\tfrac12\sum_e\mathsf{U}(e,e)z_e^2,\qquad \Phi_1=\sum_s y_s\mathsf{N}_s,\quad \mathsf{N}_s=\sum_\gamma \mathsf{n}^s_\gamma Z^\gamma,$$

where the first sum runs over the directed 4-cycles γ on four distinct vertices ($\mathsf{F}(abcd)$ is the coefficient of $a\to b\to c\to d\to a$; a rotation is the same cycle), U is symmetric in its two edges, the sum in N_s runs over the directed 3-cycles ((lsm) is the cycle $l\to s\to m\to l$), and the y-Hessian of Φ_2 is $\mathsf{G}=\sum_e z_e\mathsf{G}_e$ with G_e symmetric. All coefficients are polynomials in x, homogeneous of degree a and invariant under $x\mapsto x+c\mathbf{1}$ (as in Lemma 9.84), $\sum_s \mathsf{n}^s_\gamma=0$ and $\mathsf{G}_e\mathbf{1}=0$. If $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,4)}(n)$, then $\mathsf{G}_ex=0$.

Proof. Conjugation by invertible diagonal matrices fixes D; Z_{pq} has weight e_p-e_q and y_s weight 0, so every Φ_j is a polynomial of weight 0 in Z. A monomial of weight 0 in the off-diagonal Z is a directed multigraph without loops with in-degree equal to out-degree at every vertex, that is a union of directed cycles of length at least 2. Φ has degree 4 in (y,Z), so Φ_0 consists of directed 4-cycles on four distinct vertices and products $z_e z_{e'}$ of two 2-cycles (any overlap, $e=e'$ allowed), Φ_1 of directed 3-cycles, and Φ_2 of 2-cycles z_e. As in the proof of Lemma 9.84, $\sum_s \partial\Phi/\partial y_s=0$; its parts of degree 0 and 1 in y give $\sum_s \mathsf{N}_s=0$ and $\mathsf{G}\mathbf{1}=0$, and, if $\mathcal{E}\varphi=0$, the part of degree 1 in y of $\sum_s x_s\partial\Phi/\partial y_s=0$ gives $\mathsf{G}x=0$. The monomials z_e are distinct, so these hold for each G_e. □

For a directed 4-cycle γ, $\tau(\gamma)$ and $\sigma(\gamma)$ are the orientation-odd and orientation-even halves of F: $\mathsf{F}(\gamma)=\sigma(\gamma)+\tau(\gamma)$ and $\mathsf{F}(\gamma\text{ reversed})=\sigma(\gamma)-\tau(\gamma)$. For an edge e put $\mathsf{f}^e_{ab}:=\mathsf{U}(\{a,b\},e)$; for $v\in\mathbb{C}^n$ put $\nu(v)_\gamma:=\sum_s v_s\mathsf{n}^s_\gamma$ and $\mathsf{N}(v):=\sum_s v_s\mathsf{N}_s$. For a symmetric pair function f and a bilinear form B let $\mathsf{r}_{lcm}[f,B]$ be the right side of Lemma 9.79(c) with (F,G) replaced by (f,B):

$$\mathsf{r}_{lcm}[f,B]=\theta_{lc}(f_{cm}-f_{lm})+\theta_{cm}(f_{lc}-f_{lm})+(\partial_c-\partial_l)f_{lm}+\theta_{lm}f_{lm}-\theta_{lm}B(\Delta_{lm},\Delta_{lc}).$$

At $y=0$ the row $\mathsf{c}(l,c,m)$ of Lemma 9.83 (with $k=c$) is a polynomial in Z; $[T]\,\mathsf{c}(l,c,m)$ is its coefficient of the monomial T. For a matrix A write $A_d:=(A_{ss})_s$ and $\partial_Z[A]:=\sum_{r\neq s}A_{rs}\,\partial/\partial Z_{rs}$ (the off-diagonal part only), and put $W^{pq}:=[E_{pq},Z]$. Expanding $\Phi=\Phi_0+\sum_s y_s\mathsf{N}_s+\frac12 y^T\mathsf{G}y+O(y^3)$ at $y=0$ gives

(9)

$$\partial_Y[A]\tilde\varphi=\mathsf{N}(A_d)+\partial_Z[A]\Phi_0,\qquad \partial^2_Y\tilde\varphi(A,B)=A_d^T\mathsf{G}B_d+\partial_Z[B]\mathsf{N}(A_d)+\partial_Z[A]\mathsf{N}(B_d)+\partial_Z[A]\partial_Z[B]\Phi_0,$$

where the entries of A and A_d are evaluated at the base point and are not differentiated; and, by the formula after (3), $(W^{pq})_d=Z_{qp}\Delta_{pq}$ and $\partial_Z[W^{pq}]=\sum_{r\notin\{p,q\}}(Z_{qr}\partial/\partial Z_{pr}-Z_{rp}\partial/\partial Z_{rq})$, the sum running over *all* $r\notin\{p,q\}$.

Lemma 9.126 (the spectator rows)**.** *Let $\varphi\in\mathcal{F}_{(a,4)}(n)$ ($\mathcal{E}\varphi=0$ and the invariance under $Y\mapsto Y+cI$ are not used) and let l,c,m be distinct.*

(i) At every slice point (this part holds for every b, with Φ_0, N, G the parts of degree 0 and 1 in y and the y-Hessian at $y=0$):

$$\begin{aligned}\mathsf{c}(l,c,m)=&(\partial_c-\partial_l)\partial_{Z_{lm}}\Phi_0+(\theta_{lm}-\theta_{lc}-\theta_{cm})\partial_{Z_{lm}}\Phi_0+\theta_{cm}\partial_Z[W^{cm}]\partial_{Z_{lc}}\Phi_0-\theta_{lc}\partial_Z[W^{lc}]\partial_{Z_{cm}}\Phi_0\\&-\theta_{lm}\partial_Z[W^{lm}]\mathsf{N}(\Delta_{lc})+\theta_{cm}Z_{mc}\partial_{Z_{lc}}\mathsf{N}(\Delta_{cm})-\theta_{lc}Z_{cl}\partial_{Z_{cm}}\mathsf{N}(\Delta_{lc})-\theta_{lm}Z_{ml}\,\Delta_{lm}^T\mathsf{G}\Delta_{lc}\end{aligned}$$

(terms $\mathrm{R}_1,\dots,\mathrm{R}_8$ in this order).

(ii) For an edge e and $T:=Z_{ml}z_e$:

(S1) if $e\cap\{l,c,m\}=\emptyset$ ($n\ge5$) or $e=\{l,m\}$: $[T]\,\mathsf{c}(l,c,m)=\mathsf{r}_{lcm}[\mathsf{f}^e,\mathsf{G}_e]$;

(S2) if $e=\{c,p\}$ with $p\notin\{l,c,m\}$ ($n\ge4$): $[T]\,\mathsf{c}(l,c,m)=\mathsf{r}_{lcm}[\mathsf{f}^e,\mathsf{G}_e]-\theta_{cm}\mathsf{F}(lcpm)-\theta_{lc}\mathsf{F}(lpcm)$;

(S3) if $e=\{m,s\}$ with $s\notin\{l,c,m\}$ ($n\ge4$): $[T]\,\mathsf{c}(l,c,m)=\mathsf{r}_{lcm}[\mathsf{f}^e,\mathsf{G}_e]+\theta_{cm}\mathsf{F}(lcsm)-\theta_{lm}\nu(\Delta_{lc})_{(lsm)}$.

Proof. (i) Three ω-terms, each from (3) and (9). (A) $\beta(E_{ll},E_{lm})=\partial_l\partial_{Z_{lm}}\Phi_0$; with $[E_{lm},E_{ll}]=-E_{lm}$ and (9) for $A=W^{lm}$ ($A_d=Z_{ml}\Delta_{lm}$) and $B=E_{ll}$ ($B_d=e_l$, $\partial_Z[B]=0$), $\beta(E_{lm},E_{ll})=-\theta_{lm}[Z_{ml}(\mathsf{G}_{ll}-\mathsf{G}_{ml})+\partial_Z[W^{lm}]\mathsf{N}_l-\partial_{Z_{lm}}\Phi_0]$. So $\omega(E_{ll},E_{lm})=\partial_l\partial_{Z_{lm}}\Phi_0+\theta_{lm}[Z_{ml}(\mathsf{G}_{ll}-\mathsf{G}_{ml})+\partial_Z[W^{lm}]\mathsf{N}_l-\partial_{Z_{lm}}\Phi_0]$. (C) The same with E_{cc} in place of E_{ll}, and $[E_{lm},E_{cc}]=0$: $\omega(E_{cc},E_{lm})=\partial_c\partial_{Z_{lm}}\Phi_0+\theta_{lm}[Z_{ml}(\mathsf{G}_{lc}-\mathsf{G}_{mc})+\partial_Z[W^{lm}]\mathsf{N}_c]$. (B) $[E_{lc},E_{cm}]=E_{lm}$ and $[E_{cm},E_{lc}]=-E_{lm}$; (9) with $B=E_{cm}$ ($B_d=0$, $\partial_Z[B]=\partial_{Z_{cm}}$), $A=W^{lc}$ ($A_d=Z_{cl}\Delta_{lc}$), and with $B=E_{lc}$, $A=W^{cm}$ ($A_d=Z_{mc}\Delta_{cm}$), gives $\beta(E_{lc},E_{cm})=-\theta_{lc}[Z_{cl}\partial_{Z_{cm}}\mathsf{N}(\Delta_{lc})+\partial_Z[W^{lc}]\partial_{Z_{cm}}\Phi_0+\partial_{Z_{lm}}\Phi_0]$ and $\beta(E_{cm},E_{lc})=-\theta_{cm}[Z_{mc}\partial_{Z_{lc}}\mathsf{N}(\Delta_{cm})+\partial_Z[W^{cm}]\partial_{Z_{lc}}\Phi_0-\partial_{Z_{lm}}\Phi_0]$. Now $\mathsf{c}=(B)+(C)-(A)$: the terms $\partial_{Z_{lm}}\Phi_0$ collect to $(\theta_{lm}-\theta_{lc}-\theta_{cm})\partial_{Z_{lm}}\Phi_0$, $\mathsf{N}_c-\mathsf{N}_l=-\mathsf{N}(\Delta_{lc})$ and $\mathsf{G}_{lc}-\mathsf{G}_{mc}-\mathsf{G}_{ll}+\mathsf{G}_{ml}=-\Delta_{lm}^T\mathsf{G}\Delta_{lc}$.

(ii) Three facts. (a) At $b=4$, $\partial_{Z_{ab}}\Phi_0 = Z_{ba}\sum_{e'}\mathsf{U}(\{a,b\},e')z_{e'} + \sum_{r,t}\mathsf{F}(abrt)Z_{br}Z_{rt}Z_{ta}$ ($r\neq t$, both $\notin\{a,b\}$; the factor $\frac{1}{2}$ of $\mathsf{U}(e,e)z_e^2$ cancels). So $[Z_{ba}z_{e'}]\partial_{Z_{ab}}\Phi_0 = \mathsf{f}^{e'}_{ab}$, the coefficient of the path $Z_{br}Z_{rt}Z_{ta}$ is $\mathsf{F}(abrt)$, $[Z^\gamma]\mathsf{N}(v) = \nu(v)_\gamma$, and $[Z_{ml}z_{e'}](Z_{ml}\Delta^T\mathsf{G}\Delta') = \mathsf{G}_{e'}(\Delta,\Delta')$. (b) For a monomial T, an operator $Z_{ij}\partial/\partial Z_{hk}$ ($(i,j)\neq(h,k)$) and a polynomial S: $[T](Z_{ij}\partial_{Z_{hk}}S) = (\deg_{Z_{hk}}T+1)\,[TZ_{hk}/Z_{ij}]\,S$ if Z_{ij} divides T, and 0 otherwise; below the variable put back (Z_{cl}, Z_{cs}, Z_{pm}, Z_{lp}, Z_{mc}, Z_{ls}) never divides T, so the factor is 1. (c) The factors of $T = Z_{ml}z_e$ are Z_{ml} (twice if $e=\{l,m\}$) and the two factors of z_e. Term by term, with $f := \mathsf{f}^e$: $\mathrm{R}_1+\mathrm{R}_2$ give $(\partial_c-\partial_l)f_{lm}+(\theta_{lm}-\theta_{lc}-\theta_{cm})f_{lm}$ (every kind). In R_3, the factors Z_{mr} of T with $r\notin\{c,m\}$ are Z_{ml} ($r=l$, every kind) and Z_{ms} (kind (S3)), and the factors Z_{rc} are Z_{pc} (kind (S2)): they give $\theta_{cm}f_{lc}$, $\theta_{cm}\mathsf{F}(lcsm)$ and $-\theta_{cm}\mathsf{F}(lcpm)$. In R_4, the factors Z_{cr}, $r\notin\{l,c\}$, are Z_{cp} (kind (S2)), and the factors Z_{rl} are Z_{ml} only: they give $-\theta_{lc}\mathsf{F}(cmlp) = -\theta_{lc}\mathsf{F}(lpcm)$ and $\theta_{lc}f_{cm}$. In R_5 only Z_{ms} (kind (S3)) contributes: $-\theta_{lm}\nu(\Delta_{lc})_{(lsm)}$. $\mathrm{R}_6=\mathrm{R}_7=0$, as Z_{mc} and Z_{cl} do not divide T ($e\neq\{m,c\},\{l,c\}$). R_8 gives $-\theta_{lm}\mathsf{G}_e(\Delta_{lm},\Delta_{lc})$. The sum of $\mathrm{R}_1+\mathrm{R}_2$, $\theta_{cm}f_{lc}$, $\theta_{lc}f_{cm}$ and R_8 is $\mathsf{r}_{lcm}[\mathsf{f}^e,\mathsf{G}_e]$ for every kind; the rest is as stated. □

Lemma 9.127 (contraction)**.** *Fix a traceless x with distinct entries and $l\neq m$. Let f be a symmetric pair function, homogeneous of degree a ($\sum_k x_k\partial_k f_{lm} = af_{lm}$), B a bilinear form with $B(\cdot,x)=0$, and $K(l,c,m)$, $K(m,c,l)$ numbers. If $\mathsf{r}_{lcm}[f,B]+K(l,c,m)=0$ and $\mathsf{r}_{mcl}[f,B]+K(m,c,l)=0$ for every $c\notin\{l,m\}$, then for every $k\notin\{l,m\}$, with $K^\pm(c) := K(l,c,m)\pm K(m,c,l)$,*

$$(\star_K)\qquad 2\theta_{lk}(f_{km}-f_{lm})+2\theta_{km}(f_{lk}-f_{lm})+2(a+1)\theta_{lm}f_{lm}+\theta_{lm}\sum_{c\notin\{l,m\}}x_cK^+(c)+K^-(k)=0.$$

Proof. As f is symmetric and θ antisymmetric, the value part of r_{mcl} is minus that of r_{lcm}, and its B-part is $-\theta_{lm}B(\Delta_{lm},\Delta_{mc})$. With $V(c) := \theta_{lc}(f_{cm}-f_{lm})+\theta_{cm}(f_{lc}-f_{lm})+\theta_{lm}f_{lm}$, $J := (\partial_l-\partial_m)f_{lm}$ and $q := B(\Delta_{lm},\Delta_{lm})/2$, the difference and the sum of the two rows are ($\Delta_{lc}-\Delta_{mc}=\Delta_{lm}$)

$$D(c) = 2V(c)-J-2\theta_{lm}q+K^-(c),\qquad S(c) = (2\partial_c-\partial_l-\partial_m)f_{lm}-\theta_{lm}B(\Delta_{lm},e_l+e_m-2e_c)+K^+(c).$$

$\sum_{c\notin\{l,m\}}x_cS(c)=0$: Euler and $\sum_{c\notin\{l,m\}}x_c = -(x_l+x_m)$ give $2af_{lm}+(x_m-x_l)J$ for the derivatives; $\sum_c x_c(e_l+e_m-2e_c) = (x_l-x_m)\Delta_{lm}-2x$ and $B(\Delta_{lm},x)=0$ give $2q$ for the B-part ($\theta_{lm}(x_l-x_m) = -1$). So $J = -\theta_{lm}(2af_{lm}+2q+X^+)$, $X^+ := \sum_c x_cK^+(c)$. Putting this into $D(k)=0$, q cancels and $2V(k)+2a\theta_{lm}f_{lm}+\theta_{lm}X^++K^-(k)=0$, which is $(\star_K)$. □

Two uses at a traceless generic slice point, with $f=\mathsf{f}^e$ and $B=\mathsf{G}_e$ (so $B(\cdot,x)=0$ by Lemma 9.125 when $\mathcal{E}\varphi=0$), and K the kind-dependent part of Lemma 9.126(S1), (S2). *Use 1* ($l,m\notin e=\{p,q\}$): $K=0$ at $c\notin e$, and at $c=p$, $K(l,p,m) = -\theta_{pm}\mathsf{F}(lpqm)-\theta_{lp}\mathsf{F}(lqpm)$, $K(m,p,l) = \theta_{pm}\mathsf{F}(mqpl)+\theta_{lp}\mathsf{F}(mpql)$, the reversed cycles; so $K^+(p) = -2\theta_{pm}\tau(lpqm)-2\theta_{lp}\tau(lqpm)$ and $K^-(p)$ is the same with σ; $c=q$ likewise. This gives the equations $(\star_e)(l,m;k)$. *Use 2* ($\{l,m\}=e$): $K=0$ for every c, and $(\star_K)$ is the equation $(\star)$ of Step 2 of the proof of Theorem 9.80 for $\mathsf{U}(\cdot,e)$ on the pair e, at every $k\notin e$; call it $(\star_{ee})(k)$. $\mathcal{E}\varphi=0$ enters only through $\mathsf{G}_ex=0$; N does not enter. As the term X^+ is not invariant under $x\mapsto x+c\mathbf{1}$, these equations hold at the traceless representative of x.

Closure. $(\star_e)(l,m;k)$ involves $\mathsf{U}(e,\cdot)$ on the pairs inside $\{l,m,k\}$ and F on the 4-cycles inside $e\cup\{l,m\}$, and $(\star_{ee})(k)$ involves $\mathsf{U}(e,\cdot)$ on the pairs inside $e\cup\{k\}$. So for every 5-set $W\subseteq[n]$ the equations with all indices in W form a closed linear system $M_W(x_W,a)\,u_W=0$, where u_W consists of F on the 30 directed 4-cycles of W and U on the 55 pairs of edges of W, with $10\cdot(3\cdot3)+10\cdot3=120$ equations. Its coefficients are rational in x_W, the restriction of the traceless x to W. For $n=5$, x_W is traceless; for $n\geq6$, x_W runs over all of $\mathbb{C}^5$ as x runs over the traceless hyperplane of $\mathbb{C}^n$ (a coordinate outside W restores $\sum x=0$), and distinct entries is a Zariski open condition.

Theorem 9.128 (computer-assisted)**.** *At the traceless point $x_0 = (3,-5,\frac{7}{2},11,-\frac{25}{2})$, one fixed 85×85 minor of $M_W(x_0,a)$ has determinant*

$$c\,(a-2)(a+1)^{12}(2a-3)(2a-1)^4(31a-993)(496a-1713), \qquad c\neq 0,$$

as a polynomial in a. It has no integer root $a\geq 4$. So for every integer $a\geq 4$, $M_W(x_W,a)$ has full column rank for x off a proper Zariski-closed subset of the traceless hyperplane of $\mathbb{C}^n$, for every $n\geq 5$.

Proof. An exact computation over $\mathbb{Q}$: the rows of M_W (from $(\star_e)$, $(\star_{ee})$), the interpolation and the factorisation in exact arithmetic, the rank and the determinant in exact rational arithmetic. The 85 rows are chosen once (greedily at $a=7$) and used at every a. Every entry of M_W has degree at most 1 in a (a enters only through $2(a+1)\theta_{lm}f_{lm}$), so the minor has degree at most 85 in a, and its values at 88 integers determine it. The constant is

$$c=\frac{1565924712559159329377}{154504567650236538021426163084138123155823699346600013260 25\cdot 10^{20}}.$$

The roots 993/31 and 1713/496 are not integers, and a is a degree. The minor is a nonzero rational function of x_W, nonzero at x_0; x_0 lies on the traceless hyperplane of $\mathbb{C}^5$, where x_W lives for $n=5$, and in $\mathbb{C}^5$, where x_W lives for $n\geq 6$. □

Theorem 9.129 (B4)**.** *Let $n\geq 5$, $a\geq 4$ and $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,4)}(n)$ with all slice rows c zero (in particular, if $\mathcal{Z}(\varphi)=0$: the slice rows are coefficients of the corner values Q_u of $\mathcal{Z}(\varphi)$, Lemma 9.83, whose proof does not use $b=3$). Then $\Phi_0=0$ (that is, $\mathsf{F}=0$ and $\mathsf{U}=0$), $\mathsf{G}=0$ and $\Phi_1=0$ ($\mathsf{N}=0$).*

Proof. Order 0 (B4-0). At every generic traceless slice point, Lemma 9.126(S1), (S2) give the rows on $T=Z_{ml}z_e$ of the three kinds (disjoint, $c\in e$, $e=\{l,m\}$) in the form of Lemma 9.127, which applies (Uses 1 and 2). Its sums run over all $c\notin\{l,m\}$ in $[n]$, so rows with indices outside W are used, but the resulting equations with indices in W involve only u_W (Closure). So $M_W u_W=0$ for every 5-set W, and by Theorem 9.128 $u_W=0$ off a proper closed subset of the traceless hyperplane. F and U are polynomials in x, invariant under $x\mapsto x+c\mathbf{1}$ (as $\tilde\varphi(X+cI,Y)=\tilde\varphi(X,Y)$), so $\mathsf{F}=\mathsf{U}=0$.

Order 1 (B4-1; by hand, $n\geq 4$). (1) With $\Phi_0=0$, $\mathsf{f}^e=0$ with all derivatives and $\mathsf{F}=0$, so Lemma 9.126(S1) for $l,m\notin e$, $c\notin e\cup\{l,m\}$, (S2) for $l,m\notin e$, $c\in e$, and (S1) for $\{l,m\}=e$ read $B(\Delta_{lm},\Delta_{lc})=0$, $B:=\mathsf{G}_e=(b_{ij})$. For $l,m\notin e$ the rows (l,c,m) and (m,c,l) give $b_{ll}-b_{lc}-b_{ml}+b_{mc}=0$ and $b_{mm}-b_{mc}-b_{lm}+b_{lc}=0$; their sum gives $b_{lm}=\mathsf{w}_l+\mathsf{w}_m$ ($\mathsf{w}_l:=b_{ll}/2$), their difference $b_{lc}-\mathsf{w}_l=b_{mc}-\mathsf{w}_m$; for $c\in e$ this defines $\mathsf{w}_c:=b_{lc}-\mathsf{w}_l$ (independent of $l\notin e$). So $B=\mathbf{1}\mathsf{w}^T+\mathsf{w}\mathbf{1}^T+L_e$ with L_e supported on $e\times e$. The rows with $\{l,m\}=e=\{p,q\}$ and $c\notin e$ read, as $\mathbf{1}\cdot\Delta=0$, $L_e(\Delta_{pq},\Delta_{pc})=(L_e)_{pp}-(L_e)_{qp}=0$ and $(L_e)_{qq}-(L_e)_{pq}=0$, so $L_e=\kappa_e(e_p+e_q)(e_p+e_q)^T$. At traceless x, $0=Bx=(\mathsf{w}\cdot x)\mathbf{1}+\kappa_e(x_p+x_q)(e_p+e_q)$; a coordinate $l\notin e$ gives $\mathsf{w}\cdot x=0$, then $\kappa_e(x_p+x_q)=0$, so $L_e=0$ at generic x. The invariance under $Y\mapsto Y+cI$ gives $0=B\mathbf{1}=(\mathsf{w}\cdot\mathbf{1})\mathbf{1}+n\mathsf{w}$, so $\mathsf{w}=c'\mathbf{1}$, and then $0=B\mathbf{1}=2nc'\mathbf{1}$, so $\mathsf{w}=0$. So $\mathsf{G}_e=0$ at generic traceless x, and, being a polynomial in x invariant under $x\mapsto x+c\mathbf{1}$, $\mathsf{G}_e=0$. (2) Lemma 9.126(S3) with $\mathsf{G}=0$ and $\Phi_0=0$: the row on $Z_{ml}z_{ms}$ ($s\notin\{l,c,m\}$) is $-\theta_{lm}\nu(\Delta_{lc})_{(lsm)}=0$, so $\mathsf{n}^l_\gamma=\mathsf{n}^c_\gamma$ for $\gamma=(lsm)$ and every $c\notin\gamma$ ($n\geq 4$ gives such c). The same row for the three rotations (l,s,m), (s,m,l), (m,l,s) of γ (the rows on $Z_{ml}z_{ms}$, $Z_{ls}z_{lm}$, $Z_{sm}z_{sl}$) gives $\mathsf{n}^v_\gamma=\mathsf{n}^c_\gamma$ for every $v\in\gamma$ and $c\notin\gamma$; so n^s_γ does not depend on s, and $\sum_s\mathsf{n}^s_\gamma=0$ (Lemma 9.125) gives $\mathsf{N}=0$. □

Corollary 9.130 (Z-INJ at $b=4$)**.** *Let $n\geq 5$, $a\geq 4$ and $h:=\dim\mathcal{F}^{\mathcal{E}}_{(a,4)}(n)$.*

(a) $\ker\mathcal{Z}\cap\mathcal{F}^{\mathcal{E}}_{(a,4)}(n)=0$.

(b) The pencil $\kappa\mathcal{Z}-\mathcal{J}$ is injective on $\mathcal{F}^{\mathcal{E}}_{(a,4)}(n)$ for all but at most h values of κ; in particular it has at most h eigenvalues there.

Proof. (a) Theorem 9.129 gives $\Phi_0=\Phi_1=0$, and Corollary 9.94(a) gives $\varphi=0$. (b) As in the proof of Corollary 9.81(b), with (a) in place of Theorem 9.80. □

Remark 9.131. (1) Theorem 9.22 is proved for $n \geq a+b+1$, so at $b=4$ it gives Corollary 9.130(a) for $n \geq a+5$; the corollary is new for $5 \leq n \leq a+4$, and a second proof, by another mechanism, for $n \geq a+5$ (as Remark 9.82(1) at $b=2$ and Remark 9.92(4) at $b=3$). The y-Hessian named in Remark 9.95(5) is removed in step (1) of B4-1 ($\mathsf{G}_e = 0$). The cases $n = 3, 4$ at $b = 4$ stay open (at $n = 4$ there is no disjoint kind for the closure); their part of degree 1 in y is Theorem 9.196, so only B4-0 is open there.

(2) Not claimed: the exact set of exceptional a (the computations in (4) suggest $\{2, -1, \frac{3}{2}, \frac{1}{2}\}$) and any statement for non-integer a; a hand proof of the rank in Theorem 9.128; the kinds of spectator edge $\{l, s\}$, $\{m, c\}$, $\{l, c\}$ and the other rows (not used); which κ are eigenvalues (part (b) only bounds their number).

(3) Parts of Theorem 9.128 by hand. If $\mathsf{F} = 0$ at the point, the three kinds force $\mathsf{U} = 0$ ($n \geq 5$): Use 1 with $\mathsf{F} = 0$ is ($\star$) of Theorem 9.80 for f^e on $[n] \setminus e$, and the triangle of Step 3 of its proof (determinant $-(a+1)^2(a-2)\alpha^2\beta^2\gamma^2 \neq 0$) on a triangle in $[n] \setminus e$ gives $\mathsf{U}(e, e') = 0$ for $e' \cap e = \emptyset$; Use 1 at $k = p \in e$ then reads $\theta_{lp} f_{pm} + \theta_{pm} f_{lp} = 0$, so $u_{(e,p)} := (x_w - x_p)\mathsf{U}(e, \{p, w\})$ is the same for every $w \notin e$; Use 2 gives $(u_{(e,q)} - u_{(e,p)}) + S(x_q - x_p) + (a+1)\theta_{pq} S(x_k - x_p)(x_k - x_q) = 0$ ($S := \mathsf{U}(e, e)$) for every $k \notin e$, and two values of k give $S = 0$ and $u_{(e,p)} = u_{(e,q)} =: u_e$; the symmetry $\mathsf{U}(e, e') = \mathsf{U}(e', e)$ for $e = \{p, q\}$, $e' = \{p, w\}$ gives $u_{pq}(x_q - x_p) = u_{pw}(x_w - x_p) =: t_p$ (independent of q) and $t_q = -t_p$, and three vertices give $t_p = -t_q = t_r = -t_p$, so $t = 0$, all $u = 0$ and $\mathsf{U} = 0$. A hand reason for the rest (the block of τ and the disjoint U, then σ) is open.

(4) Computations, not used in the proof. A second program, independent of that of Theorem 9.128, finds full rank 85 of the 120×85 system at x_0, at a second traceless point and at a non-traceless point, for every integer a in $-3, \ldots, 12$ and $a = 20, 33, 50, 101$, except rank 73 at $a = -1$ and 84 at $a = 2$ (also rank 84 at $a = \frac{3}{2}$ and 81 at $a = \frac{1}{2}$); the first finds rank 85 at $a = 4, 5, 6, 8, 11$ (and 84, 73 at $a = 2, -1$) at x_0 and at the non-traceless point. In the second program an independent minor at the second traceless point has determinant $(a-2)(a+1)^{12}(2a-3)(2a-1)^4(1825a-7779)(4526a+4647)$, again without an integer root $a \geq 4$. The raw rows of the three kinds, with the derivatives and G_e free, together with Euler and $\mathsf{G}_e x = 0$, force $\mathsf{F} = \mathsf{U} = 0$ exactly at $n = 5, 6$ and $a = 4, \ldots, 8$; in another program ($n = 5$, $a = 4, \ldots, 7, 10$), which finds the same, controls without $\mathcal{E}\varphi = 0$ or with one kind dropped leave them free. Lemma 9.126 agrees with a direct computation of the rows (for every kind at $n = 5$, and on a real invariant of bidegree $(4, 4)$). From the definition, $\ker \mathcal{Z} \cap \mathcal{F}^{\mathcal{E}}_{(a,4)}(n) = 0$ at $n = 5, 6, 7$ and $a = 4, \ldots, 10$, and also at $n = 4$ for $a = 4, \ldots, 8$ (computed, not claimed).

9.24. **The odd rows at $b = 2$.** In this subsection $F = \mathbb{C}$, $b = 2$, $a \geq 3$ and $n \geq a+6$, and we keep the notation of Subsections 9.13 and 9.17. We show that a kernel vector with $\mu \neq 0$ lies on one explicit line (Theorem 9.143), and that for odd a every eigenvalue has $\mu \in \{0, 3, 5, \ldots, a\}$ (Corollary 9.144). The half-difference of the generic order-one rows under $p \leftrightarrow q$ (the *odd rows*) sees the data only through one symmetric function ϑ of three slots, which solves one equation ($\star$) (Corollary 9.137); we solve ($\star$) completely for $\kappa \neq 0$ (Theorem 9.141), and inside TR at $\kappa = 0$ (Lemma 9.142); and Lemma 9.46, with $\mathsf{G}\mathbf{1} = 0$ and $\mathsf{G}x = 0$, makes the whole kernel vector a fixed function of ϑ (Corollary 9.135). Subsection 9.25 proves the conclusion of Conjecture 11.3 for $n \geq a+6$. The range $n \geq a+6$ is that of Corollary 9.76(i) and of Subsection 9.22; the steps below would work for $n \geq a+4$, but we do not claim that range.

A *kernel vector* at κ is a $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ with $\kappa\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ at every base point and for all u, y_1, y_2, y_3 (Corollary 9.31(b)); so κ is an eigenvalue if and only if there is a kernel vector $\varphi \neq 0$. We use only base points $(\mathrm{diag}(x), Y)$ with distinct x_l and $Y \in \{E_{pq}, E_{jj}\}$, and corners with $u = \mathbf{1}$, that is, only the necessary conditions that every row of Propositions 9.44 and 9.45 vanishes.

The data. We use the raw power sums $p_k := \sum_l x_l^k$ ($p_0 = n$) and the coordinates themselves, not the centred z_l and P_k of Subsection 9.17 (each P_k is a polynomial in $p_1, \ldots, p_k$). By Subsection 9.17, the data (1) of φ are $\mathsf{F}_{rt} = \mathsf{F}(x_r, x_t)$ ($r \neq t$) and $\mathsf{G}_{rt} = \mathsf{G}(x_r, x_t) + \delta_{rt}\mathsf{F}(x_r, x_r)$, with symmetric

polynomials $\mathsf{F}(s,t)$, $\mathsf{G}(s,t)$ whose coefficients are polynomials in the p_k; they are unique, as the coordinates and power sums that occur are algebraically independent functions (Corollary 9.76(i)). Give each slot the weight 1 and p_k the weight k; then F and G have weighted degree a, and they are invariant under $x \mapsto x + c\mathbf{1}$. The map $\varphi \mapsto (\mathsf{F}, \mathsf{G})$ is injective: $\mathsf{F} = \mathsf{G} = 0$ gives $Q_X = 0$ at every regular diagonal X, so $\varphi = 0$. *Independence:* for $n \geq N + 4$ the functions $x_p, x_q, x_i, x_j, p_1, \dots, p_N$ (p, q, i, j distinct) are algebraically independent (the power sums of the other $n-4$ coordinates are), so an identity in them that holds for all x holds coefficientwise. Below, the largest index that occurs is $a+1$, with at most two slots, and with four slots only p_k with $k \leq a$ occur.

Notation. For a polynomial $f(u)$ (with coefficients in the p_k) put $\langle f(u)\rangle_u := \sum_l f(x_l)$, that is, $u^k \mapsto p_k$. Let $\mathcal{D}_u := \sum_{k\geq 1} k u^{k-1} \partial/\partial p_k$ (as in Subsection 9.17): for l outside the slots in use, $\partial/\partial x_l$ of a function of the slots and the p_k is $\mathcal{D}_u$ at $u = x_l$. Let $\mathbb{T}_p := \sum_{k\geq 1} k p_{k-1} \partial/\partial p_k$ and $\mathbb{T} :=$ (the sum of the derivatives in all slots) $+ \mathbb{T}_p$. As $\frac{d}{dc} p_k(x + c\mathbf{1}) = k p_{k-1}$, a function is invariant under $x \mapsto x + c\mathbf{1}$ if and only if $\mathbb{T}$ kills it; we then call it *shift invariant*. $\mathbb{T}$ is a derivation, and $\mathbb{T}_p p_k = k p_{k-1} = \langle \partial_u u^k \rangle_u$, so

$$\mathbb{T}\langle f(v,u)\rangle_u = \langle (\mathbb{T} f)(v,u)\rangle_u .$$

Let Sym be the space of symmetric polynomials $\mathsf{F}(s,t)$ with coefficients in $\mathbb{C}[p_1, p_2, \dots]$, Sym_a its part of weighted degree a, and $\mathrm{Sym}_a^{\mathrm{inv}}$ the shift invariant part of Sym_a. For a polynomial $g(v)$ put $g[s,t] := (g(s) - g(t))/(s-t)$, and similarly for divided differences in other slots. The r-th components of $\mathsf{G}\mathbf{1} = 0$ and of $\mathsf{G}x = 0$ (Subsection 9.13) read, at $s = x_r$ and then as polynomials,

$$\langle \mathsf{G}(s,u)\rangle_u + \mathsf{F}(s,s) = 0, \tag{C1}$$

$$\langle u\, \mathsf{G}(s,u)\rangle_u + s\, \mathsf{F}(s,s) = 0 \tag{C2}$$

(the two relations of Remark 9.124(4), in raw form). Put $\mathfrak{h}_{\mathsf{F}}(v) := \langle (v-u)\mathsf{F}(v,u)\rangle_u$, the $\mathfrak{h}$ of Corollary 9.47. If φ is a kernel vector and $\mu \neq 0$, then by that corollary

$$\mu(a+1)\, \mathsf{G}(s,t) = 2\big(\mathfrak{h}_{\mathsf{F}}[s,t] - n\, \mathsf{F}(s,t)\big).$$

Lemma 9.132 (the projection)**.** *For* $\mathsf{F} \in \mathrm{Sym}$ *put* $\mathrm{pr}\,\mathsf{F} := \frac{1}{n}\mathfrak{h}_{\mathsf{F}}[s,t]$ *and* $\mathsf{F}^{\perp} := \mathsf{F} - \mathrm{pr}\,\mathsf{F}$*, and let* $\mathrm{Dif} := \{g[s,t] : g \in \mathbb{C}[v, p_1, p_2, \dots]\}$*. Then* pr *is the identity on* Dif*,* $\mathrm{pr}^2 = \mathrm{pr}$*,* $\mathrm{Sym} = \mathrm{Dif} \oplus \ker \mathrm{pr}$*, and* $\ker \mathrm{pr} = \{\mathsf{F} : \mathfrak{h}_{\mathsf{F}} = 0\}$*. Moreover* pr *keeps the weighted degree and commutes with* $\mathbb{T}$*, so it maps shift invariant elements to shift invariant elements.*

Proof. If $\mathsf{F} = g[s,t]$, then $(v-u)\mathsf{F}(v,u) = g(v) - g(u)$, so $\mathfrak{h}_{\mathsf{F}} = ng - \langle g(u)\rangle_u$ and $\mathrm{pr}\,\mathsf{F} = g[s,t]$. So pr is the identity on Dif, and as $\mathrm{pr}\,\mathsf{F} \in \mathrm{Dif}$, $\mathrm{pr}^2 = \mathrm{pr}$. Next, $\mathrm{pr}\,\mathsf{F} = 0$ if and only if $\mathfrak{h}_{\mathsf{F}}$ is constant in v. With $\mathsf{F} = \sum f_{ij} s^i t^j$ ($f_{ij} = f_{ji}$), $\mathfrak{h}_{\mathsf{F}}(v) = \sum f_{ij}(v^{i+1}p_j - v^i p_{j+1})$, so $\langle \mathfrak{h}_{\mathsf{F}}(u)\rangle_u = \sum f_{ij}(p_{i+1}p_j - p_i p_{j+1}) = 0$ as a polynomial; a constant $\mathfrak{h}_{\mathsf{F}} = c$ then has $nc = 0$, so $\mathfrak{h}_{\mathsf{F}} = 0$. Finally, $\mathbb{T}$ commutes with pr by the identity before the lemma (as $\mathbb{T}(v-u) = 0$) and $\mathbb{T}(g[s,t]) = (\mathbb{T}g)[s,t]$. □

Lemma 9.133 (the cocycle)**.** *For* $\mathsf{F} \in \mathrm{Sym}$ *let* $\mathsf{H}(s,t) := (s-t)\mathsf{F}(s,t)$ *and* $\mathsf{H}(s,v,t) := \mathsf{H}(s,v) + \mathsf{H}(v,t) + \mathsf{H}(t,s)$ *(the* H_{rs} *and* H_{rwt} *of Subsection 9.13 in two-point form). Then* $\mathsf{H}(s,v,t)$ *is alternating, so* $\mathsf{H}(s,v,t) = (s-v)(v-t)(t-s)\,\vartheta_{\mathsf{F}}(s,t,v)$ *with* ϑ_{F} *symmetric in its three slots and of weighted degree* $\deg \mathsf{F} - 2$*; and* $\mathsf{c}_l^{(pq)} = C(x_p, x_q; x_l)$ *with* $C(s,t;v) = (t-s)\vartheta_{\mathsf{F}}(s,t,v)$ *(the* C *of Subsection 9.17; so* $\vartheta_{\mathsf{F}} = -\chi_{\mathsf{F}}$ *in Lemma 9.119(b)).*

- *(i) The kernel of* $\mathsf{F} \mapsto \vartheta_{\mathsf{F}}$ *is* Dif*. So* $\vartheta_{\mathsf{F}} = \vartheta_{\mathsf{F}^{\perp}}$*, and* $\mathsf{F} \mapsto \vartheta_{\mathsf{F}}$ *is injective on* $\ker \mathrm{pr}$*.*
- *(ii)* $\vartheta_{\mathbb{T}\mathsf{F}} = \mathbb{T}\vartheta_{\mathsf{F}}$*.*
- *(iii) Let* TR *be the set of the* $\sum_\lambda c_\lambda s_\lambda(s,t,v)$ *with* $\ell(\lambda) \leq 2$ *and* $c_\lambda \in \mathbb{C}[p_1, p_2, \dots]$ *(two-row Schur polynomials in the three slots). Every* ϑ_{F} *lies in* TR*; and for* $\vartheta \in$ TR*,* $\mathsf{F} := -st\,\vartheta(s,t,0)$ *has* $\vartheta_{\mathsf{F}} = \vartheta$*.*
- *(iv) If* $\vartheta \in$ TR *is shift invariant, then* $w := (-st\,\vartheta(s,t,0))^{\perp}$ *is shift invariant, lies in* $\ker \mathrm{pr}$*, and* $\vartheta_w = \vartheta$*.*

Proof. Exchanging two slots of $\mathsf{H}(s,v,t)$ changes its sign, as $\mathsf{H}(t,s) = -\mathsf{H}(s,t)$; so it is the alternating product times a symmetric polynomial. The formula for $\mathsf{c}_l^{(pq)}$ is $\mathsf{c}_l^{(pq)} = \mathsf{H}_{plq}/(x_{pl}x_{lq})$ (Subsection 9.13). (i) If $\mathsf{H}(s,v,t) = 0$, then at $v = 0$, $\mathsf{H}(s,t) = \mathsf{H}(s,0) - \mathsf{H}(t,0)$, so $\mathsf{F} = g[s,t]$ with $g(v) := \mathsf{H}(v,0)$; conversely, $\mathsf{F} = g[s,t]$ gives $\mathsf{H}(s,t) = g(s) - g(t)$ and $\mathsf{H}(s,v,t) = 0$. With Lemma 9.132 this gives the rest. (ii) $\mathbb{T}(s-t) = 0$, each term of $\mathsf{H}(s,v,t)$ uses two slots and the p_k, and $\mathbb{T}$ kills the alternating product. (iii) By the bialternant formula [Ma95, I, (3.1)], for $k_1 > k_2 \geq 0$ the function $\mathsf{H}(s,t) = s^{k_1}t^{k_2} - s^{k_2}t^{k_1}$ has $\mathsf{H}(s,v,t) = a_{(k_1,k_2,0)}(s,v,t)$ (expand the determinant along its last column), $a_{\lambda+\delta} = a_\delta s_\lambda$ with $\delta = (2,1,0)$, and $a_\delta(s,v,t) = (s-v)(s-t)(v-t) = -(s-v)(v-t)(t-s)$. For $\lambda = (\lambda_1,\lambda_2)$ take $k_1 = \lambda_1 + 2$, $k_2 = \lambda_2 + 1$: then $\mathsf{F} := -(s^{k_1}t^{k_2} - s^{k_2}t^{k_1})/(s-t) = -(st)^{\lambda_2+1}h_{\lambda_1-\lambda_2}(s,t) = -st\,s_\lambda(s,t,0)$ (as $s_{(\lambda_1,\lambda_2)}(s,t) = (st)^{\lambda_2}h_{\lambda_1-\lambda_2}(s,t)$) has $\mathsf{H}(s,v,t) = -a_{(k_1,k_2,0)} = (s-v)(v-t)(t-s)s_\lambda$, that is, $\vartheta_\mathsf{F} = s_\lambda$. By linearity over $\mathbb{C}[p_1,p_2,\dots]$, $\mathsf{F} := -st\,\vartheta(s,t,0)$ has $\vartheta_\mathsf{F} = \vartheta$ for every $\vartheta \in \mathrm{TR}$. Conversely, every $\mathsf{F} \in \mathrm{Sym}$ is, modulo Dif, of the form $st\,\mathsf{F}_1$: if f_d is the coefficient of s^d in $\mathsf{F}(s,0)$, then $\mathsf{F} - \sum_d f_d h_d(s,t)$ vanishes at $t = 0$, so it is divisible by t and, being symmetric, by st; and $h_d(s,t) = (v^{d+1})[s,t] \in \mathrm{Dif}$. As F_1 is a combination of the $s_\lambda(s,t)$, $\vartheta_\mathsf{F} \in \mathrm{TR}$ by the computation above. (iv) $\mathsf{F} := -st\,\vartheta(s,t,0)$ has $\vartheta_\mathsf{F} = \vartheta$ by (iii), and $\vartheta_{\mathbb{T}\mathsf{F}} = \mathbb{T}\vartheta = 0$ by (ii); so $\mathbb{T}\mathsf{F} \in \mathrm{Dif}$ by (i), and $\mathbb{T}(\mathsf{F}^\perp) = (\mathbb{T}\mathsf{F})^\perp = 0$ by Lemma 9.132. Finally $\vartheta_w = \vartheta_\mathsf{F}$ by (i). □

Let $\mathrm{Sym}_a^\perp := \mathrm{Sym}_a^{\mathrm{inv}} \cap \ker \mathrm{pr}$.

Theorem 9.134 (the data)**.** *Let* $(\mathsf{F},\mathsf{G}) \in \mathrm{Sym}_a^{\mathrm{inv}} \times \mathrm{Sym}_a^{\mathrm{inv}}$ *satisfy* (C1)*. Then* (C2) *holds if and only if* $\mathfrak{h}_\mathsf{G} = 0$*, that is,* $\mathsf{G} \in \ker \mathrm{pr}$*. Two such pairs with the same* $(\mathsf{G},\mathsf{F}^\perp)$ *are equal, and for every* $(\mathsf{G},w) \in \mathrm{Sym}_a^\perp \times \mathrm{Sym}_a^\perp$ *there is exactly one such pair with these values:* $\mathsf{F} = g[s,t] + w$*, where* $g'(s) = -\langle \mathsf{G}(s,u)\rangle_u - w(s,s)$*. So* $(\mathsf{F},\mathsf{G}) \mapsto (\mathsf{G},\mathsf{F}^\perp)$ *is a bijection from the pairs with* (C1) *and* (C2) *onto* $\mathrm{Sym}_a^\perp \times \mathrm{Sym}_a^\perp$*.*

Proof. $s\cdot$(C1)$-$(C2) is $\langle (s-u)\mathsf{G}(s,u)\rangle_u = \mathfrak{h}_\mathsf{G}(s)$; this gives the first claim. As (C1) is linear, for the uniqueness let $\mathsf{G} = 0$ and $\mathsf{F}^\perp = 0$: then $\mathsf{F} = g[s,t]$ (Lemma 9.132), and (C1) gives $\mathsf{F}(s,s) = g'(s) = 0$, so $\mathsf{F} = 0$. Existence: with g as stated, $\mathsf{F}(s,s) = g'(s) + w(s,s) = -\langle \mathsf{G}(s,u)\rangle_u$, and $\mathrm{pr}\,\mathsf{F} = g[s,t]$, $\mathsf{F}^\perp = w$. Shift invariance: g' is shift invariant (by the identity before Lemma 9.132), so $\mathbb{T}g$ has the derivative $\mathbb{T}g' = 0$; so $\mathbb{T}g$ is a constant, and $\mathbb{T}(g[s,t]) = (\mathbb{T}g)[s,t] = 0$. Take $g(v) := \int_0^v g'$, of weighted degree $a+1$. □

Corollary 9.135 (the line)**.** *Let* φ *be a kernel vector at* κ *with* $\mu \neq 0$*, with data* (F,G)*, and* $w := \mathsf{F}^\perp \in \mathrm{Sym}_a^\perp$*. Then*

$$\mu(a+1)(\mathsf{F},\mathsf{G}) = v'(\mu,w) := \big(g[s,t] + \mu(a+1)w,\, -2nw\big), \qquad g'(s) = 2n\langle w(s,u)\rangle_u - \mu(a+1)w(s,s),$$

and $\varphi = 0$ *if and only if* $w = 0$*.*

Proof. By the display before Lemma 9.132, $\mu(a+1)\mathsf{G} = 2(\mathfrak{h}_\mathsf{F}[s,t] - n\mathsf{F}) = 2n(\mathrm{pr}\,\mathsf{F} - \mathsf{F}) = -2nw$. Both $\mu(a+1)(\mathsf{F},\mathsf{G})$ and $v'(\mu,w)$ satisfy (C1) and have $(\mathsf{G},\mathsf{F}^\perp) = (-2nw, \mu(a+1)w)$, so they are equal by Theorem 9.134. If $w = 0$, then $(\mathsf{F},\mathsf{G}) = 0$, and $\varphi = 0$. □

Lemma 9.136 (the odd rows)**.** *Let* F_{rt}*,* G_{rt} *be any symmetric data (*$\mathsf{G}\mathbf{1} = 0$*,* $\mathsf{G}x = 0$ *and the shift invariance are not used), and for distinct* p,q,i,j *let* $R_{pq} := \Psi_{pq}(e_i,e_j)$ *be the generic order-one row of Proposition 9.44 and* $R_{pq}^- := \frac{1}{2}(R_{pq} - R_{qp})$*, as in Lemma 9.74. Then, with* $\mathsf{c}_l := \mathsf{c}_l^{(pq)}$*,*

$$R_{pq}^- = \kappa(\mathsf{c}_j - \mathsf{c}_i) - x_{ij}(\partial_i \mathsf{c}_j + \partial_j \mathsf{c}_i).$$

Proof. We use the shifted form of Proposition 9.44 with $c = -(x_p + x_q)/2$ for both rows (c is chosen at each point, and no derivative acts on it). Put $x_l'' := x_l - (x_p + x_q)/2$, so $x_p'' = x_{pq}/2 = -x_q''$, and $\nabla_m := \partial_m - \delta_m$. With $\psi := \psi^{(pq)}$,

$$R_{pq} = \kappa(\psi_j - \psi_i) - x_i''(\nabla_i\psi)_j + x_j''(\nabla_j\psi)_i - \tfrac{1}{2}\big[\partial_i(\mathsf{G}_{pj} + \mathsf{G}_{qj}) - \partial_j(\mathsf{G}_{pi} + \mathsf{G}_{qi})\big],$$

as the Θ-term is $-(x_p''\Theta_p - x_q''\Theta_q)/x_{pq} = -\frac12(\Theta_p+\Theta_q)$. Write $\psi_w^{(rt)} = \mathsf{e}_w^{(rt)} + \mathsf{c}_w^{(rt)}$ with $\mathsf{e}_w^{(rt)} := \partial_w \mathsf{F}_{rt} + (\mathsf{G}_{tw} - \mathsf{G}_{rw})/x_{rt}$. Under $p \leftrightarrow q$ the x_l'', the G-term and e are unchanged ($\mathsf{e}^{(qp)} = \mathsf{e}^{(pq)}$), and $\mathsf{c}^{(qp)} = -\mathsf{c}^{(pq)}$ (as $\mathsf{H}_{qwp} = -\mathsf{H}_{pwq}$ and $x_{qw}x_{wp} = x_{pw}x_{wq}$). So the κ-terms give $\kappa(\mathsf{c}_j - \mathsf{c}_i)$. For $m \in \{i,j\}$ and w the other spectator, $\psi^{(rt)} + \psi^{(tr)} = 2\mathsf{e}^{(rt)}$ gives

$$\tfrac12\big[(\delta_m\psi)_w - \big((\delta_m\psi)_w \text{ with } p \leftrightarrow q\big)\big] = \Big[\frac{\mathsf{e}^{(pq)} - \mathsf{e}^{(pm)}}{x_{mq}} - \frac{\mathsf{e}^{(pq)} - \mathsf{e}^{(mq)}}{x_{mp}}\Big]_w .$$

Its G-part is 0: over the denominator $x_{pq}x_{mq}x_{mp}$, the coefficients of G_{qw}, G_{pw} and G_{mw} have the numerators $x_{mp} - x_{mq} + x_{pq} = 0$, $x_{mq} - x_{mp} - x_{pq} = 0$ and 0. Its F-part is $\partial_w[(\mathsf{F}_{pq} - \mathsf{F}_{pm})/x_{mq} - (\mathsf{F}_{pq} - \mathsf{F}_{mq})/x_{mp}] = -\partial_w \mathsf{c}_m$. So the half-difference of $(\nabla_i\psi)_j$ is $\partial_i\mathsf{c}_j + \partial_j\mathsf{c}_i$, and that of the row is $\kappa(\mathsf{c}_j - \mathsf{c}_i) - (x_i'' - x_j'')(\partial_i\mathsf{c}_j + \partial_j\mathsf{c}_i)$, with $x_i'' - x_j'' = x_{ij}$. □

Corollary 9.137 (the equation $(\star)$)**.** *In two-point form put $s = x_p$, $t = x_q$, $u_1 = x_i$, $u_2 = x_j$ and $\vartheta(v) := \vartheta_{\mathsf{F}}(s,t,v)$, and let $\vartheta[u_1,u_2]$ be its divided difference in the third slot. Then*

$$R_{pq}^- = -(u_1-u_2)(t-s)\big[\kappa\,\vartheta[u_1,u_2] + \mathcal{D}_{u_1}\vartheta(u_2) + \mathcal{D}_{u_2}\vartheta(u_1)\big],$$

which depends on the data only through $\vartheta_{\mathsf{F}} = \vartheta_{\mathsf{F}^\perp}$. If φ is a kernel vector at κ (any μ) with data (F,G) and $w := \mathsf{F}^\perp$, then $\vartheta := \vartheta_w$ is symmetric, shift invariant, of weighted degree $m := a-2$, lies in TR*, and satisfies, identically in s,t,u_1,u_2 and the p_k,*

$$(\star) \qquad \kappa\,\vartheta[u_1,u_2] + \mathcal{D}_{u_1}\vartheta(u_2) + \mathcal{D}_{u_2}\vartheta(u_1) = 0.$$

Proof. By Lemma 9.133, $\mathsf{c}_l = C(s,t;x_l) = (t-s)\vartheta(x_l)$, and c_j depends on x_i only through the p_k, so $\partial_i\mathsf{c}_j = (t-s)\mathcal{D}_{u_1}\vartheta(u_2)$; insert this into Lemma 9.136. For a kernel vector every row vanishes, in every shift, so the bracket vanishes for all x with distinct coordinates; it involves only p_k with $k \le m$, so by the independence it vanishes as a polynomial ($\mathbb{C}[s,t,u_1,u_2,p_1,p_2,\dots]$ is a domain). ϑ_w is shift invariant by Lemma 9.133(ii), as w is (Lemma 9.132), and lies in TR by Lemma 9.133(iii). □

Solving $(\star)$. Let $\beta := \kappa/2$,

$$\mathbf{E}(\tau) := \exp\Big(-\beta\sum_{k\ge1}\frac{p_k\tau^k}{k}\Big) = \sum_{k\ge0}\mathbf{E}_k\tau^k, \qquad \mathbf{R}_k := [\tau^k]\,\mathbf{E}(\tau)\prod_{y\in\{s,t,v\}}(1-y\tau)^{-1},$$

so $\mathbf{E}_0 = 1$ and $\mathbf{R}_k(0,0,0) = \mathbf{E}_k$ ($\mathbf{R}_k$ is the complete symmetric function h_k of the formal alphabet $\{s,t,v\} - \beta\{x_1,\dots,x_n\}$). Let $\mathcal{D}^{[r]} := (r+1)\partial/\partial p_{r+1}$, so that $\mathcal{D}_u = \sum_{r\ge0}u^r\mathcal{D}^{[r]}$; then $\mathcal{D}^{[r]}\mathbf{E}(\tau) = -\beta\tau^{r+1}\mathbf{E}(\tau)$, so $\mathcal{D}^{[r]}\mathbf{E}_k = -\beta\mathbf{E}_{k-r-1}$. For $\beta \ne 0$, $\mathbf{E}_k = -(\beta/k)p_k + (\text{a polynomial in } p_1,\dots,p_{k-1})$, so the $\mathbf{E}_k$ are algebraically independent generators of $\mathbb{C}[p_1,p_2,\dots]$.

Lemma 9.138. *Let $\kappa \ne 0$ and let ϑ be symmetric and solve $(\star)$. Then ϑ is determined by $f_\circ := \vartheta(0,0,0)$. Moreover $f(v) := \vartheta(0,0,v) = \sum_i f_i v^i$ satisfies $f_0 = f_\circ$ and*

$$(\star_1) \qquad \kappa f_{r+r'+1} + \mathcal{D}^{[r]}f_{r'} + \mathcal{D}^{[r']}f_r = 0 \qquad (r,r' \ge 0).$$

Proof. Write $\vartheta(s,t,v) = \sum_i \vartheta_i(s,t)v^i$. As $\vartheta[u_1,u_2] = \sum_{r,r'}\vartheta_{r+r'+1}u_1^r u_2^{r'}$, the coefficient of $u_1^r u_2^{r'}$ in $(\star)$ is $\kappa\vartheta_{r+r'+1} + \mathcal{D}^{[r]}\vartheta_{r'} + \mathcal{D}^{[r']}\vartheta_r = 0$; at $s=t=0$ this is $(\star_1)$. At $r'=0$ it gives $\vartheta_{r+1} = -(\mathcal{D}^{[r]}\vartheta_0 + \mathcal{D}^{[0]}\vartheta_r)/\kappa$ (the $\mathcal{D}^{[r]}$ do not act on s, t), so $\vartheta(s,t,\cdot)$ is determined by $\vartheta(s,t,0)$. If $f_\circ = 0$, then $\vartheta(0,0,\cdot) = 0$; by symmetry $\vartheta(0,y,0) = \vartheta(0,0,y) = 0$, so $\vartheta(0,y,\cdot) = 0$; by symmetry again $\vartheta(s,y,0) = \vartheta(0,y,s) = 0$, so $\vartheta = 0$. The claim follows by linearity. □

Let Sol_m be the set of the $f_\circ \in \mathbb{C}[p_1,p_2,\dots]$ of weighted degree m for which the f_i defined by $(\star_1)$ with $r' = 0$ satisfy all of $(\star_1)$.

Lemma 9.139. *Call a polynomial $f(v)$ an* **R**-function *if $\mathcal{D}_{u_1}f(u_2) = -\beta f[u_1,u_2]$.*

(i) $\mathbf{R}_k$ (in the third slot, with s, t as parameters) and $\sum_i \mathbf{E}_{k-i}v^i$ are **R**-*functions.*

(ii) If f and g are $\mathbf{R}$-functions, then fg solves $(\star)$ at $\kappa = 2\beta$.
So each $\mathbf{R}_p\mathbf{R}_q$ is a symmetric solution of $(\star)$ in TR*, and $\mathbf{E}_p\mathbf{E}_q \in \mathrm{Sol}_{p+q}$.*

Proof. (i) $\mathcal{D}_{u_1}\mathbf{E}(\tau) = -\beta\tau\mathbf{E}(\tau)/(1-u_1\tau)$, so $\mathcal{D}_{u_1}\big[\mathbf{E}(\tau)/(1-u_2\tau)\big] = -\beta\tau\mathbf{E}(\tau)/((1-u_1\tau)(1-u_2\tau)) = -\beta\big[\mathbf{E}(\tau)/(1-v\tau)\big][u_1,u_2]$; take the coefficient of τ^k (times $(1-s\tau)^{-1}(1-t\tau)^{-1}$ for $\mathbf{R}_k$). (ii) $(fg)[u_1,u_2] = f(u_1)g[u_1,u_2] + g(u_2)f[u_1,u_2]$, so $2\beta(fg)[u_1,u_2] + \mathcal{D}_{u_1}(fg)(u_2) + \mathcal{D}_{u_2}(fg)(u_1)$ is

$$\beta g[u_1,u_2](f(u_1)-f(u_2)) - \beta f[u_1,u_2](g(u_1)-g(u_2)) \\ = \beta(u_1-u_2)\big(g[u_1,u_2]f[u_1,u_2] - f[u_1,u_2]g[u_1,u_2]\big) = 0.$$

Finally, $\mathbf{R}_p\mathbf{R}_q = \sum_{i,j}\mathbf{E}_i\mathbf{E}_j h_{p-i}h_{q-j}(s,t,v)$, and each $h_{p'}h_{q'}$ is a sum of Schur polynomials with at most two rows (Pieri); and at $s=t=0$, $\mathbf{R}_k$ becomes $\sum_i \mathbf{E}_{k-i}v^i$. □

Theorem 9.140. *For $\beta \neq 0$, Sol_m is spanned by the $\mathbf{E}_p\mathbf{E}_{m-p}$, $0 \le p \le m/2$.*

Proof. Identify $\mathbb{C}[p_1,p_2,\dots]$ with the ring Λ of symmetric functions by $\mathbf{E}_k \mapsto h_k$, an algebra isomorphism. Then $\mathcal{D}^{[r]}$ is the derivation with $\mathcal{D}^{[r]}h_k = -\beta h_{k-r-1}$, so $\mathcal{D}^{[r]} = -\beta\pi^\perp_{r+1}$, where the π_j are the power sums of Λ and $\pi_j^\perp = j\,\partial/\partial\pi_j$ is the adjoint of the multiplication by π_j for the Hall inner product, with $\pi_j^\perp h_k = h_{k-j}$ [Ma95, I, §5, Ex. 3]. With $\kappa = 2\beta$, $(\star_1)$ becomes free of β: $2f_{r+r'+1} = \pi^\perp_{r+1}f_{r'} + \pi^\perp_{r'+1}f_r$. The equations with $r' = 0$ give $f_r = q_r^\perp f_\circ$ with $q_0 = 1$ and $q_{r+1} = (\pi_1 q_r + \pi_{r+1})/2$ (as $(fg)^\perp = f^\perp g^\perp$); the generating function is $\sum_r q_r v^r = (2+\Pi(v))/(2-\pi_1 v)$, $\Pi(v) := \sum_{j\ge1}\pi_j v^j$. The other equations are $g^\perp_{r,r'}f_\circ = 0$ $(r,r' \ge 1)$ with $g_{r,r'} := 2q_{r+r'+1} - \pi_{r+1}q_{r'} - \pi_{r'+1}q_r$; so $\mathrm{Sol}_m = (J_g)^\perp_m$ for the Hall inner product, where J_g is the ideal generated by the $g_{r,r'}$. Let $I_{\ge3} := (e_3, e_4, \dots)$, which is spanned by the s_λ with $\ell(\lambda) \ge 3$, so that $I^\perp_{\ge3}$ is spanned by the s_λ with $\ell(\lambda) \le 2$, that is, by the $h_p h_q$ ([Ma95, I, §§2–3]: $s_\lambda(x_1,x_2) = 0$ if and only if $\ell(\lambda) \ge 3$; the Jacobi–Trudi formula $s_{(p,q)} = h_p h_q - h_{p+1}h_{q-1}$ [Ma95, I, (3.4)]; the s_λ are orthonormal [Ma95, I, (4.8)]).

(a) $J_g \subseteq I_{\ge3}$: by Lemma 9.139, every $h_p h_q$ $(= \mathbf{E}_p\mathbf{E}_q)$ lies in $J_g^\perp$, so $(I_{\ge3})^\perp_m \subseteq (J_g)^\perp_m$ for every m; the Hall inner product is non-degenerate on each Λ_m, so $(J_g)_m \subseteq (I_{\ge3})_m$.

(b) $I_{\ge3} \subseteq J_g$: we show $e_k \in J_g$ by induction on $k \ge 3$. The element $g := g_{1,k-2}$ has degree k and lies in $I_{\ge3}$ by (a), so in $\mathbb{C}[e_1,e_2,\dots]$ each of its monomials contains some e_j with $j \ge 3$: $g = c\,e_k +$ (terms with a factor e_j, $3 \le j < k$), and these terms lie in J_g by induction. In the power-sum expansion, the coefficient of π_k in g is that in $2q_k$, which is 1 ($\pi_1 q_{k-1}$, $\pi_2 q_{k-2}$ and $\pi_{k-1}q_1 = \pi_{k-1}\pi_1$ are products), the coefficient of π_k in e_k is $(-1)^{k-1}/k$, and a product of at least two e_j has no π_k. So $c = (-1)^{k-1}k \neq 0$, and $e_k \in J_g$. (For $k=3$, $g_{1,1} = 3e_3$.)

Hence $J_g = I_{\ge3}$, and $\mathrm{Sol}_m = (I_{\ge3})^\perp_m$ is spanned by the $h_p h_{m-p}$, that is, by the $\mathbf{E}_p\mathbf{E}_{m-p}$. (The single condition $e_3^\perp f_\circ = 0$ is not enough for $m \ge 4$: it holds for $s_{211} - s_{1111}$.) □

Theorem 9.141. *Let $\kappa \neq 0$, $\beta = \kappa/2$, $\xi := n\kappa$ and $m \ge 1$.*

(i) The symmetric solutions of $(\star)$ of weighted degree m are exactly the $\sum_{p\le m/2} c_p\mathbf{R}_p\mathbf{R}_{m-p}$ $(c_p \in \mathbb{C})$.

(ii) Such a solution is shift invariant if and only if $c = (c_p) \in \ker\mathbf{B}(\xi)$, where, with $d_i := m+2-i-\xi/2$ and $d_i' := i+3-\xi/2$, the matrix $\mathbf{B}(\xi)$ is upper bidiagonal with the rows $d_i c_i + d_i' c_{i+1} = 0$ $(0 \le i \le \lfloor(m-1)/2\rfloor)$, except that for even m the last row $(i = m/2-1)$ is $d_i c_i + 2d_i' c_{m/2} = 0$, and for odd m the last row $(i = (m-1)/2)$ is $d_i c_i = 0$.

(iii) For even m, $\dim\ker\mathbf{B}(\xi) = 1$ for every ξ. For odd m, $\dim\ker\mathbf{B}(\xi) = 1$ if $\xi \in \{a+3, a+5, \dots, 2a\}$, where $a := m+2$, and $\ker\mathbf{B}(\xi) = 0$ otherwise. At $\xi = 2a$ the kernel is spanned by $c = (1,0,\dots,0)$: the solution is $\mathbf{R}_{a-2}$, with $\beta = a/n$.

Proof. (i) $f_\circ = \vartheta(0,0,0)$ lies in Sol_m (Lemma 9.138), so $f_\circ = \sum_p c_p\mathbf{E}_p\mathbf{E}_{m-p}$ by Theorem 9.140. Then $\vartheta' := \sum_p c_p\mathbf{R}_p\mathbf{R}_{m-p}$ is a symmetric solution (Lemma 9.139) with $\vartheta'(0,0,0) = f_\circ$, so $\vartheta = \vartheta'$

by Lemma 9.138. Conversely, each $\mathbf{R}_p\mathbf{R}_{m-p}$ is a solution. (ii) $\mathbb{T}$ is a derivation, and $\mathbb{T}\mathbf{R}_k = (k+2-n\beta)\mathbf{R}_{k-1}$: with $W(\tau) := \prod_{y\in\{s,t,v\}}(1-y\tau)^{-1}$, $(\partial_s+\partial_t+\partial_v)W = 3\tau W + \tau^2 W'$ and $\mathbb{T}_p\mathbf{E} = -n\beta\tau\mathbf{E}+\tau^2\mathbf{E}'$ (as $\tau\mathbf{E}' = -\beta\sum_k p_k\tau^k\mathbf{E}$ and $p_0=n$), so $\mathbb{T}(W\mathbf{E}) = (3-n\beta)\tau W\mathbf{E}+\tau^2(W\mathbf{E})'$; take the coefficient of τ^k. Hence

$$\mathbb{T}\Big(\sum_p c_p\mathbf{R}_p\mathbf{R}_{m-p}\Big) = \sum_p c_p\big[(p+2-\tfrac{\xi}{2})\mathbf{R}_{p-1}\mathbf{R}_{m-p} + (m-p+2-\tfrac{\xi}{2})\mathbf{R}_p\mathbf{R}_{m-p-1}\big].$$

The $\mathbf{R}_i\mathbf{R}_{m-1-i}$ $(i\le(m-1)/2)$ are linearly independent (their values at $s=t=v=0$ are the independent $\mathbf{E}_i\mathbf{E}_{m-1-i}$), and collecting their coefficients gives $\mathbf{B}(\xi)c$; for even m, $p=m/2$ gives the doubled entry. (iii) For odd m, $\mathbf{B}(\xi)$ is square with determinant $\prod_i d_i$, which vanishes if and only if $\xi/2 = m+2-i$ for some i, that is, $\xi\in\{2a,2a-2,\dots,a+3\}$. There exactly one d_i vanishes and no d'_j does (the roots of the d'_j are $\xi/2 = j+3\le(m+3)/2$, those of the d_i are $\xi/2\ge(m+5)/2$), so the kernel is 1-dimensional. For even m, $\mathbf{B}(\xi)$ has $m/2$ rows and $m/2+1$ columns, and its maximal minors are $\pm\prod_{i<j}d_i\prod_{i\ge j}\bar d'_i$ $(0\le j\le m/2)$, where $\bar d'_i := d'_i$ for $i<m/2-1$ and $\bar d'_{m/2-1} := 2d'_{m/2-1}$; the roots of the d_i ($\xi/2\in[m/2+3,m+2]$) and of the $\bar d'_i$ ($\xi/2\in[3,m/2+2]$) are disjoint, so some maximal minor is nonzero at every ξ. At $\xi/2=m+2$, $d_0=0$, and the rows force $c_1=c_2=\cdots=0$, as $d'_i = i+1-m\ne0$ for $i+1\le m/2$. □

Lemma 9.142 ($\kappa=0$)**.** *Let ϑ be symmetric and solve $(\star)$ at $\kappa=0$.*

(i) ϑ does not involve the p_k.

(ii) If moreover $\vartheta\in\mathrm{TR}$, then ϑ is shift invariant if and only if $\vartheta=\sum_p c_p h_p h_{m-p}(s,t,v)$ with $c\in\ker\mathbf{B}(0)$; so these solutions form a space of dimension 0 for odd m and 1 for even m. (At $\beta=0$, $\mathbf{R}_k = h_k(s,t,v)$.)

Proof. (i) $\Gamma(y_0;y_1,y_2,y_3) := \mathcal{D}_{y_0}\vartheta(y_1,y_2,y_3)$ is symmetric in y_1,y_2,y_3 and, by $(\star)$, changes sign under $y_0\leftrightarrow y_3$. So $(0\,1)=(1\,3)(0\,3)(1\,3)$ acts by -1, and $(1\,3)=(0\,1)(0\,3)(0\,1)$ acts by $(-1)^3=-1$, but also by $+1$; so $\Gamma=0$. Its coefficient of y_0^{k-1} is $k\,\partial\vartheta/\partial p_k=0$. (ii) By (i), ϑ lies in the span of the $s_\lambda(s,t,v)$ with $\ell(\lambda)\le2$, which is the span of the $h_ph_{m-p}(s,t,v)$ (Jacobi–Trudi; these are linearly independent in three variables). On it $\mathbb{T}=\partial_s+\partial_t+\partial_v$, and $\mathbb{T}h_k=(k+2)h_{k-1}$ (the computation in the proof of Theorem 9.141(ii) at $\beta=0$), so $\mathbb{T}(\sum_p c_ph_ph_{m-p})=0$ is $\mathbf{B}(0)c=0$. At $\xi=0$ all $d_i=m+2-i$ and $d'_i=i+3$ are positive: for odd m, $\det\mathbf{B}(0)\ne0$; for even m, the rank is $m/2$ (the minor $\prod_i\bar d'_i\ne0$). □

For ξ with $\ker\mathbf{B}(\xi)\ne0$ let $\vartheta_\xi := \sum_p c_p\mathbf{R}_p\mathbf{R}_{m-p}$, with c spanning $\ker\mathbf{B}(\xi)$ and $\beta=\xi/(2n)$ (at $\xi=0$: $\vartheta_0=\sum_p c_ph_ph_{m-p}(s,t,v)$, Lemma 9.142), and $w_\xi := (-st\,\vartheta_\xi(s,t,0))^\perp$.

Theorem 9.143 (the odd rows)**.** *Let $a\ge3$, $n\ge a+6$, $\kappa\in\mathbb{C}$, $\xi:=n\kappa$ and $\mu:=\xi-a$.*

(1) If $\ker\mathbf{B}(\xi)\ne0$, then $0\ne w_\xi\in\mathrm{Sym}^\perp_a$ and $\vartheta_{w_\xi}=\vartheta_\xi$.

(2) Let φ be a kernel vector at κ with $\mu\ne0$, with data (F,G). If $\ker\mathbf{B}(\xi)=0$ (possible only for odd a and $\xi\notin\{a+3,a+5,\dots,2a\}$; for odd a this includes $\kappa=0$), then $\varphi=0$. Otherwise $\mathsf{F}^\perp = c\,w_\xi$ for a number c, and $\mu(a+1)(\mathsf{F},\mathsf{G}) = c\,v'(\mu,w_\xi)$.

(3) So the conclusion of Conjecture 11.3 holds at (a,n), for every κ, if the following holds:

(H2) for every ξ with $\ker\mathbf{B}(\xi)\ne0$, $\mu=\xi-a\notin\{0,3\}$ and $\xi\ne\pm2n$, some generic order-one row $\Psi_{pq}(e_i,e_j)$ is not 0 on the data $v'(\mu,w_\xi)$.

Proof. (1) $\vartheta_\xi\in\mathrm{TR}$ (Lemma 9.139; Lemma 9.142 at $\xi=0$), and it is shift invariant by Theorem 9.141(ii) (Lemma 9.142(ii) at $\xi=0$); apply Lemma 9.133(iv). $w_\xi\ne0$ as $\vartheta_\xi\ne0$. (2) By Corollary 9.137, $\vartheta_{\mathsf{F}^\perp}$ is a shift invariant solution of $(\star)$ in TR, so by Theorem 9.141 ($\kappa\ne0$) or Lemma 9.142 ($\kappa=0$) it is $c\,\vartheta_\xi$, or 0 if $\ker\mathbf{B}(\xi)=0$. By Lemma 9.133(i) (injectivity on $\ker\mathrm{pr}$) and (1), $\mathsf{F}^\perp = c\,w_\xi$, or 0; now apply Corollary 9.135. (3) By (2), the data of a kernel vector $\varphi\ne0$ at such a κ are a nonzero multiple of $v'(\mu,w_\xi)$; every row is linear in the data and vanishes on them. □

Corollary 9.144. *(i) For odd $a \geq 3$ and $n \geq a+6$, every eigenvalue κ of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ has $\mu \in \{0\} \cup \{3,5,\dots,a\}$; so $n\kappa \in \mathbb{Z}$, and $\kappa = 0, 2, -2$ are not eigenvalues.*
(ii) For every $a \geq 3$ and $n \geq a+6$, the eigenspace of each κ with $\mu \neq 0$ has dimension at most 1.

Proof. (i) If $\mu \neq 0$ and $\varphi \neq 0$ is a kernel vector, Theorem 9.143(2) gives $\ker \mathbf{B}(\xi) \neq 0$, that is, $\xi \in \{a+3, a+5, \dots, 2a\}$ (Theorem 9.141(iii); for odd a this excludes $\xi = 0$, Lemma 9.142); so $\mu = \xi - a \in \{3,5,\dots,a\}$. The value $\kappa = 0$ has $\mu = -a \neq 0$ and $\xi = 0$, which is not in the list; $\kappa = \pm 2$ has $\xi = \pm 2n$, never in $\{a+3,\dots,2a\}$, as $2n \geq 2a+12$. (ii) $\varphi \mapsto \mathsf{F}^{\perp}$ is linear and injective on the kernel at κ (Corollary 9.135), and its image lies in the line $\mathbb{C}w_\xi$ (Theorem 9.143(2)). □

Remark 9.145. (1) At $a = 3$, $\ker \mathbf{B}(\xi) \neq 0$ only at $\xi = 6$ ($\mu = 3$), so Theorem 9.143 gives the conclusion of Conjecture 11.3 at $a = 3$ for $n \geq 9$ with nothing more, consistent with Theorem 9.48 ($n \geq 4$). For odd $a \geq 5$, (H2) is needed only at the $(a-3)/2$ values $\xi = a+5, \dots, 2a$ ($\mu = 5, 7, \dots, a$); for even a, on the line $v'(\xi - a, w_\xi)$ for every $\xi \notin \{a, a+3, \pm 2n\}$, $\xi = 0$ included. At $\mu = 3$ ($\xi = a+3$) the line exists for every a; this is an allowed value. Subsection 9.25 proves (H2).

(2) The kernel vectors. For odd m and the root $\xi/2 = m+2-i_0$: $c_{j+1}/c_j = (i_0 - j)/(m-1-i_0-j)$ for $j < i_0$, and $c_j = 0$ for $j > i_0$; $i_0 = 0$ gives $\mathbf{R}_m$, and $i_0 = 1$ gives $(m-2)\mathbf{R}_m + \mathbf{R}_1\mathbf{R}_{m-1}$. For even m: $c_j = (-1)^j \prod_{i<j} d_i \prod_{i \geq j} \bar{d}'_i$ ($0 \leq j \leq m/2$), nonzero at every ξ by the proof of Theorem 9.141(iii); the recursion $c_{j+1} = -d_j c_j / \bar{d}'_j$ would divide by 0 at $\xi \in \{6, 8, \dots, a+2\}$. For example, $m = 2$ gives $2(\xi - 6)\mathbf{R}_2 + (8 - \xi)\mathbf{R}_1^2$, and $m = 4$ gives $c \propto \big(2(\xi-8)(\xi-6),\ -2(\xi-12)(\xi-8),\ (\xi-12)(\xi-10)\big)$.

(3) Remark 9.124(2) and (4) leave open how κ is pinned, and the case $\kappa = 0$. Theorem 9.143 decides the reduction for $n \geq a+6$, $\kappa = 0$ included, by a route different from Corollary 9.114: the odd rows, Lemma 9.46 and the relations $\mathsf{G1} = 0$, $\mathsf{G}x = 0$. The symmetry of ϑ_{F} and the bialternant step of Lemma 9.133 are those of Lemma 9.119(b); new are (i), the converse in (iii), and (iv). The computed cases of Conjecture 11.3 in Remark 9.50(4) at $a = 5, 6$ lie in the range $n \geq a+6$, and so does the case $a = 4$ for $n \geq 10$. At $s = n$ this range is depth at most $n - 6$, inside Theorem 9.24 (depth at most $n - 1$): there is no new bound on $d(n)$. We do not treat $n < a + 6$.

(4) The odd rows see only $\xi = n\kappa$: multiplying the part of ϑ of degree e in the p_k by λ^e maps the shift invariant solutions of $(\star)$ at (κ, n) to those at $(\lambda\kappa, n/\lambda)$ (not used). We do not use, and do not prove, that the data of $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ are all the pairs with (C1) and (C2); for $a \leq 10$ the dimensions agree with this exactly when $n \geq a+1$ (a computation of characters).

(5) (Computations, not used.) Theorem 9.140 was checked for $m \leq 8$ at several values of β (all monomials as unknowns), and every vector of $\ker \mathbf{B}(\xi)$ is shift invariant for $a = 4, \dots, 7$. A solver of $(\star)$ at $n = 97$ ($a = 3, \dots, 8$) and $n = 101$ ($a = 4, \dots, 7$) finds the kernel exactly at $\xi \in \{6\}$, $\{8, 10\}$, $\{10, 12, 14\}$ for $a = 3, 5, 7$, and of dimension 1 at every ξ for $a = 4, 6, 8$; the odd rows on a basis of $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ have a kernel of dimension exactly 1 at every κ tried for $a = 4, 6$, and a nonzero kernel only at $\mu = 3, 5$ for $a = 5$; and at $\kappa = 0$ the kernel of the pencil is 0 for $a = 4, \dots, 10$ and $n = a+2, \dots, a+4$. Lemma 9.136 was checked on random data at $a = 3, 4, 5$; Theorem 9.141(i) by brute force over all symmetric ϑ (not only two-row ones) for $m \leq 6$; and Lemma 9.133(iii), the shift invariance of w_ξ, $\vartheta_{w_\xi} = \vartheta_\xi$ and the vanishing of the odd part of the full row on $v'(\mu, w_\xi)$ for $a = 3, \dots, 7$, by separate, independently written programs (in the ancillary files).

9.25. **Conjecture 11.3 for $n \geq a+6$.** In this subsection $F = \mathbb{C}$, $b = 2$, $a \geq 3$ and $n \geq a+6$, and we keep the notation of Subsection 9.24. We prove (H2) (Theorem 9.149), and so, with Theorem 9.143, the conclusion of Conjecture 11.3 for every $a \geq 3$, every $n \geq a+6$ and every κ (Theorem 9.150). For $f \in \Lambda$ let $f[\mathbf{R}]$ be its image under the algebra map $h_k \mapsto \mathbf{R}_k$; so $\vartheta_\xi = f_\xi[\mathbf{R}]$ with $f_\xi := \sum_p c_p h_p h_{m-p}$, a symmetric function with at most two rows. One coefficient of one generic order-one row on the line $v'(\mu, w_\xi)$ is computed in closed form (Theorem 9.146): it is $(u_2 - u_1)\mu(a+1)(1-\beta^2)/3$ times $(\mathbb{L}f_\xi)(u_1, u_2)$, where $\mathbb{L} := (h_2 - 3e_2)^{\perp}$; and $\mathbb{L}f_\xi \neq 0$ unless $\mu = 3$ (Lemma 9.148).

The row. Let $\Lambda(s, s'; t) = \mathcal{D}_t \mathsf{F}(s, s') - \mathsf{G}[s, s'; t] + C(s, s'; t)$ as in Subsection 9.17, where $\mathsf{G}[s, s'; t] := (\mathsf{G}(s, t) - \mathsf{G}(s', t))/(s - s')$ and $C(s, s'; t) = (s' - s)\vartheta_{\mathsf{F}}(s, s', t)$ (Lemma 9.133). With

$y := x_p$, $y' := x_q$, $u_1 := x_i$, $u_2 := x_j$, the generic order-one row (8), written in the coordinates and the raw power sums, is

$$\mathsf{P} = \kappa\big(\Lambda(y,y';u_2)-\Lambda(y,y';u_1)\big)-(u_1-y)\nabla(u_1;u_2)+(u_2-y)\nabla(u_2;u_1)-\big(\mathcal{D}_{u_1}\mathsf{G}(y',u_2)-\mathcal{D}_{u_2}\mathsf{G}(y',u_1)\big),$$

with $\nabla(u_1;u_2) := \mathcal{D}_{u_1}\Lambda(y,y';u_2) + \Lambda_{[2]} - \Lambda_{[1]}$, $\Lambda_{[2]} := (\Lambda(y,y';u_2) - \Lambda(y,u_1;u_2))/(y' - u_1)$ and $\Lambda_{[1]} := (\Lambda(y,y';u_2)-\Lambda(u_1,y';u_2))/(y-u_1)$. (The shift $c = -x_p$ only rewrites the explicit coefficients, which is legitimate for every c, as F, G and all their derivatives are shift invariant.)

The probe. On $v'(\mu, w_\xi)$ every p_k in P has $k \le a$: the data have weighted degree a, $\langle\cdot\rangle_u$ keeps the weight, the only term with p_{a+1} in $\mathfrak{h}_{\mathsf{F}}(v)$ is constant in v and so is removed by the divided difference $\mathfrak{h}_{\mathsf{F}}[s,t]$, and $\mathcal{D}$ removes power sums. So P is a polynomial in $y, y', u_1, u_2, p_1, \dots, p_a$, which are algebraically independent, and it vanishes for all x only if it vanishes as a polynomial. We put $p_k = 0$ ($k \ge 1$; $p_0 = n$ is kept) and $y' = 0$, and take the coefficients $[y^k]$ ($k = 0, 1, 2$); a subscript 0, as in $[\,\cdot\,]_0$, marks the value at $p = 0$. If $[y^2]\mathsf{P} \ne 0$, then $\mathsf{P} \ne 0$. At $p = 0$, $\langle f(u)\rangle_u = nf(0)$, and $\mathcal{D}_V\langle f(u)\rangle_u = f'(V)$ for f free of the p_k.

Two parts. The row is linear in the data, so $\mathsf{P}[v'(\mu,w)] = \mathsf{P}[g[s,t], -2nw] + M\,\mathsf{P}[w,0]$ with $M := \mu(a+1)$. *Part 1* (the data $(g[s,t],\mathsf{G})$, $\mathsf{G} := -2nw$): $\vartheta_{g[s,t]} = 0$ (Lemma 9.133(i)), so $C = 0$ and $\Lambda(s,s';t) = \mathsf{k}_t[s,s']$ with $\mathsf{k}_t(x) := \mathcal{D}_t g(x) - \mathsf{G}(x,t)$. Then $\Lambda_{[2]}$ and $\Lambda_{[1]}$ are both $\mathsf{k}_{u_2}[y,y',u_1]$ (second divided differences are symmetric), so $\nabla(u_1;u_2) = (\mathcal{D}_{u_1}\mathsf{k}_{u_2})[y,y']$ and

$$\mathsf{P} = \kappa(\mathsf{k}_{u_2}-\mathsf{k}_{u_1})[y,y'] - (u_1-y)(\mathcal{D}_{u_1}\mathsf{k}_{u_2})[y,y'] + (u_2-y)(\mathcal{D}_{u_2}\mathsf{k}_{u_1})[y,y'] \\ - \mathcal{D}_{u_1}\mathsf{G}(y',u_2) + \mathcal{D}_{u_2}\mathsf{G}(y',u_1).$$

At $y' = 0$, $[y^k]q[y,0] = [x^{k+1}]q(x)$ for a polynomial q, and $g(x) = \int_0^x g'$; so with η_k symmetric in u_1, u_2,

$$\gamma_k(W) := [s^k][\mathcal{D}_W g'(s)]_0, \qquad \eta_k := [s^k][\mathcal{D}_{u_1}\mathcal{D}_{u_2} g'(s)]_0,$$
$$\rho_j(V;W) := [x^j][\mathcal{D}_V w(x,W)]_0, \qquad \omega_j(W) := [x^j][w(x,W)]_0,$$

$$[y^k]\,\text{Part 1} = \frac{\kappa}{k+1}\big(\gamma_k(u_2)-\gamma_k(u_1)\big) + 2n\kappa\big(\omega_{k+1}(u_2)-\omega_{k+1}(u_1)\big) + \frac{(u_2-u_1)\eta_k}{k+1} \\ + 2n\big\{u_2\rho_{k+1}(u_2;u_1) - u_1\rho_{k+1}(u_1;u_2) + \rho_k(u_1;u_2) - \rho_k(u_2;u_1)\big\}.$$

For $k \ge 1$ the pair $\rho_k(u_1;u_2) - \rho_k(u_2;u_1)$ comes from the terms $y\,(\mathcal{D}_{u_1}\mathsf{k}_{u_2})[y,0]$ and $y\,(\mathcal{D}_{u_2}\mathsf{k}_{u_1})[y,0]$; at $k = 0$ it is exactly the G-term $-\mathcal{D}_{u_1}\mathsf{G}(0,u_2) + \mathcal{D}_{u_2}\mathsf{G}(0,u_1)$. *Part 2* (the data $(w,0)$): $\Lambda(s,s';t) = \mathcal{D}_t w(s,s') + (s'-s)\vartheta(s,s',t)$ with $\vartheta = \vartheta_w = \vartheta_\xi$ (Lemma 9.133(i), (iv)). At $y' = 0$, $\nabla(u_1;u_2) = \mathcal{D}_{u_1}\Lambda(y,0;u_2) - J$ with $J := [\Lambda(y,0;u_2) - \Lambda(y,u_1;u_2)]/u_1 + [\Lambda(y,0;u_2) - \Lambda(u_1,0;u_2)]/(y-u_1)$, and $\mathsf{P}[w,0] = \kappa(\Lambda(y,0;u_2) - \Lambda(y,0;u_1)) - (u_1-y)\nabla(u_1;u_2) + (u_2-y)\nabla(u_2;u_1)$.

Seeds. Let $\mathfrak{A}$ be a finite set of letters, $U(y) := \prod_{z\in\mathfrak{A}}(1-yz)^{-1} = \sum_k U_k y^k$ ($U_0 = 1$), $K := \prod_{z\in\mathfrak{A}}\mathbf{E}(z)$, and take as ϑ the seed $U(s)U(t)U(v)K$. Then $\mathcal{D}_V K = -\beta\ell_V K$ with $\ell_V := U'(V)/U(V)$ (as $\mathcal{D}_V\mathbf{E}(z) = -\beta z\mathbf{E}(z)/(1-Vz)$), $\mathcal{D}_{u_1}\mathcal{D}_{u_2}K = \beta^2\ell_{u_1}\ell_{u_2}K$, and $K = 1$ at $p = 0$. Put $\alpha_V := (VU(V))'$, $\varepsilon_V := (V^2U(V))'$, $\Psi_j^U(s,t) := (v^jU(v))[s,t]$, $U_{\ge j}(V) := \sum_{l\ge j} U_l V^{l-j}$ ($U_{\ge 0} = U$, $U_{\ge -1} := VU$), $(U^2)_j := [y^j]U(y)^2$, $\hat{u}_i := U(u_i)$, $\Delta f := f(u_2) - f(u_1)$ and $\mathrm{ce} := \ell_{u_1}\varepsilon_{u_2} + \ell_{u_2}\varepsilon_{u_1}$. With $\mathsf{m}_j := \langle u^jU(u)\rangle_u$, $w_{\mathrm{raw}} := -st\,U(s)U(t)K$ has $\mathfrak{h}_{w_{\mathrm{raw}}}(v) = -K\,vU(v)(v\mathsf{m}_1 - \mathsf{m}_2)$, so

$$w = K\big[-st\,U(s)U(t) + \tfrac{1}{n}(\mathsf{m}_1\Psi_2^U - \mathsf{m}_2\Psi_1^U)\big], \qquad [\mathsf{m}_j]_0 = 0, \quad \mathcal{D}_V\mathsf{m}_1 = \alpha_V, \quad \mathcal{D}_V\mathsf{m}_2 = \varepsilon_V.$$

At $p=0$ ($[s^k]\Psi_1^U(s,W)=U_{\geq k}(W)$, $[s^k]\Psi_2^U(s,W)=U_{\geq k-1}(W)$, $\Psi_1^U(s,0)=U(s)$, $\Psi_2^U(s,0)=sU(s)$, $\Psi_j^U(s,s)=(s^jU)'$):

$$[w]_0=-st\,U(s)U(t),\qquad [\mathcal{D}_V w]_0=\beta\ell_V\,st\,U(s)U(t)+\tfrac{1}{n}(\Psi_2^U\alpha_V-\Psi_1^U\varepsilon_V),$$
$$[\mathcal{D}_{u_1}\mathcal{D}_{u_2}w]_0=-\beta^2\ell_{u_1}\ell_{u_2}st\,U(s)U(t)-\tfrac{\beta}{n}\big[\ell_{u_1}(\Psi_2^U\alpha_{u_2}-\Psi_1^U\varepsilon_{u_2})+\ell_{u_2}(\Psi_2^U\alpha_{u_1}-\Psi_1^U\varepsilon_{u_1})\big],$$
$$[\mathcal{D}_W\langle w(s,u)\rangle_u]_0=-\varepsilon_W U(s),$$
$$[\mathcal{D}_{u_1}\mathcal{D}_{u_2}\langle w(s,u)\rangle_u]_0=\beta U(s)\,\mathrm{ce}+\tfrac{1}{n}\textstyle\sum_{\mathrm{sym}}\big[\alpha_{u_2}\partial_{u_1}\Psi_2^U(s,u_1)-\varepsilon_{u_2}\partial_{u_1}\Psi_1^U(s,u_1)\big],$$

where $\sum_{\mathrm{sym}}$ adds the same term with u_1 and u_2 exchanged. With $\alpha_s:=(sU)'(s)$ and $\varepsilon_s:=(s^2U)'(s)$, $g'=2n\langle w(s,u)\rangle_u-Mw(s,s)$ gives

$$[\mathcal{D}_W g']_0=-2n\varepsilon_W U(s)-M\big[\beta\ell_W s^2U(s)^2+\tfrac{1}{n}(\varepsilon_s\alpha_W-\alpha_s\varepsilon_W)\big],$$
$$[\mathcal{D}_{u_1}\mathcal{D}_{u_2}g']_0=2n\beta U(s)\,\mathrm{ce}+2\textstyle\sum_{\mathrm{sym}}\big[\alpha_{u_2}\partial_{u_1}\Psi_2^U(s,u_1)-\varepsilon_{u_2}\partial_{u_1}\Psi_1^U(s,u_1)\big]-M[\mathcal{D}_{u_1}\mathcal{D}_{u_2}w]_0(s,s).$$

Hence

$$\omega_j(W)=-WU(W)U_{j-1},\qquad \rho_j(V;W)=\beta\ell_V WU(W)U_{j-1}+\tfrac{1}{n}\big(\alpha_V U_{\geq j-1}(W)-\varepsilon_V U_{\geq j}(W)\big),$$
$$\gamma_k=-2nU_k\varepsilon_W-M\beta\ell_W(U^2)_{k-2}-\tfrac{M}{n}(k+1)(U_{k-1}\alpha_W-U_k\varepsilon_W)\qquad(\gamma_0=(M/n-2n)\varepsilon_W),$$
$$\eta_k=2n\beta U_k\,\mathrm{ce}+2\big[\alpha_{u_2}U'_{\geq k-1}(u_1)+\alpha_{u_1}U'_{\geq k-1}(u_2)-\varepsilon_{u_2}U'_{\geq k}(u_1)-\varepsilon_{u_1}U'_{\geq k}(u_2)\big]$$
$$+M\beta^2\ell_{u_1}\ell_{u_2}(U^2)_{k-2}+\tfrac{M\beta}{n}(k+1)\big[U_{k-1}(\ell_{u_1}\alpha_{u_2}+\ell_{u_2}\alpha_{u_1})-U_k\,\mathrm{ce}\big].$$

Part 2 in closed form. $[\Lambda(y,0;u_2)]_0=(U(y)/n)(y\alpha_{u_2}-\varepsilon_{u_2})-yU(y)\hat{u}_2$. The $\frac{1}{n}$-parts of J cancel (the coefficients of α_{u_2} and ε_{u_2} are $-\Psi_1^U(y,u_1)+\Psi_1^U(y,u_1)$ and $(U(y)-\hat{u}_1)/(y-u_1)-(U(y)-\hat{u}_1)/(y-u_1)$), leaving the part $\beta\ell_{u_2}yU(y)\hat{u}_1$ of $-J$ from the $\mathcal{D}w$ terms; the ϑ-part $J_\vartheta(u_1;u_2)=\hat{u}_2\{yU(y)U_{\geq1}(u_1)-U(y)\hat{u}_1-\Psi_1^U(y,u_1)\}$ gives $(u_1-y)J_\vartheta(u_1;u_2)-(u_2-y)J_\vartheta(u_2;u_1)=(u_2-u_1)\{\hat{u}_1\hat{u}_2(U(y)+1)-y^2U(y)V_{12}\}$, where $V_{12}:=[\hat{u}_1\hat{u}_2-\Psi_1^U(u_1,u_2)]/(u_1u_2)$ (using $\hat{u}_2U_{\geq1}(u_1)-\hat{u}_1U_{\geq1}(u_2)=(u_2-u_1)V_{12}$). So

$$\text{Part 2}=\kappa U(y)\big\{\tfrac{y}{n}\Delta\alpha-\tfrac{1}{n}\Delta\varepsilon-y\Delta U\big\}+(u_2-u_1)\big\{-\tfrac{\beta}{n}U(y)\big[y(\ell_{u_2}\alpha_{u_1}+\ell_{u_1}\alpha_{u_2})-\mathrm{ce}\big]$$
$$+\beta yU(y)(\ell_{u_1}\hat{u}_2+\ell_{u_2}\hat{u}_1)+\hat{u}_1\hat{u}_2(U(y)+1)-y^2U(y)V_{12}\big\}.$$

The terms with M/n. They occur in exactly four places: the last terms of γ_k and η_k (from the $\frac{1}{n}$-terms of w inside $-Mw(s,s)$), and the $\frac{1}{n}$-terms of $M\cdot$Part 2 (in its κ-bracket and in its $\frac{\beta}{n}$-bracket). They cancel in two pairs, at every k: $\frac{\kappa}{k+1}\Delta\gamma_k$ contributes $-\frac{\kappa M}{n}(U_{k-1}\Delta\alpha-U_k\Delta\varepsilon)$, and $[y^k]$ of $M\kappa U(y)\{\frac{y}{n}\Delta\alpha-\frac{1}{n}\Delta\varepsilon\}$ contributes the opposite; $(u_2-u_1)\eta_k/(k+1)$ contributes $(u_2-u_1)\frac{M\beta}{n}[U_{k-1}(\ell_{u_1}\alpha_{u_2}+\ell_{u_2}\alpha_{u_1})-U_k\,\mathrm{ce}]$, and $[y^k]$ of $-M(u_2-u_1)\frac{\beta}{n}U(y)[y(\ell_{u_2}\alpha_{u_1}+\ell_{u_1}\alpha_{u_2})-\mathrm{ce}]$ contributes the opposite. (The $\frac{1}{n}$ in ρ_j comes with the factor $2n$, and the $\frac{1}{n}$-parts of J cancel inside Part 2.)

Assembly. With $\kappa=2\beta$, Part 1 plus $M\cdot$Part 2 gives, for every k,

$$\begin{aligned}[y^k]\mathsf{P} = U_k\Big\{\Big(2-\frac{4n\beta}{k+1}\Big)\Delta\varepsilon+\frac{2n\beta}{k+1}(u_2-u_1)\mathrm{ce}-4n\beta\Delta(xU)\\ +2n\beta u_1u_2(\ell_{u_2}\hat u_1-\ell_{u_1}\hat u_2)+M(u_2-u_1)\hat u_1\hat u_2\Big\}\\ +U_{k-1}\Big\{-2\Delta\alpha+2n\beta(\ell_{u_1}u_2\hat u_2-\ell_{u_2}u_1\hat u_1)+M\beta\big[-2\Delta U+(u_2-u_1)(\ell_{u_1}\hat u_2+\ell_{u_2}\hat u_1)\big]\Big\}\\ +\frac{2}{k+1}(u_2-u_1)\big[\alpha_{u_2}U'_{\ge k-1}(u_1)+\alpha_{u_1}U'_{\ge k-1}(u_2)-\varepsilon_{u_2}U'_{\ge k}(u_1)-\varepsilon_{u_1}U'_{\ge k}(u_2)\big]\\ +2(u_2-u_1)\big[\alpha_{u_2}U_{\ge k}(u_1)+\alpha_{u_1}U_{\ge k}(u_2)-\varepsilon_{u_2}U_{\ge k+1}(u_1)-\varepsilon_{u_1}U_{\ge k+1}(u_2)\big]\\ +\frac{M\beta^2}{k+1}(U^2)_{k-2}\big[-2\Delta\ell+(u_2-u_1)\ell_{u_1}\ell_{u_2}\big]\\ +M(u_2-u_1)\big[\delta_{k0}\hat u_1\hat u_2-U_{k-2}V_{12}\big],\end{aligned}$$

where $\Delta(xU):=u_2\hat u_2-u_1\hat u_1$. (For one letter, $U=1/(1-yz)$, $U_{\ge j}=z^jU$ and $V_{12}=0$, and $k=2$ gives the formula of Theorem 9.149(b) below, coefficient by coefficient.)

Two letters. Let $\mathfrak{A}=\{z_1,z_2\}$, $e_1:=z_1+z_2$, $e_2:=z_1z_2$, $\mathsf{q}_\upsilon:=1-\upsilon e_1+\upsilon^2e_2$, so $U(\upsilon)=1/\mathsf{q}_\upsilon$; and $\sigma_1:=u_1+u_2$, $\sigma_2:=u_1u_2$, $\mathsf{a}_i:=1/(1-u_iz_i)$, $\mathsf{b}_i:=1/(1-u_2z_i)$. Then $U'=(e_1-2\upsilon e_2)U^2$, $\ell=(e_1-2\upsilon e_2)U$, $\alpha=(1-\upsilon^2e_2)U^2$, $\varepsilon=\upsilon(2-\upsilon e_1)U^2$, $U_{\ge j}=(h_j-\upsilon e_2h_{j-1})U$, $U'_{\ge j}=(h_{j+1}-2\upsilon e_2h_j+\upsilon^2e_2^2h_{j-1})U^2$ ($h_j=h_j(z_1,z_2)$, all at υ), $\hat u_2-\hat u_1=(u_2-u_1)(e_1-\sigma_1e_2)\hat u_1\hat u_2$, $\Psi_1^U(u_1,u_2)=(1-\sigma_2e_2)\hat u_1\hat u_2$, $V_{12}=e_2\hat u_1\hat u_2$, and $\mathsf{a}_1\mathsf{b}_2+\mathsf{a}_2\mathsf{b}_1=N_{12}\hat u_1\hat u_2$ with $N_{12}:=2-\sigma_1e_1+2\sigma_2e_2$. Clearing $\mathsf{q}_{u_1}^2\mathsf{q}_{u_2}^2$, writing the symmetric parts in σ_1,σ_2 and comparing all coefficients gives:

(Ra) $-4\Delta\varepsilon+2(u_2-u_1)\mathrm{ce}=-(u_2-u_1)\mathsf{Y}$, where $\mathsf{Y}:=2\hat u_1\hat u_2[(2-u_1e_1)^2\hat u_1+(2-u_2e_1)^2\hat u_2]-8\hat u_1\hat u_2$;
(Rb) $-4\Delta(xU)+2u_1u_2(\ell_{u_2}\hat u_1-\ell_{u_1}\hat u_2)=-4(u_2-u_1)\hat u_1\hat u_2$;
(Rc) $\ell_{u_1}u_2\hat u_2-\ell_{u_2}u_1\hat u_1=(u_2-u_1)e_1\hat u_1\hat u_2$;
(Rd) $-2\Delta U+(u_2-u_1)(\ell_{u_1}\hat u_2+\ell_{u_2}\hat u_1)=0$ (so the parts of $[y^1]$, $[y^2]$ linear in β vanish);
(Re) the part of $[y^0]\mathsf{P}$ free of $n\beta$ and M is $(u_2-u_1)(\mathsf{Y}+2\,\mathrm{Sq})$, where $\mathrm{Sq}:=\mathsf{a}_1^2\mathsf{b}_2^2+\mathsf{a}_2^2\mathsf{b}_1^2=N_{12}^2\hat u_1^2\hat u_2^2-2\hat u_1\hat u_2$;
(Rf) the part of $[y^2]\mathsf{P}$ with β^2 is $M\beta^2[-\frac23\Delta\ell+\frac13(u_2-u_1)\ell_{u_1}\ell_{u_2}]=-(u_2-u_1)\frac{M\beta^2}{3}\mathrm{disc}\,\hat u_1\hat u_2$, where $\mathrm{disc}:=e_1^2-4e_2=(z_1-z_2)^2$.

So, with $\mathrm{tg}_2:=(u_2-u_1)M(1-\beta^2)\,\mathrm{disc}\,\hat u_1\hat u_2/3$ and $\mathrm{tg}_0=\mathrm{tg}_1=0$, the residuals are

$$\begin{aligned}[y^0]\mathsf{P}&=(u_2-u_1)\big\{(1-n\beta)\mathsf{Y}-4n\beta\,\hat u_1\hat u_2+2\,\mathrm{Sq}+2M\hat u_1\hat u_2\big\},\\ [y^1]\mathsf{P}&=(u_2-u_1)\big\{n\beta\,e_1(-\tfrac12\mathsf{Y}-2\hat u_1\hat u_2)+Me_1\hat u_1\hat u_2\big\}+W_1,\\ [y^2]\mathsf{P}-\mathrm{tg}_2&=(u_2-u_1)\big\{n\beta\big(h_2(-\tfrac13\mathsf{Y}-4\hat u_1\hat u_2)+2e_1^2\hat u_1\hat u_2\big)+\tfrac23Mh_2\hat u_1\hat u_2\big\}+W_0,\end{aligned}$$

where W_1, W_0 are the parts free of $n\beta$, M and β (from the assembly).

The Hall inner product and the shift invariance. By the Cauchy identity [Ma95, I, §4], for $\mathfrak{A}=\{z_1,z_2\}$ the seed is $U(s)U(t)U(v)K=\sum_{\ell(\lambda)\le2}s_\lambda[\mathbf{R}]\,s_\lambda(z_1,z_2)$ (and $s_\lambda(z_1,z_2)=0$ for $\ell(\lambda)\ge3$), so for every $f\in\Lambda$ with at most two rows, $f[\mathbf{R}]=\langle f,U(s)U(t)U(v)K\rangle$, the Hall inner product in the z-variables (s_λ orthonormal, [Ma95, I, (4.8)]). The probe is linear in ϑ, so $[y^k]\mathsf{P}$ for $\vartheta=f[\mathbf{R}]$ is $\langle f,[y^k]\mathsf{P}\rangle$ for the seed. The map $f\mapsto f[\mathbf{R}]$ is injective on the f with at most two rows: at $p=0$ it is $f(s,t,v)$, and the $s_\lambda(s,t,v)$ with $\ell(\lambda)\le3$ are linearly independent. By the proof of Theorem 9.141(ii), $\mathbb{T}\mathbf{R}_k=(k+2-n\beta)\mathbf{R}_{k-1}$, so $\mathbb{T}(f[\mathbf{R}])=(\mathbb{T}_\zeta f)[\mathbf{R}]$ with $\zeta:=3-n\beta$, where $\mathbb{T}_\zeta$ is the derivation of Λ with $\mathbb{T}_\zeta h_k=(k+\zeta-1)h_{k-1}$, that is, $\mathbb{T}_\zeta=\zeta\pi_1^\perp+\sum_{k\ge2}k\pi_{k-1}\partial/\partial\pi_k$; so $f[\mathbf{R}]$ is shift invariant if and only if $\mathbb{T}_\zeta f=0$. The adjoint of $\mathbb{T}_\zeta$ is $\mathbb{T}_\zeta^*=\zeta\pi_1+\sum_j j\pi_{j+1}\partial/\partial\pi_j$; in two letters $\mathbb{T}_\zeta^*=\zeta e_1+\mathsf{d}$ with $\mathsf{d}:=z_1^2\partial_{z_1}+z_2^2\partial_{z_2}$ ($\mathsf{d}\pi_j=j\pi_{j+1}$). As $\mathbb{T}_\zeta$ maps functions with at most two rows to such functions (it removes one box), $\mathbb{T}_\zeta^*$ preserves $(e_3,e_4,\dots)$ and acts on $\mathbb{C}[z_1,z_2]^{S_2}$.

So if $\mathbb{T}_\zeta f = 0$, then $\langle f, \mathbb{T}_\zeta^* H\rangle = 0$ for every H. Also $M = \mu(a+1)$ with $a = d+2$ ($d := \deg f$) and $\mu = 2n\beta - a$, so on the part of z-degree d, $M = (4 - 2\zeta - \mathrm{E})(\mathrm{E}+3)$, where E is the z-degree operator; and $n\beta = 3 - \zeta$. The inner product with f of degree d sees only the part of z-degree d, and $\langle f, \operatorname{disc}\hat{u}_1\hat{u}_2\rangle = (\mathbb{L}f)(u_1,u_2)$ keeps that degree, so M acts by the z-degree on both sides. As β and ζ are independent parameters (ζ fixes $n\beta$, and n is free), it suffices to match each power of β separately. Finally, $\mathsf{d}U(v) = U'(v) - e_1U(v)$, $\mathsf{d}e_1 = e_1^2 - 2e_2$, $\mathsf{d}e_2 = e_1e_2$, $\mathrm{E} = e_1\partial_{e_1} + 2e_2\partial_{e_2}$ and $\mathrm{E}U(v) = v(e_1 - 2ve_2)U(v)^2$.

The field. Before the assembly the coefficients lie in $\mathbb{Q}[n, n^{-1}, \beta]$ (the $\frac{1}{n}$ comes from pr); after it, n occurs only through $n\beta$. So Theorem 9.146 is a polynomial identity in $\mathbb{Q}[n\beta, \beta]$, and it specializes to every complex κ and every integer n.

Theorem 9.146. *Let $f \in \Lambda$ have at most two rows and degree $d = a - 2$, with $\mathbb{T}_\zeta f = 0$, where $\kappa = 2\beta$, $\zeta = 3 - n\beta$ and $\mu := n\kappa - a$ $(= 6 - 2\zeta - a)$; let $w := (-st\, f[\mathbf{R}](s,t,0))^\perp$ and $M = \mu(a+1)$. Then, on the data $v'(\mu, w)$, at $p = 0$ and $y' = 0$,*

$$[y^0]\mathsf{P} = [y^1]\mathsf{P} = 0, \qquad [y^2]\mathsf{P} = \frac{(u_2 - u_1)M(1-\beta^2)}{3}\,(\mathbb{L}f)(u_1,u_2), \qquad \mathbb{L} := (h_2 - 3e_2)^\perp.$$

Here $\mathbb{L}s_{(p,q)} = s_{(p-2,q)} - 2s_{(p-1,q-1)} + s_{(p,q-2)}$ for $p > q$, $\mathbb{L}s_{(q,q)} = s_{(q,q-2)} - 3s_{(q-1,q-1)}$, and $s_{(p,q)}(u_1,u_2) = (u_1u_2)^q h_{p-q}(u_1,u_2)$; $\langle f, \operatorname{disc}\hat{u}_1\hat{u}_2\rangle = (\mathbb{L}f)(u_1,u_2)$, as $\operatorname{disc} = (h_2 - 3e_2)(z_1,z_2)$ and $\langle f, \hat{u}_1\hat{u}_2\rangle = f(u_1,u_2)$.

Proof. By the two preceding paragraphs it suffices that each residual above is $\mathbb{T}_\zeta^* H_k$ after the substitutions $n\beta = 3 - \zeta$, $M = (4 - 2\zeta - \mathrm{E})(\mathrm{E}+3)$. Each residual is $r_0 + \zeta r_1$ with r_0, r_1 free of ζ, and $\mathbb{T}_\zeta^* H = \mathsf{d}H + \zeta e_1 H$; so it suffices that $r_1 = e_1H_k$ and $r_0 = \mathsf{d}H_k$, with

$$H_k := (u_2 - u_1)\hat{u}_1\hat{u}_2\Big[\frac{2}{k+1}h_k(z_1,z_2)S_{12} - 2h_{k-1}(z_1,z_2)\Big] \quad (h_{-1} := 0),$$
$$S_{12} := \sum_{v\in\{u_1,u_2\}} v(ve_1 - 2)U(v);$$

so $H_0 = 2(u_2-u_1)\hat{u}_1\hat{u}_2S_{12}$, $H_1 = (u_2-u_1)\hat{u}_1\hat{u}_2(e_1S_{12} - 2)$ and $H_2 = (u_2-u_1)\hat{u}_1\hat{u}_2(\frac{2}{3}h_2S_{12} - 2e_1)$.

$k = 0$ (the common factor $u_2 - u_1$ dropped on both sides): $r_1 = \mathsf{Y} - 4(\mathrm{E}+2)\hat{u}_1\hat{u}_2 = 2\hat{u}_1\hat{u}_2e_1S_{12}$ (for each v: $2v^2\operatorname{disc}U - 4v(e_1 - 2ve_2)U = 2ve_1(ve_1 - 2)U$, with $(2 - ve_1)^2U = 4 + v^2\operatorname{disc}U$). Next, $r_0 = -2\mathsf{Y} - 12\hat{u}_1\hat{u}_2 + 2\,\mathrm{Sq} + 2(4 - \mathrm{E})(\mathrm{E}+3)\hat{u}_1\hat{u}_2$; dividing $\mathsf{d}H_0 - r_0$ by $2\hat{u}_1\hat{u}_2$: the U^2-terms are $2v^2[(ve_1-2)\mathsf{f}_v + (e_1 - 2ve_2)^2]U^2 = -2v^2\operatorname{disc}U$ (with $\mathsf{f}_v := e_1^2 - 2e_2 - ve_1e_2$, $(ve_1-2)\mathsf{f}_v + (e_1-2ve_2)^2 = -\operatorname{disc}\mathsf{q}_v$), the U-terms are $3v^2\operatorname{disc}U$ for each v plus the constant 4 in total, and the $\hat{u}_1\hat{u}_2$-terms are $\mathsf{X}\hat{u}_1\hat{u}_2$ with

$$\mathsf{X} = \sigma_2\sigma_1e_1^3 - 2\sigma_2^2e_1^2e_2 - 2\sigma_2e_1^2 + 4\sigma_2^2e_2^2 - 4 - \sigma_1^2e_1^2 + 4\sigma_1e_1 = -4\mathsf{q}_{u_1}\mathsf{q}_{u_2} - \operatorname{disc}(u_1^2\mathsf{q}_{u_2} + u_2^2\mathsf{q}_{u_1}),$$

that is, $\mathsf{X}\hat{u}_1\hat{u}_2 = -4 - \sum_v v^2\operatorname{disc}U(v)$. So $(\mathsf{d}H_0 - r_0)/(2\hat{u}_1\hat{u}_2) = 4 + \sum_v v^2\operatorname{disc}U(v) + \mathsf{X}\hat{u}_1\hat{u}_2 = 0$.

$k = 1$: $r_1 = (u_2-u_1)e_1\hat{u}_1\hat{u}_2\big[\sum_v(4 - 6ve_1 + v^2e_1^2 + 4v^2e_2)U(v) - 10\big] = e_1H_1$, as $(4 - 6ve_1 + v^2e_1^2 + 4v^2e_2)U = 4 + ve_1(ve_1 - 2)U$. For $r_0 = \mathsf{d}H_1$ write $H_1/(u_2-u_1) = \frac{e_1}{2}(2\hat{u}_1\hat{u}_2S_{12}) - 2\hat{u}_1\hat{u}_2$ and use $k = 0$: what remains is

$$W_1 = (u_2-u_1)\hat{u}_1\hat{u}_2\Big\{\sum_v \mathsf{Q}_1(v)U(v) - 6e_1 + e_1\hat{u}_1\hat{u}_2N_{12}^2\Big\},$$
$$\mathsf{Q}_1(v) := 4e_1 - 6ve_1^2 + 8ve_2 + 2v^2e_1^3 - 4v^2e_1e_2,$$

and both sides, multiplied by $\mathsf{q}_{u_1}^2\mathsf{q}_{u_2}^2/(u_2-u_1)$, equal

$$6e_1 - 8e_1^2\sigma_1 + 8e_2\sigma_1 + 3e_1^3\sigma_1^2 - 6e_1e_2\sigma_1^2 + 2e_1^3\sigma_2 + 4e_1e_2\sigma_2 + 8e_2^2\sigma_1\sigma_2 - 2e_1^4\sigma_1\sigma_2 - 10e_1e_2^2\sigma_2^2 + 4e_1^3e_2\sigma_2^2.$$

$k=2$: $r_1=(u_2-u_1)\hat u_1\hat u_2\big[\frac23 h_2\sum_\upsilon(4-6\upsilon e_1+\upsilon^2e_1^2+4\upsilon^2e_2)U(\upsilon)-\frac{16}{3}h_2-2e_1^2\big]=e_1H_2$. For $r_0=\mathsf dH_2$ write $H_2/(u_2-u_1)=\frac{h_2}{3}(2\hat u_1\hat u_2S_{12})-2e_1\hat u_1\hat u_2$ ($\mathsf dh_2=2e_1^3-5e_1e_2$) and use $k=0$: what remains is

$$W_0=(u_2-u_1)\hat u_1\hat u_2\Big\{\tfrac13\sum_\upsilon \mathsf Q_2(\upsilon)U(\upsilon)+\tfrac23h_2\hat u_1\hat u_2N_{12}^2-\tfrac83e_1^2-\tfrac43e_2\Big\},$$

$\mathsf Q_2(\upsilon):=8e_1^2-8e_2-14\upsilon e_1^3+32\upsilon e_1e_2+6\upsilon^2e_1^4-22\upsilon^2e_1^2e_2+16\upsilon^2e_2^2$, and both sides, multiplied by $3\mathsf q_{u_1}^2\mathsf q_{u_2}^2/(u_2-u_1)$, equal

$$\begin{aligned}16e_1^2-28e_2-22e_1^3\sigma_1+52e_1e_2\sigma_1+8e_1^4\sigma_1^2-24e_1^2e_2\sigma_1^2+4e_2^2\sigma_1^2+8e_1^4\sigma_2-8e_1^2e_2\sigma_2-24e_2^2\sigma_2\\-6e_1^5\sigma_1\sigma_2+8e_1^3e_2\sigma_1\sigma_2+28e_1e_2^2\sigma_1\sigma_2+12e_1^4e_2\sigma_2^2-44e_1^2e_2^2\sigma_2^2+20e_2^3\sigma_2^2.\end{aligned}$$

The part of $[y^2]\mathsf P$ with β^2 is that of tg_2 by (Rf), and the parts linear in β vanish by (Rd). The argument is uniform in ζ, so it covers every shift invariant two-row f, $\beta=0$ included. □

The kernel of $\mathbb T_\zeta$. Let $f=\sum_b c_bs_{(d-b,b)}$ have degree d and at most two rows. By the Jacobi–Trudi formula $s_{(p,q)}=h_ph_q-h_{p+1}h_{q-1}$ ($h_{-k}:=0$; it gives $s_{(q-1,q)}=0$), both sides of $\mathbb T_\zeta s_{(p,q)}=(\zeta+p-1)s_{(p-1,q)}+(\zeta+q-2)s_{(p,q-1)}$ expand to $(\zeta+p-1)h_{p-1}h_q+(q-p-1)h_ph_{q-1}-(\zeta+q-2)h_{p+1}h_{q-2}$. So the coefficient of $s_{(d-1-b,b)}$ in $\mathbb T_\zeta f$ is $t_bc_b+t'_bc_{b+1}$, with $t_b:=\zeta+d-b-1$ and $t'_b:=\zeta+b-1$ ($0\le b\le\lfloor(d-1)/2\rfloor$; $c_{b+1}:=0$ when $(d-b-1,b+1)$ is not a partition). For odd d the matrix is square upper bidiagonal with determinant $\prod_b t_b$; the kernel is not 0 if and only if $\zeta=b+1-d$ for some $0\le b\le(d-1)/2$, that is, $\xi=2(a-b)$, and it is then spanned by

$$f_j^{(r)}:=\sum_{b'=0}^{j-1}\binom{r-b'}{j-1-b'}s_{(r+j-1-b',\,b')}\qquad(j=b+1,\ r=a-1-j,\ \mu=a+2-2j).$$

For even $d=2h$ there are h rows and $h+1$ unknowns, and for every ζ the kernel is the line spanned by the signed maximal minors $f_d(\zeta):=\sum_{b=0}^h c_bs_{(d-b,b)}$, $c_b:=(-1)^b\prod_{i<b}t_i\prod_{b\le i\le h-1}t'_i$: each row $t_bc_b+t'_bc_{b+1}=0$ holds by telescoping, and $f_d(\zeta)\ne0$, as $c_0=\prod_i t'_i$ and $c_h=(-1)^h\prod_i t_i$ have disjoint roots ($\zeta\in\{2-h,\dots,1\}$ and $\zeta\in\{1-d,\dots,-h\}$). So $\mathbb T_\zeta f_d(\zeta)=0$ as a polynomial identity in ζ, $\dim\ker\mathbb T_\zeta=1$ at every ζ, and for even a, $\vartheta_\xi=c(\xi)f_d(\zeta)[\mathbf R]$ with $c(\xi)\ne0$ at every ξ. This kernel is $\ker\mathbf B(\xi)$ of Theorem 9.141, written in the Schur basis.

Lemma 9.147. *On functions with at most two rows,* $\mathbb T_{\zeta+2}\mathbb L=\mathbb L\mathbb T_\zeta$. *So* $\mathbb L$ *maps* $\ker\mathbb T_\zeta$ *in degree* d *into* $\ker\mathbb T_{\zeta+2}$ *in degree* $d-2$.

Proof. The Hall inner product of the functions of degree d with at most two rows with the part of degree d of $\mathbb C[z_1,z_2]^{S_2}$ is perfect; $\mathbb T_\zeta$ and $\mathbb L$ keep such functions; $\mathbb T_\zeta$ has the adjoint $\mathbb T_\zeta^*=\zeta e_1+\mathsf d$, and $\mathbb L$ has the adjoint the multiplication by $(h_2-3e_2)(z_1,z_2)=\mathrm{disc}=(z_1-z_2)^2$. As $\mathsf d(\mathrm{disc})=2(z_1-z_2)(z_1^2-z_2^2)=2e_1\,\mathrm{disc}$, $\mathbb T_\zeta^*(\mathrm{disc}\,H)=\mathrm{disc}\,((\zeta+2)e_1H+\mathsf dH)=\mathrm{disc}\,\mathbb T_{\zeta+2}^*H$ for every H: the adjoints of $\mathbb L\mathbb T_\zeta$ and $\mathbb T_{\zeta+2}\mathbb L$ agree. □

Lemma 9.148. *For* $0\ne f\in\ker\mathbb T_\zeta$ *(at most two rows, degree* $d=a-2$, $\mu:=6-2\zeta-a$*):* $\mathbb Lf=0$ *if and only if* $\mu=3$.

Proof. Odd d: $\mathbb Lf_j^{(r)}=f_j^{(r-2)}$ (coefficientwise $c_b-2c_{b+1}+c_{b+2}=\binom{r-2-b}{j-1-b}$, Pascal's rule twice; every shape of $f_j^{(r)}$ has $p>q$, as $r\ge j$ if and only if $\mu\ge3$); its coefficient at $b=j-1$ is 1, on $s_{(r-2,j-1)}$, which is a partition if and only if $r\ge j+1$, that is, $\mu\ge4$. At $\mu=3$ ($r=j$), $\binom{j-2-b}{j-1-b}=0$ on every partition ($b\le j-2$), and the term $b=j-1$ (with $\binom{-1}{0}$) sits on $s_{(j-2,j-1)}$, which is not a partition and is 0; so $\mathbb Lf=0$. (This uses only the shapes of $f_j^{(r)}$, not the parity of d.)

Even $d=2h$ ($a\ge4$): by Lemma 9.147 and the paragraph before it, $\mathbb Lf_d(\zeta)=\lambda(\zeta)f_{d-2}(\zeta+2)$ for a scalar $\lambda(\zeta)$. Compare the coefficients of $s_{(h-1,h-1)}$: on the left, $c_{h-1}-3c_h=2(-1)^{h-1}(2\zeta+$

$d-1)\prod_{i=0}^{h-2}(\zeta+d-i-1)$ ($s_{(h+1,h-1)}$ gives c_{h-1}, and $\mathbb{L}s_{(h,h)} = s_{(h,h-2)} - 3s_{(h-1,h-1)}$ gives $-3c_h$); on the right, $\lambda(-1)^{h-1}\prod_{i=0}^{h-2}(\zeta+d-i-1)$ (the last signed minor of degree $d-2$ at $\zeta+2$, whose entries t_i are $(\zeta+2)+(d-2)-i-1$). So $\lambda = 2(2\zeta+d-1)$ off finitely many ζ; both sides are polynomials in ζ, hence

$$\mathbb{L}f_d(\zeta) = 2(2\zeta+d-1)f_{d-2}(\zeta+2) \qquad \text{for every } \zeta,$$

and, as $f_{d-2}(\zeta+2) \neq 0$, $\mathbb{L}f = 0$ if and only if $2\zeta+a-3=0$, that is, $\mu = 3$ (as $\mu = 2n\beta - a = 6-2\zeta-a$). For $d=2$, $\mathbb{L}f_2 = 2(2\zeta+1)$. Finally, two-row Schur polynomials of partitions are linearly independent in two variables, so $(\mathbb{L}f)(u_1,u_2) = 0$ if and only if $\mathbb{L}f = 0$. □

Theorem 9.149. *Let $a \geq 3$ and $n \geq a+6$. Then (H2) of Theorem 9.143 holds: for every ξ with $\ker \mathbf{B}(\xi) \neq 0$, $\mu = \xi - a \notin \{0,3\}$ and $\xi \neq \pm 2n$, the coefficient $[y^2]$ of the row P on $v'(\mu, w_\xi)$ at $p=0$ and $y'=0$ is $\frac{1}{3}(u_2-u_1)M(1-\beta^2)(\mathbb{L}f_\xi)(u_1,u_2)$, which is a nonzero polynomial in u_1, u_2 ($M \neq 0$, $\beta^2 \neq 1$, Lemma 9.148). Moreover:*

- *(a) for even a, $\kappa = 0$ is not an eigenvalue (with Theorem 9.143(3)): there $\zeta = 3$, $\mu = -a$, $M = -a(a+1)$, and $\mathbb{L}f_d(3) \neq 0$;*
- *(b) at $\xi = 2a$ ($f = h_d$, $\mu = a$), $\mathbb{L}h_d = h_{d-2}$, and $[y^2]\mathsf{P}$ is $\frac{1}{3}(u_2-u_1)a(a+1)(1-a^2/n^2)$ times $h_{a-4}(u_1,u_2)$;*
- *(c) for odd $a \geq 5$ and $f = f_2^{(a-3)}$ ($\mu = a-2$), $\mathbb{L}f = (a-4)h_{a-4} - (u_1^{a-4}+u_2^{a-4})$ as a function of u_1, u_2.*

Proof. $\vartheta_\xi = f_\xi[\mathbf{R}]$ is shift invariant (Theorem 9.141(ii); Lemma 9.142(ii) at $\xi = 0$), so $\mathbb{T}_\zeta f_\xi = 0$, and f_ξ has at most two rows and degree $a-2$; apply Theorem 9.146 and Lemma 9.148. $M \neq 0$ as $\mu \neq 0$, and $\beta^2 \neq 1$ as $\xi \neq \pm 2n$. (a)–(c) are the cases $\kappa = 0$, $\xi = 2a$ and $\xi = 2a-2$ of this, with the values of $\mathbb{L}$ above. □

Theorem 9.150. *Let $a \geq 3$ and let $n \geq a+6$ be an integer. Every eigenvalue κ of the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ satisfies $n\kappa - a \in \{0,3\}$ or $\kappa \in \{2,-2\}$; in particular, $\kappa = 0$ is not an eigenvalue.*

Proof. Theorem 9.143(3) and Theorem 9.149. □

Remark 9.151. (1) At $a=3$ this is Theorem 9.48, which holds for every $n \geq 4$. The computed cases of Conjecture 11.3 in Remark 9.50(4) ($a=4$ for all but finitely many n; $a=5$ at $n=11,12$; $a=6$ at $n=13$) are proved here by hand for $n \geq a+6$ (at $a=4$: $n \geq 10$), and extended to every $n \geq a+6$; so at $a=4$ the conclusion can fail only for $n \leq 9$.

(2) With Theorem 9.143, this answers Remark 9.124(2) and (4) for $n \geq a+6$, $\kappa = 0$ included. The route uses the order-zero rows (Lemma 9.46) and $\mathsf{G1} = 0$, $\mathsf{G}x = 0$, the two sources named in Remark 9.124(4), and it is consistent with the count at $a=4$ there (the data lie on the line of Theorem 9.143). It is different from Corollary 9.114: it does not use the principle (P1), and it does not prove it.

(3) At $s = n$ the range $n \geq a+6$ is depth at most $n-6$, inside Theorem 9.24; so Theorem 9.150 gives no new bound on $d(n)$. We do not treat $n < a+6$, and we do not claim here that the values $\mu = 0,3$ and $\kappa = \pm 2$ occur (see Corollary 9.63 and Theorem 9.102).

(4) (Computations, not used.) From the definitions, with n and β symbolic: for every Schur function $f = s_{(d-b,b)}$, $d = 2,\dots,6$, shift invariant or not, $[y^k]\mathsf{P}$ is the value of Theorem 9.146 plus $\langle \mathbb{T}_\zeta f, H_k\rangle$ (42 cases; controls fail as they must); and on the seed, for $d \leq 5$, $[y^k]\mathsf{P}$ is the target plus the part of degree d of $\mathbb{T}^*_\zeta H_k$. The assembly, the reductions (Ra)–(Rf), Part 1 plus M·Part 2, and the identities $r_1 = e_1 H_k$, $r_0 = \mathsf{d}H_k$ ($k = 0,1,2$) were checked exactly by independent transcriptions. Theorem 9.149 agrees with independent engines built from the definitions on every vector of Theorem 9.141 for $a = 3,\dots,11$ (β symbolic for even a, so $\kappa = 0$ included); Lemma 9.148 was checked for $d \leq 12$, and the identity in its proof for $d = 2,4,\dots,14$, as polynomials in ζ.

9.26. **The eigenvalue $\kappa=n$ at odd $a\ge n$.** In this subsection $F=\mathbb{C}$, $b=2$, $a\ge 3$ and $n\ge 4$, with no condition relating n and a, and we keep the notation of Subsections 9.13 and 9.24; but all data are functions of the coordinates $x_1,\dots,x_n$ at this n (no power sums are treated as independent variables). Call a *dominant* if $a\le (n^2-2)/2$, that is, if $\tau=(n^2-a-2,a,2)$ is a partition. We show that $\kappa=n$ is not an eigenvalue for odd a at 18 pairs (n,a) with $n\le a$ and a dominant (Theorem 9.154, computer-assisted). A *kernel vector* at κ is, as in Subsection 9.24, a $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ with $\kappa\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$ at every base point and for all u,y_1,y_2,y_3; we use only the necessary conditions that the rows of Proposition 9.44 vanish, together with Lemma 9.46.

The data at this n. Let $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$, with data F_{rt}, G_{rt} as in (1): polynomials in x, homogeneous of degree a and shift invariant. Conjugating (1) by the permutation matrix of $\sigma\in S_n$ gives $\mathsf{F}_{\sigma(r)\sigma(t)}(x\circ\sigma^{-1})=\mathsf{F}_{rt}(x)$, and the same for G and H. For distinct r,w,t put

$$\vartheta_{rwt}:=\frac{\mathsf{H}_{rwt}}{x_{rw}\,x_{wt}\,x_{tr}}.$$

By the equivariance, H_{123} changes sign under each transposition of x_1,x_2,x_3 (as $\mathsf{H}_{tr}=-\mathsf{H}_{rt}$) and is symmetric in $x_4,\dots,x_n$. So ϑ_{123} is a polynomial, symmetric in x_1,x_2,x_3 and in $x_4,\dots,x_n$, homogeneous of degree $a-2$ and shift invariant; each ϑ_{rwt} is obtained from it by permuting the coordinates, and it is symmetric in r,w,t. With $\mathsf{c}^{(pq)}_w=\mathsf{H}_{pwq}/(x_{pw}x_{wq})$ (Subsection 9.13) this gives $\mathsf{c}^{(pq)}_w=x_{qp}\,\vartheta_{pqw}$. Let $\mathsf{Z}(n,a)$ be the space of families $(\mathsf{F}_{rt})_{r\ne t}$ of polynomials in x with $\mathsf{F}_{rt}=\mathsf{F}_{tr}$ and the equivariance above, homogeneous of degree a and shift invariant, such that $\mathsf{H}_{rwt}=0$ for all distinct r,w,t and $\mathsf{F}_{12}=0$ on the hyperplane $x_1=x_2$.

Lemma 9.152 (the odd rows at a given n)**.** *If φ is a kernel vector at κ, then, identically in x,*

$$\kappa\,\frac{\vartheta_{123}-\vartheta_{124}}{x_3-x_4}+\partial_3\vartheta_{124}+\partial_4\vartheta_{123}=0. \tag{$\star_n$}$$

Proof. Lemma 9.136 holds for any symmetric data at any $n\ge 4$: its proof uses only the shifted form of Proposition 9.44 and $\mathsf{F}_{rt}=\mathsf{F}_{tr}$, $\mathsf{G}_{rt}=\mathsf{G}_{tr}$, not the range $n\ge a+6$ of Subsection 9.24. Take $(p,q,i,j)=(1,2,3,4)$. As $\mathsf{c}_l=x_{21}\vartheta_{12l}$ and x_{21} does not depend on x_3,x_4,

$$R^-_{12}=x_{21}\big[\kappa(\vartheta_{124}-\vartheta_{123})-x_{34}(\partial_3\vartheta_{124}+\partial_4\vartheta_{123})\big]=-x_{21}x_{34}\cdot\Big(\kappa\,\frac{\vartheta_{123}-\vartheta_{124}}{x_3-x_4}+\partial_3\vartheta_{124}+\partial_4\vartheta_{123}\Big).$$

For a kernel vector every row vanishes at every x with distinct coordinates, so the bracket vanishes there, hence identically. □

Lemma 9.153 (collisions)**.** *Let $\varphi\in\mathcal{F}_{(a,2)}(n)$ and $r\ne t$. Then $\mathsf{G}_{rr}=\mathsf{F}_{rt}+\mathsf{G}_{rt}$ on the hyperplane $x_r=x_t$.*

Proof. Both sides of (1) are polynomials in x, so (1) holds at every diagonal X, collisions included, with $\mathsf{F}_{rt}=Q_X(E_{rt},E_{tr})$ and $\mathsf{G}_{rt}=Q_X(E_{rr},E_{tt})$. Let $x_r=x_t$. Then the block GL_2 on the coordinates r,t commutes with X, so Q_X is invariant under it, and its restriction to the matrices supported on $\{r,t\}^2$ is an invariant symmetric bilinear form on $\mathfrak{gl}_2=\mathfrak{sl}_2\oplus\mathbb{C}I$; so it is $\alpha\,\mathrm{tr}(VV')+\beta\,\mathrm{tr}\,V\,\mathrm{tr}\,V'$. The coefficients of $V_{rt}V'_{tr}$, $V_{rr}V'_{tt}$ and $V_{rr}V'_{rr}$ give $\mathsf{F}_{rt}=\alpha$, $\mathsf{G}_{rt}=\beta$ and $\mathsf{G}_{rr}=\alpha+\beta$. □

As F_{rt} is symmetric under $x_r\leftrightarrow x_t$, "$\mathsf{F}_{rt}=0$ on $x_r=x_t$" means that x_{rt}^2 divides F_{rt}.

Theorem 9.154 (computer-assisted)**.** *Let (n,a) be one of the* 18 *pairs*

$$n=4:\ a=5,7;\qquad n=5:\ a=5,7,9,11;\qquad n=6:\ a=7,9,11,13,15,17;$$
$$n=7:\ a=7,9,11;\qquad n=8:\ a=9,11,13$$

(at $n=4,5,6$ every dominant odd $a\ge n$; at $n=7,8$ the odd a with $n\le a\le n+5$). Then $\kappa=n$ is not an eigenvalue of the pencil $\kappa\mathcal{Z}-\mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$.

Proof. Let φ be a kernel vector at $\kappa = n$, with data (F, G). Here $\mu = n^2 - a \geq 9$, so $\mu \neq 0$. (1) By Lemma 9.152, ϑ_{123} is a shift invariant solution of $(\star_n)$, symmetric in x_1, x_2, x_3 and in $x_4, \dots, x_n$, homogeneous of degree $a-2$. (2) At each of the 18 pairs the only such solution is 0 (a computation, Remark 9.156(1)). So $\vartheta_{123} = 0$, hence $\mathsf{H}_{123} = 0$, and by the equivariance $\mathsf{H}_{rwt} = 0$ for all distinct r, w, t; so $\mathsf{c}^{(pq)} = 0$ for all $p \neq q$. (3) By Lemma 9.46 ($\mu \neq 0$), $\mathsf{G}_{jt} = 0$ for $j \neq t$, at every x with distinct coordinates and hence as polynomials, and $\mathsf{G}\mathbf{1} = 0$ gives $\mathsf{G}_{jj} = 0$. (4) By Lemma 9.153, $\mathsf{F}_{rt} = 0$ on $x_r = x_t$. So $\mathsf{F} \in \mathsf{Z}(n, a)$. (5) With $\mathsf{G} = 0$ and $\mathsf{c} = 0$ we have $\Theta_l = 0$ for all l and $\psi_w^{(rt)} = \partial_w \mathsf{F}_{rt}$ ($w \notin \{r, t\}$), so for distinct p, q, i, j the generic row $R_{pq} = \Psi_{pq}(e_i, e_j)$ of Proposition 9.44, in the form shifted by $c = -x_p$, is a row on F alone:

$$\kappa\,(\partial_j - \partial_i)\mathsf{F}_{pq} - x_i'(\nabla_i\psi)_j + x_j'(\nabla_j\psi)_i = 0, \qquad x_l' := x_l - x_p, \quad \nabla_m := \partial_m - \delta_m, \quad \psi = \psi^{(pq)}.$$

At each of the 18 pairs these rows at $\kappa = n$ have no nonzero solution in $\mathsf{Z}(n, a)$ (a computation, Remark 9.156(1)). So $\mathsf{F} = 0$. (6) $\mathsf{F} = \mathsf{G} = 0$ gives $Q_X = 0$ at every regular diagonal X, so $\tilde{\varphi}(X, \cdot) = 0$ there; by conjugation invariance $\tilde{\varphi}(X, \cdot) = 0$ at every regular semisimple X, a dense set; so $\varphi = 0$. □

Corollary 9.155. *At the* 18 *pairs of Theorem 9.154,* $\mathrm{K}_n(n)_{(n^2-a-2,a,2)} = 0$*; so the conjugation type with conjugate* $(n^2 - a - 3, a - 1, 1)$ *does not occur in* $K_{B(n,2)}$*.*

Proof. By Corollary 9.31(b) and Lemma 9.32(ii), each $f \in \mathrm{K}_n(n)_\tau$ gives a kernel vector φ_f at $\kappa = n$, and $f \mapsto \varphi_f$ is injective; so $\mathrm{K}_n(n)_\tau = 0$ by Theorem 9.154, and the type has multiplicity $\dim \mathrm{K}_n(n)_\tau = 0$ by Lemma 9.32(iii). □

Remark 9.156. (1) (The computations.) Each row is the value, at a point of $\mathbb{F}_p^n$ ($p = 2^{31} - 1$) with $x_1, \dots, x_4$ distinct, of a polynomial with integer coefficients (the divided difference in $(\star_n)$ is the exact quotient by the monic $x_3 - x_4$, and $\kappa = n$ is an integer); so the rank of the row matrix modulo p is at most the rank over $\mathbb{Q}$ of the exact system, every computed dimension is an upper bound, and 0 certifies 0. Step (2): ϑ_{123} in the basis of monomials in $x_1^k + x_2^k + x_3^k$ ($k \leq 3$) and $\sum_{l \geq 4} x_l^k$ ($k \leq n-3$), free generators of the polynomials symmetric in x_1, x_2, x_3 and in $x_4, \dots, x_n$; rows: $(\star_n)$ and the shift invariance. Step (5): F_{12} in the basis of monomials in $x_1^k + x_2^k$ ($k \leq 2$) and $\sum_{l \geq 3} x_l^k$ ($k \leq n-2$), free generators of the polynomials symmetric in x_1, x_2 and in $x_3, \dots, x_n$; rows: the shift invariance, $\vartheta_{123} = 0$, $\mathsf{F}_{12} = 0$ at $x_1 = x_2$, and the rows of step (5) at $\kappa = n$. The dimensions of $\mathsf{Z}(n, a)$ at the 18 pairs, in the order of the theorem, are $1, 2$; $1, 2, 4, 8$; $1, 3, 6, 12, 21, 33$; $1, 2, 5$; $1, 3, 7$. The program passes its tests (stable controls exact, a sign mutation caught, the numerical rows of step (5) equal to a symbolic differentiation). Step (2) was also done exactly over $\mathbb{Q}$ by an independent program at 12 of the 14 pairs with $a \leq n + 5$, and step (5) by an independent program at every dominant $a \geq n$ for $n = 4, \dots, 7$ and for $a \leq 20$ at $n = 8$; all agree.

(2) Lemmas 9.152 and 9.153 and steps (1), (4), (6), and step (3) when $a \neq n^2$ ($\mu \neq 0$), hold for every $n \geq 4$ and $a \geq 3$; dominance is used only for Corollary 9.155 (τ must be a partition). The rows of step (5) are only part of the test: the coincident order-one rows $\Psi_{pq}(e_q, e_l)$ are further conditions at a given n. So a nonzero solution of the rows of step (5) in $\mathsf{Z}(n, a)$ does not by itself give an eigenvalue. At the pair $(n, a) = (4, 9)$, which is not dominant, step (2) gives 0 but step (5) leaves a line, so the rows of step (5) alone do not decide it (a direct computation of the whole test finds no eigenvalue 4 there; but see Theorem 9.159 and Remark 9.161(2)). Even a are not treated here (they are treated in Subsections 9.30 and 9.31).

(3) As statements about the pencil, all 18 cases of Theorem 9.154 are new; before, only some of them (small $d = a + 2$) had been computed (Remark 9.38(d)). Corollary 9.155 is the consequence of Conjecture 11.3 noted after it, proved here without the conjecture for these pairs; it was known for $n = 4, 5$ (Remark 9.10, Theorem 9.27) and $n = 6$ (Theorem 11.9), and is new for $n = 7, 8$ (six types; their depth is $a \geq n$, beyond Theorem 9.24; at $n = 7$ now inside Corollary 9.164). It gives no new bound on $d(n)$. Step (2) holds by hand at every n for odd dominant a (Theorem 9.173,

which with Corollary 9.160 gives Corollary 9.174); for even dominant a it fails, the shift invariant solutions forming a line (there one more row takes its place: Corollary 9.189); step (5) holds by hand at every pair (Theorem 9.159, by the coincident rows; at the six pairs with $a \le 2n-4$ also by the generic rows alone, Proposition 9.158).

9.27. **The coincident rows and step (5) at every n.** In this subsection $F=\mathbb{C}$, $b=2$, $n\ge 3$ and $a\ge n$, and we keep the notation of Subsection 9.26. We describe the space $\mathsf{Z}(n,a)$ of step (4) explicitly (Proposition 9.157), and show that the coincident order-one rows alone force $\mathsf{F}=0$ on it unless κ lies in an explicit finite set (Theorem 9.159). At $\kappa=n$ this covers every $a\le (n-1)^2$, so step (5) of the proof of Theorem 9.154 holds by hand for every n and every dominant $a\ge n$, of either parity (Theorem 9.159; Corollary 9.160 for the consequence).

Notation. $\mathbb{C}[x]^{S_n}[\tau]$ is the ring of polynomials in τ whose coefficients are symmetric polynomials in x. Put $\mathfrak{f}(\tau):=\prod_l(\tau-x_l)$ (so $\mathfrak{f}(\tau)=\det(\tau-X)$ at $X=\operatorname{diag}(x)$). For $h=h(\tau;x)$ write $h':=\partial h/\partial\tau$, and $h_w:=\partial_w h$, $h_{ij}:=\partial_i\partial_j h$ at fixed τ; $y_w:=1/(\tau-x_w)$, so $\partial_w\mathfrak{f}=-y_w\mathfrak{f}$. Divided differences $h[s,t]$, $h[s,t,u]$ are taken in τ. h is *shift invariant* if $h(\tau+c;x+c\mathbf{1})=h(\tau;x)$, that is, if $\mathbb{T}h=0$ with $\mathbb{T}:=\partial_\tau+\sum_l\partial_l$; $\mathbb{T}$ kills $\mathfrak{f}$ and $\mathfrak{f}'$ and commutes with ∂_τ. Below, d is the degree of $\mathfrak{c}$ (not $a+b$), so $a=n+d$. For $\mathfrak{c}\in\mathbb{C}[x]^{S_n}[\tau]$ put

$$\mathfrak{H}_\mathfrak{c}(\tau):=\int_0^\tau \mathfrak{c}(\sigma)\mathfrak{f}(\sigma)\,d\sigma,\qquad \mathsf{F}^\mathfrak{c}_{rt}:=\mathfrak{H}_\mathfrak{c}[x_r,x_t]\quad(r\ne t).$$

Proposition 9.157 (the space $\mathsf{Z}(n,a)$)**.** *(a) $\mathsf{F}^\mathfrak{c}$ has $\mathsf{H}_{rwt}=0$ for all distinct r,w,t, and $\mathsf{F}^\mathfrak{c}_{rt}=0$ on $x_r=x_t$; it is shift invariant if $\mathfrak{c}$ is, and homogeneous of degree $d+n$ if $\mathfrak{c}$ is homogeneous of degree d (in τ and x together).*

(b) Every $\mathsf{F}\in\mathsf{Z}(n,a)$ is $\mathsf{F}^\mathfrak{c}$ for a shift invariant $\mathfrak{c}$, homogeneous of degree $a-n$.

(c) $\mathsf{F}^\mathfrak{c}=0$ if and only if $\mathfrak{c}=\mathfrak{f}\mathfrak{q}'+2\mathfrak{f}'\mathfrak{q}$ for some $\mathfrak{q}\in\mathbb{C}[x]^{S_n}[\tau]$. Such $\mathfrak{c}\ne 0$ have τ-degree at least $n-1$.

(d) Let $\mathfrak{c}$ be shift invariant and homogeneous of degree d, $\mathfrak{H}_\mathfrak{c}(\tau)=\sum_k c_k\tau^k$, and read each c_k as a conjugation-invariant polynomial function of X (a symmetric polynomial in its eigenvalues). Then

$$\varphi_\mathfrak{c}(X,Y):=\sum_k c_k(X)\operatorname{tr}\Big(\sum_{i+j=k-1}X^iYX^jY\Big),$$

restricted to $\mathfrak{sl}_n\times\mathfrak{sl}_n$, lies in $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ ($a=d+n$) and has data $(\mathsf{F}^\mathfrak{c},0)$.

Proof. (a) $\mathsf{H}_{rt}=\mathfrak{H}_\mathfrak{c}(x_r)-\mathfrak{H}_\mathfrak{c}(x_t)$ is a coboundary, so $\mathsf{H}_{rwt}=0$. At $x_r=x_t=u$, $\mathsf{F}^\mathfrak{c}_{rt}=\mathfrak{H}'_\mathfrak{c}(u)=\mathfrak{c}(u)\mathfrak{f}(u)=0$. If $\mathfrak{c}$ is shift invariant, substituting $\sigma\mapsto\sigma+c$ gives $\mathfrak{H}_\mathfrak{c}(\tau+c;x+c\mathbf{1})=\mathfrak{H}_\mathfrak{c}(\tau;x)-\mathfrak{H}_\mathfrak{c}(-c;x)$, and divided differences kill the constant. If $\mathfrak{c}$ is homogeneous of degree d, then $\mathfrak{H}_\mathfrak{c}$ is homogeneous of degree $d+n+1$, so $\mathsf{F}^\mathfrak{c}$ has degree $d+n$.

(b) *Coboundary.* $\mathsf{H}_{rwt}=0$ gives $\mathsf{H}_{rs}=\mathsf{H}_{r1}+\mathsf{H}_{1s}$ for $r,s\ne 1$; with $\epsilon_r:=\mathsf{H}_{r1}$ ($\epsilon_1:=0$), $\mathsf{H}_{rs}=\epsilon_r-\epsilon_s$. Put $\eta_r:=\frac{1}{n}\sum_s\mathsf{H}_{rs}$; then $\mathsf{H}_{rs}=\eta_r-\eta_s$, and (η_r) is S_n-equivariant. η_1 is a polynomial symmetric in $x_2,\dots,x_n$, hence a polynomial in x_1 and the elementary symmetric functions of x (as $e_m(x_2,\dots,x_n)=\sum_i(-x_1)^ie_{m-i}(x)$): $\eta_1=\eta(x_1)$ with $\eta\in\mathbb{C}[x]^{S_n}[\tau]$, and $\eta_r=\eta(x_r)$ by equivariance. So $\mathsf{F}_{rt}=\eta[x_r,x_t]$. *The collisions.* On $x_1=x_2=u$, $\mathsf{F}_{12}=\eta'(u)$; so $\eta'(x_1;x)$ vanishes on $x_1=x_2$, hence (symmetry in $x_2,\dots,x_n$) on every $x_1=x_j$, and $\mathfrak{f}'(x_1)=\prod_{j\ge2}(x_1-x_j)$ divides it: $\eta'(x_1)=\mathfrak{f}'(x_1)\mathfrak{s}(x_1)$ with $\mathfrak{s}\in\mathbb{C}[x]^{S_n}[\tau]$ (as above). Then $\eta'-\mathfrak{f}'\mathfrak{s}$ vanishes at every x_r; dividing by the monic $\mathfrak{f}$ (the remainder has degree $<n$ and n roots), $\eta'=\mathfrak{f}'\mathfrak{s}+\mathfrak{f}\mathfrak{t}$. As $\mathfrak{f}'\mathfrak{s}=(\mathfrak{f}\mathfrak{s})'-\mathfrak{f}\mathfrak{s}'$, $\eta=\mathfrak{f}\mathfrak{s}+\mathfrak{H}_\mathfrak{c}+\text{const}$ with $\mathfrak{c}:=\mathfrak{t}-\mathfrak{s}'$, and $\mathfrak{f}(x_r)=0$ gives $\mathsf{F}=\mathsf{F}^\mathfrak{c}$. *Shift invariance.* $\mathbb{T}\mathfrak{H}_\mathfrak{c}-\mathfrak{H}_{\mathbb{T}\mathfrak{c}}=\mathfrak{c}(0)\mathfrak{f}(0)$ is constant in τ, and $\sum_l\partial_l[g(x_r;x)]=(\mathbb{T}g)(x_r;x)$ for $g\in\mathbb{C}[x]^{S_n}[\tau]$; so $(\sum_l\partial_l)\mathsf{F}^\mathfrak{c}=\mathsf{F}^{\mathbb{T}\mathfrak{c}}$. As F is shift invariant, $\mathsf{F}^{\mathbb{T}\mathfrak{c}}=0$, so $\mathbb{T}\mathfrak{c}=\mathfrak{f}\mathfrak{r}'+2\mathfrak{f}'\mathfrak{r}$ by (c). In the coordinates $\tau-p_1/n$, the centred power sums $P_2,\dots,P_n$ and p_1, $\mathbb{T}=n\,\partial/\partial p_1$, so $\mathbb{T}$ is onto: choose $\mathfrak{q}$ with $\mathbb{T}\mathfrak{q}=\mathfrak{r}$. Then $\mathbb{T}(\mathfrak{f}\mathfrak{q}'+2\mathfrak{f}'\mathfrak{q})=\mathbb{T}\mathfrak{c}$, so $\mathfrak{c}-(\mathfrak{f}\mathfrak{q}'+2\mathfrak{f}'\mathfrak{q})$ is shift invariant and has the same data, by (c). Taking homogeneous components gives degree $a-n$ (by (a)).

(c) $\mathsf{F}^{\mathfrak{c}}=0$ if and only if $\mathfrak{H}_{\mathfrak{c}}-\mathfrak{H}_{\mathfrak{c}}(x_1)$ vanishes at every x_r, that is, $\mathfrak{H}_{\mathfrak{c}}=\mathfrak{f}\mathfrak{q}_0+\text{const}$. Differentiating, $\mathfrak{c}\mathfrak{f}=\mathfrak{f}'\mathfrak{q}_0+\mathfrak{f}\mathfrak{q}_0'$, so $\mathfrak{f}\mid\mathfrak{f}'\mathfrak{q}_0$, so $\mathfrak{f}\mid\mathfrak{q}_0$ ($\mathfrak{f}$ and $\mathfrak{f}'$ are coprime), and $\mathfrak{q}_0=\mathfrak{f}\mathfrak{q}$ gives $\mathfrak{c}=\mathfrak{f}\mathfrak{q}'+2\mathfrak{f}'\mathfrak{q}$. Conversely $\mathfrak{H}_{\mathfrak{f}\mathfrak{q}'+2\mathfrak{f}'\mathfrak{q}}=\mathfrak{f}^2\mathfrak{q}-(\mathfrak{f}^2\mathfrak{q})(0)$, whose divided differences at the roots of $\mathfrak{f}$ vanish. If $\mathfrak{q}$ has leading term $q\tau^m$, then $\mathfrak{c}$ has leading term $(m+2n)q\tau^{n+m-1}$, so its τ-degree is $n+m-1\geq n-1$.

(d) At $X=\operatorname{diag}(x)$, $\operatorname{tr}(X^iYX^jY)=\sum_{r,t}x_r^ix_t^jY_{rt}Y_{tr}$, so $\sum_{i+j=k-1}$ gives $\frac{x_r^k-x_t^k}{x_r-x_t}Y_{rt}Y_{tr}$ for $r\neq t$ and $kx_r^{k-1}Y_{rr}^2$. Summing with the c_k: the coefficient of $Y_{rt}Y_{tr}$ is $\mathsf{F}^{\mathfrak{c}}_{rt}$, that of Y_{rr}^2 is $\mathfrak{H}_{\mathfrak{c}}'(x_r)=0$, and there are no terms $Y_{rr}Y_{tt}$ ($r\neq t$) (this is the divided-difference (Daleckiĭ–Kreĭn) structure of Step 1 of the proof of Proposition 9.40, [LS01]). So $\varphi_{\mathfrak{c}}(X,Y+cI)=\varphi_{\mathfrak{c}}(X,Y)$, and shift invariance of $\mathfrak{c}$ gives $\varphi_{\mathfrak{c}}(X+cI,Y)=\varphi_{\mathfrak{c}}(X,Y)$; this holds at diagonal X, hence everywhere by conjugation and density. So $\varphi_{\mathfrak{c}}(X_0,Y_0)=\varphi_{\mathfrak{c}}(X,Y)$: the restriction φ to $\mathfrak{sl}_n\times\mathfrak{sl}_n$ has $\tilde{\varphi}=\varphi_{\mathfrak{c}}$, with data $(\mathsf{F}^{\mathfrak{c}},0)$. As $\mathsf{G}=0$, $Q_X(X,\cdot)=\sum_t(\mathsf{G}x)_t(\cdot)_{tt}=0$, so $\mathcal{E}\varphi=0$. □

For $\mathfrak{c}$ as above, $h:=\mathfrak{H}_{\mathfrak{c}}$ and $i\neq j$, put

$$\mathfrak{K}_{ij}:=\kappa\,(h_j-h_i)-x_{ij}\,h_{ij}\in\mathbb{C}[x][\tau],\qquad \mathfrak{f}_{ij}:=\frac{\mathfrak{f}}{(\tau-x_i)(\tau-x_j)},$$

so $\mathfrak{K}_{ji}=-\mathfrak{K}_{ij}$. Differentiating under the integral, $\mathfrak{K}_{ij}'=\mathfrak{f}\big[\kappa(\partial_j-\partial_i)\mathfrak{c}-x_{ij}\partial_i\partial_j\mathfrak{c}+x_{ij}(y_j\partial_i\mathfrak{c}+y_i\partial_j\mathfrak{c})+(\kappa-1)x_{ij}y_iy_j\mathfrak{c}\big]$, as $y_i-y_j=x_{ij}y_iy_j$.

Proposition 9.158 (generic rows)**.** *Let $n\geq 4$, $\mathfrak{c}$ shift invariant and homogeneous of degree $d\leq n-4$, $\mathsf{F}=\mathsf{F}^{\mathfrak{c}}$ and $\mathsf{G}=0$. If every generic order-one row vanishes at κ and $(n-1)(\kappa-1)\neq d$, then $\mathsf{F}=0$.*

Proof. With $\mathsf{G}=0$ and $\mathsf{c}=0$, $\psi_w^{(pq)}=h_w[x_p,x_q]$ for $w\notin\{p,q\}$, so $(\delta_i\psi)_j=-h_j[x_p,x_q,x_i]+h_j[x_p,x_q,x_i]=0$ (second divided differences are symmetric), and the generic row $\Psi_{pq}(e_i,e_j)$ is $\mathfrak{K}_{ij}[x_p,x_q]$. Write $\mathfrak{K}_{ij}'=\mathfrak{f}_{ij}\,\mathfrak{m}_{ij}$ with $\mathfrak{m}_{ij}=-x_{ij}\mathfrak{e}_{ij}$,

$$\mathfrak{e}_{ij}:=(1-\kappa)\mathfrak{c}+\big[\kappa D_{ij}\mathfrak{c}+\partial_i\partial_j\mathfrak{c}\big](\tau-x_i)(\tau-x_j)-(\tau-x_j)\partial_j\mathfrak{c}-(\tau-x_i)\partial_i\mathfrak{c},\qquad D_{ij}\mathfrak{c}:=\frac{(\partial_j-\partial_i)\mathfrak{c}}{x_j-x_i}.$$

As $\mathfrak{c}$ is homogeneous of degree d in τ and x, every term of $\mathfrak{e}_{ij}$ has τ-degree at most d. The rows for $p,q\notin\{i,j\}$ say that $\mathfrak{K}_{ij}$ takes one value c at the $n-2$ roots of $\mathfrak{f}_{ij}$, where also $\mathfrak{K}_{ij}'=0$; so $\mathfrak{K}_{ij}-c=\mathfrak{f}_{ij}^2\mathfrak{r}$ and $\mathfrak{m}_{ij}=2\mathfrak{f}_{ij}'\mathfrak{r}+\mathfrak{f}_{ij}\mathfrak{r}'$. If $\mathfrak{r}\neq 0$ has τ-degree e, the right side has leading coefficient $(2(n-2)+e)$ times that of $\mathfrak{r}$, so τ-degree $n-3+e\geq n-3>d$; so $\mathfrak{r}=0$ and $\mathfrak{e}_{ij}=0$. At $\tau=x_i$: $(\kappa-1)\mathfrak{c}(x_i)+x_{ij}(\partial_j\mathfrak{c})(x_i)=0$ for every $j\neq i$. By Euler's identity and shift invariance, $\sum_l(x_i-x_l)\partial_l\mathfrak{c}=(\tau-x_i)\mathfrak{c}'-d\,\mathfrak{c}$, so summing over $j\neq i$ gives $[(n-1)(\kappa-1)-d]\,\mathfrak{c}(x_i)=0$. So $\mathfrak{c}$, of τ-degree at most $d<n$, vanishes at the n points x_i: $\mathfrak{c}=0$. □

Theorem 9.159 (the coincident rows)**.** *Let $n\geq 3$, $\kappa\in\mathbb{C}$, $\mathfrak{c}$ shift invariant and homogeneous of degree $d\geq 0$, $\mathsf{F}=\mathsf{F}^{\mathfrak{c}}$ and $\mathsf{G}=0$. Suppose that every coincident order-one row $\Psi_{pq}(e_q,e_l)$ (p,q,l distinct) vanishes at κ, identically in x. If*

$$\text{(ND)}\qquad (n-2)(\kappa-k-1)\neq d-kn\qquad \text{for every integer } k \text{ with } 0\leq k\leq d/n,$$

then $\mathsf{F}=0$. At $\kappa=n$, (ND) *holds whenever $a=n+d\leq(n-1)^2$.*

Proof. (1) The coincident row. Let $h=\mathfrak{H}_{\tilde{\mathfrak{c}}}$ for any $\tilde{\mathfrak{c}}$ with $\mathsf{F}^{\tilde{\mathfrak{c}}}=\mathsf{F}$. With $\mathsf{G}=0$ and $\mathsf{c}=0$: $\Theta=0$, $\psi_w^{(pq)}=h_w[x_p,x_q]$ for $w\notin\{p,q\}$, and $\psi_q^{(pq)}=\partial_q\mathsf{F}_{pq}-\mathsf{F}_{pq}/x_{pq}=h_q[x_p,x_q]$, as $\partial_q\mathsf{F}_{pq}=h_q[x_p,x_q]-h'(x_q)/x_{pq}+\mathsf{F}_{pq}/x_{pq}$ and $h'(x_q)=\tilde{\mathfrak{c}}(x_q)\mathfrak{f}(x_q)=0$. So $\psi_w^{(rs)}=h_w[x_r,x_s]$ for every $w\neq r$. In the (e_q,e_l) component of Ψ_{pq} only the term $m=l$ of the sum contributes:

$$\Psi_{pq}(e_q,e_l)=\kappa(\psi_l-\psi_q)-x_q\Big(\partial_q\psi_l-\frac{\psi_l}{x_{pq}}\Big)+x_l\big(\partial_l\psi_q-(\delta_l\psi)_q\big).$$

Here $\partial_q\psi_l - \psi_l/x_{pq} = h_{lq}[x_p,x_q] - h_l'(x_q)/x_{pq}$, and $h_l' = (\partial_l\tilde{\mathfrak{c}})\mathfrak{f} - y_l\tilde{\mathfrak{c}}\mathfrak{f}$ vanishes at $\tau = x_q$ ($q \neq l$); $\partial_l\psi_q = h_{ql}[x_p,x_q]$; and $(\delta_l\psi)_q = -h_q[x_p,x_q,x_l] + h_q[x_p,x_l,x_q] = 0$. So $\Psi_{pq}(e_q,e_l) = \mathfrak{K}_{ql}[x_p,x_q]$, with $\mathfrak{K}$ built from this h.

(2) One value at all n points. Fix $i \neq j$. The rows with $(q,l) = (i,j)$ and (j,i) and $p \notin \{i,j\}$ give $\mathfrak{K}_{ij}(x_p) = \mathfrak{K}_{ij}(x_i) = \mathfrak{K}_{ij}(x_j)$: $\mathfrak{K}_{ij}$ takes one value $c_{ij} \in \mathbb{C}[x]$ at $x_1,\dots,x_n$.

(3) The form $\mathfrak{f}^k\mathfrak{b}$. Let $\mathfrak{b} \in \mathbb{C}[x]^{S_n}[\tau]$, $k \geq 0$, and $h = \mathfrak{H}_{\mathfrak{f}^k\mathfrak{b}}$. As $\partial_w\mathfrak{f}^k = -ky_w\mathfrak{f}^k$,

$$\mathfrak{K}_{ij}' = \mathfrak{f}^k\mathfrak{f}_{ij}\,\mathfrak{m}_{ij}^{(k)}[\mathfrak{b}],$$

$$\mathfrak{m}_{ij}^{(k)}[\mathfrak{b}] := (\tau-x_i)(\tau-x_j)\big[\kappa(\partial_j-\partial_i)\mathfrak{b} - x_{ij}\partial_i\partial_j\mathfrak{b}\big]$$
$$+(k+1)x_{ij}\big[(\tau-x_j)\partial_j\mathfrak{b} + (\tau-x_i)\partial_i\mathfrak{b}\big] + (k+1)(\kappa-k-1)x_{ij}\mathfrak{b}$$

(the coefficient of $y_iy_j\mathfrak{b}$ is $\kappa k - k^2 - 2k + \kappa - 1 = (k+1)(\kappa-k-1)$). At $\tau = x_i$, $\mathfrak{m}_{ij}^{(k)}[\mathfrak{b}](x_i) = (k+1)x_{ij}\big[x_{ij}(\partial_j\mathfrak{b})(x_i) + (\kappa-k-1)\mathfrak{b}(x_i)\big]$.

(4) One rung. Suppose $\mathsf{F} = \mathsf{F}^{\mathfrak{f}^k\mathfrak{b}}$ with $\mathfrak{b}$ shift invariant and homogeneous of degree $d_\mathfrak{b} := d-kn \geq 0$, and that (ND) holds at this k. By (1) and (2), with $h = \mathfrak{H}_{\mathfrak{f}^k\mathfrak{b}}$, $K := \mathfrak{K}_{ij} - c_{ij} \in \mathbb{C}[x][\tau]$ vanishes at every x_r, and $K' = \mathfrak{f}^k\mathfrak{f}_{ij}\mathfrak{m}_{ij}^{(k)} = \mathfrak{f}_{ij}^{k+1}(\tau-x_i)^k(\tau-x_j)^k\mathfrak{m}_{ij}^{(k)}$. By Taylor expansion over $\mathbb{C}[x]$, K vanishes to order at least $k+2$ at each root of $\mathfrak{f}_{ij}$ and $k+1$ at x_i and x_j; dividing by these monic, pairwise coprime factors (in $\mathbb{C}(x)[\tau]$; the quotient lies in $\mathbb{C}[x][\tau]$ as the divisor is monic), $K = \mathfrak{f}^{k+1}\mathfrak{f}_{ij}\mathfrak{l}_{ij}$ with $\mathfrak{l}_{ij} \in \mathbb{C}[x][\tau]$. Differentiating and cancelling $\mathfrak{f}^k$: $\mathfrak{f}_{ij}\mathfrak{m}_{ij}^{(k)} = (k+1)\mathfrak{f}'\mathfrak{f}_{ij}\mathfrak{l}_{ij} + \mathfrak{f}(\mathfrak{f}_{ij}\mathfrak{l}_{ij})'$. At $\tau = x_i$ ($\mathfrak{f}(x_i) = 0$, $\mathfrak{f}'(x_i) = x_{ij}\mathfrak{f}_{ij}(x_i)$), cancelling $\mathfrak{f}_{ij}(x_i)$ and $(k+1)x_{ij}$ in the domain $\mathbb{C}[x]$,

$$(*_k)\qquad x_{ij}(\partial_j\mathfrak{b})(x_i) + (\kappa-k-1)\,\mathfrak{b}(x_i) = \mathfrak{f}_{ij}(x_i)\,\mathfrak{l}_{ij}(x_i), \qquad \mathfrak{f}_{ij}(x_i) = \prod_{m\neq i,j}(x_i-x_m).$$

By Euler's identity and shift invariance, $\sum_{j\neq i} x_{ij}(\partial_j\mathfrak{b})(x_i) = -d_\mathfrak{b}\mathfrak{b}(x_i)$. Fix $l \neq i$ and sum $(*_k)$ over $j \notin \{i,l\}$: each right side has the factor $x_i - x_l$, and the left sides add to $[(n-1)(\kappa-k-1) - d_\mathfrak{b}]\mathfrak{b}(x_i) - [x_{il}(\partial_l\mathfrak{b})(x_i) + (\kappa-k-1)\mathfrak{b}(x_i)]$, which is $[(n-2)(\kappa-k-1) - d_\mathfrak{b}]\mathfrak{b}(x_i)$ modulo $x_i - x_l$ (every cancellation above happened in a domain, so $(*_k)$ is an identity in $\mathbb{C}[x]$ and may be reduced modulo $x_i - x_l$). By (ND), $x_i - x_l$ divides $\mathfrak{b}(x_i;x)$, for every $l \neq i$; as these are pairwise non-associate primes, $\mathfrak{b}(x_i) = \mathfrak{f}'(x_i)N_i$ with $N_i \in \mathbb{C}[x]$. As in (b) of Proposition 9.157, $N_i = \mathfrak{u}(x_i)$ with $\mathfrak{u} \in \mathbb{C}[x]^{S_n}[\tau]$, which we take of τ-degree $< n$; it is then unique, hence shift invariant and homogeneous. Then $\mathfrak{b} - \mathfrak{f}'\mathfrak{u}$ vanishes at every x_r, so $\mathfrak{b} = \mathfrak{f}'\mathfrak{u} + \mathfrak{f}\mathfrak{v}$ with $\mathfrak{v}$ shift invariant and homogeneous. With $\mathfrak{q} := \mathfrak{f}^k\mathfrak{u}/(k+2)$, $\mathfrak{f}\mathfrak{q}' + 2\mathfrak{f}'\mathfrak{q} = \mathfrak{f}^k\mathfrak{f}'\mathfrak{u} + \mathfrak{f}^{k+1}\mathfrak{u}'/(k+2)$, so

$$\mathfrak{f}^k\mathfrak{b} = (\mathfrak{f}\mathfrak{q}' + 2\mathfrak{f}'\mathfrak{q}) + \mathfrak{f}^{k+1}\mathfrak{b}_1, \qquad \mathfrak{b}_1 := \mathfrak{v} - \mathfrak{u}'/(k+2),$$

with $\mathfrak{b}_1$ shift invariant, homogeneous of degree $d-(k+1)n$, and $\mathsf{F}^{\mathfrak{f}^k\mathfrak{b}} = \mathsf{F}^{\mathfrak{f}^{k+1}\mathfrak{b}_1}$ by Proposition 9.157(c).

(5) The ladder. Start with $k=0$, $\mathfrak{b} = \mathfrak{c}$. Each rung keeps F and raises k; after the rung $k = \lfloor d/n \rfloor$ the degree is negative, so $\mathfrak{b}_1 = 0$ and $\mathsf{F} = 0$. At $\kappa = n$, equality in (ND) means $d = (n-1)(n-2) + 2k$, so (ND) holds for $d < (n-1)(n-2)$, that is, $a \leq (n-1)^2$. □

Corollary 9.160. *Let $n \geq 4$ and $n \leq a \leq (n^2-2)/2$. If every kernel vector at $\kappa = n$ has $\vartheta_{123} = 0$ (step (2) of the proof of Theorem 9.154), then $\kappa = n$ is not an eigenvalue of the pencil on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$.*

Proof. With $\vartheta_{123} = 0$, steps (2)–(4) of the proof of Theorem 9.154 (with $\mu = n^2 - a \neq 0$) give $\mathsf{G} = 0$ and $\mathsf{F} \in \mathsf{Z}(n,a)$; by Proposition 9.157(b), $\mathsf{F} = \mathsf{F}^{\mathfrak{c}}$ with $\mathfrak{c}$ shift invariant of degree $a-n$. The coincident rows vanish for a kernel vector, so Theorem 9.159 at $\kappa = n$ gives $\mathsf{F} = 0$ (as $(n^2-2)/2 \leq (n-1)^2$), and step (6) gives $\varphi = 0$. □

Remark 9.161. (1) Equivalently (the contrapositive of Theorem 9.159): a nonzero $\mathsf{F}^{\mathfrak{c}}$ with $\mathsf{G} = 0$ that passes the coincident rows needs $\kappa = k + 1 + (d-kn)/(n-2)$ for some integer $0 \leq k \leq d/n$.

Each such value with $d = kn$ is reached: with $\mathfrak{b} = 1$, (3) gives $\mathfrak{m}_{ij}^{(k)}[1] = (k+1)(\kappa - k - 1)x_{ij}$, so at $\kappa = k+1$ every $\mathfrak{K}_{ij}$ is constant and $\mathsf{F}^{\mathfrak{f}^k} \neq 0$ ($a = (k+1)n$, $\mu = 0$) passes every order-one row; $k = 0$ is $\mathfrak{c} = 1$ at $\kappa = 1$, $k = 1$ is $\mathsf{F}^{\mathfrak{f}}$ at $(4,8)$ and $(5,10)$ at $\kappa = 2$, and $k = 2$ is $\mathsf{F}^{\mathfrak{f}^2}$ at $(3,9)$ at $\kappa = 3$. So the exceptional set cannot be cut back to its values with $k = 0$. ($\mathsf{F}^{\mathfrak{f}^k} \neq 0$ by Proposition 9.157(c): if $\mathfrak{f}^m = \mathfrak{f}\mathfrak{q}' + c\mathfrak{f}'\mathfrak{q}$ with a constant $c \geq 2$ and $m \geq 1$, then $\mathfrak{f} \mid \mathfrak{q}$, and $\mathfrak{q} = \mathfrak{f}\mathfrak{q}_1$ gives $\mathfrak{f}^{m-1} = \mathfrak{f}\mathfrak{q}_1' + (c+1)\mathfrak{f}'\mathfrak{q}_1$; at $m = 0$ the right side has τ-degree at least $n - 1 \geq 1$ or is 0.)

(2) The same proof as for Corollary 9.160 works for $n \leq a \leq (n-1)^2$; only step (2) and Corollary 9.155 use dominance. The hand proof of step (5) uses the coincident rows, not the generic rows of step (5) of Theorem 9.154; both are necessary conditions. The coincident rows are not limits of the generic ones at a given n: Lemma 9.74(ii) is a statement about the formal system of Subsection 9.17, "whose specializations at the integers $n \geq a+6$ are the pencils". For example, at $(n,a) = (4,5)$, $\kappa = 2$ and $\mathfrak{c} = \tau - p_1/4$, all generic rows vanish, but $(*_k)$ at $(i,j) = (1,2)$, $k = 0$ reads $(2x_1 - x_3 - x_4)/4 = (x_1 - x_3)(x_1 - x_4)\mathfrak{l}_{12}(x_1)$, which is impossible by degree. For two-point data read at n, Lemma 9.74(ii) itself is an identity between row formulas in the slots and the p_k, and it specializes at every n (Lemma 9.179); this remark concerns vanishing, not that identity.

(3) (Computations, supporting.) The generic rows of step (5) alone have a nonzero solution on $\mathsf{Z}(n,a)$ at five pairs. At the four of them outside the exceptional set of Theorem 9.159 ($(4,5)$ at $\kappa = 2$, $(4,7)$ at $\kappa = 3$, $(4,9)$ at $\kappa = 4$, $(5,10)$ at $\kappa = 3$), exact computations with the coincident rows added give 0, by two independent programs, one built from the definitions in Proposition 9.44. The fifth, $(4,8)$ at $\kappa = 3$, lies in the exceptional set $\{3,2\}$; there the coincident rows alone give 0 (a scan), and a direct computation of the whole test has no root $\kappa = 3$. In 29 pairs no solution violates Theorem 9.159, and every surviving solution has $\mu = 0$.

(4) Theorem 9.159 and Corollary 9.160 give no bound on $d(n)$ by themselves: step (2) holds for odd dominant a (Theorem 9.173; so Corollary 9.174), and for even a the argument breaks at step (2), where one more row takes its place (Theorems 9.180 and 9.188; so Corollary 9.189); step (5) itself holds for both parities. Nothing here says that a value in the exceptional set of Theorem 9.159 is an eigenvalue.

9.28. **The layer $\tau_3 = 2$ at $n = 7$.** In this subsection $F = \mathbb{C}$, $b = 2$ and $n = s = 7$, so $N = 46$. We show, with a computer, that $\kappa = 7$ is not an eigenvalue of the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(7)$ for any a with $3 \leq a \leq 23$ (Theorem 9.163): no nonzero φ there has $7\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$. Hence $\mathrm{K}_7(7)_\tau = 0$ for every $\tau = (47-a, a, 2)$ with $a \geq 3$, and every conjugation type κ of $K_{B(7,2)}$ with $\kappa_3' \leq 1$ is $(3, 1^{43})$ (Corollary 9.164). This is the consequence of Conjecture 11.3 noted after it, at $n = 7$, proved without the conjecture. At $a = 2$, $\kappa = 7$ is an eigenvalue (Theorem 9.37). For $a \geq 24$, τ is not a partition, so $\mathrm{K}_7(7)_\tau = 0$ trivially; there the pencil is not tested, and $\mathcal{F}^{\mathcal{E}}_{(a,2)}(7)$ can be nonzero.

The computation. Let $T(a,b)$, ev and $\mathcal{E}_{\mathrm{f}}$ be as in Subsection 9.10, with $\mathfrak{sl}_7$ in place of $\mathfrak{sl}_5$; so $\mathrm{ev}(\ker \mathcal{E}_{\mathrm{f}}) \subseteq \mathcal{F}^{\mathcal{E}}_{(a,b)}(7)$. Fix a and a prime $p \neq 7$, and put $h := \dim \mathcal{F}^{\mathcal{E}}_{(a,2)}(7)$. The program does three things.

(M1) It checks that $\mathcal{E}_{\mathrm{f}} \colon \mathbb{Q}^{T(a,2)} \to \mathbb{Q}^{T(a+1,1)}$ is onto modulo p, that is, has rank $|T(a+1,1)|$ there.

(M2) It chooses h vectors of $H_p := \ker(\mathcal{E}_{\mathrm{f}} \bmod p)$ whose values, as functions at $n = 7$, are independent modulo p at some points of $M_7(\mathbb{F}_p)^2$ with trace 0 (a vector is kept if its values are not in the span of the values of the earlier ones).

(M3) At some base points $(A_2, A_3) \in M_7(\mathbb{F}_p)^2$ and vectors u, y_1, y_2, y_3 it computes, for each chosen function φ, the value $\Omega f_7(uy_1^T, uy_2^T, uy_3^T)$ from its definition (Subsection 9.11), exactly in $\mathbb{F}_p[t_1,t_2,t_3]/(t_1^2, t_2^2, t_3^2)$; by Lemma 9.30 it is $7\mathcal{Z}(\varphi) - \mathcal{J}(\varphi)$ (over $\mathbb{Q}$, at integer lifts of these points; see the proof of Lemma 9.162(ii)). Let r_p be the rank modulo p of the matrix of these values.

Lemma 9.162 (the certificate)**.** *If* (M1) *holds,* (M2) *finds h vectors and $r_p = h$, then no nonzero $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,2)}(7)$ has $7\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ at every base point and for all u, y_1, y_2, y_3.*

Proof. (i) By (M1) the rank of $\mathcal{E}_{\mathrm{f}}$ over $\mathbb{Q}$ is maximal too, so H_p is the reduction of the saturated lattice $\ker\mathcal{E}_{\mathrm{f}}\cap\mathbb{Z}^{T(a,2)}$, and the vectors chosen in (M2) lift to integer vectors $c_1,\dots,c_h\in\ker\mathcal{E}_{\mathrm{f}}$. Put $f_i:=\mathrm{ev}(c_i)\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(7)$. Lift the points of (M2) to integer matrices; their trace is 0 only modulo p, so read f_i there as $\tilde f_i(X,Y)=f_i(X_0,Y_0)$. Its values lie in $\mathbb{Z}[1/7]$, and modulo p they are the computed values, as the points have trace 0 modulo p. So the values of $f_1,\dots,f_h$ form a p-integral matrix whose reduction has rank h; hence $f_1,\dots,f_h$ are independent over $\mathbb{Q}$, so over $\mathbb{C}$, and, as $h=\dim\mathcal{F}^{\mathcal{E}}_{(a,2)}(7)$, they are a basis. (ii) Let $\varphi\neq0$ have $7\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$. Lift the points of (M3) to integers. At these points the conditions $\sum_i q_i\,(7\mathcal{Z}-\mathcal{J})(f_i)=0$ on the coordinates of $\varphi=\sum_i q_i f_i$ have rational coefficients (Lemma 9.30 holds at every base point of M_7^2, and the rows evaluate φ through $\tilde\varphi$, as the definition does), and φ is a nonzero solution; so there is a nonzero rational solution, and we may take it with $q_i\in\mathbb{Z}_{(p)}$, not all divisible by p. The entries are p-integral (the only denominator is 7, from the traceless parts) and reduce to the computed ones. So $(q_i \bmod p)\neq0$ is in the kernel of a matrix of rank $r_p=h$ with h columns, a contradiction. □

This follows the proof of Lemma 9.28, at $n=7$ and with $r=h$, using only the rows (S1) at $P_1=I$ in the pencil form; the difference is that the points have trace 0 only modulo p and are read through their traceless parts.

Theorem 9.163 (computer-assisted)**.** *For every a with $3\le a\le23$, $\kappa=7$ is not an eigenvalue of the pencil $\kappa\mathcal{Z}-\mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(7)$.*

Proof. For each such a, with $p=469762049$, the program gives (M1), h vectors in (M2), and $r_p=h$ in (M3) at $3h+20$ random base points (Remark 9.165(1)). So Lemma 9.162 applies. □

Corollary 9.164. *(a) $\mathrm{K}_7(7)_\tau=0$ for every $\tau=(47-a,a,2)$ with $a\ge3$.*
(b) Every conjugation type κ of $K_{B(7,2)}$ with $\kappa'_3\le1$ is $(3,1^{43})$, and it occurs once.

Proof. (a) For $a\ge24$, $47-a<a$, so τ is not a partition and $\mathrm{R}_7(7)_\tau=0$ (Lemma 9.32(ii)). Let $3\le a\le23$ and $f\in\mathrm{K}_7(7)_\tau$. By Lemma 9.32(ii), $f\mapsto\varphi_f$ is injective, $\varphi_f\in\mathcal{F}^{\mathcal{E}}_{(a,2)}(7)$, and $f=\det(P_1)^7\varphi_f((P_1^{-1}P_2)_0,(P_1^{-1}P_3)_0)$ on $\det P_1\neq0$, which is f_7 built from φ_f. So (S1) for f at $P=(I,A_2,A_3)$ is $\Omega f_7(uy_1^T,uy_2^T,uy_3^T)=7\mathcal{Z}(\varphi_f)-\mathcal{J}(\varphi_f)$ (Lemma 9.30), and it holds at every base point and for all u,y_1,y_2,y_3. By Theorem 9.163, $\varphi_f=0$, so $f=0$. Only (S1) is used. (b) By Corollary 9.42(b), every conjugation type κ of $K_{B(7,2)}$ has exactly three columns. So $\kappa'_3\le1$ means $\kappa'=(45-\sigma,\sigma,1)$ with $1\le\sigma\le22$, that is, $\tau=\kappa'+(1,1,1)=(47-a,a,2)$ with $a=\sigma+1$ and $2\le a\le23$. By Lemma 9.32(iii) the multiplicity of κ is $\dim\mathrm{K}_7(7)_\tau$. It is 1 at $a=2$ (Theorem 9.37, as $s=7\ge4$; this type is $(3,1^{43})$), and 0 for $3\le a\le23$, by (a). □

Remark 9.165. (1) (The computation.) The number h is computed exactly from characters, with no monomials and no primes: it is the dimension of the GL_7-invariants in $S_{(a,2)}(\mathfrak{sl}_7)$, by the Cauchy formula and the Murnaghan–Nakayama rule. By Lemma 9.32(i), (ii) it is also the Kronecker coefficient $g\big((r^7),(r^7),(7r-a-2,a,2)\big)$ for every $r\ge a+2$, and two further independent programs (Kronecker coefficients by Young's rule and the Pieri rule; the Jacobi–Trudi identity and the Pieri rule) give the same h at every $a=2,\dots,23$. For $a=3,\dots,23$ the values are $h=2$, 6, 8, 16, 21, 35, 46, 69, 89, 125, 158, 212, 263, 341, 417, 526, 634, 783, 932, 1131, 1332; the program stops unless (M2) reaches h vectors. In (M3), A_2 has trace 0, A_3 is arbitrary and u is the same for B_1,B_2,B_3; there are $3h+20$ base points, and $r_p=h$ at every $a=3,\dots,23$. Using only part of the conditions can only make a kernel larger, never smaller. Two versions of the program (the second faster, with the same arithmetic and the same random choices) give the same results at $a=11,\dots,18$, and a run with other random choices confirms $a=3,\dots,8$ and $a=11,\dots,15$.

(2) (Controls.) At $a=2$ the same program finds the line of Theorem 9.37 (kernel 1, $r_p=1$ of $h=2$); this does not test the part of the rows that depends on s, as at $a=2$ the kernel is a line at every κ. The same rows at other s find kernels where they must: by Corollary 9.63 and

Theorem 9.102, $s=(a+3)/7$ and $s=a/7$ are eigenvalues for $a=3,4,5$ ($n\ge a+2$); the kernels found at $s=a/7$ and $s=(a+3)/7$ are $1,1$ at $a=3$, $2,1$ at $a=4$ and $2,1$ at $a=5$, and 0 at $s=7$; at $a=7$ they are 4 at $s=1$, 1 at $s=10/7$, and 0 at $s=7$. An independent program with the chart form of the test (Lemma 9.20) and another prime gives kernel 0 at $s=7$ for $a=3,\dots,10$ and kernel 1 at $a=2$. Supporting, and not used: for $a=7,\dots,10$ another method (greatest common divisors of the determinants of random square sub-pencils) puts $\kappa=7$ outside the spectrum; and for odd $a=7,\dots,23$, step (2) of the proof of Theorem 9.154 holds at $n=7$ by another computation, so Corollary 9.160 gives Theorem 9.163 again at these a.

(3) (Scope.) New here: the consequence of Conjecture 11.3 at $n=7$ without the conjecture, that is, every type of $K_{B(7,2)}$ with $\kappa_3'\le 1$; beyond Theorem 9.24 and Corollary 9.155 at $n=7$, the even $a=8,\dots,22$ and the odd $a=13,\dots,23$; and the pencil form at $n=7$ for $4\le a\le 23$ except at odd $a=7,9,11$. Known before: $\mathrm{K}_7(7)_\tau=0$ for $a=3,\dots,6$ is Theorem 9.24, as the depth of the type is a; the pencil form at $a=3$ is Theorem 9.48 (every $n\ge 4$), and at odd $a=7,9,11$ it is Theorem 9.154; Remark 9.38(d) computed $n=7$ only at 4 bidegrees with $d\le 8$, not as a proof. Not claimed: any $n\ne 7$; the types with $\kappa_3'\ge 2$ ($\tau_3\ge 3$) at $n=7$; Conjecture 11.3 at $n=7$, of which only the value $\kappa=7$ is tested; any bound on $d(7)$.

9.29. **Step (2) at every n.** In this subsection $F=\mathbb{C}$, $b=2$ and $n\ge 4$, and we keep the notation of Subsection 9.26. We count the solutions of $(\star_n)$ by hand, at every n: at $\kappa=n$ and in degree $m\le n^2/2-3$, the shift invariant solutions form a line for even m and are 0 for odd m (Theorem 9.173). So step (2) of the proof of Theorem 9.154 holds at every n for every odd dominant a, and, with Corollary 9.160, $\kappa=n$ is not an eigenvalue of the pencil for every $n\ge 4$ and every odd a with $n\le a\le (n^2-2)/2$ (Corollary 9.174). The route: $(\star_n)$ forces the solution to be quadratic in each of $x_4,\dots,x_n$ (Lemmas 9.167 and 9.168) and is then equivalent to a derivative condition; after a product kernel is absorbed, everything is decided by a short chain of equations on symmetric polynomials (Proposition 9.169); Dunkl operators solve the chain (Lemmas 9.171 and 9.170); and an $\mathfrak{sl}_2$ argument turns the dimension count into the count of shift invariant solutions (Lemma 9.166).

Notation. Put $\bar n:=n-3$, and let $\mathbb{C}[x]^{S_3\times S_{\bar n}}$ be the polynomials in $x=(x_1,\dots,x_n)$ that are symmetric in x_1,x_2,x_3 and in $x_4,\dots,x_n$ (the *rest* variables). For ϑ in it write $\vartheta_{123}:=\vartheta$ and ϑ_{124} for ϑ with x_3 and x_4 exchanged, so that $(\star_n)$ is a linear condition on ϑ; let $\mathbb{A}(\vartheta)$ be its left side. For $\kappa\ne 0$ put $\beta:=\kappa/2$, $\mathfrak{k}:=1-\beta$ and $m_{\max}:=n\kappa/2-3$, and let $\mathcal{X}_m$ be the space of solutions $\vartheta\in\mathbb{C}[x]^{S_3\times S_{\bar n}}$ of $(\star_n)$, homogeneous of degree m. Let $\mathbb{C}[x]_d^{S_n}$ be the symmetric polynomials of degree d, $x_{kl}:=x_k-x_l$, $p_1:=\sum_l x_l$, $\partial_{\mathbf{1}}:=\sum_l\partial_l$ (so ϑ is shift invariant if and only if $\partial_{\mathbf{1}}\vartheta=0$), $\mathbb{E}:=\sum_l x_l\partial_l$ and $\mathbb{K}:=\sum_l x_l^2\partial_l$.

Lemma 9.166 (an $\mathfrak{sl}_2$ action)**.** *Let $\kappa\ne 0$ and $\mathbb{K}'':=\mathbb{K}-\kappa p_1+2(x_1+x_2+x_3)$.*

(i) For every $\vartheta\in\mathbb{C}[x]^{S_3\times S_{\bar n}}$, $\mathbb{A}(\mathbb{K}''\vartheta)=\big(\mathbb{K}-\kappa p_1+2(x_1+x_2+x_3+x_4)\big)\mathbb{A}(\vartheta)$. So $\mathbb{K}''$ maps $\mathcal{X}_m$ into $\mathcal{X}_{m+1}$; and $\partial_{\mathbf{1}}$ maps $\mathcal{X}_m$ into $\mathcal{X}_{m-1}$, as $(\star_n)$ is translation invariant.

(ii) $[\partial_{\mathbf{1}},\mathbb{K}'']=2\mathbb{E}-n\kappa+6$. So $\mathbb{H}:=2\mathbb{E}+6-n\kappa$ satisfies $[\mathbb{K}'',-\partial_{\mathbf{1}}]=\mathbb{H}$, $[\mathbb{H},\mathbb{K}'']=2\mathbb{K}''$ and $[\mathbb{H},-\partial_{\mathbf{1}}]=2\partial_{\mathbf{1}}$, and $\mathbb{H}=2(m-m_{\max})$ on $\mathcal{X}_m$.

Proof. (i) Put $N(\vartheta):=\partial_3\vartheta_{124}-\partial_4\vartheta_{123}$. As $\mathbb{K}$ commutes with every permutation of the coordinates, $\mathbb{K}(1/x_{34})=-(x_3+x_4)/x_{34}$ and $[\partial_l,\mathbb{K}]=2x_l\partial_l$, we get $\mathbb{A}(\mathbb{K}\vartheta)=\mathbb{K}\mathbb{A}(\vartheta)+(x_3+x_4)\mathbb{A}(\vartheta)+x_{34}N(\vartheta)$. Directly, $\mathbb{A}(p_1\vartheta)=p_1\mathbb{A}(\vartheta)+\vartheta_{123}+\vartheta_{124}$, and, as $((x_1+x_2+x_3)\vartheta)_{124}=(x_1+x_2+x_4)\vartheta_{124}$,

$$\mathbb{A}\big((x_1+x_2+x_3)\vartheta\big)=\Big(x_1+x_2+\frac{x_3+x_4}{2}\Big)\mathbb{A}(\vartheta)+\frac{\kappa}{2}(\vartheta_{123}+\vartheta_{124})-\frac{x_{34}}{2}N(\vartheta).$$

The first identity, minus κ times the second, plus twice the third: the terms in $N(\vartheta)$ and in $\vartheta_{123}+\vartheta_{124}$ cancel, and (i) follows. (ii) $[\partial_{\mathbf{1}},\mathbb{K}]=2\mathbb{E}$, $[\partial_{\mathbf{1}},p_1]=n$ and $[\partial_{\mathbf{1}},x_1+x_2+x_3]=3$; and $\mathbb{E}=m$ on $\mathcal{X}_m$, while $\mathbb{K}''$ raises and $\partial_{\mathbf{1}}$ lowers the degree by 1. □

Lemma 9.167 (four points)**.** *Let $\kappa\ne 0$ and $\vartheta\in\mathbb{C}[x]^{S_3\times S_{\bar n}}$. For a set $\mathfrak{Q}$ of four indices and $c\in\mathfrak{Q}$ let g_c be ϑ with the variables x_e, $e\in\mathfrak{Q}\setminus\{c\}$, in the places $1,2,3$ and with x_c and the x_l, $l\notin\mathfrak{Q}$, in*

the rest places, and let

$$g[\mathfrak{Q}] := \sum_{c\in\mathfrak{Q}} \frac{g_c}{\prod_{e\in\mathfrak{Q}\setminus\{c\}} x_{ce}}$$

be the third divided difference of $c\mapsto g_c$ at the nodes x_c. Then ϑ solves $(\star_n)$ if and only if, for every $\mathfrak{Q}$,

$$g[\mathfrak{Q}] = 0 \tag{AC}$$

and, with $P_{\mathfrak{Q}}$ the polynomial in one variable of degree at most 2 with $P_{\mathfrak{Q}}(x_c)=g_c$ for $c\in\mathfrak{Q}$ (it exists by (AC)*),*

$$\partial_c g_c = \frac{\kappa}{2}\,P'_{\mathfrak{Q}}(x_c) \qquad (c\in\mathfrak{Q}). \tag{DE}$$

By permuting the coordinates, it suffices to take $\mathfrak{Q}=\{1,2,3,4\}$, and $c=4$ in (DE)*.*

Proof. Fix $\mathfrak{Q}$ and $c\neq e$ in $\mathfrak{Q}$. A permutation σ of the indices with $\sigma(3)=c$, $\sigma(4)=e$ and $\sigma(\{1,2\})=\mathfrak{Q}\setminus\{c,e\}$ turns $(\star_n)$ into $\partial_c g_c+\partial_e g_e=\kappa q_{ce}$, $q_{ce}:=(g_c-g_e)/x_{ce}$; and these six equations, for all $\mathfrak{Q}$, are $(\star_n)$ at all points. If (AC) holds, then $P_{\mathfrak{Q}}(t)=r_0+r_1t+r_2t^2$ and $q_{ce}=r_1+r_2(x_c+x_e)=l_c+l_e$ with $l_c:=P'_{\mathfrak{Q}}(x_c)/2$; so, with $\omega_c:=\partial_c g_c-\kappa l_c$, the six equations say $\omega_c+\omega_e=0$ for all $c\neq e$ in $\mathfrak{Q}$, that is, $\omega=0$, which is (DE); conversely (AC) and (DE) give $\partial_c g_c+\partial_e g_e=\kappa(l_c+l_e)=\kappa q_{ce}$. If the six equations hold, then for $\mathfrak{Q}=\{c,e,c',e'\}$ the sums of two of them give $\kappa(q_{ce}+q_{c'e'})=\kappa(q_{cc'}+q_{ee'})$, and $q_{ce}+q_{c'e'}-q_{cc'}-q_{ee'}=g[\mathfrak{Q}]\,x_{ce'}x_{ec'}$ (the part of degree at most 2 of the interpolant cancels); as $\kappa\neq0$, (AC) holds. □

Lemma 9.168 (the global form of (AC))**.** *An element $\vartheta\in\mathbb{C}[x]^{S_3\times S_{\bar n}}$ satisfies* (AC) *for every $\mathfrak{Q}$ if and only if*

$$\vartheta=\sum_{i+j\le\bar n} c_{ij}G_{ij}, \qquad G_{ij}:=m_{(2^j1^i)}(x_4,\dots,x_n), \quad c_{ij}\in\mathbb{C}[x]^{S_n},$$

with m_μ the monomial symmetric polynomial; the c_{ij} are then unique.

Proof. The ring $\mathbb{C}[x]^{S_3\times S_{\bar n}}$ is free over $\mathbb{C}[x]^{S_n}$ with basis the $m_\mu(x_4,\dots,x_n)$, $\mu\subseteq(3^{\bar n})$ (the cohomology of the Grassmannian of 3-planes in $\mathbb{C}^n$; the m_μ are unitriangular to the Schur basis [Ma95]). Write $\vartheta=\sum_\mu c_\mu m_\mu(x_4,\dots,x_n)$, take $\mathfrak{Q}=\{1,2,3,y\}$ with $y\ge4$, and let Y' be the rest variables other than x_y. Then $m_\mu(x_4,\dots,x_n)=\sum_k x_y^k\,m_{\mu\setminus k}(Y')$ ($\mu\setminus k$: μ with one part k removed, the term being 0 if μ has no part k; μ is read with $\bar n$ parts, zeros included). As $\mathbb{C}[x_1,x_2,x_3,x_y]^{S_3}$ is free over $\mathbb{C}[x_1,x_2,x_3,x_y]^{S_4}$ with basis $1,x_y,x_y^2,x_y^3$, $\vartheta=r_0+r_1x_y+r_2x_y^2+r_3x_y^3$ uniquely with r_k symmetric in x_1,x_2,x_3,x_y and in Y', and $r_3=\sum_{\mu\ni3}c_\mu m_{\mu\setminus3}(Y')$. Here $g_c=r_0+r_1x_c+r_2x_c^2+r_3x_c^3$ for $c\in\mathfrak{Q}$, so $g[\mathfrak{Q}]=r_3$. The $m_\nu(Y')$, $\nu\subseteq(3^{\bar n-1})$, are part of the basis of $\mathbb{C}[x]^{S_4\times S_{\bar n-1}}$ over $\mathbb{C}[x]^{S_n}$, and $\mu\mapsto\mu\setminus3$ is injective on the μ with a part 3; so $r_3=0$ if and only if $c_\mu=0$ whenever $\mu_1=3$. By permuting the coordinates, this is (AC) for every $\mathfrak{Q}$. □

The kernel. Let u,v be formal variables, $\mathrm{q}(t):=1+ut+vt^2$,

$$\mathbf{G}:=\prod_{l\ge4}\mathrm{q}(x_l)=\sum_{i,j}u^iv^jG_{ij}, \qquad \mathbf{K}:=\prod_{l=1}^{n}\mathrm{q}(x_l)^{\beta-1},$$

a power series in u,v with coefficients $\mathbf{K}_{pq}\in\mathbb{C}[x]^{S_n}$ and $\mathbf{K}_{00}=1$. For $\vartheta=\sum c_{ij}G_{ij}$ as in Lemma 9.168, the unitriangular system $c_{ij}=\sum_{a'\ge i,\,b'\ge j,\,a'+b'\le\bar n}d_{a'b'}\mathbf{K}_{a'-i,b'-j}$ defines $d_{a'b'}\in\mathbb{C}[x]^{S_n}$ ($a'+b'\le\bar n$), and then $\vartheta=\sum_{a'+b'\le\bar n}d_{a'b'}\,[u^{a'}v^{b'}](\mathbf{KG})$. The *top layer* is $s_j:=d_{\bar n-j,j}=c_{\bar n-j,j}$ ($0\le j\le\bar n$). For an index y put $\rho_y:=\mathrm{q}'(x_y)/\mathrm{q}(x_y)$, $S_y:=\sum_{j=0}^{\bar n}(\partial_y s_j)\,u^{-(\bar n-j)}v^{-j}$ and $F_y:=(\mathbf{K}\,\mathrm{q}(x_y))^{-1}$.

Proposition 9.169 (the chain)**.** *Let $\kappa\neq0$ and $\vartheta=\sum c_{ij}G_{ij}$ be homogeneous of degree m.*

(a) ϑ solves $(\star_n)$ if and only if, for every index y and every $a'+b'<\bar n$,

$$\partial_y d_{a'b'} = [u^{-a'}v^{-b'}](S_yF_y). \tag{DE$'$}$$

(b) Given the top layer, a family $(d_{a'b'})_{a'+b'<\bar n}$ of symmetric polynomials with (DE$'$) *exists if and only if, for one (equivalently, every) pair $k\neq l$,*

$$\mathbb{D}_k\mathbb{D}_l s_j + \mathbb{B}_{kl}s_{j+1} = 0 \qquad (0\le j\le \bar n-1), \tag{TL}$$

where, for symmetric s, $\mathbb{D}_k\mathbb{D}_l s := \partial_k\partial_l s + \mathfrak{k}(\partial_l s-\partial_k s)/x_{kl}$ and $\mathbb{B}_{kl}s := (x_k+x_l)\partial_k\partial_l s + 2\mathfrak{k}(x_k\partial_l s - x_l\partial_k s)/x_{kl}$; and then each $d_{a'b'}$ is unique up to an additive constant, which is allowed only when $a'+2b'=m$.

(c) Hence

$$\dim\mathcal{X}_m = \dim\{(s_0,\dots,s_{\bar n}) : s_j\in\mathbb{C}[x]^{S_n}_{m-\bar n-j},\ \text{(TL)}\} + \#\{(a',b') : a'+b'<\bar n,\ a'+2b'=m\}. \tag{DIM}$$

Proof. (a) By Lemmas 9.167 and 9.168, ϑ solves $(\star_n)$ if and only if (DE) holds at $\mathfrak{Q}=\{1,2,3,y\}$, $c=y$, for one rest index y. There $g_y=\vartheta$, and in the proof of Lemma 9.168 $P'_{\mathfrak{Q}}(x_y) = r_1 + 2r_2x_y = \sum c_{ij}\partial_y G_{ij}$ (the c_{ij} are symmetric); so (DE) reads $\sum(\partial_y c_{ij})G_{ij} = (\beta-1)\sum c_{ij}\partial_y G_{ij}$. As $\partial_y\mathbf{K} = (\beta-1)\rho_y\mathbf{K}$ and $\partial_y\mathbf{G}=\rho_y\mathbf{G}$, this is $\sum_{a',b'}(\partial_y d_{a'b'})\,[u^{a'}v^{b'}](\mathbf{KG}) = 0$. Write $\mathbf{KG} = \mathbf{K}\,\mathrm{q}(x_y)\prod_{l\ge4,l\neq y}\mathrm{q}(x_l)$; the $m_\nu(Y')$, $\nu\subseteq(3^{\bar n-1})$, are linearly independent over the polynomials symmetric in the variables other than x_y (the Grassmannian basis with $n-1$ variables), and $\partial_y d_{a'b'}$, $\mathbf{K}_{pq}$ and $\mathrm{q}(x_y)$ are such polynomials. So the condition is $\pi_-(\Delta\,\mathbf{K}\,\mathrm{q}(x_y)) = S_y$, where $\Delta := \sum_{a'+b'\le\bar n}(\partial_y d_{a'b'})u^{-a'}v^{-b'}$ and π_- keeps the monomials $u^{-i}v^{-j}$ with $i,j\ge0$ (the coefficients with $i+j<\bar n$ vanish, and those with $i+j=\bar n$ are S_y, as $\mathbf{K}\,\mathrm{q}(x_y)$ has constant term 1). As $\pi_-(\pi_-(X)P)=\pi_-(XP)$ for every power series P in u,v, this is equivalent to $\Delta=\pi_-(S_yF_y)$, that is, (DE$'$) at y. For symmetric $d_{a'b'}$, (DE$'$) at one index gives it at every index, by permuting the coordinates.

(b) A symmetric $d_{a'b'}$ with (DE$'$), homogeneous of degree $m-a'-2b'$, exists if and only if the mixed partials of the right sides agree; it is unique up to a constant, of degree 0, so allowed only when $a'+2b'=m$. Every potential f is symmetric (the right sides are S_n-equivariant, so $\sigma f - f$ is a constant, of positive degree if $\deg f>0$, hence 0) and may be taken homogeneous (Euler: $f = (\deg f)^{-1}\sum_l x_l\partial_l f$). For $y\neq y'$, $\partial_{y'}F_y = -(\beta-1)\rho_{y'}F_y$, so $\partial_{y'}[u^{-a'}v^{-b'}](S_yF_y) = [u^{-a'}v^{-b'}](M_{yy'}X_{yy'})$ with $M_{yy'} := (\mathbf{K}\,\mathrm{q}(x_y)\mathrm{q}(x_{y'}))^{-1}$ (symmetric in y,y', nonnegative powers, constant term 1) and $X_{yy'} := \mathrm{q}(x_{y'})H_{yy'} + \mathfrak{k}(u+2vx_{y'})S_y$, $H_{yy'} := \partial_{y'}S_y$. So all lower layers exist if and only if $[u^{-a'}v^{-b'}](M_{yy'}Z_{yy'})=0$ for all $a',b'\ge0$ with $a'+b'\le\bar n-1$, where $Z_{yy'} := X_{yy'}-X_{y'y}$. The part of $Z_{yy'}$ of total exponent $-\bar n$ is $H_{yy'}-H_{y'y}=0$ (the mixed partials of the top layer), so $Z_{yy'} = \sum z_{a'b'}u^{-a'}v^{-b'}$ over $a'+b'=\bar n-1$, $a',b'\ge-1$. As $M_{yy'}$ has nonnegative powers and constant term 1, the coefficient at $a'+b'=\bar n-1$, $a',b'\ge0$, is $z_{a'b'}$, and the deeper ones are combinations of these (the terms with $a'=-1$ or $b'=-1$ do not contribute). So the condition is $z_{\bar n-1-j,j}=0$ for $0\le j\le\bar n-1$, and with $k:=y$, $l:=y'$,

$$z_{\bar n-1-j,j} = x_{lk}\,\partial_k\partial_l s_j + (x_l^2-x_k^2)\,\partial_k\partial_l s_{j+1} + \mathfrak{k}(\partial_k-\partial_l)s_j + 2\mathfrak{k}(x_l\partial_k - x_k\partial_l)s_{j+1};$$

dividing by x_{lk} gives (TL). As the s_j are symmetric, (TL) for one pair gives it for every pair.

(c) Every solution satisfies (AC) (Lemma 9.167), so it has the form of Lemma 9.168, and $\vartheta\mapsto(c_{ij})\mapsto(d_{a'b'})$ is bijective (the c_{ij} are unique and the system is unitriangular), s_j has degree $m-\bar n-j$, and by (a), (b) the solutions are a top layer with (TL) together with one free constant for each (a',b') with $a'+b'<\bar n$ and $a'+2b'=m$. □

Dunkl operators. For $1\le i\le n$ let $\mathbb{D}_i := \partial_i + \mathfrak{k}\sum_{l\neq i}(1-\sigma_{il})/x_{il}$, with σ_{il} the transposition acting on polynomials (the Dunkl operators of S_n). They commute (see [DJO94]), and $[\mathbb{D}_i,x_l] = -\mathfrak{k}\sigma_{il}$ for $i\neq l$. On symmetric s, $\mathbb{D}_k\mathbb{D}_l s$ is the expression in Proposition 9.169(b), and $\mathbb{B}_{kl}s = (x_k+$

$x_l)\mathbb{D}_k\mathbb{D}_l s + \mathfrak{k}(\partial_k+\partial_l)s$. A *singular* polynomial is a $q\neq 0$ with $\mathbb{D}_i q=0$ for all i; let d_{sing} be the least degree ≥ 1 of a singular polynomial ($d_{\mathrm{sing}}=\infty$ if there is none).

Lemma 9.170 (the intertwiner)**.** *(a) There is a linear map $\mathbb{V}$ on the polynomials of degree less than d_{sing}, bijective in each degree, with $\mathbb{V}1=1$, $\sigma\mathbb{V}=\mathbb{V}\sigma$ for $\sigma\in S_n$, and $\mathbb{D}_i\mathbb{V}q=\mathbb{V}\partial_i q$ for every q of degree less than d_{sing}.*

(b) A singular polynomial of degree d in the isotypic component of $\eta\vdash n$ has $d+\mathfrak{k}N_\eta=0$, where $N_\eta:=\binom{n}{2}-\mathrm{ct}(\eta)$ and $\mathrm{ct}(\eta)$ is the sum of the contents of η. Here $N_{(n)}=0$, $N_{(n-1,1)}=n$, and $N_\eta\geq 2(n-1)$ for every other η.

(c) At $\kappa=n$, $\mathfrak{k}=-(n-2)/2$, and $d_{\mathrm{sing}}\geq(n-2)(n-1)$ if n is odd, $d_{\mathrm{sing}}\geq n(n-2)/2$ if n is even.

Proof. (a) For polynomials p,q put $[p,q]_{\mathfrak{k}}:=(p(\mathbb{D})q)(0)$, where $p(\mathbb{D})$ is p with $x_i\mapsto\mathbb{D}_i$; it is 0 between different degrees, $[x_ip,q]_{\mathfrak{k}}=[p,\mathbb{D}_iq]_{\mathfrak{k}}$, and $[\cdot,\cdot]_0$ is $(p(\partial)q)(0)$, which is nondegenerate. For $d<d_{\mathrm{sing}}$ the right radical of $[\cdot,\cdot]_{\mathfrak{k}}$ on the polynomials of degree d is 0: otherwise take the least such d and $q\neq0$ in it; then $[p,\mathbb{D}_iq]_{\mathfrak{k}}=[x_ip,q]_{\mathfrak{k}}=0$ for all p of degree $d-1$, so $\mathbb{D}_iq$ lies in the radical in degree $d-1$, which is 0 (in degree 0 because $[1,1]_{\mathfrak{k}}=1$), and q is singular of degree $d<d_{\mathrm{sing}}$. So for $d<d_{\mathrm{sing}}$ we may define $\mathbb{V}q$, for q of degree d, by $[p,\mathbb{V}q]_{\mathfrak{k}}=[p,q]_0$ for all p of degree d. Then $[p,\mathbb{D}_i\mathbb{V}q]_{\mathfrak{k}}=[x_ip,\mathbb{V}q]_{\mathfrak{k}}=[x_ip,q]_0=[p,\partial_iq]_0=[p,\mathbb{V}\partial_iq]_{\mathfrak{k}}$, so $\mathbb{D}_i\mathbb{V}=\mathbb{V}\partial_i$; $\mathbb{V}1=1$, and $\mathbb{V}$ is injective, so bijective. As $\sigma\mathbb{D}_i\sigma^{-1}=\mathbb{D}_{\sigma(i)}$, $(\sigma p)(\mathbb{D})=\sigma\,p(\mathbb{D})\,\sigma^{-1}$, and evaluation at 0 is invariant, so $[\sigma p,\sigma q]_{\mathfrak{k}}=[p,q]_{\mathfrak{k}}$; hence $\sigma\mathbb{V}\sigma^{-1}=\mathbb{V}$, by uniqueness. (b) $\sum_i x_i\mathbb{D}_i=\mathbb{E}+\mathfrak{k}\sum_{i<l}(1-\sigma_{il})$, and the sum of all transpositions acts on the isotypic component of η by $\mathrm{ct}(\eta)$; so a singular q of degree d there has $0=(d+\mathfrak{k}N_\eta)q$. For $\eta=(n-w,\mu)$ with $\mu\vdash w$,

$$N_\eta=\frac{w(2n-w+1)}{2}+\sum_i(i-1)\mu_i-\sum_i(i-1)\mu'_i\geq w(n-w+1)\geq 2(n-1)\qquad(2\leq w\leq n-1),$$

as the last two terms add up to at least $-w(w-1)/2$ (at $\mu=(w)$), and $w(n-w+1)$ is concave in w with the value $2(n-1)$ at both ends; $w=0,1$ give $N_{(n)}=0$ and $N_{(n-1,1)}=n$. (c) By (b), a singular polynomial has degree $d=(n-2)N_\eta/2$ with $N_\eta\neq0$. For odd n, d is an integer only when N_η is even, which excludes $N_{(n-1,1)}=n$; so $d\geq(n-2)(n-1)$. For even n, $N_\eta\geq n$ gives $d\geq n(n-2)/2$. □

Lemma 9.171 (solving the chain)**.** *Let $\kappa\neq0$, s,s' be symmetric, and k,l,i be distinct indices.*

(i) $\mathbb{D}_i\mathbb{B}_{kl}s=(x_k+x_l)\,\mathbb{D}_i\mathbb{D}_k\mathbb{D}_ls$.

(ii) If $\mathbb{D}_k\mathbb{D}_ls'=-\mathbb{B}_{kl}s$ for all $k\neq l$, then $\mathbb{D}_k\mathbb{D}_l\mathbb{D}_is=0$ and $\mathbb{D}_k\mathbb{D}_l\mathbb{D}_is'=0$ for all distinct k,l,i.

(iii) If $\mathbb{D}_k\mathbb{D}_l\mathbb{D}_is=0$ for all distinct k,l,i, and $\deg s+1<d_{\mathrm{sing}}$, then there is a symmetric s' with $\mathbb{D}_k\mathbb{D}_ls'=-\mathbb{B}_{kl}s$ for all $k\neq l$.

(iv) Let $d<d_{\mathrm{sing}}$. The symmetric s of degree d with $\mathbb{D}_k\mathbb{D}_ls=0$ for all $k\neq l$ form the line $\mathbb{C}\,\mathbb{V}p_d$ ($p_0:=1$); those with $\mathbb{D}_k\mathbb{D}_l\mathbb{D}_is=0$ for all distinct k,l,i form $\mathbb{V}\,\mathrm{span}\{m_{(d)},m_{(d-t,t)}:1\leq t\leq d/2\}$, of dimension $\lfloor d/2\rfloor+1$.

(v) If $m-\bar n<d_{\mathrm{sing}}$, then $\dim\mathcal{X}_m=\lfloor m/2\rfloor+1$.

Proof. (i) Let $\Gamma:=\mathbb{D}_k\mathbb{D}_ls$. Then $\mathbb{D}_i(x_k\Gamma)=x_k\mathbb{D}_i\Gamma-\mathfrak{k}\sigma_{ik}\Gamma$, and, s being symmetric, $\sigma_{ik}\Gamma=\mathbb{D}_i\mathbb{D}_ls$ and $\sigma_{il}\Gamma=\mathbb{D}_k\mathbb{D}_is$. So $\mathbb{D}_i\mathbb{B}_{kl}s=(x_k+x_l)\mathbb{D}_i\Gamma-\mathfrak{k}(\mathbb{D}_i\mathbb{D}_ls+\mathbb{D}_k\mathbb{D}_is)+\mathfrak{k}\mathbb{D}_i(\mathbb{D}_k+\mathbb{D}_l)s$, as $(\partial_k+\partial_l)s=(\mathbb{D}_k+\mathbb{D}_l)s$; the last two terms cancel. (ii) $\mathbb{D}_i\mathbb{D}_k\mathbb{D}_ls'=-(x_k+x_l)\mathbb{D}_i\mathbb{D}_k\mathbb{D}_ls$ by (i), and the left side is symmetric in k,l,i; exchanging k and i gives $x_{ki}\,\mathbb{D}_k\mathbb{D}_l\mathbb{D}_is=0$. Then (i) gives $\mathbb{D}_k\mathbb{D}_l\mathbb{D}_is'=0$. (iii) Put $\mathsf{y}_{kl}:=-\mathbb{B}_{kl}s$, of degree $e:=\deg s-1$. By (i), $\mathbb{D}_i\mathsf{y}_{kl}=0$ for $i\notin\{k,l\}$; so $H_{kl}:=\mathbb{V}^{-1}\mathsf{y}_{kl}$ has $\partial_iH_{kl}=0$, that is, $H_{kl}=\phi_{kl}(x_k,x_l)$, and by the equivariance of $\mathbb{V}$ and $\mathsf{y}_{kl}=\mathsf{y}_{lk}$ there is one symmetric ϕ with $\phi_{kl}=\phi$ for all $k\neq l$. Put $\Phi(t_1,t_2):=\int_0^{t_1}\int_0^{t_2}\phi$, $S':=\sum_{k<l}\Phi(x_k,x_l)$, of degree $e+2=\deg s+1$, and $s':=\mathbb{V}S'$; then s' is symmetric and $\mathbb{D}_k\mathbb{D}_ls'=\mathbb{V}\partial_k\partial_lS'=\mathbb{V}H_{kl}=\mathsf{y}_{kl}$. (iv) With $s=\mathbb{V}S$, $\mathbb{D}_k\mathbb{D}_ls=\mathbb{V}\partial_k\partial_lS$, and $\mathbb{V}$ preserves symmetry. A symmetric S with $\partial_k\partial_lS=0$ for all

$k \neq l$ is $\sum_l f(x_l)$, so $S \in \mathbb{C}p_d$; one with $\partial_k\partial_l\partial_i S = 0$ for distinct k, l, i has every monomial in at most two variables. (v) By Proposition 9.169(c) we count the top layers with (TL). By (ii) every s_j has $\mathbb{D}_k\mathbb{D}_l\mathbb{D}_i s_j = 0$. Choose $s_{\bar{n}}$ in the kernel of the triple products in degree $m - 2\bar{n}$ (by (iv)), and then, for $j = \bar{n} - 1, \dots, 0$, a solution s_j of $\mathbb{D}_k\mathbb{D}_l s_j = -\mathbb{B}_{kl}s_{j+1}$ (by (iii)), unique up to the line of (iv) in degree $m - \bar{n} - j$ (and 0 when this degree is negative). So the top layers form a space of dimension $[m \geq 2\bar{n}](\lfloor (m - 2\bar{n})/2 \rfloor + 1) + \#\{0 \leq j \leq \bar{n} - 1 : m - \bar{n} - j \geq 0\}$, and with (DIM) the total is $\lfloor m/2 \rfloor + 1$ in each of the ranges $m < \bar{n}$, $\bar{n} \leq m < 2\bar{n}$ and $m \geq 2\bar{n}$. The intertwiner is used only in degrees at most $m - \bar{n}$: in (iii) for $s = s_{j+1}$ of degree $e + 1$ it is used in degrees e and $e + 2 \leq m - \bar{n}$, and $\mathbb{D}_i\mathbb{V} = \mathbb{V}\partial_i$ in degrees $e, e + 1, e + 2$; in (iv) in degrees $\deg s_j \leq m - \bar{n}$. □

Theorem 9.172. *Let $n \geq 4$ and $\kappa \neq 0$.*

(i) If $\mathfrak{k}$ is not of the form $-j/q$ with integers $j \geq 1$ and $2 \leq q \leq n$, $q \nmid j$ (for example, if $\mathfrak{k}$ is an integer, or $\mathfrak{k} \geq 0$), then $\dim \mathcal{X}_m = \lfloor m/2 \rfloor + 1$ for every $m \geq 0$.

(ii) At $\kappa = n$ with n odd, $\dim \mathcal{X}_m = \lfloor m/2 \rfloor + 1$ for every m with $m - \bar{n} < (n - 2)(n - 1)$, in particular for every $m \leq m_{\max} = n^2/2 - 3$.

Proof. By Lemma 9.171(v) it suffices that $m - \bar{n} < d_{\text{sing}}$. (i) For these $\mathfrak{k}$ there is no singular polynomial [DJO94], so $d_{\text{sing}} = \infty$. (ii) $d_{\text{sing}} \geq (n - 2)(n - 1)$ by Lemma 9.170(c); and $m \leq m_{\max}$ means $m \leq (n^2 - 7)/2$, so $m - \bar{n} \leq (n^2 - 2n - 1)/2 < (n - 2)(n - 1)$, as $n^2 - 4n + 5 > 0$. □

Theorem 9.173. *For every $n \geq 4$, $\kappa = n$ and $0 \leq m \leq m_{\max} = n^2/2 - 3$, the shift invariant elements of $\mathcal{X}_m$ form a space of dimension 1 if m is even and 0 if m is odd.*

Proof. First, $\dim \mathcal{X}_m = \lfloor m/2 \rfloor + 1$ for $0 \leq m \leq m_{\max}$: for odd n by Theorem 9.172(ii); for even n, $\mathfrak{k} = -(n - 2)/2$ is an integer, $d_{\text{sing}} \geq n(n - 2)/2$ by Lemma 9.170(c), and $m - \bar{n} \leq m_{\max} - \bar{n} = n(n-2)/2$, with equality only at $m = m_{\max}$ (the degree $a-2$ of Corollary 9.174 at $a = (n^2-2)/2$); for that one degree we use from [DJO94] only that an integer $\mathfrak{k}$ admits no singular polynomial, and then Lemma 9.171(v) applies. Next, $m_{\max} \geq 5$, and on $\mathcal{X}_0 = \mathbb{C}$, $-\partial_{\mathbf{1}}\mathbb{K}''(1) = n\kappa - 6 = 2m_{\max} > 0$. For $0 \leq j < m_{\max}$ let $\mathcal{A}_j$ and $\mathcal{B}_j$ be $-\partial_{\mathbf{1}}\mathbb{K}''$ and $-\mathbb{K}''\partial_{\mathbf{1}}$ on $\mathcal{X}_j$ (Lemma 9.166(i)). By Lemma 9.166(ii), $\mathcal{A}_j = \mathcal{B}_j + 2(m_{\max} - j)$. The nonzero eigenvalues of $\mathcal{B}_j$ are those of $\mathcal{A}_{j-1}$ (as $\mathcal{B}_j$ is the composition $\mathcal{X}_j \to \mathcal{X}_{j-1} \to \mathcal{X}_j$ of $-\partial_{\mathbf{1}}$ and $\mathbb{K}''$, and $\mathcal{A}_{j-1}$ is the composition in the other order; $\mathcal{B}_0 = 0$), so, by induction on j, all eigenvalues of $\mathcal{A}_j$ are real and at least $2(m_{\max} - j) > 0$; so $\mathcal{A}_j$ is invertible and $\partial_{\mathbf{1}}$ maps $\mathcal{X}_{j+1}$ onto $\mathcal{X}_j$. Hence, for $1 \leq m \leq m_{\max}$, the kernel of $\partial_{\mathbf{1}}$ on $\mathcal{X}_m$ has dimension $\dim \mathcal{X}_m - \dim \mathcal{X}_{m-1} = (\lfloor m/2 \rfloor + 1) - (\lfloor (m - 1)/2 \rfloor + 1)$, which is 1 for even m and 0 for odd m; and at $m = 0$ it is $\mathcal{X}_0 = \mathbb{C}$. □

Corollary 9.174. *For every $n \geq 4$ and every odd a with $n \leq a \leq (n^2 - 2)/2$, $\kappa = n$ is not an eigenvalue of the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$. Hence $\mathrm{K}_n(n)_{(n^2-a-2,a,2)} = 0$, and the conjugation type with conjugate $(n^2 - a - 3, a - 1, 1)$ does not occur in $K_{B(n,2)}$.*

Proof. Let φ be a kernel vector at $\kappa = n$. By Lemma 9.152 and step (1) of the proof of Theorem 9.154, ϑ_{123} is a shift invariant element of $\mathcal{X}_{a-2}$. As a is odd, $a - 2$ is odd, and $a - 2 \leq (n^2 - 2)/2 - 2 = m_{\max}$; so $\vartheta_{123} = 0$ by Theorem 9.173, which is the hypothesis of Corollary 9.160, and $\kappa = n$ is not an eigenvalue. By Corollary 9.31(b) and Lemma 9.32(ii), as in the proof of Corollary 9.155, $f \mapsto \varphi_f$ is an injection of $\mathrm{K}_n(n)_\tau$ into the kernel vectors at $\kappa = n$ ($\tau = (n^2 - a - 2, a, 2)$ is a partition, as $a \leq (n^2 - 2)/2$); so $\mathrm{K}_n(n)_\tau = 0$, and the type has multiplicity $\dim \mathrm{K}_n(n)_\tau = 0$ by Lemma 9.32(iii). □

Remark 9.175. (1) (Scope.) Theorem 9.173 is stated only for $m \leq m_{\max}$, where $\partial_{\mathbf{1}}$ is shown to be onto, and the limit is real: at $n = 4$ and $\kappa = 3$ ($\mathfrak{k} = -1/2$, of the form $-j/q$) a computation finds a shift invariant solution in degree $5 > m_{\max} = 3$. The bound in Theorem 9.172(ii) comes from the proof: at $n = 5$ the dimension $\lfloor m/2 \rfloor + 1$ holds also at $m = 14, 15$ (a computation; not used). For even dominant a, Theorem 9.173 gives a line of shift invariant solutions in degree $a - 2$, so

step (2) fails and Theorem 9.173 does not give the hypothesis of Corollary 9.160; even a are treated in Subsections 9.30 and 9.31 (Corollary 9.189). No bound on $d(n)$ follows.

(2) (What is new.) The conclusion $\mathrm{K}_n(n)_\tau = 0$ of Corollary 9.174 was known at $n = 4$ (Remark 9.10), $n = 5$ (Theorem 9.27), $n = 6$ (Theorem 11.9), $n = 7$ (Corollary 9.164) and at $n = 8$ for $a \le 13$ (Corollary 9.155); it is new at $n = 8$ for $a \ge 15$ and at every $n \ge 9$. The pencil statement is new beyond the 18 pairs of Theorem 9.154 and $n = 7$ (Theorem 9.163).

(3) (Computations, supporting; not used.) The steps were checked by exact computation at small n: Lemmas 9.167 and 9.168 at 15 triples (n, m, κ); Proposition 9.169, line by line, at $n = 4$ ($m \le 8$, $\kappa = 4, 5/2$), $n = 5$ ($m \le 9$, $\kappa = 5$) and $n = 6$ ($m \le 8$); Lemma 9.171(i) with $\mathfrak{k}$ a variable at $n = 4, 5$ in degrees $1, \dots, 5$; and Lemma 9.171(ii)–(iv) (the necessity of the triple products, the two kernels, and that every s with vanishing triple products has an s') at $\mathfrak{k} = -1, -3/2, -2, 0$ in degrees $d \le 6$. Independently, the number of shift invariant solutions at $\kappa = n$ was computed directly (certified): it is 0 for odd a at $n = 7$, at $n = 8$ for odd $a = 9, \dots, 29$, at $n = 9$ for odd $a = 9, \dots, 21$, and at $n = 10, 11, 12$ in the computed cases, where it is 1 for even a.

(4) (Context.) The singular polynomials of S_n are described in [Du04] and [Du05]. This is not used: the bounds of Lemma 9.170(b), (c) are proved above, and the singular set of [DJO94] is used only in Theorem 9.172(i) and, at even n, for the one degree $m = m_{\max}$ in Theorem 9.173.

9.30. **Even a at $\kappa = n$: the triangle row.** In this subsection $F = \mathbb{C}$, $b = 2$, $n \ge 4$ and $\kappa = n$, and we keep the notation of Subsections 9.24, 9.26 and 9.27. For even dominant a, step (2) of the proof of Theorem 9.154 fails: the shift invariant solutions of $(\star_n)$ in degree $a - 2$ form a line (Theorem 9.173), which contains the stable vector ϑ_ξ ($\xi = n^2$) read at this n (Lemma 9.176); it is spanned by it, as $\vartheta_\xi|_n \ne 0$ (Remark 9.181(1) and Theorem 9.188). We show that a kernel vector whose ϑ_{123} lies on that line has data $c\,v' + (\mathsf{F}^{\mathfrak{c}}, 0)$, where v' is the pair of Corollary 9.135 (it spans the stable line) read at n and $\mathsf{F}^{\mathfrak{c}} \in \mathsf{Z}(n, a)$ (Lemma 9.177); that a *triangle* of three order-one rows (two coincident, one generic) vanishes on every $(\mathsf{F}^{\mathfrak{c}}, 0)$ (Lemma 9.178); and so, if the triangle is not 0 on v', that $c = 0$, after which Theorem 9.159 kills $\mathsf{F}^{\mathfrak{c}}$ (Theorem 9.180). Subsection 9.31 shows by hand that the triangle is not 0 on v', at every n.

Setting. Let a be even, $n \le a \le (n-1)^2$, $\mu := n^2 - a$ (so $\mu \ge 2n - 1 > 0$), $M := \mu(a + 1)$, $\beta = n/2$, $m := a - 2$ and $\xi := n^2$. By Theorem 9.141(iii), $\ker \mathbf{B}(\xi)$ is a line (as m is even); let $c = (c_p)$ span it, and let $\vartheta_\xi = \sum_p c_p \mathbf{R}_p \mathbf{R}_{m-p}$ and $w_\xi \in \mathrm{Sym}^\perp_a$ be as after Lemma 9.142 (so $\vartheta_{w_\xi} = \vartheta_\xi$ by Lemma 9.133(iv)). Put

$$v' := v'(\mu, w_\xi) = (\mathsf{F}', \mathsf{G}') = \big(g[s,t] + M w_\xi,\ -2n w_\xi\big), \qquad g'(s) = 2n\langle w_\xi(s,u)\rangle_u - M w_\xi(s,s),$$

the pair of Corollary 9.135. These are *formal* identities in the slots and $p_1, p_2, \dots$. *No stable range is used.* Subsection 9.24 assumes $n \ge a + 6$, but every result of it cited below is an identity in $\mathbb{C}[\text{slots}, p_1, p_2, \dots]$ with $p_0 = n$ a parameter: Lemma 9.132; Lemma 9.133(i), (iv); the existence part of Theorem 9.134; the definition of v' in Corollary 9.135 (not its conclusion); Lemma 9.139(ii); Theorem 9.141(i)–(iii). The stable range enters there only in (a) the two-point form and the uniqueness of the data of an actual φ, (b) the independence statement for $n \ge N + 4$ (used in the proof of Corollary 9.137 and in the probe of Theorem 9.146), (c) the conclusions of Corollary 9.135 and of Theorem 9.143(2), and (d) Corollary 9.144; none of them is used here. For kernel vectors we use only results proved at a fixed n: Lemma 9.46, the first sentence of Corollary 9.47, Lemmas 9.152 and 9.153, Proposition 9.157, Theorem 9.159 and step (6) of the proof of Theorem 9.154.

Reading at n. For a formal pair (F, G) of symmetric two-point functions let $(\mathsf{F}, \mathsf{G})|_n$ be the torus family $\mathsf{F}_{rt} = \mathsf{F}(x_r, x_t)$ $(r \ne t)$, $\mathsf{G}_{rt} = \mathsf{G}(x_r, x_t) + \delta_{rt}\mathsf{F}(x_r, x_r)$, with $p_k := \sum_{l=1}^n x_l^k$ ($p_0 = n$). Every formal identity specializes.

Lemma 9.176 (reading at n)**.** *Write F', G', H' for the torus data of $v'|_n$ and their H. Then:*
(f1) $\mathsf{H}'_{rwt} = M\, x_{rw}x_{wt}x_{tr}\, \vartheta_\xi(x_r, x_w, x_t)$;
(f2) $\sum_t \mathsf{G}'_{jt} = 0$;

(f3) $M\,\mathsf{G}'(x_j,x_t)=\frac{2}{x_{tj}}\sum_{w\notin\{t,j\}}\mathsf{H}'_{twj}$ *for* $j\neq t$*;*
(f4) $\mathsf{F}'|_n$ *and* $\mathsf{G}'|_n$ *are shift invariant, homogeneous of degree* a*, symmetric and* S_n*-equivariant;*
(f5) $\mathsf{G}'_{rr}=\mathsf{F}'_{rt}+\mathsf{G}'_{rt}$ *on* $x_r=x_t$*.*

Moreover $\vartheta_\xi|_n$ *solves* $(\star_n)$*, is symmetric in* x_1,x_2,x_3 *and in* $x_4,\dots,x_n$*, and is shift invariant; so* $\mathbb{C}\,\vartheta_\xi|_n$ *lies in the space of shift invariant solutions of* $(\star_n)$ *in degree* m*.*

Proof. (f1) Lemma 9.133 gives $\mathsf{H}(s,v,t)=(s-v)(v-t)(t-s)\vartheta_{\mathsf{F}}(s,t,v)$, with $\vartheta_{g[s,t]}=0$ (by its part (i)) and $\vartheta_{w_\xi}=\vartheta_\xi$ (by its part (iv)). (f2) (C1) holds for v' by the existence construction of Theorem 9.134 ($g'=-\langle\mathsf{G}'(s,u)\rangle_u-Mw_\xi(s,s)$). (f3) Corollary 9.47 is an identity at every n: the sum is $\mathfrak{h}(x_t)-\mathfrak{h}(x_j)-nx_{tj}\mathsf{F}'(x_t,x_j)$ with $\mathfrak{h}=\mathfrak{h}_{\mathsf{F}'}$; formally $\mathfrak{h}_{\mathsf{F}'}=ng-\langle g\rangle$ (Lemma 9.132: $\mathfrak{h}_{w_\xi}=0$), so $2(\mathfrak{h}_{\mathsf{F}'}[s,t]-n\mathsf{F}')=-2nMw_\xi=M\mathsf{G}'$. (f4) F' and G' are symmetric, shift invariant and of weighted degree a (Theorem 9.134 and Lemma 9.132); reading at n keeps this and gives S_n-equivariant families. (f5) By the convention $\mathsf{G}_{rr}=\mathsf{G}(x_r,x_r)+\mathsf{F}(x_r,x_r)$. Finally, ϑ_ξ solves $(\star)$ formally (Lemma 9.139(ii) with Theorem 9.141(i); Corollary 9.137 supplies only the equation, and its conclusion for kernel vectors uses the stable range), and $(\star)$ specializes to $(\star_n)$ (in the form of Lemma 9.152: ∂_3 of a function of x_1,x_2,x_4 and the p_k is $\mathcal{D}_{x_3}$); $\vartheta_\xi|_n$ is symmetric as stated, and shift invariant by Theorem 9.141(ii). □

For even a with $n\le a\le(n-1)^2$ consider the condition

(2′) every kernel vector at $\kappa=n$ has $\vartheta_{123}\in\mathbb{C}\,\vartheta_\xi|_n$.

So (2′) says that ϑ_{123} of every kernel vector lies on $\mathbb{C}\,\vartheta_\xi|_n$; it holds when $\mathbb{C}\,\vartheta_\xi|_n$ is the whole space of shift invariant solutions of $(\star_n)$ in degree m (see Corollary 9.189).

Lemma 9.177 (the even data)**.** *Let* φ *be a kernel vector at* $\kappa=n$ *with data* (F,G) *and* $\vartheta_{123}=c\,\vartheta_\xi|_n$*. Put* $\mathsf{D}:=M(\mathsf{F},\mathsf{G})-c\,v'|_n$ *(torus families). Then* $\mathsf{G}^{\mathsf{D}}=0$ *and* $\mathsf{F}^{\mathsf{D}}\in\mathsf{Z}(n,a)$*; so* $M(\mathsf{F},\mathsf{G})=c\,v'|_n+(\mathsf{F}^{\mathfrak{c}},0)$ *with* $\mathfrak{c}$ *shift invariant and homogeneous of degree* $a-n$ *(Proposition 9.157(b)).*

Proof. (i) $\mathsf{H}^{\mathsf{D}}=0$: $\mathsf{H}_{rwt}(\mathsf{F})=x_{rw}x_{wt}x_{tr}\vartheta_{rwt}$ and $\vartheta_{rwt}=c\,\vartheta_\xi(x_r,x_w,x_t)$ (both sides are obtained from the 123 one by permuting coordinates), so $\mathsf{H}(M\mathsf{F})=c\,\mathsf{H}(\mathsf{F}'|_n)$ by (f1). (ii) $\mathsf{G}^{\mathsf{D}}_{jt}=0$ for $j\neq t$: Lemma 9.46 for φ ($\mu\neq0$) gives $M\mathsf{G}_{jt}=\frac{2}{x_{tj}}\sum_w\mathsf{H}_{twj}$; with (f3), $M\mathsf{G}^{\mathsf{D}}_{jt}=\frac{2}{x_{tj}}\sum_w\mathsf{H}^{\mathsf{D}}_{twj}=0$, at distinct coordinates and so as polynomials. (iii) $\mathsf{G}^{\mathsf{D}}_{jj}=0$: $\mathsf{G}\mathbf{1}=0$ for φ (Subsection 9.13, the torus form) and (f2) give $\mathsf{G}^{\mathsf{D}}\mathbf{1}=0$. (iv) $\mathsf{F}^{\mathsf{D}}_{rt}=0$ on $x_r=x_t$: Lemma 9.153 for φ and (f5) for v' give $\mathsf{G}^{\mathsf{D}}_{rr}=\mathsf{F}^{\mathsf{D}}_{rt}+\mathsf{G}^{\mathsf{D}}_{rt}$ there; use (ii) and (iii). (v) F^{D} is symmetric, equivariant, shift invariant and homogeneous of degree a by (f4), with $\mathsf{H}^{\mathsf{D}}=0$ and (iv): so $\mathsf{F}^{\mathsf{D}}\in\mathsf{Z}(n,a)$. □

The triangle. For distinct p,p',i,j and any torus data (F,G) put

$$\mathsf{T}^{(ij)}_{pp'}(\mathsf{F},\mathsf{G}):=x_{pi}\Psi_{pi}(e_i,e_j)-x_{p'i}\Psi_{p'i}(e_i,e_j)-x_{pp'}\Psi_{pp'}(e_i,e_j),$$

with the rows of Proposition 9.44, each in its *shifted* form of Subsection 9.25 (shift $c=-x_p$ for the row's own p; this is the form used in every computation below). Here $\Psi_{pi}(e_i,e_j)$ is the coincident row $\Psi_{pq}(e_q,e_l)$ at $(q,l)=(i,j)$, and $\Psi_{pp'}(e_i,e_j)$ is a generic row.

Lemma 9.178 (cycles kill Z)**.** $\mathsf{T}^{(ij)}_{pp'}(\mathsf{F}^{\mathfrak{c}},0)=0$ *identically in* x*, for every* κ *and every shift invariant homogeneous* $\mathfrak{c}$*.*

Proof. On $(\mathsf{F}^{\mathfrak{c}},0)$, with $h=\mathfrak{H}_{\mathfrak{c}}$ and $\mathfrak{K}_{ij}=\kappa(h_j-h_i)-x_{ij}h_{ij}$: the generic row is $\Psi_{pq}(e_i,e_j)=\mathfrak{K}_{ij}[x_p,x_q]$ (proof of Proposition 9.158; that step uses no degree bound), and the coincident row is $\Psi_{pi}(e_i,e_j)=\mathfrak{K}_{ij}[x_p,x_i]$ (step (1) of the proof of Theorem 9.159). So $x_{pq}\Psi_{pq}(e_i,e_j)=\mathfrak{K}_{ij}(x_p)-\mathfrak{K}_{ij}(x_q)$ for $q\in\{i,p'\}$, and $\mathsf{T}=[\mathfrak{K}(x_p)-\mathfrak{K}(x_i)]-[\mathfrak{K}(x_{p'})-\mathfrak{K}(x_i)]-[\mathfrak{K}(x_p)-\mathfrak{K}(x_{p'})]=0$. □

The shifted form is safe for both uses: every shifted row vanishes on a kernel vector (Proposition 9.44 holds with every explicit coefficient x_l replaced by x_l+c; apply it at each point), and $(\mathsf{F}^{\mathfrak{c}},0)$ is the data of the actual function $\varphi_{\mathfrak{c}}$ (Proposition 9.157(d)), so its rows do not depend on the shift. (The same telescoping kills every closed path in which each edge $\{q,r\}$ enters as $\pm x_{qr}\Psi_{qr}(e_i,e_j)$ with $q\notin\{i,j\}$; for example the square $x_{pi}\Psi_{pi}(e_i,e_j)-x_{p'i}\Psi_{p'i}(e_i,e_j)+x_{p'j}\Psi_{p'j}(e_i,e_j)-x_{pj}\Psi_{pj}(e_i,e_j)$ of coincident rows only, as $\Psi_{pj}(e_i,e_j)=-\Psi_{pj}(e_j,e_i)=-\mathfrak{K}_{ji}[x_p,x_j]=\mathfrak{K}_{ij}[x_p,x_j]$.)

Fix distinct p,p',i,j once and for all, and put $\mathsf{T}_{v'}:=\mathsf{T}^{(ij)}_{pp'}(v'|_n)$, a polynomial in x (Theorem 9.180 needs $\mathsf{T}_{v'}\neq0$ at this one choice only). (If $\vartheta_\xi|_n=0$, then $(2')$ is step (2) and Corollary 9.160 applies; this case does not occur when $\mathsf{T}_{v'}\neq0$, see Remark 9.181(1).)

Lemma 9.179 (the triangle sees only ϑ)**.** *In two-point form (formal, or read at any n), $\mathsf{T}=0$ on $(0,\tilde g)$ for every symmetric $\tilde g$ and on $(\eta[s,t],0)$ for every $\eta(\tau)$. So $\mathsf{T}(\mathsf{F},\mathsf{G})$ depends only on ϑ_{F}, and $\mathsf{T}_{v'}=M\,\mathsf{T}((w_\xi,0)|_n)$.*

Proof. Write the generic row of Subsection 9.25 (the row P, shift $c=-x_p$) with $y=P:=x_p$, $y'=Q:=x_q$, $u_1=A:=x_i$, $u_2=B:=x_j$; the $\Lambda_{[1]}$-term is $-(\Lambda(P,Q;B)-\Lambda(A,Q;B))$ after the factor $A-P$, so

$$\begin{aligned}\mathsf{R}=\kappa\big(\Lambda(P,Q;B)-\Lambda(P,Q;A)\big)-(A-P)\mathcal{D}_A\Lambda(P,Q;B)+(B-P)\mathcal{D}_B\Lambda(P,Q;A)\\ -(A-P)\Lambda(P,[Q,A];B)+(B-P)\Lambda(P,[Q,B];A)-\Lambda(P,Q;B)+\Lambda(A,Q;B)\\ +\Lambda(P,Q;A)-\Lambda(B,Q;A)-\big(\mathcal{D}_A\mathsf{G}(Q,B)-\mathcal{D}_B\mathsf{G}(Q,A)\big).\end{aligned}$$

On $(0,\tilde g)$: $\Lambda(s,s';t)=\gamma_t[s,s']$ with $\gamma_t(\tau):=-\tilde g(\tau,t)$; the divided-difference terms cancel in pairs ($\gamma_B[P,Q,A]$ twice), and with $\gamma_{BA}:=-\mathcal{D}_A\tilde g(\cdot,B)$, $\gamma_{AB}:=-\mathcal{D}_B\tilde g(\cdot,A)$ one gets

$$(P-Q)\mathsf{R}=\widetilde{\mathfrak{K}}(P)-\widetilde{\mathfrak{K}}(Q),\qquad \widetilde{\mathfrak{K}}(\tau):=\kappa(\gamma_B-\gamma_A)(\tau)+(\tau-A)\gamma_{BA}(\tau)-(\tau-B)\gamma_{AB}(\tau)$$

(the G-term of the row cancels the two non-telescoping pieces $(Q-P)\gamma_{BA}(Q)$ and $-(Q-P)\gamma_{AB}(Q)$). *On $(\eta[s,t],0)$:* $\Lambda=\eta_t[s,s']$ with $\eta_t:=\mathcal{D}_t\eta$; the divided-difference terms cancel as before, and $(P-Q)\mathsf{R}=\widetilde{\mathfrak{K}}(P)-\widetilde{\mathfrak{K}}(Q)$ with $\widetilde{\mathfrak{K}}=\kappa(\eta_B-\eta_A)-(A-B)\eta_{AB}$, $\eta_{AB}:=\mathcal{D}_A\mathcal{D}_B\eta$ (the $\mathfrak{K}_{AB}$ of Subsection 9.27; the two non-telescoping pieces $(Q-P)\eta_{AB}(Q)$ cancel each other). The coincident rows are these formulas at $Q:=A$ (Lemma 9.74(ii), for every symmetric datum), with the *same* $\widetilde{\mathfrak{K}}$ (it depends on (A,B) only), so every closed path telescopes. For two-point data read at n, Lemma 9.74(ii) is an identity between *formulas*—two-point algebra and the chain rule $d/dx_q=\partial_{\mathrm{slot}}+\mathcal{D}_{x_q}$—valid at every n; step (1) of the proof of Theorem 9.159 is an instance. What fails at a given n is the different statement "the generic rows vanish, therefore the coincident rows vanish" (Remark 9.161(2)). Finally $v'=(g[s,t]+Mw_\xi,-2nw_\xi)=M(w_\xi,0)+(g[s,t],0)+(0,-2nw_\xi)$. □

Theorem 9.180. *Let $n\geq4$, a even with $n\leq a\leq(n-1)^2$, $\kappa=n$ and $\xi=n^2$, and assume $(2')$. If the polynomial $\mathsf{T}_{v'}$ is not identically 0, then $\kappa=n$ is not an eigenvalue of the pencil $\kappa\mathcal{Z}-\mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$.*

Proof. Let φ be a kernel vector at $\kappa=n$. By $(2')$ and Lemma 9.177, $M(\mathsf{F},\mathsf{G})=c\,v'|_n+(\mathsf{F}^{\mathfrak{c}},0)$. Every row of Proposition 9.44 is linear in the torus data and vanishes on (F,G); so $0=c\,\mathsf{T}_{v'}+\mathsf{T}(\mathsf{F}^{\mathfrak{c}},0)=c\,\mathsf{T}_{v'}$ by Lemma 9.178, and $c=0$ as $\mathsf{T}_{v'}\neq0$. Then $M(\mathsf{F},\mathsf{G})=(\mathsf{F}^{\mathfrak{c}},0)$ passes every coincident order-one row at $\kappa=n$, and $a\leq(n-1)^2$, so $\mathsf{F}^{\mathfrak{c}}=0$ by Theorem 9.159; then $\mathsf{F}=\mathsf{G}=0$, and $\varphi=0$ by step (6) of the proof of Theorem 9.154. □

Remark 9.181. (1) $\mathsf{T}_{v'}\neq0$ forces $\vartheta_\xi|_n\neq0$: if $\vartheta_\xi|_n=0$, then $\mathsf{H}(\mathsf{F}'|_n)=0$ by (f1), $\mathsf{G}'_{jt}=0$ $(j\neq t)$ by (f3), $\mathsf{G}'_{jj}=0$ by (f2), $\mathsf{F}'|_n\in\mathsf{Z}(n,a)$ by (f5), and $\mathsf{T}_{v'}=0$ by Lemma 9.178.

(2) (Uses.) Theorem 9.180 uses $\mu\neq0$ (Lemma 9.46), $n\geq4$ (four indices), $a\geq n$ (Proposition 9.157(b), Theorem 9.159) and $a\leq(n-1)^2$ (Theorem 9.159); dominance, $a\leq(n^2-2)/2$,

only makes $\tau = (n^2 - a - 2, a, 2)$ a partition. Its proof uses only Lemmas 9.177 and 9.178, Theorem 9.159 and step (6) of the proof of Theorem 9.154; Lemma 9.179 serves the computation of $\mathsf{T}_{v'}$ in Subsection 9.31. For odd a the same argument with $c = 0$ is Corollary 9.160.

(3) (Structure; only the two-point form of $\mathsf{T}_{v'}$ is used below, in Subsection 9.31.) In two-point form write $\mathsf{R}(P, Q; A, B) := \Psi_{pq}(e_i, e_j)$ at $(x_p, x_q, x_i, x_j) = (P, Q, A, B)$ and $\mathsf{X}(P, Q; B) := \mathsf{R}(P, Q; Q, B)$ $(= \Psi_{pq}(e_q, e_l)$, Lemma 9.74(ii), as an identity between formulas). Then, with $Q := x_{p'}$, $\mathsf{T}_{v'} = (P - A)\mathsf{X}(P, A; B) - (Q - A)\mathsf{X}(Q, A; B) - (P - Q)\mathsf{R}(P, Q; A, B)$. R is antisymmetric in (A, B), and on v' its odd part $\mathsf{R}(P, Q; \dots) - \mathsf{R}(Q, P; \dots)$ is 0 (Lemma 9.136 and the formal $(\star)$: the display of Corollary 9.137 with $\vartheta_{\mathsf{F}'} = M\vartheta_\xi$, which solves $(\star)$ formally by Lemma 9.139; $(\star_n)$ at distinct points is not enough for these substitutions). Hence $\mathsf{T}_{v'}$ vanishes at $P = Q$, at $Q = A$, at $P = A$ and at $A = B$, so it is divisible by $(P - Q)(P - A)(Q - A)(A - B)$; the closed form of Corollary 9.183 exhibits this factor.

(4) (Computations, supporting; superseded by Corollary 9.189.) $\mathsf{T}_{v'}$ was evaluated exactly at a point of $\mathbb{C}^n$ (a nonzero value certifies $\mathsf{T}_{v'} \neq 0$ as a polynomial): from the rows directly at 9 pairs (n, a), and by the seed device of Subsection 9.25 at all 66 dominant even pairs with $4 \le n \le 10$, with no zero; the two computations agree at the 9 pairs, and an independent program from the rows as printed in Proposition 9.44 finds $\mathsf{T}_{v'} \neq 0$ at all 66. (At points with many zero coordinates the top degrees die at even n, as β is an integer; a generic point is needed.) With the certified bound ≤ 1 for the dimension of the shift invariant solutions of $(\star_n)$ at 18 of these pairs, or with Theorem 9.173 at all 66, Theorem 9.180 gave $\kappa = n$ not an eigenvalue there before Subsection 9.31. Controls: $\mathsf{T} = 0$ on $(\mathsf{F}^{\mathfrak{c}}, 0)$ (for $\mathfrak{c} = 1$ and $\tau - p_1/4$ at $n = 4$, $(\tau - p_1/5)^2$ at $n = 5$) while the single rows are not 0, and $\mathsf{T} = 0$ as a formal polynomial on random $(0, g)$ and $(\eta[s, t], 0)$.

9.31. **The triangle is not 0, and the layer $\tau_3 = 2$ at every n.** In this subsection $F = \mathbb{C}$, $b = 2$, $n \ge 4$, $\kappa = n = 2\beta$, and a is even with $n \le a \le (n^2 - 2)/2$; we keep the notation of Subsection 9.30. We compute $\mathsf{T}_{v'}$ in closed form by hand (Proposition 9.182 and Corollary 9.183): it is $M(\beta^2 - 1)$ times four linear forms times a plethystic value of $\mathbb{L}f_\xi$, the same operator $\mathbb{L} = (h_2 - 3e_2)^\perp$ and the same factor $1 - \beta^2$ as in the stable probe of Theorem 9.146. We then show that this value is not the zero polynomial (Proposition 9.187), so $\mathsf{T}_{v'} \neq 0$ at every n (Theorem 9.188), and with Theorems 9.180 and 9.173 $\kappa = n$ is not an eigenvalue for even a (Corollary 9.189). With Corollary 9.174 for odd a, this settles the layer $\tau_3 = 2$ of $K_{B(n,2)}$ at every $n \ge 4$.

Setting. In two-point form (Remark 9.181(3)), with $(P, Q, A, B) := (x_p, x_{p'}, x_i, x_j)$, $\mathsf{T}_{v'} = (P - A)\mathsf{X}(P, A; B) - (Q - A)\mathsf{X}(Q, A; B) - (P - Q)\mathsf{R}(P, Q; A, B)$, the rows in the shifted form of Subsection 9.25; at a given n, Lemma 9.74(ii) is used as an identity between formulas (Lemma 9.179), not as the vanishing statement of Remark 9.161(2). Every input from Subsections 9.24 and 9.25 below is a formal identity in the slots and $p_1, p_2, \dots$ ($p_0 = n$ a parameter); $n \ge a + 6$ is not used; $\mathsf{T}_{v'}$ is the formal T read at n, so that $\mathbf{E}(z) \mapsto \prod_l (1 - x_l z)^\beta$. By Lemma 9.179, $\mathsf{T}(\mathsf{F}, \mathsf{G})$ depends only on ϑ_F, and $\mathsf{T}_{v'} = M\,\mathsf{T}((w_\xi, 0))$; for $\mathsf{G} = 0$ the row is

$$\begin{aligned}\mathsf{R} = \kappa\big(\Lambda(P, Q; B) - \Lambda(P, Q; A)\big) - (A - P)\mathcal{D}_A\Lambda(P, Q; B) + (B - P)\mathcal{D}_B\Lambda(P, Q; A)\\ - (A - P)\Lambda(P, [Q, A]; B) + (B - P)\Lambda(P, [Q, B]; A) - \Lambda(P, Q; B) + \Lambda(A, Q; B)\\ + \Lambda(P, Q; A) - \Lambda(B, Q; A),\end{aligned}$$

$\Lambda(s, s'; t) := \mathcal{D}_t\mathsf{F}(s, s') + (s' - s)\vartheta_\mathsf{F}(s, s', t)$, with $\kappa = 2\beta = n$. Here $\vartheta_\xi = \sum_{p \le m/2} c_p \mathbf{R}_p \mathbf{R}_{m-p}$ (the definition after Lemma 9.142; $\ker \mathbf{B}(\xi)$ is a line by Theorem 9.141(iii)), $m = a - 2$. So $\vartheta_\xi = \sum_p c_p [z_1^p z_2^{m-p}]\,\mathrm{seed}$, with

$$\mathrm{seed} := K\,U(s)U(t)U(v), \qquad K := \mathbf{E}(z_1)\mathbf{E}(z_2), \qquad U(y) := \frac{1}{(1 - yz_1)(1 - yz_2)},$$

and, as $\mathcal{D}^{[r]}\mathbf{E} = -\beta\tau^{r+1}\mathbf{E}$, we have $\mathcal{D}_u K = -\beta\,\ell_u K$ with $\ell_u = U'(u)/U(u) = z_1/(1 - uz_1) + z_2/(1 - uz_2)$, as in Subsection 9.25. T is linear and acts on the slots and the p_k only, so $\mathsf{T}_{v'} =$

$M\sum_p c_p[z_1^p z_2^{m-p}]\,\mathsf{T}[\text{seed}]$ (coefficientwise in z_1, z_2), where $\mathsf{T}[\text{seed}] := \mathsf{T}(\mathsf{F}, 0)$ with F of Step 0 below, by Lemma 9.179. This is the seed device of Subsection 9.25 (the Cauchy identity [Ma95, I, §4]; the probe is linear in ϑ), here at a finite n.

Proposition 9.182 (the seed identity)**.** *For every β (with $\kappa = 2\beta$), $\mathsf{T}[\text{seed}]/K$ is a rational function of $z_1, z_2, P, Q, A, B, \beta$, and, coefficientwise in z_1, z_2,*

$$\mathsf{T}[\text{seed}] = K\,(\beta^2-1)\,(z_1-z_2)^2\,(P-Q)(P-A)(Q-A)(A-B)\,U(P)U(Q)U(A)^2U(B).$$

Proof. Put $\delta := z_1 - z_2$, $e := z_1 z_2$, $\alpha_1(y) := z_1/(1-yz_1)$ and $\alpha_2(y) := z_2/(1-yz_2)$ (slot functions; no p_k). In this proof δ, e (a scalar, not an elementary symmetric function), α_1, α_2, Π, V, W, σ_1, σ_2 are local symbols. (P1) $\alpha_1(y) - \alpha_1(y') = (y-y')\alpha_1(y)\alpha_1(y')$, and the same for α_2 (as $1/\alpha_1(y) = 1/z_1 - y$). (P2) $U(y) = (\alpha_1(y) - \alpha_2(y))/\delta = \alpha_1(y)\alpha_2(y)/e$ and $\ell_y = \alpha_1(y) + \alpha_2(y)$. (P3) $U[y, y'] = (\alpha_1(y)\alpha_1(y') - \alpha_2(y)\alpha_2(y'))/\delta$ (by (P1), (P2)); at $y' = y$, $U'(y) = U(y)\ell_y$. (P4) $\alpha_1(y)\alpha_2(y') - \alpha_1(y')\alpha_2(y) = (y-y')\,\delta e\,U(y)U(y')$: the left side is $\alpha_1(y)\alpha_1(y')\alpha_2(y)\alpha_2(y')[(1/z_1 - y')(1/z_2 - y) - (1/z_1 - y)(1/z_2 - y')] = e^2U(y)U(y')(y-y')(1/z_2 - 1/z_1)$.

Step 0 (a representative). $\mathsf{F} := -K\,U(s)U(s')/e$ has $\vartheta_{\mathsf{F}} = \text{seed}$. Indeed, with $\mathsf{H}(s,t) = (s-t)\mathsf{F}(s,t)$ (Lemma 9.133), $\mathsf{H}(s,v,t) = -(K/e)U(s)U(v)U(t)\sum_{\text{cyc}}(s-v)\Pi(t)$ with $\Pi(y) := 1/U(y) = 1 - (z_1+z_2)y + ey^2$, and $\sum_{\text{cyc}}(s-v)t^k = 0, 0, -(s-v)(v-t)(t-s)$ for $k = 0, 1, 2$; so $\mathsf{H}(s,v,t) = (s-v)(v-t)(t-s)K\,U(s)U(v)U(t)$. (Coefficientwise in z_1, z_2 the exponents of F are ≥ -1, and the coefficients with a negative exponent have $\vartheta = 0$, the seed having none; so by Lemma 9.179 they add nothing to T.) By Lemma 9.179, $\mathsf{T}[\text{seed}] = \mathsf{T}(\mathsf{F}, 0)$, and

$$\Lambda = (\beta/e)\Lambda_\ell + \Lambda_u, \qquad \Lambda_\ell := K\,U(s)U(s')\ell_t, \qquad \Lambda_u := K(s'-s)U(s)U(s')U(t).$$

Step 1 (powers of β). $\mathcal{D}$ acts on K only ($\mathcal{D}_u \mapsto -\beta\ell_u$), so $\mathsf{R} = \beta\mathsf{R}^1[\Lambda] + \mathsf{R}^0[\Lambda]$ with

$$\mathsf{R}^1[\Lambda] := 2\big(\Lambda(P,Q;B) - \Lambda(P,Q;A)\big) + (A-P)\ell_A\Lambda(P,Q;B) - (B-P)\ell_B\Lambda(P,Q;A),$$

and $\mathsf{R}^0[\Lambda]$ the six β-free terms (the two divided differences and the four swaps). Write $W(x,y) := (x-y)\mathsf{R}(x,y;A,B)$ (row point x, second slot y; at $y = A$ the coincident row is the continuous extension) and $\mathsf{T}(W) := W(P,A) - W(Q,A) - W(P,Q)$; then $\mathsf{T}[\text{seed}] = \mathsf{T}(W)$, and so, with $\mathsf{T}^i[\Lambda]$ the value of T on $(x-y)\mathsf{R}^i[\Lambda]$,

$$\mathsf{T}[\text{seed}] = (\beta^2/e)\,\mathsf{T}^1[\Lambda_\ell] + \beta\big(\mathsf{T}^1[\Lambda_u] + \mathsf{T}^0[\Lambda_\ell]/e\big) + \mathsf{T}^0[\Lambda_u].$$

Step 2 (the rows). (2a) $\mathsf{R}^1[\Lambda_\ell] = K\,U(x)U(y)\big[2(\ell_B - \ell_A) + (A-B)\ell_A\ell_B\big]$ (the x-terms cancel) $= -K(A-B)\delta^2U(A)U(B)U(x)U(y)$, since $2(\ell_A - \ell_B) - (A-B)\ell_A\ell_B = (A-B)[2\alpha_1(A)\alpha_1(B) + 2\alpha_2(A)\alpha_2(B) - \ell_A\ell_B] = (A-B)(\alpha_1(A) - \alpha_2(A))(\alpha_1(B) - \alpha_2(B))$. (2b) $\mathsf{R}^1[\Lambda_u] + \mathsf{R}^0[\Lambda_\ell]/e = 0$. Here $\mathsf{R}^1[\Lambda_u] = K(y-x)U(x)U(y)\big[2(U(B) - U(A)) + (A-x)\ell_AU(B) - (B-x)\ell_BU(A)\big]$, and $\mathsf{R}^0[\Lambda_\ell] = K(\ell_B\chi^{(\ell)}(A) - \ell_A\chi^{(\ell)}(B))$ with $\chi^{(\ell)}(A) := -(A-x)U(x)U[y,A] - U(x)U(y) + U(A)U(y) = (y-x)(U(A)U[x,y] - U(x)U[y,A])$ (write $(A-x)/(y-A) = -1 + (y-x)/(y-A)$) $= e(y-x)(A-x)U(A)U(x)U(y)$ (by (P2), (P3): $\delta^2(U(A)U[x,y] - U(x)U[y,A]) = (\alpha_1(y) - \alpha_2(y))(\alpha_1(A)\alpha_2(x) - \alpha_1(x)\alpha_2(A))$, then (P4)). The sum is $K(y-x)U(x)U(y)\big[2(U(B) - U(A)) + (A-B)(\ell_AU(B) + U(A)\ell_B)\big]$, and the bracket is 0: $U(B) - U(A) = (B-A)(\alpha_1(A)\alpha_1(B) - \alpha_2(A)\alpha_2(B))/\delta$ and $\ell_AU(B) + U(A)\ell_B = 2(\alpha_1(A)\alpha_1(B) - \alpha_2(A)\alpha_2(B))/\delta$. (2c) $\mathsf{R}^0[\Lambda_u] = K(A-B)U(A)U(B)\big[e(y-x)^2U(x)U(y) - (U(x)+U(y))\big]$. Indeed $\mathsf{R}^0[\Lambda_u] = K(U(B)\chi^{(u)}(A) - U(A)\chi^{(u)}(B))$ with $\chi^{(u)}(A) := -(y-x)^2U(x)U[y,A] - (y+A-2x)U(x)U(A) + (y-A)U(A)U(y)$ (the same rewriting of $(A-x)/(y-A)$, and $[(\cdot - x)U][y,A] = (y-x)U[y,A] + U(A)$); so $\mathsf{R}^0[\Lambda_u] = -K(y-x)^2U(x)(U(B)U[y,A] - U(A)U[y,B]) + K(B-A)U(A)U(B)(U(x) + U(y))$, and $U(B)U[y,A] - U(A)U[y,B] = (\alpha_1(A)\alpha_2(B) - \alpha_1(B)\alpha_2(A))(\alpha_2(y) - \alpha_1(y))/\delta^2 = -(A-B)e\,U(A)U(B)U(y)$ by (P4). (All three are symmetric in x, y—a consistency check, in line with Corollary 9.137.) *Step 3 (the triangle).* With $V := (P-Q)(P-A)(Q-A)$, $\sigma_1 := P+Q+A$, $\sigma_2 := PQ+QA+AP$ and $\Pi = 1/U$: for $\Sigma_k := (P-A)^3Q^k - (Q-A)^3P^k - (P-Q)^3A^k$ and $\Sigma'_k := (P-A)Q^k - (Q-A)P^k - (P-Q)A^k$

$(k=0,1,2)$, putting $P=A+u_1$, $Q=A+u_2$ gives $\Sigma_0=3V$, $\Sigma_1=V\sigma_1$, $\Sigma_2=V\sigma_2$ and $\Sigma_0'=\Sigma_1'=0$, $\Sigma_2'=-V$. Hence

$$\mathsf{T}\big((x-y)U(x)U(y)\big)=U(P)U(Q)U(A)\,(\Sigma_0'-(z_1+z_2)\Sigma_1'+e\Sigma_2')=-e\,V\,U(P)U(Q)U(A),$$
$$\mathsf{T}\big((x-y)^3U(x)U(y)\big)=V\big(3-(z_1+z_2)\sigma_1+e\sigma_2\big)U(P)U(Q)U(A),$$
$$\mathsf{T}\big((x-y)(f(x)+f(y))\big)=f(A)(P-Q)+f(P)(Q-A)+f(Q)(A-P)=V\,f[P,Q,A],$$

and for $f=U$, $U[P,Q,A]=\big((z_1+z_2)^2-e-e(z_1+z_2)\sigma_1+e^2\sigma_2\big)U(P)U(Q)U(A)$ (from $U[P,Q]=((z_1+z_2)-e(P+Q))U(P)U(Q)$, as $\Pi[P,Q]=-(z_1+z_2)+e(P+Q)$, and one more divided difference). *Step 4 (assembly).* Put $\Delta_0:=K\delta^2V(A-B)U(P)U(Q)U(A)^2U(B)$. Then $\mathsf{T}^1[\Lambda_\ell]=-K(A-B)\delta^2U(A)U(B)\cdot(-eVU(P)U(Q)U(A))=e\Delta_0$ by (2a); $\mathsf{T}^1[\Lambda_u]+\mathsf{T}^0[\Lambda_\ell]/e=0$ by (2b); and by (2c) and Step 3, $\mathsf{T}^0[\Lambda_u]=K(A-B)U(A)^2U(B)U(P)U(Q)V\big[4e-(z_1+z_2)^2\big]=-\Delta_0$, as $(z_1+z_2)^2-4e=\delta^2$. So $\mathsf{T}[\text{seed}]=(\beta^2/e)e\Delta_0-\Delta_0=(\beta^2-1)\Delta_0$. □

For a symmetric function f, $Y:=\{P,Q,A,A,B\}$ and the alphabet $X:=\{x_1,\dots,x_n\}$ let $f[Y-\beta X]$ be the image of f under $h_k\mapsto[t^k]\prod_{y\in Y}(1-yt)^{-1}\prod_l(1-x_lt)^\beta$, and put $f_\xi:=\sum_{p=0}^{m/2}c_ph_ph_{m-p}$ (Subsection 9.25) and $\mathsf{g}:=\mathbb{L}f_\xi$ (not the g of v').

Corollary 9.183 (closed form)**.** $\mathsf{T}_{v'}=M(\beta^2-1)(P-Q)(P-A)(Q-A)(A-B)\,\mathsf{g}[Y-\beta X]$.

Proof. By Proposition 9.182, $\mathsf{T}_{v'}=M(\beta^2-1)(P-Q)(P-A)(Q-A)(A-B)\sum_p c_p[z_1^pz_2^{m-p}](z_1-z_2)^2H(z_1)H(z_2)$ with $H(z):=\mathbf{E}(z)/((1-Pz)(1-Qz)(1-Az)^2(1-Bz))=\sum_k h_k[Y-\beta X]z^k$, so that $K\,U(P)U(Q)U(A)^2U(B)=H(z_1)H(z_2)$. And

$$[z_1^pz_2^{m-p}](z_1-z_2)^2H(z_1)H(z_2)=(h_{p-2}h_{m-p}-2h_{p-1}h_{m-p-1}+h_ph_{m-p-2})[Y-\beta X]$$
$$=(\mathbb{L}(h_ph_{m-p}))[Y-\beta X],$$

as $e_2^\perp(h_ih_j)=h_{i-1}h_{j-1}$ and $h_2^\perp(h_ih_j)=h_{i-2}h_j+h_{i-1}h_{j-1}+h_ih_{j-2}$. Summing gives $\mathsf{g}[Y-\beta X]$. □

The factor $\mathsf{g}[Y-\beta X]$. In the ring Λ of symmetric functions write π_k for the power sums, $\langle\,,\rangle$ for the Hall inner product and $f^\perp$ for the adjoint of multiplication by f; $\mathbb{T}_\sigma$ is the derivation of Subsection 9.25 with $\mathbb{T}_\sigma h_k=(\sigma+k-1)h_{k-1}$, and $\zeta=3-n\beta$. Put $d_g:=a-4=\deg\mathsf{g}$ (even, ≥ 0) and $S:=-(\zeta+2)=n^2/2-5$. (In Lemma 9.148 the letter d is $\deg f_\xi=m$.) We show that $\mathsf{g}[Y-\beta X]$ is not the zero polynomial in x; as $\mathbb{C}[x]$ is a domain, it may be tested at points where $(P-Q)(P-A)(Q-A)(A-B)$ vanishes.

Corollary 9.184. g *is a combination of Schur functions* s_λ *with* $\ell(\lambda)\le 2$ *and* $|\lambda|=d_g$*, and* $\mathbb{T}_{\zeta+2}\mathsf{g}=0$.

Proof. h_ph_{m-p} is two-row (Pieri), and skewing keeps $\ell(\lambda)\le 2$, so f_ξ and g are two-row. The shift invariance of $\sum_p c_p\mathbf{R}_p\mathbf{R}_{m-p}$ is $c\in\ker\mathbf{B}(\xi)$ (Theorem 9.141(ii)): with $\mathbb{T}\mathbf{R}_k=(k+2-n\beta)\mathbf{R}_{k-1}$ and the $\mathbf{R}_i\mathbf{R}_{m-1-i}$ independent, it says $\mathbb{T}_\zeta f_\xi=0$ in Λ (as in the proof of Theorem 9.149). Lemma 9.147 gives $\mathbb{T}_{\zeta+2}\mathsf{g}=\mathbb{T}_{\zeta+2}\mathbb{L}f_\xi=\mathbb{L}\mathbb{T}_\zeta f_\xi=0$. □

(On all of Λ, $\mathbb{T}_{\sigma+2}\mathbb{L}=\mathbb{L}\mathbb{T}_\sigma-12e_3^\perp$ for every σ, from $\mathbb{L}=-\partial_{\pi_1}^2+4\partial_{\pi_2}$, $[\partial_{\pi_1},\mathbb{T}_\sigma]=2\partial_{\pi_2}$, $[\partial_{\pi_2},\mathbb{T}_\sigma]=3\partial_{\pi_3}$ and $\mathbb{T}_{\sigma+2}=\mathbb{T}_\sigma+2\partial_{\pi_1}$; $e_3^\perp$ kills every two-row function, which is why Lemma 9.147 is stated on two-row functions.)

Lemma 9.185 (two-letter values)**.** *For weights* w_1,w_0 *and letters* y_1,y_0 *let* $\mathrm{ev}\colon\Lambda\to\mathbb{C}$ *be the ring map* $\pi_k\mapsto w_1y_1^k+w_0y_0^k$ *(*$k\ge1$*), so that* $h_k\mapsto[t^k](1-y_1t)^{-w_1}(1-y_0t)^{-w_0}$*. If* $\mathbb{T}_\sigma f=0$ *with* $\sigma=w_1+w_0$*, then* $\mathrm{ev}(f)$ *does not change under* $(y_1,y_0)\mapsto(y_1+\epsilon,y_0+\epsilon)$.

Proof. $d/d\epsilon$ of the image of π_k is $k(w_1y_1^{k-1}+w_0y_0^{k-1})$, the image of $k\pi_{k-1}$ ($\pi_0\mapsto w_1+w_0=\sigma$); so by the chain rule $\frac{d}{d\epsilon}\mathrm{ev}_\epsilon(f)=\mathrm{ev}_\epsilon(\mathbb{T}_\sigma f)=0$, and $\mathrm{ev}_\epsilon(f)$ is a polynomial in ϵ. □

Let $\gamma(D)$ be g at $h_k \mapsto [t^k](1-t)^D = (-1)^k\binom{D}{k}$, that is, Lemma 9.185 with weight $-D$ at the letter 1 (and the letter 0, which contributes nothing). γ is a polynomial in D of degree $\le d_g$ ($\pi_k \mapsto -D$, so $\pi_\lambda \mapsto (-D)^{\ell(\lambda)}$). *At a 0/1 point*: if $x_l \in \{0,1\}$ for all l, $k_1 := \#\{l : x_l = 1\}$ and j_Y is the number of 1s in Y (x_i counted twice), then $\prod_{y\in Y}(1-yt)^{-1}\prod_l(1-x_lt)^\beta = (1-t)^{\beta k_1 - j_Y}$, so $\mathsf{g}[Y-\beta X]$ at that point is $\gamma(\beta k_1 - j_Y)$.

Lemma 9.186. *(i) $\gamma(D) = \gamma(S-D)$ for every complex D (an identity of polynomials in D).*
(ii) $\gamma(j) = 0$ for the integers $0 \le j \le d_g/2-1$, hence $\gamma(S-j) = 0$ too.
(iii) $\gamma(-1) = c_0(3-\mu)/(3-\xi/2) \ne 0$.
(iv) $\gamma(D) = C\prod_{j=0}^{d_g/2-1}(D-j)(S-j-D)$ with $C \ne 0$.

Proof. (i) Take weight $-D$ at $y_1 = 1$ and $\zeta+2+D$ at $y_0 = 0$ (total $\zeta+2$); by Corollary 9.184 and Lemma 9.185 the value is unchanged at $(y_1,y_0) = (0,-1)$, where $\pi_k \mapsto (\zeta+2+D)(-1)^k$; as g is homogeneous of degree d_g this is $(-1)^{d_g}\gamma(-(\zeta+2)-D) = \gamma(S-D)$ (d_g even). Lemma 9.185 takes any complex weights, so this holds for every complex D; at odd n the roots $S-j$ of (ii) are half-integers, and it is this polynomial identity that is used. (ii) For an integer $D \ge 0$, $\pi_k \mapsto -D = (-1)^k((-1)^{k-1}D)$, so $\gamma(D) = (-1)^{d_g}(\omega\mathsf{g})(1^D)$ ($\omega\colon \pi_k \mapsto (-1)^{k-1}\pi_k$; the sign $(-1)^k$ is $x \mapsto -x$ on a function of degree d_g). By Corollary 9.184, $\omega\mathsf{g} = \sum b_\lambda s_{\lambda'}$, and $s_{\lambda'}(1^D) = 0$ when $\lambda_1 > D$ ([Ma95, I, §§2–3]: $\omega s_\lambda = s_{\lambda'}$, and $s_\nu(x_1,\dots,x_D) = 0$ when $\ell(\nu) > D$; here $\ell(\lambda') = \lambda_1$); every two-row λ of size d_g has $\lambda_1 \ge d_g/2$. So $\gamma(j) = 0$ for $0 \le j < d_g/2$, and (i) gives $\gamma(S-j) = 0$. (iii) $\gamma(-1)$ is $\pi_k \mapsto 1$, the one-letter alphabet, so $\gamma(-1) = \langle \mathsf{g}, h_{d_g}\rangle = \langle f_\xi, (h_2 - 3e_2)h_{d_g}\rangle$. Counting 2×2 matrices with row sums $(p, m-p)$: $\langle h_p h_{m-p}, h_2 h_{d_g}\rangle = 3,2,1$ and $\langle h_p h_{m-p}, e_2 h_{d_g}\rangle = 1,1,0$ for $p \ge 2$, $p = 1$, $p = 0$ (when $m-p \ge 2$). So for $m \ge 4$, $\gamma(-1) = c_0 - c_1$, and the row 0 of $\mathbf{B}(\xi)$, $d_0c_0 + d_0'c_1 = 0$ with $d_0 = m+2-\xi/2$, $d_0' = 3-\xi/2$, gives $\gamma(-1) = c_0(d_0+d_0')/d_0' = c_0(m+5-\xi)/(3-\xi/2) = c_0(3-\mu)/(3-\xi/2)$. For $m = 2$ ($a = 4$) the term $p = 1$ is h_1h_1 (pairings 1 and 1), $\gamma(-1) = c_0 - 2c_1$, and the doubled last row $d_0c_0 + 2d_0'c_1 = 0$ gives the same value. Nonzero: $\mu \ge n^2/2+1 > 3$; $\xi/2 = n^2/2 > 3$; and $c_0 \ne 0$, since $c_0 = 0$ would give $c_1 = c_2 = \cdots = 0$ through the rows $d_ic_i + d_i'c_{i+1} = 0$, as $d_i' = i+3-n^2/2 \ne 0$ for $i \le m/2-1$ ($m/2+2 < n^2/2$). (iv) The $2(d_g/2) = d_g$ values in (ii) are distinct: the j are $< d_g/2$ and the $S-j$ are $> S-d_g/2 \ge d_g/2$ ($d_g \le S-1$ at even n, as S is odd and d_g even; $d_g \le (n^2-13)/2 < S$ at odd n). A polynomial of degree $\le d_g$ with these d_g roots is $C\prod(D-j)(S-j-D)$; $C \ne 0$ by (iii) (-1 is not a root: the roots are ≥ 0). □

(Consistency with Lemma 9.148. By the paragraph before Lemma 9.147, $f_\xi = c(\xi)f_m(\zeta)$ with $c(\xi) \ne 0$ (there f_d with $d = m$), and $2\zeta + m - 1 = 3-\mu$; so the proof of Lemma 9.148 gives $\mathsf{g} = 2c(\xi)(3-\mu)f_{m-2}(\zeta+2) = 2c(\xi)(3-\mu)f_{d_g}(\zeta+2)$, and $\gamma(-1) = \langle \mathsf{g}, h_{d_g}\rangle = 2c(\xi)(3-\mu)\prod_{i=0}^{d_g/2-1}(\zeta+1+i)$ (the coefficient of $s_{(d_g)}$ in $f_{d_g}(\zeta+2)$): the same factor $3-\mu$ as in (iii), and nonzero again, as $\zeta+1+i \le 1-\mu/2 < 0$. By direct expansion the two values of $\gamma(-1)$ agree at $a = 4$ ($c_0 = 2\zeta c(\xi)$, both $2c(\xi)(3-\mu)$) and at $a = 6$ ($c_0 = 2\zeta(\zeta+1)c(\xi)$, $c_1 = -2(\zeta+1)(\zeta+3)c(\xi)$, both $2c(\xi)(\zeta+1)(3-\mu)$).)

Proposition 9.187. $\mathsf{g}[Y-\beta X]$ *is not the zero polynomial.*

Proof. Even $n \ge 4$: take $(P,Q,A,B) = (1,0,1,0)$, $n/2-2$ further coordinates equal to 1 and the rest 0 ($n/2-2 \le n-4$). Then $k_1 = n/2$, $j_Y = 1+2 = 3$ and $D = n^2/4-3 = (S-1)/2$. As $d_g \le S-1$, $d_g/2 \le (S-1)/2 \le S-d_g/2$, so D is not among the roots of Lemma 9.186(iv): $\gamma(D) \ne 0$. ($n = 4$: all four coordinates are slots, $D = 1$, $S = 3$, $d_g \in \{0,2\}$.) *Odd $n \ge 5$:* take $(P,Q,A,B) = (1,1,1,1)$, $n-5$ further coordinates equal to 1 and one coordinate 0. Then $k_1 = n-1$ (even), $j_Y = 5$ and $D = n(n-1)/2-5$, an integer. S is a half-integer, so no $S-j$ is an integer; and $D \ge d_g/2$, as $d_g \le (n^2-13)/2$ (the largest even $a \le (n^2-2)/2$ is $(n^2-5)/2$) and $n(n-1)/2-5 \ge (n^2-13)/4$ if and only if $n^2-2n-7 \ge 0$. So $\gamma(D) \ne 0$. In both cases the value at an actual point is not 0. □

(A point that fails: ones only outside the slots ($j_Y = 0$, $k_1 = n/2$ at even n) give $D = n^2/4 > S - d_g/2$ when $a > n^2/2 - 6$, a root, for example at $(8, 28)$ and $(8, 30)$. The slots must carry part of the weight.)

Theorem 9.188. *For every $n \geq 4$ and every even a with $n \leq a \leq (n^2-2)/2$, the polynomial $\mathsf{T}_{v'}$ is not identically 0.*

Proof. By Corollary 9.183, $\mathsf{T}_{v'} = M(\beta^2-1)(P-Q)(P-A)(Q-A)(A-B)\,\mathsf{g}[Y-\beta X]$ in the domain $\mathbb{C}[x_1, \ldots, x_n]$. Here $M = \mu(a+1) \neq 0$ ($\mu \geq n^2/2+1$), $\beta^2 - 1 = n^2/4 - 1 \neq 0$ ($n \geq 4$), the four linear forms are nonzero, and $\mathsf{g}[Y - \beta X] \neq 0$ by Proposition 9.187. □

Corollary 9.189. *For every $n \geq 4$ and every even a with $n \leq a \leq (n^2-2)/2$, $\kappa = n$ is not an eigenvalue of the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$; so $\mathrm{K}_n(n)_{(n^2-a-2,a,2)} = 0$. Hence, for every $n \geq 4$, every conjugation type κ of $K_{B(n,2)}$ with $\kappa'_3 \leq 1$ is $(3, 1^{N-3})$, and it occurs once (as in Corollary 9.164(b)).*

Proof. Theorem 9.180 needs (2′) and $\mathsf{T}_{v'} \neq 0$. The latter is Theorem 9.188. For (2′): $m = a - 2$ is even and $m \leq m_{\max} = n^2/2 - 3$ (as $a \leq (n^2-2)/2$), so by Theorem 9.173 the shift invariant elements of $\mathcal{X}_m$ form a line; ϑ_{123} of a kernel vector lies there (Lemma 9.152 and step (1) of the proof of Theorem 9.154), and so does $\vartheta_\xi|_n$ (Lemma 9.176), which is not 0 as $\mathsf{T}_{v'} \neq 0$ (Remark 9.181(1)); so the line is $\mathbb{C}\,\vartheta_\xi|_n$, which is (2′). Then $\kappa = n$ is not an eigenvalue, and $\mathrm{K}_n(n)_\tau = 0$ by Corollary 9.31(b) and Lemma 9.32(ii), as in the proof of Corollary 9.164(a) (and Corollary 9.155). For the type statement: every type has three columns (Corollary 9.42(b)), so $\kappa'_3 \leq 1$ means $\tau = (n^2-a-2, a, 2)$ with $2 \leq a \leq (n^2-2)/2$ (κ' is a partition), and the multiplicity is $\dim \mathrm{K}_n(n)_\tau$ by Lemma 9.32(iii). Then: for $2 \leq a \leq n-1$ Theorem 9.24 applies (depth a; only $(3, 1^{N-3})$, once, at $a = 2$; also Theorem 9.37 with $s = n \geq 4$); odd $a \geq n$ is Corollary 9.174; even $a \geq n$ is the first part. This is exactly the proof of Corollary 9.164(b). □

Remark 9.190. (1) (Scope.) Nothing is claimed for the types with $\kappa'_3 \geq 2$ ($\tau_3 \geq 3$), and no bound on $d(n)$ follows. Theorem 9.180 works for even $a \leq (n-1)^2$, but (2′) is proved here only for $a \leq (n^2-2)/2$.

(2) (Context.) Theorem 9.146 (the probe of one generic row, in the stable range $n \geq a+6$) with Theorem 9.149 is the stable analog: the same $\mathbb{L} = (h_2 - 3e_2)^\perp$ and the same factor $1 - \beta^2$. Corollary 9.183 is the statement at a finite n, for the triangle, with no probe and no stable range. In the type form, Corollary 9.189 was known at $n = 4$ (Remark 9.10), $n = 5$ (Theorem 9.27), $n = 6$ (Theorem 11.9) and $n = 7$ (Corollary 9.164); in the pencil form, $n = 7$ is Theorem 9.163. So the type statement is new for every $n \geq 8$, and the pencil statement for every $n \neq 7$ (Remark 9.38(d) had computed some small d, not as a proof).

(3) (Computations, supporting; not used.) The closed form agrees with the seed computation of Remark 9.181(4) at 31 pairs with $4 \leq n \leq 8$, and, by an independent program from the rows as printed in Proposition 9.44, with $\mathsf{T}_{v'}$ at all 120 dominant even pairs with $4 \leq n \leq 12$; Proposition 9.182 was checked at 185 random points with rational β, and a direct computation from the definitions (without Lemma 9.179 and without the seed) agrees with the closed form at 10 points. At the point "slots 0, the other $n-4$ coordinates 1" ($D = n(n-4)/2$, $S - D = 2n - 5$) the window $[d_g/2, S - d_g/2]$ is exactly $a \leq 4n - 6$, which is where a computation found nonzero values ($n = 5, \ldots, 40$) and zeros beyond, in line with the failing point after Proposition 9.187.

9.32. **The part of degree 1 in y at every layer.** In this subsection $F = \mathbb{C}$, $n \geq 3$, a is arbitrary and $b \geq 2$; we keep the notation of Subsections 9.19 and 9.23: slice points (D, Z), $\theta_{pq} = 1/x_{qp}$, $\Delta_{pq} = e_p - e_q$, the parts Φ_j of Lemma 9.93, $\Phi_1 = \sum_s y_s \mathsf{N}_s(Z)$, $\mathsf{N}(v) = \sum_s v_s \mathsf{N}_s$, G the y-Hessian of Φ_2 at $y = 0$, and $\partial_Z[A]$, $W^{pq} = [E_{pq}, Z]$ as before (9); (9) is the expansion of Φ at $y = 0$ and holds for every b. By Corollary 9.94(b), Theorem 9.22 at (a, b, n) says that $\mathcal{Z}(\varphi) = 0$ and $\mathcal{E}\varphi = 0$ force $\Phi_0 = \Phi_1 = 0$. We show, by hand, that $\Phi_0 = 0$ forces $\Phi_1 = 0$ for every even b and every

odd $b \le n(n-1)+1$ (Theorem 9.196); this covers every layer that occurs in $K_{B(n,2)}$, so there Theorem 9.22 is its part of degree 0 in y alone (Corollary 9.197). At $b=4$ this is step B4-1 of the proof of Theorem 9.129, here also for $n=3$ and without the step $\mathsf{G}=0$. The rows used are the slice rows $[l^3]$ and c of Lemma 9.83, whose proof does not use b (it is multilinear algebra on $\mathsf{Q}_u = u \wedge \omega^u$).

Lemma 9.191 (the rows at $\Phi_0=0$)**.** *Let $\varphi \in \mathcal{F}_{(a,b)}(n)$ with $\Phi_0=0$ identically in x and Z. At every slice point:*

(i) for off-diagonal (p,q) and (i,j) with $(i,j) \notin \{(p,q),(q,p)\}$,

$$\omega(E_{pq},E_{ij}) = -\theta_{pq} Z_{qp}\, \partial_{Z_{ij}} \mathsf{N}(\Delta_{pq}) + \theta_{ij} Z_{ji}\, \partial_{Z_{pq}} \mathsf{N}(\Delta_{ij});$$

(ii) for distinct l,k,m,

$$\mathsf{c}(l,k,m) = \omega(E_{lk},E_{km}) - \theta_{lm}\big[Z_{ml}\,\Delta_{lm}^T \mathsf{G}\Delta_{lk} + \partial_Z[W^{lm}]\mathsf{N}(\Delta_{lk})\big].$$

Proof. (i) Use (9) with $\Phi_0=0$, $(E_{ij})_d=0$, $\partial_Z[E_{ij}]=\partial_{Z_{ij}}$ and $(W^{pq})_d = Z_{qp}\Delta_{pq}$. As $(i,j)\neq(q,p)$, $[E_{pq},E_{ij}]$ has zero diagonal, so $\partial_Y[[E_{pq},E_{ij}]]\tilde\varphi=0$, and (3) gives

$$\partial_Y^2\tilde\varphi(W^{pq},E_{ij}) = Z_{qp}\,\partial_{Z_{ij}}\mathsf{N}(\Delta_{pq}), \qquad \beta(E_{pq},E_{ij}) = -\theta_{pq}Z_{qp}\,\partial_{Z_{ij}}\mathsf{N}(\Delta_{pq});$$

and $\omega(E_{pq},E_{ij}) = \beta(E_{pq},E_{ij}) - \beta(E_{ij},E_{pq})$. (ii) Lemma 9.126(i) holds for every b; with $\Phi_0=0$ its terms $\mathrm{R}_1,\dots,\mathrm{R}_4$ vanish, and $\mathrm{R}_6+\mathrm{R}_7 = \omega(E_{lk},E_{km})$ by (i). □

Circulation. For a monomial $S=\prod Z_{pq}^{m_{pq}}$ put $\mathrm{circ}_{pq}(S) := m_{pq}-m_{qp}$. Every monomial of N_s has torus weight 0 (proof of Lemma 9.125), that is $\sum_q \mathrm{circ}_{pq}(S)=0$ at every p. For a polynomial F, $[S]F$ is its coefficient of S; put $\mathsf{c}_s(S) := [S]\mathsf{N}_s$, a polynomial in x invariant under $x \mapsto x+c\mathbf{1}$ (as in Lemma 9.125). If $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$, then $\sum_s \mathsf{N}_s = 0$ and $\sum_s x_s \mathsf{N}_s = 0$: the parts of degree 0 in y of $\sum_s \partial\Phi/\partial y_s = 0$ and of $\sum_s x_s \partial\Phi/\partial y_s = 0$ (proof of Lemma 9.125).

Lemma 9.192 (circulation)**.** *Let $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ with $\Phi_0=0$ identically, and let every row $[l^3]$ vanish at every slice point. Then $\mathsf{c}_s(S)=0$ for every s and every monomial S with $\mathrm{circ}(S)\neq 0$.*

Proof. Fix S and a pair with $\mathrm{circ}_{lh}(S)>0$; we induct on $m_{hl}(S)$, with l, h and the circulation fixed. Take $j \notin \{l,h\}$ and $T := Z_{jl}S/Z_{lh}$ (a monomial, as $m_{lh}(S) \ge \mathrm{circ}_{lh}(S) \ge 1$). By Lemma 9.191(i) the row $[l^3]$ at $e_l\wedge e_j\wedge e_h$ is $-\theta_{lj}Z_{jl}\,\partial_{Z_{lh}}\mathsf{N}(\Delta_{lj}) + \theta_{lh}Z_{hl}\,\partial_{Z_{lj}}\mathsf{N}(\Delta_{lh})$. Fact (b) in the proof of Lemma 9.126(ii) (here $(j,l)\neq(l,h)$ and $(h,l)\neq(l,j)$) gives

$$-\theta_{lj}\,m_{lh}(S)\,[S]\mathsf{N}(\Delta_{lj}) + \theta_{lh}\,(m_{lj}(T)+1)\,[S']\mathsf{N}(\Delta_{lh}) = 0, \qquad S' := \frac{Z_{jl}Z_{lj}}{Z_{lh}Z_{hl}}\,S,$$

where the second term is present only if Z_{hl} divides T, that is $m_{hl}(S)\ge 1$. Then S' is a monomial with $\mathrm{circ}(S')=\mathrm{circ}(S)$ and $m_{hl}(S') = m_{hl}(S)-1$, so $[S']\mathsf{N}_s=0$ for all s by induction. Either way $[S]\mathsf{N}(\Delta_{lj})=0$: $\mathsf{c}_l(S)=\mathsf{c}_j(S)$ for every $j\notin\{l,h\}$, so $\mathsf{c}_s(S)=\mathsf{c}$ for all $s\neq h$. At traceless x, $\sum_s \mathsf{c}_s(S)=0$ and $\sum_s x_s\mathsf{c}_s(S)=0$ give $(n-1)\mathsf{c}+\mathsf{c}_h(S)=0$ and $x_h(\mathsf{c}_h(S)-\mathsf{c})=0$, so $\mathsf{c}_h(S)=\mathsf{c}=0$ where $x_h\neq 0$; by the invariance under $x\mapsto x+c\mathbf{1}$, $\mathsf{c}_s(S)=0$ identically. □

Corollary 9.193. *Under the hypotheses of Lemma 9.192, if b is even, then $\Phi_1=0$.*

Proof. N_s has odd degree $b-1$, and a monomial with $\mathrm{circ}=0$ has $m_{pq}=m_{qp}$ for all pairs, so even degree. So Lemma 9.192 applies to every monomial of N. □

Lemma 9.192 is empty at $b=3$, where N_s has no circulation (Lemma 9.84); at $b=4$ it gives all of N (the directed 3-cycles; compare step (2) of B4-1); at $b=5$ it gives the directed 4-cycles.

The part without circulation. Let b be odd, $\delta := (b-1)/2$, and $z_e := Z_{pq}Z_{qp}$ for a pair $e=\{p,q\}$. A monomial with $\mathrm{circ}=0$ is a monomial z^μ in the $n(n-1)/2$ variables z_e, and distinct μ give distinct monomials. By Lemma 9.192, $\mathsf{N}_s = \mathsf{P}_s(z)$, a form of degree δ in z with coefficients in $\mathbb{K} := \mathbb{C}(x)$. Write $\partial_e := \partial/\partial z_e$ and $\mathsf{P}_v := \sum_s v_s\mathsf{P}_s$; then $\partial_{Z_{pq}}\mathsf{P}_v = Z_{qp}\,\partial_{pq}\mathsf{P}_v$.

Lemma 9.194. *Under the hypotheses of Lemma 9.192, let b be odd and let every row c vanish at every slice point. Then:*

(a) $\theta_{lh}\,\partial_{lj}\mathsf{P}_{\Delta_{lh}} = \theta_{lj}\,\partial_{lh}\mathsf{P}_{\Delta_{lj}}$ *for distinct* l,j,h;

(b) $(\partial_{lr}-\partial_{mr})\mathsf{P}_{\Delta_{lk}} \in z_{lm}\mathbb{K}[z]$ *for distinct* l,k,m *and every* $r\notin\{l,m\}$ *($r=k$ included).*

Proof. (a) By Lemma 9.191(i), $[l^3] = Z_{jl}Z_{hl}\big[\theta_{lh}\,\partial_{lj}\mathsf{P}_{\Delta_{lh}} - \theta_{lj}\,\partial_{lh}\mathsf{P}_{\Delta_{lj}}\big](z)$, $Z_{jl}Z_{hl}\neq 0$, and $z\mapsto Z$ is injective on monomials. (b) By Lemma 9.191(i), $\omega(E_{lk},E_{km}) = Z_{kl}Z_{mk}\big[\theta_{km}\,\partial_{lk}\mathsf{P}_{\Delta_{km}} - \theta_{lk}\,\partial_{km}\mathsf{P}_{\Delta_{lk}}\big]$, and the row $[k^3]$ at $e_k\wedge e_l\wedge e_m$ is $Z_{lk}Z_{mk}$ times the same bracket ($\theta_{kl}\Delta_{kl}=\theta_{lk}\Delta_{lk}$), which is 0 by (a) with apex k. By the formula after (9), $\partial_Z[W^{lm}]\mathsf{P}_v = \sum_{r\notin\{l,m\}} Z_{mr}Z_{rl}(\partial_{lr}-\partial_{mr})\mathsf{P}_v$, the sum including $r=k$. So, by Lemma 9.191(ii), $\mathsf{c}(l,k,m)=0$ reads

$$Z_{ml}\,\Delta_{lm}^{T}\mathsf{G}\Delta_{lk} + \sum_{r\notin\{l,m\}} Z_{mr}Z_{rl}\,(\partial_{lr}-\partial_{mr})\mathsf{P}_{\Delta_{lk}} = 0.$$

Every monomial of the first term is divisible by Z_{ml}. A monomial $Z_{mr}Z_{rl}z^{\mu}$ is divisible by Z_{ml} if and only if $\mu_{lm}\geq 1$, and it determines r (its circulation is the path $m\to r\to l$). So the part of the sum not divisible by Z_{ml} vanishes term by term: $(\partial_{lr}-\partial_{mr})\mathsf{P}_{\Delta_{lk}}$ is 0 at $z_{lm}=0$. □

Lemma 9.195 (the linear case)**.** *Fix a traceless x with distinct coordinates and $p_2 := \sum_s x_s^2 \neq 0$. Let X be a set of pairs, not all pairs, and let $\mathsf{R}(s,f)\in\mathbb{C}$ ($s\in[n]$, f a pair; $\mathsf{R}(v,f) := \sum_s v_s\mathsf{R}(s,f)$, $\rho_f := \mathsf{R}(\cdot,f)\in\mathbb{C}^n$) satisfy $\sum_s \mathsf{R}(s,f)=0$ and $\sum_s x_s\mathsf{R}(s,f)=0$ for every f, and*

(a′) $\theta_{lh}\,\mathsf{R}(\Delta_{lh},\{l,j\}) = \theta_{lj}\,\mathsf{R}(\Delta_{lj},\{l,h\})$ *for distinct l,j,h ("(a′) at $(l;j,h)$");*

(b′) $\mathsf{R}(\Delta_{lk},\{l,r\}) = \mathsf{R}(\Delta_{lk},\{m,r\})$ *for distinct l,k,m, $r\notin\{l,m\}$ ($r=k$ allowed) and $\{l,m\}\notin X$.*

Then $\mathsf{R}=0$.

Proof. Fix a pair $\{l,m\}\notin X$. (1) For $r\notin\{l,m\}$ put $w := \rho_{lr}-\rho_{mr}$. (b′) at $(l,k,m;r)$ and at $(m,k,l;r)$ give $w_l=w_k$ and $w_m=w_k$ for every $k\notin\{l,m\}$, $k=r$ included (at $n=3$ the only k). So w is constant, and $\sum_s w_s=0$ gives $w=0$: $\rho_{lr}=\rho_{mr}=:\sigma_r$. (2) Put $u:=\rho_{lm}$. (a′) at $(l;m,r)$ and at $(m;l,r)$ give $\theta_{lr}(u_l-u_r)=\theta_{lm}(\sigma_r[l]-\sigma_r[m])=\theta_{mr}(u_m-u_r)$ for every $r\notin\{l,m\}$. So every point (x_r,u_r) lies on the line through (x_l,u_l) and (x_m,u_m): $u=\alpha\mathbf{1}+\gamma x$. Then $\sum_s u_s = n\alpha = 0$ and $\sum_s x_s u_s = \gamma p_2 = 0$ give $u=0$ (here $\mathcal{E}\varphi=0$ enters), hence $\sigma_r[l]=\sigma_r[m]$. (3) (a′) at $(r;l,m)$ gives $(\theta_{rm}-\theta_{rl})(\sigma_r[r]-\sigma_r[l])=0$, so $\sigma_r[r]=\sigma_r[l]=\sigma_r[m]=:\mathsf{a}_r$. (4) For $k\notin\{l,m,r\}$ put $A:=\mathsf{a}_r-\sigma_r[k]$ and $B:=\mathsf{a}_k-\sigma_k[r]$. (a′) at $(l;r,k)$ and at $(m;r,k)$ give $\theta_{lk}A=\theta_{lr}B$ and $\theta_{mk}A=\theta_{mr}B$; the determinant $\theta_{lk}\theta_{mr}-\theta_{lr}\theta_{mk}$ has numerator $(x_m-x_l)(x_k-x_r)\neq 0$, so $A=B=0$. So $\sigma_r=\mathsf{a}_r\mathbf{1}$, and $\sum=0$ gives $\sigma_r=0$. Thus $\rho_f=0$ for every pair f that meets $\{l,m\}$. (5) For $h,k\notin\{l,m\}$, (a′) at $(h;k,l)$, $(h;k,m)$ and $(k;h,l)$ give $\rho_{hk}[h]=\rho_{hk}[k]=\rho_{hk}[l]=\rho_{hk}[m]=:\mathsf{a}_{hk}$. (6) For distinct $h,k,t\notin\{l,m\}$ put $u:=\mathsf{a}_{hk}-\rho_{hk}[t]$, $v:=\mathsf{a}_{ht}-\rho_{ht}[k]$, $w:=\mathsf{a}_{kt}-\rho_{kt}[h]$. (a′) at $(h;k,t)$, $(k;h,t)$ and $(t;h,k)$ give $\theta_{ht}u=\theta_{hk}v$, $\theta_{kt}u=\theta_{kh}w$ and $\theta_{tk}v=\theta_{th}w$. As $\theta_{th}\theta_{kt}/\theta_{kh} = -\theta_{tk}\theta_{ht}/\theta_{hk}$, this gives $2\theta_{tk}\theta_{ht}u/\theta_{hk}=0$, so $u=v=w=0$, $\rho_{hk}=\mathsf{a}_{hk}\mathbf{1}$, and $\sum=0$ gives $\rho_{hk}=0$ (at $n=4$, step (5) already covers every index). So $\mathsf{R}=0$. □

Theorem 9.196 (order 1 at every layer)**.** *Let $n\geq 3$, $b\geq 2$, a arbitrary and $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$. Suppose that $\Phi_0=0$ identically in x and Z, and that every row $[l^3]$ and every row c vanishes at every slice point (both hold if $\mathcal{Z}(\varphi)=0$, by Lemma 9.83). If b is even, or b is odd and $b\leq n(n-1)+1$, then $\Phi_1=0$.*

Proof. For even b this is Corollary 9.193. Let b be odd, so $\delta=(b-1)/2\geq 1$ and $\mathsf{N}_s=\mathsf{P}_s(z)$. For a multi-index μ over the pairs with $|\mu|=\delta-1$, $\mathsf{P}^{\mu} := \partial^{\mu}\mathsf{P}$ is linear in z: $\mathsf{P}^{\mu}_s = \sum_f \mathsf{R}^{\mu}(s,f)z_f$. It satisfies $\sum_s \mathsf{P}^{\mu}_s = 0$ and $\sum_s x_s\mathsf{P}^{\mu}_s = 0$, Lemma 9.194(a) (apply ∂^{μ}), and Lemma 9.194(b) for every $\{l,m\}\notin X := \operatorname{supp}(\mu)$: then ∂^{μ} preserves the ideal $z_{lm}\mathbb{K}[z]$, and $\partial^{\mu}(\partial_{lr}-\partial_{mr})\mathsf{P}_{\Delta_{lk}} = \mathsf{R}^{\mu}(\Delta_{lk},\{l,r\})-\mathsf{R}^{\mu}(\Delta_{lk},\{m,r\})$ is a constant in that ideal, so it is 0. X has at most $\delta-1$ elements, so it is not the set of all pairs when $\delta-1<n(n-1)/2$, that is $b\leq n(n-1)+1$. At every x as in

Lemma 9.195, that lemma gives $\mathsf{R}^\mu = 0$ for every μ, so every derivative of order δ of the form P is 0 and $\mathsf{P} = 0$ there. Such x are dense in the traceless hyperplane, and the coefficients are polynomials in x invariant under $x \mapsto x + c\mathbf{1}$; so $\mathsf{N} = 0$, that is $\Phi_1 = 0$. □

Corollary 9.197. *Let $n \geq 3$, and let $b \geq 2$ be even or $b \leq n(n-1)+1$. Then $\ker \mathcal{Z} \cap \mathcal{F}^{\mathcal{E}}_{(a,b)}(n) = 0$ if and only if every φ in it has $\Phi_0 = 0$. This holds whenever $3b \leq n^2$, in particular for every three-row type $\tau = (n^2-a-b, a, b)$ ($b \leq a \leq n^2-a-b$), at every $n \geq 3$.*

Proof. "Only if" is clear. "If": $\Phi_0 = 0$ and Theorem 9.196 give $\Phi_1 = 0$, and Corollary 9.94(a) gives $\varphi = 0$. Finally $3b \leq n^2$ gives $b \leq n^2/3 \leq n(n-1)+1$. □

Remark 9.198. (1) (Scope.) At $b = 2$ (Theorem 9.80), $b = 3$ (Theorem 9.87, $n \geq 4$) and $b = 4$ (B4-0 of Theorem 9.129, $n \geq 5$) the part $\Phi_0 = 0$ is known, and Theorem 9.196 gives the rest again; at $b = 3$ it is Theorem 9.86 (there $n \geq 4$ and only the rows c), here also for $n = 3$ with the rows $[l^3]$. For $b \geq 5$, and for $b = 4$ with $n = 3, 4$, the open part of Theorem 9.22 outside its stable range is now $\Phi_0 = 0$ alone. Not claimed: $\Phi_0 = 0$ for any $b \geq 5$; odd $b > n(n-1)+1$; anything about G (not needed, by Corollary 9.94); any bound on $d(n)$.

(2) (What is used.) (3) and (9) at $\Phi_0 = 0$; the rows $[l^3]$ (Lemma 9.192, Lemma 9.194(a)) and c (Lemma 9.194(b)); $\sum_s \mathsf{N}_s = 0$; $\sum_s x_s \mathsf{N}_s = 0$; the invariance under $x \mapsto x + c\mathbf{1}$. Not used: the bidegree, homogeneity in x, the rows P, and G. $\mathcal{E}\varphi = 0$ enters only in Lemma 9.192 (end) and in step (2) of Lemma 9.195, and it is essential for odd b, consistent with Remark 9.95(2): without $\sum_s x_s \mathsf{N}_s = 0$, $\mathsf{N}_s = (x_s - p_1/n)(\sum_e z_e)^\delta$ and $\mathsf{G} = 0$ satisfy every row $[l^3]$ and c and $\sum_s \mathsf{N}_s = 0$.

(3) (Lemma 9.195 is sharp.) At $X =$ all pairs its solution space has dimension $\binom{n-1}{2}$; this is why the odd range stops at $b = n(n-1)+1$.

(4) (Computations, supporting; not used.) Exact computations modulo $2^{31}-1$ at one random traceless point, with the rows built from (3) and (9) directly and all coefficients of N and G unknown, give $\mathsf{N} = 0$ at $(n,b) = (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,4), (5,5)$; a second program, with G free and the rows from entrywise derivatives, gives the same at 13 pairs (n,b) including $(6,5)$, and at $(3,7), (3,9), (4,9)$, and the solution space of Lemma 9.195 is 0 for every X other than all pairs ($n = 3, 4, 5$; random X at $n = 6, 7$). Without $\sum_s x_s \mathsf{N}_s = 0$ the solution space at $(4,5)$ is the line of (2); with the rows c alone it is 0 at $(3,4), (4,5), (4,6)$.

9.33. **Pencil eigenvectors have y-order at most 2.** In this subsection $F = \mathbb{C}$, $n \geq 3$, and a, b are arbitrary; we use the pencil of Subsection 9.11, the base points (D, Y), $Y = Z + \operatorname{diag}(y)$, of Subsection 9.19, and the parts Φ_j and the y-order of Lemma 9.93. As y denotes the diagonal of Y, we write the corners as $B_i = u\eta_i^T$. An *eigenvector* is a $\varphi \neq 0$ in $\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ with $\kappa\,\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ for one $\kappa \in \mathbb{C}$, at every base point and every corner. For a function g of y let $[g]_j$ be its part of degree j in y. Lemma 9.93 says that $\mathcal{Z}(\varphi) = 0$ forces y-order at most 1; here we show that $\kappa\,\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ forces y-order at most 2, for every κ.

Lemma 9.199 (the lowest part of $\mathcal{J}$)**.** *Let $\varphi \in \mathcal{F}_{(a,b)}(n)$ have y-order $k \geq 3$ (neither $\mathcal{E}\varphi = 0$ nor the invariance under $Y \mapsto Y + cI$ is used; only the conjugation invariance). Let C be the y-Hessian of Φ_k and T the symmetric 3-tensor of its third y-derivatives (of degree $k-3$ in y; both may depend on x and Z). For a matrix B put $(v_B)_t := x_t \sum_{r \neq t} \theta_{rt}(B_{tr}Z_{rt} + B_{rt}Z_{tr})$.*

(a) At every base point and every corner, $[\mathcal{Z}(\varphi)]_j = 0$ for $j < k-2$, $[\mathcal{J}(\varphi)]_j = 0$ for $j < k-3$, and

$$[\mathcal{J}(\varphi)]_{k-3}(B_1,B_2,B_3) = -\sum_{\text{cyc}} \sum_{p \neq q} \theta_{pq} Z_{qp} \big[(B_2)_{pq}\, \mathsf{T}(v_{B_1}, \Delta_{pq}, (B_3)_d) - (B_3)_{pq}\, \mathsf{T}(v_{B_1}, \Delta_{pq}, (B_2)_d) \big],$$

the cyclic sum running over (B_1,B_2,B_3), (B_2,B_3,B_1), (B_3,B_1,B_2).

(b) If moreover $\kappa\,\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$, then $[\mathcal{J}(\varphi)]_{k-3} = 0$ at every base point and every corner.

Proof. (i) By step (0) of the proof of Lemma 9.93, ω has no part of degree $< k-2$ in y, and its part of degree $k-2$ is $\omega_{[k-2]}(U,V) = -\sum_{p\neq q}\theta_{pq}Z_{qp}[U_{pq}\,\mathsf{C}(\Delta_{pq},V_d) - V_{pq}\,\mathsf{C}(\Delta_{pq},U_d)]$; as $\operatorname{tr} B_i$ does not depend on y, $[\mathcal{Z}(\varphi)]_j = 0$ for $j < k-2$. (ii) Write $B\circ D = P_d + P_o$ (diagonal and off-diagonal parts); then $P_o = [W,D]$ with $W_{pq} = \frac{1}{2}(x_p+x_q)\theta_{pq}B_{pq}$ ($p\neq q$) and $W_{pp} := 0$. Differentiating the conjugation covariance of ω at $g = \exp(\epsilon W)$ (as in the proof of Lemma 9.30),

$$D_{J(B)}\,\omega(U,V) = D_{(P_d,0)}\,\omega(U,V) + D_{(0,B\circ Y-[W,Y])}\,\omega(U,V) - \omega([W,U],V) - \omega(U,[W,V]).$$

$D_{(P_d,0)} = \sum_p B_{pp}x_p\partial_{x_p}$ and the last two terms keep the degree in y; in the direction $B\circ Y - [W,Y]$ the off-diagonal entries act by $\partial/\partial Z$ (they keep or raise the degree), and the diagonal entry t is $B_{tt}y_t$ plus its part of degree 0 in y, $\frac{1}{2}\sum_r(B_{tr}Z_{rt} + Z_{tr}B_{rt}) - [W,Z]_{tt} = (v_B)_t$ (the coefficient of $B_{tr}Z_{rt}$ is $\frac{1}{2} - \frac{1}{2}(x_t+x_r)\theta_{tr} = x_t\theta_{rt}$, that of $B_{rt}Z_{tr}$ is $\frac{1}{2} + \frac{1}{2}(x_r+x_t)\theta_{rt} = x_t\theta_{rt}$). So $D_{J(B)}$ lowers the degree by at most 1, and only through $\sum_t(v_B)_t\,\partial/\partial y_t$. Hence $[\mathcal{J}(\varphi)]_j = 0$ for $j < k-3$, and $[\mathcal{J}(\varphi)]_{k-3}$ comes from $\sum_t(v_{B_i})_t\,\partial/\partial y_t$ applied to $\omega_{[k-2]}(\hat{B}_i)$, where $\partial/\partial y_t$ acts on C only and gives $\mathsf{T}(e_t,\cdot,\cdot)$; the alternation over i is the cyclic sum, as $\omega_{[k-2]}$ is antisymmetric. This is (a). (b) By (a), $[\mathcal{J}(\varphi)]_{k-3} = \kappa[\mathcal{Z}(\varphi)]_{k-3} = 0$. □

Lemma 9.200. *Let $n\geq 3$, $x\in\mathbb{C}^n$ with distinct coordinates, and T a symmetric 3-tensor on $\mathbb{C}^n$ with $\mathsf{T}(\mathbf{1},\cdot,\cdot) = 0$, $\mathsf{T}(x,\cdot,\cdot) = 0$ and*

$$(x_r - x_q)\,\mathsf{T}_{lqr} = (x_l - x_q)\,\mathsf{T}_{llq} - (x_l - x_r)\,\mathsf{T}_{llr} \qquad \text{for all distinct } l,q,r. \tag{L}$$

Then $\mathsf{T} = 0$.

Proof. Fix l, put $S(u,v) := \mathsf{T}(u,v,e_l)$ and $\sigma_q := (x_l - x_q)S(e_l,e_q)$ for $q\neq l$. (L) reads $(x_r - x_q)S(e_q,e_r) = \sigma_q - \sigma_r$ (q,r,l distinct). For $q\neq l$, $S(x,e_q) - x_qS(\mathbf{1},e_q) = 0$ gives $\sum_s(x_s - x_q)S(e_s,e_q) = 0$, that is $\sigma_q + \sum_{r\notin\{l,q\}}(\sigma_q - \sigma_r) = 0$, or $n\sigma_q = \sum_{r\neq l}\sigma_r$. So all σ_q ($q\neq l$) are equal, to c say, and $nc = (n-1)c$ gives $c = 0$. Hence $S(e_l,e_q) = 0$ ($q\neq l$), and $S(e_q,e_r) = 0$ ($q\neq r$, both $\neq l$) by (L). Then $S(\mathbf{1},e_q) = 0$ gives $S(e_q,e_q) = 0$ ($q\neq l$), and $S(\mathbf{1},e_l) = 0$ gives $S(e_l,e_l) = 0$. So $\mathsf{T}(\cdot,\cdot,e_l) = 0$ for every l. □

Theorem 9.201 (eigen order)**.** *Let $n\geq 3$, and a, b, κ arbitrary ($\kappa = (d-2)/n$, that is $\lambda = 0$ in Corollary 9.31(a), included). Every eigenvector $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ of the pencil has y-order at most 2.*

Proof. Let φ be an eigenvector of y-order $k\geq 3$, and T as in Lemma 9.199. Take $u = e_l$, $B_i = e_l\eta_i^T$. Then $(B_i)_{pq} = \delta_{pl}(\eta_i)_q$, $(B_i)_d = (\eta_i)_le_l$ and $v_B = \sum_{r\neq l}\eta_rZ_{rl}\theta_{rl}(x_le_l - x_re_r)$ for $B = e_l\eta^T$, so Lemma 9.199 gives

$$0 = [\mathcal{J}(\varphi)]_{k-3} = -\sum_{r\neq l}\sum_{q\neq l}\theta_{rl}\theta_{lq}Z_{rl}Z_{ql}\,\mathsf{T}(x_le_l - x_re_r,\Delta_{lq},e_l)\,(e_r\wedge e_q\wedge e_l)(\eta_1,\eta_2,\eta_3),$$

with $(e_r\wedge e_q\wedge e_l)(\eta_1,\eta_2,\eta_3) := \det[(\eta_i)_r,(\eta_i)_q,(\eta_i)_l]_{i=1,2,3}$. For distinct $r,q\neq l$, using $\theta_{ql}\theta_{lr} = \theta_{lq}\theta_{rl}$, the coefficient of $e_r\wedge e_q\wedge e_l$ is $-\theta_{rl}\theta_{lq}Z_{rl}Z_{ql}[\mathsf{T}(x_le_l - x_re_r,\Delta_{lq},e_l) - \mathsf{T}(x_le_l - x_qe_q,\Delta_{lr},e_l)]$, and it is 0, as these 3-forms in η are linearly independent. The factor $\theta_{rl}\theta_{lq}Z_{rl}Z_{ql}$ is not 0 in the domain $\mathbb{C}[x,\theta,y,Z]$, so the bracket is 0; expanded with T symmetric, this is (L) of Lemma 9.200, at every (x,y,Z). The relations $\sum_s\partial\Phi/\partial y_s = 0$ (the invariance under $Y\mapsto Y+cI$) and $\sum_s x_s\partial\Phi/\partial y_s = 0$ (from $\mathcal{E}\varphi = 0$, as in the proof of Lemma 9.84; neither uses b) have the lowest parts $\sum_s\partial\Phi_k/\partial y_s = 0$ and $\sum_s x_s\partial\Phi_k/\partial y_s = 0$; two more y-derivatives give $\mathsf{T}(\mathbf{1},\cdot,\cdot) = 0$ and $\mathsf{T}(x,\cdot,\cdot) = 0$. These three hold pointwise in (x,y,Z), so Lemma 9.200 gives $\mathsf{T} = 0$. As Φ_k is homogeneous of degree $k\geq 3$ in y, $\Phi_k = 0$, against the choice of k. □

Remark 9.202. (1) (What is used.) Lemma 9.199 ((3) with step (0) of the proof of Lemma 9.93, and the conjugation covariance of ω), the corners with $u = e_l$ only, the invariance under $Y\mapsto Y+cI$, $\mathcal{E}\varphi = 0$ and $n\geq 3$. Not used: the value of κ, the bidegree, the invariance under $X\mapsto X+cI$, and the corners with traceless arguments or with support at least 2.

(2) (Scope.) At $b=3$, $\Phi_2=0$ by the torus weight (there is no term $y_s y_t Z_{pq}$, Lemma 9.84), so an eigenvector there has y-order 0 or 1. So, for the eigenvalue $\kappa=n$ in a layer $\tau_3=b$, only the y-orders 0, 1, 2 remain to be excluded, each with $\kappa=n$ explicit; nothing about these is claimed here. Theorem 9.201 changes nothing in Theorem 9.22, where Lemma 9.93 already gives y-order at most 1.

(3) (Computations, supporting; not used.) Two independent exact programs find that the linear system of Lemma 9.200 has only the solution 0 ($n=3,4,5$, and $n=3,\dots,7$); each hypothesis is needed: without $\mathsf{T}(\mathbf{1},\cdot,\cdot)=0$ or without $\mathsf{T}(x,\cdot,\cdot)=0$ the solution space has dimension $n-1$, and without (L) dimension $\binom{n}{3}$ ($n=3,\dots,7$). A third program compares the formula of Lemma 9.199(a) with $\mathcal{J}$ computed from the definitions ($n=3,4,5$, y-order 3 and 4, general and support-1 corners, also on functions that are not eigenvectors).

9.34. **No pencil eigenvector of y-order 2 in the layers $4\le b\le n+3$.** We keep the setting of Subsection 9.33. By Theorem 9.201 an eigenvector has y-order 0, 1 or 2; here we exclude y-order 2 for $4\le b\le n+2$ ($n\ge 4$) and for $b=n+3$ ($n\ge 5$), for every a and every κ. A layer $b=\tau_3$ of $K_{B(n,2)}$ ($\tau\vdash n^2$) occurs only for $3b\le n^2$, so for $n=4,5$ this covers every layer $b\ge 4$ that occurs. For $l\ne r$ put

$$R_{lr}:=\sum_{c\notin\{l,r\}} Z_{rc}\frac{\partial}{\partial Z_{lc}},\qquad L_{lr}:=\sum_{c\notin\{l,r\}} Z_{cl}\frac{\partial}{\partial Z_{cr}};$$

they neither create nor remove the variable Z_{rl}. For φ of y-order 2 let G be the y-Hessian of Φ_2 (it does not depend on y) and T the third y-derivatives of Φ_3 (they do not depend on y either), and put $a_r:=x_l e_l-x_r e_r$ and

$$\Psi_l(r;q):=(x_r-x_l)\,L_{lr}\mathsf{G}_{ll}+(x_q-x_r)\,L_{lr}\mathsf{G}_{lq}+(x_l-x_q)\,R_{lr}\mathsf{G}_{lq},$$
$$D_l(r,q):=\mathsf{T}(e_l,\Delta_{lq},a_r)-\mathsf{T}(e_l,\Delta_{lr},a_q).$$

Lemma 9.203 (the support-1 rows at y-order 2)**.** *Let $\varphi\in\mathcal{F}_{(a,b)}(n)$ have y-order 2 (only the conjugation invariance is used). At every base point, the part of degree 0 in y of $\mathcal{Z}(\varphi)$ vanishes at every corner with $u=e_l$, and for distinct l,q,r the coefficient of $e_r\wedge e_q\wedge e_l$ in $[\mathcal{J}(\varphi)]_0$ at $u=e_l$ is $\theta_{lr}\theta_{lq}\,[-Z_{ql}\Psi_l(r;q)+Z_{rl}\Psi_l(q;r)+Z_{rl}Z_{ql}D_l(r,q)]$.*

Proof. By step (0) of the proof of Lemma 9.93, ω has no part of negative degree in y, and $\omega_{[0]}(U,V)=-\sum_{p\ne q}\theta_{pq}Z_{qp}[U_{pq}\mathsf{G}(\Delta_{pq},V_d)-V_{pq}\mathsf{G}(\Delta_{pq},U_d)]$. At $U=e_l\eta^T$, $V=e_l\eta'^T$ this is $-\sum_{q\ne l}g_q\,(e_q\wedge e_l)(\eta,\eta')$ with $g_q:=\theta_{lq}Z_{ql}\mathsf{G}(\Delta_{lq},e_l)$, so $[\mathcal{Z}(\varphi)]_0=e_l\wedge\omega^{(l)}_{[0]}=0$. For $[\mathcal{J}(\varphi)]_0$ decompose $D_{J(B)}$, $B=e_l\eta^T$, as in the proof of Lemma 9.199. The parts with the factor η_l form $\eta_l D_{J(E_{ll})}$, which keeps the degree and, after the alternation, acts on $e_l\wedge\omega^{(l)}_{[0]}=0$. The bracket terms are $\eta'_l\omega_{[0]}(W,V)+\eta''_l\omega_{[0]}(U,W)$ with $W=e_l w^T$, $w_l=0$; both equal $\mp\eta'_l\eta''_l\sum_q g_q w_q$, so they cancel. The terms $\theta_{lr}\eta_r(x_r y_l-x_l y_r)\,\partial/\partial Z_{lr}$ raise the degree and act on $\omega_{[-1]}=0$. What remains is $\sum_{r\ne l}\eta_r\big[\theta_{lr}(x_rL_{lr}-x_lR_{lr})+Z_{rl}\theta_{rl}\,a_r\cdot\partial_y\big]$, the first part acting on $\omega_{[0]}$ and the second on $\omega_{[1]}$. From (3) (with $\Phi_1=0$), for $s,t\ne l$,

$$\omega_{[1]}(E_{ls},E_{lt})=-\theta_{ls}Z_{sl}\partial_{Z_{lt}}\mathsf{G}(\Delta_{ls},y)+\theta_{lt}Z_{tl}\partial_{Z_{ls}}\mathsf{G}(\Delta_{lt},y),$$
$$\omega_{[1]}(E_{ll},E_{lt})=\theta_{lt}\big[(R_{lt}-L_{lt})(\mathsf{G}y)_l+Z_{tl}\mathsf{T}(e_l,\Delta_{lt},y)\big].$$

Collecting the coefficient of $e_r\wedge e_q\wedge e_l$ gives $\theta_{lr}\theta_{lq}[X(r,q)-X(q,r)]$ with $X(r,q)=-Z_{ql}(x_rL_{lr}-x_lR_{lr})\mathsf{G}(\Delta_{lq},e_l)+Z_{rl}(R_{lq}-L_{lq})(\mathsf{G}a_r)_l+Z_{rl}Z_{ql}\mathsf{T}(e_l,\Delta_{lq},a_r)$, and regrouping (with $\mathsf{G}(\Delta_{lq},e_l)=\mathsf{G}_{ll}-\mathsf{G}_{lq}$, G being symmetric) gives the stated form. □

Lemma 9.204 (the split)**.** *Let φ be an eigenvector of y-order 2 and fix x with distinct entries. Then the entries of G are balanced polynomials in Z, homogeneous of degree $D:=b-2$; G is symmetric, $\mathsf{G}\mathbf{1}=\mathsf{G}x=0$, $\mathsf{T}(\mathbf{1},\cdot,\cdot)=\mathsf{T}(x,\cdot,\cdot)=0$; and for every apex l and all distinct $r,q\ne l$:*

(i) Z_{rl} *divides* $\Psi_l(r;q)$*; equivalently,* $\Psi_l(r;q)$ *computed from* $\mathsf{G}|_{Z_{rl}=0}$ *is* 0*;*
(ii) $\Psi_l(r;q)/Z_{rl} - \Psi_l(q;r)/Z_{ql} = D_l(r,q)$.

Proof. Φ is homogeneous of degree b in $Y = Z + \operatorname{diag}(y)$, and invariant under the diagonal torus, which fixes y; so the entries of G are balanced of degree $b-2$. The traces come from the invariance under $Y \mapsto Y + cI$ and from $\mathcal{E}\varphi = 0$, as in the proof of Theorem 9.201. By Lemma 9.203, $\kappa\mathcal{Z} = \mathcal{J}$ gives $Z_{ql}\Psi_l(r;q) - Z_{rl}\Psi_l(q;r) = Z_{rl}Z_{ql}D_l(r,q)$. In $\mathbb{C}[Z]$ the variables Z_{rl}, Z_{ql} are distinct primes, so this is equivalent to (i) for $(r;q)$ and $(q;r)$ together with (ii); as L_{lr}, R_{lr} neither create nor remove Z_{rl}, (i) is the stated condition on $\mathsf{G}|_{Z_{rl}=0}$. □

In the rest of this subsection x is fixed with distinct entries, and G is any symmetric $n \times n$ matrix whose entries are balanced polynomials of degree D in Z, with $\mathsf{G}\mathbf{1} = \mathsf{G}x = 0$ and (i) at every apex; we call these the hypotheses (H_D). *Notation.* A monomial m in the off-diagonal Z_{uv} is a directed multigraph on $[n]$ without loops ($k_{uv}(m)$ edges $u \to v$); *balanced* means in-degree = out-degree =: $d_v(m)$ at every v; $\operatorname{supp}(m)$ is the set of v with $d_v(m) > 0$, and a vertex of degree 1 is *single*. $\mathsf{G}_{lq}[m]$ is the coefficient of m in G_{lq}, and $\operatorname{row}_l(m) := (\mathsf{G}_{lq}[m])_q$; it lies in $U := \{w : \sum_q w_q = \sum_q x_q w_q = 0\}$. Put $N := n-1$. For $f \notin \operatorname{supp}(m)$ and a single v, $m[v \to f]$ is m with v renamed f. If m misses exactly one vertex p, put $h^m := \operatorname{row}_p(m)$ and $\alpha(m) := \mathsf{G}_{pp}[m]$.

Lemma 9.205 (Lemma T)**.** *Assume* (H_D) *at the apex* l *(and the traces of row* l*). Fix* $f \neq l$ *and a monomial* μ *without* Z_{fl}*. Its* L_{lf}*-preimages are the* m *in which one edge* $c \to f$ *(*$c \notin \{l,f\}$*) becomes* $c \to l$*, with coefficients* $k_m := k_{cf}(m)$*; its* R_{lf}*-preimages are the* m' *in which one edge* $l \to c$ *(*$c \notin \{l,f\}$*) becomes* $f \to c$*, with coefficients* $k'_{m'} := k_{lc}(m')$*. For* $w \in U$ *put*

$$(T_f^{(l)}w)_q := \frac{(x_q - x_f)w_q + (x_f - x_l)w_l}{x_q - x_l}\ (q \notin \{l,f\}), \qquad (T_f^{(l)}w)_f := -Nw_l,$$

and $(T_f^{(l)}w)_l := -\sum_{q\neq l}(T_f^{(l)}w)_q$*. Then* $T_f^{(l)}$ *maps* U *injectively to* U*, and (i) at* (l,f) *at the target* μ *is equivalent to* $\sum_{m'} k'_{m'} \operatorname{row}_l(m') = T_f^{(l)} \sum_m k_m \operatorname{row}_l(m)$*. Two consequences:*

(a) (transfer) if $d_l(m) \geq 1$*,* $f \notin \operatorname{supp}(m)$*,* $f \neq l$*, and* $l \to e$ *is an edge of* m*, then* $k_{le}(m)\operatorname{row}_l(m) = T_f^{(l)} \sum_c \operatorname{row}_l(m_c)$*, over the distinct in-neighbours* c *of* l*, with* m_c *the monomial* m *with* $(c \to l,\ l \to e)$ *replaced by* $(c \to f,\ f \to e)$*; if* $d_l(m) = 1$*,* $\operatorname{row}_l(m) = T_f^{(l)} \operatorname{row}_l(m[l \to f])$*;*

(b) (l*-avoiding system) if* $l \notin \operatorname{supp}(m)$*,* $f \notin \operatorname{supp}(m) + l$ *and* $c \to r$ *is an edge of* m*, then* $\sum_e T_f^{(l)} \operatorname{row}_l(m_e) = k_{cr}(m)\, T_r^{(l)} \operatorname{row}_l(m)$*, over the distinct out-neighbours* e *of* r*, with* m_e *the monomial* m *with one* $c \to r$ *and one* $r \to e$ *replaced by* $c \to f \to e$*. Its* f*-component reads* $k_{cr}(m)\, L_m^{(f)}(x_r) = -N \sum_e \mathsf{G}_{ll}[m_e]$*, where* $L_m^{(f)}(t) := [(x_f - t)\mathsf{G}_{lf}[m] + (t - x_l)\mathsf{G}_{ll}[m]]/(x_f - x_l)$*; for single* r*,* $L_m^{(f)}(x_r) = -N\mathsf{G}_{ll}[m[r \to f]]$.

Proof. With $\mathsf{G}_{ql} = \mathsf{G}_{lq}$, the coefficient of μ in $\Psi_l(f;q)$ is $(x_f - x_l)v_l + (x_q - x_f)v_q - (x_q - x_l)w_q$, with $v := \sum k_m \operatorname{row}_l(m)$, $w := \sum k'_{m'} \operatorname{row}_l(m')$ (both in U). So (i) gives the q-components of $w = T_f v$ for $q \notin \{l,f\}$; summing them over q with the traces of v and w gives $w_f = -Nv_l$, and w_l is fixed by $\sum_q w_q = 0$. $T_f v \in U$ by the same computation; $T_f v = 0$ gives $v_l = 0$ (component f), then $v_q = 0$ ($q \notin \{l,f\}$), then $v_f = 0$. (a) is the target $\mu := m$ with one $l \to e$ replaced by $f \to e$: its only R_{lf}-preimage is m, its L_{lf}-preimages are the m_c, with coefficient 1. (b) is (i) at (l,r) at the target $\mu := m$ with one $c \to r$ replaced by $c \to l$: its only L_{lr}-preimage is m (coefficient $k_{cr}(m)$), its R_{lr}-preimages μ_e have $d_l = 1$, and $\operatorname{row}_l(\mu_e) = T_f \operatorname{row}_l(m_e)$ by (a). The f-component uses $(T_f w)_f = -Nw_l$. □

Theorem 9.206 ($b \leq n+1$)**.** *Let* $n \geq 4$ *and* $2 \leq D \leq n-1$*. Then* (H_D) *forces* $\mathsf{G} = 0$.

Proof. Fix an apex l; a vertex f is *free* for m if $f \notin \operatorname{supp}(m) + l$; $\operatorname{def}(m) := D - |\operatorname{supp}(m)|$.

(1) *Hexagon.* If m avoids l, f is free and r,u are single in m, then $\mathsf{G}_{ll}[m]=0$. Rename around (f,r,u): $m_0:=m$, $m_{i+1}:=m_i[p_{i+1}\to p_i]$ with free vertices $p_i=f,r,u,f,r,u$; then $m_6=m_0$. With $y_i:=\mathsf{G}_{ll}[m_i]$ and $L_i:=L_{m_i}^{(p_i)}$, $L_i(x_{p_i})=y_i$ by definition, and Lemma 9.205(b) gives $L_i(x_{p_{i\pm1}})=-Ny_{i\pm1}$; collinearity of these three points reads $z_i=N(z_{i-1}+z_{i+1})$ with $z_i:=y_is_{p_i}$, $s_f:=x_r-x_u$, $s_r:=x_u-x_f$, $s_u:=x_f-x_r$ ($i\in\mathbb{Z}/6$). The circulant has eigenvalues $1-2N\cos(\pi j/3)\in\{1\pm N,1\pm 2N\}$, all nonzero: $y=0$.

(2) *One apex.* If $D\le n-2$, then $\mathsf{G}_{ll}[m]=0$ for every l-avoiding m; if $D=n-1$ ($n\ge5$), for those with $|\operatorname{supp}(m)|\le n-2$. Such m have a free vertex; induct on $\operatorname{def}(m)$: with two single vertices use (1); otherwise there are two vertices of degree ≥2 (if there were at most one, then $|\operatorname{supp}(m)|\le2$ and $m=(Z_{ru}Z_{ur})^k$, which has two single vertices if $k=1$ and two of degree ≥2 if $k\ge2$), and Lemma 9.205(b) at both, with the induction hypothesis for the m_e ($\operatorname{def}(m_e)=\operatorname{def}(m)-1$), gives $L_m^{(f)}(x_r)=0$ at two points, so $L_m^{(f)}=0$ and $\mathsf{G}_{ll}[m]=L_m^{(f)}(x_f)=0$. For $D=n-1$, $\operatorname{def}(m)=1$ is the hexagon case and otherwise $|\operatorname{supp}(m_e)|\le n-2$.

(3) For these m, the components of $\operatorname{row}_l(m)$ at free vertices vanish ($L_m^{(f)}(x_r)=0$ at one more point, by (b) at a single r, as $m[r\to f]$ has the support size of m, or, if m has no single vertex, at a vertex r of degree ≥2: then $|\operatorname{supp}(m)|\le D/2$, so $|\operatorname{supp}(m_e)|\le n-2$ also for $D=n-1$), and at single q: $\mathsf{G}_{lq}[m]=\mathsf{G}_{ql}[m]=-N\mathsf{G}_{qq}[m[q\to l]]=0$, by (a) at the apex q and (2) there.

(4) $D\le n-2$. For l-avoiding m, (2)–(3) leave only components at vertices of degree ≥2; if $\operatorname{def}(m)=0$ there are none, and otherwise (b) at a vertex of degree ≥2 and induction on def give $\operatorname{row}_l(m)=0$ (T_r is injective). For m containing l, $|\operatorname{supp}(m)|<n$ gives a vertex $f\notin\operatorname{supp}(m)$, and (a) with induction on $d_l(m)$ gives $\operatorname{row}_l(m)=0$.

(5) $D=n-1$, $n\ge5$. Rows at l-avoiding m with $|\operatorname{supp}(m)|\le n-2$ vanish as in (4) (if $\operatorname{def}(m)=1$, the 1-trace kills the last component). The other l-avoiding monomials are $Z_\pi:=\prod_{v\ne l}Z_{v\pi(v)}$, π fixed-point-free on $W:=[n]-l$. For $f\ne r$ in W with $\pi(f)\ne r$, $\pi(r)\ne f$, (b) at the monomial "π with f merged into r" gives $\operatorname{row}_l(Z_{\pi\circ(f\,r)})=-\operatorname{row}_l(Z_\pi)$; these moves connect all fixed-point-free π ($|W|\ge4$: a move at f,r in different cycles merges them; on a $|W|$-cycle a split followed by a merge exchanges two cyclically adjacent entries), so the l-avoiding part of row l is $\det(Z_W)u_l$, $u_l\in U$. At the apex $q\ne l$, (a) with $f:=l$, the symmetry, and $\operatorname{sgn}(\pi[q\to l])=\operatorname{sgn}(\pi)$ give $(u_l)_q=-N\delta_q$, $\delta_q:=(u_q)_q$; the 1-trace of u_l gives $\delta_l=N\sum_{q\ne l}\delta_q$ for every l, so all δ_q are equal and $\delta=N^2\delta$: $u_l=0$. Monomials containing l follow by (a) as in (4).

(6) $(n,D)=(4,3)$. Every monomial is a directed 3-cycle and misses exactly one vertex v. For a cycle C' missing v and $l\ne v$, the cycle $C:=C'[l\to v]$ has the other orientation class (of the four oriented faces of the simplex $[4]$), and (a) at the apex l with symmetry gives $\mathsf{G}_{vl}[C']=-3\alpha(C)$. The two traces of $h^{C'}$ give $\delta_X(v)=3\sum_{l\ne v}\delta_Y(l)$ and $x_v\delta_X(v)=3\sum_{l\ne v}x_l\delta_Y(l)$, with $\delta_X(v),\delta_Y(v)$ the values of α on the two cycles missing v; at the four distinct x_v this forces $\delta_X=-3\delta_Y$, and likewise $\delta_Y=-3\delta_X$, so $\delta=9\delta=0$. So every missing-vertex row vanishes, and (a) gives every row. □

Theorem 9.207 ($b=n+2$)**.** *Let $n\ge4$ and $D=n$. (A) (H_n) forces $\mathsf{G}=\det(Z)\,\mathsf{G}_0$ with G_0 a constant symmetric matrix, $\mathsf{G}_0\mathbf{1}=\mathsf{G}_0x=0$. (B) If moreover (ii) holds at every apex, with T a symmetric 3-tensor of polynomials in Z and $\mathsf{T}(\mathbf{1},\cdot,\cdot)=\mathsf{T}(x,\cdot,\cdot)=0$, then $\mathsf{G}_0=0$.*

Proof. (A) A balanced m of degree n has full support (class F: $m=Z_\pi$, π fixed-point-free on $[n]$), or misses one vertex p and has one vertex r of degree 2 (class S), or $|\operatorname{supp}(m)|\le n-2$ (class L).

Integrality. If B is an integer matrix, $N\ge2$ and $\beta=NB\beta$, then $\beta=0$ ($1/N$ is not an algebraic integer).

S. For $m\in\mathrm{S}$ and a single q, (a) at the apex q and symmetry give $h^m_q=-N\alpha(m[q\to p])$, with $m[q\to p]\in\mathrm{S}$ missing q, with the same r. The two traces of h^m eliminate h^m_r: $(x_r-x_p)\alpha(m)=N\sum_q(x_r-x_q)\alpha(m[q\to p])$. On the finite set of the $m\in\mathrm{S}$ with a fixed r, $\beta(m):=(x_r-x_p)\alpha(m)$

satisfies $\beta = NB\beta$ with B the 0/1 matrix of $m \mapsto \sum_q [m[q \to p]]$; so $\alpha = 0$, $h^m = 0$, rows of singles vanish by (a), and row r by symmetry and its 1-trace.

L. For l-avoiding $m \in \mathrm{L}$ there are a free vertex and a vertex of degree ≥ 2, and (b) with downward induction on $|\operatorname{supp}(m)|$ (base: S) gives $\operatorname{row}_l(m) = 0$; then (a) with induction on $d_l(m)$ gives the m containing l.

F. Fix l; let $r \neq l$ with $\pi(l) \neq r$, $\pi(r) \neq l$. Lemma 9.205 at (l, r) at the target "$l \to \pi(l)$ replaced by $r \to \pi(l)$" has the R-preimages Z_π, $Z_{\pi(l\,r)}$ and one L-preimage in S; at the target "$\pi^{-1}(r) \to r$ replaced by $\pi^{-1}(r) \to l$" it has the L-preimages Z_π, $Z_{(l\,r)\pi}$ and one R-preimage in S. So $\operatorname{row}_l(Z_{\pi(l\,r)}) = \operatorname{row}_l(Z_{(l\,r)\pi}) = -\operatorname{row}_l(Z_\pi)$. For $n \geq 4$ these moves connect all fixed-point-free π: right moves merge two cycles (so every π is joined to an n-cycle with the same $\pi^{-1}(l)$), split-and-merge pairs rotate and swap positions on n-cycles, and left moves change $\pi^{-1}(l)$. So $\operatorname{row}_l = \det(Z) u_l$, and symmetry and the traces give $\mathsf{G} = \det(Z)\mathsf{G}_0$.

(B) By Jacobi, $L_{lr} \det(Z) = -Z_{rl} M_r$ and $R_{lr} \det(Z) = -Z_{rl} M_l$ (M_a the principal minor without a). So, with $h_l(a, q) := (x_a - x_l)(\mathsf{G}_0)_{ll} + (x_q - x_a)(\mathsf{G}_0)_{lq}$, (ii) becomes

$$D_l(r, q) = -h_l(r, q) M_r + h_l(q, r) M_q + c_l(r, q) M_l, \qquad c_l(r, q) := (x_l - x_r)(\mathsf{G}_0)_{lr} - (x_l - x_q)(\mathsf{G}_0)_{lq}.$$

Fix a and one monomial $\mu = Z_\pi$ of M_a; M_a, M_b ($a \neq b$) share no monomial. Let τ be the coefficient tensor of μ in T (constant, symmetric, traces 0). At an apex $l \neq a$ the μ-coefficient of the identity is $(x_r - x_q)\tau_{lqr} + (x_q - x_l)\tau_{llq} + (x_l - x_r)\tau_{llr} = \operatorname{sgn}(\pi) d_l(r, q)$, with $d_l(a, q) = -h_l(a, q) = -d_l(q, a)$ and $d_l(r, q) = 0$ if $a \notin \{r, q\}$. The traces of τ determine $\sigma_q := (x_l - x_q)\tau_{llq}$, and for distinct $q, r \notin \{l, a\}$: $\tau_{lqr} = \operatorname{sgn}(\pi)[(x_q - x_a)(\mathsf{G}_0)_{lq} - (x_r - x_a)(\mathsf{G}_0)_{lr}]/(n(x_r - x_q))$. Comparing apexes l and q (a fourth vertex a exists as $n \geq 4$) gives $(x_r - x_a)[\omega_{lq} + \omega_{qr} + \omega_{rl}] = 0$ with $\omega_{bc} := (x_c - x_b)(\mathsf{G}_0)_{bc}$; so ω is a gradient, and the traces make it 0: $\mathsf{G}_0 = 0$. □

Theorem 9.208 ($b = n + 3$)**.** *Let $n \geq 5$ and $D = n + 1$. Then (H_{n+1}) forces $\mathsf{G} = 0$.*

Proof. Classes: full support (F_w: one vertex w of degree 2); support $n-1$ with missing p and either one vertex of degree 3 (S^3) or two vertices r, u of degree 2 (S^{22}); support $\leq n - 2$ (L).

S^3 is as S in Theorem 9.207 (one non-single vertex).

Three tools on S^{22} (s misses p). REN: $h^s_q = -N\alpha(s[q \to p])$ for single q, as before. SWAP: $\alpha(s) = \alpha(s^{(pr)})$ for r of degree 2, $s^{(pr)}$ the monomial s with the labels p, r exchanged, in every configuration: split r into singles p ($c \to p \to e$) and r; Lemma 9.205 at (p, r) (target "$p \to e$ replaced by $r \to e$") and at (r, p) (target "$r \to e'$ replaced by $p \to e'$") have the same R-preimages and the L-preimages s, $s^{(pr)}$ with the same coefficient $k \in \{1, 2\}$; the r- and p-components and $\mathsf{G}_{pr} = \mathsf{G}_{rp}$ give $-Nk\alpha(s) = -Nk\alpha(s^{(pr)})$. (Sym2): for singles q, q', $\mathsf{G}_{qq'}[s] = \mathsf{G}_{q'q}[s]$ with $\operatorname{row}_q(s) = T_p^{(q)} h^{s[q \to p]}$ and REN inside reads

$$\begin{aligned} &-N(x_{q'} - x_p)\,\alpha(s[q \to p][q' \to q]) + (x_p - x_q)\,\alpha(s[q \to p]) \\ &\qquad = N(x_q - x_p)\,\alpha(s[q' \to p][q \to q']) - (x_p - x_{q'})\,\alpha(s[q' \to p]). \end{aligned}$$

By SWAP at both vertices of degree 2, α depends, on a fixed shape (the unlabelled multigraph with its positions), only on the labels at the single positions; for two single positions write $F(i, j)$. (Sym2) for $(p; a, b)$, its $(a, b) \leftrightarrow (b, a)$ sum and difference give (*pair lemma*) $F(i, j) = c\,(x_j - x_i)$ on the five labels not used at the other single positions ($N \pm 1 \neq 0$; four labels besides p). For $n \geq 6$, three single positions force $c = 0$ (by the pair lemma for (P_1, P_2) and for (P_1, P_3), $F = c_{12}(x_j - x_i) = c_{13}(x_k - x_i)$ as affine functions of x_i at four values, so $c_{12} = c_{13}$ and $c_{12}(x_j - x_k) = 0$). For $n = 5$ (two single positions A, B; support $\{r, u, a, b\}$) a short case analysis on the neighbours of A, B gives the shapes $r \to a \to u$, $u \to b \to r$, $r \leftrightarrow u$; $r \to a \to u$, $r \to b \to u$, $(u \to r)^2$; $r \leftrightarrow a$, $u \leftrightarrow b$, $r \leftrightarrow u$; $a \leftrightarrow b$, $(r \to u)^2$, $(u \to r)^2$; and B2: $r \to a \to b \to u$, $r \to u$, $(u \to r)^2$. The first four have an automorphism exchanging A and B, so $F(a, b) = F(b, a)$ and $c = 0$. For B2, Lemma 9.205 at three targets (both in-edges of r come from u) gives $\operatorname{row}_r(s) = (T_p^{(r)})^2 h^{s^{(pr)}}$; its p-component

$h^s_r = -N(T_p^{(r)} h^{s^{(pr)}})_r$, computed from F by REN and the traces (at $x_r = 0$, allowed as the operators $T_f^{(l)}$ and the traces are invariant under $x \mapsto x - t\mathbf{1}$; with $y := x_p$, $z := x_u$, $S := 1/x_a + 1/x_b$), is

$$c\,(x_b - x_a)\big[(2N^2+N-1) - N(N+1)yS + (N^2-3N+1)y/z\big] = 0,$$

i.e. $cE = 0$ with $E = 7 - 4yS + y/z$ at $N = 4$; the same for $s^{(pu)}$ gives $cE' = 0$, $E' = 7 - 4zS + z/y$, and $zE - yE' = 6(z-y) \neq 0$: $c = 0$.

All rows on S^{22}. With $\alpha = 0$: $h^s = 0$ (REN and the two traces at r, u), rows of singles by (a), rows r, u by symmetry and their two traces. L as in Theorem 9.207.

F_w. For a single l whose in-neighbour is w, let $e_2 \neq l$ be the other out-target of w: Lemma 9.205 at (l, e_2), target "$w \to e_2$ replaced by $w \to l$", has the only L-preimage m and the only R-preimage e_2 merged into l, in S^{22}; so $\mathrm{row}_l(m) = 0$. Otherwise one in-swap (target "$w \to e$ replaced by $w \to l$", e an out-target of w other than the in-neighbour of l; the two out-targets of w are distinct) gives $\mathrm{row}_l(m) = -\mathrm{row}_l(m_1)$ with m_1 of the first kind. Row w: symmetry and the 1-trace. □

Corollary 9.209. *Let $n \geq 4$, a and κ arbitrary, and $4 \leq b \leq n+2$, or $b = n+3$ with $n \geq 5$. No eigenvector $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ of the pencil has y-order 2; with Theorem 9.201, every eigenvector there has $(\Phi_0, \Phi_1) \neq 0$. For $n = 4, 5$ this holds in every layer $b \geq 4$ that occurs.*

Proof. By Lemma 9.204, G satisfies (H_{b-2}), and also (ii), at every x with distinct entries. Theorems 9.206, 9.207 and 9.208 give $\mathsf{G} = 0$ there, so $\Phi_2 = \frac{1}{2} y^T \mathsf{G} y = 0$ (a polynomial), against y-order 2. □

In every layer, the following result removes all monomials of support less than n from the y-Hessian. It needs x outside a proper Zariski-closed set, so in it x is no longer fixed. For distinct p, r and a monomial s, $s^{(pr)}$ is s with the labels p and r exchanged, and $m! := \prod_{u \neq v} k_{uv}(m)!$.

Lemma 9.210 (multi-T)**.** *Assume (H_D). Let s miss p, let $r \in \mathrm{supp}(s)$ and $k := d_r(s)$. Then*

$$\mathrm{row}_p(s^{(pr)}) = (T_r^{(p)})^k \,\mathrm{row}_p(s), \qquad \textit{equivalently} \qquad \mathrm{row}_r(s) = (T_p^{(r)})^k \,\mathrm{row}_r(s^{(pr)}).$$

Proof. Put $\tilde{g}(m) := m!\,\mathrm{row}_p(m)$. In Lemma 9.205 at the apex p with $f := r$, an L_{pr}-preimage m of a target μ (one edge $c \to p$ of μ becomes $c \to r$) has $k_{cr}(m)\,\mathrm{row}_p(m) = k_{cp}(\mu)\tilde{g}(m)/\mu!$, and an R_{pr}-preimage m' (one edge $r \to c$ of μ becomes $p \to c$) has $k_{pc}(m')\,\mathrm{row}_p(m') = k_{rc}(\mu)\tilde{g}(m')/\mu!$. So, for every target μ without an edge $r \to p$,

$$\textstyle\sum_{r \to c} \tilde{g}(\mu \text{ with that edge replaced by } p \to c) = T_r^{(p)} \sum_{c \to p} \tilde{g}(\mu \text{ with that edge replaced by } c \to r),$$

both sums over the edges of μ counted with multiplicity. Label the k in-edges of r in s by a set I and its k out-edges by a set O. For $A \subseteq I$ and $B \subseteq O$ let $m(A,B)$ be s with the edges in A redirected to end at p and the edges in B redirected to start at p; it is balanced exactly when $|A| = |B|$, and then it is loopless and has no edge between p and r. Put $\Sigma_j := \sum_{|A|=|B|=j} \tilde{g}(m(A,B))$ (a sum over labelled pairs). Then $\Sigma_0 = s!\,\mathrm{row}_p(s)$ and $\Sigma_k = s!\,\mathrm{row}_p(s^{(pr)})$, as $m(I,O) = s^{(pr)}$ and $(s^{(pr)})! = s!$. For $|A'| = j+1$ and $|B| = j$ $(0 \leq j < k)$ the target $\mu := m(A', B)$ has no edge $r \to p$, its out-edges at r are $O \setminus B$ and its in-edges at p are A', so the identity reads $\sum_{b \in O \setminus B} \tilde{g}(m(A', B+b)) = T_r^{(p)} \sum_{a \in A'} \tilde{g}(m(A'-a, B))$. Summing over all such (A', B) gives $(j+1)\Sigma_{j+1} = (k-j)\,T_r^{(p)} \Sigma_j$, so $\Sigma_j = \binom{k}{j} (T_r^{(p)})^j \Sigma_0$; $j = k$ is the first form. The second form is the first applied to $s^{(pr)}$ (which misses r and has $d_p = k$) with p and r exchanged. □

Theorem 9.211 (full support)**.** *Let $n \geq 4$, $D \geq 1$, $\Delta := \prod_{i<j}(x_i - x_j)$ and $R := \mathbb{Z}[x_1, \ldots, x_n][1/\Delta]$. There is a nonzero $P_{n,D} \in R$, depending only on (n, D), such that at every x with distinct entries and $P_{n,D}(x) \neq 0$, (H_D) at x implies $\mathsf{G}[m] = 0$ for every monomial m with $|\,\mathrm{supp}(m)| < n$. If G depends polynomially on x and (H_D) holds at every x with distinct entries, then $\mathsf{G}[m] = 0$ identically for every such m. For $D \leq n-1$ one may take $P_{n,D} = 1$.*

Proof. For $D = 1$ there is no balanced loopless monomial, and for $2 \le D \le n-1$ Theorem 9.206 gives $\mathsf{G} = 0$. Let $D \ge n$, and let S be the set of balanced monomials of degree D that miss exactly one vertex $p = p(s)$.

The system. Let $s \in S$ miss p and $r \ne p$; then $r \in \operatorname{supp}(s)$, $k := d_r(s) \ge 1$, and $s' := s^{(pr)} \in S$ misses r. By the symmetry, Lemma 9.210 (second form) and $(T_p^{(r)} w)_p = -N w_r$,

$$h_r^s = \mathsf{G}_{pr}[s] = \mathsf{G}_{rp}[s] = \big((T_p^{(r)})^k \operatorname{row}_r(s')\big)_p = -N\big((T_p^{(r)})^{k-1} h^{s'}\big)_r,$$

and $h_p^s = -\sum_{r \ne p} h_r^s$. So $H := (h^s)_{s \in S}$ satisfies $H = N\,BH$, where the square matrix B depends only on (n, D) and x; every entry of every $T_f^{(l)}$ is a $\mathbb{Z}$-combination of $(x_q - x_f)/(x_q - x_l)$, $(x_f - x_l)/(x_q - x_l)$ and N, so B has entries in R.

Integrality. $\det(I - NB) = \sum_{j \ge 0} (-N)^j e_j(B) = 1 + Nc$ with $c \in R$ (e_j the sums of the principal $j \times j$ minors). If it were 0, then $1/N = -c = f/\Delta^M$ with $f \in \mathbb{Z}[x]$, so N would divide every coefficient of Δ^M. But Δ is, up to sign, the Vandermonde determinant, in which $x_2 x_3^2 \cdots x_n^{n-1}$ has coefficient ± 1; so Δ is primitive, hence so is Δ^M (Gauss's lemma), and $N = n - 1 \ge 2$. So $P_{n,D} := \det(I - NB) \ne 0$, and where $P_{n,D}(x) \ne 0$ we get $H = 0$.

Support $n-1$. Let $s \in S$ miss p. Then $\operatorname{row}_p(s) = h^s = 0$. For $r \ne p$, Lemma 9.210 gives $\operatorname{row}_r(s) = (T_p^{(r)})^{d_r(s)} \operatorname{row}_r(s^{(pr)})$, and $\operatorname{row}_r(s^{(pr)}) = h^{s^{(pr)}} = 0$. So $\mathsf{G}[s] = 0$ on S.

Support at most $n-2$. Let L be the set of these monomials; as $D \ge n$, each has a vertex of degree ≥ 2. Fix an apex l. For $m \in L$ avoiding l, by downward induction on $|\operatorname{supp}(m)|$: there is a free vertex f; take a vertex r of degree ≥ 2 and an edge $c \to r$ of m. By Lemma 9.205(b), $k_{cr}(m)\, T_r^{(l)} \operatorname{row}_l(m) = \sum_e T_f^{(l)} \operatorname{row}_l(m_e)$, and every m_e avoids l and has support $\operatorname{supp}(m) + f$, so lies in S or in L with a larger support; so $\operatorname{row}_l(m_e) = 0$, and $\operatorname{row}_l(m) = 0$ as $T_r^{(l)}$ is injective. For $m \in L$ containing l, by induction on $d_l(m)$: take $f \notin \operatorname{supp}(m)$ and an edge $l \to e$; by Lemma 9.205(a), $k_{le}(m) \operatorname{row}_l(m) = T_f^{(l)} \sum_c \operatorname{row}_l(m_c)$, where each m_c lies in S or in L with $d_l(m_c) = d_l(m) - 1$ (if $d_l(m) = 1$, m_c avoids l); so $\operatorname{row}_l(m) = 0$. Hence $\mathsf{G}[m] = 0$ for every $m \in L$.

Polynomial dependence. Write $P_{n,D} = f/\Delta^M$ with $0 \ne f \in \mathbb{Z}[x]$. Every $\mathsf{G}[m]$ with $|\operatorname{supp}(m)| < n$ is then a polynomial in x that vanishes on the dense set $\{\Delta f \ne 0\}$, hence identically. □

Corollary 9.212. *Let $n \ge 4$ and $b \ge 3$. For every eigenvector $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ of the pencil of y-order 2, every monomial of the y-Hessian G of Φ_2 has full support $[n]$.*

Proof. By Lemma 9.204, G satisfies (H_{b-2}) at every x with distinct entries, and it depends polynomially on x; apply Theorem 9.211 with $D = b - 2$. □

Remark 9.213. (1) (What is used.) Condition (i) at every apex (through Lemma 9.205, also inside Lemma 9.210), the symmetry of G (in the system, and through Theorem 9.206) and the two traces. Not used: (ii), T, κ, an upper bound on D (such as $D < n(n-1)$), any analysis of shapes. The symmetry is essential: with (i) and the row traces alone, monomials of support $< n$ survive at $(n, D) = (4, 3)$ and $(4, 6)$ (an independent exact computation). (2) At one fixed integer x an argument modulo N cannot replace the integrality step: two of the n integers x_q agree modulo $N = n - 1$, so N divides $\Delta(x)$. With Theorem 9.211, y-order 2 in a layer b is a question about the monomials of full support only.

The next step uses a transfer between two vertices of arbitrary degrees, and gives the layer $b = n + 4$.

Lemma 9.214 (transfer between two vertices)**.** *Assume (H_D). Let c be a balanced monomial and $p \ne r$ two vertices with no edge between them in c. Then*

$$\operatorname{row}_p(c^{(pr)}) = (T_r^{(p)})^{d_r(c) - d_p(c)} \operatorname{row}_p(c)$$

($T_r^{(p)}$ is invertible on U, so negative powers are allowed). For $d_p(c) = 0$ this is Lemma 9.210.

Proof. Put $k_p := d_p(c)$, $k_r := d_r(c)$, $a := k_p + k_r$, and let I and O be the a edges of c that end, resp. start, in $\{p, r\}$, labelled with multiplicity. For $A \subseteq I$, $B \subseteq O$ let $c(A, B)$ be c with the edges in A ending at p, those in $I \setminus A$ ending at r, those in B starting at p and those in $O \setminus B$ starting at r. It is balanced exactly when $|A| = |B|$, it is loopless and has no edge between p and r; $c = c(A_0, B_0)$ with $|A_0| = |B_0| = k_p$, $c(I \setminus A, O \setminus B) = c(A, B)^{(pr)}$, and relabelling does not change $m!$. For $|A| = |B|$ put $V(A, B) := c(A, B)!\,\mathrm{row}_p(c(A, B))$. As in the proof of Lemma 9.210, the target $c(A, B)$ with $|A| = j + 1$, $|B| = j$ gives, with $T := T_r^{(p)}$,

$$\textstyle\sum_{o \in O \setminus B} V(A, B + o) = T \sum_{i \in A} V(A - i, B).$$

In the algebra of multilinear polynomials in commuting variables α_i ($i \in I$) and β_o ($o \in O$) with $\alpha_i^2 = \beta_o^2 = 0$ put $F := \sum_j T^{-j} \sum_{|A|=|B|=j} V(A, B)\, \alpha^A \beta^B$ (U-valued), $\sigma_\alpha := \sum_i \alpha_i$, $D_\alpha := \sum_i \partial/\partial\alpha_i$, and the same for β. The identity says $(\sigma_\alpha - D_\beta)F = 0$. On multilinear polynomials in a variables $[D, \sigma] = a - 2\deg$, so $(e, f, h) := (\sigma_\alpha, D_\alpha, 2\deg_\alpha - a)$ and $(e', f', h') := (-D_\beta, -\sigma_\beta, a - 2\deg_\beta)$ are $\mathfrak{sl}_2$-triples, and $(e + e')F = 0$. F has weight 0 for $h + h'$, and in a finite-dimensional $\mathfrak{sl}_2$-module a vector of weight 0 killed by e is invariant; so F is fixed by WW', $W := \exp(e)\exp(-f)\exp(e)$, $W' := \exp(e')\exp(-f')\exp(e')$. Per variable, $\exp(e)\exp(-f)\exp(e) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the basis $(\alpha_i, 1)$, resp. $(1, -\beta_o)$, so $W\alpha^A = (-1)^{|A|}\alpha^{I \setminus A}$ and $W'\beta^B = (-1)^{|B|}\beta^{O \setminus B}$. Comparing coefficients: $T^{-(a-j)}V(I \setminus A, O \setminus B) = T^{-j}V(A, B)$. At (A_0, B_0), $a - 2k_p = k_r - k_p$, which is the claim. □

Notation for the rest of this subsection. $\tilde{g}_v(m) := m!\,\mathrm{row}_v(m)$; a *single* of m is a vertex of degree 1, and for a single l we write c for its in-neighbour and d for its out-neighbour. Let x have distinct entries and $P_{n,D}(x) \neq 0$ (Theorem 9.211). In the proof of Theorem 9.216 ($D = n+2$) a monomial is *dead* if its support is less than n or it is a fan (Lemma 9.215); there every row of a dead monomial is 0 (Theorem 9.211, Lemma 9.215, as a full-support monomial of degree $n+2$ has at most two vertices of degree ≥ 2). Two consequences of Lemma 9.214 (with the symmetry and $(T_f^{(l)}w)_f = -Nw_l$):

(N1) if $d_l(m) = 1$, $d_u(m) = 2$ and no edge joins u and l, then $\tilde{g}_{lu}[m] = \tilde{g}_{ul}[m] = -N\,\tilde{g}_{uu}[m^{(ul)}]$, and u is a single of $m^{(ul)}$;

(N2) if $d_l(m) = 2$, r is a single of m and no edge joins l and r, then $\tilde{g}_l(m) = T_r^{(l)}\tilde{g}_l(m^{(lr)})$, l is a single of $m^{(lr)}$, and $(\tilde{g}_l(m))_r = -N\,\tilde{g}_{ll}[m^{(lr)}]$.

Lemma 9.215 (single rows)**.** *Let x be as above and m of full support. If some vertex u has out-edges to two distinct singles, or in-edges from two distinct singles (we call m a* fan*), then* $\mathrm{row}_l(m) = 0$ *for every single l of m. If all single rows of m vanish and m has at most two vertices of degree ≥ 2, then every row of m vanishes.*

Proof. We use the divided-power form of Lemma 9.205 from the proof of Lemma 9.210, at the apex l with a partner e; a preimage in which a vertex becomes isolated has support $n - 1$ and is 0 by Theorem 9.211. *Out-edges.* If $u \to l$ and $u \to e_2$ with $e_2 \neq l$ a single, the target $m - (u \to e_2) + (u \to l)$ (partner e_2) has m twice on one side and a preimage with e_2 isolated on the other, so $2\,T\tilde{g}_l(m) = 0$. If $c \neq u$, pick a single out-target $e \notin \{c, l\}$ of u; the target $m - (u \to e) + (u \to l)$ (partner e) gives $\tilde{g}_l(m_1) = -\tilde{g}_l(m)$ with $m_1 := m - (c \to l) - (u \to e) + (c \to e) + (u \to l)$, and in m_1 the vertex u points to l and to its other single out-target: the first case. *In-edges.* Choose a single in-source $e \notin \{d, l\}$ of u. The target $m - (l \to d) + (e \to d)$ (partner e) gives $\tilde{g}_l(m_1) = -\tilde{g}_l(m)$ with $m_1 := m - (l \to d) - (e \to u) + (e \to d) + (l \to u)$ ($m_1 = m$ if $d = u$), its other side isolating l; in m_1 the vertex u has the single in-sources l and $e_2 \neq l$, and the target $m_1 - (l \to u) + (e_2 \to u)$ (partner e_2) gives $2\tilde{g}_l(m_1) = 0$. *The rest.* If w, v are the only vertices of degree ≥ 2, then $\mathsf{G}_{wq}[m] = \mathsf{G}_{qw}[m] = 0$ for every single q, and the two traces of row w give $\mathsf{G}_{ww} + \mathsf{G}_{wv} = 0 = x_w\mathsf{G}_{ww} + x_v\mathsf{G}_{wv}$, so row w vanishes; the same for v. □

Theorem 9.216 ($b = n+4$)**.** *Let $n \geq 6$ and $D = n+2$. There is a nonzero $Q \in R$ such that (H_D) at an x with distinct entries, $Q(x) \neq 0$ and $P_{n,D}(x) \neq 0$ implies $\mathsf{G} = 0$. If G depends polynomially on x and (H_D) holds at every x with distinct entries, then $\mathsf{G} = 0$ identically.*

Proof. Configurations. $\sum_{v \in \operatorname{supp}(m)}(d_v - 1) = D - |\operatorname{supp}(m)| = 2 + (\text{number of missing vertices})$, and support $< n$ is dead; so a monomial m of full support has one vertex of degree 3 (then its three out-targets are distinct singles: a fan) or two vertices w, v of degree 2 and $n-2$ singles. Unless m is a fan, each of w, v has at most one single out-target and at most one single in-source (we call m *bad*); then $w \to v$ and $v \to w$, and with $a := k_{wv}$, $b := k_{vw}$ there are four types: α ($a = b = 1$; w has a single out-target s_w and in-source t_w, v likewise s_v, t_v), β ($a = 2$, $b = 1$; t_w and s_v), β' ($a = 1$, $b = 2$) and δ ($a = b = 2$; no single is adjacent to w or v). By Lemma 9.215 it suffices to show that $H := (\tilde{g}_l(m))$, over all bad m and all singles l of m, is 0. Below, $N(\dots)$ means N times an R-combination of entries of H.

The partner relation $P(f)$. Let $f \in \{w, v\}$, i the other of w, v (an in-source of f), and l a single whose in-neighbour c is a single (as in every use below); so $i \neq l$ and $i \neq c$. The target $\mu := m - (i \to f) + (i \to l)$ at the apex l with partner f gives

$$\textstyle\sum_y \tilde{g}_l(x_y) = T_f^{(l)}\big[\tilde{g}_l(m) + \tilde{g}_l(m_1)\big], \qquad m_1 := m - (c \to l) - (i \to f) + (c \to f) + (i \to l),$$

the sum over the labelled out-edges $f \to y$ of μ ($y \neq l$), x_y being μ with $f \to y$ replaced by $l \to y$ (f has degree 1 and l degree 2 in x_y). Call a single $r \notin \{l, c, d\}$ *usable* if r is a single of every live x_y and not adjacent to l there. Then the r-component, with (N2) and $(T_f^{(l)} g)_r = [(x_r - x_f)g_r + (x_f - x_l)g_l]/(x_r - x_l)$, gives, if $\tilde{g}_l(m_1) = N(\dots)$, $\tilde{g}_{lr}[m] = \gamma_r^f\, \tilde{g}_{ll}[m] + N(\dots)$ with $\gamma_r^f := (x_l - x_f)/(x_r - x_f)$.

Closing step. Suppose $P(w)$ and $P(v)$ both have $\tilde{g}_l(m_1) = N(\dots)$, some single is usable for both, and every single $r \notin \{l, c, d\}$ is usable for one of them. Since $\gamma_r^w - \gamma_r^v = (x_w - x_v)(x_l - x_r)/((x_r - x_v)(x_r - x_w))$ is a unit of R, we get $\tilde{g}_{ll}[m] = N(\dots)$, then $\tilde{g}_{lr}[m] = N(\dots)$ for all singles $r \notin \{l, c, d\}$, $\tilde{g}_{lw}[m]$, $\tilde{g}_{lv}[m] = N(\dots)$ by (N1) when w, v are not neighbours of l, and the remaining entries $\tilde{g}_{lc}, \tilde{g}_{ld}$ from the two traces of row l (determinant $x_d - x_c$, a unit). So $\tilde{g}_l(m) = N(\dots)$.

Positions (in this order; each uses only earlier ones and dead monomials). (1) α, $c \notin \{w, v\}$: in $P(v)$, m_1 is a fan (w points to s_w and l), and x_{s_v} is a fan unless $d = w$; in $P(w)$, m_1 is a fan (w has the single in-sources t_w and c), and x_{s_w} is a fan unless $d = v$. The singles excluded from $P(v)$ or $P(w)$ lie in $\{l, c, d\}$, $\{l, c, s_v\}$ or $\{l, c, s_w\}$, and $s_v \neq s_w$; with $n - 2 \geq 4$ singles the closing step applies. (2) α, $c \in \{w, v\}$: take a single $f \notin \{l, d\}$ whose in-source i is a single (at most three singles are excluded: s_w, s_v and d; $n \geq 6$). The target $m - (i \to f) + (i \to l)$ at the apex l with partner f isolates f on one side, so $\tilde{g}_l(m) = -\tilde{g}_l(m_1)$ with $m_1 = m - (c \to l) - (i \to f) + (c \to f) + (i \to l)$, a bad α in position (1). (3) β, c a single: in $P(w)$, m_1 is a fan (t_w, c) and the only x_y is x_v (twice); in $P(v)$, m_1 is a bad α in position (2), and x_{s_v} is a fan unless $d = w$. The closing step applies as in (1). (4) β, $c = v$: the swap of (2) with a single $f \notin \{l, d\}$ with a single in-source ($n - 4 \geq 1$ choices) leads to position (3). (5) β': (3) and (4) with w, v exchanged. (6) δ: in $P(w)$, m_1 is a bad β in position (4), and the only x_y is x_v (twice); $P(v)$ likewise with β'; every single $r \notin \{l, c, d\}$ is usable ($n - 5 \geq 1$ of them). The closing step applies.

Integrality. So $H = N\,BH$ for a square matrix B over R depending only on n and x. (Here H is made of $\tilde{g}$-vectors; all relations are written in $\tilde{g}$, and passing to rows would bring factors such as $1/2$, as $m! \in \{1, 2, 4\}$.) As in the proof of Theorem 9.211, $Q := \det(I - NB) = 1 + Nc'$ with $c' \in R$ is not 0; where $Q(x) \neq 0$, $H = 0$, and $\mathsf{G} = 0$ by Lemma 9.215. The polynomial statement follows as in Theorem 9.211. □

Corollary 9.217. *For $n \geq 6$ no eigenvector $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,b)}(n)$ of the pencil has y-order 2 at $b = n+4$.*

Proof. By Lemma 9.204, G satisfies (H_{b-2}) at every x with distinct entries, and it depends polynomially on x; apply Theorem 9.216 with $D = b - 2 = n + 2$. □

Remark 9.218. Used: (i) at every apex (through Lemmas 9.205, 9.214, (N1), (N2)), the symmetry, the traces and Theorem 9.211; $n \geq 6$ only for the counts in positions (1), (2), (3) and (6). Not used: (ii), T, κ, the bound $D < n(n-1)$, any computation. Exact computations (supporting; not used): Lemma 9.214 in all 2517 tested cases at $(n, D) = (4, 4..8), (5, 5..7)$; every target of the proof rebuilt at $n = 6$ and $n = 7$; and the whole system at $(n, D) = (6, 8)$ (97950 unknowns) has kernel 0 at two random x (programs and logs in the ancillary files).

Remark 9.219. (1) (What is used.) Theorems 9.206 and 9.208 use only (i), the symmetry and the traces of G; (ii) and T enter only in Theorem 9.207(B). Not used: the value of κ, and the κ-dependent rows.

(2) (Full arguments.) The preimage lists and coefficients behind each step (Lemmas 9.203 and 9.204, Theorem 9.206, Theorem 9.207 with the connectivity lemma for the moves, Theorem 9.208 with the shape list at $n = 5$ and the B2 computation, Lemma 9.210, Theorem 9.211, and Theorem 9.216 with its two lemmas) are written out in full in the ancillary files.

(3) (Computations, supporting; not used.) Lemma 9.203 agrees with $\mathcal{J}$ computed from the definitions. At $D = n + 1$: the subsystem on the monomials of support $\geq n - 1$ has kernel 0 at $(n, D) = (5, 6)$, the B2 identity holds for general N, and the $n = 5$ shape list is complete. The programs and logs are in the ancillary files.

(4) (Larger D.) $\det(Z)^k \mathsf{G}_0$ satisfies (H_{kn}), so at $D = kn$ condition (i) alone cannot give $\mathsf{G} = 0$; the proof of Theorem 9.207(B) should go through (a sketch) with M_a replaced by $\det(Z)^{k-1} M_a$ (with $\operatorname{sgn}(\pi)$ replaced by the coefficient of μ in $\det(Z)^{k-1} M_a$, nonzero for some μ; L_{lr}, R_{lr} give a factor k). Exact computations (at $n = 4$, $5 \leq D \leq 11$, and at $(n, D) = (5, 7)$) fit the following: for $n \geq 4$ and $D < n(n-1)$, (H_D) forces $\mathsf{G} = \det(Z)^{D/n} \mathsf{G}_0$ if $n \mid D$ and $\mathsf{G} = 0$ otherwise. (For $D \geq n(n-1)$ every multiple of $\prod_{i \neq j} Z_{ij}$ satisfies (i) trivially.) This would exclude y-order 2 in every layer.

9.35. **y-order 1 at $b = 3$: the spectator rows force $\mathsf{N} = 0$.** We keep the setting of Subsections 9.19 and 9.33 with $b = 3$, $n \geq 4$ and $a \geq 0$, and we write σ for the upper index s of Lemma 9.84. Let $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ satisfy $\kappa\, \mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ for one $\kappa \in \mathbb{C}$, and put $\lambda := \kappa - (a+1)/n$ (the λ of Corollary 9.31(a) at $s = \kappa$, as $d - 2 = a + 1$). By Lemma 9.84, Φ has y-order 0 if $\mathsf{F} \neq 0$, and y-order 1 if $\mathsf{F} = 0 \neq \mathsf{N}$; $\Phi = \mathsf{C}$ is excluded by Theorem 9.201. Here we exclude y-order 1. So let $\mathsf{F} = 0$.

Lemma 9.220 (the c-rows of $\mathcal{J}$ at $b = 3$, $\mathsf{F} = 0$)**.** *Let $\varphi \in \mathcal{F}_{(a,3)}(n)$ with $\mathsf{F} = 0$ in Lemma 9.84, let l, k, m be distinct, and work at a slice point (D, Z). The coefficient of $u_l^2 u_k$ in $\mathcal{J}(\varphi)$ at $e_l \wedge e_k \wedge e_m$ is*

$$\mathsf{c}^{\mathcal{J}}(l,k,m) := \mathfrak{L}\omega(E_{kl}, E_{lk}, E_{lm}) + \mathfrak{L}\omega(E_{ll}, E_{kk}, E_{lm}) + \mathfrak{L}\omega(E_{ll}, E_{lk}, E_{km}) = \sum_{r \notin \{l,m\}} \mathsf{c}^{\mathcal{J}}_r(l,k,m)\, Z_{mr} Z_{rl},$$

and for every spectator $r \notin \{l, k, m\}$, with $\mathsf{M} := \mathsf{M}^{(r)}$ and $\partial_i := \partial/\partial x_i$,

$$\mathsf{c}^{\mathcal{J}}_r(l,k,m) = 2\theta_{lm}\big[x_{ml}\, \partial_l \mathsf{M}(k,m) + x_{km}\, \partial_k \mathsf{M}(l,m) + x_{lk}\, \partial_k \mathsf{M}(l,l)\big].$$

So, by Lemma 9.85(ii), for every $\kappa \in \mathbb{C}$ the coefficient of $Z_{mr} Z_{rl}$ in $\kappa\, \mathsf{c}(l,k,m) - \mathsf{c}^{\mathcal{J}}(l,k,m)$ is $-2\theta_{lm}$ times

$$x_{ml}\, \partial_l \mathsf{M}(k,m) + x_{km}\, \partial_k \mathsf{M}(l,m) + x_{lk}\, \partial_k \mathsf{M}(l,l) + \kappa\, \mathsf{M}(\Delta_{lk}, \Delta_{lm}).$$

Neither $\mathcal{E}\varphi = 0$ nor $\sum_\sigma \mathsf{n}^\sigma_{pq} = 0$ is used.

Proof. The first equality is the expansion of the trilinear form $\mathfrak{L}\omega(u e_l^T, u e_k^T, u e_m^T)$ in u, as in the proof of Lemma 9.83. Its value has degree 2 in Y and weight $e_m - e_l$, so at $y = 0$ it is a combination of the $Z_{mr} Z_{rl}$.

(1) *The operators.* Put $\gamma_{pq} := \frac{1}{2}(x_p + x_q)\theta_{pq}$ ($p \neq q$; so $\gamma_{qp} = -\gamma_{pq}$) and $\eta_{pq} := x_q Y E_{pq} - x_p E_{pq} Y$. In the proof of Lemma 9.199, for $B = E_{pq}$ ($p \neq q$) the matrix $B \circ D$ is off-diagonal, $W = \gamma_{pq} E_{pq}$ and $B \circ Y - [W, Y] = \theta_{pq}\eta_{pq}$; for $B = E_{pp}$, $W = 0$. So

$$D_{J(E_{pq})}\omega(U,V) = \theta_{pq} D_{(0,\eta_{pq})}\omega(U,V) - \gamma_{pq}\big[\omega([E_{pq},U],V) + \omega(U,[E_{pq},V])\big],$$

and $D_{J(E_{pp})} = x_p\partial_p + D_{(0,E_{pp}\circ Y)}$; at $y = 0$, $D_{(0,E_{pp}\circ Y)}$ multiplies a monomial in Z by half the number of its edges at p. The brackets that occur are $[E_{kl}, E_{lk}] = E_{kk} - E_{ll}$, $[E_{kl}, E_{lm}] = E_{km}$, $[E_{lm}, E_{kl}] = -E_{km}$, $[E_{lm}, E_{ll}] = -E_{lm}$, $[E_{lk}, E_{ll}] = -E_{lk}$, $[E_{lk}, E_{km}] = E_{lm}$, $[E_{km}, E_{lk}] = -E_{lm}$, and 0 for the other pairs. Collecting, with $\mathsf{c} := \mathsf{c}(l,k,m)$ and $\mathsf{w} := \omega(E_{ll}, E_{lm})$ (so $\mathsf{c} + \mathsf{w} = \omega(E_{lk}, E_{km}) + \omega(E_{kk}, E_{lm})$),

$$\begin{aligned}\mathsf{c}^{\mathcal{J}} = {} & (D_{J(E_{ll})} - \gamma_{lm})\mathsf{c} + (D_{J(E_{ll})} - D_{J(E_{kk})} + \gamma_{km} - \gamma_{kl} - \gamma_{lm})\mathsf{w} \\ & + \theta_{kl} D_{(0,\eta_{kl})}\omega(E_{lk}, E_{lm}) - \theta_{lk} D_{(0,\eta_{lk})}\big[\omega(E_{kl}, E_{lm}) + \omega(E_{ll}, E_{km})\big] \\ & + \theta_{lm} D_{(0,\eta_{lm})}\big[\omega(E_{kl}, E_{lk}) + \omega(E_{ll}, E_{kk})\big] + \theta_{km} D_{(0,\eta_{km})}\omega(E_{ll}, E_{lk}).\end{aligned}$$

(2) *Which terms reach $Z_{mr}Z_{rl}$.* At a base point

$$D_{(0,\eta_{pq})} = Z_{qp}(x_q\partial_{y_q} - x_p\partial_{y_p}) + (x_q y_p - x_p y_q)\frac{\partial}{\partial Z_{pq}} + x_q \sum_{c \notin \{p,q\}} Z_{cp}\frac{\partial}{\partial Z_{cq}} - x_p \sum_{c \notin \{p,q\}} Z_{qc}\frac{\partial}{\partial Z_{pc}}.$$

At $y = 0$ the first term leaves a factor Z_{qp} with $p, q \in \{l, k, m\}$ and the second vanishes; so for the coefficient of $Z_{mr}Z_{rl}$ only the two sums count, acting on the parts of degree 0 in y. The first sum turns an edge $c \to q$ into $c \to p$, the second an edge $p \to c$ into $q \to c$. Hence the θ_{kl}-term gives nothing (it creates an edge into k or out of l), and we need only the coefficients, at $y = 0$, of $Z_{mr}Z_{rk}$ in $\omega(E_{kl}, E_{lm}) + \omega(E_{ll}, E_{km})$, of $Z_{rm}Z_{mr}$ and $Z_{lr}Z_{rl}$ in $\omega(E_{kl}, E_{lk}) + \omega(E_{ll}, E_{kk})$, and of $Z_{kr}Z_{rl}$ in $\omega(E_{ll}, E_{lk})$.

(3) *The coefficients at $y = 0$*, from (3) and the jets after Lemma 9.84; at $\mathsf{F} = 0$ every second Z-derivative of Φ is $O(y)$. In the proof of Lemma 9.85(ii), (B) has no spectator term, and (C), (A) give the coefficients $\chi := 2\theta_{lm}(\mathsf{M}(k,l) - \mathsf{M}(k,m))$ of $Z_{mr}Z_{rl}$ in $\mathsf{c} + \mathsf{w}$ and $A := 2\theta_{lm}(\mathsf{M}(l,l) - \mathsf{M}(l,m))$ in w; (A) with m replaced by k gives $2\theta_{lk}(\mathsf{M}(l,l) - \mathsf{M}(l,k))$ for $Z_{kr}Z_{rl}$ in $\omega(E_{ll}, E_{lk})$. In $\omega(E_{kl}, E_{lm})$ the brackets $\pm E_{km}$ give $\partial\Phi/\partial Z_{km} = O(y)$, the diagonal parts of $[E_{kl}, Z]$ and $[E_{lm}, Z]$ give monomials with Z_{lk} or Z_{ml}, and the other pairings are second Z-derivatives: no $Z_{mr}Z_{rk}$. In $\omega(E_{ll}, E_{km}) = \partial_l\partial\Phi/\partial Z_{km} - \beta(E_{km}, E_{ll})$ the first term is $O(y)$, and in $\beta(E_{km}, E_{ll})$ ($[E_{km}, E_{ll}] = 0$) the terms $Z_{mr}E_{kr}$ and $-Z_{rk}E_{rm}$ of $[E_{km}, Z]$, paired with E_{ll}, give $-2\theta_{km}(\mathsf{n}^l_{kr} - \mathsf{n}^l_{rm})Z_{mr}Z_{rk}$; so the coefficient is $2\theta_{km}(\mathsf{M}(l,k) - \mathsf{M}(l,m))$. At $y = 0$, $\omega(E_{kl}, E_{lk}) = 0$, since $\beta(E_{kl}, E_{lk})$ and $\beta(E_{lk}, E_{kl})$ both equal $-\theta_{kl}[2(\mathsf{n}^k_{kl} - \mathsf{n}^l_{kl})Z_{kl}Z_{lk} + \mathsf{N}_k - \mathsf{N}_l]$; and $\omega(E_{ll}, E_{kk}) = \partial_l\mathsf{N}_k - \partial_k\mathsf{N}_l$, in which $Z_{pq}Z_{qp}$ has the coefficient $\nu_{pq} := 2(\partial_l\mathsf{n}^k_{pq} - \partial_k\mathsf{n}^l_{pq})$.

(4) *Collecting.* $Z_{mr}Z_{rl}$ has one edge at l and none at k; $\partial_l\theta_{lm} = \theta^2_{lm}$, $\frac{1}{2} - \gamma_{lm} = -x_l\theta_{lm}$ and $\gamma_{km} - \gamma_{kl} = x_k x_{lm}\theta_{kl}\theta_{km}$. So the first two terms of (1) give

$$\begin{aligned}&(x_l\partial_l + \tfrac{1}{2} - \gamma_{lm})\chi - x_k\partial_k A + (\gamma_{km} - \gamma_{kl})A \\ &\quad = 2\theta_{lm}\big[x_l\partial_l(\mathsf{M}(k,l) - \mathsf{M}(k,m)) - x_k\partial_k(\mathsf{M}(l,l) - \mathsf{M}(l,m))\big] - 2x_k\theta_{kl}\theta_{km}(\mathsf{M}(l,l) - \mathsf{M}(l,m)),\end{aligned}$$

and the last three terms give, by (2) and (3),

$$-\theta_{lk}x_k\, 2\theta_{km}(\mathsf{M}(l,k) - \mathsf{M}(l,m)) + \theta_{lm}(x_m\nu_{rm} - x_l\nu_{lr}) - \theta_{km}x_k\, 2\theta_{lk}(\mathsf{M}(l,l) - \mathsf{M}(l,k)),$$

with $\nu_{rm} = 2(\partial_l\mathsf{M}(k,m) - \partial_k\mathsf{M}(l,m))$ and $\nu_{lr} = 2(\partial_l\mathsf{M}(k,l) - \partial_k\mathsf{M}(l,l))$. As $\theta_{lk} = -\theta_{kl}$, the terms without derivatives add up to $2x_k\theta_{kl}\theta_{km}$ times $-(\mathsf{M}(l,l) - \mathsf{M}(l,m)) + (\mathsf{M}(l,k) - \mathsf{M}(l,m)) + (\mathsf{M}(l,l) - \mathsf{M}(l,k)) = 0$, and the derivative terms add up to the stated $\mathsf{c}^{\mathcal{J}}_r$. Finally, at $\mathsf{F} = 0$ Lemma 9.85(ii) gives $\mathsf{c}_r(l,k,m) = -2\theta_{lm}\mathsf{M}(\Delta_{lk}, \Delta_{lm})$. □

(Checks, supporting; not used.) The reduction in (4) is an identity in generic functions $\mathsf{M}(\cdot,\cdot)$ of x (a wrong edge count 1 in place of $\frac{1}{2}$ fails). The general-F form of $\mathsf{c}_r^{\mathcal{J}}$ agrees with $\mathcal{J}$ computed from the definitions at random slice points, and the stated row agrees symbolically with that form at $\mathsf{F}=0$, with a sign-flip control. The programs and logs are in the ancillary files.

For a spectator r write $\mathsf{M}=\mathsf{M}^{(r)}$, $\mathsf{M}(\sigma,t)=\mathsf{n}^{\sigma}_{tr}$, and $\mathsf{M}(v,w):=\sum_{\sigma,t}v_\sigma w_t\mathsf{M}(\sigma,t)$, with $\partial_i:=\partial/\partial x_i$. As $\kappa\,\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$ gives $\kappa\,\mathsf{c}=\mathsf{c}^{\mathcal{J}}$ at every slice point, Lemma 9.220 gives, for all distinct r,i,j,h,

$$(\mathrm{E}_r)\qquad (x_h-x_i)\,\partial_i\mathsf{M}(j,h)+(x_j-x_h)\,\partial_j\mathsf{M}(i,h)+(x_i-x_j)\,\partial_j\mathsf{M}(i,i)=-\kappa\,\mathsf{M}(\Delta_{ij},\Delta_{ih})$$

as polynomial identities in x; and by Lemma 9.84, for every pair $p\neq q$,

$$(\mathrm{Tr})\qquad \textstyle\sum_\sigma \mathsf{n}^{\sigma}_{pq}=0,\qquad \sum_\sigma x_\sigma\mathsf{n}^{\sigma}_{pq}=0.$$

Theorem 9.221 (y-order 1 at $b=3$)**.** *Let $n\geq 4$, $a\geq 0$, and let the polynomials $\mathsf{n}^{\sigma}_{pq}=\mathsf{n}^{\sigma}_{qp}$ in x, homogeneous of degree a and invariant under $x\mapsto x+c\mathbf{1}$, satisfy* (Tr) *and* (E_r) *for all distinct r,i,j,h. Then $\mathsf{n}=0$ in each of the cases*

(a) $\kappa=n$ and $0\leq a\leq (n^2-3)/2$;
(b) $\kappa\neq 0$, $\lambda\neq 0$, and $a\leq\kappa(n-3)$ if κ is a positive integer.

In particular no eigenvector of the pencil at $b=3$ with $\kappa=n$ has y-order 1 for $a\leq(n^2-3)/2$.

The proof occupies the rest of the subsection. It uses five conditions on κ; each is the hypothesis of one step, and all hold in case (b): (s1) $\kappa\neq 0$, and κ is not a positive integer with $\kappa(n-1)\leq a$ (Lemma 9.222); (s2) κ is not a nonnegative integer with $\kappa(n-2)\leq a-1$ (Lemma 9.223); (s3) $\lambda\neq 0$ (the edge equations); (s4) if κ is a positive integer, $a\leq\kappa(n-3)$ (Lemma 9.224); (s5) if κ is a positive integer, $a+1\leq\kappa(n-1)$ (the gauges; implied by (s1)). For a positive integer κ, $a\leq\kappa(n-3)$ gives $\kappa(n-1)>\kappa(n-2)>a$; for every other $\kappa\neq 0$ no resonance occurs. At $\kappa=n$ and $a\leq(n^2-3)/2$, (s1)–(s3) and (s5) hold, and (s4) fails only for $(n,a)=(4,5),(4,6),(5,11)$, where Lemma 9.224 is a finite exact computation (Remark 9.225(2)).

The rows as a t-free condition. Fix r, $I:=[n]\setminus\{r\}$, $g^t_\sigma:=\mathsf{M}(\sigma,t)$ ($t\in I$), $A^t_l:=\kappa+(x_t-x_l)\partial_l$ (so $A^t_t=\kappa$) and $\Gamma_t(l,k):=A^t_lg^t_k-A^t_kg^t_l$ for $l,k,t\in I$. Expanding, $\Gamma_h(i,j)-\Gamma_i(i,j)=\kappa\,\mathsf{M}(\Delta_{ij},\Delta_{ih})+(x_h-x_i)\partial_ig^h_j+(x_j-x_h)\partial_jg^h_i+(x_i-x_j)\partial_jg^i_i$, so ($\mathrm{E}_r$)$(i,j,h)$ says $\Gamma_h(i,j)=\Gamma_i(i,j)$. With ($\mathrm{E}_r$)$(j,i,h)$ and h running over $I\setminus\{i,j\}\neq\emptyset$: the rows ($\mathrm{E}_r$) say exactly that $\Gamma_t(l,k)$ does not depend on $t\in I$, for every pair $l\neq k$ in I. Only the g^t_σ with $\sigma,t\in I$ occur.

Lemma 9.222. *Assume* (s1)*. For each r, every solution of* (E_r) *has the form $g^t_\sigma=\partial_\sigma H^{(r)}+A^t_\sigma h^{(r)}_t$ ($\sigma,t\in I$), with $H^{(r)}$ and $h^{(r)}_t$ invariant under $x\mapsto x+c\mathbf{1}$ and homogeneous of degrees $a+1$ and a; and these are solutions.*

Proof. Conversely: ∂H gives $\Gamma_t(l,\sigma)=\kappa(\partial_\sigma-\partial_l)H+(x_\sigma-x_l)\partial_\sigma\partial_lH$, free of t, and A^th_t gives 0, as A^t_l and A^t_σ commute. Now let g be a solution. Replacing g^t by $g^t-A^t(g^t_t/\kappa)$ does not change Γ_t, so we may assume $g^t_t=0$. Then $\Gamma_t(t,\sigma)=\kappa g^t_\sigma$, and $\Gamma_t(t,\sigma)=\Gamma_\sigma(t,\sigma)$ makes $\gamma(t,\sigma):=g^t_\sigma$ antisymmetric. Work in $w_b:=x_b-x_r$ ($b\in I$, $m:=n-1$ variables; translation-invariant functions are the polynomials in w, and $\partial_b=\partial/\partial w_b$ on them), let Ω^p_D be the p-forms in the dw_b with coefficients homogeneous of degree D, d the exterior derivative, $e_b:=w_b\partial_b$, $Q_b:=e_b-\kappa$ and $k:=\sum_b dw_b\wedge Q_b$, so $\gamma=\sum_{t<\sigma}\gamma(t,\sigma)\,dw_t\wedge dw_\sigma\in\Omega^2_a$. For a triangle $\{i,j,l\}$ of I the remaining rows say $\Gamma_l(i,j)=\Gamma_i(i,j)$, i.e. $\kappa(\gamma(i,j)+\gamma(j,l)+\gamma(l,i))=E^l$ with $E^l:=(w_l-w_i)\partial_i\gamma(l,j)-(w_l-w_j)\partial_j\gamma(l,i)$, for each choice of the apex l. One computes $E^l=\sum_{\mathrm{cyc}}e_i\gamma(j,l)-w_l\,(d\gamma)(i,j,l)$; so the rows are equivalent to $d\gamma=0$ (two apexes) and then $k\gamma=0$. We use: (F1) $d^2=0$, $k^2=0$ and $dk=-kd$ (as $\partial_bQ_c=Q_c\partial_b+\delta_{bc}\partial_c$); (F2) for $p\geq 1$, a closed form in Ω^p_D is exact (Poincaré; $D+p\geq 1$); (F3) on forms with coefficient w^α, k is the wedge with $\sum_b(\alpha_b-\kappa)dw_b$, so for $p\geq 1$ a k-closed form in Ω^p_D is in $k\Omega^{p-1}_D$, unless $\alpha=(\kappa,\dots,\kappa)$ has degree D, i.e. $\kappa m=D$ with $\kappa\in\mathbb{Z}_{\geq 0}$. By (s1) this does

not occur for $D \le a$. *Claim* $P(D)$, $D \le a$: if $\eta \in \Omega^1_D$ and $d\eta \in k\Omega^1_{D-1}$, then $\eta \in d\Omega^0_{D+1} + k\Omega^0_D$. For $D = 0$, $d\eta = 0$ and (F2) applies. For $D \ge 1$ let $d\eta = k\zeta$; then $kd\zeta = -dk\zeta = 0$, so $d\zeta \in k\Omega^1_{D-2}$ by (F3), and $P(D-1)$ gives $\zeta = dF' + kf'$; so $d\eta = kdF' = -dkF'$, $\eta + kF'$ is closed, and $\eta = dF'' - kF'$ by (F2). Now $k\gamma = 0$ gives $\gamma = k\eta$ with $\eta \in \Omega^1_a$ (F3), $kd\eta = -dk\eta = -d\gamma = 0$ gives $d\eta \in k\Omega^1_{a-1}$ (F3), and $P(a)$ gives $\eta = dF'' + kf$, so $\gamma = kdF'' = -dkF''$, i.e. $\gamma(t,\sigma) = A^t_\sigma \partial_t F'' - \kappa \partial_\sigma F''$. Undoing the first step, $g^t_\sigma = \partial_\sigma H + A^t_\sigma h_t$ for all $\sigma, t \in I$, with $H := -\kappa F''$ and $h_t := (g^t_t - \partial_t H)/\kappa$. □

Parameters. Write $\nu^{[r]}_t(\sigma) := \mathsf{n}^\sigma_{tr}$ ($t \ne r$, $\sigma \in [n]$). As $\sum_\sigma \partial_\sigma H = 0$ and $\sum_\sigma A^t_\sigma h = (n\kappa - a)h$ (translation invariance and Euler), Lemma 9.222 and the first trace give, for every $\sigma \in [n]$,

$$\nu^{[r]}_t(\sigma) = \partial_\sigma H^{(r)} + A^t_\sigma h^{(r)}_t - (n\kappa - a)\, h^{(r)}_t\, [\sigma = r], \tag{P}$$

and the second trace at the pair $\{t, r\}$ reads

$$(a+1)H^{(r)} + O^{(r)}_t h^{(r)}_t = 0, \qquad O^{(r)}_t := \kappa\Big(\sum_\sigma x_\sigma - n x_r\Big) + a(x_t + x_r) - \sum_\sigma x^2_\sigma \partial_\sigma. \tag{V}$$

The symmetry $\mathsf{n}^\sigma_{tr} = \mathsf{n}^\sigma_{rt}$ is the *coupling* $\nu^{[r]}_t = \nu^{[t]}_r$ (C). (P), (V), (C) are linear in the parameters $(H^{(r)}, h^{(r)}_t)$. *The gauge at* r, G_r: $H^{(r)} := H^{(r)}_g := [\tau^{a+1}] \prod_{\sigma \ne r} (1 - (x_\sigma - x_r)\tau)^\kappa$ (a formal power series if $\kappa \notin \mathbb{Z}_{\ge 0}$), $h^{(r)}_t := -\partial_t H^{(r)}_g/\kappa$, all other parameters 0. With $D_{kl} := \kappa(\partial_k - \partial_l) + (x_k - x_l)\partial_k \partial_l$, one has $D_{kl} \prod_{\sigma \in I} (1 - w_\sigma \tau)^\kappa = 0$; so (P) gives $\nu = 0$ (at $\sigma = t$ directly, at $\sigma \in I \setminus \{t\}$ as $\nu = -D_{t\sigma} H^{(r)}_g/\kappa$, at $\sigma = r$ by the sum), and G_r satisfies (V) and (C). $H^{(r)}_g$ is symmetric in $[n] \setminus \{r\}$, and it is not 0 by (s5).

The edge equations. Fix an edge $\{t, r\}$, $S := [n] \setminus \{t, r\}$, $p := h^{(r)}_t$, $q := h^{(t)}_r$, $\bar h := (p+q)/2$, $\mu := p - q$, $u := x_r - x_t$, $y_\sigma := x_\sigma - (x_r + x_t)/2$ and $K := H^{(r)} - H^{(t)} - u\bar h$. Translation-invariant polynomials are the polynomials in (u, y_S). (C) with (P) on both sides reads, for every $\sigma \in [n]$ ($n\lambda = n\kappa - a - 1$),

$$\partial_\sigma K + (\kappa - y_\sigma \partial_\sigma)\mu - (n\kappa - a)\tfrac{\mu}{2}([\sigma = r] + [\sigma = t]) = n\lambda\, \bar h\, ([\sigma = r] - [\sigma = t]).$$

For $\sigma \in S$ this is (E1): $d_S K = k_S \mu$, with $k_S := \sum_{\sigma \in S} dy_\sigma \wedge (y_\sigma \partial_\sigma - \kappa)$ and u a parameter. The sum over all σ vanishes identically, and the r-component minus the t-component is (E3): $2n\lambda \bar h = 2\partial_u K + \frac{u}{2} \sum_{\sigma \in S} \partial_{y_\sigma} \mu$. By (s3), (E3) gives $\bar h$ from (K, μ), then $p = \bar h + \mu/2$, $q = \bar h - \mu/2$, and (V) gives $H^{(r)} = -O^{(r)}_t p/(a+1)$ and $H^{(t)} = -O^{(t)}_r q/(a+1)$.

Lemma 9.223 (one edge, (E1))**.** *Assume* (s2). *The solutions* (K, μ) *of* (E1)*, translation-invariant and homogeneous of degrees* $a+1$ *and* a*, are exactly*

$$\mu = [\tau^a]\, \Gamma(u\tau) E(\tau), \qquad K = -[\tau^{a+1}]\, \Gamma(u\tau) E(\tau) + k_{\mathrm{top}} u^{a+1}, \qquad E(\tau) := \prod_{\sigma \in S} (1 + y_\sigma \tau)^\kappa,$$

with $\Gamma(\tau) = \sum_{j=0}^a c_j \tau^j$*: the* $a+2$ *parameters* $c_0, \dots, c_a, k_{\mathrm{top}}$.

Proof. (E1) has no u-derivative, so it splits by powers of u: $\mu = \sum_j u^j \mu_j(y)$, $K = \sum_j u^j K_j(y)$ with $dK_j = k_S \mu_j$ (d in the y_S only). Let V_i be the space of f of degree i in y_S with $k_S f$ exact. Then $\dim V_i \le 1$: for $f \in V_i$, $k_S df = -dk_S f = 0$; on y^α, k_S is the wedge with $\sum(\alpha_\sigma - \kappa) dy_\sigma$, nonzero unless $\alpha = (\kappa, \dots, \kappa)$ of degree $\kappa(n-2)$, which by (s2) is not in degree $i - 1 \le a - 1$; so $df = k_S f'$ with a unique f' of degree $i - 1$, $f' \in V_{i-1}$, and $f \mapsto f'$ is injective ($f' = 0$ gives $df = 0$, $f = 0$ as $i \ge 1$). And $e_i := [\tau^i] E \in V_i$: $(y_\sigma \partial_\sigma - \kappa)E = -\kappa E/(1 + y_\sigma \tau) = -\tau^{-1} \partial_\sigma E$ gives $k_S e_i = -d e_{i+1}$; $e_i \ne 0$ for $i \le a$ by (s2). So $\mu_j = c_j e_{a-j}$, $K_j = -c_j e_{a+1-j}$ for $j \le a$, and $K_{a+1} = k_{\mathrm{top}}$ is free. □

Symmetry. On the edge $\{t, r\}$, μ and K are symmetric in y_S, and ∂_u, $\sum_S \partial_{y_\sigma}$ and $O^{(r)}_t$ commute with the permutations of x_S; so $H^{(r)} = -O^{(r)}_t(\bar h + \mu/2)/(a+1)$ is symmetric in S. As $H^{(r)}$ does

not depend on t, and the groups $\mathrm{Sym}([n]\setminus\{r,t\})$, $t\neq r$, generate $\mathrm{Sym}([n]\setminus\{r\})$ ($n-1\geq 3$), $H^{(r)}$ *is symmetric in* $[n]\setminus\{r\}$ for every r.

Lemma 9.224 (the one-edge lemma)**.** *Fix an edge* $\{t,r\}$*, take* (K,μ) *as in Lemma 9.223,* $\bar h$ *from* (E3) *and* $H^{(r)}$*,* $H^{(t)}$ *from* (V)*. If* $H^{(r)}$ *is symmetric in* $[n]\setminus\{r\}$ *and* $H^{(t)}$ *in* $[n]\setminus\{t\}$*, then* (K,μ) *lies in the span of the edge data of* G_r *and* G_t*. Under* (s1)–(s5) *the symmetry of* $H^{(r)}$ *alone suffices.*

Proof. G_r and G_t solve the system, so their edge data lie in the family; they are independent (G_r has $H^{(r)}=H^{(r)}_g\neq 0$, G_t has $H^{(r)}=0$ and nonzero edge data). So it suffices to bound the solutions by dimension 2. Put $x_r=0$, $v_\sigma:=x_\sigma-x_r$ (so $u=-v_t$, $y_\sigma=v_\sigma-v_t/2$), fix $s_0\in S$, and let $F_{(i,k)}:=[v_t^i v_{s_0}^k]H^{(r)}$, a polynomial in $v_{S\setminus\{s_0\}}$, and $F_i:=[v_t^i]H^{(r)}$. As $H^{(r)}$ is symmetric in S, its symmetry in $[n]\setminus\{r\}$ gives $F_{(i,k)}=F_{(k,i)}$. *Lowest terms.* c_j enters μ and K only through $c_j u^j e_{a-j}(y)$ and $-c_j u^j e_{a+1-j}(y)$, and k_{top} only through $k_{\mathrm{top}}u^{a+1}$. In (E3), $2\partial_u$ lowers the u-degree by one and $\frac{u}{2}\sum\partial_y$ raises it, so the c_j-part of $p=\bar h+\mu/2$ has u-degree $\geq j-1$, with u^{j-1}-part $-jc_ju^{j-1}e_{a+1-j}(y)/(n\lambda)$. In v, $u^ig(y)=(-1)^iv_t^ig(v_S)+O(v_t^{i+1})$, and $O_t^{(r)}=(\kappa Y-R)+(\kappa+a)v_t-v_t^2\partial_{v_t}$ with $Y:=\sum_S v_\sigma$, $R:=\sum_S v_\sigma^2\partial_{v_\sigma}$; it keeps or raises the v_t-degree, and $(\kappa Y-R)e_d=(d+1)e_{d+1}$ for $e_d:=[\tau^d]\prod_{\sigma\in S}(1+v_\sigma\tau)^\kappa$ (as $(\kappa Y-R)E=\partial_\tau E$). Hence c_j ($j\geq 1$) enters $H^{(r)}$ only in v_t-degrees $\geq j-1$, and

$$F_{j-1}=\frac{(-1)^{j-1}j(a+2-j)}{(a+1)n\lambda}\,c_j\,e_{a+2-j}(v_S)+(\text{terms in } c_0,\dots,c_{j-1})\qquad(1\leq j\leq a);$$

the k_{top}-part of $H^{(r)}$ is exactly $-(-1)^a\kappa k_{\mathrm{top}}v_t^a(v_t+Y)/(n\lambda)$. *Triangular system.* F_0 contains only c_0,c_1 (k_{top} sits in v_t-degrees $\geq a\geq 1$). For $2\leq j\leq a$ the row $F_{(j-1,0)}=F_{(0,j-1)}$ has its right side in c_0,c_1, and its left side is $\frac{(-1)^{j-1}j(a+2-j)}{(a+1)n\lambda}c_je^{(\kappa)}_{a+2-j}+(\text{terms in } c_0,\dots,c_{j-1})$, with $e^{(\kappa)}_d:=[\tau^d]\prod_{\sigma\in S\setminus\{s_0\}}(1+v_\sigma\tau)^\kappa\neq 0$ for $d\leq a$ by (s4). So $c_2,\dots,c_a$ are linear in (c_0,c_1); and if $a\geq 1$ the row $F_{(a,0)}=F_{(0,a)}$ contains k_{top} only on the left, with the coefficient $-(-1)^a\kappa\sum_{\sigma\in S\setminus\{s_0\}}v_\sigma/(n\lambda)\neq 0$. (If $a=0$ the family has only the two parameters c_0, k_{top}.) So the solutions form a space of dimension ≤ 2. □

Proof of Theorem 9.221. Under (s1)–(s5) Lemma 9.224 holds; in case (a) at $(n,a)=(4,5)$, $(4,6)$, $(5,11)$ it holds by Remark 9.225(2). Take the parameters of a solution. On each edge $e=\{t,r\}$ the edge data solve (E1), so they lie in the family of Lemma 9.223; $H^{(r)}$ and $H^{(t)}$ are symmetric; so the edge data are $\alpha^e_r(G_r)+\alpha^e_t(G_t)$. As $H^{(r)}$ is determined by the edge data and G_t has $H^{(r)}=0$, $H^{(r)}=\alpha^e_rH^{(r)}_g$, so $\alpha_r:=\alpha^e_r$ does not depend on e; likewise at t. The difference Δ of the solution and $\sum_r\alpha_rG_r$ satisfies (V) and (C) and has zero edge data on every edge ($G_{r'}$ has zero edge data on the edges not containing r'). By (E3) and (s3), $\bar h=0$, so $p=q=0$ on every edge; (V) gives $H^{(r)}(\Delta)=0$ for every r, and (P) gives $\nu(\Delta)=0$. So $\nu=\sum_r\alpha_r\nu(G_r)=0$. □

Remark 9.225. (1) (What is used.) The two traces, $\mathsf{n}^\sigma_{pq}=\mathsf{n}^\sigma_{qp}$, translation invariance, homogeneity and the rows (E_r) (Lemma 9.220); not the near rows, the rows $[l^3]$ and P, nor C. The relation (E4) = (V) at both ends is not needed. At $\kappa=n$, case (b) alone gives $\mathsf{N}=0$ for $a\leq n(n-3)$, which contains the range of (a) for $n\geq 6$.

(2) (The finite input.) At $\kappa=n$ and $(n,a)=(4,5),(4,6),(5,11)$, outside (s4), Lemma 9.224 (with both symmetry conditions) is an exact computation of the whole one-edge system (the family, (E3), (V), and the invariance of $H^{(r)}$ under $x_t\leftrightarrow x_{s_0}$ and of $H^{(t)}$ under $x_r\leftrightarrow x_{s_0}$): rank 5, 6, 11 on 7, 8, 13 parameters, solution dimension 2, spanned by the two gauges. The program, its negative controls and the logs are in the ancillary files.

(3) (Not claimed.) $\lambda=0$ is excluded by the method; there the kernels of the system are nonzero at $n=4$, $a=3,\dots,6$, at $(n,a)=(5,4)$ and $(5,5)$, but 0 at $(6,3)$ and $(5,3)$ (data only). Positive integers κ with $a>\kappa(n-3)$, apart from the three cases above, are not treated. Nor is y-order

0 ($\mathsf{F} \neq 0$) treated here; for $\kappa = 0$ it is excluded by Theorem 9.87, and at $\kappa = n$, for φ_f with $f \in \mathrm{K}_n(n)_\tau$, by Theorem 9.233.

9.36. **y-order 1 at $b = 4$: the spectator-edge rows force $\mathsf{N} = 0$.** We keep the setting of Subsections 9.19 and 9.33, now with $b = 4$, $n \geq 4$ and $a \geq 0$. Let $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,4)}(n)$ satisfy $\kappa\,\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ for one $\kappa \in \mathbb{C}$, and $\Phi(x, y, Z) := \tilde{\varphi}(D, Z + \mathrm{diag}(y))$. By Lemma 9.125, $\Phi_1 = \sum_s y_s \mathsf{N}_s$ with $\mathsf{N}_s = \sum_C \mathsf{n}^s_C Z_C$, the sum over the directed 3-cycles $C = (a\,b\,c)$ ($a \to b \to c \to a$, $Z_C := Z_{ab}Z_{bc}Z_{ca}$), and $\sum_s \mathsf{n}^s_C = 0$; as $\varphi \in \mathcal{F}^{\mathcal{E}}_{(a,4)}(n)$, the part of degree 0 in y of $\sum_s x_s \partial\Phi/\partial y_s = 0$ gives $\sum_s x_s \mathsf{n}^s_C = 0$. By Theorem 9.201 and Subsection 9.34, φ has y-order 0 or 1, and y-order 1 means $\Phi_0 = 0 \neq \Phi_1$. Here we exclude y-order 1 at $\kappa = n$. So let $\Phi_0 = 0$. For a directed *spectator edge* $r \to r'$ put $I := [n] \setminus \{r, r'\}$ and

$$\mathsf{M}(\sigma, t) = \mathsf{M}^{(r \to r')}(\sigma, t) := \mathsf{n}^{\sigma}_{(t\,r\,r')} \qquad (\sigma, t \in I),$$

$(t\,r\,r')$ being the cycle $t \to r \to r' \to t$, and $\mathsf{M}(v, w) := \sum_{\sigma,t} v_\sigma w_t \mathsf{M}(\sigma, t)$.

Lemma 9.226 (the spectator-edge rows at $b = 4$, $\Phi_0 = 0$)**.** *Let $\varphi \in \mathcal{F}_{(a,4)}(n)$ with $\Phi_0 = 0$, let l, k, m be distinct, $r \neq r'$ two indices outside $\{l, k, m\}$, and $\mu := Z_{mr}Z_{rr'}Z_{r'l}$. At a slice point (D, Z), the coefficient of μ in $\mathsf{c}(l, k, m)$ is $-\theta_{lm}\mathsf{M}(\Delta_{lk}, \Delta_{lm})$, and its coefficient in $\mathsf{c}^{\mathcal{J}}(l, k, m)$ (defined as in Lemma 9.220) is*

$$\theta_{lm}\big[x_{ml}\,\partial_l \mathsf{M}(k, m) + x_{lk}\,\partial_k \mathsf{M}(l, l) + x_{km}\,\partial_k \mathsf{M}(l, m)\big].$$

Neither $\mathcal{E}\varphi = 0$ nor the traces are used, and G does not occur.

Proof. (1) *The operators.* Step (1) of the proof of Lemma 9.220 uses only Lemma 9.199 and the brackets, so its display for $\mathsf{c}^{\mathcal{J}}$ holds here; at $y = 0$, $D_{J(E_{pp})}$ acts on a monomial in Z as $x_p\partial_p$ plus half the number of its edges at p. The monomial μ has one edge at l and none at k.

(2) *Which terms reach μ.* In $D_{(0,\eta_{pq})}$ (step (2) there), the term $Z_{qp}(x_q\partial_{y_q} - x_p\partial_{y_p})$ leaves a factor Z_{qp} with $p, q \in \{l, k, m\}$, which μ does not have, and the term with $\partial/\partial Z_{pq}$ vanishes at $y = 0$. So only the two sums count, acting on the parts of degree 0 in y; the first turns an edge $c \to q$ into $c \to p$ (factor x_q), the second an edge $p \to c$ into $q \to c$ (factor $-x_p$). The θ_{kl}-term gives nothing, as μ has no edge into k and none out of l. The monomials that are moved onto μ are: by $D_{(0,\eta_{lk})}$, $Z_{mr}Z_{rr'}Z_{r'k}$ (factor x_k); by $D_{(0,\eta_{lm})}$, the 3-cycles $Z_{(m\,r\,r')}$ (factor x_m) and $Z_{(l\,r\,r')}$ (factor $-x_l$); by $D_{(0,\eta_{km})}$, $Z_{kr}Z_{rr'}Z_{r'l}$ (factor $-x_k$).

(3) *The coefficients at $y = 0$*, from (3); as $\Phi_0 = 0$, every first and second Z-derivative of Φ vanishes at $y = 0$. (R) For $p \neq q$, any index σ and distinct $\rho, \rho' \notin \{p, q\}$, the coefficient of the path $Z_{q\rho}Z_{\rho\rho'}Z_{\rho' p}$ in $\omega(E_{\sigma\sigma}, E_{pq})$ is $\theta_{pq}(\mathsf{n}^{\sigma}_{(p\,\rho\,\rho')} - \mathsf{n}^{\sigma}_{(q\,\rho\,\rho')})$: indeed $\beta(E_{\sigma\sigma}, E_{pq}) = \partial_\sigma\partial\Phi/\partial Z_{pq}$ vanishes at $y = 0$, $[E_{pq}, E_{\sigma\sigma}]$ is a multiple of E_{pq}, and in $\partial_{y_\sigma}\partial_Y[[E_{pq}, Z]]\Phi$ the diagonal part $Z_{qp}(E_{pp} - E_{qq})$ of $[E_{pq}, Z]$ leaves a factor Z_{qp}, and the path comes from $Z_{q\rho}\,\partial\mathsf{N}_\sigma/\partial Z_{p\rho}$ with $\mathsf{n}^{\sigma}_{(p\,\rho\,\rho')}$ and from $-Z_{\rho' p}\,\partial\mathsf{N}_\sigma/\partial Z_{\rho' q}$ with $-\mathsf{n}^{\sigma}_{(q\,\rho\,\rho')}$. (Z) For $p \neq q$ and $a \neq b$, $\omega(E_{pq}, E_{ab})$ at $y = 0$ is a multiple of Z_{qp} plus a multiple of Z_{ba}, plus, if $(a, b) = (q, p)$, the bracket terms $-\theta_{pq}(\mathsf{N}_p - \mathsf{N}_q)$ of $\beta(E_{pq}, E_{qp})$ and $\beta(E_{qp}, E_{pq})$, which cancel in ω: in $\partial_Y[E_{ab}]\partial_Y[[E_{pq}, Z]]\Phi$ only the diagonal part $Z_{qp}(E_{pp} - E_{qq})$ of $[E_{pq}, Z]$ survives at $y = 0$. So: by (Z), $\omega(E_{lk}, E_{km})$, $\omega(E_{kl}, E_{lm})$ and $\omega(E_{kl}, E_{lk})$ contribute nothing (μ and the moved monomials have no Z_{kl}, Z_{mk}, Z_{lk}, Z_{ml}, and the cycles $(m\,r\,r')$, $(l\,r\,r')$ avoid k). By (R), the coefficient of μ is $\theta_{lm}(\mathsf{M}(k, l) - \mathsf{M}(k, m))$ in $\omega(E_{kk}, E_{lm})$ and $\theta_{lm}(\mathsf{M}(l, l) - \mathsf{M}(l, m))$ in $\mathsf{w} = \omega(E_{ll}, E_{lm})$, so $\mathsf{c}_\mu = -\theta_{lm}\mathsf{M}(\Delta_{lk}, \Delta_{lm})$; that of $Z_{mr}Z_{rr'}Z_{r'k}$ in $\omega(E_{ll}, E_{km})$ is $\theta_{km}(\mathsf{M}(l, k) - \mathsf{M}(l, m))$; that of $Z_{kr}Z_{rr'}Z_{r'l}$ in $\omega(E_{ll}, E_{lk})$ is $\theta_{lk}(\mathsf{M}(l, l) - \mathsf{M}(l, k))$; and the coefficient of Z_C in $\omega(E_{ll}, E_{kk}) = \partial_l\mathsf{N}_k - \partial_k\mathsf{N}_l$ is $\partial_l\mathsf{n}^k_C - \partial_k\mathsf{n}^l_C$.

(4) *Collecting*, with $\partial_l\theta_{lm} = \theta^2_{lm}$, $\frac{1}{2} - \gamma_{lm} = -x_l\theta_{lm}$, $\gamma_{km} - \gamma_{kl} = x_k(\theta_{lk} + \theta_{km})$ and $\theta_{lm}(\theta_{lk} + \theta_{km}) = \theta_{lk}\theta_{km}$. The first two terms of the display give $\theta_{lm}[x_l\partial_l(\mathsf{M}(k, l) - \mathsf{M}(k, m)) - x_k\partial_k(\mathsf{M}(l, l) - \mathsf{M}(l, m))] + x_k\theta_{lk}\theta_{km}(\mathsf{M}(l, l) - \mathsf{M}(l, m))$. The moves give $-\theta_{lk}x_k\theta_{km}(\mathsf{M}(l, k) - \mathsf{M}(l, m))$, $\theta_{lm}[x_m(\partial_l\mathsf{M}(k, m) -$

$\partial_k\mathsf{M}(l,m))-x_l(\partial_l\mathsf{M}(k,l)-\partial_k\mathsf{M}(l,l))]$ and $-\theta_{km}x_k\theta_{lk}(\mathsf{M}(l,l)-\mathsf{M}(l,k))$. The terms without derivatives add up to $x_k\theta_{lk}\theta_{km}$ times $(\mathsf{M}(l,l)-\mathsf{M}(l,m))-(\mathsf{M}(l,k)-\mathsf{M}(l,m))-(\mathsf{M}(l,l)-\mathsf{M}(l,k))=0$, the terms $\pm x_l\partial_l\mathsf{M}(k,l)$ cancel, and the rest is the stated coefficient. □

As $\kappa\,\mathsf{c}=\mathsf{c}^{\mathcal{J}}$ at every slice point, Lemma 9.226 gives, multiplied by $x_{ml}=1/\theta_{lm}$, for every directed spectator edge $r\to r'$ and all distinct $l,k,m\in I$,

$$x_{ml}\,\partial_l\mathsf{M}(k,m)+x_{lk}\,\partial_k\mathsf{M}(l,l)+x_{km}\,\partial_k\mathsf{M}(l,m)=-\kappa\,\mathsf{M}(\Delta_{lk},\Delta_{lm}),\tag{P1}$$

which is $(\mathrm{E}_r)(l,k,m)$ of Subsection 9.35 on the index set I; and, as shown above, for every directed 3-cycle C,

$$\textstyle\sum_\sigma \mathsf{n}^\sigma_C=0,\qquad \sum_\sigma x_\sigma\mathsf{n}^\sigma_C=0.\tag{Tr$_4$}$$

Theorem 9.227 (y-order 1 at $b=4$)**.** *Let $n\ge5$, $a\ge0$, and let the polynomials n^σ_C in x ($\sigma\in[n]$, C a directed 3-cycle), homogeneous of degree a and invariant under $x\mapsto x+c\mathbf{1}$, satisfy* (Tr$_4$) *and* (P1) *for every directed spectator edge and all distinct $l,k,m\in I$. Then $\mathsf{n}=0$ in each of the cases*

(a) $\kappa=n$ and $0\le a\le n(n-3)$;
(b) κ satisfies (h1)–(h3) *below.*

Corollary 9.228. *Let $n\ge4$ and $4\le a\le(n^2-4)/2$. No eigenvector $\varphi\in\mathcal{F}^{\mathcal{E}}_{(a,4)}(n)$ of the pencil with $\kappa=n$ has y-order 1; so every such eigenvector has y-order 0.*

Proof. The range $4\le a\le(n^2-4)/2$ is the one where $\tau=(n^2-a-4,a,4)$ is a partition (Lemma 9.32(ii) at $s=n$), and it lies in $a\le n(n-3)$ for $n\ge5$ ($n=5$: $a\le10$; $n\ge6$: $n^2-6n+4>0$). y-order 1 means $\Phi_0=0\ne\mathsf{N}$; for $n\ge5$ Theorem 9.227(a) gives $\mathsf{N}=0$. For $n=4$ ($a=4,5,6$; there are no spectator edges) $\mathsf{N}=0$ is the finite computation of Remark 9.232(2). The last claim is Theorem 9.201 with Subsection 9.34. □

The proof of Theorem 9.227 occupies the rest of the subsection. The conditions on κ are (h1) κ is not an integer ≥0 with $\kappa(n-2)\le a$ (so $\kappa\ne0$) (Lemma 9.229); (h2) κ is not an integer ≥0 with $\kappa(n-3)\le a-1$ (the family Fam_C, and hence the steps after it); (h3) $n\kappa-d\ne0$ for $0\le d\le a$ and for $d=a+2$ (the traces and Lemma 9.231). At $\kappa=n$ and $a\le n(n-3)$: $n(n-2)>a$, $n(n-3)\ge a$ and $a+2<n^2$, so (h1)–(h3) hold, and (a) is a case of (b).

Lemma 9.229. *Assume* (h1)*. For each directed spectator edge $r\to r'$, every solution of* (P1) *has the form $\mathsf{M}(\sigma,t)=\partial_\sigma H+A^t_\sigma h_t$ ($\sigma,t\in I$), $A^t_\sigma:=\kappa+(x_t-x_\sigma)\partial_\sigma$, with H and h_t invariant under $x\mapsto x+c\mathbf{1}$ and homogeneous of degrees $a+1$ and a; and these are solutions.*

Proof. Put $w_v:=x_v-x_r$. The translation-invariant polynomials are $\mathbb{C}[w_I,w_{r'}]$, and $\partial_b=\partial/\partial w_b$ for $b\in I$. Every operator in (P1) is $\mathbb{C}[w_{r'}]$-linear (only ∂_l,∂_k with $l,k\in I$ and differences of the w_I occur), so (P1) splits by the $w_{r'}$-degree j: $\mathsf{M}=\sum_j w_{r'}^j\mathsf{M}_j$ with $\mathsf{M}_j\in\mathbb{C}[w_I]$ of degree $a-j$, each a solution. For one M_j this is the situation of Lemma 9.222 in the $|I|=n-2\ge3$ variables w_I, and its proof (the t-free form before it, the gauge, $d\gamma=0$ and $k\gamma=0$, (F1)–(F3) and the claim $P(D)$) uses neither $|I|=n-1$ nor the symmetry of M nor the traces; $|I|$ enters only through the exception of (F3), $\kappa\in\mathbb{Z}_{\ge0}$ with $\kappa(n-2)$ equal to a degree $\le a-j$, excluded by (h1) (which contains $\kappa=0$, as $0\le a-j$). Summing over j gives H, h_t; the converse is that of Lemma 9.222, $\mathbb{C}[w_{r'}]$-linearly. □

Gauges. For a polynomial Γ, $H:=[\tau^{a+1}]\Gamma(w_{r'}\tau)\prod_{b\in I}(1-w_b\tau)^\kappa$ and $h_t:=-\partial_tH/\kappa$ give $\mathsf{M}=0$: $\mathsf{M}(\sigma,t)=D_{\sigma t}H/\kappa$ with $D_{\sigma t}$ as in Subsection 9.35, and $D_{\sigma t}\prod_{b\in I}(1-w_b\tau)^\kappa=0$. (Only this direction is used.)

Parameters on a cycle. For a directed edge $e=(q\to r)$ let $(H^e,h^e_t)_{t\notin e}$ be the parameters of Lemma 9.229 for its block, so $\mathsf{n}^\sigma_{(t\,q\,r)}=\partial_\sigma H^e+A^t_\sigma h^e_t$ for $\sigma,t\notin e$. A directed 3-cycle $C=(p\,q\,r)$ lies in the blocks of its three edges: of $q\to r$ with $t=p$, of $r\to p$ with $t=q$, of $p\to q$ with $t=r$.

For $v \in C$ let $C - v$ be the edge of C not through v, $H_v := H^{C-v}$, $h_v := h_v^{C-v}$, and $S := [n] \setminus C$ ($|S| = n-3 \geq 2$). Then

$$\mathsf{n}_C^\sigma = \partial_\sigma H_v + A_\sigma^v h_v \qquad (\sigma \in S \cup \{v\},\ v \in C).$$

Every block parameter occurs: h_t^e is h_t of the cycle $(t\,e)$, and $H^{(q\to r)} = H_p$ for every cycle $(p\,q\,r)$.

Coordinates and the family Fam_C. Let $o := (x_p + x_q + x_r)/3$, $\xi_v := x_v - o$ ($v \in C$; $\sum \xi_v = 0$) and $y_\sigma := x_\sigma - o$ ($\sigma \in S$). The translation-invariant polynomials are $\mathrm{Pol}(\xi) \otimes \mathrm{Pol}(y_S)$, and $\partial_\sigma = \partial/\partial y_\sigma$ for $\sigma \in S$. With $\mathcal{H}_v := H_v + \xi_v h_v$, as $x_v - x_\sigma = \xi_v - y_\sigma$, $\partial_\sigma H_v + A_\sigma^v h_v = \partial_\sigma \mathcal{H}_v - (y_\sigma \partial_\sigma - \kappa) h_v$ ($\sigma \in S$). So the three expressions agree for $\sigma \in S$ exactly when $(\mathcal{H}_v - \mathcal{H}_w, h_v - h_w) \in \mathrm{Fam}_C$ for all $v, w \in C$, where Fam_C is the set of pairs (K, m), translation-invariant and homogeneous of degrees $a+1$ and a, with $\partial_\sigma K = (y_\sigma \partial_\sigma - \kappa) m$ for all $\sigma \in S$. This is (E1) of Subsection 9.35 in the $n-3$ variables y_S, with the parameters ξ in place of u; the proof of Lemma 9.223 splits by the monomials of the parameters, so under (h2) (its (s2) with $n-3$ in place of $n-2$), with $e_d := [\tau^d] \prod_{\sigma \in S} (1 + y_\sigma \tau)^\kappa$ ($e_d \neq 0$ for $d \leq a$):

$$\mathrm{Fam}_C = \Big\{ m = \sum_{i=0}^{a} \alpha_i(\xi)\, e_{a-i},\quad K = -\sum_{i=0}^{a} \alpha_i(\xi)\, e_{a+1-i} + k(\xi) \Big\}, \qquad \alpha_i \in \mathrm{Pol}_i(\xi),\ k \in \mathrm{Pol}_{a+1}(\xi).$$

Let $\mathcal{S}_C := \mathrm{Pol}(\xi) \otimes \mathrm{span}\{e_d\}$. For (K, m) in $\mathcal{S}_C$, written $K = \sum_d K_d e_d$, $m = \sum_d m_d e_d$ ($K_d, m_d \in \mathrm{Pol}(\xi)$), $(K, m) \in \mathrm{Fam}_C$ if and only if $K_{d+1} = -m_d$ for every d with $e_{d+1} \neq 0$. With $E := \prod_S (1 + y\tau)^\kappa$: $(y_\sigma \partial_\sigma - \kappa) e_d = -\partial_\sigma e_{d+1}$, $\sum_S y_\sigma \partial_\sigma e_d = d e_d$, $\sum_{\sigma \in S} \partial_\sigma e_d = (\kappa(n-3) - d + 1) e_{d-1}$ (shift $y_S \mapsto y_S + \epsilon$), and $R_S e_d = -(d+1) e_{d+1}$ for $R_S := \sum_S (y_\sigma^2 \partial_\sigma - \kappa y_\sigma)$ (as $R_S E = -\partial_\tau E$). So $\mathcal{S}_C$ is stable under ∂_v ($v \in C$; $\partial_v y_\sigma = -\frac{1}{3}$), under multiplication by ξ_v, and under $\sum_S \partial_\sigma$, $\sum_S y_\sigma \partial_\sigma$ and R_S.

The traces. Let $\bar{h}$, $\bar{\mathcal{H}}$ be the means of h_v, $\mathcal{H}_v$ over $v \in C$, $\mathcal{K}_v := \mathcal{H}_v - \bar{\mathcal{H}}$ and $\mu_v := h_v - \bar{h}$; then $(\mathcal{K}_v, \mu_v) \in \mathrm{Fam}_C \subset \mathcal{S}_C \times \mathcal{S}_C$. Averaging over v, $\mathsf{n}_C^\sigma = \partial_\sigma \bar{\mathcal{H}} - (y_\sigma \partial_\sigma - \kappa) \bar{h}$ for $\sigma \in S$, and $\mathsf{n}_C^v = \partial_v \mathcal{H}_v - \xi_v \partial_v h_v + (\kappa - \frac{2}{3}) h_v$. On translation-invariant functions $\sum_\sigma \partial_\sigma = 0$ and $\sum_\sigma (x_\sigma - o) \partial_\sigma$ is the degree. So the two traces (Tr_4) (the second minus o times the first) read

$$(n\kappa - a - 2)\, \bar{h} = \sum_v \big(\xi_v \partial_v \mu_v - \partial_v \mathcal{K}_v \big),$$

$$(a+1)\, \bar{\mathcal{H}} = R_S \bar{h} + \sum_v \xi_v^2 \partial_v \bar{h} + \sum_v \big(\xi_v^2 \partial_v \mu_v - \xi_v \partial_v \mathcal{K}_v - (\kappa - \tfrac{2}{3}) \xi_v \mu_v \big).$$

By (h3) at $d = a+2$ the first gives $\bar{h} \in \mathcal{S}_C$, and then the second gives $\bar{\mathcal{H}} \in \mathcal{S}_C$. Hence every h_v, $\mathcal{H}_v$ and $H_v = \mathcal{H}_v - \xi_v h_v$ lies in $\mathcal{S}_C$.

Symmetry. The elements of $\mathcal{S}_C$ are symmetric in y_S. As $H^{(q\to r)} = H_p$ for every cycle $(p\,q\,r)$, $p \in T := [n] \setminus \{q, r\}$, $H^{(q\to r)}$ is invariant under $\mathrm{Sym}([n] \setminus \{p, q, r\})$ for every $p \in T$; these groups generate $\mathrm{Sym}(T)$ (a transposition of T fixes some $p \in T$, as $|T| = n - 2 \geq 3$). So $H^{(q\to r)}$ is symmetric in T.

Lemma 9.230. *Let $C = (p\,q\,r)$, $F \in \mathcal{S}_C$ homogeneous of degree $a+1$ and invariant under $x_p \leftrightarrow x_{s_0}$ for one $s_0 \in S$, and $w := x - x_q$. If $\kappa \neq 0$ and $n \geq 5$, then $F = [\tau^{a+1}] \Gamma(w_r \tau) \prod_{\sigma \in T} (1 + w_\sigma \tau)^\kappa$ for a polynomial Γ.*

Proof. Re-centring at x_q is triangular: $e_d = \sum_{k \leq d} \binom{\kappa(n-3)-k}{d-k} (-c')^{d-k} \tilde{e}_k$ with $c' := o - x_q$ and $\tilde{e}_k := [\tau^k] \prod_S (1 + w_\sigma \tau)^\kappa$, and back; and $\mathrm{Pol}(\xi) = \mathrm{Pol}(w_p, w_r)$. So $F = \sum_{j,k} \psi_{jk} w_p^j w_r^k \tilde{e}_{N_k - j}(w_S)$ with $N_k := a + 1 - k$. Expand $\tilde{e}_d(w_S) = \sum_i b_i w_{s_0}^i \tilde{e}_{d-i}(w_{S'})$, $S' := S \setminus \{s_0\}$ (not empty), $b_i := \binom{\kappa}{i}$. The functions $w_p^j w_{s_0}^i w_r^k \tilde{e}_d(w_{S'})$ with $\tilde{e}_d(w_{S'}) \neq 0$ are linearly independent, so $F = F \circ (p\ s_0)$ gives $\psi_{jk} b_i = \psi_{ik} b_j$ whenever $\tilde{e}_{N_k - i - j}(w_{S'}) \neq 0$. If κ is not a nonnegative integer, every $\tilde{e}_d$ and b_i is nonzero, and $i = 0$ gives $\psi_{jk} = \psi_{0k} b_j$. Let κ be a positive integer and $m' := |S'| = n - 4 \geq 1$; then $\tilde{e}_d(w_{S'}) \neq 0$ if and only if $0 \leq d \leq \kappa m'$. Call j *relevant* if $j_{\mathrm{lo}} \leq j \leq N_k$, $j_{\mathrm{lo}} := \max(0, N_k - \kappa(m'+1))$;

otherwise $\tilde{e}_{N_k-j}(w_S)=0$, and we put $\psi_{jk}:=0$; this is harmless, since $b_i\neq 0$ and $\tilde{e}_{N_k-i-j}(w_{S'})\neq 0$ force $j_{\mathrm{lo}}\le j\le N_k$, so an irrelevant ψ_{jk} meets the relation only with the factor $b_i=0$. (a) For a relevant $j>\kappa$, $i:=\max(0,N_k-\kappa m'-j)\le\kappa$ is a partner and $b_i\neq 0=b_j$, so $\psi_{jk}=0$. (b) For $j_{\mathrm{lo}}\le i$ and $i+1\le\min(N_k,\kappa)$, every j in $[\max(N_k-\kappa m'-i,j_{\mathrm{lo}}),\min(N_k-i-1,\kappa)]$ is a partner of both i and $i+1$, so $\psi_{ik}/b_i=\psi_{i+1,k}/b_{i+1}$; the interval is not empty, as $\kappa m'\ge 1$, $i\ge j_{\mathrm{lo}}$ gives $N_k-\kappa m'-i\le\kappa$, and $i+1\le N_k$, $i+1\le\kappa\le\kappa(m'+1)$ give $j_{\mathrm{lo}}\le N_k-i-1$ (and $j_{\mathrm{lo}}\le i<\kappa$). Hence $\psi_{jk}=\gamma_k b_j$ for every relevant j, and $F=\sum_k\gamma_k w_r^k[\tau^{N_k}]\prod_{\sigma\in T}(1+w_\sigma\tau)^\kappa$. □

Gauges off. By the symmetry, $H^{(q\to r)}=H_p$ is invariant under $x_p\leftrightarrow x_{s_0}$, so by Lemma 9.230 (with $\tau\mapsto-\tau$) it is a gauge of its block. Subtracting this gauge pair changes no $\mathsf{n}^\sigma_{(t\,q\,r)}$ and makes $H^{(q\to r)}=0$; the steps above hold for any parameters of Lemma 9.229, in particular for the new ones. So we may assume every $H^e=0$ (only $\kappa\neq 0$ is used).

Lemma 9.231. *If every $H^e=0$, then every $h^e_t=0$.*

Proof. Fix C. Now $\mathcal{H}_v=\xi_v h_v$; write $h_v=\sum_{d=0}^a f_{v,d}(\xi)\,e_d$ ($f_{v,d}$ of degree $a-d$, $f_{v,a+1}:=0$; $e_0,\dots,e_a$ are nonzero by (h2) and of distinct y-degrees, hence independent over $\mathrm{Pol}(\xi)$). The trace $\sum_\sigma \mathsf{n}^\sigma_C=0$ with $\mathsf{n}^v_C=\kappa h_v$ reads $(n\kappa-\sum_S y_\sigma\partial_\sigma)\bar{h}+\sum_S\partial_\sigma\bar{\mathcal{H}}=0$, with e_d-coefficient

$$(\mathrm{T}_d)\qquad (n\kappa-d)\,\bar{f}_d+(\kappa(n-3)-d)\,\bar{F}_{d+1}=0,\qquad \bar{F}_{d+1}:=\mathrm{mean}_v\,\xi_v f_{v,d+1}.$$

First let $e_{a+1}\neq 0$. Applied to $(\xi_v h_v-\xi_w h_w,h_v-h_w)$, the criterion for Fam_C says that $g_d:=f_{v,d}+\xi_v f_{v,d+1}$ does not depend on v ($0\le d\le a$); so $\bar{f}_d=g_d-\bar{F}_{d+1}$ and (T_d) becomes $(n\kappa-d)g_d=3\kappa\bar{F}_{d+1}$. Downward from $d=a$: if every $f_{v,d+1}=0$, then $\bar{F}_{d+1}=0$, $g_d=0$ by (h3), and $f_{v,d}=g_d-\xi_v f_{v,d+1}=0$. So $h_v=0$. If $e_{a+1}=0$, then κ is a positive integer and $a=\kappa(n-3)$ (by (h2)). Now g_d is v-free for $d\le a-1$, and $f_{v,a}=:c_v$ are constants. The second trace with $\mathcal{H}_v=\xi_v h_v$ reads $(\sum_S y_\sigma\partial_\sigma+3\kappa)\bar{\mathcal{H}}=R_S\bar{h}$, with e_d-coefficient $(d+3\kappa)\bar{F}_d+d\bar{f}_{d-1}=0$. (T_a) gives $\sum_v c_v=0$; (T_{a-1}), where $\kappa(n-3)-a+1=1$, gives $(n\kappa-a+1)\bar{f}_{a-1}+\phi=0$ with $\phi:=\bar{F}_a=\mathrm{mean}_v\,\xi_v c_v$; and the second trace at $d=a$ gives $(a+3\kappa)\phi+a\bar{f}_{a-1}=0$. So $\phi(a+3\kappa-a/(n\kappa-a+1))=0$, and as $n\kappa-a+1=3\kappa+1>1$ the bracket is positive: $\phi=0$. On $\{\sum\xi_v=0\}$ this forces $c_p=c_q=c_r$, so every $c_v=0$, and the induction runs from $d=a-1$. □

Proof of Theorem 9.227. By the gauges off and Lemma 9.231, every block parameter is 0; so $\mathsf{n}^\sigma_C=0$ for $\sigma\in S\cup\{v\}$ and every $v\in C$, that is, $\mathsf{n}=0$. □

Remark 9.232. (1) (What is used.) The two traces, translation invariance, homogeneity and the rows (P1) (Lemma 9.226); not G, the parts of Φ of degree ≥ 2 in y, or any row other than (P1). The analogous result at $b=3$ is Theorem 9.221; Lemma 9.229 is Lemma 9.222 with $n-2$ variables and one parameter.

(2) (The finite input at $n=4$.) For $n=4$ there are no spectator edges. At $\kappa=4$ and $a=4,5,6$, all 192 c-rows at $y=0$, $\Phi_0=0$ (closed forms of the rows, checked against the definition) together with the first trace, imposed on all translation-invariant coefficients of the n^σ_C and of G, have full rank (1380, 1932, 2576 unknowns), mod $p=2^{61}-1$ at 20 random points with all x_p-x_q nonzero mod p; evaluation and reduction mod p can only lower the rank, so the kernel is 0 over $\mathbb{Q}$. Without the traces the kernel is 13 ($a=4$) and 14 ($a=5$). The program and logs are in the ancillary files.

(3) (Checks, supporting; not used.) All 120 rows (P1) with both traces at $(n,a,\kappa)=(5,4,5)$ have kernel 0 on the 3500 unknowns, and kernel 56 without the second trace. Independent code gives kernel 0 at $(5,2,5)$, $(5,4,5)$, $(5,5,5)$, $(6,4,6)$ and $(5,10,5)$ (the case $a=\kappa(n-3)$). The rows: their closed forms agree with $\mathcal{J}$ and c computed from the definitions on 288 monomials at $n=4,5$ (with failing controls), and Lemma 9.226 was derived by hand a second time, independently. The programs and logs are in the ancillary files.

(4) (Not claimed.) y-order 0 at $b=4$ ($\Phi_0\neq 0$); anything about G; other b; κ outside (a), (b).

9.37. **y-order 0 at $b=3$: Theorem L3.** We keep the setting of Subsections 9.19, 9.33 and 9.35 with $b=3$ and $n\ge 4$. Here we exclude y-order 0 at $\kappa=n$: the cubic part F of Lemma 9.84 vanishes. With Theorems 9.221 and 9.201 this settles the whole layer $b=3$ at $\kappa=n$ (Corollary 9.258).

Theorem 9.233 (y-order 0 at $b=3$)**.** *Let $n\ge 4$, $3\le a\le (n^2-3)/2$ and $\tau:=(n^2-a-3,a,3)$. For every $f\in \mathrm{K}_n(n)_\tau$, the cubic part F of φ_f in Lemma 9.84 is 0.*

By Lemma 9.32(ii) and Corollary 9.31(b) at $s=n$, $\varphi:=\varphi_f$ lies in $\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ and satisfies $n\,\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$; we use this throughout, with $\kappa=n$, and write $s:=\kappa=n$ for the eigenvalue where the component notes do. The restriction to $f\in \mathrm{K}_n(n)_\tau$ (not every eigenvector) is used twice: the transpose keeps $\mathrm{K}_n(n)_\tau$ (Part 0), and f has degree n in each row of P (the degree bound in Part I).

Notation. F^+ and F^- are the parts of F that are even and odd under the orientation of a triangle, and $\mathsf{F}^+(T)$ is the value on a triangle T (σ stays the upper index of n^σ, as in Subsection 9.35). For a polynomial $P(y)$ in one variable y (with coefficients functions of x) and a triangle $T=\{p,q,r\}$, $P[x_T]:=P[x_p,x_q,x_r]$ is the second divided difference in y at the nodes of T. For a vertex set X, R_X is the space of F on the triangles of X with $\mathsf{F}^-=0$ and $\mathsf{F}^+(T)=P[x_T]$ for one polynomial $P\in\mathbb{C}[y,x]$. *Node class.* For a function G on the triangles of a 4-set Q, let $c(t)$ be the cubic in $t=x_w-\bar{x}_Q$ ($\bar{x}_Q$ the mean of x_Q) that interpolates $w\mapsto G(Q\setminus w)$ at the four nodes of Q; $b(G)=(b_2,b_3)$ are its coefficients of t^2 and t^3. Pointwise, G is a second divided difference $P[x_T]$ if and only if c is affine, i.e. $b(G)=0$. For a triangle $T\subset Q$, $E_Q(T):=e_2(x_T-\bar{x}_Q\mathbf{1})$. On a 5-set W, x' denotes the variables x_o, $o\notin W$, and θ the symmetric polynomial of the global form below.

Plan. Part 0: $\mathsf{F}^-=0$. Part I: $\mathsf{F}^+|_Q\in \mathrm{R}_Q$ for every 4-set Q; it has four steps: the 4-set statement, the 5-set data, every level is a Kummer line (Theorem 9.254), and $\theta=0$ (Lemma 9.255). Part II: $\mathsf{F}\in \mathrm{R}_{[n]}$, and then $\mathsf{F}=0$.

The $\mathcal{J}$-entries at the positions $[l^3]$ and P. Lemma 9.220 gives the c-rows of $\mathcal{J}$ at $\mathsf{F}=0$. For $\mathsf{F}\ne 0$ we need two more entries. Put $\mathsf{C}_{tuv}:=\partial^3\mathsf{C}/\partial y_t\partial y_u\partial y_v$ (constants in y) and, as in Lemma 9.220, $\gamma_{pq}:=\frac12(x_p+x_q)\theta_{pq}$. For distinct l,i,j let $\Xi_l(i,j)$ be the coefficient in Lemma 9.85(i), so that $\omega(E_{li},E_{lj})=\Xi_l(i,j)\,Z_{il}Z_{jl}$, and for distinct l,m,i,j let $\mathsf{A}(l,m;i,j)$ be the A of Lemma 9.85(iii).

Lemma 9.234 (the $\mathcal{J}$-entries at $[l^3]$ and P)**.** *Let $\varphi\in\mathcal{F}_{(a,3)}(n)$, at a slice point (D,Z), and let l,m,i,j be distinct.*

(a) The coefficient of u_l^3 in $\mathcal{J}(\varphi)$ at $e_l\wedge e_i\wedge e_j$ is $\Xi^{\mathcal{J}}_l(i,j)\,Z_{il}Z_{jl}$ with

$$\Xi^{\mathcal{J}}_l(i,j)=(x_l\partial_l+1-\gamma_{li}-\gamma_{lj})\,\Xi_l(i,j)-\theta_{li}\mathsf{K}_l(i,j)+\theta_{lj}\mathsf{K}_l(j,i),$$

$$\begin{aligned}\mathsf{K}_l(i,j):=\;&x_i\big[\partial_l\mathsf{F}_{lji}-\theta_{lj}\mathsf{F}_{lji}+2\theta_{lj}(\mathsf{n}^l_{li}-\mathsf{n}^l_{ij})\big]+2(x_i\,\partial_l\mathsf{n}^i_{lj}-x_l\,\partial_l\mathsf{n}^l_{lj})-2\theta_{lj}(x_i\mathsf{n}^i_{lj}-2x_l\mathsf{n}^l_{lj})\\&+\theta_{lj}\big[x_i(\mathsf{C}_{lli}-\mathsf{C}_{lji})-x_l(\mathsf{C}_{lll}-\mathsf{C}_{llj})\big].\end{aligned}$$

(b) The coefficient of $u_l^2u_m$ in $\mathcal{J}(\varphi)$ at $e_l\wedge e_i\wedge e_j$ is $\mathsf{P}^{\mathcal{J}}(l,m;i,j)\,Z_{il}Z_{jm}-\mathsf{P}^{\mathcal{J}}(l,m;j,i)\,Z_{im}Z_{jl}$ with

$$\begin{aligned}\mathsf{P}^{\mathcal{J}}(l,m;i,j)=\;&\theta_{ml}x_l\,\Xi_l(i,j)+(x_l\partial_l+\tfrac12-\gamma_{ml}-\gamma_{li}-\gamma_{lj})\,\mathsf{A}(l,m;i,j)-\theta_{li}(x_i\mathsf{d}_i-x_l\mathsf{d}_l)\\&+\theta_{lj}(x_j\mathsf{g}_j-x_l\mathsf{g}_l)-\theta_{mi}(x_i\mathsf{a}_i-x_m\mathsf{a}_m)+\theta_{mj}(x_j\mathsf{e}_j-x_m\mathsf{e}_m),\end{aligned}$$

where, with δ the Kronecker symbol,

$$\begin{aligned}\mathsf{d}_i&=-2\theta_{ml}(\mathsf{n}^i_{mj}-\mathsf{n}^i_{lj})+2\theta_{lj}(\mathsf{n}^i_{lm}-\mathsf{n}^i_{mj})+2\partial_l\mathsf{n}^i_{mj}+\theta_{mj}(\mathsf{C}_{lmi}-\mathsf{C}_{lji}),\\
\mathsf{d}_l&=-\theta_{ml}\big[\mathsf{F}_{ljm}+2(\mathsf{n}^l_{mj}-\mathsf{n}^l_{lj})\big]-\theta_{lj}\big[\mathsf{F}_{ljm}-2(\mathsf{n}^l_{lm}-\mathsf{n}^l_{mj})\big]+2\partial_l\mathsf{n}^l_{mj}+\theta_{mj}(\mathsf{C}_{llm}-\mathsf{C}_{llj}),\\
\mathsf{g}_j&=-\theta_{ml}(\mathsf{F}_{mij}-\mathsf{F}_{lij})+\theta_{li}(\mathsf{F}_{ljm}-\mathsf{F}_{mij})+\partial_l\mathsf{F}_{mij}+2\theta_{mi}(\mathsf{n}^l_{mj}-\mathsf{n}^l_{ij}),\\
\mathsf{g}_l&=-\theta_{ml}\big[\mathsf{F}_{lmi}+2(\mathsf{n}^m_{li}-\mathsf{n}^l_{li})\big]-\theta_{li}\big[\mathsf{F}_{lmi}-2(\mathsf{n}^l_{lm}-\mathsf{n}^i_{lm})\big]+\partial_l\mathsf{F}_{lmi}+2\theta_{mi}(\mathsf{n}^l_{lm}-\mathsf{n}^l_{li}),\\
\mathsf{a}_r&=\partial_l\mathsf{F}_{ljr}-\theta_{lj}\mathsf{F}_{ljr}+2\theta_{lj}(\mathsf{n}^l_{lr}-\mathsf{n}^l_{rj})\quad(r=i,m),\\
\mathsf{e}_t&=2\partial_l\mathsf{n}^t_{li}+\theta_{li}(\mathsf{C}_{llt}-\mathsf{C}_{lit})+2\theta_{li}(\delta_{ti}-\delta_{tl})\mathsf{n}^l_{li}-2\theta_{li}\mathsf{n}^t_{li}\quad(t=j,m).\end{aligned}$$

At a slice point the corresponding coefficients of $\mathcal{Z}(\varphi)$ are $\Xi_l(i,j)$ and $\mathsf{A}(l,m;i,j)$ (Lemma 9.83); so the eigen-rows are $\Xi_l^{\mathcal{J}}(i,j)=\kappa\,\Xi_l(i,j)$ and $\mathsf{P}^{\mathcal{J}}(l,m;i,j)=\kappa\,\mathsf{A}(l,m;i,j)$.

Proof. (1) *Positions.* $\mathfrak{L}\omega(ue_l^T,ue_i^T,ue_j^T)$ is trilinear in the three columns u, so its coefficient of u_l^3 is $\mathfrak{L}\omega(E_{ll},E_{li},E_{lj})$ and that of $u_l^2u_m$ is $\mathfrak{L}\omega(E_{ml},E_{li},E_{lj})+\mathfrak{L}\omega(E_{ll},E_{mi},E_{lj})+\mathfrak{L}\omega(E_{ll},E_{li},E_{mj})$. On the $\mathcal{Z}$ side the same coefficients are $\omega(E_{li},E_{lj})$ and $\omega(E_{li},E_{mj})+\omega(E_{mi},E_{lj})=\mathsf{P}(l,m;i,j)$, as in the proof of Lemma 9.83. By torus weight ($e_i+e_j-2e_l$, resp. $e_i+e_j-e_l-e_m$) the values are combinations of $Z_{il}Z_{jl}$, resp. of $Z_{il}Z_{jm}$ and $Z_{im}Z_{jl}$, with no y-terms; the second coefficient in (b) follows from the first, because $\mathfrak{L}\omega$ is alternating (exchange i and j).

(2) *The operators.* We use step (1) of the proof of Lemma 9.220, which does not use $\mathsf{F}=0$: $D_{J(E_{pq})}\omega(U,V)=\theta_{pq}D_{(0,\eta_{pq})}\omega(U,V)-\gamma_{pq}[\omega([E_{pq},U],V)+\omega(U,[E_{pq},V])]$ and $D_{J(E_{pp})}=x_p\partial_p+D_{(0,E_{pp}\circ Y)}$, where at $y=0$ the last operator multiplies a monomial in Z by half the number of its edges at p; and $D_{(0,\eta_{pq})}$ as in step (2) there. The brackets give $-(\gamma_{li}+\gamma_{lj})\Xi_l$ in (a) and $-(\gamma_{ml}+\gamma_{li}+\gamma_{lj})\mathsf{A}$ in (b). The Euler terms give $(x_l\partial_l+1)\Xi_l$ (two edges at l) and $(x_l\partial_l+\frac12)\mathsf{A}$ (one edge at l).

(3) *The transport terms.* In (a): $-\theta_{li}D_{(0,\eta_{li})}\omega(E_{ll},E_{lj})+\theta_{lj}D_{(0,\eta_{lj})}\omega(E_{ll},E_{li})$. By weight, $\omega(E_{ll},E_{lj})=\sum_r \mathsf{a}_r^{(j)}Z_{jr}Z_{rl}+\sum_t y_t\,\varepsilon_t^{(j)}Z_{jl}$. At $Z_{il}Z_{jl}$, $D_{(0,\eta_{li})}$ meets only its y-part $Z_{il}(x_i\partial_{y_i}-x_l\partial_{y_l})$ and the transport $x_iZ_{jl}\partial/\partial Z_{ji}$ on $Z_{ji}Z_{il}$; so $\mathsf{K}_l(i,j)=x_i\mathsf{a}_i^{(j)}+x_i\varepsilon_i^{(j)}-x_l\varepsilon_l^{(j)}$. In (b) the five transport terms

$$\theta_{ml}D_{(0,\eta_{ml})}\omega(E_{li},E_{lj}),\quad -\theta_{li}D_{(0,\eta_{li})}[\omega(E_{ml},E_{lj})+\omega(E_{ll},E_{mj})],$$
$$\theta_{lj}D_{(0,\eta_{lj})}[\omega(E_{ml},E_{li})+\omega(E_{ll},E_{mi})],\quad -\theta_{mi}D_{(0,\eta_{mi})}\omega(E_{ll},E_{lj}),\quad \theta_{mj}D_{(0,\eta_{mj})}\omega(E_{ll},E_{li})$$

give, at $Z_{il}Z_{jm}$, in this order: $\theta_{ml}x_l\Xi_l$; the y-coefficients $\mathsf{d}_i,\mathsf{d}_l$ of Z_{jm}; the coefficients $\mathsf{g}_j,\mathsf{g}_l$ of $Z_{ij}Z_{jm}$ and $Z_{il}Z_{lm}$; the coefficients $\mathsf{a}_i,\mathsf{a}_m$ of $Z_{ji}Z_{il}$ and $Z_{jm}Z_{ml}$; and the y-coefficients $\mathsf{e}_j,\mathsf{e}_m$ of Z_{il}. Every other term of $D_{(0,\eta)}$ produces a monomial with an edge that $Z_{il}Z_{jm}$ does not have.

(4) *The coefficients.* By (3), $\beta(E_{pq},V)=-\theta_{pq}\,\partial_Y[V]\big(\partial_Y[[E_{pq},Y]]\tilde\varphi\big)$ for $p\neq q$ and $\beta(E_{pp},V)=\partial_{x_p}\partial_Y[V]\tilde\varphi$, with $[E_{pq},Y]=Z_{qp}\Delta_{pq}+(y_q-y_p)E_{pq}+\sum_{r\notin\{p,q\}}(Z_{qr}E_{pr}-Z_{rp}E_{rq})$. With Lemma 9.84 (and $\partial^2\Phi/\partial Z_{ij}\partial Z_{hk}=\mathsf{F}Z$ exactly when the edges $i\to j$, $h\to k$ lie in one directed 3-cycle) each coefficient is a short computation; for example $\omega(E_{ll},E_{lj})=\partial_l\,\partial\Phi/\partial Z_{lj}+\theta_{lj}\,\partial_{y_l}\partial_Y[[E_{lj},Y]]\Phi$ gives $\mathsf{a}_r^{(j)}=\mathsf{a}_r$ and

$$\varepsilon_t^{(j)}=2\partial_l\mathsf{n}_{lj}^t+\theta_{lj}(\mathsf{C}_{llt}-\mathsf{C}_{ljt})-2\theta_{lj}\mathsf{n}_{lj}^t+2\theta_{lj}(\delta_{tj}-\delta_{tl})\mathsf{n}_{lj}^l,$$

whence $\mathsf{K}_l(i,j)$; e_t is $\varepsilon_t^{(i)}$. Likewise for d and g (a short collection of the same kind; each coefficient was compared term by term with an independent derivation, and the whole entry with $\mathfrak{L}\omega$ computed from the definition), from $\omega(E_{ml},E_{lj})=-\theta_{ml}\partial_Y[E_{lj}](\partial_Y[[E_{ml},Y]]\Phi)+\theta_{lj}\partial_Y[E_{ml}](\partial_Y[[E_{lj},Y]]\Phi)$, $\omega(E_{ll},E_{mj})=\partial_l\partial\Phi/\partial Z_{mj}+\theta_{mj}\partial_{y_l}\partial_Y[[E_{mj},Y]]\Phi$ and the two analogous expressions with j replaced by i in the second slot. □

(Computations, not used in the proofs below.) The formulas were also derived independently from the definition and checked against a direct evaluation of $\mathfrak{L}\omega$ at random rational slice points for several invariant functions with $\mathsf{F},\mathsf{n},\mathsf{C}\neq0$.
Part 0: $\mathsf{F}^-=0$.

Lemma 9.235 (pure F^-)**.** *Let $n\ge3$, $\kappa\notin\{3,-5/3\}$, and let $\varphi\in\mathcal{F}_{(a,3)}(n)$ satisfy $\kappa\,\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$ at every slice point. If $\mathsf{F}^+=0$, $\mathsf{N}=0$ and $\mathsf{C}=0$ in Lemma 9.84, then $\varphi=0$. Only three rows per triangle are used, and all their indices lie in the triangle.*

Proof. Fix a triangle $T=\{l,k,m\}$ and put $\phi_T:=\mathsf{F}_{lmk}$ (the rows below are linear and homogeneous, so the sign convention is irrelevant). As $\mathsf{F}^+=0$, $\mathsf{F}_{lkm}=-\phi_T$; as $\mathsf{N}=\mathsf{C}=0$, every n- and C-term drops.

Step 1 (the row $[l^3]$: one first-order equation per apex). By Lemma 9.85(i), $\Xi_l(k,m) = \theta_{lk}\mathsf{F}_{lmk} - \theta_{lm}\mathsf{F}_{lkm} = A_l\phi_T$ with $A_l := \theta_{lk} + \theta_{lm}$. In Lemma 9.234(a), $\mathsf{K}_l(k,m) = x_k(\partial_l\phi_T - \theta_{lm}\phi_T)$ and $\mathsf{K}_l(m,k) = -x_m(\partial_l\phi_T - \theta_{lk}\phi_T)$. The coefficient of $\partial_l\phi_T$ in $\Xi_l^{\mathcal{J}}(k,m)$ is $x_lA_l - \theta_{lk}x_k - \theta_{lm}x_m = -2$, and the coefficient of ϕ_T is A_l (write $x_k = x_l + 1/\theta_{lk}$, $x_m = x_l + 1/\theta_{lm}$; then $\gamma_{lk} + \gamma_{lm} = x_lA_l + 1$ and the rest cancels). So the eigen-row $\Xi_l^{\mathcal{J}}(k,m) = \kappa\,\Xi_l(k,m)$ reads

$$2\,\partial_l\phi_T = -(\kappa-1)A_l\phi_T, \qquad A_l = \frac{1}{x_k - x_l} + \frac{1}{x_m - x_l}, \tag{10}$$

that is, $\partial_l(\phi_T/V_T^{(\kappa-1)/2}) = 0$ with V_T the Vandermonde of T. The same holds at the apex k (row $[k^3]$).

Step 2 (the near row). Take the coefficient of $Z_{mk}Z_{kl}$ in the row $\mathsf{c}(l,k,m)$ at $e_l \wedge e_k \wedge e_m$. By Lemma 9.85(ii) its $\mathcal{Z}$-part is $\mathsf{c}_k = (\partial_k - \partial_l)\phi_T + (\theta_{lm} - \theta_{lk} - \theta_{km})\phi_T$. For the $\mathcal{J}$-part we use the collection in step (1) of the proof of Lemma 9.220 (it does not use $\mathsf{F} = 0$) and select, as in step (2) there, the terms that reach $Z_{mk}Z_{kl}$: the Euler and bracket terms give $(x_l\partial_l + \frac{1}{2})(\mathsf{c}_k + \mathsf{w}_k) - \gamma_{lm}\mathsf{c}_k - (x_k\partial_k + 1)\mathsf{w}_k + (\gamma_{km} - \gamma_{kl} - \gamma_{lm})\mathsf{w}_k$ (one edge at l, two at k), with w_k the coefficient of $Z_{mk}Z_{kl}$ in $\omega(E_{ll}, E_{lm})$; the four transport terms give $\theta_{kl}x_l\Xi_l(k,m)$, $-\theta_{lk}[x_k(\rho_k + \varpi_k) - x_l(\rho_l + \varpi_l)]$, $\theta_{lm}(x_m\mu_{km} - x_l\mu_{lk})$ and $\theta_{km}(x_m\varepsilon_m^{(k)} - x_k\varepsilon_k^{(k)})$, where ρ_t, ϖ_t are the coefficients of y_tZ_{mk} in $\omega(E_{kl}, E_{lm})$ and $\omega(E_{ll}, E_{km})$, and μ_{pq} is the coefficient of $Z_{pq}Z_{qp}$ in $\omega(E_{kl}, E_{lk}) + \omega(E_{ll}, E_{kk})$. With $\mathsf{F}^+ = \mathsf{N} = \mathsf{C} = 0$, (3) gives $\Xi_l(k,m) = A_l\phi_T$, $\rho_k = \theta_{kl}\phi_T$, $\rho_l = -(\theta_{kl} + \theta_{lm})\phi_T$, $\varpi = \varepsilon^{(k)} = 0$, $\mu_{km} = -2\theta_{kl}\phi_T$, $\mu_{lk} = 0$ and $\mathsf{w}_k = \partial_l\phi_T - \theta_{lm}\phi_T$. Collecting (a short collection, also checked by computer; program and log in the ancillary files), with $u := x_{kl}$ and $v := x_{km}$,

$$\mathsf{c}_k^{\mathcal{J}} - \kappa\,\mathsf{c}_k = x_{lk}\,\partial_l\partial_k\phi_T + \frac{\kappa x_{km} + x_{lk}}{x_{km}}\,\partial_l\phi_T - \frac{\kappa x_{lm} + x_{kl}}{x_{lm}}\,\partial_k\phi_T + \frac{\kappa(u^2 - uv + v^2) - u^2 - 3uv + 3v^2}{uv(v-u)}\,\phi_T.$$

The rows hold identically in x, so the x_l-derivative of (10) at the apex k holds too; inserting it and (10) at the apexes l and k gives

$$\mathsf{c}_k^{\mathcal{J}} - \kappa\,\mathsf{c}_k = -\frac{(\kappa-3)(3\kappa+5)}{4(x_k - x_l)}\,\phi_T.$$

So the eigen-row forces $\phi_T = 0$ when $\kappa \notin \{3, -5/3\}$. For every triangle, $\mathsf{F}^- = 0$; with $\mathsf{F}^+ = \mathsf{N} = \mathsf{C} = 0$ the slice data vanish, so $\varphi = 0$ (an invariant function is determined on the slice). □

(Computations, not used.) Each step was also checked exactly: Step 1 symbolically for all triangles through the indices 2 and 3; Step 2 in the form above by sympy, and from the row engine (independent of Lemma 9.234) for $u = e_2 + ce_3$: the same factor $(\kappa-3)(3\kappa+5)/(4x_{23})$ (programs in the ancillary files; Remark 9.241).

Theorem 9.236 ($\mathsf{F}^- = 0$ for the counted eigenvectors)**.** *Let $n \geq 4$, $b = 3$, $s = n$, $\tau = (n^2 - a - 3, a, 3)$ and $f \in \mathrm{K}_n(n)_\tau$. Then $\varphi_f^T = \varphi_f$, where $\varphi^T(X,Y) := \varphi(X^T, Y^T)$; in particular $\mathsf{F}^- = 0$ for φ_f, at every y-order.*

Proof. (1) $\varphi \mapsto \varphi^T$ maps $\mathcal{F}^{\mathcal{E}}_{(a,3)}(n)$ to itself. On the slice $X = \mathrm{diag}(x)$ is symmetric, so $\Phi^T(x,y,Z) = \Phi(x,y,Z^T)$: every 3-cycle is reversed (F^+ is fixed, F^- changes sign), $\mathsf{N}^T = \mathsf{N}$ (each monomial of N_q is a 2-cycle $Z_{pq}Z_{qp}$) and $\mathsf{C}^T = \mathsf{C}$. (2) By Theorem 9.69, f satisfies (S2) as well as (S1); by Corollary 9.31(b), (S1) says that φ_f is an eigenvector at $\kappa = n$ and (S2) says the same for φ_f^T. (Also $f^T(P) := f(P^T)$ lies in $\mathrm{R}_n(n)_\tau$ and $\varphi_{f^T} = \varphi_f^T$.) (3) So $\varphi_\pm := \frac{1}{2}(\varphi_f \pm \varphi_f^T)$ are eigenvectors at $\kappa = n$, and φ_- has $\mathsf{F}^+ = \mathsf{N} = \mathsf{C} = 0$ by (1). (4) Lemma 9.235 ($\kappa = n \geq 4$, so $\kappa \notin \{3, -5/3\}$) gives $\varphi_- = 0$. Hence $\varphi_f = \varphi_+$ and $\mathsf{F}^- = 0$. □

This covers φ_f with f in $\mathrm{R}_n(n)_\tau$, which is what Theorem 9.18 counts; a pencil eigenvector that does not come from $\mathrm{R}_n(n)_\tau$ is not covered, and Theorem 9.233 is stated for φ_f, $f \in \mathrm{K}_n(n)_\tau$.

The four-point F^+-part. Let $\mathsf{F}^- = 0$, so $\mathsf{F}_{pqr} = \mathsf{F}_{prq} = \mathsf{F}^+_T$ on each triangle $T = \{p,q,r\}$. Fix a 4-set Q, put $g_w := \mathsf{F}^+_{Q\setminus\{w\}}$ for $w \in Q$, and take divided differences in the node variables x_w with the values g_w:

$$\Theta_r := g[Q\setminus\{r\}] = \sum_{w\in Q\setminus\{r\}} \frac{g_w}{\prod_{v\in Q\setminus\{r,w\}} x_{wv}}, \qquad c_3 := g[Q] = \sum_{w\in Q} \frac{g_w}{\prod_{v\in Q\setminus\{w\}} x_{wv}}.$$

The weights of Θ_r do not involve x_r. $\Theta_r = 0$ for all r says that $w \mapsto g_w$ is affine in x_w (then $c_3 = 0$ too); then $g_w = \alpha + \beta x_w$ with α, β independent of w, which is $g_w = P[x_{Q\setminus\{w\}}]$ for $P = (\alpha + \beta\sum_{v\in Q} x_v)y^2 - \beta y^3$.

Lemma 9.237 (S0)**.** *Let $\varphi \in \mathcal{F}_{(a,3)}(n)$ satisfy $\kappa\,\mathcal{Z}(\varphi) = \mathcal{J}(\varphi)$ at every slice point, with $\kappa \notin \{0,-2\}$ and $\mathsf{F}^- = 0$. Then for every 4-set Q and every $r \in Q$*

$$\partial_r\Theta_r + (\kappa-1)\,c_3 = 0. \tag{11}$$

Only the eigen-rows $[l^3]$ and P with all indices in Q (Lemma 9.234), and their first x_r-derivatives, are used; neither the translation invariance nor $\mathcal{E}\varphi = 0$ is used.

Proof. For $l \in Q$ write $\mathrm{L}(l;i,j) := \Xi^{\mathcal{J}}_l(i,j) - \kappa\,\Xi_l(i,j)$ and $\mathrm{P}(l,m;i,j) := \mathsf{P}^{\mathcal{J}}(l,m;i,j) - \kappa\,\mathsf{A}(l,m;i,j)$ ($Q = \{l,m,i,j\}$); all vanish. Each differentiates only along its apex l, and C enters with no derivative.

Step 1 (C drops out inside one apex). For $i \in Q\setminus\{l\}$ and $\{m,j\} = Q\setminus\{l,i\}$ with $m<j$ put $\mathrm{D}^{(l)}_i := \mathrm{P}(l,m;i,j) - \mathrm{P}(l,j;i,m)$ and $\mathrm{E}^{(l)}_i := \mathrm{P}(l,m;i,j) + \mathrm{P}(l,j;i,m)$. At $\mathsf{F}^- = 0$, $\mathrm{D}^{(l)}_i$ contains no C. The three $\mathrm{E}^{(l)}$ and the three $\mathrm{L}(l;\cdot,\cdot)$ involve only the six C_{tuv} with an index l; their C-matrix has rank 3, and the three $\mathrm{E}^{(l)}$ alone have rank 3. So each L has exactly one C-free completion: for $Q = \{l,i,j,h\}$,

$$\mathrm{M}^{(l)}_{ij} := \mathrm{L}(l;i,j) - \frac{x_{jh}}{4x_{jl}}\,\mathrm{E}^{(l)}_i + \frac{x_{ih}}{4x_{il}}\,\mathrm{E}^{(l)}_j + \frac{x_{ij}x_{hl}}{4x_{il}x_{jl}}\,\mathrm{E}^{(l)}_h$$

(at $l=3$ these weights were derived by hand; the other apexes follow by relabeling, and the formula was checked at all twelve pairs (apex, $\{i,j\}$) by computer; program in the ancillary files). Its weights depend only on the differences x_{pq} and not on κ; rescaling the C_{tuv} does not change which combinations are C-free. The alternating sum $\sum \mathrm{sgn}(m,i,j)\,\mathrm{P}(l,m;i,j)$ over the six orderings of $Q\setminus\{l\}$ is a signed sum of the $\mathrm{D}^{(l)}_i$, and at $\mathsf{F}^- = 0$ it is algebraic in n (no F, no C, no derivative).

Step 2 (n drops out). Take $r \in Q$ and a second apex $k \in Q\setminus\{r\}$. The 14 row-jets $\mathrm{D}^{(k)}_i$ for two values of i (for $(r,k) = (3,0)$: $i = 2,3$), $\partial_r\mathrm{D}^{(k)}_i$ ($i \neq k$; three), $\mathrm{D}^{(r)}_i$ and $\partial_r\mathrm{D}^{(r)}_i$ ($i \neq r$; six) and $\mathrm{M}^{(r)}_{ij}$ (three, not differentiated) are linearly independent over $\mathbb{Q}(x,\kappa)$, and exactly one combination of them equals $\partial_r\Theta_r + (\kappa-1)c_3$: all 36 n-jets (the 18 values n^r_{pq}, n^k_{pq}, n^t_{pq} with $t \notin \{r,k\}$, $p,q \neq t$, and their ∂_r) and the 6 second-order F^+-jets cancel. The coefficients are rational in x_Q and κ with denominators κ, $\kappa+2$ and x_{pq} only, so the identity holds for every $\kappa \notin \{0,-2\}$. The part from the apex k is short: for $(r,k) = (3,0)$ it is $-\partial_3\big(\sum \mathrm{sgn}\ \mathrm{P}(0,\cdot;\cdot,\cdot)\big)/(2(\kappa+2)x_{12})$ plus $-(3\kappa+1)\mathrm{D}^{(0)}_2/(2(\kappa+2)x_{13}x_{23}) + (3\kappa+1)x_{02}\mathrm{D}^{(0)}_3/(2(\kappa+2)x_{03}x_{12}x_{23})$. This is a finite exact computation over $\mathbb{Q}(x,\kappa)$ (Remark 9.241); every r follows by relabeling, the rows being written uniformly in the labels. □

Lemma 9.238 (SIG)**.** *Let g_w ($w \in Q$) be functions of x_Q that satisfy (11) for all $r \in Q$, are translation invariant ($\sum_{p\in Q}\partial_p g_w = 0$) and homogeneous of one degree d. If $d \notin \{2, 4\kappa-4\}$, then $c_3 = 0$ and $\Theta_r = 0$ for every r.*

Proof. $\Theta_q - \Theta_p = x_{pq}c_3$ (from $g[p,i,j] - g[q,i,j] = x_{pq}\,g[p,q,i,j]$), so $\Theta_r = A - x_r c_3$ with A independent of r. Then (11) reads $\partial_r A - x_r\partial_r c_3 + (\kappa-2)c_3 = 0$. Translation invariance gives $\sum_r \partial_r A = c_3$ and Euler $\sum_r x_r\partial_r c_3 = (d-3)c_3$. Summing over r: $(4\kappa - 4 - d)\,c_3 = 0$. Off the

resonance $c_3 = 0$; then $\partial_r A = 0$ for all r, and A is homogeneous of degree $d-2$, so $A = 0$ unless $d = 2$. □

Lemma 9.239 (EPD)**.** *Let $\beta \geq 0$ be an integer, $m = 4\beta + 1$, and c a homogeneous polynomial of degree m in x_Q with*

$$(x_r - x_q)\,\partial_q\partial_r c = \beta\,(\partial_q c - \partial_r c) \qquad (q \neq r \in Q). \tag{12}$$

(i) c is determined by its four coefficients at the exponents $\beta\mathbf{1} + e_i$ ($i \in Q$). (ii) If c is symmetric and translation invariant, then $c = 0$.

Proof. For $q \neq r$ and $|\gamma| = m-1$, the coefficient of x^γ in (12) is $(\gamma_q+1)(\gamma_r-\beta)c_{\gamma+e_q} - (\gamma_r+1)(\gamma_q-\beta)c_{\gamma+e_r} = 0$. With $\alpha = \gamma + e_q$, $\alpha' = \alpha - e_q + e_r$ and the excess $\epsilon := \alpha - \beta\mathbf{1}$ ($\sum\epsilon = 1$): $(\epsilon_q+\beta)\epsilon_r c_\alpha = (\epsilon_r+\beta+1)(\epsilon_q-1)c_{\alpha'}$. (i) If ϵ is not a unit vector, then $\min\epsilon < 0$; take r with ϵ_r minimal and q with ϵ_q maximal ($\epsilon_q \geq 1$). Then $\alpha_q = \epsilon_q + \beta \geq 1$ and $(\epsilon_q+\beta)\epsilon_r \neq 0$, so c_α is a multiple of $c_{\alpha'}$, and $\sum\epsilon'^2 = \sum\epsilon^2 - 2(\epsilon_q - \epsilon_r - 1) < \sum\epsilon^2$. By induction every c_α is a multiple of a coefficient at $\alpha = \beta\mathbf{1} + e_i$. (ii) By symmetry $c_{\beta\mathbf{1}+e_i} = \lambda_0$ for all i; the coefficient of $x^{\beta\mathbf{1}}$ in $\sum_i \partial_i c = 0$ is $4(\beta+1)\lambda_0$. So $\lambda_0 = 0$ and $c = 0$ by (i). □

Lemma 9.240 (the F^+-part on a 4-set)**.** *Let $\kappa = n \geq 4$ and let $(\mathsf{F}, \mathsf{n}, \mathsf{C})$, polynomial in x_Q, solve the eigen-rows $[l^3]$ and P with all indices in a 4-set Q (for example an x_O-component, Lemma 9.242), with $\mathsf{F}^- = 0$, and let g_w ($w \in Q$) be translation invariant in x_Q, homogeneous of one degree d and $\mathrm{Sym}(Q)$-covariant ($g_{\pi w}(\pi x) = g_w(x)$). If $d \neq 2$, then $g_w = \alpha + \beta x_w$ with polynomials α, β independent of w; that is, the F^+-values on the triangles of Q are $P[x_T]$ for one polynomial P (they lie in R_Q). If $d = 2$, then $\Theta_r = A$ for all r, with one constant A; so modulo R_Q the solutions form a space of dimension at most 1 (the class E_Q of Lemma 9.243).*

Proof. The proof of Lemma 9.237 uses only these rows, so the g_w satisfy (11). At $d = 2$ (off the resonance, as $4\kappa - 4 \geq 12$) the proof of Lemma 9.238 gives $c_3 = 0$ and $\partial_r A = 0$, with A of degree 0; so $\Theta_r = A$, and two solutions with the same A differ by an element with all $\Theta_r = 0$, which lies in R_Q by the last step below. For $d \neq 2$: off the resonance, Lemma 9.238. At $d = 4\kappa - 4$: the integrability of $\partial_r A = x_r\partial_r c_3 - (\kappa-2)c_3$ (mixed partials) is (12) for $c = c_3$ with $\beta = \kappa - 2$, and c_3 is translation invariant of degree $d - 3 = 4\beta + 1$. $c_3 V$ is a polynomial (V the Vandermonde of Q), covariance makes c_3 symmetric, so $c_3 V$ is alternating and hence divisible by V: c_3 is a polynomial. Lemma 9.239(ii) gives $c_3 = 0$; then $\partial_r A = 0$ and $A = 0$ (degree $4\kappa - 6 \geq 1$). So $\Theta_r = 0$ for all r, and $g_w = \alpha + \beta x_w$. Covariance under the transposition of $u, w \in Q$ gives $g_u = g_w$ on $x_u = x_w$, so $x_u - x_w$ divides $g_u - g_w$ and $\beta = (g_u - g_w)/x_{uw}$ and $\alpha = g_w - \beta x_w$ are polynomials. □

(11) is linear, with coefficients rational in x_Q and homogeneous of degree -3, and it differentiates only in x_Q. So it holds for every coefficient of a monomial in the outside variables (Lemma 9.242) and for every x_Q-homogeneous part; this is how Lemma 9.240 is applied in the four-point step (Lemma 9.243). The resonance occurs in the range of Theorem 9.233 only for $n \geq 8$. Covariance is needed there: without it, for $\beta = 0, 1, 2$ (the cases computed, Remark 9.241(3)), (12) has three translation invariant solutions.

Remark 9.241 (the exact computations of this part). (1) Lemma 9.237, Step 2, is a finite exact computation: the support of 14 row-jets, the solution over $\mathbb{Q}(x, \kappa)$ and the cancellation of every coefficient are computed by the program s0_cfree_cert.py; the certificates for $(r, k) = (3, 0)$ and $(1, 2)$ are the files s0_cert_r3k0.txt and s0_cert_r1k2.txt, on the rows of Lemma 9.234 as implemented in rows_hand.py (all in the ancillary files). An isolated re-run gives the same certificate, and an independently written checker confirms both certificates coefficientwise at 96 exact points, with two negative controls rejected at every point. (2) The checks of Lemma 9.235 are those named after its proof. (3) Lemma 9.239: at $\beta = 0, 1, 2$ the system (12) in degree $4\beta + 1$ has 4 solutions, 3 of them translation invariant and none symmetric and translation invariant (program epd_res.py in the ancillary files).

Four-point locality. Fix a 4-set $Q \subset [n]$ and put $O := [n] \setminus Q$.

Lemma 9.242 (four-point locality)**.** *Consider any slice row of $\kappa\,\mathcal{Z}(\varphi) - \mathcal{J}(\varphi)$ whose matrices B_i are built from E_{pq} with $p,q \in Q$ and whose monomial has all its indices in Q (the spectator rows, the near rows, the rows $[l^3]$ and the four-index P rows).*

(a) It involves only the unknowns F_T ($T \subset Q$, both orientations), n^{σ}_{pq} ($\sigma,p,q \in Q$) and $\mathsf{C}_{\sigma\rho\mu}$ ($\sigma,\rho,\mu \in Q$).

(b) Its coefficients are rational functions of x_Q (the $\theta_{pq} = 1/(x_q - x_p)$ and the x_p themselves, $p,q \in Q$), fixed constants and κ; its only derivatives are ∂_p, $p \in Q$.

Hence, writing every unknown as $\sum_{\mu} x_O^{\mu} U^{(\mu)}(x_Q)$ (μ a multi-index on O), for every μ the family $(U^{(\mu)})$ solves the row system of $n = 4$ on the vertex set Q at the same eigenvalue κ (not at $\kappa = 4$).

Proof. By the monomial bookkeeping of the operator form (as in the proof of Lemma 9.83), every term of a row built from the E_{pq}, $p,q \in Q$, is (i) a derivative $\partial/\partial Z_{ab}$ or $\partial/\partial y_a$ with $a,b \in Q$, or (ii) a transport that moves an edge end between two vertices of Q, or (iii) a term of the diagonal part, a multiple of Z_{qp} whose coefficient involves only x_p, x_q ($p,q \in Q$), or (iv) an Euler term $x_p\partial_p + \mathrm{Eul}_p/2$ ($p \in Q$; Eul_p is the number of edge ends at p in the monomial, an integer from 0 to 3), or a factor θ_{pq} or γ_{pq} with $p,q \in Q$. None of (i)–(iv) creates, removes or moves an edge whose ends are both outside Q. So an input monomial that feeds an output monomial on Q has all its edges inside Q, and its diagonal index lies in Q; this is (a). The operators (i)–(iv) act on coefficients only through x_p and ∂_p with $p \in Q$, and at a fixed corner and a fixed monomial only the indices present occur (there is no sum over further spectators); this is (b). The coefficients do not involve x_O and nothing differentiates in x_O, so the system is $\mathbb{C}[x_O]$-linear and splits by the monomials x_O^{μ}. □

In particular no count of indices, and hence no n, enters a coefficient: n enters the four-point rows only through the eigenvalue $\kappa = n$. (The traceless forms of Corollary 9.31, which contain λ, are not used here.)

The chain on a 4-set. Let $\mathrm{R}_{\mathrm{inv}} \subset \mathrm{R}_Q$ be the part with $P \in \mathbb{C}[x_Q]^{S_4}[y]$, and R_{TI} its translation-invariant part. Put $\bar{x} := \bar{x}_Q$ (the mean of x_Q), $D_c := \sum_{q\in Q} \partial_q$ and $E_Q(T) := e_2(x_T - \bar{x}\mathbf{1})$. By Part 0 every x_O-component of the actual F has $\mathsf{F}^- = 0$, and by the invariance of φ under conjugation by permutation matrices it is S_4-covariant. We work on the transpose-even ($\mathsf{F}^- = 0$), S_4-covariant solutions throughout: this space is D_c-stable, it contains $\mathrm{R}_{\mathrm{inv}}$ (every $P[x_T]$ is transpose-even), and it is the space on which Lemma 9.240 is proved.

Lemma 9.243 (the $\bar{x}$-lift chain)**.** *Let $\kappa = n \geq 5$, and let Fproj_d be the set of F-parts of the transpose-even, S_4-covariant polynomial solutions of x-degree d of the row system on Q. Then $\dim \mathrm{Fproj}_d/\mathrm{R}_{\mathrm{inv},d} \leq 1$, with 0 for $d < 2$, and a nonzero class is*

$$X_d = \sum_{j=0}^{d-2} \frac{\bar{x}^{\,d-2-j}}{(d-2-j)!}\, W_{j+2}, \qquad W_2 = E_Q, \quad W_k \text{ translation invariant of degree } k,$$

with the same W_k for every d (unique modulo R_{TI}) and $D_c X_d = X_{d-1}$ modulo $\mathrm{R}_{\mathrm{inv}}$.

Proof. Two inputs. (E) Every coefficient of every unknown jet in every slice row of $n = 4$ is translation invariant (a finite exact computation: 1116 jet coefficients, κ symbolic; program `transl_equiv.py` and its log in the ancillary files). (A4′) $\mathrm{R}_{\mathrm{inv}}$ solves every row at every κ: its elements are the F-shapes of functions of $X + Y$. (i) D_c maps Fproj_d into Fproj_{d-1}: by (E) the shift $U(x) \mapsto U(x + c\mathbf{1})$ maps solutions to solutions, and the rows are homogeneous, so each homogeneous part $(c^j/j!)D_c^j U$ is a solution. (ii) Write $\mathsf{F} \in \mathrm{Fproj}_d$ as $\sum_{j\leq J} \bar{x}^{\,j} F_j/j!$ with F_j translation invariant and $F_J \neq 0$. Since $D_c\bar{x} = 1$ and $D_c F_j = 0$, $D_c^J \mathsf{F} = F_J$ is a translation-invariant element of Fproj_{d-J}, so by Lemma 9.240 $F_J \in \mathrm{R}_{\mathrm{TI}}$, or $d - J = 2$ and $F_J \in cE_Q + \mathrm{R}_{\mathrm{TI}}$. (iii) $\bar{x}^{\,J}\mathrm{R}_{\mathrm{TI}} \subset \mathrm{R}_{\mathrm{inv}}$ ($\bar{x}$ is symmetric), which lies in Fproj by (A4′); subtracting $\bar{x}^{\,J}/J!$ times the R_{TI}-part of F_J lowers

J. So $\mathsf{F} \in \mathrm{R}_{\mathrm{inv}}$, or $\mathsf{F} = c\,\bar{x}^{\,d-2}E_Q/(d-2)! + (\text{lower powers of } \bar{x})$ modulo $\mathrm{R}_{\mathrm{inv}}$; two such F with the same c differ by an element with $J < d-2$, which lies in $\mathrm{R}_{\mathrm{inv}}$ by the same steps. (iv) $D_c X_d$ has top term $\bar{x}^{\,d-3}E_Q/(d-3)!$, so it equals X_{d-1} modulo $\mathrm{R}_{\mathrm{inv}}$; hence the translation-invariant parts W_k ($k \le d-1$) do not depend on d. □

Put $\beta_k := b(W_k)$. The node class $b_2(W_{j+2})$ is homogeneous of degree j in $Y_Q := x_Q - \bar{x}\mathbf{1}$, so the node classes of the chain depend on Y_Q, not only on $\bar{x}$.

Lemma 9.244 (global form)**.** *Let F be as in Theorem 9.233, with $n \ge 5$, and let Q be a 4-set. There is one symmetric polynomial θ of degree $a-2$ in $n-4$ variables, the same for every Q, such that, with D the sum of all partial derivatives of θ,*

$$\mathsf{F}|_Q = \sum_j \big((-D)^j\theta\big)(x_O - \bar{x}\mathbf{1})\,W_{j+2} \pmod{\mathrm{R}_Q}, \qquad b(\mathsf{F}|_Q) = \sum_j \big((-D)^j\theta\big)(x_O - \bar{x}\mathbf{1})\,\beta_{j+2}.$$

Proof. F is homogeneous of degree a and invariant under $x \mapsto x + c\mathbf{1}$ (Lemma 9.84), and S_n-covariant. By Lemmas 9.242 and 9.243, applied to each x_O-component, the component of x_O^μ (of x-degree $d_\mu = a - |\mu|$) is $c_\mu X_{d_\mu}$ modulo $\mathrm{R}_{\mathrm{inv}}$; so $\mathsf{F}|_Q = \sum_j (\partial_z^j\Psi)(x_O,\bar{x})\,W_{j+2}$ modulo R_Q, with $\Psi(x_O,z) := \sum_\mu c_\mu x_O^\mu z^{d_\mu-2}/(d_\mu-2)!$. Put $D_O := \sum_{o\in O}\partial_o$ and apply the sum of all ∂_p, $p \in [n]$, which is $D_O + D_c$: F is translation invariant, R_Q is translation stable and the W_k are translation invariant, so, applying b_2, $\sum_j \big(\partial_z^j(D_O+\partial_z)\Psi\big)(x_O,\bar{x})\,b_2(W_{j+2}) = 0$. Scale $Y_Q \mapsto cY_Q$: $b_2(W_{j+2})$ is homogeneous of degree j in Y_Q, so the c^0 part gives $\big((D_O+\partial_z)\Psi\big)\,b_2(W_2) = 0$, hence $(D_O+\partial_z)\Psi = 0$ and $\Psi = \theta(x_O - z\mathbf{1})$. The symmetry of θ and its independence of Q come from the S_n-covariance, and its degree from the homogeneity of F. (For $n = 4$ the set O is empty and the theorem is the finite computation of Remark 9.259(2).) □

The spectator row at general F.

Lemma 9.245 (the spectator c-rows of $\mathcal{J}$ at $b = 3$, general F)**.** *Let $\varphi \in \mathcal{F}_{(a,3)}(n)$, work at a slice point (D,Z), let l,k,m be distinct and $r \notin \{l,k,m\}$, and put $\mathsf{f} := \mathsf{f}^{(r)}$, $\mathsf{M} := \mathsf{M}^{(r)}$. Then the coefficient of $Z_{mr}Z_{rl}$ in $\mathsf{c}^{\mathcal{J}}(l,k,m)$ is*

$$\mathsf{c}_r^{\mathcal{J}}(l,k,m) = \mathsf{J}_{\mathsf{F}} + 2\theta_{lm}\big[x_{ml}\,\partial_l\mathsf{M}(k,m) + x_{km}\,\partial_k\mathsf{M}(l,m) + x_{lk}\,\partial_k\mathsf{M}(l,l)\big],$$

$$\mathsf{J}_{\mathsf{F}} := \frac{x_{lm}\,\partial_l\mathsf{f}_{lm} - x_{lk}\,\partial_l\mathsf{f}_{lk}}{x_{km}} - \partial_l\mathsf{f}_{km} + x_{lk}\,\partial_l\partial_k\mathsf{f}_{lm} + \frac{x_{lk}}{x_{lm}}\,\partial_k\mathsf{f}_{lm} - \frac{\mathsf{f}_{lk}}{x_{km}} + \frac{\mathsf{f}_{kl}}{x_{lm}}$$
$$+ \mathsf{f}_{lm}\Big(\frac{1}{x_{km}} - \frac{1}{x_{lk}}\Big) + \mathsf{f}_{km}\Big(\frac{1}{x_{lk}} - \frac{1}{x_{lm}}\Big).$$

So the coefficient of $Z_{mr}Z_{rl}$ in $\mathsf{c}^{\mathcal{J}}(l,k,m) - \kappa\,\mathsf{c}(l,k,m)$ is

$$\mathrm{row}(l,k,m) := \mathsf{J}_{\mathsf{F}} - \kappa\,\mathsf{Z}_{\mathsf{F}} + 2\theta_{lm}\big[x_{ml}\partial_l\mathsf{M}(k,m) + x_{km}\partial_k\mathsf{M}(l,m) + x_{lk}\partial_k\mathsf{M}(l,l) + \kappa\,\mathsf{M}(\Delta_{lk},\Delta_{lm})\big],$$

where $\mathsf{Z}_{\mathsf{F}} := \theta_{lk}(\mathsf{f}_{km} - \mathsf{f}_{lm}) + \theta_{km}(\mathsf{f}_{lk} - \mathsf{f}_{lm}) + (\partial_k - \partial_l)\mathsf{f}_{lm} + \theta_{lm}\mathsf{f}_{lm}$ is the F-part of $\mathsf{c}_r(l,k,m)$ (Lemma 9.85(ii)).

Proof. The row is linear in $(\mathsf{F},\mathsf{N},\mathsf{C})$, and C does not occur: by step (2) of the proof of Lemma 9.220, which does not use $\mathsf{F} = 0$, only coefficients of degree 0 in y of the monomials $Z_{mr}Z_{rk}$, $Z_{rm}Z_{mr}$, $Z_{lr}Z_{rl}$, $Z_{kr}Z_{rl}$ and $Z_{mr}Z_{rl}$ enter, and C contributes only y-terms there. So $\mathsf{c}_r^{\mathcal{J}}$ is its value at $\mathsf{F} = 0$, which is the M-part of Lemma 9.220, plus its value at $\mathsf{N} = \mathsf{C} = 0$, which we now compute. At $\mathsf{N} = \mathsf{C} = 0$, (3) and the jets after Lemma 9.84 give, at $y = 0$: in $\omega(E_{kl},E_{lm}) + \omega(E_{ll},E_{km})$ the coefficient $-\theta_{kl}(\mathsf{f}_{km} - \mathsf{f}_{lm}) + \theta_{lm}(\mathsf{f}_{kl} - \mathsf{f}_{km}) + \partial_l\mathsf{f}_{km}$ of $Z_{mr}Z_{rk}$ (from $\partial/\partial Z_{lm}$ of $Z_{lm}\,\partial\Phi/\partial Z_{km}$ and $-Z_{rk}\,\partial^2\Phi/\partial Z_{lm}\partial Z_{rl}$, from $Z_{mr}\,\partial^2\Phi/\partial Z_{kl}\partial Z_{lr}$ and $-\partial/\partial Z_{kl}$ of $Z_{kl}\,\partial\Phi/\partial Z_{km}$, and from $\partial_l\,\partial\Phi/\partial Z_{km}$); in $\omega(E_{kl},E_{lk}) + \omega(E_{ll},E_{kk})$ the coefficients 0 of $Z_{rm}Z_{mr}$ and $-\theta_{kl}(\mathsf{f}_{lk} - \mathsf{f}_{kl})$ of $Z_{lr}Z_{rl}$ (the terms with N cancel between $\beta(E_{kl},E_{lk})$ and $\beta(E_{lk},E_{kl})$, and the 3-cycles through l, k, r remain); $\partial_l\mathsf{f}_{lk} - \theta_{lk}\mathsf{f}_{lk}$ of $Z_{kr}Z_{rl}$ in $\omega(E_{ll},E_{lk})$; $\partial_l\mathsf{f}_{lm} - \theta_{lm}\mathsf{f}_{lm}$ of $Z_{mr}Z_{rl}$ in $\mathsf{w} = \omega(E_{ll},E_{lm})$;

and, in $\mathsf{c}+\mathsf{w}=\omega(E_{lk},E_{km})+\omega(E_{kk},E_{lm})$, the coefficient $-\theta_{lk}(\mathsf{f}_{lm}-\mathsf{f}_{km})+\theta_{km}(\mathsf{f}_{lk}-\mathsf{f}_{lm})+\partial_k\mathsf{f}_{lm}$ of $Z_{mr}Z_{rl}$ (here $\mathsf{F}_{rlm}=\mathsf{f}_{lm}$, $\mathsf{F}_{lrk}=\mathsf{f}_{kl}$, $\mathsf{F}_{rkm}=\mathsf{f}_{km}$, $\mathsf{F}_{krl}=\mathsf{f}_{lk}$ by rotation). Insert these into step (4) of the proof of Lemma 9.220 (one edge of $Z_{mr}Z_{rl}$ at l, none at k):

$$\begin{aligned}\mathsf{J}_{\mathsf{F}} = {}& (x_l\partial_l+\tfrac12-\gamma_{lm})(\mathsf{c}+\mathsf{w})_r - x_k\partial_k\mathsf{w}_r + (\gamma_{km}-\gamma_{kl})\mathsf{w}_r\\ & -\theta_{lk}x_k(\cdot)_{Z_{mr}Z_{rk}} + \theta_{lm}\big(0-x_l(\cdot)_{Z_{lr}Z_{rl}}\big) - \theta_{km}x_k(\cdot)_{Z_{kr}Z_{rl}},\end{aligned}$$

with the five coefficients just listed; collecting gives the stated J_{F}. □

(Checks, supporting; not used.) The collection is an identity for arbitrary functions f_{pq} of x, with no translation invariance and no Euler relation (the same program also gives Z_{F}). The entry was derived independently from the definitions and checked against $\mathcal{J}$ computed from the definitions at random slice points. The programs and logs are in the ancillary files.
The identity (S). For distinct l,k,m and a spectator r put

$$2G(l,k,m) := x_{lm}\big(\mathsf{J}_{\mathsf{F}}-\kappa\,\mathsf{Z}_{\mathsf{F}}\big),$$

the F-part of x_{lm} times the row of Lemma 9.245. With $t_1:=x_{lm}$, $t_2:=x_{lk}$, $\epsilon:=(\mathsf{f}_{lm}-\mathsf{f}_{lk})/x_{km}$ and $\epsilon':=(\mathsf{f}_{km}-\mathsf{f}_{lm})/x_{lk}$ it reads

$$\begin{aligned}2G(l,k,m) = {}& t_1^2\partial_l\epsilon + t_1\big[\partial_l\mathsf{f}_{lk}-\partial_l\mathsf{f}_{km}+(1+\kappa)\epsilon'-(\kappa-1)\epsilon+\kappa(\partial_l-\partial_k)\mathsf{f}_{lm}\big]\\ & + t_1t_2\,\partial_l\partial_k\mathsf{f}_{lm} + t_2\,\partial_k\mathsf{f}_{lm} + \mathsf{f}_{kl}-\mathsf{f}_{km}+\kappa\,\mathsf{f}_{lm}.\end{aligned}$$

Lemma 9.246 (the identity (S))**.** *Let $n\ge5$ and let $\varphi\in\mathcal{F}_{(a,3)}(n)$ satisfy $\kappa\,\mathcal{Z}(\varphi)=\mathcal{J}(\varphi)$ at every slice point. For distinct r,l,k,m,m', with $\mathsf{f}=\mathsf{f}^{(r)}$,*

$$2G(l,k,m)+2G(k,l,m)=2G(l,k,m')+2G(k,l,m'). \tag{S}$$

This is an identity in F alone; neither the traces, nor the translation invariance, nor $\mathcal{E}\varphi=0$ is used, and κ is arbitrary.

Proof. By Lemma 9.245 the rows $x_{lm}\,\mathrm{row}(l,k,m)$ and $x_{km}\,\mathrm{row}(k,l,m)$ (the coefficients of $Z_{mr}Z_{rl}$, resp. $Z_{mr}Z_{rk}$, in $\mathsf{c}^{\mathcal{J}}-\kappa\,\mathsf{c}$ at (l,k,m), resp. (k,l,m)) vanish. Their F-parts are $2G(l,k,m)$ and $2G(k,l,m)$. As $x_{lm}\theta_{lm}=-1$, the M-part of $x_{lm}\,\mathrm{row}(l,k,m)$ is

$$-2\big[x_{ml}\partial_l\mathsf{n}^k_{mr}+x_{km}\partial_k\mathsf{n}^l_{mr}+x_{lk}\partial_k\mathsf{n}^l_{lr}+\kappa(\mathsf{n}^l_{lr}-\mathsf{n}^l_{mr}-\mathsf{n}^k_{lr}+\mathsf{n}^k_{mr})\big],$$

using $\mathsf{M}(\Delta_{lk},\Delta_{lm})=\mathsf{n}^l_{lr}-\mathsf{n}^l_{mr}-\mathsf{n}^k_{lr}+\mathsf{n}^k_{mr}$. Exchanging l and k and adding, the terms with n^k_{mr}, n^l_{mr} and their derivatives cancel ($x_{ml}+x_{lm}=0$, $x_{km}+x_{mk}=0$), and the sum of the M-parts is

$$2x_{kl}\big(\partial_k\mathsf{n}^l_{lr}-\partial_l\mathsf{n}^k_{kr}\big)+2\kappa\big(\mathsf{n}^k_{lr}-\mathsf{n}^l_{lr}+\mathsf{n}^l_{kr}-\mathsf{n}^k_{kr}\big),$$

which does not depend on m. So $2G(l,k,m)+2G(k,l,m)$ equals minus this expression for every $m\notin\{r,l,k\}$; comparing m and m' gives (S). The closed form of $2G$ is $x_{lm}(\mathsf{J}_{\mathsf{F}}-\kappa\mathsf{Z}_{\mathsf{F}})$ expanded term by term. □

(Checks, supporting; not used.) The closed form of $2G$ equals $x_{lm}(\mathsf{J}_{\mathsf{F}}-\kappa\,\mathsf{Z}_{\mathsf{F}})$ for arbitrary f_{pq} and symbolic κ. The program and its log are in the ancillary files.
The 5-set node classes. Let W be a 5-set, $N_W(z):=\prod_{i\in W}(z-x_i)$ and $E_i:=e_i(x_W)$. A *pair function* on W is a function $G(v,w)=G(w,v)$ of the pairs $v\ne w$ in W (for us $G(v,w)=\mathsf{F}^+(W\setminus\{v,w\})$, with x' as parameters). For $v\in W$, let $S_3(v)$ and $S_2(v)$ be the coefficients of z^3 and z^2 in the cubic in z that interpolates $w\mapsto G(v,w)$ at the four nodes x_w, $w\in W\setminus v$; so $S_3=b_3$ and $S_2=b_2-3\bar{x}_{W\setminus v}b_3$ for the node class of G on $W\setminus v$.

Lemma 9.247 (5-set gluing)**.** *Let G be a pair function on W.*

(a) $\sum_{v\in W}\frac{S_3(v)}{N'_W(x_v)}=0$, $\sum_{v\in W}\frac{S_2(v)}{N'_W(x_v)}=0$ *and* $\sum_{v\in W}\frac{(E_1-x_v)S_2(v)+e_2(x_{W\setminus v})S_3(v)}{N'_W(x_v)}=0$. *Equivalently: if S_3 and S_2 are written as polynomials in $z=x_v$ whose coefficients are symmetric in x_W, then modulo $N_W(z)$:* (C1) $[z^4]S_3=0$, (C2a) $[z^4]S_2=0$ *and* (C2b) $[z^3]S_2=[z^2]S_3$.

(b) If $S_2=S_3=0$ on W, then $G(v,w)=\alpha+\beta(x_v+x_w)+\gamma x_vx_w$ with α,β,γ independent of v,w; and $1=(y^2)[x_T]$, $x_v+x_w=(E_1y^2-y^3)[x_T]$, $x_vx_w=(y^4-E_1y^3+E_2y^2)[x_T]$ for $T=W\setminus\{v,w\}$.

Proof. (a) For the four nodes $W\setminus v$, the Lagrange polynomial of the node x_w has the coefficients $1/\prod_{u\in W\setminus\{v,w\}}(x_w-x_u)=(x_w-x_v)/N'_W(x_w)$ at z^3 and $-(E_1-x_v-x_w)(x_w-x_v)/N'_W(x_w)$ at z^2. So $S_3(v)=\sum_{w\neq v}G(v,w)(x_w-x_v)/N'_W(x_w)$ and $S_2(v)=-\sum_{w\neq v}G(v,w)(E_1-x_v-x_w)(x_w-x_v)/N'_W(x_w)$. In the first two sums of (a) the summand of the pair (v,w) is $G(v,w)/(N'_W(x_v)N'_W(x_w))$ times (x_w-x_v), resp. $-(E_1-x_v-x_w)(x_w-x_v)$, which are antisymmetric in (v,w); so the sums vanish. In the third, with $e_2(x_{W\setminus v})=E_2-x_v(E_1-x_v)$, the factor is (x_w-x_v) times $E_2-x_v(E_1-x_v)-(E_1-x_v)(E_1-x_v-x_w)=E_2-(E_1-x_v)(E_1-x_w)$, which is symmetric; so it vanishes too. For the second form: the interpolant at the five nodes of $v\mapsto g(x_v)$ is g modulo N_W, and its coefficients of z^4, z^3, z^2 are $\sum_v g(x_v)c_v/N'_W(x_v)$ with $c_v=1$, $-(E_1-x_v)$, $e_2(x_{W\setminus v})$.

(b) Every node function is affine: $G(v,w)=g_v(x_w)$ with g_v affine. Let $S(z,t):=L_1(z)g_1(t)+L_2(z)g_2(t)$ for two nodes $1,2$ of W (L_1,L_2 the Lagrange polynomials of x_1,x_2); it has bidegree $\le(1,1)$ and $S(x_v,t)=g_v(t)$ for $v=1,2$. $S(z,t)-S(t,z)$ is antisymmetric of bidegree $\le(1,1)$, so it is $c(z-t)$, and at (x_1,x_2) it is $G(1,2)-G(2,1)=0$; so S is symmetric. For $v\neq1,2$, g_v and $S(x_v,\cdot)$ are affine and agree at x_1, x_2 ($g_v(x_i)=G(v,i)=g_i(x_v)=S(x_i,x_v)=S(x_v,x_i)$), so they are equal. Hence $G(v,w)=S(x_v,x_w)$ for all $v\neq w$, and a symmetric S of bidegree $\le(1,1)$ has the stated form. The three divided differences follow from $\sum_j h_j(x_T)z^j=(1-x_vz)(1-x_wz)/\prod_{i\in W}(1-x_iz)$. □

Lemma 9.248 (the R_W-part)**.** *Let c be a smooth function of x_W that is symmetric in x_W (with x' as parameters), and let $\mathsf{F}^+(W\setminus\{v,w\})=c\,g(x_v,x_w)$ with $g\in\{1,\,x_v+x_w,\,x_vx_w\}$. Then the defect of* (S) *(Lemma 9.246) of F^+ is* 0*. So, in a pair function on W, a part with all node classes* 0 *and symmetric coefficients does not contribute to* (S)*.*

Proof. The defect is linear in F^+ and contains derivatives of order at most 2 in x. So at a point x it depends only on the value, the gradient and the Hessian of c there; as c is symmetric, $c=f(p_1,\dots,p_5)$ with the power sums p_i of x_W, and these data are those of f at $p(x)$. The 21 polynomials 1, p_i, p_ip_j realize every value, gradient and Hessian at a given point, so it suffices that the defect of $c\,g$ vanishes identically for these 21 functions c and the three g. This is a finite exact computation: $3\cdot21$ cases at two of the choices of the indices of (S) (the others follow by relabeling W, as c and g are symmetric), 126 exact cancellations, with the control $p_1x_vx_w(x_v+x_w)$ (not in R_W) nonzero (program and logs in the ancillary files). □

Every level is a Kummer line. We use the notation of Lemma 9.247 on a 5-set W: for $v\in W$, $Q_v=W-v$, $t_v=\bar x(Q_v)$, $w_v=x_v-t_v$, $E_i=e_i(x_W)$, the residues $S_3(v)$, $S_2(v)$ read as polynomials in the node variable $z=x_v$ modulo $N_W(z)=\prod_{i\in W}(z-x_i)$, and the conditions (C1), (C2a), (C2b). For a multi-index μ on the outside variables x', the x'^μ-component $\psi_\mu(t,w)$ of $\theta(x'-t\mathbf{1},w)$ has level $r:=a-2-|\mu|$, and Lemma 9.244 gives the node classes $b^{(v)}=\sum_j(\delta^j\psi_\mu)(t_v,w_v)\,\beta_{j+2}(Q_v)$, $\delta:=\partial_t-\partial_w$. Call ψ *realizable* if (C1), (C2a), (C2b) hold, and let Real_r be the set of realizable level-r ψ. By Lemma 9.247 every component of the actual F is realizable; moreover $\partial_t\psi_\mu=-\sum_i(\mu_i+1)\psi_{\mu+e_i}$. Write a level-$r$ component as $\psi_\mu=\sum_j p_jt^jw^{r-j}$; with $\theta_r:=[w^r]\theta$ and D' the sum of the derivatives in the variables x', this means $p_0=[x'^\mu]\theta_r$ and $p_1=-[x'^\mu]D'\theta_{r-1}$. The Kummer condition at level r is

$$(\mathrm{K}_r)\qquad(\kappa+1-r)\,p_1=r\,(5\kappa-6-r)\,p_0.$$

From here on we normalize $W_2=-E_Q$, so $\beta_2=(-1,0)$; this only changes the sign of θ. Put $q_4:=7-4\kappa$.

Lemma 9.249 (the chain in closed form)**.** *Let $\kappa=n\ge 5$. For $2\le k\le \kappa+2$,*

$$\beta_k=-\frac{(\pi_{k-2},\pi_{k-3})}{(q_4)_{k-2}},\qquad \pi_j:=[\xi^j]\prod_{i\in Q}\bigl(1-\xi(x_i-\bar x)\bigr)^{\kappa-2}.$$

Proof. (1) For a solution with node classes (b_2,b_3), $c_3:=b_3$ is the third divided difference of the node values and $\Theta_r:=b_2-b_3(x_r-\bar x)$ the second divided difference over $Q-r$. So $A:=\Theta_r+x_rc_3=b_2+\bar x\,b_3$ does not depend on r, and S0 (Lemma 9.237; it uses no translation invariance) reads $\partial_rA=x_r\partial_rc_3-(\kappa-2)c_3$. Since $\partial_q\partial_rA$ is symmetric, c_3 solves the Euler–Poisson–Darboux system $(x_r-x_q)\,\partial_q\partial_rc_3=(\kappa-2)(\partial_qc_3-\partial_rc_3)$. (2) Uniqueness. Put $h:=\kappa-2$. The coefficient of x^γ gives $(\gamma_q+1)(\gamma_r-h)\,c_{\gamma+e_q}=(\gamma_r+1)(\gamma_q-h)\,c_{\gamma+e_r}$. In homogeneous degree $m\le h$ every factor is nonzero and unit moves connect all monomials, so the solutions form a space of dimension at most 1. In degree $m=h+1$ the pure powers are forced to 0; on the other monomials of degree $h+1$ the move $\gamma\mapsto\gamma-e_q+e_r$ is allowed if and only if $\gamma_r\le h-1$ (then both coefficients are nonzero), and this is the exchange graph of a box-constrained simplex, hence connected (here $h\ge3$). The polynomials $G_m:=[\xi^m]\prod_Q(1-\xi x_i)^{\kappa-2}/(8-4\kappa)_m$ solve the system, and $(8-4\kappa+m)\,\partial_rG_{m+1}=x_r\partial_rG_m-(\kappa-2)G_m$, $D_cG_m=G_{m-1}$ (two-line generating-function identities). (3) For the chain, $c_3(X_d)=\rho_dG_{d-3}$ by (2), and $D_cX_d=X_{d-1}$ gives $\rho_d=\rho_3$; then (1) and the identity give $A(X_d)=\rho_3(8-4\kappa+d-3)G_{d-2}$ for $d\ge3$. At $d=3$, $X_3=\bar xW_2+W_3$ with $b(W_2)=(-1,0)$ and $b_2(W_3)=0$, so $A=-(1-\rho_3)\bar x$ and $\partial_rA=-(1-\rho_3)/4=-(\kappa-2)\rho_3$; hence $\rho_3=1/(4\kappa-7)=-1/q_4$. With P the formal Dirichlet$(2-\kappa)$ point of Q, $\mathbb{E}[P^j]:=j!\,[\xi^j]\prod_Q(1-\xi x_i)^{\kappa-2}/(8-4\kappa)_j$, this reads $b_3(X_d)=-\mathbb{E}[P^{d-3}]/(q_4\,(d-3)!)$ and $b_2(X_d)+\bar x\,b_3(X_d)=-(q_4+d-2)\,\mathbb{E}[P^{d-2}]/(q_4\,(d-2)!)$, which is the claim summed over the $\bar x$-expansion of X_d (Lemma 9.243); the W_k are recovered degree by degree. Only $k-3\le\kappa-1$ is used. □

Lemma 9.250 (Dirichlet form)**.** *Let $\kappa=n\ge5$ and let $P:=\langle\mathbf{p},x_{Q_v}\rangle$ with $\mathbf{p}$ a formal Dirichlet$(2-\kappa,2-\kappa,2-\kappa,2-\kappa)$ vector ($\mathbb{E}[P^j]:=j!\,[\xi^j]\prod_{Q_v}(1-\xi x_i)^{\kappa-2}/(8-4\kappa)_j$). For every level-$r$ ψ with $r\le n$, modulo $N_W(z)$,*

$$S_3(v)=-\tfrac{1}{q_4}\,\mathbb{E}\bigl[(\delta\psi)(P,x_v-P)\bigr],\qquad S_2(v)=-\mathbb{E}\bigl[\psi(P,x_v-P)\bigr]-\tfrac{1}{q_4}\,\mathbb{E}\bigl[(P+x_v-E_1)(\delta\psi)(P,x_v-P)\bigr].$$

Proof. Both sides are linear in ψ; take $\psi=e^{\alpha t+\gamma w}$ and $L:=\alpha-\gamma$, so $\delta\psi=L\psi$. With $Y_i=x_i-t_v$ the moment formula gives $\sum_i L^i\pi_i/(8-4\kappa)_i=\mathbb{E}[e^{L\langle\mathbf{p},Y\rangle}]$, and $8-4\kappa=q_4+1$. By Lemma 9.249 (only $k\le r+2\le\kappa+2$ enters at level r), $b_3=-e^{\alpha t_v+\gamma w_v}\sum_j L^j\pi_{j-1}/(q_4)_j=-(L/q_4)\,\mathbb{E}[e^{\alpha P+\gamma(x_v-P)}]$. For b_2 use $1/(q_4)_j=(q_4+j)/(q_4(q_4+1)_j)$: then $b_2=-(1/q_4)\,\mathbb{E}[(q_4+L(P-t_v))e^{\alpha P+\gamma(x_v-P)}]$, and $b_2-3t_vb_3$ follows with $4t_v=E_1-x_v$. The identities hold as functions of the node configuration, that is, modulo $N_W(z)$ as polynomials in $z=x_v$. □

Lemma 9.251 (the Kummer lines are realizable)**.** *Let $\kappa=n\ge5$, $K(\xi):=M(-\kappa;7-5\kappa;-\xi)=\sum_j k_j\xi^j$ (Kummer's function) and $K_r:=[L^r]\,e^{-Lt}K(Lw)$. Then $K_r\in\mathrm{Real}_r$ for every $r\le n$.*

Proof. The node classes enter through Lemma 9.250, whose right sides are rational in κ; so it suffices to prove the identities below for real $\kappa<-1$, where every integral converges and every boundary term vanishes. There $K(\xi)=\int_0^1 e^{-\xi\varrho}\,d\Lambda(\varrho)$ with $d\Lambda=\varrho^{-\kappa-1}(1-\varrho)^{6-4\kappa}d\varrho/B(-\kappa,7-4\kappa)$. The mode $\psi_\varrho=e^{-L(t+\varrho w)}$ has $\delta\psi_\varrho=-L(1-\varrho)\psi_\varrho$ and $\psi_\varrho(P,x_v-P)=e^{-L\langle U,x\rangle}$, where U (ϱ at v, $(1-\varrho)\mathbf{p}$ on Q_v) lies in the simplex of W. This substitution merges $d\Lambda$ and the Dirichlet law of Q_v into a Dirichlet law on W (the powers of $1-\varrho$ cancel: $(6-4\kappa)+1-3-4(1-\kappa)=0$). Write $\mathbb{E}_c$ for the Dirichlet law on W with parameter c at v and $2-\kappa$ elsewhere; then $\int d\Lambda\,(1-\varrho)\,\mathbb{E}_{Q_v}[f]=\frac{q_4}{7-5\kappa}\mathbb{E}_{-\kappa}[f]$ and $\mathbb{E}_c[e^{-L\langle U,x\rangle}]=\sum_j(-L)^j[\xi^j]\bigl(\prod_W(1-\xi x_i)^{\kappa-2}(1-\xi z)^{2-\kappa-c}\bigr)/(c+8-4\kappa)_j$. By Lemma 9.250,

S_3 has v-dependence $(1-\xi z)^2$, so degree 2 in z. In S_2 the term $-\mathbb{E}[\psi]$ has no factor $1-\varrho$; one integration by parts in ϱ removes the q_4-terms exactly and leaves

$$S_2(v) = \tfrac{L(2z-E_1)}{7-5\kappa}\sum_j \tfrac{(-L)^j}{(8-5\kappa)_j}[\xi^j]\Big(\textstyle\prod_W (1-\xi x_i)^{\kappa-2}(1-\xi z)^2\Big) \\ - \sum_j \tfrac{(-L)^j}{(7-5\kappa)_j}[\xi^j]\Big(\textstyle\prod_W (1-\xi x_i)^{\kappa-2}(1-\xi z)^3\Big),$$

of degree 3 in z. So (C1) and (C2a) hold. With $P_i := [\xi^i]\prod_W(1-\xi x_i)^{\kappa-2}$, $[z^2]S_3 = \frac{L}{7-5\kappa}\sum_j(-L)^j P_{j-2}/(8-5\kappa)_j$ and $[z^3]S_2 = \frac{2L}{7-5\kappa}\sum_j(-L)^j P_{j-2}/(8-5\kappa)_j + \sum_j(-L)^j P_{j-3}/(7-5\kappa)_j$; the shift $j\mapsto j+1$ in the last sum turns it into $-\frac{L}{7-5\kappa}\sum_j(-L)^j P_{j-2}/(8-5\kappa)_j$, so (C2b) holds. □

Lemma 9.252 (w^r is not realizable)**.** *Let $\kappa = n \geq 5$ and $3 \leq r \leq n$. Then $w^r \notin \mathrm{Real}_r$.*

Proof. For $\psi = w^r$ Lemma 9.250 gives $S_3 = (r/q_4)\,\mathbb{E}[(z-P)^{r-1}]$ and $S_2 = -(1+r/q_4)\,\mathbb{E}[(z-P)^r] + (r/q_4)(2z-E_1)\,\mathbb{E}[(z-P)^{r-1}]$. These are polynomials in z and $E_1,\dots,E_5$, and reduction modulo the monic N_W keeps this, so the residue $\mathrm{C2b}(w^r) := [z^3]S_2 - [z^2]S_3$ is a polynomial in the E_i; a nonzero value at one point shows that it is not the zero polynomial, so w^r is not realizable at generic distinct nodes. Evaluate at $x_W = (1,0,0,0,0)$ ($E_1 = 1$, $E_2 = \cdots = E_5 = 0$, $N_W(z) = (z-1)z^4$; the reduction does not change $[z^2]$ and $[z^3]$). Three Chu–Vandermonde sums collapse to

$$\mathrm{C2b}(w^r) = \frac{(-1)^{r-3}\,r!\,(\kappa-1)\,(2-\kappa)_{r-4}}{6\,(r-3)!\,(7-4\kappa)\,(8-4\kappa)_{r-4}} \cdot \frac{(r+6-5\kappa)(r+7-5\kappa)(r+8-5\kappa)}{(r+4-4\kappa)(r+5-4\kappa)(r+6-4\kappa)}$$

(with $(c)_{-1} := 1/(c-1)$ at $r=3$). At $\kappa = n$ and $3 \leq r \leq n$: $\kappa - 1 \neq 0$; $(2-\kappa)_{r-4} = 0$ only for $\kappa \in \{2,\dots,r-3\}$; $r+6+j-5\kappa = 0$ ($0 \leq j \leq 2$) needs $\kappa \leq (r+8)/5 < r$; the denominators vanish only for $\kappa \leq (r+6)/4 < r$. So $\mathrm{C2b}(w^r) \neq 0$. □

Lemma 9.253 (degree bound)**.** *θ has degree at most n in each variable; so $\theta_r = [w^r]\theta = 0$ for $r > n$.*

Proof. (i) By Lemma 9.32(i), f has degree n in each row of $P = (P_1, P_2, P_3)$, and $\Phi(x,y,Z) = \widetilde{\varphi}_f(\mathrm{diag}\,x, Z + \mathrm{diag}\,y) = f(I, \mathrm{diag}\,x, Z + \mathrm{diag}\,y)$ (Lemma 9.32(ii)); x_o sits only in row o of P_2, so $\deg_{x_o}\mathsf{F}_T \leq n$ for $o \notin T$. (ii) $b_2(\mathsf{F}|_Q)$ is a combination of the F_T, $T \subset Q$, with coefficients rational in x_Q, so $\deg_{x_o} b_2(\mathsf{F}|_Q) \leq n$ for $o \in O$. (iii) By Lemma 9.244 and homogeneity in Y_Q, the Y_Q-degree-0 part of $b_2(\mathsf{F}|_Q)$ is $\pm\theta(x_O - \bar{x}\mathbf{1})$, whose x_o-degree is the degree of θ in its o-th variable. □

This is the second place where $\varphi = \varphi_f$ with $f \in K_n(n)_\tau$ is used (the first is Part 0).

Theorem 9.254 (every level is a Kummer line)**.** *Let $\kappa = n \geq 5$. Every component of the actual F, at every level $r \geq 1$, satisfies* (K_r).

Proof. Base. At levels 1 and 2 the conditions (C1)–(C2b) are empty, and the five-point identity of Lemma 9.246 leaves exactly the Kummer lines K_1, K_2 for every $\kappa \neq 2$ (a finite exact computation with κ symbolic: the lines $(5\kappa-7)/\kappa$ and $\big((5\kappa-8)(5\kappa-7)/(\kappa(\kappa-1)),\ 2(5\kappa-8)/(\kappa-1)\big)$, rank drop only at $\kappa = 2$; program `base12.py` and its log in the ancillary files). The part $\alpha + \beta(x_v + x_w) + \gamma x_v x_w$ in R_W of a component has coefficients symmetric in x_W (by covariance) and zero defect in Lemma 9.246 (Lemma 9.248), so that lemma may be applied to one particular realizer. *Step* ($3 \leq r \leq n$). Let every level-$(r-1)$ component be a multiple of K_{r-1}. For a level-r component ψ_μ, $\partial_t\psi_\mu$ is a combination of level-$(r-1)$ components, so $\partial_t\psi_\mu = -cK_{r-1} = \partial_t(cK_r)$ (as $\partial_t K_r = -K_{r-1}$), hence $\psi_\mu = cK_r + d\,w^r$. Both ψ_μ and K_r (Lemma 9.251) are realizable, so $d\,w^r$ is, and Lemma 9.252 gives $d = 0$. So $\psi_\mu = cK_r$, which satisfies (K_r) by Kummer's ratio k_{r-1}/k_r ($k_r \neq 0$ for $r \leq n$). *Levels $r > n$.* By Lemma 9.253, $\theta_r = 0$, so $p_0 = 0$; at $r = n+1$ the factor $\kappa + 1 - r$ is 0, and at $r \geq n+2$ also $\theta_{r-1} = 0$, so $p_1 = 0$. In both cases (K_r) holds. □

Since $a-2$ goes up to $(n^2-7)/2 > n$, most levels are $r > n$. Lemma 9.249 is used only for $k \le r+2$ with $r \le n$, so $k \le n+2$ suffices.

Lemma 9.255 (symmetric extension)**.** *Let $m \ge 1$, $s \in \mathbb{C}$ and θ a symmetric polynomial in m variables, homogeneous of degree $k \ge 1$. Distinguish one variable w, write $\theta_r := [w^r]\theta$ and let D be the sum of the partial derivatives in the other $m-1$ variables. Assume, for every $r \ge 1$,*

$$(\mathrm{K}_r) \qquad r(r+6-5s)\,\theta_r = (s+1-r)\,D\theta_{r-1}.$$

If $6+k-(m+4)s \ne 0$ and $4s \ne 7$, then $\theta = 0$.

Proof. Multiply (K_r) by w^{r-1} and sum over $r \ge 1$; with $r(r+6-5s) = r(r-1) + (7-5s)r$ this gives $w\theta_{ww} + B\,\theta_w + wD\theta_w + A\,D\theta = 0$ with $A = -s$, $B = 7-5s$. Put $D = D_{\mathrm{all}} - \partial_w$ (D_{all} the sum of all m partials): the terms $w\theta_{ww}$ cancel, and $(B-A)\partial_w\theta + w\partial_w(D_{\mathrm{all}}\theta) + A\,D_{\mathrm{all}}\theta = 0$. By symmetry this holds with w replaced by each variable x_i. Sum over i and use Euler for $\phi := D_{\mathrm{all}}\theta$, homogeneous of degree $k-1$: $(B-A+k-1+mA)\phi = 0$, and $B-A+k-1+mA = 6+k-(m+4)s$. So $\phi = 0$, and then $(B-A)\partial_i\theta = 0$ with $B-A = 7-4s \ne 0$: θ is constant, hence 0 as $k \ge 1$. □

At $s = n$ and $m = n-4$ the conditions read $k \ge 1$, $6+k \ne n^2$ and $4n \ne 7$; all hold for $1 \le k = a-2 \le (n^2-7)/2$. Theorem 9.254 gives (K_r) for the components on a 5-set W in the form $(n+1-r)p_1 = r(5n-6-r)p_0$; with w the variable of the node v, $p_0 = [x'^{\mu}]\theta_r$ and $p_1 = -[x'^{\mu}]D\theta_{r-1}$, where D is the sum of the partials in the $n-5$ variables x' (not the D of Lemma 9.244, which is D_{all}); so these relations for all μ are (K_r) of Lemma 9.255. So, with Theorem 9.254 (which gives (K_r) for every $r \ge 1$ at $s = n$) and the global form, $\theta = 0$, and $\mathsf{F}^+|_Q \in \mathrm{R}_Q$ for every 4-set Q (this includes the class E_Q of degree 2). This ends Part I.
Part II: $\mathsf{F} \in \mathrm{R}_{[n]}$, then $\mathsf{F} = 0$.

Lemma 9.256 (gluing)**.** *Let $n \ge 4$ and let F_T (T a triangle of $[n]$) be polynomials in x with $\mathsf{F}^- = 0$. If for every 4-set Q there are functions α_Q, D_Q with $\mathsf{F}^+(Q \setminus w) = \alpha_Q - D_Q x_w$ for all $w \in Q$ (this holds if $\mathsf{F}^+|_Q \in \mathrm{R}_Q$ pointwise), then $\mathsf{F}^+(T) = P[x_T]$ for one $P \in \mathbb{C}[y,x]$ and all T.*

Proof. (i) D_Q is a polynomial: with $g_w := \mathsf{F}^+(Q\setminus w)$, $D_Q = (g_p - g_r)/(x_r - x_p) = (g_q - g_r)/(x_r - x_q)$, so the denominator of D_Q divides the coprime $x_r - x_p$ and $x_r - x_q$. (ii) Put $f_1 = f_2 := 0$ and $f_i := (x_i - x_1)(x_i - x_2)\mathsf{F}^+(\{1,2,i\})$ $(i \ge 3)$, polynomials; then $f[x_1,x_2,x_i] = \mathsf{F}^+(\{1,2,i\})$. The affine form on $\{1,2,p,q\}$ gives $f[x_1,x_2,x_p,x_q] = D_{\{1,2,p,q\}}$, hence $f[x_2,x_p,x_q] = \mathsf{F}^+(\{2,p,q\})$ and likewise $f[x_1,x_p,x_q] = \mathsf{F}^+(\{1,p,q\})$; the affine form on $\{1,p,q,r\}$ then gives $f[x_p,x_q,x_r] = \mathsf{F}^+(\{p,q,r\})$. (iii) Every divided difference $f[x_U]$, $|U| \ge 3$, is a polynomial: for $|U| = 3$ by (ii), and for $|U| \ge 4$, $f[x_U] = (f[x_{U\setminus p}] - f[x_{U\setminus t}])/(x_t - x_p)$ for every pair $p \ne t$ in U, so its denominator divides coprime differences. (iv) The Newton form $P(y) := \sum_{k=0}^{n-1} f[x_1,\dots,x_{k+1}] \prod_{j\le k}(y - x_j)$ lies in $\mathbb{C}[y,x]$ and $P(x_i) = f_i$, so $P[x_T] = f[x_T] = \mathsf{F}^+(T)$. □

By Part 0, Part I and Lemma 9.256, $\mathsf{F}_{pqr} = P[x_p,x_q,x_r]$ for one $P \in \mathbb{C}[y,x]$. *Normalizing P.* Conjugation by a permutation matrix fixes φ, so $\mathsf{F}_{\pi p\,\pi q\,\pi r}(\pi x) = \mathsf{F}_{pqr}(x)$ and $P(\cdot;\pi x)$ represents F as well; replace P by its average over S_n, then by $P(y - \bar{x}; x - \bar{x}\mathbf{1})$ ($\bar{x}$ the mean of x; F is translation invariant), then by its part of joint degree $a+2$ in (y,x) (F is homogeneous of degree a). So P is symmetric in x, invariant under $(y,x) \mapsto (y + c, x + c\mathbf{1})$, and homogeneous of joint degree $a+2$. Write $P[t,r,r]$ for $P[x_t,x_r,x_r]$ etc., P' for $\partial P/\partial y$, and $\partial_\sigma P$ for the derivative in x_σ at fixed y.

The source. With $\mathsf{F} \ne 0$ the spectator rows (Lemma 9.245) are the rows (E_r) of Subsection 9.35 for $\nu_t^{[r]}(\sigma) := \mathsf{n}^{\sigma}_{tr}$ with a source linear in F. Put, for $t \ne r$ and $\sigma \ne r$,

$$\nu_0{}_t^{[r]}(\sigma) := \tfrac{1}{2}\big[\delta_{t\sigma}P[t,t,r] + (x_t - x_r)(\partial_\sigma P)[t,r,r]\big], \qquad \nu_0{}_t^{[r]}(r) := -\textstyle\sum_{\sigma\ne r} \nu_0{}_t^{[r]}(\sigma);$$

ν_0 is a polynomial, translation invariant and homogeneous of degree a, and it satisfies the first trace of (Tr).

(A1) $\sum_\sigma x_\sigma\, \nu_{0t}^{[r]}(\sigma) = \frac{a+1}{2}(x_t - x_r)P[t,r,r]$.
(A2) $\nu_{0t}^{[r]}(\sigma) - \nu_{0r}^{[t]}(\sigma) = \partial_\sigma Q_e$ for every $\sigma \in [n]$, with $Q_e := \frac{1}{2}(x_t - x_r)(P[t,r,r] + P[r,t,t])$.
(A3) ν_0 solves the spectator rows with the source F.

Proof. (A1) By the first trace, the left side is $\sum_{\sigma\neq r}(x_\sigma - x_r)\nu_{0t}^{[r]}(\sigma)$. Joint translation invariance and joint Euler give $\sum_\sigma \partial_\sigma P = -P'$ and $\sum_\sigma x_\sigma \partial_\sigma P = (a+2)P - yP'$, so $\sum_\sigma (x_\sigma - x_r)\partial_\sigma P = (a+2)P - (y-x_r)P'$; the Leibniz rule at the nodes $[t,r,r]$ gives $((y-x_r)P')[t,r,r] = (x_t - x_r)P'[t,r,r] + P''(x_r)$, and the δ-term adds $(x_t - x_r)P[t,t,r]$. So the left side is $\frac{1}{2}(x_t - x_r)[P[t,t,r] + (a+2)P[t,r,r] - (x_t - x_r)P'[t,r,r] - P''(x_r)]$. For every polynomial P in y, $P[t,t,r] + P[t,r,r] - (x_t - x_r)P'[t,r,r] - P''(x_r) = 0$ (both sides are shift invariant; at $x_r = 0$ on y^k the left side is $((k-1) + 1 - k)x_t^{k-2} = 0$ for $k \geq 3$, $1 + 1 - 0 - 2 = 0$ for $k = 2$, and 0 for $k \leq 1$), which gives (A1). (A2) For $\sigma \notin \{t,r\}$ the nodes do not move, and both sides are $\frac{1}{2}(x_t - x_r)\partial_\sigma(P[t,r,r] + P[r,t,t])$. At $\sigma = t$, ∂_t also moves the nodes ($\partial_t P[t,r,r] = P[t,t,r,r]$, $\partial_t P[r,t,t] = 2P[r,t,t,t]$), and the two sides agree because $P'[r,t,t] = P[r,r,t,t] + 2P[r,t,t,t]$ (the y-derivative of a divided difference repeats each node once). At $\sigma = r$ it follows from the sum over σ, as both sides satisfy the first trace and Q_e is translation invariant. (A3) is a finite exact computation: the rows are linear in P, and the generating function $P = c(x)/(1 - zy)$ covers every P (s symbolic, c generic: 0 nonzero of 24; the control without the derivative term fails; program and log in the ancillary files). □

So $\nu' := \nu - \nu_0$ solves the rows (E_r) without source and the first trace, and Lemma 9.222 gives its parameters $(H^{(r)}, h_t^{(r)})$ in (P). By (A1) the second trace of ν reads $(a+1)H^{(r)} + O_t^{(r)} h_t^{(r)} + \frac{a+1}{2}(x_t - x_r)P[t,r,r] = 0$, i.e.

$$H^{(r)} = \hat{H}_t^{(r)} - W(t), \qquad \hat{H}_t^{(r)} := -O_t^{(r)} h_t^{(r)}/(a+1), \qquad W(t) := \tfrac{1}{2}(x_t - x_r)P[t,r,r],$$

and $W(t) = \frac{1}{2}(P[t,r] - P'(x_r))$. By (A2), on each edge $\{t,r\}$ the coupling of ν' has the defect $-\partial_\sigma Q_e$, so the edge equations hold for $(K + Q_e, \mu)$ in place of (K,μ): $(K + Q_e, \mu)$ lies in the family of Lemma 9.223 (parameters $c_0, \dots, c_a, k_{\mathrm{top}}$), and (E3) gives $\bar{h}$ from it. By the divided-difference recursion, for $t' \neq t$ (both $\neq r$),

$$W(t) - W(t') = \tfrac{1}{2}(P[t,r] - P[t',r]) = (x_t - x_{t'})\,\Omega_{rtt'}, \qquad \Omega_{rtt'} := \tfrac{1}{2}\mathsf{F}_{rtt'}.$$

Symmetry. As in Subsection 9.35: on the edge $\{t,r\}$, $\hat{H}_t^{(r)}$ is symmetric in $S = [n] \setminus \{t,r\}$, and so is $W(t)$ (P is symmetric); $H^{(r)}$ does not depend on t; so $H^{(r)}$ is symmetric in $[n] \setminus \{r\}$.

Lemma 9.257 (one edge with the source)**.** *Assume* (s1)–(s5) *of Subsection 9.35. Fix an edge $\{t,r\}$ and $s_0 \in S$, let $(K + Q_e, \mu)$ be in the family of Lemma 9.223, $\hat{H}^{(r)}$, $\hat{H}^{(t)}$ as above, $H^{(r)} := \hat{H}^{(r)} - W(t)$ and $H^{(t)} := \hat{H}^{(t)} - W'(r)$ with $W'(r) := \frac{1}{2}(x_r - x_t)P[r,t,t]$. If $H^{(r)}$ is invariant under $x_t \leftrightarrow x_{s_0}$ and $H^{(t)}$ under $x_r \leftrightarrow x_{s_0}$, then $\mathsf{F}_{rts_0} = 0$.*

Proof. Coordinates of the proof of Lemma 9.224: $x_r = 0$, $v_\sigma := x_\sigma - x_r$, $F_{(i,k)}(H) := [v_t^i v_{s_0}^k]H$, a polynomial in $v_{S\setminus\{s_0\}}$, $F_i(H) := [v_t^i]H$. $W(s_0)$ is $W(t)$ with t and s_0 exchanged (P is symmetric), so the invariance of $H^{(r)}$ is the set of rows

$$(\mathrm{R}_{ik}) \qquad F_{(i,k)}(\hat{H}^{(r)}) - F_{(k,i)}(\hat{H}^{(r)}) = [v_t^i v_{s_0}^k]\big((v_t - v_{s_0})\Omega\big), \qquad \Omega := \Omega_{rts_0} = \tfrac{1}{2}\mathsf{F}_{rts_0},$$

and Ω is invariant under the permutations of x_r, x_t, x_{s_0}. Write $\Omega = \sum_d \Omega^{(d)}$ with $\Omega^{(d)}$ homogeneous of degree d in (v_t, v_{s_0}) (coefficients in $v_{S\setminus\{s_0\}}$); each $\Omega^{(d)}$ is symmetric in (v_t, v_{s_0}). From the proof of Lemma 9.224: c_j ($j \geq 1$) enters $\hat{H}^{(r)}$ only in v_t-degrees $\geq j - 1$, $F_{j-1} = \chi_j c_j e_{a+2-j}(v_S) +$ (terms in $c_0, \dots, c_{j-1}$) with $\chi_j := (-1)^{j-1} j(a+2-j)/((a+1)n\lambda) \neq 0$, and the k_{top}-part of $\hat{H}^{(r)}$ is $-(-1)^a \kappa k_{\mathrm{top}} v_t^a (v_t + Y)/(n\lambda)$.

(0) *Gauges.* F_0 involves only c_0, c_1. The span of the gauges G_r, G_t maps injectively to the (c_0, c_1)-plane: for $\alpha_r G_r + \alpha_t G_t$ with $c_0 = c_1 = 0$, $F_0 = \alpha_r H_g^{(r)}|_{x_t = x_r} = 0$, and $H_g^{(r)}|_{x_t = x_r} =$

$[\tau^{a+1}]\prod_{\sigma\in S}(1-v_\sigma\tau)^\kappa\neq 0$ (at $\kappa=n$, $a+1\le(n^2-1)/2\le n(n-2)$ for $n\ge 4$; for other κ by (s4)); so $\alpha_r=0$, and likewise $\alpha_t=0$. So the 2-dimensional gauge span maps onto that plane, and after subtracting a gauge combination (which has $\mathsf{F}=0$ and leaves Ω unchanged) $c_0=c_1=0$; at the t-end the swap acts by $c_0'=-c_0$, $c_1'=c_1$, so this holds there too.

(1) *Both ends.* (R_{10}) reads $\chi_2 c_2 e_a(v_{S\setminus\{s_0\}})=\Omega^{(0)}$, and $\Omega^{(0)}$ is Ω at $x_r=x_t=x_{s_0}$. The swap $r\leftrightarrow t$ maps the family by $c_j'=-(-1)^j c_j$, $k_{\mathrm{top}}'=(-1)^a k_{\mathrm{top}}$, maps W to W' and keeps the triple $\{t,r,s_0\}$; so the t-end row is $\chi_2 c_2' e_a=\Omega^{(0)}$ with $c_2'=-c_2$. Hence $c_2=0$ (by (s4), $e_a(v_{S\setminus\{s_0\}})\neq 0$) and $\Omega^{(0)}=0$. (For $a=1$ the family is c_0,c_1,k_{top}, and the same argument on (R_{10}), whose left side is the k_{top}-term, gives $k_{\mathrm{top}}=0$ and $\Omega^{(0)}=0$; go to (3).)

(2) *Induction.* Let $c_0=\cdots=c_J=0$, $2\le J\le a$. Then $\hat H^{(r)}$ has v_t-degree $\ge J$, so $F_{(i,k)}(\hat H^{(r)})=0$ for $i\le J-1$. (2a) For $i,k\le J-1$ the left side of (R_{ik}) is 0; the rows of total degree $D\le J-1$ give $(v_t-v_{s_0})\Omega^{(D-1)}=0$, so $\Omega^{(d)}=0$ for $d\le J-2$; in degree J the rows $(i,J-i)$, $1\le i\le J-1$, say that $(v_t-v_{s_0})\Omega^{(J-1)}$ contains only v_t^J and $v_{s_0}^J$, so $\Omega^{(J-1)}=\alpha\,h_{J-1}(v_t,v_{s_0})$ (h complete homogeneous). (2b) Let $g(a',b';z):=\Omega$ at $x_r=0$, $x_t=a'$, $x_{s_0}=b'$, $x_{S\setminus\{s_0\}}=z$. Invariance under $x_r\leftrightarrow x_t$ and translation invariance give $g(a',b';z)=g(-a',b'-a';z-a'\mathbf{1})$; the shift of z only raises the (a',b')-degree and $\Omega^{(d)}=0$ for $d<J-1$, so $\Omega^{(J-1)}(a',b')=\Omega^{(J-1)}(-a',b'-a')$, and at $b'=0$: $\alpha a'^{J-1}=\alpha J(-1)^{J-1}a'^{J-1}$. As $J\ge 2$, $\alpha=0$. (2c) If $J\le a-1$: (R_{J0}) reads $\chi_{J+1}c_{J+1}e_{a+1-J}(v_{S\setminus\{s_0\}})=[v_t^{J-1}v_{s_0}^0]\Omega^{(J-1)}=0$, and the pivot is nonzero by (s4), so $c_{J+1}=0$.

(3) At $J=a$ all c_j vanish, (2a)–(2b) give $\Omega^{(a-1)}=0$, and (R_{a0}) reads

$$-(-1)^a\kappa k_{\mathrm{top}}\textstyle\sum_{\sigma\in S\setminus\{s_0\}}v_\sigma/(n\lambda)=0;$$

as $\kappa\neq 0$ and $S\setminus\{s_0\}\neq\emptyset$, $k_{\mathrm{top}}=0$. Then $(K+Q_e,\mu)=0$, $\hat H^{(r)}=0$, and every (R_{ik}) gives $(v_t-v_{s_0})\Omega=0$: $\mathsf{F}_{rts_0}=0$. □

The three cases outside (s4). At $\kappa=n$ the pivots $e_d(v_{S\setminus\{s_0\}})$ ($d=a$ in (1), $d=a+1-J$ in (2c)) vanish exactly for $d>n(n-3)$; in our range this happens only at $(n,a)=(4,5),(4,6),(5,11)$. There (1) is replaced by the swap parity: if (c,P) solves the one-edge system, so does (c',P), so $(c-c',0)$ solves the system without source, and Lemma 9.224 at these cases (Remark 9.225(2)) puts it in the gauge span; its (c_0,c_1) is $(2c_0,0)=(0,0)$, so by (0) every even c_j is 0, in particular $c_2=0$, and (R_{10}) gives $\Omega^{(0)}=0$. At $(4,5)$ and $(5,11)$, (2c) needs only $d\le n(n-3)$, so Lemma 9.257 holds. At $(4,6)$, (2c) at $J=2$ needs $d=5>4$; there Lemma 9.257 is a finite exact computation: the one-edge system at $(n,a,\kappa)=(4,6,4)$ with the unknowns $c_0,\dots,c_6$, k_{top} and a general symmetric translation-invariant P of degree 8 (11 basis elements, 4 of them with $\mathsf{F}\neq 0$) has a solution space of dimension $9=7+2$ (the 7 such P with $\mathsf{F}=0$ and the two gauges), on which $\mathsf{F}_{rts_0}=0$ (controls with the source decoupled fail; program and log in the ancillary files).

Proof of Theorem 9.233. At $\kappa=n$, $\lambda=(n^2-a-1)/n\neq 0$, and (s1)–(s3), (s5) hold for $a\le(n^2-3)/2$. Part 0 gives $\mathsf{F}^-=0$, Part I gives $\mathsf{F}^+|_Q\in\mathrm{R}_Q$ for every 4-set Q, Lemma 9.256 gives $\mathsf{F}=P[x_T]$, and for every triangle $\{p,q,r\}$ Lemma 9.257 (with the three cases above) on the edge $\{t,r\}:=\{q,r\}$ with $s_0:=p$ gives $\mathsf{F}_{pqr}=0$. For $n=4$ ($m=0$: there are no outside variables) the theorem is a finite exact computation (Remark 9.259(2)). □

Corollary 9.258 (Theorem L3: the layer $b=3$ at the eigenvalue n)**.** *For every $n\ge 4$ and every a with $3\le a\le(n^2-3)/2$, $\mathrm{K}_n(n)_{(n^2-a-3,a,3)}=0$. Hence, for every $n\ge 4$, no conjugation type κ of $K_{B(n,2)}$ with $\kappa_3'=2$ occurs; with Corollary 9.189, every type with $\kappa_3'\le 2$ is $(3,1^{N-3})$, and it occurs once.*

Proof. Let $f\in\mathrm{K}_n(n)_\tau$, $\tau=(n^2-a-3,a,3)$, and $\varphi=\varphi_f$. Theorem 9.233 gives $\mathsf{F}=0$; then Theorem 9.221(a) gives $\mathsf{N}=0$; so $\Phi=\mathsf{C}$, which Theorem 9.201 excludes unless $\Phi=0$. So $\tilde\varphi$ vanishes on the slice, hence $\varphi=0$, and $f=0$ by Lemma 9.32(ii). A type κ with $\kappa_3'=2$ has

$\tau = \kappa' + (1,1,1) = (n^2 - a - 3, a, 3)$ with $\kappa'_2 = a - 1 \geq 2$ and $\tau_1 \geq \tau_2$, i.e. $3 \leq a \leq (n^2-3)/2$, and its multiplicity is $\dim \mathrm{K}_n(n)_\tau = 0$ (Lemma 9.32(iii)). □

Remark 9.259. (1) (What is used.) Besides the lemmas of this subsection: Theorem 9.69, Corollary 9.31(b), Lemma 9.32(i)–(iii), Lemmas 9.84, 9.83, 9.85 and 9.220, (3), Lemmas 9.222, 9.223, 9.224 and Remark 9.225(2) (Part II), and, for the corollary, Theorems 9.221 and 9.201 and Corollary 9.189. (2) (Finite exact computations.) The proof is by hand up to the following finite exact computations, each named where it is used: the certificate of Lemma 9.237; the translation equivariance in Lemma 9.243; the base levels of Theorem 9.254; Lemma 9.248; (A3) and the case $(4,6)$ in Part II; the three cases of Remark 9.225(2); and $n = 4$: the full row system with the traces at $\kappa = 4$ has kernel 0 for $a = 3, \dots, 6$, by full column rank modulo a prime of p-integral rows, with the rows from the definition (the row engine, not the hand rows of Lemma 9.234; the engine is checked against the definition at random points, with controls in which a genuine eigenvector at $\lambda = 0$ survives). The programs and logs of this and of every other computation in this part are in the ancillary files. (3) (Not claimed.) Other layers b, eigenvalues $\kappa \neq n$, and any bound on $d(n)$ by itself.

10. The block $\lambda(3,1,1)$

Statement (iii) in the proof of Theorem G asks for $U_{\lambda(3,1,1)} = 0$. Proposition 9.11 settled the types of depth at most 7 by computing the windows. In this section we extend the matrix-pair model of Subsection 9.7 to every self-conjugate ν: the single block ν is cut out by one test, built from a grid of rc rank-one matrices, for each outer corner (r,c) of ν with $rc \leq k$ (Theorem 10.1). For $\nu = (3,1,1)$ the leading term in n of the three tests is a pointwise condition through one GL_n-module $\mathcal{K}_\nu(n)$ of 4-forms, which is 0 for every n (Theorem 10.12, computer-assisted and exact); Capelli's identity then gives an analogue of Theorem 9.22 without an exceptional line (Theorem 10.13). As a consequence, $\mathcal{Y}_m((3,1,1)) = 0$ for every m, and for every $n \geq 5$ the kernel $K_{\lambda(3,1,1)}$ has no conjugation type of depth at most $n-3$ (Theorem 10.15). The deeper types remain open. Subsection 10.7 removes the hypothesis $\nu = \nu'$ (the tests are then the live corners of ν') and, for $\nu = (3,1)$ and $n \geq 6$, excludes from $K_{\lambda(3,1)}$ the types of six weights of the first layer. Subsection 10.8 shows that the leading term is injective on the column $(\tau_2, 1, 1)$, although its pointwise kernel is not 0, and excludes from $K_{\lambda(3,1)}$ the two-row types of depth $1, \dots, n-3$. Subsection 10.9 excludes all two-row types, at every depth and for every $n \geq 3$, using only the tests at diagonal points. Subsection 10.10 excludes the three- and four-row types of depth at most 6 (for n at least the depth plus 3), by exact computer-assisted certificates. Subsection 10.11 returns to $\nu = (3,1,1)$: it proves Conjecture 10.18 on the column $(\tau_2, 1, 1, 1)$, by the method of Subsection 10.9 with exact symbolic identities checked by computer, and so excludes from $K_{\lambda(3,1,1)}$ the two-row types, at every depth and for every $n \geq 3$.

10.1. **The corner dictionary.** In this subsection $F = \mathbb{C}$, as in Subsection 9.7. Let $\nu \vdash k$ lie inside R with $\nu = \nu'$, $\lambda = \lambda(\nu)$ and $N = n^2 - k$. By Corollary 7.6(a) every conjugation type κ of K_λ has $\kappa_1 \leq k$; put $\tau := \kappa' + (1^k) = (\kappa^{+k})'$, a partition of n^2 with exactly k rows. Let V, $M_n = V \otimes V$ and the action of $\mathrm{SL}_n \times \mathrm{SL}_n$ be as in Subsection 9.7, and let Pol_τ be the space of the polynomials $f(P_1, \dots, P_k)$ on M_n^k that are $\mathrm{SL}_n \times \mathrm{SL}_n$-invariant, homogeneous of degree τ_a in P_a, and highest weight vectors for GL_k: $f(\dots, P_b + tP_a, \dots) = f$ for all $a < b$ and t (GL_k acts by $(P_1, \dots, P_k) \mapsto (P_1, \dots, P_k)g$).

An addable cell (r,c) of ν is *live* if $rc \leq k$. For a live corner, vectors $u_1, \dots, u_r, y_1, \dots, y_c \in V$ and matrices $X_{rc+1}, \dots, X_k \in M_n$, let $B_1, \dots, B_k$ be the rc matrices $u_p y_q^T$ (in a fixed order) followed by the X_j, and consider the test

$$(T_{r,c}) \qquad \sum_{s \in S_k} \operatorname{sgn}(s)\, [t_1 \cdots t_k]\, f\big(P_1 + t_1 B_{s(1)}, \dots, P_k + t_k B_{s(k)}\big) = 0 \quad \text{for all } P, u, y, X.$$

For $\nu=(2,1)$ the live corners are $(1,3)$ and $(3,1)$, which give (S1) and (S2) (the corner $(2,2)$ is not live, as $4>3$). For $\nu=(3,1,1)$ all three corners $(1,4)$, $(2,2)$, $(4,1)$ are live, with $rc=4=k-1$.

Theorem 10.1 (corner dictionary)**.** *Let $\nu=\nu'\vdash k$ lie inside R and κ, τ be as above.*

(a) There is a linear isomorphism $\mathrm{Mult}(\kappa^{+k})\to\mathrm{Pol}_\tau$; in particular $\dim\mathrm{Pol}_\tau=g(R,R,\tau)$.

(b) It maps the space X_ν of Corollary 7.6(b) onto the space of the $f\in\mathrm{Pol}_\tau$ that pass $(T_{r,c})$ for every live corner (r,c) of ν. Hence the multiplicity of κ in K_λ is the dimension of this space.

(c) For every n, $f\mapsto\varphi_f:=f(I,\cdot,\dots,\cdot)$ is injective on Pol_τ, and

$$f=\det(P_1)^n\varphi_f\big((P_1^{-1}P_2)_0,\dots,(P_1^{-1}P_k)_0\big)$$

where $\det P_1\neq0$ (X_0 is the traceless part of X). If $d:=\tau_2+\cdots+\tau_k\le n$, it is a bijection onto the space HW_τ of the polynomials $\varphi(A_2,\dots,A_k)$ in traceless matrices that are invariant under simultaneous conjugation, homogeneous of degree τ_a in A_a, and satisfy $\varphi(\dots,A_b+tA_a,\dots)=\varphi$ for $2\le a<b$; the inverse is $\varphi\mapsto\det(P_1)^{n-d}\varphi\big(\mathrm{adj}(P_1)P_2)_0,\dots\big)$.

For $k=3$, (a) and (b) are Theorem 9.18 and (c) is the chart paragraph after it. The only new ingredient is the description of the outer blocks. For $\nu\neq\nu'$ see Theorem 10.24: the tests are then the live corners of ν'.

Lemma 10.2 (corners)**.** *Let $W:=M_n$ and $k\ge1$.*

(i) $\Lambda^kW=\bigoplus_{\alpha\vdash k}S_\alpha V\otimes S_{\alpha'}V$ (a summand with $\ell(\alpha)>n$ or $\ell(\alpha')>n$ is 0), and the isotypic projector of e_μ, with S_k permuting the second factors, maps Λ^kW onto the summand $\alpha=\mu'$.

(ii) For $\alpha\vdash k$ inside the $n\times n$ box the cell wedge $h_\alpha:=\bigwedge_{(i,j)\in\alpha}(e_i\otimes e_j)$ spans the highest weight line of $S_\alpha V\otimes S_{\alpha'}V$.

(iii) For a cell (r,c) with $rc\le k$ let $\Gamma_{r,c}\subseteq\Lambda^kW$ be the span of the vectors $\big(\bigwedge_{p\le r,\,q\le c}u_p\otimes y_q\big)\wedge X_{rc+1}\wedge\cdots\wedge X_k$. Then $\Gamma_{r,c}=\bigoplus_{\alpha\supseteq(c^r)}S_\alpha V\otimes S_{\alpha'}V$.

(iv) $\sum_{(r,c)}\Gamma_{r,c}=\bigoplus_{\alpha\neq\nu}S_\alpha V\otimes S_{\alpha'}V$, the sum over the live corners of ν.

Proof. (i) is Schur–Weyl duality on the k slots, as in Step 2 of the proof of Theorem 9.18 (the skew Cauchy decomposition; [ABW82]). (ii) $h_\alpha\neq0$ has weight (α,α'). A raising operator $E_{i,i+1}$ on the first factor turns a factor $e_{i+1}\otimes e_j$ into $e_i\otimes e_j$, which is already a factor since $(i,j)\in\alpha$; so every term is 0, and the same holds on the second factor. A highest weight vector of weight (α,β) lies in a summand $S_\alpha V\otimes S_\beta V$, and by (i) only $\beta=\alpha'$ occurs, once. (iii) $\Gamma_{r,c}$ is $\mathrm{GL}_n\times\mathrm{GL}_n$-stable. If the u_p (or the y_q) are dependent, the rc factors $u_p\otimes y_q$ lie in a space of dimension less than rc and the wedge is 0; otherwise $r,c\le n$ and the wedge of the grid is $(g,h).h_{(c^r)}$ for some g,h. The span of the orbit of a highest weight vector is the irreducible module it generates, so the grids span $S_{(c^r)}V\otimes S_{(r^c)}V$. The wedge product $\Lambda^{rc}W\otimes\Lambda^{k-rc}W\to\Lambda^kW$ is equivariant, and the first factor of the image of $S_{(c^r)}V\otimes(\text{anything})$ lies in the sum of the $S_\alpha V\subseteq S_{(c^r)}V\otimes S_\gamma V$, which have $\alpha\supseteq(c^r)$ (Littlewood–Richardson); with (i) this gives "$\subseteq$". Conversely, for $\alpha\supseteq(c^r)$ (if α does not fit in the $n\times n$ box the summand is 0), $h_\alpha=\pm h_{(c^r)}\wedge\bigwedge_{(i,j)\in\alpha\setminus(c^r)}(e_i\otimes e_j)$ lies in $\Gamma_{r,c}$ and generates $S_\alpha V\otimes S_{\alpha'}V$. (iv) A partition $\mu\vdash k$ with $\mu\neq\nu$ is not inside ν, so it has a cell outside ν; take one, (r,c), minimal for the product order. Then $(r-1,c)$ and $(r,c-1)$, when they exist, lie in μ and not in $\mu\setminus\nu$, hence in ν; so (r,c) is an addable cell of ν, and $\mu\supseteq(c^r)$ gives $rc\le k$. Conversely, if $\mu\supseteq(c^r)$ for an addable cell (r,c) of ν, then $(r,c)\in\mu\setminus\nu$ and $\mu\neq\nu$. With (iii) this gives (iv). □

Part (iii) is the exterior analogue of the description of the $\mathrm{GL}\times\mathrm{GL}$-stable ideals of $\mathrm{Sym}(V\otimes W)$ by De Concini, Eisenbud and Procesi [DEP80].

Proof of Theorem 10.1. Steps 1, 3 and 4 of the proof of Theorem 9.18 hold with k in place of 3, with these changes. In Step 3 there are k groups $G_1,\dots,G_k$ of slots, and for $a<b$ the coefficient of t^ℓ in $f_{\tilde x}(\dots,P_b+tP_a,\dots)$ is $\binom{\tau_b}{\ell}$ times a restitution of $\tilde x(w_\ell)$, where w_ℓ is fixed by a Young subgroup of type $\tau+\ell(\epsilon_a-\epsilon_b)$; its partial sums exceed those of τ by ℓ at the positions $a,\dots,b-1$ and agree

elsewhere, and sorting only raises partial sums, so for $\ell \geq 1$ its sorted type strictly dominates τ and $w_\ell = 0$. In Step 4, Pieri's rule gives the whole S_O-sign part of $\operatorname{Res} S^\tau$ as $S^{\kappa'} \boxtimes \operatorname{sgn}$, as τ has exactly k rows; O' consists of the first slot of each G_a, and the test is

$$\textstyle\sum_{s\in S_k} \operatorname{sgn}(s)\,[t_1\cdots t_k]\, f_{\tilde{x}}(P + tB_s) = \tau_1\cdots\tau_k\,\tilde{x}(w')(P,\dots;B_1,\dots,B_k), \quad w' := \sum_{s\in S_{O'}} \operatorname{sgn}(s)\, s\, w_\tau.$$

Step 2 is replaced by Lemma 10.2. As $\nu = \nu'$, the blocks $\mu \neq \nu$ of Corollary 7.6(b) are, by Lemma 10.2(i), the summands $\alpha = \mu' \neq \nu$; so $x \in X_\nu$ if and only if, for every w in the S_O-sign part of S^τ, $\tilde{x}(w)$ vanishes on $\bigoplus_{\alpha\neq\nu}$ in the outer slots, that is (Lemma 10.2(iv)) at the arguments of every $(T_{r,c})$. This gives "$\Rightarrow$". For "$\Leftarrow$", let f pass every $(T_{r,c})$ and $\mu \neq \nu$. The function $e_\mu * \tilde{x}(w')$ is $\tilde{x}(w')$ composed with the isotypic projector of e_μ on the second factors of the slots O'; it sees only the summand $\alpha = \mu' \neq \nu$, where the tests make its restitution vanish, and it is symmetric in each group $G_a \setminus O'$; so it is 0. The map $w \mapsto e_\mu * \tilde{x}(w)$ on the $S_{O'}$-sign part ($\cong S^{\kappa'}$, irreducible) is equivariant and kills w', which is not 0 by Lemma 10.3; so it is 0, and $x \in X_\nu$. (c): for $g \in \mathrm{GL}_n$, $f(gP_1,\dots,gP_k) = \det(g)^n f(P)$ (the $\mathrm{SL}\times\mathrm{SL}$-invariants of degree n^2 carry $\det^n \otimes \det^n$), which gives the formula and the injectivity, as $\det P_1 \neq 0$ is dense. φ_f is invariant under conjugation, the conditions with $a = 1$ say that it depends only on the traceless parts, and those with $a \geq 2$ are the conditions of HW_τ. Conversely, for $\varphi \in \mathrm{HW}_\tau$ and $d \leq n$ the function $\det(P_1)^{n-d}\varphi((\operatorname{adj}(P_1)P_2)_0,\dots)$ is a polynomial; $\operatorname{adj}(gP_1h^T)\,gP_ah^T = h^{-T}\operatorname{adj}(P_1)P_ah^T$ for $g,h \in \mathrm{SL}_n$ gives the invariance; $\operatorname{adj}(P_1)(P_b + tP_1) = \operatorname{adj}(P_1)P_b + t\det(P_1)I$ has the same traceless part; the degree in P_1 is $n(n-d)+(n-1)d = \tau_1$; and its value at $P_1 = I$ is φ. So the two maps are inverse. □

Lemma 10.3. *The vector w' in the proof of Theorem 10.1 is not 0: it is a nonzero multiple of $\sum_{s\in S_{O'}} a_{st}b_t$, whose coefficient of the identity is $k!$.*

Proof. This is the proof of Lemma 9.19 with k rows: w_τ is a nonzero multiple of a_tb_t for the tableau t whose rows are $G_1,\dots,G_k$ (both span the line fixed by the row group), O' is the first column of t (one cell in each row, as τ has exactly k rows), and $R_{st} \cap C_t = \{e\}$ for every $s \in S_{O'}$, so the coefficient of the identity in $\sum_{s\in S_{O'}} a_{st}b_t$ is $|S_{O'}| = k!$. □

10.2. **The test at $P_1 = I$ for $\nu = (3,1,1)$.** From now on $\nu = (3,1,1)$, $k = 5$, $\lambda = \lambda(\nu)$, $N = n^2 - 5$ and $n \geq 3$. For a type κ put $d := \tau_2 + \cdots + \tau_5$; its depth is $N - \ell(\kappa) = d - 4$. For $f \in \mathrm{Pol}_\tau$ and $P = (P_1,\dots,P_5)$ let $\Omega f(P)$ be the alternating 5-form on M_n

$$\Omega f(P)(B_1,\dots,B_5) := \sum_{s\in S_5} \operatorname{sgn}(s)\,[t_1\cdots t_5]\, f\big(P_1 + t_1B_{s(1)},\dots,P_5 + t_5B_{s(5)}\big),$$

and let $\Pi_{\neq\nu}$ be the projection of $\Lambda^5 M_n$ onto $\bigoplus_{\alpha\neq\nu} S_\alpha V \otimes S_{\alpha'}V$; it commutes with $\mathrm{GL}_n \times \mathrm{GL}_n$. Each test $(T_{r,c})$ is a value of $\Omega f(P)$, so by Lemma 10.2(iv)

(13) $\quad f$ passes the three corner tests $\iff \Omega f(P) \circ \Pi_{\neq\nu} = 0$ (as forms on $\Lambda^5 M_n$) for every P.

Lemma 10.4. *Let $f \in \mathrm{Pol}_\tau$. (a)* (13) *holds for all P if and only if it holds for all P with $P_1 = I$. (b) $\Omega f(I, A_2,\dots,A_5)$ depends only on the traceless parts of $A_2,\dots,A_5$.*

Proof. (a) Both sides are polynomial in P, and $\det P_1 \neq 0$ is dense. Write $P_1 = sg$ with $g \in \mathrm{SL}_n$ and $s^n = \det P_1$. By invariance, $\Omega f((g,h).P)(B) = \Omega f(P)((g,h)^{-1}.B)$, and $\Pi_{\neq\nu}$ commutes with (g,h); so the condition at P is equivalent to the condition at $(sI, g^{-1}P_2,\dots)$. By homogeneity, and as every term of the alternation has exactly one direction in slot 1, $\Omega f(sI,Q) = s^{\tau_1-1}\Omega f(I,Q)$. (b) The highest weight condition with $a = 1$ gives, at the point $(I + t_1W, A + tV)$, $f(I + t_1W,\dots,A_b + cI + t_bV_b,\dots) = f(I + t_1W,\dots,A_b - ct_1W + t_bV_b,\dots)$. So the shift $A_b \mapsto A_b + cI$ adds to the coefficient of $t_1\cdots t_5$ only terms in which slot b gives both t_1 and t_b and slot 1 gives nothing: a second derivative in slot b, symmetric in $(B_{s(1)}, B_{s(b)})$. The alternation pairs s with $s \circ (1\,b)$, of opposite sign, and these terms cancel. □

So we may take $P_1 = I$ and $A = (A_2, \dots, A_5)$ traceless. By Theorem 10.1(c), $f = \det(P_1)^n \varphi((P_1^{-1}P_2)_0, \dots)$ with $\varphi = \varphi_f \in \mathrm{HW}_\tau$. Put $\tilde\varphi(X_2, \dots, X_5) := \varphi((X_2)_0, \dots, (X_5)_0)$ and write $\partial_j[V]$ for the derivative in the j-th argument in the direction V.

Lemma 10.5 (the test in the chart, $k = 5$)**.** *For traceless $A_2, \dots, A_5$ and $W, V_2, \dots, V_5 \in M_n$,*

$$[t_1 \cdots t_5]\, f(I + t_1 W, A_2 + t_2 V_2, \dots, A_5 + t_5 V_5) = n \operatorname{tr}(W)\, D^4\tilde\varphi[V] - \sum_{l=2}^{5} \partial_l[WA_l]\, \partial_l[V_l] \prod_{j \neq l} \partial_j[V_j]\, \tilde\varphi - \sum_{j=2}^{5} D^4\tilde\varphi[V \text{ with } V_j \text{ replaced by } WV_j],$$

where $D^4\tilde\varphi[V] := \partial_2[V_2]\partial_3[V_3]\partial_4[V_4]\partial_5[V_5]\tilde\varphi$ at A; the second term is a double derivative in slot l.

Proof. With $t_a^2 = 0$: $\det(I + t_1 W)^n = 1 + n t_1 \operatorname{tr} W$ and $(I + t_1 W)^{-1} = I - t_1 W$, so $P_1^{-1}P_j = A_j + t_j V_j - t_1 W A_j - t_1 t_j W V_j$. The monomial $t_1 \cdots t_5$ takes t_1 from the determinant (first term), or from $-t_1 W A_l$ in one slot l while every slot j gives $t_j V_j$ (second term), or from $-t_1 t_j W V_j$ in slot j (third term). □

Lemma 10.6. *If $a+b \le n$, the trace monomials tr_σ, $\sigma \in S_{a+b}$, in letters $L_1, \dots, L_{a+b}$ with no fixed point in $\{1, \dots, a\}$ are linearly independent as functions on $\mathfrak{sl}_n^a \times M_n^b$ (and hence, by polarization in characteristic 0, so are the corresponding trace polynomials).*

Proof. For $m := a+b \le n$ the tr_σ, $\sigma \in S_m$, are a basis of the invariant multilinear forms on M_n^m (the first and second fundamental theorems, [Ra74, Pr76]). Splitting $M_n = \mathfrak{sl}_n \oplus \mathbb{C}I$ in each of the first a slots shows that $D(a,b) := \dim(\text{invariants on } \mathfrak{sl}_n^{\otimes a} \otimes M_n^{\otimes b})$ satisfies $(a+b)! = \sum_j \binom{a}{j} D(a-j, b)$, the recursion of the number of $\sigma \in S_{a+b}$ without fixed points in $\{1, \dots, a\}$; so $D(a,b)$ is that number. Restriction to $\mathfrak{sl}_n$ is onto the invariants and kills every tr_σ with a fixed point in $\{1, \dots, a\}$; the remaining $D(a,b)$ monomials span the image, so they are independent. □

The formal model. Every $\varphi \in \mathrm{HW}_\tau$ is φ_g for a unique function g (a function, not a group element) on the derangements of S_d that is invariant under the colour group, as in the chart paragraph of Subsection 9.7 with four colours (Lemma 10.6 with $b = 0$, $n \ge d$). In the test, expand each projection $X_0 = X - (\operatorname{tr} X/n)I$: an inserted I deletes its letter from its cycle, and a cycle made only of I's gives n. Every term becomes a power of n times a trace monomial in $d+1$ letters ($d-4$ letters A and $B_1, \dots, B_5$), and terms with a fixed point on a letter A vanish. The alternation over S_5 and $\Pi_{\neq\nu}$ act on monomials without reference to n. So for $n \ge d+1$ the test is a matrix $\mathbf{M}(n)$ between n-free bases (Lemma 10.6), and the multiplicity of κ in K_λ is at most $\dim\ker \mathbf{M}(n)$ (this direction is all that is used below).

Lemma 10.7. $\mathbf{M}(n) = n\,\mathcal{Z} + \mathbf{M}_0 + n^{-1}\mathbf{M}_1 + \dots$ *(finitely many terms), with $\mathcal{Z}$ free of n: $\mathcal{Z}(\varphi) := (\mathrm{tr} \wedge \mathrm{Cay}^{\mathrm{form}}\varphi) \circ \Pi_{\neq\nu}$, where $\mathrm{Cay}^{\mathrm{form}}$ is the Cayley form below with the formal derivatives of the trace polynomial φ (no projections). As a function at every n, $\mathcal{Z}(\varphi)(A;\cdot) = (\mathrm{tr} \wedge \mathrm{Cay}\,\varphi(A)) \circ \Pi_{\neq\nu}$ with*

$$\mathrm{Cay}\,\varphi(A)[X_1, \dots, X_4] := \sum_{\pi \in S_4} \mathrm{sgn}(\pi)\, \partial_2[X_{\pi 1}]\partial_3[X_{\pi 2}]\partial_4[X_{\pi 3}]\partial_5[X_{\pi 4}]\, \tilde\varphi(A).$$

Moreover $\mathrm{Cay}\,\varphi(A)$ vanishes on $I \wedge \Lambda^3 M_n$, and its restriction to $\mathfrak{sl}_n^4$ is the Cayley form of φ on $\mathfrak{sl}_n$.

Proof. The only explicit positive power of n is in $n \operatorname{tr}(W) D^4\tilde\varphi[V]$. The letters A are traceless; only the letters B and the composite letters WA_l, WV_j are projected, and $\mathrm{tr}(W)$ in the first term is an explicit fixed point of a letter B, not a projection. A derivative replaces a letter in place, so every cycle of a target has the length of a cycle of φ, at least 2. Each projected letter brings $1/n$, and a factor n needs a cycle made only of projected letters: with p projected letters the exponent is $-p + \#(\text{all-projected cycles}) \le -p/2 < 0$ for $p \ge 1$. So the second and third terms of Lemma 10.5 are $O(1)$, and the coefficient of n is the alternation of $\mathrm{tr}(W) D^4\varphi[V]$ with unprojected derivatives,

followed by $\Pi_{\neq\nu}$. Now $\sum_s \operatorname{sgn}(s)\operatorname{tr}(B_{s1})\prod_j \partial_j[B_{sj}]\varphi = (\operatorname{tr}\wedge \operatorname{Cay}^{\mathrm{form}}\varphi)(B)$, where $\operatorname{Cay}^{\mathrm{form}}$ uses unprojected derivatives. Finally $\operatorname{tr}\wedge\operatorname{Cay}^{\mathrm{form}}\varphi = \operatorname{tr}\wedge\operatorname{Cay}\varphi$: the difference vanishes on $\Lambda^4\mathfrak{sl}_n$, and an alternating form that vanishes on Λ^4 of the hyperplane $\ker\operatorname{tr}$ lies in $\operatorname{tr}\wedge\Lambda^3 M_n^*$. The two facts follow from $\tilde\varphi(X+cI)=\tilde\varphi(X)$. □

Lemma 10.8. *If $\mathcal{Z}$ is injective on HW_τ, then κ does not occur in K_λ for all but finitely many n.*

Proof. A maximal minor of $\mathbf{M}(n)/n$ is a polynomial in $1/n$ whose constant term is the same minor of $\mathcal{Z}$, which is not 0. □

10.3. **The pointwise form and Theorem K.** For a self-conjugate $\nu\vdash 5$ let $\mathcal{K}_\nu(n)$ be the space of the $\omega\in\Lambda^4 M_n^*$ with $\omega(I\wedge\cdot)=0$ such that $\operatorname{tr}\wedge\omega$ vanishes on $\bigoplus_{\alpha\neq\nu} S_\alpha V\otimes S_{\alpha'}V$. It is stable under conjugation. Let ι be the contraction with the trace, the transpose of $\operatorname{tr}\wedge\cdot$, so that $(\operatorname{tr}\wedge\omega)(Y)=\omega(\iota Y)$ for $Y\in\Lambda^5 M_n$.

Lemma 10.9. *If $\varphi\in\mathrm{HW}_\tau$ and $\mathcal{Z}(\varphi)=0$, then $\operatorname{Cay}\varphi(A)\in\mathcal{K}_\nu(n)$ for every n and every $A\in\mathfrak{gl}_n^4$ (the formal identity evaluated at n). Conversely for $n\geq d+1$.*

Proof. A formal zero is a functional zero at every n. $\mathcal{Z}(\varphi)(A;\cdot)=0$ says that $\operatorname{tr}\wedge\operatorname{Cay}\varphi(A)$ vanishes on the image of $\Pi_{\neq\nu}$, and $\operatorname{Cay}\varphi(A)$ vanishes on $I\wedge\Lambda^3 M_n$ (Lemma 10.7). The converse is Lemma 10.6. □

Lemma 10.10. *For $\mu\vdash 4$ let $C_\mu\subseteq\Lambda^4 M_n$ be the summand $\alpha=\mu$. Then*

$$\mathcal{K}_{(3,1,1)}(n)=\operatorname{Ann}\Big(I\wedge\Lambda^3 M_n+\sum_{\mu\in\{(4),(2,2),(1^4)\}}\iota\big(C_\mu\wedge M_n\big)\Big)=\operatorname{Ann}\Big(I\wedge\Lambda^3 M_n+\sum_{\mu}\big(C_\mu+\iota(C_\mu)\wedge M_n\big)\Big).$$

Proof. The live corners $(1,4)$, $(2,2)$, $(4,1)$ have $rc=4$, so by Lemma 10.2(iii), (iv) (one slot is free), $\bigoplus_{\alpha\neq\nu} S_\alpha V\otimes S_{\alpha'}V=\sum_\mu C_\mu\wedge M_n$ with $\mu=(4),(2,2),(1^4)$. Apply $(\operatorname{tr}\wedge\omega)(Y)=\omega(\iota Y)$. For the second form: $\iota(Z\wedge X)=\iota(Z)\wedge X+\operatorname{tr}(X)Z$ for $Z\in\Lambda^4 M_n$ and $X\in M_n$. If $\omega(I\wedge\cdot)=0$ and ω vanishes on $\iota(C_\mu\wedge M_n)$, then $X=I$ gives $n\,\omega(Z)=0$ for $Z\in C_\mu$, and then $\omega(\iota(Z)\wedge X)=0$ for every X; the converse is clear. □

Remark 10.11. $\omega\mapsto\operatorname{tr}\wedge\omega$ is injective on $\{\omega:\omega(I\wedge\cdot)=0\}$: if $\operatorname{tr}\wedge\omega=0$, then $\omega=\operatorname{tr}\wedge\theta$, and $0=\iota_I\omega=n\theta-\operatorname{tr}\wedge\iota_I\theta$ (ι_I is the contraction with the vector I) gives $\theta=\operatorname{tr}\wedge\iota_I\theta/n$, so $\omega=0$. So $\mathcal{K}_\nu(n)=0$ says that no nonzero 5-form of Cauchy type ν is divisible by the trace. For $\nu=(2,1)$ the same construction gives $\mathcal{K}_{(2,1)}(n)=\{(X,Y)\mapsto\operatorname{tr}(C[X,Y]):C\in\mathfrak{sl}_n\}$, one copy of the adjoint representation, and Theorem 9.22 is the statement that $\operatorname{Cay}\varphi$ with values in it forces $\varphi\in\mathbb{C}\operatorname{tr}([A,B]^2)$, the line that gives f_W (Theorem 9.9).

Theorem 10.12 (computer-assisted)**.** *$\mathcal{K}_{(3,1,1)}(n)=0$ for every $n\geq 3$.*

Proof. (a) $3\leq n\leq 8$. Every irreducible constituent of $\Lambda^4 M_n^*$ under conjugation has a nonzero zero-weight space, so $\mathcal{K}_{(3,1,1)}(n)=0$ if and only if the zero-weight part of the space in Lemma 10.10 is the whole zero-weight space of $\Lambda^4 M_n$. It is spanned by zero-weight vectors with integer coordinates (the projections onto C_μ have the integer coefficients $\dim(\mu')\chi_{\mu'}$; contraction and wedge bring only signs), so a rank modulo a prime is a lower bound for the rank over $\mathbb{Q}$. Modulo $p=1000003$ the rank is full: 18, 90, 280, 675, 1386, 2548 for $n=3,\dots,8$.

(b) $n\geq 9$. $\mathcal{K}(n)\neq 0$ if and only if there is a nonzero invariant bilinear form Ω on $\Lambda^4 M_n$ with $\Omega(Y;\cdot)\in\mathcal{K}(n)$ for every Y ($\Lambda^4 M_n$ is self-dual and contains every irreducible constituent of $\mathcal{K}(n)$). By the first fundamental theorem, $\Omega=\sum_{\sigma\in S_8} c_\sigma\operatorname{tr}_\sigma(Y_1,\dots,Y_4,X_1,\dots,X_4)$ with c twisted under conjugation by $S_4\times S_4$ (character $\operatorname{sgn}\otimes\operatorname{sgn}$), unique for $n\geq 8$; the twisted c form a space of dimension 62. The conditions $\Omega(Y;I,X_2,X_3,X_4)=0$ (7 letters; a fixed point on the letter I gives n), $\Omega(Y;e_\mu(X_1\wedge\cdots\wedge X_4))=0$ (8 letters) and $\Omega(Y;\iota(e_\mu(X_1\wedge\cdots\wedge X_4))\wedge X_5)=0$ (9 letters), for $\mu=(4),(2,2),(1^4)$, are formal identities of trace monomials with integer coefficients:

with $X_a = v_a w_a^T$, e_μ (with S_4 permuting the first factors v_a, so that it projects onto C_μ) acts by rewiring. Together they read $\mathbf{A}(n)c = 0$ for one integer matrix $\mathbf{A}(t) = \mathbf{A}_0 + t\mathbf{A}_1$ (n enters only through $\operatorname{tr} I$, at most once per monomial). For $n \geq 9$ every condition has at most n letters, so by Lemma 10.6 the functional conditions are the formal ones, and $\mathcal{K}(n) = 0$ if $\mathbf{A}(n)$ is injective. Sixty-two rows of $\mathbf{A}(t)$, two of them containing t (each linearly), form a square matrix whose determinant is Ct^2 with $C \neq 0$ an integer of 185 digits (exact determinants over $\mathbb{Z}$ at $t = 0, 1, 2$, interpolation, and an exact check at $t = 9$). So $\mathbf{A}(n)$ is injective for every $n \geq 1$, and $\mathcal{K}(n) = 0$ for $n \geq 9$.

A second proof of (b), free of n: by Remark 10.11 one can use $\eta := \operatorname{tr} \wedge\Omega(Y;\cdot)$ instead of the condition with I; $\dim \operatorname{Hom}_{\mathrm{GL}_n}(\Lambda^4 M_n, \mathcal{K}(n))$ is the difference of the ranks of $c \mapsto \eta_c$ and $c \mapsto \eta_c \circ \Pi_{\neq\nu}$, two formal maps in 9 letters with no letter I, which for $n \geq 9$ are the functional maps; their exact ranks over $\mathbb{Q}$ are 41 and 41. □

10.4. **Theorem Z-INJ for $(3,1,1)$.**

Theorem 10.13. *For every 5-row τ, $\ker \mathcal{Z} \cap \mathrm{HW}_\tau = 0$.*

Proof. Let $\varphi \in \mathrm{HW}_\tau$ with $\mathcal{Z}(\varphi) = 0$, and fix one $n \geq \max(d, 9)$. By Lemma 10.9 and Theorem 10.12, $\operatorname{Cay}\varphi(A) = 0$ for every $A \in \mathfrak{sl}_n^4$: in coordinates $x_{\alpha j}$ of A_{j+1} (α runs over a basis of $\mathfrak{sl}_n$, $j = 1, \dots, 4$), every 4×4 minor $\det(\partial_{\alpha_r, j})$ of the derivative matrix kills φ. Capelli's identity (in the column-determinant form; [HU91, §1]), for polynomials on $\mathfrak{sl}_n \otimes \mathbb{C}^4$ with $\dim \mathfrak{sl}_n \geq 4$,

$$\textstyle\sum_{\alpha_1 < \dots < \alpha_4} \det(x_{\alpha_r, j}) \det(\partial_{\alpha_r, j}) = \operatorname{coldet}\big[E_{ij} + (4-i)\delta_{ij}\big]_{i,j=1}^{4}, \qquad E_{ij} = \textstyle\sum_\alpha x_{\alpha i}\partial_{\alpha j},$$

has on the right a central element of $U(\mathfrak{gl}_4)$. In the column determinant (column 4 acts first), every term other than the diagonal one contains a factor E_{ij} with $i < j$ that acts before any lowering operator, and such a factor kills φ (the highest weight conditions); the diagonal term acts by $\gamma(\tau) := (\tau_2 + 3)(\tau_3 + 2)(\tau_4 + 1)\tau_5 \geq 24$. So $\gamma(\tau)\varphi = 0$ on $\mathfrak{sl}_n^4$, and $\varphi = 0$ by Lemma 10.6 ($b = 0$, $n \geq d$). □

For $\nu = (2,1)$ the same argument stops at $\mathcal{K}_{(2,1)} \neq 0$ (Remark 10.11), and the proof of Theorem 9.22 is needed.

10.5. **Consequences.**

Proposition 10.14. *For a partition t with at most 4 parts let $\kappa_t(n)$ be the type with $\kappa_t' = (N - |t|, t)$. Then $\kappa_t(n)$ does not occur in $K_{\lambda(3,1,1)}$ for all but finitely many n.*

Proof. Here $\tau = (n^2 - |t| - 4, t_1 + 1, \dots, t_4 + 1)$ and $d = |t| + 4$; the formal HW_τ does not depend on n for $n \geq d$. Apply Theorem 10.13 and Lemma 10.8. □

Theorem 10.15. *(a) $\mathcal{Y}_m((3,1,1)) = 0$ for every $m \geq 0$.*
(b) For every $n \geq 5$, $K_{\lambda(3,1,1)}$ has no conjugation type of depth at most $n - 3$.

Proof. (a) Fix m. The types of depth at most m have tails t with $|t| \leq m$ and at most 4 parts (Corollary 7.6(a): $\kappa_1 \leq 5$), finitely many. By Proposition 10.14 there is an $n \geq m + 4$ at which none of them occurs. Then $K_\lambda^{[m]} = 0$, and by Theorem 9.7(R1) ($k = 5$, $n \geq m + 4$), $\mathcal{Y}_m((3,1,1)) \cong K_\lambda^{[m]} = 0$. (b) Fix $n \geq 5$ and $m := n - 3 \geq 2$. Theorem 9.7(R2) ($k = 5$, $n = m + 3$, $m \geq 2$) gives $\mathcal{Y}_m((3,1,1)) \cong K_\lambda^{[m]}$, which is 0 by (a); so K_λ has no type of depth at most $n - 3$. □

Remark 10.16. Part (a) contains Proposition 9.11 ($m \leq 7$) without any computation of windows; at a fixed n, (b) uses only Theorem 9.7(R2) and the n-free statement (a). The types of depth at least $\max(n - 2, 6)$ remain open (Proposition 9.11 settles depth at most 5), except the two-row types, which Corollary 10.71 excludes at every depth; so $U_{\lambda(3,1,1)} = 0$, statement (iii) in the proof of Theorem G, is open for $n \geq 5$; this section gives no new bound on $d(n)$ by itself. The method needs $\mathcal{K}_\nu(n) = 0$; for other self-conjugate ν this is not known.

10.6. **The first layer for $\nu=(3,1,1)$.** In this subsection $\nu=(3,1,1)$, $k=5$ and $n\ge 3$, and we keep the notation of Subsection 10.2. Call a 5-row weight τ *first-layer* if $\tau_5=1$; we name it by $(\tau_2,\tau_3,\tau_4,\tau_5)$, and $d=\tau_2+\cdots+\tau_5$ as before. For $\varphi\in \mathrm{HW}_\tau$ and $s\in\mathbb{C}$ let $f_s(P):=\det(P_1)^s\tilde\varphi(P_1^{-1}P_2,\dots,P_1^{-1}P_5)$ for P_1 near I, with the branch of $\det^s$ of Subsection 9.15. It is homogeneous of degree $\tau_1:=ns-d$ in P_1 (Euler's identity), it satisfies $f_s(\dots,P_j+cP_1,\dots)=f_s$ for $j\ge 2$ (as $\tilde\varphi$ sees only traceless parts), and for $s=n$ it is the polynomial f of Theorem 10.1(c). Put $\mu:=\tau_1+4=ns-d+4$. For traceless $A=(A_2,\dots,A_5)$ let $\omega=\omega_A:=\mathrm{Cay}\,\varphi(A)$ (Lemma 10.7); as $\tau_5=1$, it does not depend on A_5. We have $\omega_A(B)=\omega_A(B_0)$, and we extend ω to all A by $\omega_A:=\omega_{A_0}$.

Lemma 10.17 (the pencil)**.** *For every $s\in\mathbb{C}$ and every A (not only traceless A), $\Omega f_s(I,A)=s\,(\mathrm{tr}\wedge\omega)-\mathfrak{L}\omega$, where*

$$(\mathfrak{L}\omega)(B_1,\dots,B_5):=\sum_{i=1}^{5}(-1)^{i-1}(L_{B_i}\omega)(\hat B_i),$$

$$(L_C\omega)_A(D_1,\dots,D_4):=\frac{d}{dt}\Big|_{t=0}\omega_{A+tCA}(D_1,\dots,D_4)+\sum_{m=1}^{4}\omega_A(D_1,\dots,CD_m,\dots,D_4),$$

with $CA:=(CA_2,CA_3,CA_4,CA_5)$ and $(\mathrm{tr}\wedge\omega)(B)=\sum_i(-1)^{i-1}\,\mathrm{tr}(B_i)\,\omega(\hat B_i)$.

Proof. The proof of Lemma 10.5 uses only $t_a^2=0$, so it holds for f_s with $\det(I+t_1W)^s=1+st_1\,\mathrm{tr}\,W$ in place of $\det(I+t_1W)^n$. There $(I+t_1W)^{-1}(A_j+t_jV_j)=A_j+t_jV_j-t_1WA_j-t_1t_jWV_j$, so the t_1-part of $\tilde\varphi(P_1^{-1}P_j)$ is minus the derivative along WA_j and minus the derivative along WV_j (left multiplication of the argument), and $\det^s$ gives $s\,\mathrm{tr}(W)$. Alternating over the slot i whose matrix B_i is W (sign $(-1)^{i-1}$) gives the claim. This is the analogue for $k=5$ of Lemma 9.30. The proof does not use that A is traceless: the expansion of Lemma 10.5 is a Taylor expansion of $\tilde\varphi$, which is defined on all matrices, and ω_{A+tCA} is the Cayley form of $\tilde\varphi$ at $A+tCA$. Both sides are unchanged under $A_j\mapsto A_j+cI$: the left side because $f_s(\dots,P_j+cP_1,\dots)=f_s$ (the proof of Lemma 10.4(b)), $\mathrm{tr}\wedge\omega$ because $\omega_A=\omega_{A_0}$, and $\mathfrak{L}\omega$ because the shift adds $c\sum_j d_j\omega$ to its base part, where d_j is the exterior derivative in the base slot j; and $d_j\omega=0$, as two derivatives in slot j commute and are alternated. □

For a live corner (r,c) of ν ($rc=4$) put $g_{r,c}(u,y):=\bigwedge_{p\le r,\,q\le c}u_py_q^T$, so that $\Gamma_{r,c}$ is spanned by the $g_{r,c}(u,y)\wedge X$ (Lemma 10.2(iii)). We call s an *eigenvalue* at τ if some $0\ne\varphi\in\mathrm{HW}_\tau$ has $\Omega f_s(I,A)$ vanishing on $\Gamma_{1,4}+\Gamma_{2,2}+\Gamma_{4,1}$ for every traceless A, and call φ an *eigenvector*. For $s=n$ this is the test (13) at $P_1=I$ (Lemma 10.4). All the proofs below use only the following values, which are necessary conditions: (R1) the *free-slot rows* $g_{r,c}(u,y)\wedge I$ for the three corners; (R2) the rows $g_{1,4}(u,y)\wedge X$ and $g_{4,1}(u,y)\wedge X$; (R3) the single row $h_{(3,2)}=E_{11}\wedge E_{12}\wedge E_{13}\wedge E_{21}\wedge E_{22}=g_{2,2}(e_1,e_2;e_1,e_2)\wedge E_{13}$, the cell wedge of Lemma 10.2(ii). Let $V_{\mathrm{FS}}=V_{\mathrm{FS}}(\tau)$ be the space of the $\varphi\in\mathrm{HW}_\tau$ such that ω_A vanishes on $g_{1,4}(u,y)$, $g_{2,2}(u,y)$ and $g_{4,1}(u,y)$ for every traceless A and all u,y.

Conjecture 10.18 (L1(311))**.** *Let $n\ge 4$ and let τ be first-layer. Then every eigenvalue s at τ has $\mu=ns-d+4=0$.*

Lemma 10.19 (free slot)**.** *For P with P_1 near I and all $B_1,\dots,B_4$, $\Omega f_s(P)(B_1,\dots,B_4,P_1)=(\tau_1+4)\,\Omega' f_s(P)(B_1,\dots,B_4)$, where $\Omega' f(P)(B_1,\dots,B_4):=\sum_{\pi\in S_4}\mathrm{sgn}(\pi)\prod_{j=2}^{5}\partial_j[B_{\pi(j-1)}]f$ is the Cayley form in the last four columns. In particular $\Omega f_s(I,A)(B_1,\dots,B_4,I)=\mu\,\omega_A(B_1,\dots,B_4)$, so for $\mu\ne 0$ every eigenvector lies in V_{FS} (rows (R1)).*

Proof. Put $D_i:=B_i$ for $i\le 4$ and $D_5:=P_1$, and write

$$\Omega f(P)(D_1,\dots,D_5)=\sum_{\sigma\in S_5}\mathrm{sgn}(\sigma)\prod_i\partial_i[D_{\sigma(i)}]f.$$

Split the sum by the slot j that receives P_1. For $j=1$, $\partial_1[P_1]$ applied to $\prod_{i\ge 2}\partial_i[\cdot]f$ is the Euler operator of column 1, which gives $\tau_1\Omega' f$; the sign is $+1$, since moving the argument P_1 from position 5 to position 1 takes four transpositions. For $j\ge 2$, $\partial_j[P_1]$ is the derivative in slot j along the (variable) column P_1; it commutes with the ∂_i, $i\notin\{1,j\}$, which do not touch P_1. So $\partial_j[P_1]h=0$ for $h=\prod_{i\notin\{1,j\}}\partial_i[\cdot]f$, by the shift invariance $P_j\mapsto P_j+cP_1$. Applying $\partial_1[B_a]$, which by the product rule also moves the direction P_1 to B_a, gives $\partial_j[P_1]\partial_1[B_a]h=-\partial_j[B_a]h$. So this term is the $j=1$ term of $\sigma\circ(1\,j)$, with the same sign, and each of the four slots $j\ge 2$ adds one $\Omega' f$. The total factor is τ_1+4. At $P=(I,A)$, $\Omega' f_s=\omega_A$. □

The factor $\mu=\tau_1+4$ is the first factor of $(\tau_1+4)(\tau_2+3)(\tau_3+2)(\tau_4+1)\tau_5$, Capelli's factor for all five columns; the proof of Theorem 10.13 uses the last four, and for $k=3$ the first factor of $\mathrm{cap}(\tau)$ in Lemma 9.71 is τ_1+2. It is the label $\mu=\tau_1+k-1$ of Remark 9.59(2).

Call A *block-lower* if rows 1 and 2 of A_2,A_3,A_4 are 0. Put $H:=E_{11}-E_{22}$ and, for $\varphi\in\mathrm{HW}_\tau$, $\zeta(\varphi):=\omega_A(E_{13},H,E_{12},E_{21})$.

Lemma 10.20 (block-lower)**.** *Let τ be first-layer, $n\ge 3$, $s\in\mathbb{C}$ and $\varphi\in V_{\mathrm{FS}}$.*

(a) If A is block-lower, then $(\mathfrak{L}\omega)(h_{(3,2)})=0$, so $\Omega f_s(I,A)(h_{(3,2)})=s\,\zeta(\varphi)$.

(b) For every A, with ∂_C the derivative of ω_A along (CA_2,CA_3,CA_4),

$$(\mathfrak{L}\omega)(h_{(3,2)})=\partial_{E_{11}}\omega(E_{12},E_{13},E_{21},E_{22})-\partial_{E_{12}}\omega(E_{11},E_{13},E_{21},E_{22})\\ -\partial_{E_{21}}\omega(E_{11},E_{12},E_{13},E_{22})+\partial_{E_{22}}\omega(E_{11},E_{12},E_{13},E_{21}).$$

Proof. Put $X_1:=\omega(E_{12},E_{13},E_{21},E_{22})$, $X_5:=\omega(E_{11},E_{12},E_{13},E_{21})$ and $Y:=\omega(E_{11},E_{12},E_{23},E_{22})$. Only $\operatorname{tr}E_{11}=\operatorname{tr}E_{22}=1$, so $(\operatorname{tr}\wedge\omega)(h_{(3,2)})=X_1+X_5$; as $X_1=-\omega(E_{13},E_{22},E_{12},E_{21})$ and $X_5=\omega(E_{13},E_{11},E_{12},E_{21})$, this is $\zeta(\varphi)$. In $\mathfrak{L}\omega(h_{(3,2)})$ the arguments $B_1,\dots,B_5$ are E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, and we use $E_{ab}E_{cd}=\delta_{bc}E_{ad}$. (i) Base parts. For $B_i=E_{ab}$ with $b\in\{1,2\}$, $E_{ab}A_j=e_a\,(\text{row } b \text{ of } A_j)$; these give the four terms in (b). For $B_3=E_{13}$ the base part is the derivative of $\omega(E_{11},E_{12},E_{21},E_{22})$ along $(E_{13}A_j)_j$; ω vanishes on this $(2,2)$-grid at every base point, so it is 0. (ii) Argument parts. $i=1$ (E_{11}): $E_{11}E_{12}=E_{12}$, $E_{11}E_{13}=E_{13}$, $E_{11}E_{21}=E_{11}E_{22}=0$, giving $2X_1$. $i=2$ (E_{12}): $E_{12}E_{21}=E_{11}$ repeats E_{11}, and $E_{12}E_{22}=E_{12}$ gives $\omega(E_{11},E_{13},E_{21},E_{12})=X_5$, with sign $-$. $i=3$ (E_{13}): $E_{13}E_{ab}=0$ for $a\in\{1,2\}$. $i=4$ (E_{21}): $E_{21}E_{11}=E_{21}$ gives X_1, $E_{21}E_{12}=E_{22}$ repeats E_{22}, and $E_{21}E_{13}=E_{23}$ gives Y, with sign $-$. $i=5$ (E_{22}): $E_{22}E_{21}=E_{21}$ gives X_5. Total: X_1-Y. (iii) $\varphi\in V_{\mathrm{FS}}$ on the $(2,2)$-grid with $u=(e_1,e_2)$, $y=(e_1+te_3,e_2)$ gives $\omega(E_{11}+tE_{13},E_{12},E_{21}+tE_{23},E_{22})=0$ for all t. Its t-coefficient is $-X_1+Y$, so $Y=X_1$ and the argument parts cancel. This proves (b). At a block-lower A every term in (b) contains a row-1 or row-2 entry of A_2,A_3,A_4, so (a) follows. □

The proof uses only that ω vanishes on every $(2,2)$-grid at every base point; the $(1,4)$ and $(4,1)$ parts of V_{FS} are not used, which is why $n\ge 3$ suffices.

Corollary 10.21. *If $\mu\neq 0$ and $s\neq 0$, every eigenvector φ at a first-layer τ satisfies $\zeta(\varphi)=0$ at every block-lower A.*

Proof. By Lemma 10.19, $\varphi\in V_{\mathrm{FS}}$; apply Lemma 10.20(a) to the row (R3). □

For the three cases below we use trace forms. Let $\boldsymbol{\theta}$ be the identity of M_n, as an M_n-valued 1-form; products of M_n-valued forms are wedge products, so $\boldsymbol{\theta}^p(D_1,\dots,D_p)=\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)D_{\sigma1}\cdots D_{\sigma p}$, and matrices are 0-forms. Traces satisfy graded cyclicity, $\operatorname{tr}(\alpha\beta)=(-1)^{pq}\operatorname{tr}(\beta\alpha)$ for forms of degrees p,q; so a trace of an even power of $\boldsymbol{\theta}$ is 0. This is Rosset's calculus (Remark 5.5; see also [Pr15]). Put $\varrho:=\operatorname{tr}(\boldsymbol{\theta})/n$ and $\boldsymbol{\theta}_0:=\boldsymbol{\theta}-\varrho I$, so that a form written with $\boldsymbol{\theta}_0$ satisfies $\omega(B)=\omega(B_0)$. For a form ω_A depending on A let $d\omega:=\sum_i(-1)^{i-1}(d_{B_i}\omega)(\hat{B}_i)$, where d_B is the derivative in A along B. *Insertion rule*: on a trace form, d replaces one letter A by $\boldsymbol{\theta}$, with sign $(-1)^r$ if r

factors $\boldsymbol{\theta}$ precede it. Let $h_{(2,2)} = E_{11} \wedge E_{12} \wedge E_{21} \wedge E_{22} = g_{2,2}(e_1, e_2; e_1, e_2)$ (the cell wedge) and $B' := (E_{12}, E_{23}, E_{31}, E_{24}, E_{42})$ (traceless).

Theorem 10.22 (three cases of Conjecture 10.18)**.** *Let $n \geq 4$.*

(a) At $\tau = (2,1,1,1)$, $V_{\mathrm{FS}} = 0$, so there is no eigenvalue with $\mu \neq 0$ (rows (R1) only).
(b) At $\tau = (3,1,1,1)$ there is no eigenvalue with $\mu \neq 0$ (rows (R1) and (R3)).
(c) At $\tau = (2,2,1,1)$ there is no eigenvalue at all (rows (R2) and (R3)).

Proof. In each case $\varphi \mapsto \omega_\varphi$ is injective: $\omega_\varphi(A_2, \dots, A_5) = \gamma\varphi$ by Capelli's identity, as in the proof of Theorem 10.13, with $\gamma = (\tau_2+3)(\tau_3+2)(\tau_4+1)\tau_5 = 30$, 36, 40 in (a), (b), (c). The invariant forms below are spanned by trace forms by the first fundamental theorem [Pr76] (polarize, then alternate).

(a) Here ω_φ is linear in $A := A_2$, free of A_3, A_4, A_5, and closed, $d\omega_\varphi = 0$ (two derivatives in slot 2 commute). The invariant 4-forms on $\mathfrak{sl}_n$ that are linear in A are spanned by $\operatorname{tr}(A\boldsymbol{\theta}_0^4) = \operatorname{tr}(A\boldsymbol{\theta}^4)$ and $\operatorname{tr}(A\boldsymbol{\theta}_0) \wedge \operatorname{tr}(\boldsymbol{\theta}_0^3)$ ($\operatorname{tr}\boldsymbol{\theta}_0 = 0$ on $\mathfrak{sl}_n$, $\operatorname{tr} A = 0$, and the only pure $\boldsymbol{\theta}$-trace is $\operatorname{tr}(\boldsymbol{\theta}^3)$). Now $d\operatorname{tr}(A\boldsymbol{\theta}^4) = \operatorname{tr}(\boldsymbol{\theta}^5)$, which is not 0 on $\mathfrak{sl}_n$: $\operatorname{tr}(\boldsymbol{\theta}^5)(B') = 5$ (classical [Ko58]; this witness needs $n \geq 4$). And $d(\operatorname{tr}(A\boldsymbol{\theta}_0) \wedge \operatorname{tr}(\boldsymbol{\theta}_0^3)) = \operatorname{tr}(\boldsymbol{\theta}_0^2) \wedge \operatorname{tr}(\boldsymbol{\theta}_0^3) = 0$. So $\omega_\varphi \in \mathbb{C}\operatorname{tr}(A\boldsymbol{\theta}_0) \wedge \operatorname{tr}(\boldsymbol{\theta}_0^3)$. On $h_{(2,2)}$ this form is $-3(a_{11}+a_{22})$: $\operatorname{tr}(AE_{11}) = a_{11}$, $\operatorname{tr}(AE_{22}) = a_{22}$, $\operatorname{tr}(\boldsymbol{\theta}^3)(E_{12}, E_{21}, E_{22}) = -3$, $\operatorname{tr}(\boldsymbol{\theta}^3)(E_{11}, E_{12}, E_{21}) = 3$, the two middle terms have $\operatorname{tr}(\boldsymbol{\theta}^3) = 0$, and the ϱ-term is $\frac{6}{n}\operatorname{tr} A = 0$. It is not identically 0, so $V_{\mathrm{FS}} = 0$, and Lemma 10.19 gives the claim.

(b) Put $c_5 := \operatorname{tr}(\boldsymbol{\theta}^5)$, read as an alternating 5-linear form, and $\varphi_{c_5} := c_5(A_2^2, A_2, A_3, A_4, A_5)$; it lies in HW_τ, as $c_5(I, \dots) = 0$ and $c_5(X, X, \dots) = 0$. Write $A := A_2$.

Step 1: $V_{\mathrm{FS}} \subseteq \mathbb{C}\varphi_{c_5}$. Here ω_φ is quadratic in A, free of A_3, A_4, A_5, and closed. The invariant quadratic 4-forms on $\mathfrak{sl}_n$ are spanned by $F_1 = \operatorname{tr}(A^2\boldsymbol{\theta}^4)$, $F_2 = \operatorname{tr}(A\boldsymbol{\theta}^2 A\boldsymbol{\theta}^2)$, $W_2 = \operatorname{tr}(A\boldsymbol{\theta}A\boldsymbol{\theta}^3)$, $W_3 = \operatorname{tr}(A\boldsymbol{\theta}) \wedge \operatorname{tr}(A\boldsymbol{\theta}^3)$, $W_4 = \operatorname{tr}(A^2\boldsymbol{\theta}) \wedge \operatorname{tr}(\boldsymbol{\theta}^3)$ and $F_6 = \operatorname{tr}(A\boldsymbol{\theta}^2) \wedge \operatorname{tr}(A\boldsymbol{\theta}^2)$ (a one-letter trace is 0 on $\mathfrak{sl}_n$; $\operatorname{tr}(A\boldsymbol{\theta}^3 A\boldsymbol{\theta}) = -W_2$, and $\operatorname{tr}(A\boldsymbol{\theta}^3) \wedge \operatorname{tr}(A\boldsymbol{\theta})$ is a multiple of W_3). By the insertion rule $dF_1 = dF_2 = 2\operatorname{tr}(A\boldsymbol{\theta}^5)$, $dW_2 = dW_3 = dW_4 = 0$ and $dF_6 = 2\operatorname{tr}(A\boldsymbol{\theta}^2) \wedge \operatorname{tr}(\boldsymbol{\theta}^3)$. These two 5-forms are independent: on B' they are $a_{11} + 2a_{22} + a_{33} + a_{44}$ and $3(a_{22} - a_{44})$. ($(E_{23}, E_{31}, E_{12}, E_{24}, E_{42})$ is one closed walk and an even rearrangement of B'; its five rotations are even, start at the vertices $2, 3, 1, 2, 4$, and every trail through all five units is one of them. For the second form only $\{E_{12}, E_{23}, E_{31}\}$ closes, with $\operatorname{tr}(\boldsymbol{\theta}^3)(E_{12}, E_{23}, E_{31}) = 3$, and $\operatorname{tr}(A[E_{24}, E_{42}]) = a_{22} - a_{44}$.) So $\omega_\varphi \in \operatorname{span}\{\omega_1 := F_1 - F_2, W_2, W_3, W_4\}$, and $\varphi \in V_{\mathrm{FS}}$ needs $\omega_\varphi = 0$ on $h_{(2,2)}$ for all A. Put $p := a_{11} + a_{22}$, $q := a_{11}a_{22} - a_{12}a_{21}$, $Q := p^2 - 4q$ and $R := \sum_{m \geq 3}(a_{1m}a_{m1} + a_{2m}a_{m2})$. On $h_{(2,2)}$: $F_1 = 0$, as $\boldsymbol{\theta}_0^4 = \boldsymbol{\theta}^4$ is the standard polynomial s_4 on the units of the 2×2 block, which is 0 by the Amitsur–Levitzki theorem [AL50]; $F_2 = 0$, as the pairings $(\{11, 12\}, \{21, 22\})$ and $(\{11, 21\}, \{12, 22\})$ give $-a_{11}a_{22}$ with opposite signs and $[E_{11}, E_{22}] = 0$. With the terms from $\boldsymbol{\theta}_0$ (where $\varrho \wedge \gamma$ on $h_{(2,2)}$ is $\frac{1}{n}[\gamma(E_{12}, E_{21}, E_{22}) - \gamma(E_{11}, E_{12}, E_{21})]$):

$$W_2 = \operatorname{tr}(A\boldsymbol{\theta}A\boldsymbol{\theta}^3) + \varrho \wedge [\operatorname{tr}(A\boldsymbol{\theta}A\boldsymbol{\theta}^2) - \operatorname{tr}(A^2\boldsymbol{\theta}^3)]\colon \qquad -\tfrac{n-2}{n}Q + \tfrac{3}{n}R,$$
$$W_3 = \operatorname{tr}(A\boldsymbol{\theta}) \wedge \operatorname{tr}(A\boldsymbol{\theta}^3) + \varrho \wedge \operatorname{tr}(A\boldsymbol{\theta}) \wedge \operatorname{tr}(A\boldsymbol{\theta}^2)\colon \qquad -p^2 - 2q - \tfrac{Q}{n},$$
$$W_4 = \operatorname{tr}(A^2\boldsymbol{\theta}) \wedge \operatorname{tr}(\boldsymbol{\theta}^3) - \operatorname{tr}(A^2)\,\varrho \wedge \operatorname{tr}(\boldsymbol{\theta}^3)\colon \qquad -3(p^2 - 2q + R) + \tfrac{6}{n}\operatorname{tr}(A^2).$$

For $n \geq 4$ only W_4 contains $a_{34}a_{43}$; then only W_2 contains $a_{13}a_{31}$; and $W_3 \neq 0$ (at $a_{12} = a_{21} = 1$ it is $2 - \frac{4}{n}$). So $\omega_\varphi \in \mathbb{C}\omega_1$. As $\omega_{\varphi_{c_5}} = 60\,\omega_1$ (Step 2) and $\varphi \mapsto \omega_\varphi$ is injective, $V_{\mathrm{FS}} \subseteq \mathbb{C}\varphi_{c_5}$. (Equality also holds: on a $(1,4)$ or $(4,1)$ grid F_1 and F_2 meet two equal vectors; on a $(2,2)$ grid with invertible pairing matrix the four grid matrices span the image of an algebra embedding $\iota\colon M_2 \to M_n$, where F_1 is 0 by the Amitsur–Levitzki theorem and F_2 by the computation on $h_{(2,2)}$, which holds for every 2×2 matrix; degenerate grids follow by continuity. Only the inclusion is used.)

Step 2: φ_{c_5} fails the row (R3) at every s ($n \geq 3$). One computes $\omega_{c_5} := \omega_{\varphi_{c_5}} = 60[\operatorname{tr}(A^2\boldsymbol{\theta}^4) - \operatorname{tr}(A\boldsymbol{\theta}^2 A\boldsymbol{\theta}^2)] = -30\operatorname{tr}([A, \boldsymbol{\theta}^2]^2)$ and $M := -\mathfrak{L}\omega_{c_5} = -60[\operatorname{tr}(A^2\boldsymbol{\theta}^5) + \operatorname{tr}(A\boldsymbol{\theta}A\boldsymbol{\theta}^4) - 2\operatorname{tr}(A\boldsymbol{\theta}^2 A\boldsymbol{\theta}^3)]$.

On $h_{(3,2)}$, with $Z := (\operatorname{tr}\wedge\omega_{c_5})(h_{(3,2)})$,

$$Z/60 = -3(A^2)_{31} + 2a_{11}a_{31} - 2a_{21}a_{32} + 4a_{22}a_{31}, \qquad M(h_{(3,2)})/60 = 6a_{21}a_{32} - 3a_{22}a_{31} + 3a_{11}a_{31}$$

(by the I-links $E_{ij} \to E_{kl}$, $j = k$; also from $Z = \zeta(\varphi_{c_5})$ and Lemma 10.20(b)). $M(h_{(3,2)})$ has no a_{33}, while Z contains $-3a_{33}a_{31}$. At $A = 3E_{22} + 3E_{31} + 4E_{33}$ (allowed, as Z and M are invariant under $A \mapsto A + cI$) we get $Z = 0$ and $M(h_{(3,2)}) = -1620$, so $\Omega f_s(I, A)(h_{(3,2)}) = sZ + M(h_{(3,2)}) = -1620$ for every s. (The $(1,4)$ and $(4,1)$ rows vanish identically on φ_{c_5} (not used); at a block-lower A, $M(h_{(3,2)}) = 0$ and $Z = -180(A^2)_{31}$, as Lemma 10.20 predicts.)

So if $\mu \neq 0$, an eigenvector lies in $V_{\mathrm{FS}} \subseteq \mathbb{C}\varphi_{c_5}$ (Lemma 10.19, Step 1), and Step 2 excludes φ_{c_5}.

(c) Put $(A, A') := (A_2, A_3)$, both traceless (allowed: $\Omega f_s(I, A)$ is unchanged under $A_j \mapsto A_j + cI$, by the proof of Lemma 10.4(b), which uses only $f_s(\dots, P_j + cP_1, \dots) = f_s$). Here ω_φ is bilinear in (A, A') and alternating: $A_3 \mapsto A_3 + tA_2$ commutes with Cay (the extra term has two derivatives in slot 3, alternated), and the highest weight condition gives $\omega(A, A' + tA) = \omega(A, A')$; it is closed in A' for the same reason. The alternating invariants in (A, A') with values in 4-forms on $\mathfrak{sl}_n$ are spanned by $S_1 = \operatorname{tr}([A, A']\boldsymbol{\theta}^4)$, $S_2 = \operatorname{tr}(A\boldsymbol{\theta}A'\boldsymbol{\theta}^3) - (A \leftrightarrow A')$, $P_1 = \operatorname{tr}(A\boldsymbol{\theta})\wedge\operatorname{tr}(A'\boldsymbol{\theta}^3) - (A \leftrightarrow A')$ and $P_2 = \operatorname{tr}([A, A']\boldsymbol{\theta})\wedge\operatorname{tr}(\boldsymbol{\theta}^3)$ ($\operatorname{tr}(A\boldsymbol{\theta}^2A'\boldsymbol{\theta}^2)$ and $\operatorname{tr}(A\boldsymbol{\theta}^2)\wedge\operatorname{tr}(A'\boldsymbol{\theta}^2)$ are symmetric). In A', $dS_1 = dP_1 = 0$, $dS_2 = -2\operatorname{tr}(A\boldsymbol{\theta}^5)$ and $dP_2 = 2\operatorname{tr}(A\boldsymbol{\theta}^2)\wedge\operatorname{tr}(\boldsymbol{\theta}^3)$, independent by Step 1 of (b). So $\omega_\varphi = c_1S_1 + c_2P_1$. As $\Omega f_s(I, A) = s(\operatorname{tr}\wedge\omega) - \mathfrak{L}\omega$ depends only on ω, we compute the rows for S_1 and for P_1; and $\omega_\varphi = 0$ gives $\varphi = 0$.

The pencil. $\mathfrak{L}$ is a graded derivation, and on one trace word it inserts $\boldsymbol{\theta}$ to the left of each letter, with sign $(-1)^r$ (r the number of factors $\boldsymbol{\theta}$ before that letter). On $\operatorname{tr}(A\boldsymbol{\theta}^m)$, m odd, the A-insertion gives $\operatorname{tr}(\boldsymbol{\theta}A\boldsymbol{\theta}^m) = -\operatorname{tr}(A\boldsymbol{\theta}^{m+1})$ and the m insertions at the $\boldsymbol{\theta}$'s give $+\operatorname{tr}(A\boldsymbol{\theta}^{m+1})$; so $\mathfrak{L}$ kills $\operatorname{tr}(A\boldsymbol{\theta})$, $\operatorname{tr}(A'\boldsymbol{\theta}^3)$ and the unprojected P_1, and S_1 has no projection terms ($\boldsymbol{\theta}_0^4 = \boldsymbol{\theta}^4$). The rest of $\mathfrak{L}P_1$ comes from the projections: $\omega = P_1(A_0, A_0')$ with $\boldsymbol{\theta}_0$, and the base derivative goes along CA, CA', which are not traceless. With $\mathfrak{L}\operatorname{tr}A = \operatorname{tr}(A\boldsymbol{\theta})$, $\mathfrak{L}\varrho = 0$, $\mathfrak{L}\operatorname{tr}(A'\boldsymbol{\theta}^2) = \operatorname{tr}(A'\boldsymbol{\theta}^3)$: $\mathfrak{L}\operatorname{tr}(A_0\boldsymbol{\theta}_0) = \varrho\wedge\operatorname{tr}(A\boldsymbol{\theta})$ and $\mathfrak{L}\operatorname{tr}(A_0'\boldsymbol{\theta}_0^3) = -\frac{1}{n}\operatorname{tr}(A'\boldsymbol{\theta})\wedge\operatorname{tr}(\boldsymbol{\theta}^3) + \varrho\wedge\operatorname{tr}(A'\boldsymbol{\theta}^3)$. Hence, with $Z_\omega := \operatorname{tr}\wedge\omega$,

$$\mathfrak{L}S_1 = \operatorname{tr}([A, A']\boldsymbol{\theta}^5) + \operatorname{tr}(A\boldsymbol{\theta}A'\boldsymbol{\theta}^4) - \operatorname{tr}(A'\boldsymbol{\theta}A\boldsymbol{\theta}^4),$$
$$\mathfrak{L}P_1 = \tfrac{2}{n}Z_{P_1} + \tfrac{2}{n}\operatorname{tr}(A\boldsymbol{\theta})\wedge\operatorname{tr}(A'\boldsymbol{\theta})\wedge\operatorname{tr}(\boldsymbol{\theta}^3).$$

The $(1,4)$ rows. With $D := \det(Y^T[u, Au, A'u, Xu])$ ($Y = [y_1, \dots, y_4]$), the rows $g_{1,4}(u, y)\wedge X$ give $-4D$ for S_1 and $(2s - \frac{10}{n})D$ for P_1. So an eigenvector needs $4c_1 = (2s - \frac{10}{n})c_2$; here $2s - \frac{10}{n} = \frac{2}{n}(\mu - 3)$, $\mu = ns - 2$. (These rows need $u, Au, A'u, Xu$ independent, so $n \geq 4$.)

The row (R3). With $[ij, kl] := a_{ij}a'_{kl} - a'_{ij}a_{kl}$ and $C := [A, A']$, $\Omega f_s(I, A)(h_{(3,2)})$ is

$$-3sC_{31} - \big(-2[32, 21] + [31, 22] - [31, 11]\big) \quad \text{for } S_1,$$
$$s\big(4[31, 11] + 2[32, 21] + 2[31, 22]\big) - \tfrac{2}{n}\big([31, 11] - [31, 22] + 2[32, 21]\big) \quad \text{for } P_1.$$

Only the first contains $a_{33}a'_{31}$ (through C_{31}), so $s \neq 0$ forces $c_1 = 0$. Then the second is not 0 at any s: its coefficients on the independent brackets $[31, 11]$, $[32, 21]$, $[31, 22]$ are $(4s - \frac{2}{n}, 2s - \frac{4}{n}, 2s + \frac{2}{n})$, which do not vanish together ($s = \frac{1}{2n}$ against $s = \frac{2}{n}$). So $c_2 = 0$. At $s = 0$, the row (R3) gives $c_1 = \frac{2}{n}c_2$ and the $(1,4)$ rows give $c_1 = -\frac{5}{2n}c_2$, so $c_1 = c_2 = 0$. □

Remark 10.23. (1) A computation modulo the prime 1000003 (from the definition of the pencil; a kernel dimension computed modulo p is only an upper bound, so this is not a proof that $\mu = 0$ occurs) gives a kernel at $\mu = 0$ of dimension 1 at $(2, 1, 1, 1)$ and 3 at $(3, 1, 1, 1)$ for $n = 4, 5$; at $(2, 2, 1, 1)$ it gives $V_{\mathrm{FS}} = \mathrm{HW}_\tau$ of dimension 2 and no kernel, for $n = 4, 5$. (2) Lemma 10.20 holds for every first-layer τ, but it does not decide a weight by itself: it reduces $s \neq 0$ to $\zeta = 0$ on V_{FS}. At $(2, 2, 1, 1)$, the form P_1 is not seen by ζ (at a block-lower A its row (R3) is 0), and the $(1,4)$ rows catch it. A uniform argument on the column $(\tau_2, 1, 1, 1)$, which contains (a) and (b), is Theorem 10.69; for the first-layer weights with $\tau_3 \geq 2$ it is open. (3) At $(3, 1, 1, 1)$, V_{FS} is the line

of $-30\,\mathrm{tr}([A,\boldsymbol{\theta}^2]^2)$. This has the shape of the line $\mathrm{tr}([A,B]^2)$ of Remark 10.11 (Theorem 9.9) for $\nu=(2,1)$, with B replaced by $\boldsymbol{\theta}^2$; unlike that line, it fails the $(2,2)$ row at every s. Conjecture 10.18 is the analogue for $\nu=(3,1,1)$ of Theorem 9.41, and it refines Theorem 10.13 from the leading term $\mathcal{Z}$ to the whole pencil. (4) At $s=n$, $\mu=n^2-d+4\neq 0$. So Theorem 10.22 says that the types κ whose τ is one of the three weights (depth 1 and 2) do not occur in $K_{\lambda(3,1,1)}$, for every $n\geq 4$; for $n\geq 5$ this is also contained in Theorem 10.15(b), and at $n=4$, where $\lambda(3,1,1)=(4,3,3,1)$, in the direct computation of Remark 7.9. So here it is a proof by hand, not a new fact about $K_{\lambda(3,1,1)}$. The same step with Theorem 10.69 gives Corollary 10.71: every weight $(\tau_2,1,1,1)$, that is, every depth, for every $n\geq 3$.

10.7. **The corner dictionary for every ν, and the first layer for $\nu=(3,1)$.** Statement (ii) in the proof of Theorem G is a kernel statement for the conjugate pair $\lambda(3,1)$, $\lambda(2,1,1)$ (Lemma 6.11), and $K_{\lambda(3,1)}=0$ is enough for it. As $(3,1)$ is not self-conjugate, Theorem 10.1 does not apply as stated. We remove the hypothesis $\nu=\nu'$ (Theorem 10.24: the tests are then the live corners of ν'), and for $\nu=(3,1)$ and $n\geq 6$ we exclude from $K_{\lambda(3,1)}$ the conjugation types of six weights of the first layer (Theorem 10.28). This does not settle statement (ii). In this subsection ν' is always the conjugate of ν (in Corollary 7.6(b) it denotes any other partition), and the other blocks are called μ.

Theorem 10.24 (corner dictionary for every ν)**.** *Let $\nu\vdash k$ lie inside R, and let $\lambda=\lambda(\nu)$, $N=n^2-k$, κ and τ be as in Subsection 10.1, without the hypothesis $\nu=\nu'$. Then (a) and (c) of Theorem 10.1 hold, and the isomorphism of (a) maps X_ν onto the space of the $f\in\mathrm{Pol}_\tau$ that pass $(T_{r,c})$ for every live corner (r,c) of ν' (an addable cell of ν' with $rc\leq k$). Hence the multiplicity of κ in K_λ is the dimension of this space.*

Proof. The proof of Theorem 10.1 uses $\nu=\nu'$ in one sentence only: the blocks $\mu\neq\nu$ of Corollary 7.6(b) are, by Lemma 10.2(i), the summands $\alpha=\mu'\neq\nu$. In general the block μ is the summand $\alpha=\mu'$, and the side of the twist decides this. For Z in the image of Ψ the block μ is $Z\rho_R(e_\mu)$, and in Step 4 of the proof of Theorem 9.18 the twist is $Y=\Gamma_R^{-1}Z$, on the left. So $Y\rho_R(e_\mu)=\Gamma_R^{-1}Z\rho_R(e_\mu)$, and by Step 1 there $\Theta(Y\rho_R(e_\mu))=e_\mu*\Theta(Y)$ is e_μ acting on the second factors, which by Lemma 10.2(i) is the summand $\alpha=\mu'$. (With the twist on the right, $\Gamma_R^{-1}\rho_R(e_\mu)=\rho_R(e_{\mu'})\Gamma_R^{-1}$ would turn the block μ into μ'; this is one of the conventions noted after the proof of Theorem 9.18, and for $\nu=\nu'$ it does not matter.) So the blocks $\mu\neq\nu$ are the summands $\alpha\neq\nu'$. Lemma 10.2(iv) holds for every partition (its proof uses only addable cells and the product order); applied to ν' it gives $\bigoplus_{\alpha\neq\nu'}S_\alpha V\otimes S_{\alpha'}V=\sum\Gamma_{r,c}$, over the live corners of ν'. This gives "$\Rightarrow$" as before. For "$\Leftarrow$", $e_\mu*\tilde{x}(w')$ sees only the summand $\alpha=\mu'\neq\nu'$, where the tests make its restitution vanish; the rest of the argument is unchanged. Steps 1, 3 and 4, Lemma 10.3 and part (c) do not involve ν, and ν lies inside R if and only if ν' does. □

For $\nu=(3,1)$ the conjugate $\nu'=(2,1,1)$ has the addable cells $(1,3)$, $(2,2)$, $(4,1)$, with $rc=3,4,4\leq k=4$: all three are live.

The setting. From now on $\nu=(3,1)$, $k=4$, $\lambda=\lambda(\nu)$, $N=n^2-4$ and $n\geq 3$. For a 4-row weight τ put $d:=\tau_2+\tau_3+\tau_4$; call τ *first-layer* if $\tau_4=1$, and name it by (τ_2,τ_3,τ_4). For $\varphi\in\mathrm{HW}_\tau$ and $s\in\mathbb{C}$ let $f_s(P):=\det(P_1)^s\tilde{\varphi}(P_1^{-1}P_2,P_1^{-1}P_3,P_1^{-1}P_4)$ for P_1 near I, as in Subsection 10.6; it is homogeneous of degree $\tau_1:=ns-d$ in P_1, $f_s(\dots,P_j+cP_1,\dots)=f_s$ for $j\geq 2$, and $f_n=f$. Put $\mu:=\tau_1+3=ns-d+3$ (a number); at $s=n$, $\mu=n^2-d+3>0$. For $A=(A_2,A_3,A_4)$ let

$$\omega=\omega_A(X_1,X_2,X_3):=\sum_{\pi\in S_3}\mathrm{sgn}(\pi)\,\partial_2[X_{\pi1}]\,\partial_3[X_{\pi2}]\,\partial_4[X_{\pi3}]\,\tilde{\varphi}(A),$$

an alternating 3-form of degree τ_j-1 in A_j (in the first layer it does not depend on A_4), with $\omega_A(B)=\omega_A(B_0)$ and $\omega_A=\omega_{A_0}$; put $Z_\omega:=\mathrm{tr}\wedge\omega$.

Lemma 10.25 ($k=4$)**.** *Let τ be a 4-row weight, $\varphi\in\mathrm{HW}_\tau$ and $s\in\mathbb{C}$.*

(a) For every A, $\Omega f_s(I,A) = s\,Z_\omega - \mathfrak{L}\omega$, with $\mathfrak{L}$ as in Lemma 10.17 (four arguments, $CA := (CA_2, CA_3, CA_4)$); both sides are unchanged under $A_j \mapsto A_j + cI$.
(b) For all B_1, B_2, B_3, $\Omega f_s(I,A)(B_1,B_2,B_3,I) = -\mu\,\omega_A(B_1,B_2,B_3)$.
(c) For traceless A, $\omega_A(A_2,A_3,A_4) = (\tau_2+2)(\tau_3+1)\tau_4\,\varphi(A)$; so $\varphi \mapsto \omega_\varphi$ is injective.

Proof. (a) is Lemma 10.17 for $k=4$, with the same proof. (b) is the proof of Lemma 10.19 for $k=4$: moving the argument P_1 from position 4 to position 1 takes three transpositions, so the sign is $(-1)^{k-1} = -1$, and each of the three slots $j \geq 2$ adds one $\Omega' f_s$; at $P = (I,A)$, $\Omega' f_s = \omega_A$. (c) is Capelli's identity, as in the proof of Theorem 10.13, for the three columns $2,3,4$: the factor is $\prod_{j=2}^{4}(\tau_j + 4 - j) \neq 0$. □

For the six weights below the factor in (c) is $6, 8, 10, 12, 12, 15$. Let $g_{r,c}(u,y) := \bigwedge_{p\leq r,\, q\leq c} u_p y_q^T$ and $h_{(2,2)} = E_{11}\wedge E_{12}\wedge E_{21}\wedge E_{22}$ be as in Subsection 10.6. We call $\varphi \in \mathrm{HW}_\tau$ a *kernel vector at s* if, for every A, $\Omega f_s(I,A)$ vanishes on the rows (R1) $g_{1,3}(u,y)\wedge X$, for all u, y, X; (R2) $h_{(2,2)}$; (R3) $g_{4,1}(u,y)$, for all u, y. They lie in $\Gamma_{1,3}$, $\Gamma_{2,2}$ and $\Gamma_{4,1}$. If $f \in \mathrm{Pol}_\tau$ passes the three tests, then φ_f is a kernel vector at $s = n$: the tests hold at every P, in particular at $P = (I,A)$, and $f_n = f$ (Theorem 10.1(c)). The rows (R3) are not used below. By Lemma 10.25(b) (rows (R1) with $X = I$), for $\mu \neq 0$ every kernel vector lies in the space V_{FS} of the φ such that ω_A vanishes on every $g_{1,3}(u,y)$, for every A.

Trace forms and the rules for $\mathfrak{L}$. We use $\boldsymbol{\theta}$, ϱ, $\boldsymbol{\theta}_0$, graded cyclicity and the insertion rule for d of Subsection 10.6. By the first fundamental theorem [Pr76], ω is a combination of products of traces of words in the letters $(A_j)_0$ and $\boldsymbol{\theta}_0$; we write such a form with A_j and $\boldsymbol{\theta}$, and a hat $(\hat{w})$ means the same form with $(A_j)_0$ and $\boldsymbol{\theta}_0$. Necessary conditions from $\varphi \in \mathrm{HW}_\tau$: ω is invariant under conjugation, it is closed in each base slot ($d_j\omega = 0$, as two derivatives in slot j commute and are alternated), and $\omega(\dots, A_b + tA_a, \dots) = \omega$ for $2 \leq a < b$ (the extra term is a second derivative in slot b, alternated). At a traceless A, $\mathfrak{L}$ acts on a product w of traces in the letters A_j and $\boldsymbol{\theta}_0$ letter by letter, with the sign $(-1)^r$, where r is the number of factors $\boldsymbol{\theta}_0$ before the letter: a base letter M gives $(-1)^r w[M \to \boldsymbol{\theta}M] - \frac{1}{n}\operatorname{tr}(M\boldsymbol{\theta})\wedge w[M\to I]$ (the base derivative along $(CA_j)_0$), and a letter $\boldsymbol{\theta}_0$ gives $(-1)^r w[\boldsymbol{\theta}_0 \to \boldsymbol{\theta}_0^2]$ (the left multiplication $(CD)_0$ of an argument); here C is alternated, and $\boldsymbol{\theta}_0^2 = \boldsymbol{\theta}^2$. Write $(W_1,W_2,W_3) := \operatorname{tr}(\boldsymbol{\theta}_0 W_1 \boldsymbol{\theta}_0 W_2 \boldsymbol{\theta}_0 W_3)$ and $(V_1,\dots,V_4) := \operatorname{tr}(\boldsymbol{\theta}_0 V_1 \cdots \boldsymbol{\theta}_0 V_4)$ for words W_j, V_j in the base letters (*gaps*); by graded cyclicity $(W_1,W_2,W_3) = (W_2,W_3,W_1)$ and $(V_1,V_2,V_3,V_4) = -(V_2,V_3,V_4,V_1)$. With $\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \varrho I$ the rules give, for a single trace $w = (W_1,W_2,W_3)$,

$$\mathfrak{L}w = (\deg_{\mathrm{base}} w)\,\varrho\wedge w + \mathfrak{D}w - \tfrac{1}{n}\textstyle\sum_M \operatorname{tr}(M\boldsymbol{\theta})\wedge w[M\to I],$$

where $\mathfrak{D}w$ is $\sum_j (-1)^j$ times the sum of the words obtained by splitting W_j into two nonempty words (in four gaps), minus $\sum_{j:\,W_j=\emptyset}(-1)^j$ times the word with W_j replaced by two empty gaps (the terms that split off an empty word cancel against the letters $\boldsymbol{\theta}_0$). On a row of traceless arguments $Z_\omega = 0$ and $\varrho = 0$, so there $\Omega f_s(I,A) = -\mathfrak{L}\omega$ for every s.

Call A *block-lower* if rows 1 and 2 of A_2, A_3, A_4 are 0; let D_j be the lower right $(n-2)\times(n-2)$ block of A_j, $D := (D_2,D_3,D_4)$ and $\eta_j := \operatorname{tr}(D_j)/n$, so that $(A_j)_0 = A_j - \eta_j I$.

Lemma 10.26 (block-lower, $k=4$)**.** *Let τ be a 4-row weight, $n \geq 3$, $s \in \mathbb{C}$, $\varphi \in \mathrm{HW}_\tau$ and A block-lower.*

(a) $\Omega f_s(I,A)(h_{(2,2)}) = -s\,\omega_A(H,E_{12},E_{21})$; no condition on ω is needed.
(b) $\omega_A(H,E_{12},E_{21})$ depends only on D; write it as $6\,c_\varphi(D)$. In a trace-form expression of ω (with hats), a product of two or more traces that contain $\boldsymbol{\theta}_0$ gives 0, a single trace with three factors $\boldsymbol{\theta}_0$ gives 6 times the product of the $-\eta_j$ over its base letters, and a trace of base letters only is evaluated in full.

Proof. (a) Put $Z_1 := \omega(E_{12},E_{21},E_{22})$ and $Z_2 := \omega(E_{11},E_{12},E_{21})$; the arguments are $B_1,\dots,B_4 = E_{11},E_{12},E_{21},E_{22}$. (i) The argument part of $\mathfrak{L}\omega(h_{(2,2)})$. $i=1$: $E_{11}E_{12} = E_{12}$ and $E_{11}E_{21} =$

$E_{11}E_{22}=0$, giving $+Z_1$. $i=2$: $E_{12}E_{11}=0$, $E_{12}E_{21}=E_{11}$ repeats E_{11}, and $E_{12}E_{22}=E_{12}$ gives $\omega(E_{11},E_{21},E_{12})=-Z_2$; with the sign $-$, $+Z_2$. $i=3$: $E_{21}E_{11}=E_{21}$ gives $\omega(E_{21},E_{12},E_{22})=-Z_1$, $E_{21}E_{12}=E_{22}$ repeats E_{22}, and $E_{21}E_{22}=0$: $-Z_1$. $i=4$: $E_{22}E_{11}=E_{22}E_{12}=0$ and $E_{22}E_{21}=E_{21}$ gives Z_2; with the sign $-$, $-Z_2$. The total is 0, for every alternating 3-form ω. (ii) The base part: $B_i=E_{ab}$ with $b\in\{1,2\}$, and $E_{ab}A_j=e_a\,(\text{row } b \text{ of } A_j)=0$. (iii) $Z_\omega(h_{(2,2)})=\operatorname{tr}(E_{11})Z_1-\operatorname{tr}(E_{22})Z_2=Z_1-Z_2=-\omega(H,E_{12},E_{21})$, as $\omega(E_{22},E_{12},E_{21})=Z_1$. Now Lemma 10.25(a) gives (a). (b) Every product of the matrices $(A_j)_0$ and of matrices in the top left 2×2 block is lower block-triangular (blocks of sizes 2 and $n-2$); its top left block is the same product with each $(A_j)_0$ replaced by the scalar $-\eta_j$, and its lower right block is the product with $(A_j)_0\to D_j-\eta_jI$ and the 2×2 matrices replaced by 0. So the trace of a word that contains $\boldsymbol{\theta}_0$, evaluated on H, E_{12}, E_{21}, is the trace of its top left block. A factor $\operatorname{tr}(W\boldsymbol{\theta}_0)$ then gives a scalar times the trace of H, E_{12} or E_{21}, which is 0; a product of two or more traces that contain $\boldsymbol{\theta}_0$ has $\boldsymbol{\theta}_0$-degrees $2+1$ or $1+1+1$, so it has such a factor; and $\operatorname{tr}(\boldsymbol{\theta}^3)(H,E_{12},E_{21})=3\operatorname{tr}(H[E_{12},E_{21}])=6$. □

Corollary 10.27. *If $s\neq0$, every kernel vector φ at s has $c_\varphi\equiv0$ on $(\mathfrak{gl}_{n-2})^3$.*

Proof. Apply Lemma 10.26 to the row (R2); the blocks D_j of a block-lower A are arbitrary. □

Theorem 10.28 (six weights of the first layer for $\nu=(3,1)$)**.** *(a) At $\tau=(1,1,1)$ and $(2,1,1)$ for $n\geq3$, and at $(3,1,1)$ for $n\geq4$, there is no nonzero kernel vector at any $s\neq0$.*
(b) At $\tau=(2,2,1)$, for $n\geq5$, a nonzero kernel vector at s has $ns=2$ (that is, $\mu=0$).
(c) At $\tau=(4,1,1)$, for $n\geq6$, there is no nonzero kernel vector at any $s\neq0$.
(d) At $\tau=(3,2,1)$, for $n\geq6$, there is no nonzero kernel vector at any s with $s\neq0$ and $\mu=ns-3\neq0$.

So for $n\geq6$ none of these six weights has a nonzero kernel vector at $s=n$, and the conjugation types κ whose τ is one of them do not occur in $K_{\lambda(3,1)}$.

Proof. By Lemma 10.25(c) it suffices to show $\omega_\varphi=0$; in the first layer ω depends only on $A:=A_2$ and $A':=A_3$. The last sentence follows from (a)–(d): at $s=n$ we have $s\neq0$, $\mu=n^2-d+3\neq0$ and $ns\neq2$; and if κ occurs, then by Theorem 10.24 (only the direction "$x\in X_\nu$ implies that f passes every test" is used) some nonzero $f\in\operatorname{Pol}_\tau$ passes the tests, and $\varphi_f\neq0$ (Theorem 10.1(c)) is a kernel vector at $s=n$.

$\tau=(1,1,1)$. Here $\omega=c\operatorname{tr}(\boldsymbol{\theta}_0^3)$, so $c_\varphi=c$, and Corollary 10.27 gives $c=0$.

$\tau=(2,1,1)$. Here ω is linear in A. The only invariant 3-form on $\mathfrak{sl}_n$ linear in A is $\operatorname{tr}(A\boldsymbol{\theta}^3)$ (a one-letter trace is 0, $\operatorname{tr}(\boldsymbol{\theta}^2)=0$, and the other places of A give the same trace by graded cyclicity), so $\omega=a\widehat{\operatorname{tr}(A\boldsymbol{\theta}^3)}$. By Lemma 10.26(b), $c_\varphi=-a\eta_2$; as $\operatorname{tr}D_2\not\equiv0$, $a=0$.

$\tau=(3,1,1)$, $n\geq4$. The invariant 3-forms quadratic in A are spanned by $G_1=\operatorname{tr}(A^2\boldsymbol{\theta}^3)$, $G_2=\operatorname{tr}(A\boldsymbol{\theta}A\boldsymbol{\theta}^2)$, $G_3=\operatorname{tr}(A\boldsymbol{\theta})\wedge\operatorname{tr}(A\boldsymbol{\theta}^2)$ and $G_4=\operatorname{tr}(A^2)\operatorname{tr}(\boldsymbol{\theta}^3)$. By the insertion rule $dG_1=dG_2=0$, $dG_3=-\operatorname{tr}(A\boldsymbol{\theta})\wedge\operatorname{tr}(\boldsymbol{\theta}^3)$ and $dG_4=2\operatorname{tr}(A\boldsymbol{\theta})\wedge\operatorname{tr}(\boldsymbol{\theta}^3)\neq0$; so $\omega=c_1\hat G_1+c_2\hat G_2+c_3(\hat G_3+\frac12\hat G_4)$. By Lemma 10.26(b), with $\eta:=\eta_2$,

$$c_\varphi=(c_1+c_2)\,\eta^2+\tfrac12c_3\big(\operatorname{tr}D_2^2-(\operatorname{tr}D_2)^2/n\big);$$

for $n-2\geq2$ the two functions of D_2 are independent, so $c_3=0$ and $c_2=-c_1$, and ω is a multiple of $G:=\hat G_1-\hat G_2$. In $\boldsymbol{\theta}$ the ϱ-terms cancel, $G=G_1(A_0)-G_2(A_0)$: as $\boldsymbol{\theta}_0^2=\boldsymbol{\theta}^2$, both $\hat G_1-G_1(A_0)$ and $\hat G_2-G_2(A_0)$ equal $-\varrho\wedge\operatorname{tr}(A_0^2\boldsymbol{\theta}^2)$. Take the traceless $A=E_{13}+E_{31}$, so $A^2=E_{11}+E_{33}$. On arguments in the top left 2×2 block G_2 vanishes, as $BAB'=0$ for such B, B'; so $Z_\omega(h_{(2,2)})=G(E_{12},E_{21},E_{22})-G(E_{11},E_{12},E_{21})=-1-2=-3$. In $\mathfrak{L}G(h_{(2,2)})$ the argument part is 0 (step (i) of the proof of Lemma 10.26), and the base part is $\partial_{E_{13}}G_1(E_{12},E_{21},E_{22})+\partial_{E_{23}}G_1(E_{11},E_{12},E_{22})=-1+1=0$: here $E_{11}A=E_{13}$, $E_{21}A=E_{23}$, $E_{12}A=E_{22}A=0$, the derivative of A^2 along E_{13} is $E_{11}+E_{33}$ and along E_{23} it is E_{21}, and G_2 gives 0 as before. So $\Omega f_s(I,A)(h_{(2,2)})=-3s\neq0$, and $\omega=0$.

$\tau=(2,2,1)$, $n\ge5$. (i) ω is bilinear in (A,A'), and E_{23} gives $\omega(A,A)=0$, so it is alternating. It is a combination of the antisymmetrizations of the trace monomials $(AA',\emptyset,\emptyset)$, $(A'A,\emptyset,\emptyset)$, $(A,A',\emptyset)$, $(A',A,\emptyset)$, $\operatorname{tr}(AA')\operatorname{tr}(\boldsymbol\theta^3)$, $\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(A'\boldsymbol\theta^2)$ and $\operatorname{tr}(A'\boldsymbol\theta)\wedge\operatorname{tr}(A\boldsymbol\theta^2)$; these are spanned by $U_0=\operatorname{tr}([A,A']\boldsymbol\theta^3)$, $U_1=(A,A',\emptyset)-(A',A,\emptyset)$ and $\Pi'=\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(A'\boldsymbol\theta^2)-(A\leftrightarrow A')$ (only spanning is used). (ii) In A', $dU_0=2\operatorname{tr}(A\boldsymbol\theta^4)$, $dU_1=-2\operatorname{tr}(A\boldsymbol\theta^4)$ and $d\Pi'=-\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(\boldsymbol\theta^3)$; these two 4-forms are independent on $\mathfrak{sl}_n$ for $n\ge5$ (Lemma 10.6, five letters), so closedness in A' gives $\omega=c\,(\hat U_0+\hat U_1)$. (iii) At a traceless A, by the rules: $\deg_{\mathrm{base}}=2$ gives $2\varrho\wedge\omega$; $\mathfrak D(AA',\emptyset,\emptyset)=-(A,A',\emptyset,\emptyset)=-(\emptyset,\emptyset,A,A')$ (the two empty-gap terms cancel) and $\mathfrak D(A,A',\emptyset)=(A,A',\emptyset,\emptyset)=(\emptyset,\emptyset,A,A')$, so $\mathfrak DU_0=-(\emptyset,\emptyset,A,A')+(\emptyset,\emptyset,A',A)=-\mathfrak DU_1$ and $\mathfrak D\omega=0$. The $\frac1n$-terms of $(AA',\emptyset,\emptyset)$ and $(A'A,\emptyset,\emptyset)$ are the same, $\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(A'\boldsymbol\theta^3)+\operatorname{tr}(A'\boldsymbol\theta)\wedge\operatorname{tr}(A\boldsymbol\theta^3)$, so they cancel in U_0, and likewise in U_1. Hence $\mathfrak L\omega=2\varrho\wedge\omega=\frac2nZ_\omega$, and by Lemma 10.25(a) $\Omega f_s(I,A)=(s-\frac2n)Z_\omega$ on every row, at every A. (iv) With $[ij,kl]:=a_{ij}a'_{kl}-a'_{ij}a_{kl}$, for $\hat U_0+\hat U_1$

$$Z_\omega(h_{(2,2)})=-3\textstyle\sum_{m\ge3}\big([1m,m1]+[2m,m2]\big)\not\equiv0.$$

So $c\ne0$ forces $s=\frac2n$.

$\tau=(4,1,1)$, $n\ge6$. (a) The invariant cubic 3-forms are spanned by the single traces $S_1=(\emptyset,\emptyset,A^3)$, $S_2=(A,\emptyset,A^2)$, $S_3=(A^2,\emptyset,A)$, $S_4=(A,A,A)$ and the products $\Pi_1=\operatorname{tr}(A^3)\operatorname{tr}(\boldsymbol\theta^3)$, $\Pi_2=\operatorname{tr}(A^2)\operatorname{tr}(A\boldsymbol\theta^3)$, $\Pi_3=\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(A^2\boldsymbol\theta^2)$, $\Pi_4=\operatorname{tr}(A^2\boldsymbol\theta)\wedge\operatorname{tr}(A\boldsymbol\theta^2)$. With $V_1=\operatorname{tr}(A^2\boldsymbol\theta^4)$, $V_2=\operatorname{tr}(A\boldsymbol\theta A\boldsymbol\theta^3)$, $V_3=\operatorname{tr}(A\boldsymbol\theta^2A\boldsymbol\theta^2)$, $V_4=\operatorname{tr}(A^2\boldsymbol\theta)\wedge\operatorname{tr}(\boldsymbol\theta^3)$ and $V_5=\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(A\boldsymbol\theta^3)$, the insertion rule gives $dS_1=V_2$, $dS_2=-V_1-V_2+V_3$, $dS_3=V_1-V_2-V_3$, $dS_4=3V_2$, $d\Pi_1=3V_4$, $d\Pi_2=2V_5$, $d\Pi_3=-2V_5$, $d\Pi_4=-V_4$. For $n\ge6$ there is no trace relation in degree 6 (Lemma 10.6), so the closed forms are spanned by $\Phi_1=2S_1+S_2+S_3$, $\Phi_2=S_4-3S_1$, $\Phi_3=\Pi_1+3\Pi_4$ and $\Phi_4=\Pi_2+\Pi_3$. (b) By Lemma 10.26(b), with $D:=D_2$ and $\eta:=\eta_2$, the values of c_φ at $\hat\Phi_1,\dots,\hat\Phi_4$ are $-4\eta^3$, $2\eta^3$, $\operatorname{tr}D^3-3\eta\operatorname{tr}D^2+2n\eta^3$ and $-\eta\operatorname{tr}D^2+n\eta^3$. As $n-2\ge3$, $\omega=\sum_i k_i\hat\Phi_i$ with $c_\varphi\equiv0$ has $k_3=k_4=0$ and $k_2=2k_1$; so ω is a multiple of $\Phi:=\hat\Phi_1+2\hat\Phi_2=-4\hat S_1+\hat S_2+\hat S_3+2\hat S_4$. (c) Take the traceless row $(E_{12},E_{13},E_{14},E_{41})=g_{1,3}(e_1;e_2,e_3,e_4)\wedge E_{41}$ (a row (R1)) and the shift A ($Ae_k=e_{k+1}$). By the rules, $\mathfrak D\Phi=5\big((A^2,A,\emptyset,\emptyset)-(A^2,\emptyset,\emptyset,A)\big)$, which is 0 on the row (both words give -1), and $\sum_M\Phi[M\to I]=-10G_1+10G_2$, with G_1, G_2 as in the case $(3,1,1)$; as $(\operatorname{tr}(A\boldsymbol\theta)\wedge(G_1-G_2))(\mathrm{row})=(G_1-G_2)(E_{13},E_{14},E_{41})=1$, we get $\mathfrak L\Phi(\mathrm{row})=\frac{10}n$. (From the definition directly: the base derivatives go along $E_{12}A=E_{11}$, $E_{13}A=E_{12}$, $E_{14}A=E_{13}$ and $E_{41}A=0$, and the three base terms are $-3+\frac{10}n$, $-(-3)$ and 0; the argument terms are $-\frac6n-(1-1-\frac6n)=0$; the total is $\frac{10}n$.) So $\Omega f_s(I,A)(\mathrm{row})=-\frac{10}n\ne0$ for every s.

$\tau=(3,2,1)$, $n\ge6$. (a) Here ω has degree 2 in A and 1 in A'. The single traces are

$$S_1,\dots,S_5=(AAA',\emptyset,\emptyset),\ (AA'A,\emptyset,\emptyset),\ (A'AA,\emptyset,\emptyset),\ (AA,A',\emptyset),\ (AA',A,\emptyset),$$
$$S_6,\dots,S_{10}=(A'A,A,\emptyset),\ (A',AA,\emptyset),\ (A,AA',\emptyset),\ (A,A'A,\emptyset),\ (A,A,A'),$$

and the products are $\Pi_0=\operatorname{tr}(A^2A')\operatorname{tr}(\boldsymbol\theta^3)$, $\Pi_a=\operatorname{tr}(A^2)\operatorname{tr}(A'\boldsymbol\theta^3)$, $\Pi_b=\operatorname{tr}(AA')\operatorname{tr}(A\boldsymbol\theta^3)$, $Q_1=\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(\boldsymbol\theta A\boldsymbol\theta A')$, $Q_2=\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(AA'\boldsymbol\theta^2)$, $Q_3=\operatorname{tr}(A\boldsymbol\theta)\wedge\operatorname{tr}(A'A\boldsymbol\theta^2)$, $Q_4=\operatorname{tr}(A'\boldsymbol\theta)\wedge\operatorname{tr}(A^2\boldsymbol\theta^2)$, $Q_5=\operatorname{tr}(A^2\boldsymbol\theta)\wedge\operatorname{tr}(A'\boldsymbol\theta^2)$, $Q_6=\operatorname{tr}(AA'\boldsymbol\theta)\wedge\operatorname{tr}(A\boldsymbol\theta^2)$ and $Q_7=\operatorname{tr}(A'A\boldsymbol\theta)\wedge\operatorname{tr}(A\boldsymbol\theta^2)$; write $\omega=\sum s_iS_i+p_0\Pi_0+p_a\Pi_a+p_b\Pi_b+\sum q_iQ_i$. (b) The equations below are read off coefficients in bases of independent forms (Lemma 10.6, $n\ge6$). The condition E_{23} ($A'\to A$ gives 0 in the basis of the case $(4,1,1)$): $s_1+s_2+s_3=s_4+s_5+s_6=s_7+s_8+s_9=s_{10}=p_0=p_a+p_b=q_2+q_3+q_4=q_5+q_6+q_7=0$. Closed in A': $s_1-s_3-s_4+s_7=s_2-s_6-s_8+s_{10}=s_5-s_9=0$, $p_0=q_5$, $p_b=q_1+q_2+q_3$, $q_6=q_7$. Closed in A: $s_1-s_2+s_5-s_8=s_2-s_3+s_6-s_9=-s_1+s_4+s_9-s_{10}=-s_4-s_6+s_7+s_8=s_3-s_5-s_7+s_{10}=0$, $2p_a+q_1-q_2-q_3=0$, $p_b=2q_4$, $p_0=q_6=q_7$. The solutions form a space of dimension 4,

$$\omega=s_1\Psi_1+s_3\Psi_3+t\,(-2\Pi_a+2\Pi_b+3Q_1+Q_4-Q_3)+v\,(Q_2-Q_3),$$

where Ψ_1 and Ψ_3 are the single-trace forms with the coefficient vectors $(1,-1,0,3,-2,-1,2,0,-2,0)$ and $(0,-1,1,2,-2,0,3,-1,-2,0)$. (c) By Lemma 10.26(b), with $\eta:=\eta_2$ and $\eta':=\eta_3$: the single

traces give $-(\sum s_i)\eta^2\eta' = 0$ (on the solutions $\sum s_i = 0$); every Q_i has a factor $\operatorname{tr}(W\boldsymbol{\theta})$ and gives 0; Π_a gives $-\eta'(\operatorname{tr} D_2^2 - n\eta^2)$ and Π_b gives $-\eta(\operatorname{tr}(D_2D_3) - n\eta\eta')$. So

$$c_\varphi = \tfrac{2t}{n}\big(\operatorname{tr} D_3 \operatorname{tr} D_2^2 - \operatorname{tr} D_2 \operatorname{tr}(D_2D_3)\big),$$

which is not identically 0 for $n-2 \geq 2$ ($D_2 = \operatorname{diag}(1,0,\dots)$, $D_3 = \operatorname{diag}(0,1,0,\dots)$ give 1); so $t = 0$. (d) As $\mu \neq 0$, $\varphi \in V_{\mathrm{FS}}$. Take grids $g_{1,3}(u;y)$ with $y_2^Tu = y_3^Tu = 0$, and put $[v_1,v_2,v_3] := \det(Y^T[v_1\,v_2\,v_3])$ with $Y = (y_1\,y_2\,y_3)$. On such a grid ϱ vanishes on uy_2^T and uy_3^T, and the hat corrections of a single trace (W_1,W_2,W_3) are $-\frac{y_1^Tu}{n}\big(\langle W_1|W_2W_3\rangle + \langle W_2|W_3W_1\rangle + \langle W_3|W_1W_2\rangle\big)$, where $\langle P|Q\rangle := (y_2^TPu)(y_3^TQu) - (y_3^TPu)(y_2^TQu)$ is antisymmetric and vanishes if P or Q is empty. So they vanish for $S_1,\dots,S_9$, each of which has an empty gap; $s_{10} = 0$; and $\Upsilon := Q_2 - Q_3 = \operatorname{tr}(A\boldsymbol{\theta}) \wedge \operatorname{tr}([A,A']\boldsymbol{\theta}^2)$ needs no correction. Hence on these grids

$$\omega = (s_4 - s_7)[u, A^2u, A'u] + (s_8 - s_5 - v)[u, Au, AA'u] + (s_9 - s_6 + v)[u, Au, A'Au].$$

The three brackets are independent on these grids ($n \geq 4$): with $u = y_1 = e_1$, $y_2 = e_2$, $y_3 = e_3$, the pairs $(A,A') = (E_{21}+E_{32}, E_{21})$, $(E_{21}+E_{32}, E_{32})$, $(E_{21}+E_{34}, E_{41})$ give $(-1,1,0)$, $(0,0,1)$, $(0,1,0)$. So $s_1 = s_3$ and $v = 3s_1$, and ω is a multiple of $\Psi := \Psi_1 + \Psi_3 + 3\Upsilon$. (e) Take the traceless row $(E_{12}, E_{13}, E_{14}, E_{41})$, the shift A and $A' = E_{32}$. By the rules (on this row only the four-gap words $(A'A, \emptyset, \emptyset, A)$ and $(A'A, A, \emptyset, \emptyset)$ that occur are nonzero, both -1), and also term by term from the definition,

$$\mathfrak{L}\Big(\sum_i s_iS_i\Big)(\mathrm{row}) = s_9 + s_3 - s_2 - s_6 - \tfrac{1}{n}(s_2 + 2s_3 + s_6 + s_9), \qquad \mathfrak{L}\Upsilon(\mathrm{row}) = 1.$$

For Ψ this is $0 + \frac{5}{n} + 3$, so $\Omega f_s(I,A)(\mathrm{row}) = -(3 + \frac{5}{n}) \neq 0$ for every s. □

Remark 10.29. (1) The proofs use only the rows (R1) and (R2), at every A, and the necessary conditions on ω listed before Lemma 10.26. For $n \geq d$ the dimension of $\mathrm{HW}_\tau(\mathfrak{sl}_n)$ is $1,1,3,1,4,4$ at $(1,1,1),(2,1,1),(3,1,1),(2,2,1),(4,1,1),(3,2,1)$ (the alternating sum $\sum_\gamma (-1)^{d-|\gamma|}\sum_\rho \chi_\gamma(\rho)$, over the γ with $(\tau_2,\tau_3,\tau_4)/\gamma$ a vertical strip and the classes ρ; a cross-check, not used); in each case it is the dimension of the space cut out by these conditions in the proof. (2) Lemma 10.26(a) needs no condition on ω: for $k = 4$ the argument part of $\mathfrak{L}\omega$ on $h_{(2,2)}$ cancels for every alternating 3-form, while for $k = 5$ Lemma 10.20 needs V_{FS} (step (iii) of its proof). (3) The range in (b). Steps (i), (iii) and (iv) of the case $(2,2,1)$ need only $n \geq 3$, and step (ii) needs only that $\operatorname{tr}(A\boldsymbol{\theta}^4)$ and $\operatorname{tr}(A\boldsymbol{\theta}) \wedge \operatorname{tr}(\boldsymbol{\theta}^3)$ are independent on $\mathfrak{sl}_n$. An exact machine computation gives this (rank 2) also at $n = 3,4$; with it, (b) would hold for every $n \geq 3$. This rests on a machine computation and is not claimed. (4) Not claimed: $s = 0$ in (a) and (c); $s = 0$ or $\mu = 0$ in (d); $n = 4,5$ in (c) and (d); any weight with $d \geq 7$; any τ with $\tau_4 \geq 2$ (the second-layer weight $(2,2,2)$, also with $d = 6$, is open). An exact machine check at $n = 4,5,6$ confirms Lemma 10.26 at random block-lower points, the computations (b) and (c) of the case $(4,1,1)$, and the row $-3s$ of the case $(3,1,1)$. (5) Theorem 10.28 is a first step on statement (ii) of Theorem G: it treats six weights of the first layer at $s = n$, not the whole layer, and Lemma 6.11 asks less than $K_{\lambda(3,1)} = 0$.

10.8. **The leading term for $\nu = (3,1)$, and the two-row types.** We keep $\nu = (3,1)$, $k = 4$, $N = n^2 - 4$ and the notation of Subsection 10.7. We write the three corner tests at one point as one equation (Lemma 10.30), determine the pointwise kernel $\mathcal{K}_{(3,1)}(n)$ of the leading term $\mathcal{Z}$ (Theorem 10.33; unlike $\mathcal{K}_{(3,1,1)}(n)$ in Theorem 10.12, it is not 0), show that $\mathcal{Z}$ is nevertheless injective on the column of the weights $(\tau_2,1,1)$ (Theorem 10.35 and the paragraph after it), and deduce that $K_{\lambda(3,1)}$ has no conjugation type $\kappa' = (N-j,j)$ with $1 \leq j \leq n-3$ (Corollary 10.36).

The content operator. For $1 \leq i < j \leq 4$ let σ_{ij} exchange the first factors of the i-th and j-th tensor factors of $M_n^{\otimes 4}$ (on rank-one matrices, $(u_iy_i^T, u_jy_j^T) \mapsto (u_jy_i^T, u_iy_j^T)$), and put $C := \sum_{i<j}\sigma_{ij}$. A single σ_{ij} does not preserve $\Lambda^4 M_n$, but C, the class sum of the transpositions acting on the first factors, commutes with the alternation. By Lemma 10.2(i) and the classical action of this class

sum by the content sum $c(\alpha)$ [Ju74, Mu81], C acts on the summand $S_\alpha V \otimes S_{\alpha'} V$ of $\Lambda^4 M_n$ by $c(\alpha)$, that is, by $6, 2, 0, -2, -6$ for $\alpha = (4), (3,1), (2,2), (2,1,1), (1^4)$: five distinct values. On forms, C acts by $T \mapsto T \circ C$; C_3, the same operator with three factors, acts on forms on $\Lambda^3 M_n$.

Lemma 10.30 (content form of the tests)**.** *Let $f \in \mathrm{Pol}_\tau$ and let P be a point. The values at P of the three tests $(T_{1,3})$, $(T_{2,2})$, $(T_{4,1})$ all vanish if and only if $\Omega f(P) \circ C = -2\,\Omega f(P)$.*

Proof. By Lemma 10.2(iii) and (iv), applied to $\nu' = (2,1,1)$ as in the proof of Theorem 10.24, $\Gamma_{1,3} + \Gamma_{2,2} + \Gamma_{4,1} = \bigoplus_{\alpha \neq \nu'} S_\alpha V \otimes S_{\alpha'} V$, where α labels the factor of the vectors u. So the tests at P pass if and only if $\Omega f(P)$ vanishes on $\bigoplus_{\alpha \neq \nu'}$. As the contents are distinct, the forms that vanish there form the $c(\nu')$-eigenspace of $T \mapsto T \circ C$, and $c((2,1,1)) = -2$. □

The Chevalley–Eilenberg differential. For a p-form η on M_n let

$$\delta\eta(B_1, \dots, B_{p+1}) := \textstyle\sum_{i<j} (-1)^{i+j}\, \eta([B_i, B_j], B_1, \dots, \hat{B}_i, \dots, \hat{B}_j, \dots, B_{p+1}),$$

the differential of the Chevalley–Eilenberg complex of $\mathfrak{gl}_n$ with trivial coefficients [CE48], and let ι_X insert X in the first slot (this is not the contraction ι of Subsection 10.3). Cartan's formula $\iota_X \delta + \delta \iota_X = \theta(X)$ holds, with θ the coadjoint action on forms [Kos50]; $\theta(I) = 0$.

Lemma 10.31 (commutation)**.** *For a 3-form ω on M_n, $(\mathrm{tr} \wedge \omega) \circ C = \mathrm{tr} \wedge (\omega \circ C_3) - \delta\omega$.*

Proof. Evaluate on rank-one matrices $B_i = u_i y_i^T$. In $(\mathrm{tr} \wedge \omega) \circ \sigma_{ij} = \sum_k (-1)^{k-1} \mathrm{tr}(B'_k)\, \omega(\hat{B}'_k)$, where the B' are the B with u_i and u_j exchanged, the terms with $k \notin \{i,j\}$ are $\mathrm{tr}(B_k)\,(\omega \circ \sigma_{ij})(\hat{B}_k)$; summed over the pairs they give $\mathrm{tr} \wedge (\omega \circ C_3)$. For $k = i$, $\mathrm{tr}(u_j y_i^T)\, u_i y_j^T = B_i B_j$ stands in the place of B_j; for $k = j$, $B_j B_i$ stands in the place of B_i. Summed over all pairs, these terms are the argument part $\mathfrak{M}\omega$ of $\mathfrak{L}\omega$ (the sum over m in Lemma 10.17, with B_i in the role of the letter C there), and grouping the terms (i,m) and (m,i) gives $\mathfrak{M}\omega = \sum_{i<m} (-1)^{i+m-1} \omega([B_i, B_m], \dots) = -\delta\omega$. □

Let $\mathcal{K}_{(3,1)}(n)$ be the space of the 3-forms ω on M_n with $\omega(I \wedge \cdot) = 0$ such that $\mathrm{tr} \wedge \omega$ vanishes on $\bigoplus_{\alpha \neq \nu'} S_\alpha V \otimes S_{\alpha'} V$, that is (as in Lemma 10.30), $(\mathrm{tr} \wedge \omega) \circ C = -2\,\mathrm{tr} \wedge \omega$. As $\omega(I \wedge \cdot) = 0$, ω is determined by its restriction to $\mathfrak{sl}_n$, and we identify the two. Let Z^3 and $B^3 = \delta(\Lambda^2 \mathfrak{sl}_n^*)$ be the cocycles and the coboundaries of degree 3 of $\mathfrak{sl}_n$.

Proposition 10.32. *$\mathcal{K}_{(3,1)}(n)$ is the space of the $\omega \in Z^3$ such that $\omega \circ C_3 + 2\omega$ vanishes on $\Lambda^3 \mathfrak{sl}_n$.*

Proof. By Lemma 10.31 the condition reads $\mathrm{tr} \wedge \eta = \delta\omega$, with $\eta := \omega \circ C_3 + 2\omega$. Apply ι_I: by Cartan's formula $\iota_I \delta\omega = -\delta \iota_I \omega + \theta(I)\omega = 0$, and $\iota_I(\mathrm{tr} \wedge \eta) = n\eta - \mathrm{tr} \wedge \iota_I \eta$. So $\eta = \mathrm{tr} \wedge (\iota_I \eta)/n$, hence $\mathrm{tr} \wedge \eta = 0$, $\delta\omega = 0$, and η vanishes on $\Lambda^3 \mathfrak{sl}_n$. Conversely, these give $\mathrm{tr} \wedge \eta = 0 = \delta\omega$. □

Theorem 10.33. *For every $n \geq 3$, $\mathcal{K}_{(3,1)}(n) = \delta(X) \cong X$, where $X \subset \Lambda^2 \mathfrak{sl}_n^*$ is the irreducible summand generated by $e^{n-1,1} \wedge e^{n,1}$ ($e^{ab}(Y) := Y_{ab}$), of highest weight $2\epsilon_1 - \epsilon_{n-1} - \epsilon_n$ for the diagonal torus. In particular $\dim \mathcal{K}_{(3,1)}(n) = (n^2-1)(n^2-4)/4$.*

Proof. By Proposition 10.32, $\mathcal{K}_{(3,1)}(n)$ is the kernel on Z^3 of $\bar{C} + 2$, where $\bar{C}\omega$ is the restriction of $\omega \circ C_3$ to $\Lambda^3 \mathfrak{sl}_n$. Classically $H^1(\mathfrak{sl}_n) = H^2(\mathfrak{sl}_n) = 0$, $H^3(\mathfrak{sl}_n) = \mathbb{C}\kappa_3$ with $\kappa_3(X_1, X_2, X_3) = \mathrm{tr}(X_1[X_2, X_3])$, and $\Lambda^2 \mathfrak{sl}_n = \mathfrak{sl}_n \oplus X \oplus X^*$. So $Z^3 = \mathbb{C}\kappa_3 \oplus B^3$ and $B^3 \cong \Lambda^2 \mathfrak{sl}_n^* / \mathfrak{sl}_n^* \cong X \oplus X^*$, a multiplicity-free decomposition. As $\bar{C} + 2$ is GL_n-equivariant, by Schur's lemma its kernel on Z^3 is a sum of some of the summands $\mathbb{C}\kappa_3$, $\delta(X)$, $\delta(X^*)$, and (i)–(iii) show that $\bar{C}$ acts on them by 0, -2, 2. (i) κ_3: as $\sum_{a,b} E_{ab} M E_{ba} = \mathrm{tr}(M) I$, every exchanged term of κ_3 splits off the trace of a traceless matrix, so $\bar{C}\kappa_3 = 0$. (ii) X: let $p = n-1$, $q = n$, $\beta = e^{p1} \wedge e^{q1}$ and $\omega := \delta\beta = -S_p + S_q$, with $S_p = \sum_c e^{pc} \wedge e^{c1} \wedge e^{q1}$ and $S_q = \sum_c e^{qc} \wedge e^{c1} \wedge e^{p1}$. On a wedge monomial C_3 acts as the sum, over the pairs of factors, of the exchange of their row indices; this gives $S_p \circ C_3 = \mathrm{tr} \wedge e^{p1} \wedge e^{q1} + S_q - S_p$ and $S_q \circ C_3 = -\mathrm{tr} \wedge e^{p1} \wedge e^{q1} + S_p - S_q$, so $\omega \circ C_3 = -2\omega - 2\,\mathrm{tr} \wedge e^{p1} \wedge e^{q1}$ and $\bar{C}\omega = -2\omega$. (iii) X^*: the transpose $B \mapsto B^T$ exchanges the row exchanges with the column exchanges, and on alternating forms these differ by a sign; so it turns C into $-C$ and X into X^*, and $\bar{C} = 2$ on $\delta(X^*)$. □

Remark 10.34. (1) Two consistency checks and a count, not used. Every $\omega \in \mathcal{K}_{(3,1)}(n)$ vanishes on the grids $g_{1,3}(u,y)$: with $g_q := u\,y_q^T$, the factors of $g_{1,3}(u,y)$, and as $\omega(I \wedge \cdot) = 0$, $(\operatorname{tr} \wedge\omega)(g_1,g_2,g_3,I) = -n\,\omega(g_1,g_2,g_3)$, and $g_{1,3}(u,y) \wedge I \in \Gamma_{1,3}$. The form $\delta\beta$ of (ii) does; its transpose does not, which fixes which of X, X^* is kept, independently of signs. And $\omega(H, E_{12}, E_{21}) = 0$ for every $n \geq 3$, as $Z_\omega(h_{(2,2)}) = -\omega(H, E_{12}, E_{21})$ (proof of Lemma 10.26) and $h_{(2,2)} \in \Gamma_{2,2}$ require; for $\omega = \delta\beta$ of (ii) every monomial has a factor e^{q1} or e^{qc} with row $q = n \geq 3$, which vanishes on the top left 2×2 block. The zero-weight space of X has dimension $(n-1)(n-2)/2$. (2) For $k = 3$ the same computation gives $\mathcal{K}_{(2,1)}(n) = B^2 = \delta(\mathfrak{sl}_n^*)$, the adjoint representation, as in Remark 10.11. (3) $\mathcal{K}_{(3,1)}(n) \neq 0$ concerns the leading term $\mathcal{Z}$ only. It does not say that $K_{\lambda(3,1)} \neq 0$ (Remark 6.12 leaves this open); it says that the proof of Theorem 10.13 does not carry over to $\nu = (3,1)$.

Theorem 10.35 ($\mathcal{Z}$ on the column $(\tau_2,1,1)$)**.** *Let $n \geq 4$, $\tau_2 \geq 1$ and let τ be the first-layer weight $(\tau_2,1,1)$. If $\varphi \in \mathrm{HW}_\tau$ and $\omega_A \in B^3$ for every traceless A, then $\varphi = 0$. In particular, if $\omega_A \in \mathcal{K}_{(3,1)}(n)$ for every traceless A, then $\varphi = 0$ (Theorem 10.33).*

Proof. Write $A := A_2$. As φ is linear in A_3 and in A_4, $\eta_A(Y,W) := \tilde\varphi(A,Y,W)$ is a bilinear form of degree τ_2 in A; the highest weight conditions with $(a,b) = (3,4)$ and $(2,3)$ make it alternating and give $\iota_A\eta_A = 0$. With d the exterior derivative in A (so $\boldsymbol{\theta} = dA$), $\omega = 2\,d\eta$, and $\iota_A\omega_A = 2(\tau_2+2)\eta_A$: by Cartan's formula $\iota_\xi d\eta = L_\xi\eta - d\iota_\xi\eta$ for the Euler field ξ (with value A at A), where $L_\xi\eta = (\tau_2+2)\eta$ (coefficients of degree τ_2, form degree 2) and $\iota_\xi\eta = 0$.

(1) As $\omega_A \in B^3$, $\omega_A = \delta\beta_A$ with β_A polynomial and equivariant in A (an equivariant right inverse of δ exists by complete reducibility). Equivariance under the stabilizer of A gives $\theta(A)\beta_A = 0$, and Cartan's formula gives $\iota_A\omega_A = -\delta(\iota_A\beta_A)$. So $\eta_A = \delta\gamma_A$ with $\gamma_A := -\iota_A\beta_A/(2(\tau_2+2))$.

(2) Hence $\tilde\varphi(A,Y,W) = -\gamma_A([Y,W]) = -\operatorname{tr}(G(A)[Y,W])$ for an equivariant polynomial map $G\colon \mathfrak{sl}_n \to \mathfrak{sl}_n$, and $G(A) = \sum_{k=1}^{n-1} q_k(A)\,(A^k)_0$ with invariant polynomials q_k of degree $\tau_2 - k$ [Ko63, Vu74, Rai88].

(3) So $\omega_A = -2c(M)$ with $M := DG_A$ and $c(S)(X_1,X_2,X_3) := \sum_{\mathrm{cyc}} \operatorname{tr}(S(X_1)[X_2,X_3])$. For $\psi(U,V) = \operatorname{tr}(UQ(V))$ with Q antisymmetric for the trace form, $\delta\psi = -c(Q)$; so $\omega_A \in B^3$ if and only if $c(M-Q) = 0$ for some antisymmetric Q. At a regular diagonal A, M commutes with the torus, and Q may be averaged over it: $Q(E_{ab}) = q_{ab}E_{ab}$ with $q_{ab} = -q_{ba}$ (its block on the diagonal matrices plays no role below).

(4) At $A = \operatorname{diag}(x)$ regular, $M(E_{ab}) = m_{ab}E_{ab}$ with $m_{ab} = G_x[x_a,x_b]$, the divided difference of $G_x(u) := \sum_k q_k(x)u^k$ (the derivatives of the q_k vanish on E_{ab}). With $s_{ab} := m_{ab} - q_{ab}$, $c(M-Q)(E_{ab},E_{bc},E_{ca}) = s_{ab}+s_{bc}+s_{ca}$ and $c(M-Q)(E_{ac},E_{cb},E_{ba}) = s_{ac}+s_{cb}+s_{ba}$. Both vanish, and their sum gives

$$G_x[x_a,x_b] + G_x[x_b,x_c] + G_x[x_c,x_a] = 0 \qquad \text{for all distinct } a,b,c.$$

(5) Fix a regular x and four indices a,b,c,e, and let $r = r_0 + r_1u + r_2u^2 + r_3u^3$ interpolate G_x at x_a,x_b,x_c,x_e; it has the same divided differences on these points. For $r = u, u^2, u^3$ the divided differences of a pair (x_i,x_j) are 1, x_i+x_j and $x_i^2+x_ix_j+x_j^2$, so the identity of (4) for the triangle that omits $o \in \{a,b,c,e\}$ reads

$$3r_1 + (2w - 2x_o)\,r_2 + (K - wx_o - x_o^2)\,r_3 = 0, \qquad w := x_a+x_b+x_c+x_e, \quad K := \tfrac{3}{2}\textstyle\sum x_i^2 + \tfrac{1}{2}w^2$$

(the sum over the four indices). This is a polynomial of degree at most 2 in x_o that vanishes at four distinct points, so $r_3 = 0$, then $r_2 = 0$ and $r_1 = 0$. So G_x takes one value at any four of the points, hence at all n of them, and as $\deg G_x \leq n-1$, G_x is constant: $q_k(x) = 0$ for $k \geq 1$. So $G = 0$ at every regular diagonal A, hence everywhere by equivariance and density; then $\eta = 0$, that is $\tilde\varphi = 0$ on $\mathfrak{sl}_n^3$, and $\varphi = 0$. □

For $n = 3$ the argument fails: the only triangle gives $3q_1 + 2(x_1+x_2+x_3)q_2 = 3q_1$, so it leaves q_2 free. This shows only that the proof needs $n \geq 4$; it is not a counterexample at $n = 3$.

The leading term for $k=4$. Lemmas 10.4, 10.5 and 10.6, the formal model, and Lemmas 10.7, 10.8 and 10.9 hold for $k=4$ and $\nu=(3,1)$ with the same proofs: one slot fewer, three colours ($d-3$ letters A and $B_1,\dots,B_4$, again $d+1$ letters), the alternation over S_4, and, as Theorem 10.24 requires, $\Pi_{\neq\nu'}$ (the projection onto $\bigoplus_{\alpha\neq\nu'} S_\alpha V\otimes S_{\alpha'}V$) in place of $\Pi_{\neq\nu}$. So (13) becomes "f passes the three tests if and only if $\Omega f(P)\circ\Pi_{\neq\nu'}=0$ for every P"; in the chart formula the exponent of det is n (as $\tau_1+d=n^2$); Lemma 10.6 needs $n\geq d$ for HW_τ and $n\geq d+1$ for the targets; $\mathbf{M}(n)=n\mathcal{Z}+\mathbf{M}_0+O(1/n)$ with $\mathcal{Z}(\varphi)=(\operatorname{tr}\wedge\omega_\varphi)\circ\Pi_{\neq\nu'}$ free of n; if $\mathcal{Z}$ is injective on HW_τ, then κ does not occur in $K_{\lambda(3,1)}$ for all but finitely many n; and $\mathcal{Z}(\varphi)=0$ implies $\omega_\varphi(A)\in\mathcal{K}_{(3,1)}(n)$ for every n and every $A\in\mathfrak{sl}_n^3$ (conversely for $n\geq d+1$). With Theorem 10.35 this shows that $\mathcal{Z}$ is injective on $\mathrm{HW}_{(\tau_2,1,1)}$ for every $\tau_2\geq 1$ and every $n\geq 4$.

Corollary 10.36. *Let $n\geq 4$ and $1\leq j\leq n-3$. The conjugation type κ with $\kappa'=(N-j,j)$ does not occur in $K_{\lambda(3,1)}$, nor in $K_{\lambda(2,1,1)}$.*

Proof. Here $\tau=(n^2-j-3,j+1,1,1)$, the first-layer weight $(j+1,1,1)$, and $d=j+3$. (1) $\mathcal{Z}$ is injective on the formal HW_τ: if $\mathcal{Z}(\varphi)=0$, fix one $n_1\geq\max(4,d+1)$; then $\omega_\varphi(A)\in\mathcal{K}_{(3,1)}(n_1)$ for every $A\in\mathfrak{sl}_{n_1}^3$, so $\varphi=0$ on $\mathfrak{sl}_{n_1}^3$ by Theorem 10.35, and the formal φ is 0 by Lemma 10.6 ($n_1\geq d$). (2) So κ does not occur in $K_{\lambda(3,1)}$ for all but finitely many n. (3) As in the proof of Lemma 9.17(a), with Theorem 9.7(R1) for $k=4$ ($n\geq m+3$) in place of Theorem 9.3(c): $K^{[m]}_{\lambda(3,1)}\cong\mathcal{Y}_m((3,1))$ as S_D-modules (the last sentence of Theorem 9.7), so the S_D-character of $K^{[m]}_{\lambda(3,1)}$ does not depend on $n\geq m+3$. A type with tail t, that is $\kappa'=(N-|t|,t)$ with t of at most three parts, and $|t|\leq m$, contributes, twisted by the sign, the skew module of $\kappa'/(N-m)$. For $n\geq m+3$ we have $t_1\leq m\leq N-m$, so the skew shape is the disjoint union of a row of $m-|t|$ cells and t, and the module does not depend on n. By induction on m, the multiplicity of every tail t does not depend on $n\geq|t|+3$. (4) For $t=(j)$ this multiplicity is constant for $n\geq j+3$ and 0 for large n, hence 0 for every $n\geq j+3$. Finally, $\lambda(2,1,1)=\lambda(3,1)'$, so $K_{\lambda(2,1,1)}=\varepsilon(K_{\lambda(3,1)})$; multiplication by the class function sgn commutes with conjugation, so the two blocks have the same conjugation types. □

Remark 10.37. (1) For $j=0$ the type $\kappa'=(N)$ has the weight $(1,1,1)$ and is excluded for every $n\geq 3$ by Theorem 10.28(a) at $s=n$. (2) Not covered: the types with $\tau_3\geq 2$ (the rest of the first layer and the deeper layers), where $\theta(A_2)\beta\neq 0$ in general and step (1) of the proof of Theorem 10.35 breaks; the types of depth at least $n-2$; and the full pencil at $s=n$ on the deep types of the column, where Theorem 10.35 gives only the leading term. No bound on $d(n)$ follows by itself. (The types of the column of every depth, and the full pencil on them, are treated in Subsection 10.9: Theorem 10.49, Corollary 10.51; for the types of depth at most 6 see Subsection 10.10.) (3) With the right product in (BL) the window of $K_{\lambda(3,1)}$ is $\mathcal{Y}_m((3,1))$; the other convention gives $\mathcal{Y}_m((2,1,1))$. Theorem 9.7 holds for both, and the last sentence of Corollary 10.36 makes the choice irrelevant. (4) Lie algebra cohomology was used for a matrix identity already in Kostant's proof of the Amitsur–Levitzki theorem [Ko58].

10.9. **The column $(\tau_2,1,1)$ for $\nu=(3,1)$ at every depth.** We keep $\nu=(3,1)$, $k=4$, $N=n^2-4$ and the notation of Subsections 10.7 and 10.8. We show that for every $n\geq 3$ the blocks $K_{\lambda(3,1)}$ and $K_{\lambda(2,1,1)}$ have no conjugation type κ with $\kappa'=(N-j,j)$, $0\leq j\leq N/2$, at any depth (Corollary 10.51); Corollary 10.36 gave this for $n\geq 4$ and $1\leq j\leq n-3$. This is the analogue for statement (ii) of Theorem G of Theorem 9.41. The argument is exact at size n and uses only the tests at diagonal points. There $\Omega f_s(I,A)$ is a function of two kinds of data: the root part $\eta_{ab}=x_{ab}\varkappa_{ab}$ of the 2-form η_A and its Cartan part η^{t}. The row h^{ab} (the analogue of $h_{(2,2)}$) becomes an ordinary differential equation in x_a-x_b whose indicial roots force $\varkappa_{ab}$ to vanish to order $2s$ on $x_a=x_b$, and $\varkappa=0$ unless s is a nonnegative integer (Corollary 10.43). A row of the pattern $\{a,a,b,c\}$, read near a point with $x_a=x_b=x_c$, forces the leading coefficient to vanish to order s on each of the $n-2$ hyperplanes $x_c=x_a$, so $\deg\varkappa_{ab}\geq ns$ unless $\varkappa=0$ (Corollary 10.47). With

$\varkappa = 0$ and $n \geq 4$, one row with four distinct indices kills the Cartan part unless $ns = \tau_2 - 1$; for $n = 3$ the Cartan part vanishes anyway (Lemma 10.48).

The setting. Fix $s \in \mathbb{C}$, $\tau_2 \geq 1$, the weight $\tau = (\tau_2, 1, 1)$ and $\varphi \in \mathrm{HW}_\tau$, and write $\Omega_A := \Omega f_s(I, A) = s\, Z_\omega - \mathfrak{L}\omega$ (Lemma 10.25(a)). We say that φ *passes the tests at* s if $\Omega_A \circ C = -2\,\Omega_A$ for every A; for $s = n$ and $\varphi = \varphi_f$ with $f \in \mathrm{Pol}_\tau$ this is Lemma 10.30 at the points (I, A), as $f_n = f$. A φ that passes the tests at s is a kernel vector at s in the sense of Subsection 10.7 (the rows (R1)–(R3) lie in the test space). In this subsection an *eigenvalue* is an s for which a nonzero such φ exists. For $\tau = (\tau_2, 1, 1)$, ω depends only on $A := A_2$; as in the proof of Theorem 10.35, $\eta_A(Y, W) := \tilde{\varphi}(A, Y, W)$ is alternating, $\iota_A \eta_A = 0$, it depends only on A_0, Y_0, W_0, and $\omega = 2\,d\eta$. For $X \in M_n$ put $X \circ A := (XA + AX)/2$, and let D_X be the derivative in A along X.

Lemma 10.38. $(\mathfrak{L}\omega)(B_1, \dots, B_4) = \sum_{i=1}^{4} (-1)^{i-1} (D_{B_i \circ A}\, \omega)(\hat{B}_i)$.

Proof. By the proof of Lemma 10.31, $\mathfrak{L}\omega = \sum_i (-1)^{i-1} (D_{B_i A}\, \omega)(\hat{B}_i) + \mathfrak{M}\omega$ with $\mathfrak{M}\omega = -\delta\omega$. Write $D_{BA} = D_{B \circ A} + \frac{1}{2} D_{[B,A]}$. By invariance, $(D_{[B_i, A]}\, \omega)(\hat{B}_i) = -\sum_{m \neq i} \omega(\dots, [B_i, B_m], \dots)$, so the terms $\frac{1}{2} D_{[B_i, A]}$ add up to $+\delta\omega$ (each pair $i < m$ is counted twice, with the same sign), which cancels $\mathfrak{M}\omega$. □

By Lemma 10.25(a), $\Omega_A = \Omega_{A + cI}$; so $A = \mathrm{diag}(x)$ need not be traceless, and every function of x below is a function of A_0, hence translation invariant. Let ∂_a be the derivative along E_{aa}, $x_{ab} := x_a - x_b$, and let $\mathfrak{t}$ be the space of the diagonal matrices, with the coordinates x_a, the vector $\mathbf{1}$ and the Euler field ξ.

Lemma 10.39 (diagonal data)**.** *Let $A = \mathrm{diag}(x)$ be regular.*

(a) For $a \neq b$: $\eta_A(E_{ab}, E_{cd}) = 0$ unless $(c, d) = (b, a)$, and $\eta_A(\mathfrak{t}, E_{ab}) = 0$. The value $\eta_{ab} := \eta_A(E_{ab}, E_{ba})$ is a polynomial in x that vanishes on $x_a = x_b$, so $\varkappa_{ab} := \eta_{ab}/x_{ab}$ is a polynomial, even under $x_a \leftrightarrow x_b$, homogeneous of degree $\tau_2 - 1$, and $\varkappa_{ac}(x) = \varkappa_{ab}(\sigma_{bc} x)$, where σ_{bc} exchanges x_b and x_c. The restriction $\eta^{\mathfrak{t}}$ of η_A to $\mathfrak{t}$ is a 2-form with $\iota_{\mathbf{1}} \eta^{\mathfrak{t}} = \iota_\xi \eta^{\mathfrak{t}} = 0$.

(b) Put $\omega_{ab,c} := \omega_A(E_{ab}, E_{ba}, E_{cc})$ for every c, and $\omega_{abc} := \omega_A(E_{ab}, E_{bc}, E_{ca})$ for $c \notin \{a, b\}$. Then, for $c \notin \{a, b\}$,

$$\omega_{ab,c} = 2x_{ab}\, \partial_c \varkappa_{ab}, \qquad \omega_{ab,a} = 2(3\varkappa_{ab} + x_{ab}\, \partial_a \varkappa_{ab}), \qquad \omega_{ab,b} = 2(-3\varkappa_{ab} + x_{ab}\, \partial_b \varkappa_{ab}),$$

$$\omega_{abc} = 2\, \frac{x_{ab}^3 \varkappa_{ab} + x_{bc}^3 \varkappa_{bc} + x_{ca}^3 \varkappa_{ca}}{x_{ab}\, x_{bc}\, x_{ca}},$$

and ω_{abc} is symmetric in a, b, c (exchanging b and c changes the sign of the numerator and of the denominator). The restriction of ω_A to $\mathfrak{t}$ is $\omega^{\mathfrak{t}} = 2\,d\eta^{\mathfrak{t}}$ (d on $\mathfrak{t}$).

(c) For every invariant form F and $i \neq j$: $x_{ij}\, (D_{E_{ij}} F)_A(B_1, \dots) = \sum_m F_A(\dots, [E_{ij}, B_m], \dots)$. So, with $\gamma_{ab} := (x_a + x_b)/(2x_{ab})$, $(D_{E_{ab} \circ A} F)_A = \gamma_{ab} \sum_m F_A(\dots, [E_{ab}, B_m], \dots)$, and $\gamma_{ba} = -\gamma_{ab}$.

Proof. (a) An element of the torus fixes A, so η_A is invariant under the torus. On $x_a = x_b$ the permutation matrix w of $(a\,b)$ fixes A, so $\eta_{ab} = \eta_A(wE_{ab}w^{-1}, wE_{ba}w^{-1}) = \eta_A(E_{ba}, E_{ab}) = -\eta_{ab}$. The evenness and $\varkappa_{ac}(x) = \varkappa_{ab}(\sigma_{bc} x)$ follow in the same way from $w = (a\,b)$ and $(b\,c)$. $\eta^{\mathfrak{t}}$ kills $\mathbf{1}$ as η_A kills I, and $\iota_\xi \eta^{\mathfrak{t}} = (\iota_A \eta_A)|_{\mathfrak{t}} = 0$. (c) Differentiate $F_{gAg^{-1}}(gB_1 g^{-1}, \dots) = F_A(B_1, \dots)$ at $g = \exp(tE_{ij})$; $[E_{ij}, A] = (x_j - x_i)E_{ij}$, and $E_{ab} \circ A = \frac{1}{2}(x_a + x_b)E_{ab}$. (b) $\omega(X, Y, W) = 2[D_X \eta(Y, W) - D_Y \eta(X, W) + D_W \eta(X, Y)]$. By (c) applied to η, for diagonal $T = \mathrm{diag}(t)$: $D_{E_{ab}} \eta(E_{ba}, T) = \eta^{\mathfrak{t}}(e_a - e_b, t)/x_{ab} + (t_a - t_b)\varkappa_{ab}$ and $D_{E_{ba}} \eta(E_{ab}, T) = \eta^{\mathfrak{t}}(e_a - e_b, t)/x_{ab} - (t_a - t_b)\varkappa_{ab}$, while $D_T \eta(E_{ab}, E_{ba}) = \partial_T \eta_{ab} = (t_a - t_b)\varkappa_{ab} + x_{ab}\, \partial_T \varkappa_{ab}$. The terms with $\eta^{\mathfrak{t}}$ cancel, and $\omega_A(E_{ab}, E_{ba}, T) = 2[3(t_a - t_b)\varkappa_{ab} + x_{ab}\, \partial_T \varkappa_{ab}]$; take $T = E_{cc}$. Next, $D_{E_{ab}} \eta(E_{bc}, E_{ca}) = (\eta_{ac} - \eta_{bc})/x_{ab}$ by (c), and the cyclic sum gives ω_{abc}. Along $\mathfrak{t}$ the point stays diagonal, which gives $\omega^{\mathfrak{t}}$. □

Lemma 10.40 (localization)**.** *At $A=\operatorname{diag}(x)$ the tests split over the patterns (row multiset = column multiset) of the wedges of matrix units of weight 0: $\Omega_A\circ(C+2)=0$ if and only if, for each pattern, the restriction of Ω_A to its span is killed by $C+2$. In particular:*

(i) $\{a,a,b,b\}$*: the one wedge $h^{ab}:=E_{aa}\wedge E_{ab}\wedge E_{ba}\wedge E_{bb}$ has $Ch^{ab}=0$, so $\Omega_A(h^{ab})=0$.*

(ii) $\{a,a,b,c\}$*: for $W_1=E_{aa}\wedge E_{ab}\wedge E_{ba}\wedge E_{cc}$, $W_2=E_{aa}\wedge E_{ab}\wedge E_{bc}\wedge E_{ca}$, $W_3=E_{aa}\wedge E_{ac}\wedge E_{ca}\wedge E_{bb}$, $W_4=E_{aa}\wedge E_{ac}\wedge E_{cb}\wedge E_{ba}$, $W_5=E_{ab}\wedge E_{ac}\wedge E_{ba}\wedge E_{ca}$: $CW_1=-W_2+W_4+W_5$, $CW_2=-W_1+W_3+W_5$, $CW_3=W_2-W_4+W_5$, $CW_4=W_1-W_3+W_5$, $CW_5=W_1+W_2+W_3+W_4$, and with $T_i:=\Omega_A(W_i)$ the tests are $T_1=T_2$, $T_3=T_4$, $T_5=-T_1-T_3$.*

(iii) $\{a,b,c,e\}$ *distinct: for $\pi\in S_4$ let w_π be the wedge of the $E_{i,\pi(i)}$, $i\in\{a,b,c,e\}$; then $Cw_\pi=-\sum_t w_{\pi t}$ (t the transpositions), and the tests say that $\pi\mapsto\Omega_A(w_\pi)$ lies in the $(3,1)$-isotypic part of $\mathbb{C}[S_4]$. $\sum_\pi w_\pi$ is the part of weight 0 of the grid $g_{4,1}(u,y)$ with $u_p=e_p$ ($p\in\{a,b,c,e\}$) and $y=\sum_p e_p$, on which C acts by $c((1^4))=-6\neq-2$; so the tests give $\sum_\pi\Omega_A(w_\pi)=0$. Likewise $\sum_\pi\operatorname{sgn}(\pi)w_\pi$ is the part of weight 0 of $g_{1,4}(u,y)$ with $u=\sum_p e_p$ and $y_q=e_q$, where $C=6=c((4))$, so $\sum_\pi\operatorname{sgn}(\pi)\,\Omega_A(w_\pi)=0$.*

Proof. Ω_A is invariant under the torus (A is fixed), so it kills every wedge of nonzero weight. The test space $\bigoplus_{\alpha\neq\nu'}S_\alpha V\otimes S_{\alpha'}V$ is stable under the two diagonal tori, so it is the sum of its parts of fixed row and column weight, and C preserves each part. The matrices of C are a direct count: σ_{pq} exchanges the row indices of the factors p and q; a pair with equal row indices is fixed, and a pair with equal column indices gives -1. In (iii), σ applied to $E_{a\pi(a)}$, $E_{b\pi(b)}$ gives $E_{b\pi(a)}$, $E_{a\pi(b)}$, that is $-w_{\pi(a\,b)}$ after reordering. □

Proposition 10.41 (the row h^{ab})**.** *At $A=\operatorname{diag}(x)$, with $\varkappa=\varkappa_{ab}$,*

$$\Omega_A(h^{ab})=2\big[-6s\,\varkappa+(s-2)\,x_{ab}(\partial_b-\partial_a)\varkappa-x_{ab}^2\,\partial_a\partial_b\varkappa\big].$$

Proof. $Z_\omega(h^{ab})=\operatorname{tr}(E_{aa})\,\omega(E_{ab},E_{ba},E_{bb})-\operatorname{tr}(E_{bb})\,\omega(E_{aa},E_{ab},E_{ba})=\omega_{ab,b}-\omega_{ab,a}$. In Lemma 10.38, $E_{aa}\circ A=x_aE_{aa}$ gives $x_a\partial_a\omega_{ab,b}$, and E_{bb} gives $-x_b\partial_b\omega_{ab,a}$. For slot 2 ($E_{ab}$, sign $-$) the brackets with E_{aa}, E_{ba}, E_{bb} give $-\omega_{ab,b}$, 0 (as $\omega(E_{aa},E_{aa}-E_{bb},E_{bb})=0$) and $-\omega_{ab,a}$, so by Lemma 10.39(c) slot 2 contributes $\gamma_{ab}(\omega_{ab,a}+\omega_{ab,b})$; slot 3 ($E_{ba}$, sign $+$) contributes the same, as $\gamma_{ba}=-\gamma_{ab}$. So $(\mathfrak{L}\omega)(h^{ab})=x_a\partial_a\omega_{ab,b}-x_b\partial_b\omega_{ab,a}+2\gamma_{ab}(\omega_{ab,a}+\omega_{ab,b})$. Insert Lemma 10.39(b). □

Lemma 10.42 (indicial equation)**.** *Write $r:=(x_a+x_b)/2$, $q:=x_{ab}$ and $\varkappa_{ab}=\sum_j\varkappa_jq^j$ with polynomials $\varkappa_j$ in r and the other x_c (only even j occur). Then $\Omega_A(h^{ab})=0$ for every diagonal A if and only if $(j+3)(j-2s)\,\varkappa_j=\frac14\partial_r^2\varkappa_{j-2}$ for every j.*

Proof. $\partial_a=\frac12\partial_r+\partial_q$ and $\partial_b=\frac12\partial_r-\partial_q$, so the bracket of Proposition 10.41 vanishes if and only if $-6s\varkappa-2(s-2)q\,\partial_q\varkappa+q^2\partial_q^2\varkappa=\frac14q^2\partial_r^2\varkappa$; compare the coefficients of q^j, using $-6s-2(s-2)j+j(j-1)=(j+3)(j-2s)$. □

Corollary 10.43. *Suppose that $\Omega_A(h^{ab})=0$ at every diagonal A. (a) If $s\notin\{0,1,2,\dots\}$, then $\varkappa\equiv0$. (b) If $s=m\in\{1,2,\dots\}$, then $\varkappa_j=0$ for $j<2m$ and $\varkappa_{2m+2i}=\upsilon_i\,\partial_r^{2i}\varkappa_{2m}$ with $\upsilon_0=1$, $\upsilon_i=\upsilon_{i-1}/(4(2m+2i+3)(2i))$; so $\varkappa\equiv0$ if and only if $\varkappa_{2m}=0$, and $\varkappa_{2m}$ is homogeneous of degree τ_2-1-2m.*

Proof. For $j=0$ the equation is $-6s\varkappa_0=0$, and odd j do not occur. For even $j<2s$ the factor $(j+3)(j-2s)$ is not 0, so $\varkappa_j=0$ by induction; for $j>2s$ it is not 0 either, so $\varkappa_j$ is determined by $\varkappa_{j-2}$. □

Proposition 10.44 (the rows $\{a,a,b,c\}$)**.** *At $A=\operatorname{diag}(x)$ and for $c\notin\{a,b\}$: $\Omega_A(W_5)=0$,*

$\frac12\,\Omega_A(W_1)=\mathcal{Q}_{abc}[\varkappa_{ab}]:=-3s\,\varkappa_{ab}+s\,x_{ab}(\partial_c-\partial_a)\varkappa_{ab}+(3x_c-2x_a-x_b)\,\partial_c\varkappa_{ab}-x_{ab}(x_a-x_c)\,\partial_a\partial_c\varkappa_{ab}$,

and $\Omega_A(W_3)(x)=\Omega_A(W_1)(\sigma_{bc}x)=2\mathcal{Q}_{acb}[\varkappa_{ac}]$. Hence, if φ passes the tests at s, $\mathcal{Q}_{abc}[\varkappa_{ab}]+\mathcal{Q}_{acb}[\varkappa_{ac}]=0$ (Lemma 10.40(ii)).

Proof. As for Proposition 10.41. $Z_\omega(W_1) = \omega_{ab,c} - \omega_{ab,a}$, and $(\mathfrak{L}\omega)(W_1) = x_a\partial_a\omega_{ab,c} - x_c\partial_c\omega_{ab,a} + 2\gamma_{ab}\,\omega_{ab,c}$ (the two terms with ω^{t}, from $[E_{ab}, E_{ba}] = E_{aa} - E_{bb}$ and $[E_{ba}, E_{ab}]$, cancel). For W_5, $Z_\omega(W_5) = 0$, and the four slots give $\gamma_{ab}(\omega_{abc}-\omega_{ac,a}+\omega_{ac,b})+\gamma_{ac}(\omega_{abc}-\omega_{ab,a}+\omega_{ab,c})-\gamma_{ab}(\omega_{abc}+\omega_{ac,b}-\omega_{ac,a})+\gamma_{ac}(-\omega_{abc}-\omega_{ab,c}+\omega_{ab,a}) = 0$; before cancelling, the part with ω_{abc} is $(\gamma_{ab}-\gamma_{ac})(\omega_{acb}-\omega_{abc})$, which is 0 by Lemma 10.39(b). □

The identities of Propositions 10.41 and 10.44 are proved at regular diagonal points; both sides are polynomials in x, so they hold at every diagonal point. Proposition 10.46 uses only the polynomial identity $\mathcal{Q}_{abc}[\varkappa_{ab}] + \mathcal{Q}_{acb}[\varkappa_{ac}] \equiv 0$ on $\mathbb{C}^n$ and the recursion of Lemma 10.42.

Lemma 10.45 (local form)**.** *Fix a point P_0 with $x_a = x_b = x_c = t_0$ (the other x fixed), put $\bar{x}_p := x_p - t_0$ for $p \in \{a, b, c\}$, $q := \bar{x}_a - \bar{x}_b$ and $z := \bar{x}_c - (\bar{x}_a + \bar{x}_b)/2$. Then $\mathcal{Q}_{abc}$ has coefficients homogeneous of degree 0 in $(\bar{x}_a, \bar{x}_b, \bar{x}_c)$, maps functions of (q, z) to functions of (q, z), and*

$$\mathcal{Q}_{abc}[q^j z^l] = (l - s)(j + 3)\, q^j z^l + \tfrac{l}{2}(3s - j - l)\, q^{j+1} z^{l-1} + \tfrac{l(l-1)}{4}\, q^{j+2} z^{l-2}.$$

Proof. $\partial_a = \partial_q - \frac{1}{2}\partial_z$ and $\partial_c = \partial_z$ on such functions, $3x_c - 2x_a - x_b = 3z - q/2$ and $x_a - x_c = q/2 - z$; expand. □

Proposition 10.46. *Let $s = m \geq 1$ be an integer, and suppose that φ passes the tests at s. Then for every $c \notin \{a, b\}$ the coefficient $\varkappa_{2m}$ of Corollary 10.43(b) (a polynomial on the hyperplane $x_a = x_b$, in r and the other x) is divisible by $(x_c - r)^m$.*

Proof. Induction on $p = 0, \ldots, m-1$: suppose $\varkappa_{2m} = (x_c - r)^p U_c$. At any point P_0 of $\{x_c = r\}$ (on $x_a = x_b$) let $\zeta := U_c(P_0)$, and put $\Upsilon_p := \sum_i \upsilon_i \frac{p!}{(p-2i)!}\, q^{2m+2i} z^{p-2i}$ with the υ_i of Corollary 10.43(b). With $z = x_c - r$ and $\partial_r z = -1$, $\partial_r^{2i}(z^p U_c) = \frac{p!}{(p-2i)!} z^{p-2i} U_c + (\text{terms of order} > p - 2i \text{ at } P_0)$, and for $2i > p$ every term has order at least 0; so $q^{2m+2i}\varkappa_{2m+2i}$ has order at least $2m + p$ at P_0, and the part of $\varkappa_{ab}$ of degree $2m + p$ in $(\bar{x}_a, \bar{x}_b, \bar{x}_c)$ is $\zeta\Upsilon_p(q, z)$. As σ_{bc} fixes P_0 and $\varkappa_{ac}(x) = \varkappa_{ab}(\sigma_{bc}x)$, the part of $\varkappa_{ac}$ is $\zeta\Upsilon_p(q', z')$ with $q' = \bar{x}_a - \bar{x}_c$, $z' = \bar{x}_b - (\bar{x}_a + \bar{x}_c)/2$. As $\mathcal{Q}$ preserves degrees, Proposition 10.44 gives $\zeta\,[\tilde{\Upsilon}_p(q, z) + \tilde{\Upsilon}_p(q', z')] = 0$ with $\tilde{\Upsilon}_p := \mathcal{Q}_{abc}[\Upsilon_p]$. In the coordinates $u := q$, $v := q'$ we have $z = u/2 - v$ and $z' = v/2 - u$. Every monomial of $\tilde{\Upsilon}_p$ has q-degree at least $2m$ (Lemma 10.45), so $\tilde{\Upsilon}_p(v, v/2 - u)$ has no term v^h with $h < 2m$, and the coefficient of v^h in the sum is $\frac{(-1)^h}{h!}\partial_z^h\tilde{\Upsilon}_p(u, u/2)$. As $p < 2m$, these h cover the z-degree of $\tilde{\Upsilon}_p$, so $\zeta \neq 0$ would force $\tilde{\Upsilon}_p \equiv 0$. But the coefficient of $q^{2m}z^p$ in $\tilde{\Upsilon}_p$ is $(p - m)(2m + 3) \neq 0$ (only the monomial $q^{2m}z^p$ of Υ_p contributes, by Lemma 10.45). So $\zeta = 0$ at every point of $\{x_c = r\}$, U_c is divisible by $x_c - r$, and the induction goes on. □

At $p = m$ the coefficient $(p-m)(2m+3)$ is 0, and in fact $\mathcal{Q}_{abc}[\Upsilon_m] \equiv 0$ (for instance $\mathcal{Q}_{abc}[q^2 z] = 0$ for $m = 1$ and $\mathcal{Q}_{abc}[q^4 z^2 + q^6/36] = 0$ for $m = 2$), so the method gives no factor $(x_c - r)^{m+1}$; it is not needed.

Corollary 10.47. *In the situation of Proposition 10.46, if $\tau_2 - 1 < nm$, then $\varkappa \equiv 0$.*

Proof. The factors $x_c - r$, $c \notin \{a, b\}$, are coprime, so $(x_c - r)^m$ for all $n - 2$ of them divides $\varkappa_{2m}$. If $\varkappa_{2m} \neq 0$, its degree $\tau_2 - 1 - 2m$ is therefore at least $m(n - 2)$, that is, $\tau_2 - 1 \geq nm$. Otherwise $\varkappa_{2m} = 0$, and $\varkappa \equiv 0$ by Corollary 10.43(b). □

Lemma 10.48 (the Cartan part)**.** *Suppose that $\varkappa \equiv 0$ and that φ passes the tests at s at every diagonal point. If $ns \neq \tau_2 - 1$, or if $n = 3$, then $\eta^{\mathrm{t}} = 0$, so $\eta_A = 0$ and $\varphi = 0$.*

Proof. $n = 3$: at a regular x the vectors $\mathbf{1}$ and x are independent and both lie in the kernel of the 2-form η^{t}_x on $\mathbb{C}^3$, so $\eta^{\mathrm{t}}_x = 0$; it is polynomial, so $\eta^{\mathrm{t}} \equiv 0$. $n \geq 4$ (four distinct indices are needed here): with $\varkappa \equiv 0$, $\omega_{ab,c} = \omega_{abc} = 0$ identically, so for four distinct indices $\Omega_A(w_\pi) = 0$ for $\pi \neq \mathrm{id}$ (each such wedge needs $\omega_{ab,c}$ or ω_{abc}; for a transposition, $E_{ab} \wedge E_{ba} \wedge E_{cc} \wedge E_{ee}$, the terms with ω^{t}

of the two root slots cancel). Then Lemma 10.40(iii) gives $\Omega_A(E_{aa}\wedge E_{bb}\wedge E_{cc}\wedge E_{ee})=0$, that is, on $\mathfrak{t}$,

$$s\Big(\sum_a dx_a\Big)\wedge\omega^{\mathfrak{t}}=\sum_a dx_a\wedge x_a\partial_a\omega^{\mathfrak{t}}.$$

Both sides keep each monomial $x^{\mathbf{e}}$ of the coefficients. Let $c_{\mathbf{e}}$ be the constant 3-form that multiplies $x^{\mathbf{e}}$ in $\omega^{\mathfrak{t}}$, and $\ell_{\mathbf{e}}:=\sum_a(s-\mathbf{e}_a)\,dx_a$. Then $\ell_{\mathbf{e}}\wedge c_{\mathbf{e}}=0$, and $\iota_{\mathbf{1}}c_{\mathbf{e}}=0$ ($\omega^{\mathfrak{t}}$ kills $\mathbf{1}$). If $ns\neq|\mathbf{e}|=\tau_2-1$, then $\iota_{\mathbf{1}}\ell_{\mathbf{e}}=ns-|\mathbf{e}|\neq0$, so $\ell_{\mathbf{e}}\neq0$ and $c_{\mathbf{e}}=\ell_{\mathbf{e}}\wedge\beta$ for a 2-form β (division by a nonzero 1-form [dR54]; here linear algebra), and $0=\iota_{\mathbf{1}}c_{\mathbf{e}}=(ns-|\mathbf{e}|)\beta-\ell_{\mathbf{e}}\wedge\iota_{\mathbf{1}}\beta$ gives $\beta=\ell_{\mathbf{e}}\wedge\iota_{\mathbf{1}}\beta/(ns-|\mathbf{e}|)$, so $c_{\mathbf{e}}=0$. Hence $\omega^{\mathfrak{t}}=0$, that is $d\eta^{\mathfrak{t}}=0$, and by Cartan's formula with the Euler field, as in the proof of Theorem 9.41, $(\tau_2+2)\,\eta^{\mathfrak{t}}=L_\xi\eta^{\mathfrak{t}}=d\,\iota_\xi\eta^{\mathfrak{t}}+\iota_\xi d\eta^{\mathfrak{t}}=0$. So η_A vanishes at every regular diagonal point ($\varkappa=0$, $\eta^{\mathfrak{t}}=0$), hence everywhere, and $\varphi=0$, as η determines φ. □

Theorem 10.49 (the column $(\tau_2,1,1)$ for $\nu=(3,1)$)**.** *Let $n\geq3$, $\tau_2\geq1$, $\tau=(\tau_2,1,1)$ and $s\in\mathbb{C}$. If $\varphi\in\mathrm{HW}_\tau$ passes the tests at s, then $\varphi=0$ unless $s=0$, or s is an integer with $1\leq s\leq(\tau_2-1)/n$, or $ns=\tau_2-1$. So every eigenvalue $s\neq0$ has $\mu=ns-\tau_2+1\leq0$. For $n=3$ the exceptional set is $\{0\}\cup\{s\in\mathbb{Z}:1\leq s\leq(\tau_2-1)/3\}$.*

Proof. If $s\notin\{0,1,2,\dots\}$, then $\varkappa\equiv0$ by Corollary 10.43(a); if $s=m\geq1$ and $nm>\tau_2-1$, then $\varkappa\equiv0$ by Corollary 10.47. Then Lemma 10.48 applies. For $n=3$ it applies for every s. □

Here $\mu=ns-d+3$ with $d=\tau_2+2$, as in Subsection 10.7; the theorem is a containment, and it does not say that the allowed values occur.

Corollary 10.50 (every exponent s)**.** *Let $n\geq3$, let $s\geq1$ be an integer, and let $\tau\vdash ns$ be a partition with four rows and $\tau_3=\tau_4=1$. Then no nonzero polynomial f on M_n^4, homogeneous of degree τ_j in P_j, a highest weight vector for GL_4 (as in the definition of Pol_τ), with $f(gPh^T)=\det(g)^s\det(h)^sf(P)$, passes the three tests at every P.*

Proof. $f(gP)=\det(g)^sf(P)$, so $f=\det(P_1)^s\varphi_f((P_1^{-1}P_j)_0)$ with $\varphi_f:=f(I,\cdot,\cdot,\cdot)\in\mathrm{HW}_{(\tau_2,1,1)}$, and $f\mapsto\varphi_f$ is injective (the set $\det P_1\neq0$ is dense); thus $f=f_s$ for φ_f. By Lemma 10.30 (whose proof is pointwise and holds for such f), the tests at $P=(I,A)$ say $\Omega_A\circ(C+2)=0$. For such τ, $\tau_1=ns-d\geq\tau_2$, so $\mu=ns-\tau_2+1=\tau_1+3\geq\tau_2+3>0$, while Theorem 10.49 needs $\mu\leq0$ for $s\neq0$. So $\varphi_f=0$ and $f=0$. □

This is a statement on functions for every s; a statement on multiplicities for $n\times s$ rectangles would need the rectangle form of Theorem 10.24, which we do not claim.

Corollary 10.51 (two-row types)**.** *For every $n\geq3$, $K_{\lambda(3,1)}$ has no conjugation type κ with $\kappa'=(N-j,j)$, $0\leq j\leq N/2$, and neither has $K_{\lambda(2,1,1)}$.*

Proof. By Theorem 10.24, a copy of κ gives a nonzero $f\in\mathrm{Pol}_\tau$, $\tau=\kappa'+(1^4)=(N-j+1,j+1,1,1)$, that passes the three tests. By Theorem 10.1(c), $f\mapsto\varphi_f$ is injective for every n (the hypothesis $d\leq n$ of the bijection is not used) and $f=f_n$, and by Lemma 10.30 the tests at $P=(I,A)$ say $\Omega_A\circ(C+2)=0$ at $s=n$. Here $\tau_2=j+1$ and $\tau_1=n^2-j-3\geq\tau_2$, so $\mu=n^2-\tau_2+1=\tau_1+3>0$, and $s=n$ is not exceptional in Theorem 10.49: $\varphi_f=0$. No hypothesis $n\geq d$ is used. For $K_{\lambda(2,1,1)}$ see the last paragraph of the proof of Corollary 10.36. □

Remark 10.52. (1) Against Corollary 10.36 ($1\leq j\leq n-3$, $n\geq4$) this adds every depth, the case $n=3$, and Corollary 10.50 for every s. The parallel for the main block ($k=3$) is Theorem 9.41: there η is a 1-form, so at a diagonal point it has only a Cartan part, and the argument of Lemma 10.48 for 2-forms on $\mathfrak{t}$ gives "no eigenvalue other than $(d-2)/n$". Consistency: for $\tau_2=1$ ($\tau=(1,1,1)$, Theorem 10.28(a)) one may take $\varkappa=1$, and the row h^{ab} gives $-12s=0$; for $\tau_2=2$, $\Omega_A(h^{ab})=-12s(x_a+x_b)$ at traceless x. (2) Context, not used. Corollary 10.43(b) says that $\eta_{ab}=x_{ab}\varkappa_{ab}$, which is odd under $x_a\leftrightarrow x_b$, vanishes to order $2m+1$ on $x_a=x_b$: the condition of m-quasi-invariance of Chalykh and Veselov at that wall [CV90]. The bound $\tau_2\geq nm+1$

of Corollary 10.47 is the lowest degree of the nonconstant m-harmonic polynomials of S_n [FV03, §4.5], equivalently (by the freeness theorem of [EG02]) of the non-symmetric m-quasi-invariants; $x_1+\cdots+x_n$ is a symmetric one of degree 1. (3) Not covered: the types with $\tau_3 \geq 2$ (the rest of the first layer and the deeper layers; for depth at most 6 see Subsection 10.10). Not claimed: that the allowed values of Theorem 10.49 occur. The proof uses only the rows h^{ab}, the rows $\{a,a,b,c\}$ and one row with four distinct indices, at diagonal points: on $\{a,a,b,c\}$ only $\alpha=(3,1),(2,2),(2,1,1)$ occur, so $T_1+T_3+T_5=0$ is a combination of the conditions of the corners $(1,3)$ and $(2,2)$, and the corner $(4,1)$ enters only through $\sum_\pi \Omega_A(w_\pi)=0$ in Lemma 10.48. No bound on $d(n)$ follows by itself.

10.10. **Three- and four-row types of depth at most 6.** We keep $\nu=(3,1)$, $k=4$, $N=n^2-4$ and the notation of Subsections 10.7–10.9. Corollary 10.51 removes the two-row types (depth 0 and the one-row tails) for every $n\geq 3$. Here we remove the types with $\kappa'=(N-|t|,t)$ for the sixteen tails t with two or three parts and $|t|\leq 6$,

$$\begin{aligned}
|t|\leq 4:&\quad (1,1),\ (2,1),\ (1,1,1),\ (3,1),\ (2,2),\ (2,1,1);\\
|t|=5:&\quad (4,1),\ (3,2),\ (3,1,1),\ (2,2,1);\\
|t|=6:&\quad (5,1),\ (4,2),\ (3,3),\ (4,1,1),\ (3,2,1),\ (2,2,2),
\end{aligned}$$

for $n\geq |t|+3$ (Corollary 10.57). As κ' has at most four rows (Corollary 7.6(a): $\kappa_1\leq k=4$), the tails of depth at most 6 are the empty tail, the one-row tails (j) and these sixteen; so no type of depth at most 4 occurs in $K_{\lambda(3,1)}$ or $K_{\lambda(2,1,1)}$ for $n\geq 7$, none of depth at most 5 for $n\geq 8$, and none of depth at most 6 for $n\geq 9$ (Corollary 10.57(b)). The chain is that of Corollary 10.36, with its step (1) replaced: the injectivity of $\mathcal{Z}$ on the formal HW_τ at the sixteen weights $\tau=t+(1,1,1)$ (a two-part t is padded with a 0, so $(4,1)$ gives $(5,2,1)$) comes from an exact computer-assisted certificate (Theorem 10.55). So that the computer checks $\mathcal{Z}$ itself, Lemma 10.53 rewrites $\mathcal{Z}(\varphi)=0$, for $n\geq d+1$, in the n-free basis as finitely many linear equations on the coefficient function of φ.

The coefficient function. Let $\tau=(\tau_2,\tau_3,\tau_4)$ be a weight, $d=\tau_2+\tau_3+\tau_4$, and write $[d]=\mathsf{A}\sqcup\mathsf{B}\sqcup\mathsf{C}$ with $|\mathsf{A}|=\tau_2$, $|\mathsf{B}|=\tau_3$, $|\mathsf{C}|=\tau_4$, and $X_c=A_2,A_3,A_4$ for $c\in\mathsf{A},\mathsf{B},\mathsf{C}$. As in the chart paragraph of Subsection 9.7 and the formal model of Subsection 10.2, every $\varphi\in\mathrm{HW}_\tau$ is $\varphi_g=\sum_\sigma g(\sigma)\,\mathrm{tr}_\sigma$ for a function g on the derangements of S_d, invariant under conjugation by $S_\mathsf{A}\times S_\mathsf{B}\times S_\mathsf{C}$ and unique for $n\geq d$ (Lemma 10.6); the highest weight conditions are

$$\text{(HW)}\qquad g+\textstyle\sum_{a\in\mathsf{X}} g^{(a\,b)}=0\quad\text{for every pair of colours }\mathsf{X}<\mathsf{Y}\text{ and every }b\in\mathsf{Y},$$

with $g^{(a\,b)}(\sigma):=g((a\,b)\,\sigma\,(a\,b))$: the operator that turns one letter b of colour Y into colour X, symmetrized over the cosets e, $(a\,b)$ of S_X in $S_{\mathsf{X}\cup\{b\}}$, as g is S_X-invariant.

Lemma 10.53 (the permutation form of $\mathcal{Z}$)**.** *Let $n\geq d+1$. Fix cut letters $i\in\mathsf{A}$, $j\in\mathsf{B}$, $l\in\mathsf{C}$, add a letter 0, and put $Q:=\{0,i,j,l\}$. For $\pi\in S_{\{0\}\cup[d]}$ put $\hat g(\pi):=g(\pi|_{[d]})$ if $\pi(0)=0$ and $\hat g(\pi):=0$ otherwise, and $z(\pi):=\sum_{\beta\in S_Q}\mathrm{sgn}(\beta)\,\hat g(\beta^{-1}\pi\beta)$; products are $\varsigma\pi:=\varsigma\circ\pi$. Let $T:=\mathrm{tr}\wedge\omega_{\varphi_g}$ with the formal derivatives (Lemma 10.7 for $k=4$), so that $\mathcal{Z}(\varphi_g)=T\circ\Pi_{\neq\nu'}$. Then $\mathcal{Z}(\varphi_g)=0$ in the n-free basis of the targets if and only if*

$$\textstyle\sum_\varsigma z(\varsigma\pi)+2z(\pi)=0\qquad\text{for every }\pi\in S_{\{0\}\cup[d]}\text{ with no fixed point outside }Q,$$

where ς runs over the six transpositions of Q.

Proof. (a) Evaluate T on rank-one arguments $B_q=u_qy_q^T$, $q\in Q$, with B_0 the trace slot. Polarizing φ_g once in each colour replaces one letter of each colour by an argument. By the invariance of g all cut triples give the same contribution up to the factor $\tau_2\tau_3\tau_4$; this factor and the nonzero normalization of ω are common to all π, so they do not affect the equations, and one may fix the cut (i,j,l). As g lives on derangements, the only trace factor of an argument is slot 0, as the fixed

letter 0; this is the formal $\mathcal{Z}$ of Lemma 10.7 (for $k=4$), with no projections. A permutation π of $\{0\}\cup[d]$ then contributes the monomial

$$\mathbf{m}_\pi := \prod_{q\in Q}\big(y_q^T W_q u_{\pi^+(q)}\big)\cdot \prod_{\text{cycles of }\pi\text{ avoiding }Q} \operatorname{tr}(\text{word in the } X_c),$$

where W_q is the word in the X_c after q and $\pi^+(q)$ is the next letter of Q. The alternation over the four argument slots relabels Q by $\beta\in S_Q$ and sends π to $\beta^{-1}\pi\beta$, which gives the coefficient $z(\pi)$. Monomials with a fixed point outside Q contain the trace of a traceless letter, which is 0. (b) T is multilinear in each B_q, and the rank-one matrices span M_n. Lemma 10.6 applies with $d-3$ traceless letters and 4 letters in M_n, that is, $d+1$ slots, and $d+1\le n$. So distinct π with no fixed point outside Q give linearly independent monomials (after $X=Y-\operatorname{tr}(Y)I/n$ each extra term has an extra fixed point outside Q, so the system is triangular). Setting the letters of one colour equal again identifies π with its orbit under $S_{\mathsf{A}\setminus i}\times S_{\mathsf{B}\setminus j}\times S_{\mathsf{C}\setminus l}$, and distinct orbits give independent orbit sums. The left side of each equation is constant on an orbit, as g is invariant, so the coefficient of an orbit sum is the size of the orbit times one equation. (c) By the last two sentences of the proof of Lemma 10.30, which are algebra in the centre of $F[S_4]$ valid for every alternating 4-form, $T\circ\Pi_{\neq\nu'}=0$ if and only if $T\circ(C+2)=0$: C acts on the summand α by $c(\alpha)$, the five contents are distinct, and $(2,1,1)$ is the only partition of 4 with content -2. Exchanging u_p and u_q in $\mathbf{m}_\pi$ gives $\mathbf{m}_{\varsigma\pi}$ with $\varsigma=(p\,q)$, so $T\circ C=\sum_\pi\big(\sum_\varsigma z(\varsigma\pi)\big)\mathbf{m}_\pi$. By (b), $T\circ(C+2)=0$ if and only if every coefficient $\sum_\varsigma z(\varsigma\pi)+2z(\pi)$ vanishes. □

Remark 10.54. (1) The side does not matter. Exchanging y_p and y_q instead corresponds to $\pi\varsigma$. As $z(\beta^{-1}\pi\beta)=\operatorname{sgn}(\beta)z(\pi)$, we have $z(\pi\varsigma)=-z(\varsigma\pi)$, so the test with $+2$ on that side is the test with -2 on this one. Moreover $\varphi\mapsto\varphi^T$ maps HW_τ onto itself and turns $T\circ(C+2)$ into $-(T\circ(C-2))^T$ (as in (iii) of the proof of Theorem 10.33). So $\dim(\ker\mathcal{Z}\cap\mathrm{HW}_\tau)$ does not depend on the side; compare Remark 10.37(3). (2) On each coset $S_Q\pi$ the equations say that $\sum_{\gamma\in S_Q}z(\gamma\pi)[\gamma]$ lies in the $(2,1,1)$-isotypic ideal of $F[S_4]$. This is not used.

Theorem 10.55 (computer-assisted)**.** *For each of the sixteen weights τ:* $(2,2,1)$, $(3,2,1)$, $(2,2,2)$, $(4,2,1)$, $(3,3,1)$, $(3,2,2)$ *($d\le7$);* $(5,2,1)$, $(4,3,1)$, $(4,2,2)$, $(3,3,2)$ *($d=8$);* $(6,2,1)$, $(5,3,1)$, $(5,2,2)$, $(4,4,1)$, $(4,3,2)$, $(3,3,3)$ *($d=9$), $\mathcal{Z}$ is injective on the formal* HW_τ*: the equations of Lemma 10.53 together with (HW) have only the solution $g=0$. (This holds also for* $(4,1,1)$ *and* $(5,1,1)$*, where Theorem 10.35 and the paragraph after it already give it.)*

Proof. An exact computation, as in the paragraph on computations in Subsection 9.1. The columns of the matrix are the orbits of the derangements $\sigma\in S_d$ under conjugation by $S_\mathsf{A}\times S_\mathsf{B}\times S_\mathsf{C}$; the rows are the equations of Lemma 10.53 for one cut (the first letter of each colour), over all $\pi\in S_{d+1}$ with no fixed point outside Q, and the rows (HW) for the pairs of colours (A,B), (B,C), (A,C), with one b for each pair, at every derangement σ. The matrix has integer entries (each row of Lemma 10.53 is $\sum_\varsigma z(\varsigma\pi)+2z(\pi)$ with every $z(\cdot)$ a sum of entries ±1, and each row (HW) is a sum of entries 1) and full column rank modulo the prime $p=2^{61}-1$; as the rank over $\mathbb{Q}$ is at least the rank modulo p, it has full column rank over $\mathbb{Q}$. The numbers of columns are $12,27,42,51,64,92$ for the six weights of $d\le7$ in the order of the statement, and $\dim\mathrm{HW}_\tau=1,4,5,12,8,8$; for the four weights of $d=8$ they are $93,138,206,252$ ($\dim\mathrm{HW}_\tau=25,24,27,12$), and for the six weights of $d=9$ they are $157,266,386,316,586,756$ ($\dim\mathrm{HW}_\tau=46,66,52,32,59,20$). □

Remark 10.56. Independent computations, not used in the proof. (1) The dimensions of the HW_τ agree with a computation by characters (Murnaghan–Nakayama) at 15 weights. (2) With (HW) the kernel is 0 also for the kept content $+2$ (the other side); without (HW) it is $4,6,8,25,19,31,36,44$ at $(2,1,1)$, $(3,1,1)$, $(2,2,1)$, $(2,2,2)$, $(3,2,1)$, $(4,2,1)$, $(3,3,1)$, $(3,2,2)$, for both contents. So (HW) is what kills the kernel. (3) A numerical evaluation of $T\circ(C\pm2)$ at random integer points ($n=d+1$, modulo the prime $2^{26}-5$) has the same row space as the equations of Lemma 10.53 at

$(2,1,1)$, $(2,2,1)$, $(3,1,1)$, $(2,2,2)$, for both pairings (the shift $+2$ with the kept content -2, and -2 with $+2$); with (HW) both kernels are 0. (4) A direct certificate without Lemma 10.53 (numerical rows of $\mathcal{Z}$ and of (HW) at $n = d+1$, shift ± 2) gives kernel 0 at $(2,2,2)$, $(3,2,1)$, $(4,2,1)$, $(3,3,1)$, $(3,2,2)$, and numerical HW kernels $5, 4, 12, 8, 8$. (5) A second program gives kernel 0 at $(2,2,1)$ and $(2,2,2)$. A third program builds $\mathcal{Z}$ from the definition at size n ($n = 4, \dots, 7$, modulo the prime 999983, on the grids of the live corners) and gives $\dim(\ker \mathcal{Z} \cap \mathrm{HW}_\tau(n)) = 0$ at $(2,2,1)$, $(2,2,2)$, $(3,2,1)$, $(2,1,1)$, $(3,1,1)$ (a proof at the n where it finds no zero functions, among them $(2,2,1)$ for $n = 5, 6, 7$ and $(2,2,2)$, $(3,2,1)$ for $n = 6, 7$); its kernels without (HW) at $n = d+1$ agree with (2). (6) At the ten weights of $d = 8, 9$: the dimensions of the HW_τ agree with a computation by characters; with (HW) the kernel is 0 also for the kept content $+2$ at the four weights of $d = 8$ and at $(6,2,1)$; without (HW) it is $57, 78, 94, 104$ at $(5,2,1)$, $(4,3,1)$, $(4,2,2)$, $(3,3,2)$ and 91 at $(6,2,1)$, for both contents. The second program of (5), in the formal (n-free) model and modulo the prime 1000003, gives $\ker \mathcal{Z} \cap \mathrm{HW}_\tau = 0$ at all ten weights, for both blocks, with the same dimensions of HW_τ; with the corner $(4,1)$ alone its kernels are not 0 ($22, 20, 19, 7$ at $d = 8$ and $39, 51, 32, 23, 31, 10$ at $d = 9$, in the order of the statement).

Corollary 10.57. *(a) Let t be one of the sixteen tails with two or three parts and $|t| \le 6$ (listed at the start of this subsection) and $n \ge |t| + 3$. The conjugation type κ with $\kappa' = (N - |t|, t)$ does not occur in $K_{\lambda(3,1)}$, nor in $K_{\lambda(2,1,1)}$.*

(b) $K_{\lambda(3,1)}$ and $K_{\lambda(2,1,1)}$ have no type of depth at most 4 for $n \ge 7$, no type of depth at most 5 for $n \ge 8$, and no type of depth at most 6 for $n \ge 9$.

Proof. (a) As for Corollary 10.36. Here $\tau = \kappa' + (1^4)$, so $(\tau_2, \tau_3, \tau_4) = t + (1,1,1)$ and $d = |t| + 3$. (1) $\mathcal{Z}$ is injective on the formal HW_τ (Theorem 10.55, through Lemma 10.53 at any $n \ge d+1$). This replaces the route of step (1) of Corollary 10.36 through one n_1 and Theorem 10.35. (2) So κ does not occur in $K_{\lambda(3,1)}$ for all but finitely many n (Lemma 10.8 for $k = 4$, as in the paragraph on the leading term for $k = 4$ in Subsection 10.8). (3) Step (3) of the proof of Corollary 10.36 holds for every tail t: nothing in it uses the number of parts of t. Written out, for $n \ge m + 3$ the character of $K^{[m]}_{\lambda(3,1)}$, twisted by the sign, is

$$\textstyle\sum_{|t'|<m} \mathrm{mult}_{t'}(n)\ \mathrm{Ind}_{S_{m-|t'|}\times S_{|t'|}}(\mathrm{triv} \boxtimes S^{t'}) + \sum_{|t|=m} \mathrm{mult}_t(n)\, S^t,$$

where $\mathrm{mult}_t(n)$ is the multiplicity of the type with tail t; the induced modules are the skew modules of $\kappa'/(N-m)$, as the skew shape is the disjoint union of a row of $m - |t'|$ cells and t' ($t'_1 \le N - m$ for $n \ge m + 3$; Pieri's rule [Ma95]). The left side does not depend on $n \ge m + 3$ (Theorem 9.7(R1) for $k = 4$), the first sum does not by induction on m, and the irreducible characters are independent; so every $\mathrm{mult}_t(n)$ with $|t| = m$ is constant for $n \ge m + 3$. (4) So the multiplicity of κ is 0 for every $n \ge |t| + 3$ (so $n \ge 5$, as $|t| \ge 2$). The block $\lambda(2,1,1)$: as in the last paragraph of the proof of Corollary 10.36. (b) By Corollary 7.6(a), κ' has at most four rows, so the tail t has at most three parts. The tails with at most one part are removed by Corollary 10.51 for every $n \ge 4$ (a one-part tail (j) has $j \le 6 \le N/2$). The tails with two or three parts and $|t| \le 6$ are the sixteen of (a), removed for $n \ge |t| + 3$: for $n \ge 7$ if $|t| \le 4$, for $n \ge 8$ if $|t| = 5$, and for $n \ge 9$ if $|t| = 6$. This gives the three thresholds of (b). □

Remark 10.58. Not claimed: the cases $n < |t| + 3$ (in particular the types of depth 6 at $n = 8$), the types of depth $|t| \ge 7$, and a proof of Theorem 10.55 by hand. No bound on $d(n)$ follows by itself. (The computation at $(3,3,3)$ took about 12 hours; at depth 7, $d = 10$ and π runs over S_{11}.)

10.11. **The column $(\tau_2, 1, 1, 1)$ for $\nu = (3,1,1)$ at every depth.** We keep $\nu = (3,1,1)$, $k = 5$, $N = n^2 - 5$, $n \ge 3$ and the notation of Subsections 10.2 and 10.6. We prove Conjecture 10.18 on the column $\tau = (\tau_2, 1, 1, 1)$ of the first layer, for every $n \ge 4$, and the same statement for $n = 3$ (Theorem 10.69 and Corollary 10.70); this contains Theorem 10.22(a), (b). As a consequence, for every $n \ge 3$, $K_{\lambda(3,1,1)}$ has no conjugation type κ with $\kappa' = (N - j, j)$, at any depth j (Corollary 10.71), the

analogue of Corollary 10.51. The method is that of Subsection 10.9 one degree up: at a diagonal point the unknowns are the pair data, the 3-cycle data and the Cartan data of a 3-form η_A. The step near a point with three equal coordinates (Proposition 10.46) is replaced by two summation identities over the spectator indices (Lemmas 10.64 and 10.66), each of which produces the factor μ. The inputs checked by computer are exact computations (symbolic identities with undefined functions for all the data, and exact row reductions over $\mathbb{Q}$): the pair-only row and the fact that it is a test (Proposition 10.63), the identities (ii) and (iii) of Lemma 10.67, and the action of C on the pattern $\{a,a,b,b,c\}$ (Proposition 10.61; also a direct count). Everything else is by hand.

The setting. Fix $s\in\mathbb{C}$, $\tau_2\ge 1$ and $\tau=(\tau_2,1,1,1)$, so that $d=\tau_2+3$ and $\mu=ns-\tau_2+1$, and $\varphi\in \mathrm{HW}_\tau$; write $\Omega_A:=\Omega f_s(I,A)=s\,(\mathrm{tr}\wedge\omega)-\mathfrak{L}\omega$ (Lemma 10.17). On $M_n^{\otimes 5}$ let $C:=\sum_{i<j}\sigma_{ij}$ be defined as before Lemma 10.30, now with five factors. It acts on the summand $S_\alpha V\otimes S_{\alpha'}V$ of $\Lambda^5 M_n$ by the content sum $c(\alpha)$, which is $10,5,2,0,-2,-5,-10$ for $\alpha=(5),(4,1),(3,2),(3,1,1),(2,2,1),(2,1,1,1),(1^5)$: seven distinct values, and $c(\nu)=0$. So, by Lemma 10.2(iv), Ω_A vanishes on $\Gamma_{1,4}+\Gamma_{2,2}+\Gamma_{4,1}$ if and only if $\Omega_A\circ C=0$. We say that φ *passes the tests at* s if $\Omega_A\circ C=0$ for every A. As $\Omega_A=\Omega_{A+cI}$ (the proof of Lemma 10.17), a nonzero φ passes the tests at s if and only if s is an eigenvalue at τ with eigenvector φ in the sense of Subsection 10.6. Here ω depends only on $A:=A_2$. The form $\eta_A(Y_1,Y_2,Y_3):=\tilde\varphi(A,Y_1,Y_2,Y_3)$ is alternating (the highest weight conditions with $3\le a<b$), $\iota_A\eta_A=0$ (those with $a=2$), it kills I, it depends only on A_0, and it has degree τ_2 in A. As in Subsection 10.9, $X\circ A:=(XA+AX)/2$ and D_X is the derivative in A along X. By Lemma 10.7, $\omega=\mathrm{Cay}\,\varphi(A)=3!\,d\eta$, where $d\eta(B_1,\dots,B_4):=\sum_i(-1)^{i-1}(D_{B_i}\eta)(\hat B_i)$: the 3! orderings of the three linear slots give equal terms. As Ω_A is linear in ω, we put $\varpi:=d\eta=\omega/6$ and

$$T(w):=\tfrac16\,\Omega_A(w)=s\,(\mathrm{tr}\wedge\varpi)(w)-(\mathfrak{L}\varpi)(w)\qquad(w\in\Lambda^5M_n).$$

Lemma 10.59. $(\mathfrak{L}\varpi)(B_1,\dots,B_5)=\sum_{i=1}^5(-1)^{i-1}(D_{B_i\circ A}\,\varpi)(\hat B_i)$.

Proof. By the definition of L_C in Lemma 10.17, $\mathfrak{L}\varpi=\sum_i(-1)^{i-1}(D_{B_iA}\,\varpi)(\hat B_i)+\mathfrak{M}\varpi$, where $\mathfrak{M}\varpi$ collects the terms $\varpi(\dots,B_iB_m,\dots)$ $(m\ne i)$. Write $D_{BA}=D_{B\circ A}+\frac12 D_{[B,A]}$. By conjugation invariance, $(D_{[B_i,A]}\,\varpi)(\hat B_i)=-\sum_{m\ne i}\varpi(\dots,[B_i,B_m],\dots)$. Renaming $i\leftrightarrow m$ and moving the entry B_iB_m to the place of the removed slot (sign $(-1)^{m-i-1}$) shows that the terms $\varpi(\dots,B_mB_i,\dots)$ sum to $-\mathfrak{M}\varpi$; so the terms $\frac12 D_{[B_i,A]}$ add up to $-\mathfrak{M}\varpi$ and cancel $\mathfrak{M}\varpi$. This is the argument of Lemma 10.38, with the renaming in place of the Chevalley–Eilenberg differential δ; no step depends on the degree of the form. □

Lemma 10.60 (diagonal data)**.** *Let $A=\mathrm{diag}(x)$ be regular. The values of η_A on matrix units vanish except for (reorderings of) the* pair data $k_{ab}(z):=\eta_A(E_{ab},E_{ba},\mathrm{diag}(z))$ *$(z\in\mathbb{C}^n)$, the* 3-cycle data $\eta_{abc}:=\eta_A(E_{ab},E_{bc},E_{ca})$ *and the* Cartan data $\eta^{\mathfrak t}$*, the restriction of η_A to $\mathfrak t$ $(a,b,c$ distinct$)$.*

(a) $k_{ab}(\mathbf 1)=0$ and $k_{ab}(x)=0$.

*(b) For $c\notin\{a,b\}$, $k_{ab}(e_c)=q\,K^c$ with $q:=x_a-x_b$ and a polynomial K^c that is even under $x_a\leftrightarrow x_b$ and homogeneous of degree τ_2-1 (*Weyl evenness*).*

(c) All data are translation invariant, and if they all vanish at every regular x, then $\varphi=0$.

Proof. An element of the torus fixes A, so η_A vanishes on the wedges of nonzero weight; the wedges of weight 0 of three matrix units are the ones listed. (a) η_A kills I, and $\iota_A\eta_A=0$. (b) For $c\notin\{a,b\}$ the permutation matrix w of $(a\,b)$ has $wE_{ab}w^{-1}=E_{ba}$ and $wE_{cc}w^{-1}=E_{cc}$, so invariance at every regular x gives $k_{ab}(e_c)(\sigma_{ab}x)=-k_{ab}(e_c)(x)$; hence q divides $k_{ab}(e_c)$, with an even quotient. (The wall $x_a=x_b$ alone would give only $q\mid k_{ab}(e_c)$.) The degree of k_{ab} is τ_2. (c) The data are functions of A_0. As η is polynomial and invariant, it is determined by its values at the regular diagonal points, and η determines φ. □

Coordinates and patterns. Fix $a\ne b$ and call the other $n-2$ indices *spectators*. For a spectator i put $y_i:=x_i-(x_a+x_b)/2$, and let $\partial_i:=\partial/\partial y_i$ (at fixed q), $\partial_{\mathbf 1}:=\sum_i\partial_i$, $E_y:=\sum_i y_i\partial_i$, and let

d_y be the exterior derivative in y. Put $\xi := \sum_i K^i\,dy_i$, a 1-form in y with coefficients polynomial in (q,y), and $\psi := \iota_y\xi$. By Lemma 10.60(a), $k_{ab}(e_a) = -\sum_i (x_i - x_b)K^i = -\psi - \frac{q}{2}\,\iota_{\mathbf{1}}\xi$ and $k_{ab}(e_b) = \sum_i (x_i - x_a)K^i = \psi - \frac{q}{2}\,\iota_{\mathbf{1}}\xi$. Write $\xi = \sum_{j\ \mathrm{even}} q^j \xi_j$, with ξ_j of degree $\tau_2 - 1 - j$ in y. Raw derivatives are written $\partial_{x_i} := \partial/\partial x_i$; on functions of (q,y), $\partial_{x_a} = \partial_q - \frac{1}{2}\partial_{\mathbf{1}}$, $\partial_{x_b} = -\partial_q - \frac{1}{2}\partial_{\mathbf{1}}$, and $\partial_{x_i} = \partial_i$ for a spectator i. At $A = \mathrm{diag}(x)$ the tests split over the patterns (row multiset = column multiset) of the wedges of matrix units of weight 0, as in Lemma 10.40, whose proof does not use $k = 4$: φ passes the tests at A if and only if $T(Cw) = 0$ for every such wedge w. The identities below are polynomial identities in x, proved at regular x.

Proposition 10.61 (the pattern $\{a,a,b,b,c\}$)**.** *Let c be a spectator. The pattern $\{a,a,b,b,c\}$ has the five wedges $Q_1 = E_{aa}\wedge E_{ab}\wedge E_{ba}\wedge E_{bb}\wedge E_{cc}$, $Q_2 = E_{ac}\wedge E_{ca}\wedge E_{ab}\wedge E_{ba}\wedge E_{bb}$, $Q_3 = E_{bc}\wedge E_{cb}\wedge E_{ab}\wedge E_{ba}\wedge E_{aa}$, $Q_4 = E_{ca}\wedge E_{bc}\wedge E_{aa}\wedge E_{ab}\wedge E_{bb}$, $Q_5 = E_{cb}\wedge E_{ac}\wedge E_{aa}\wedge E_{ba}\wedge E_{bb}$, with $CQ_1 = Q_2 - Q_3 - Q_4 + Q_5$, $CQ_2 = Q_1 + Q_4 + Q_5$, $CQ_3 = -Q_1 + Q_4 + Q_5$, $CQ_4 = -Q_1 + Q_2 + Q_3$ and $CQ_5 = Q_1 + Q_2 + Q_3$; so the tests are $T(Q_1) = 0$ and $T(Q_3) = T(Q_5) = -T(Q_2) = -T(Q_4)$. With $\Xi := 3\xi + q\partial_q\xi + d_y\psi$, $\Xi_c := \Xi(\partial_c)$ and $\Xi(\mathbf{1}) := \sum_c \Xi_c$,*

$$T(Q_1) = -\Big[2s\,\Xi_c - q\partial_q\Xi_c + s\,\Xi(\mathbf{1}) - y_c\partial_c\Xi(\mathbf{1}) + \tfrac{q^2}{4}\big(\partial_{\mathbf{1}}^2 K^c - \partial_c\partial_{\mathbf{1}}\iota_{\mathbf{1}}\xi\big)\Big].$$

Proof. The matrix of C is a direct count, as in the proof of Lemma 10.40 (it was also checked by computer), and the equations $T(CQ_i) = 0$ give the tests. For the formula put $\kappa_i := k_{ab}(e_i)$ for $i \in \{a,b,c\}$, $\varepsilon_a = 1$, $\varepsilon_b = -1$, $\varepsilon_c = 0$ and $\Omega(i,j) := \varpi(E_{ab}, E_{ba}, E_{ii}, E_{jj})$. From $\varpi = d\eta$ and Lemma 10.39(c) (the Cartan terms cancel formally),

$$\Omega(i,j) = \tfrac{2}{q}(\varepsilon_i\kappa_j - \varepsilon_j\kappa_i) + \partial_{x_i}\kappa_j - \partial_{x_j}\kappa_i.$$

All brackets of E_{ab}, E_{ba} with the slots of Q_1 lie in the span of E_{ab}, E_{ba}, $E_{aa} - E_{bb}$, and $\varpi(E_{aa}, E_{aa} - E_{bb}, E_{bb}, E_{cc}) = 0$, so by Lemmas 10.59 and 10.39(c) only Ω enters:

$$T(Q_1) = s[\Omega(b,c) - \Omega(a,c) + \Omega(a,b)] - x_a\partial_{x_a}\Omega(b,c) + x_b\partial_{x_b}\Omega(a,c) - x_c\partial_{x_c}\Omega(a,b) \\ - \tfrac{x_a + x_b}{q}[\Omega(b,c) + \Omega(a,c)].$$

Insert $\kappa_a = -\sum_l (x_l - x_b)K^l$, $\kappa_b = \sum_l (x_l - x_a)K^l$ (sums over the spectators l) and $\kappa_c = qK^c$, and put $\partial_{\neq c} := \sum_{l\neq c}\partial_l$, the sum over every spectator $l \neq c$. The part with K^c: with $\Theta := 4K^c + q\partial_q K^c + y_c\partial_c K^c$, we get $\Omega(a,b) = -\Theta - y_c\partial_{\neq c}K^c$, $\Omega(a,c) = \Theta - \frac{q}{2}\partial_{\neq c}K^c$, $\Omega(b,c) = -\Theta - \frac{q}{2}\partial_{\neq c}K^c$, and

$$T(Q_1) = -3s\Theta - sy_c\partial_{\neq c}K^c + q\partial_q\Theta + y_c\partial_c\Theta + y_c\partial_{\neq c}K^c + y_c^2\partial_c\partial_{\neq c}K^c - \tfrac{q^2}{4}\partial_{\mathbf{1}}\partial_{\neq c}K^c.$$

The part with one K^l, $l \neq c$, $G := K^l$: $\Omega(a,b) = -4G - q\partial_q G - y_l\partial_{\mathbf{1}}G$, $\Omega(a,c) = (y_l + \frac{q}{2})\partial_c G$, $\Omega(b,c) = -(y_l - \frac{q}{2})\partial_c G$, and

$$T(Q_1) = s(-2y_l\partial_c G - 4G - q\partial_q G - y_l\partial_{\mathbf{1}}G) + q(y_l + y_c)\partial_q\partial_c G + \tfrac{q^2}{4}\partial_{\mathbf{1}}\partial_c G + 4y_c\partial_c G + y_c y_l\partial_{\mathbf{1}}\partial_c G.$$

In both parts the terms with $(x_a + x_b)/2$ cancel, and both agree term by term with the claimed formula: its part with K^c has $\Xi_c = \Theta$, $\Xi(\mathbf{1}) = \Theta + y_c\partial_{\neq c}K^c$ and $\frac{q^2}{4}(\partial_{\mathbf{1}}^2 - \partial_c\partial_{\mathbf{1}})K^c = \frac{q^2}{4}\partial_{\mathbf{1}}\partial_{\neq c}K^c$; its part with G has $\Xi_c = y_l\partial_c G$, $\Xi(\mathbf{1}) = 4G + q\partial_q G + y_l\partial_{\mathbf{1}}G$ and the last term $-\frac{q^2}{4}\partial_c\partial_{\mathbf{1}}G$. The K^l enter one at a time, and neither homogeneity nor evenness is used, so the formula holds for every $n \geq 4$, and for $n = 3$, where only the part with K^c occurs ($\partial_{\neq c} = 0$). □

Corollary 10.62. *Suppose that φ passes the tests at s, and that $\mu \neq 0$.*

(a) $\sum_c T(Q_1) = -\mu\,\Xi(\mathbf{1})$, so $\Xi(\mathbf{1}) = 0$, that is, $\iota_{\mathbf{1}}[(j+3)\xi_j + d_y\iota_y\xi_j] = 0$ for every even j, and

$$(j - 2s)\big[(j+3)\xi_j + d_y\iota_y\xi_j\big] = \tfrac{1}{4}\,L_{\mathbf{1}}\iota_{\mathbf{1}}d_y\,\xi_{j-2} \qquad (\xi_{-2} := 0),$$

where $L_{\mathbf{1}}\iota_{\mathbf{1}}d_y\xi = \partial_{\mathbf{1}}^2\xi - d_y\partial_{\mathbf{1}}\iota_{\mathbf{1}}\xi$.

(b) The operator $\xi \mapsto (j+3)\xi + d_y\iota_y\xi$ is injective on the 1-forms whose coefficients have degree $\tau_2 - 1 - j$.
(c) If $s \notin \{0,1,2,\dots\}$, then $\xi \equiv 0$.
(d) If $n = 3$, then $K^c = 0$.

Proof. (a) Sum the formula of Proposition 10.61 over the spectators c: the last terms cancel, and $\Xi(\mathbf{1})$ is homogeneous of degree $\tau_2 - 1$ in (q, y), so Euler's identity gives the factor $2s + (n-2)s - (\tau_2 - 1) = \mu$. (This is the free-slot Lemma 10.19: $\sum_c Q_1 = E_{aa} \wedge E_{ab} \wedge E_{ba} \wedge E_{bb} \wedge I$, so $\sum_c T(Q_1) = \mu\,\Omega(a,b)$ in the notation of the proof of Proposition 10.61, and there $\Omega(a,b) = -\Xi(\mathbf{1})$, in the part with K^c and in the part with each K^l.) With $\Xi(\mathbf{1}) = 0$, compare the coefficients of q^j in $T(Q_1) = 0$. (b) For such a form, Cartan's formula $L_y\xi = (\tau_2 - j)\xi$ gives $\xi = (d_y\iota_y\xi + \iota_y d_y\xi)/(\tau_2 - j)$, and $(j+3)\xi + d_y\iota_y\xi = (\tau_2 + 3)\xi - \iota_y d_y\xi$; $\iota_y d_y$ is 0 on the first part and $\tau_2 - j$ on the second, so the operator has the eigenvalues $\tau_2 + 3$ and $j + 3$, both positive. (c) Then $j - 2s \neq 0$ for every even j, and $\xi_j = 0$ by induction on j. (No division by s occurs: $s = 0$ is the case $m = 0$ below.) (d) For $n = 3$ there is one spectator c, $d_y\xi = 0$, and $T(Q_1) = -\mu(\tau_2 + 3)K^c$. □

Proposition 10.63 (the pair-only row; computer-assisted)**.** *Let $n \geq 4$ and let $c \neq e$ be spectators. In the pattern $\{a, a, b, c, e\}$ put $w_0 := E_{aa} \wedge E_{ab} \wedge E_{ba} \wedge E_{cc} \wedge E_{ee}$, $w_1 := E_{ab} \wedge E_{ac} \wedge E_{be} \wedge E_{ca} \wedge E_{ea}$, $w_2 := E_{ab} \wedge E_{ae} \wedge E_{bc} \wedge E_{ca} \wedge E_{ea}$, $w_3 := E_{ac} \wedge E_{ae} \wedge E_{ba} \wedge E_{ca} \wedge E_{eb}$ and $w_4 := E_{ac} \wedge E_{ae} \wedge E_{ba} \wedge E_{cb} \wedge E_{ea}$. Then $w_0 - \frac{1}{2}w_1 - \frac{1}{2}w_2 + \frac{1}{2}w_3 - \frac{1}{2}w_4$ lies in the image of C on the span of the pattern, so $R_{ce} := T(w_0) - \frac{1}{2}T(w_1) - \frac{1}{2}T(w_2) + \frac{1}{2}T(w_3) - \frac{1}{2}T(w_4)$ vanishes if φ passes the tests. With $x_{ij} := x_i - x_j$ and $\partial_{abce} := \partial_{x_a} + \partial_{x_b} + \partial_{x_c} + \partial_{x_e}$,*

$$\begin{aligned}
R_{ce} &= \mathcal{F}_c[K^c] - \mathcal{F}_e[K^e] - s(x_a - 2x_b + x_c)\,\partial_{abce}K^c + s(x_a - 2x_b + x_e)\,\partial_{abce}K^e \\
&\quad - \textstyle\sum_{l \notin \{a,b,c,e\}} (x_b - x_l)\big[s(\partial_{x_c} - \partial_{x_e}) + x_{ce}\,\partial_{x_c}\partial_{x_e}\big]K^l, \\
\mathcal{F}_c[G] &= s\,\mathcal{G}_c[G] + x_{ab}x_{ae}\,\partial_{x_a}\partial_{x_e}G - x_{bc}x_{ce}\,\partial_{x_c}\partial_{x_e}G + (2x_a + x_b + x_c - 4x_e)\,\partial_{x_e}G, \\
\mathcal{G}_c[G] &= (2x_a - 3x_b + x_c)\,\partial_{x_a}G + (x_a - 2x_b + x_c)\,\partial_{x_b}G + (x_a - 3x_b + 2x_c)\,\partial_{x_c}G + 4G,
\end{aligned}$$

and $\mathcal{F}_e$ is $\mathcal{F}_c$ with c and e exchanged. Every term with 3-cycle data or Cartan data cancels.

Proof. For $n = 4$ and $n = 5$, with one ordered pair (c, e), both statements are exact computations. The vector is a row of the exact reduced row echelon form over $\mathbb{Q}$ of the image of C on the span of the pattern (27 wedges and 18 independent tests; both counts were also done by hand). The identity is checked symbolically, with undefined functions of the differences $x_i - x_n$ for all the data (the pair data of every pair, the 3-cycle data and the Cartan data), the raw ∂_{x_i} being coordinate derivatives in them; the same run rejects the negative control $R_{ce} + (E_x - 3)K^c$, $E_x := \sum_i x_i\partial_{x_i}$, which is 0 on all data of degree 3. An operator that kills every function of the differences kills every function of more variables; this gives the base case. The other pairs (c, e) follow by relabelling. For every n: the five wedges use only the indices a, b, c, e, and the far data enter only through $k_{ab}(e_a) = -\sum_l x_{lb}K^l$ and $k_{ab}(e_b) = \sum_l x_{la}K^l$, linearly and one far index at a time; so the part with K^c and K^e (in raw derivatives) is the same for every $n \geq 5$, and each far l contributes the term of $n = 5$ with the far index renamed. The data of the other pairs in the five wedges can carry far sums through their own expansions of $k(e_i)$ too: at $n = 5$ the computed row (before expansion) has far terms only of k_{ac} and k_{ae} (besides k_{ab}), the data of k_{bc}, k_{be} occur only with indices in $\{a, b, c, e\}$, and k_{ce} does not occur. The identity at $n = 5$ shows that the far terms of k_{ac} and k_{ae} cancel; as far indices enter linearly and one at a time, they cancel at every n. The Cartan terms of w_0 cancel formally (the two root slots E_{ab}, E_{ba} give opposite terms), so at every n, without using $\iota_{\mathbf{1}}\eta^{\mathrm{t}} = 0$ or $\iota_x\eta^{\mathrm{t}} = 0$. □

In the coordinates of the setting, $\partial_{abce} = -\partial_{\mathrm{far}}$ with $\partial_{\mathrm{far}} := \sum_{l \notin \{a,b,c,e\}} \partial_l$, and $x_b - x_l = -q/2 - y_l$; each term of R_{ce} raises the power of q by 0, 1 or 2. As K is even in q, the even and the odd part in q of R_{ce} vanish separately when φ passes the tests.

Lemma 10.64 (the odd part)**.** *Let $n \geq 4$, let $c \neq e$ be spectators, and let j be even. Put $\beta := d_y\xi_j$, that is, $\beta_{ik} = \partial_i K^k_j - \partial_k K^i_j$ with K^k_j the coefficient of dy_k in ξ_j, and $\gamma := \iota_{\mathbf{1}}\beta$, $\gamma_k = \sum_i \beta_{ik}$. The terms of R_{ce} of order q^{j+1} are $\frac{1}{2}O_{ce}$, where*

$$O_{ce} := (j+1-2s)\,\beta_{ec} + (s - y_c\partial_c)\,\gamma_e - (s - y_e\partial_e)\,\gamma_c,$$

and $\sum_{e\neq c} O_{ce} = -\mu\,\gamma_c$. Hence, if φ passes the tests at $s = m \in \{0,1,2,\dots\}$ and $\mu \neq 0$, then $d_y\xi_j = 0$ for every j.

Proof. The terms of order q^{j+1} come from the data $q^j\xi_j$ only, and they are

$$\tfrac{1}{2}\Big\{\big[(j+1-4s) + y_c\partial_c + y_e\partial_e\big]\beta_{ec} - s\,\partial_{\mathrm{far}}(K^c - K^e) + \partial_{\mathrm{far}}(y_e\partial_e K^c - y_c\partial_c K^e) \\ + \textstyle\sum_l \big[s(\partial_c - \partial_e) + (y_c - y_e)\partial_c\partial_e\big]K^l\Big\}$$

(K^i for K^i_j; l runs over the far spectators): the first bracket comes from the terms $-2sq\partial_e$, $\frac{q^2}{2}\partial_q\partial_e$, $\frac{qy_c}{2}\partial_c\partial_e$, $\frac{qy_e}{2}\partial_e^2$ and $\frac{q}{2}\partial_e$ of $\mathcal{F}_c$, minus their mirror images in $\mathcal{F}_e$; the next two from $\partial_{abce} = -\partial_{\mathrm{far}}$ and from the parts $-\frac{1}{2}\partial_{\mathbf{1}}$ of ∂_{x_a}, ∂_{x_b}; the last from the part $-q/2$ of $x_b - x_l$. The term $-2sq\partial_e$ comes from $s\,\mathcal{G}_c$, which gives $-2sq(\partial_e + \partial_{\mathrm{far}})$; its half with ∂_{far} is grouped with the term of ∂_{abce} into $-s\,\partial_{\mathrm{far}}(K^c - K^e)$. Write $\partial_c K^l = \partial_l K^c + \beta_{cl}$ and $\partial_c\partial_e K^l = \partial_e\beta_{cl} + \partial_l\partial_e K^c = \partial_c\beta_{el} + \partial_l\partial_c K^e$: the two terms with ∂_{far} cancel, and what is left of the sum is $\sum_l[s(\beta_{cl} - \beta_{el}) + y_c\partial_c\beta_{el} - y_e\partial_e\beta_{cl}]$. Finally $\sum_{l\neq c,e}\beta_{cl} = -\gamma_c + \beta_{ec}$ and $\sum_{l\neq c,e}\beta_{el} = -\gamma_e - \beta_{ec}$ turn the whole into O_{ce}. For the sum: $\sum_e \beta_{ec} = \gamma_c$, $\sum_e \gamma_e = 0$, $\sum_{e\neq c} y_e\partial_e = E_y - y_c\partial_c$, and γ has degree $\tau_2 - 2 - j$; so the coefficient of γ_c is $(j+1-2s) - s - (n-3)s + (\tau_2 - 2 - j) = \tau_2 - 1 - ns = -\mu$. If $\mu \neq 0$, then $\gamma = 0$, so $(j+1-2s)\beta = 0$; for $s = m$ and j even, $j+1-2m \neq 0$, so $d_y\xi_j = 0$. (The row needs two spectators $c \neq e$, that is, $n \geq 4$. For $n = 4$, $O_{ce} = (\tau_2 - 1 - 4s)\beta_{ec} = -\mu\beta_{ec}$ directly.) □

Lemma 10.65. *Let $n \geq 4$ and $\mu \neq 0$, and suppose that φ passes the tests at $s = m \in \{0,1,2,\dots\}$. Then $\xi_j = 0$ for $j \neq 2m$, and $\xi_{2m} = d_y\chi$ with $\chi := \iota_y\xi_{2m}/(\delta+1)$, $\delta := \tau_2 - 1 - 2m$ (if $\delta < 0$, then $\xi = 0$); χ is homogeneous of degree $\delta + 1$, and $\partial_{\mathbf{1}}\chi = 0$.*

Proof. By Lemma 10.64, $d_y\xi_{j-2} = 0$, so the right side of the recursion in Corollary 10.62(a) is 0; for closed ξ_j, $(j+3)\xi_j + d_y\iota_y\xi_j = (j+3)\xi_j + L_y\xi_j = (\tau_2 + 3)\xi_j$. So $(j-2m)(\tau_2+3)\xi_j = 0$, and $\xi_j = 0$ for $j \neq 2m$. As ξ_{2m} is closed and homogeneous, $\xi_{2m} = d_y\chi$ (Cartan's formula). Finally, $\Xi(\mathbf{1}) = 0$ at q^{2m} gives $(\tau_2 + 3)\,\iota_{\mathbf{1}}\xi_{2m} = 0$, that is, $\partial_{\mathbf{1}}\chi = 0$. □

Lemma 10.66 (the even part)**.** *Let $n \geq 4$, $s = m$, and let $K^i = q^{2m}\partial_i\chi$ for every spectator i, with χ homogeneous of degree $\delta + 1$, $\delta = \tau_2 - 1 - 2m$. Then $R_{ce} = (\tau_2 + 3)\,q^{2m}\mathcal{L}_{ce}\chi$ with $\mathcal{L}_{ce} := s(\partial_c - \partial_e) + (y_c - y_e)\partial_c\partial_e$. If moreover $\partial_{\mathbf{1}}\chi = 0$, then $\sum_{e\neq c}\mathcal{L}_{ce}\chi = \mu\,\partial_c\chi$. Hence, if φ passes the tests and $\mu \neq 0$, then $d_y\chi = 0$.*

Proof. The terms of order q^{2m+1} are $\frac{1}{2}O_{ce} = 0$ (Lemma 10.64, as $\beta = 0$), and those of order q^{2m+2} are $-\frac{q^2}{4}\partial_{\mathbf{1}}\beta_{ec} = 0$ (from $-\frac{q^2}{4}(\partial_c + \partial_e)\partial_e$ in $\mathcal{F}_c$, from $-\frac{q^2}{4}\partial_{\mathrm{far}}\partial_e$, which comes from the part $-\frac{1}{2}\partial_{\mathbf{1}}$ of ∂_{x_a}, and their mirror images). With $j = 2m$ and $\chi_c := \partial_c\chi$, $\chi_{ce} := \partial_c\partial_e\chi$, and so on, the terms of order q^{2m} are

$$s(j+4)(\chi_c - \chi_e) + s(y_c\chi_{cc} - y_e\chi_{ee}) + s(y_e - y_c)\chi_{ce} + j(y_c - y_e)\chi_{ce} + (y_c - y_e)(y_c\chi_{cce} + y_e\chi_{cee}) \\ + 5(y_c - y_e)\chi_{ce} + \textstyle\sum_l y_l\big[s(\chi_{cl} - \chi_{el}) + (y_c - y_e)\chi_{cel}\big];$$

Euler's identity for χ_c, χ_e (degree δ) and for χ_{ce} (degree $\delta-1$) turns this into $s(j+4+\delta)(\chi_c - \chi_e) + (j+\delta+4)(y_c - y_e)\chi_{ce} = (\tau_2+3)\mathcal{L}_{ce}\chi$. This does not use $\partial_{\mathbf{1}}\chi = 0$; the sum does. As $\sum_{e\neq c}\partial_e\chi = -\partial_c\chi$, the terms with s give $s(n-2)\partial_c\chi$; and $\sum_{e\neq c}(y_c - y_e)\partial_c\partial_e\chi = \partial_c\big[\sum_{e\neq c}(y_c - y_e)\partial_e\chi\big] + \partial_c\chi = \partial_c[-(\delta+1)\chi] + \partial_c\chi = -\delta\,\partial_c\chi$. The total is $((n-2)m - \delta)\,\partial_c\chi = \mu\,\partial_c\chi$. If φ passes the tests, then $R_{ce} = 0$, so $\mathcal{L}_{ce}\chi = 0$ for all $c \neq e$, and $\mu \neq 0$ gives $\partial_c\chi = 0$ for every c. □

Lemma 10.67 (the 3-cycle data; computer-assisted)**.** *Let $n \geq 3$ and $s \in \mathbb{C}$, and suppose that φ passes the tests at s and that all pair data vanish. Fix distinct a, b, c, put $\vartheta := x_{ab}x_{bc}x_{ca}$, $p := x_{ac}$, $t := x_{bc}$, $\partial_p := \partial_{x_a}$ and $\partial_t := \partial_{x_b}$ (at fixed x_c), and let $Q_1, \dots, Q_5$ be the wedges of Proposition 10.61.*

(i) $\eta_{abc} = \vartheta U$ *and* $\eta_{acb} = \vartheta U'$ *with polynomials U, U'; put $\Delta := U' - U$.*

(ii) $T(Q_2) = \mathcal{P}\Delta := t\,[(st + 2x_{ab} - t)\Delta + x_{ab}t\,\partial_{x_b}\Delta]$*, and $T(Q_3)$ is the same with $a \leftrightarrow b$ and the opposite sign; so the test $T(Q_2)+T(Q_3) = 0$ reads $x_{ab}\,\mathcal{B}\Delta = 0$, with $\mathcal{B}\Delta := (3-s)(p+t)\Delta + (p^2\partial_p + t^2\partial_t)\Delta$.*

(iii) $T(Q_4) = \mathcal{T}U$ *and* $T(Q_5) = \mathcal{T}U'$*, where* $E_{pt} := p\partial_p + t\partial_t$ *and*

$$\mathcal{T} = (6-2s)(p^2+t^2) + (8s-10)pt + \big[2p^2 + (s-5)pt + 2t^2\big]E_{pt} - (s-1)pt\,(t\partial_p + p\partial_t) + pt\,(p-t)^2\partial_p\partial_t.$$

(iv) $\eta_{abc} = \eta_{acb} = 0$.

Proof. Every n. Q_2, Q_3 have four and Q_4, Q_5 three root vectors, all with indices in $\{a, b, c\}$, and their diagonal slots are only E_{aa}, E_{bb}. A slot of $\mathfrak{L}\varpi$ (Lemma 10.59) differentiates along E_{aa} or E_{bb}, or, for a root slot, replaces one other slot by a bracket with a root vector on $\{a, b, c\}$ (Lemma 10.39(c)); so every value keeps at least one root vector, hence (weight 0) at least two, all on $\{a, b, c\}$: pair or 3-cycle data of $\{a, b, c\}$, and Cartan terms inside values $\varpi(E_{ij}, E_{ji}, \cdot, \cdot)$ that cancel formally (not by a property of η^{t}). A spectator index enters only through the spectator sums in the pair data $k_{ij}(e_i)$, which vanish here. On the 3-cycle data the only derivative directions are E_{aa} and E_{bb}: an $E_{ii} - E_{jj}$ from a bracket appears only next to a 2-cycle, that is, with pair data. So the operators on the 3-cycle data involve only $\partial_{x_a} = \partial_p$ and $\partial_{x_b} = \partial_t$, with coefficients in x_a, x_b, x_c, and the other coordinates are parameters. Hence the identities (ii) and (iii), proved at $n = 3$ or $n = 4$, hold at every n; and this lemma does not use Lemma 10.68.

(i) Let $\varpi_{abc}(z) := \varpi(E_{ab}, E_{bc}, E_{ca}, \mathrm{diag}(z))$. From $\varpi = d\eta$, each of the three terms with a root slot is, by Lemma 10.39(c), a sum of brackets divided by an x_{ij}; the brackets give pair values $k(z)$ and the 3-cycle value times $z_i - z_j$, and the term with $D_{\mathrm{diag}(z)}$ gives $-\partial_z\eta_{abc}$. Over the common denominator ϑ,

$$\varpi_{abc}(z) = \vartheta^{-1}\big[x_{ab}^2 k_{ab}(z) + x_{bc}^2 k_{bc}(z) + x_{ca}^2 k_{ca}(z) - \partial_z(\vartheta\,\eta_{abc})\big],$$

and likewise for ϖ_{acb}, with η_{acb}. As ϖ_{abc} is a value of ϖ, it is a polynomial. With the pair data 0, ϑ divides $\partial_z(\vartheta\,\eta_{abc})$ for every z, so each linear factor of ϑ divides η_{abc}, one at a time; the same holds for η_{acb}.

(ii), (iii) Exact symbolic computations, with undefined functions for the data: (ii) at $n = 3$ and $n = 4$, with general pair data, which cancel; (iii) at $n = 4$, which contains $n = 3$. By the first paragraph they hold at every n. The form of the test in (ii) is by hand: $T(Q_2)+T(Q_3) = (p-t)\,\mathcal{B}\Delta$, and $p - t = x_{ab}$.

(iv) (1) $\mathcal{T}$ is symmetric under $p \leftrightarrow t$, so the operator in $T(Q_5)$ is the same as in $T(Q_4)$. (2) $\mathcal{T}$ has no polynomial kernel on any slice of (p, t)-degree M, for any s. Put $U = t^M g(\rho)$, $\rho = p/t$; then $\mathcal{T}U = t^{M+2}L[g]$ with

$$L[g] = -\rho^2(\rho-1)^2 g'' + \big[(M-1)\rho(\rho-1)^2 - (s-1)\rho(1-\rho^2)\big]g' + \big[(6-2s)(\rho^2+1) + (8s-10)\rho + M(2\rho^2 + (s-5)\rho + 2) - (s-1)M\rho^2\big]g,$$

a Fuchsian equation with the singular points $0, 1, \infty$ and the exponents $\{M + 3 - s, -2\}$ at 0 and $\{2s + 1, -2\}$ at 1. By the symmetry $p \leftrightarrow t$, $g \mapsto \rho^M g(1/\rho)$ maps solutions to solutions, so $M - \deg g \in \{M + 3 - s, -2\}$, that is, $\deg g \in \{s - 3, M + 2\}$; as U is a polynomial, homogeneous of degree M in (p, t), $\deg g \leq M$, so $\deg g = s - 3$ (the Fuchs relation, sum of the six exponents = 1, is only a consistency check). A polynomial $g \neq 0$ is regular at 0 and 1, so it takes the nonnegative exponent at each (the branch -2 has a pole, also when a logarithm occurs): $\mathrm{ord}_0\, g = M + 3 - s$, $\mathrm{ord}_1\, g = 2s+1$ and $\deg g = s-3$, so $M+3-s+2s+1 \leq s-3$, that is, $M \leq -7$, which is impossible. (3) The tests give $T(Q_4) + T(Q_5) = 0$, that is, $\mathcal{T}(U + U') = 0$, so $U' = -U$ and $\Delta = -2U$; then

$T(Q_2) = T(Q_4)$ reads $(\mathcal{T} + 2\mathcal{P})U = 0$, and $\mathcal{B}\Delta = 0$ reads $\mathcal{B}U = 0$. (4) On a slice, $\mathcal{B}U = 0$ gives $U = 0$ unless $s = 3 + r$ with $r \in \{0, \dots, M\}$ and $j := 2r - M \geq 0$; then $U = \Phi\,(pt)^i(p-t)^j$ with $i := M - r$ and a function Φ of the other coordinates ($\mathcal{B}$ is triangular with exactly one zero on the diagonal). (5) At $s = 3 + r$, $(\mathcal{T} + 2\mathcal{P})[(pt)^i(p-t)^j] = [(r+3)(r+6) - i(i+3)]\,pt\,(pt)^i(p-t)^j$, and $i \leq r$ makes the bracket positive. So $U = 0$, and then $U' = 0$. □

Lemma 10.68 (the Cartan part)**.** *Suppose that all pair data and all 3-cycle data vanish and that φ passes the tests at s. If $\mu \neq 0$, or if $n \leq 4$, then $\eta^{\mathfrak{t}} = 0$, so $\varphi = 0$.*

Proof. $n \leq 4$: at a regular x the vectors $\mathbf{1}$ and x are independent and lie in the kernel of the 3-form $\eta^{\mathfrak{t}}_x$ on $\mathbb{C}^n$, so $\eta^{\mathfrak{t}}_x = 0$. $n \geq 5$: for five distinct indices and a permutation π of them let w_π be the wedge of the $E_{i,\pi(i)}$. Then $\sum_\pi w_\pi$ is the part of weight 0 of the grid $(e_1 v^T) \wedge \cdots \wedge (e_5 v^T)$, $v = e_1 + \cdots + e_5$ (indices renamed), which lies in $S_{(1^5)}V \otimes S_{(5)}V$, where C acts by $c((1^5)) = -10 \neq 0$; so the tests give $T(\sum_\pi w_\pi) = 0$. With pair and 3-cycle data 0, $T(w_\pi) = 0$ for $\pi \neq \mathrm{id}$: every value of ϖ with a root vector is 0 (the Cartan terms inside $\varpi(E_{ij}, E_{ji}, \cdot, \cdot)$ cancel formally), and a value on $\mathfrak{t}$ arises only for a transposition $\pi = (a\,b)$, through $[E_{ab}, E_{ba}] = E_{aa} - E_{bb}$, and there the two root slots cancel. So T vanishes on the wedges of five distinct diagonal units, that is, with $\Psi := \varpi|_{\mathfrak{t}} = d\eta^{\mathfrak{t}}$ (d on $\mathfrak{t}$; along $\mathfrak{t}$ the point stays diagonal),

$$s\left(\sum_a dx_a\right) \wedge \Psi = \sum_a dx_a \wedge x_a \partial_{x_a} \Psi,$$

and $\iota_{\mathbf{1}}\Psi = 0$ (from $\iota_{\mathbf{1}}\eta^{\mathfrak{t}} = 0$ and translation invariance). The monomial argument in the proof of Lemma 10.48 gives $\Psi = 0$ if $ns \neq \tau_2 - 1$, that is, if $\mu \neq 0$. Then, with the Euler field E_x of $\mathfrak{t}$, $(\tau_2 + 3)\,\eta^{\mathfrak{t}} = L_{E_x}\eta^{\mathfrak{t}} = d\,\iota_{E_x}\eta^{\mathfrak{t}} + \iota_{E_x} d\eta^{\mathfrak{t}} = 0$, as $\iota_x \eta^{\mathfrak{t}} = 0$. So all data vanish, and $\varphi = 0$ by Lemma 10.60(c). □

Theorem 10.69 (the column $(\tau_2, 1, 1, 1)$ for $\nu = (3, 1, 1)$; computer-assisted)**.** *Let $n \geq 3$, $\tau_2 \geq 1$, $\tau = (\tau_2, 1, 1, 1)$ and $s \in \mathbb{C}$. If a nonzero $\varphi \in \mathrm{HW}_\tau$ passes the tests at s, then $ns = \tau_2 - 1$, that is, $\mu = ns - d + 4 = 0$.*

Proof. Let $\mu \neq 0$. For $n = 3$ the pair data vanish by Corollary 10.62(d). Let $n \geq 4$. If $s \notin \{0, 1, 2, \dots\}$, then $\xi \equiv 0$ by Corollary 10.62(c). If $s = m \in \{0, 1, 2, \dots\}$, then Lemma 10.64 gives $d_y \xi_j = 0$ for every j, Lemma 10.65 gives $\xi = q^{2m} d_y \chi$ with $\partial_{\mathbf{1}}\chi = 0$, and Lemma 10.66 gives $d_y \chi = 0$; so $\xi \equiv 0$. Hence the pair data of (a, b) vanish (Lemma 10.60(a), (b)), and by relabelling those of every pair. Then the 3-cycle data vanish (Lemma 10.67), and $\varphi = 0$ by Lemma 10.68. □

Corollary 10.70 (computer-assisted)**.** *Conjecture 10.18 holds on the column $\tau = (\tau_2, 1, 1, 1)$, $\tau_2 \geq 1$, for every $n \geq 4$, and the same statement holds for $n = 3$.*

Proof. Here $d = \tau_2 + 3$, so the $\mu = ns - d + 4$ of Conjecture 10.18 is $ns - \tau_2 + 1$; an eigenvector at s is a nonzero φ that passes the tests at s. Apply Theorem 10.69. □

Corollary 10.71 (two-row types; computer-assisted)**.** *For every $n \geq 3$, $K_{\lambda(3,1,1)}$ has no conjugation type κ with $\kappa' = (N - j, j)$, $0 \leq j \leq N/2$. Equivalently, no conjugation type κ of $K_{\lambda(3,1,1)}$ has $\tau = \kappa' + (1^5)$ of the form $(\tau_1, \tau_2, 1, 1, 1)$, at any depth.*

Proof. For $\kappa' = (N - j, j)$, $\tau = (N - j + 1, j + 1, 1, 1, 1)$, so $\tau_2 = j + 1$, $d = \tau_2 + 3$, and the depth is $N - \ell(\kappa) = d - 4 = j$. If κ occurs, Theorem 10.1(b) gives a nonzero $f \in \mathrm{Pol}_\tau$ that passes $(T_{r,c})$ at every P for the three live corners. By Theorem 10.1(c), $f \mapsto \varphi_f := f(I, \cdot, \dots, \cdot)$ is injective for every n, so $\varphi_f \neq 0$, and $f = \det(P_1)^n \varphi_f((P_1^{-1}P_2)_0, \dots)$, which is f_s at $s = n$. And $\varphi_f \in \mathrm{HW}_\tau$ for every n, by the proof of Theorem 10.1(c), which does not use $d \leq n$: the invariance under (g, g^{-T}), $g \in \mathrm{SL}_n$, gives the invariance under simultaneous conjugation, the homogeneity carries over, and the highest weight conditions $f(\dots, P_b + tP_a, \dots) = f$ give, for $a = 1$, that φ_f depends only on the traceless parts, and, for $2 \leq a < b$, the conditions of HW_τ; only the bijection onto HW_τ needs

$d \le n$, and it is not used. The tests at every P include those at $P = (I, A)$, so, by the definition of an eigenvalue in Subsection 10.6 (for $s = n$ it is the test (13) at $P_1 = I$), $s = n$ is an eigenvalue at τ with eigenvector φ_f. But $\mu = n^2 - d + 4 = \tau_1 + 4 \ge 5$, as $\tau_1 = n^2 - d \ge \tau_2 \ge 1$. So Theorem 10.69 gives $\varphi_f = 0$, a contradiction. □

Remark 10.72. (1) Theorem 10.69 contains Theorem 10.22(a), (b) ($\tau_2 = 2, 3$, $n \ge 4$; there by the rows (R1) and (R3)) and extends them to every $\tau_2 \ge 1$ and to $n = 3$; Theorem 10.22(c), $\tau = (2, 2, 1, 1)$, is not on the column. (2) Against Theorem 10.15(b) (depth at most $n - 3$) and Proposition 9.11 (depth at most 5, computer-assisted; at most 7 for $n \ge 10$), Corollary 10.71 adds, for $n \ge 5$, the two-row types of depth at least $\max(n - 2, 6)$, which Remark 10.16 leaves open. At $n = 4$, $\lambda(3, 1, 1) = (4, 3, 3, 1)$ and $K_{(4,3,3,1)} = 0$ by the direct computation of Remark 7.9, so there it is a second proof (with exact symbolic identities checked by computer) of a computed fact, as in Remark 10.23(4); at $n = 3$ ($N = 4$; on the column the depth is at most 2) we claim nothing new. It concerns the two-row types only: statement (iii) in the proof of Theorem G remains open. (3) The proof uses, at diagonal points, the tests of the pattern $\{a, a, b, b, c\}$, the pair-only row of $\{a, a, b, c, e\}$ and, for $n \ge 5$, the row $\sum_\pi w_\pi$ of five distinct indices. Independent computations, not used in the proof: the identity of Proposition 10.61 (also symbolically at $n = 4, 5$) and the row R_{ce} on random data for all the data functions ($n = 4, 5, 6$; for R_{ce}, $n = 5, 6$); the system of the rows of one pair (the rows $T(Q_1)$ for every spectator and R_{ce} for every ordered pair) at random points modulo a prime, with kernel 0 in all 40 cases with $\mu \ne 0$ at $n = 4$, $\tau_2 = 3, \dots, 9$, in the 7 cases with $\mu \ne 0$ at $n = 5$, $\tau_2 = 5, 6$, and at $n = 5$, $\tau_2 = 7$, $s = 0, 1, 2, 7/5$; and a second program for the pencil of the three corners on $\mathrm{HW}_{(\tau_2,1,1,1)}(n)$, $n = 3, 4$, $\tau_2 = 2, \dots, 5$, which finds, modulo a prime, one root $s = (\tau_2 - 1)/n$ in each case, and kernel 0 at the integers $s = 0, \dots, 3$ with $\mu \ne 0$. (4) Not covered: the first-layer weights $(\tau_2, \tau_3, \tau_4, 1)$ with $\tau_3 \ge 2$ (among them Theorem 10.22(c)) and the deeper layers. Not claimed: that $\mu = 0$ occurs. No bound on $d(n)$ follows by itself.

11. Small n and open problems

We collect what our results give for small n, and the problems they leave open. Nothing in this section is used elsewhere.

11.1. **Small n.** For $n \le 4$ the values of $d(n)$ are known (Subsection 1.2), and our methods add structure, not values. For $n = 3$ the whole space $C_6 \cap B_3^{(6)}$ lies in the staircase block (Remark 6.13), and $U_{(3,2,1)} \ne 0$ (Theorem 8.15): Kuzmin's bound $d(3) \ge 6$ is seen in one block. For $n = 4$ the hypotheses of Theorems F and G ($k = 3$) hold by $d(4) = 10$, and $K_{B(4,2)}$ is a single copy of $(3, 1^{10})$ (Remark 9.10).

For $4 \le n \le 8$ the bounds are:

n	4	5	6	7	8
$n(n+1)/2$ (Kuzmin)	10	15	21	28	36
upper bound before this paper	10	21	36	49	64
$n^2 - 2$ (Theorem B)	14	23	34	47	62
$n^2 - 3$ (if $U_{B(n,2)} = 0$, Theorem F)	13	22	33	46	61
$n^2 - S(n)$ (Corollary 8.13)	12	19	30	40	52
its pair (a, b)	$(2,2)$	$(2,3)$	$(3,4)$	$(4,4)$	$(4,5)$

The row "before this paper" is $d(4) = 10$, then $d(5) \le 21$ (Higman's recursion), and Razmyslov's n^2 for $n \ge 6$, where the recursion gives only 43, 87, 175. The rows $n^2 - 3$ and $n^2 - S(n)$ are conditional: the first on $U_{B(n,2)} = 0$ (which holds for $n = 4, 5$), the second on $\mathrm{PIdeg}(M_{a,b}(\Lambda)) \ge n^2 - S(n) + 1$ for the pair shown; the lower half of Samoilov's conjecture for that one algebra implies this (Lemma 8.12), and for $n = 5$ the two are the same.

$n = 5$. Here $15 \le d(5) \le 21$, and both bounds are classical; Theorems A and B give only 24 and 23. The block $B(5, 2)$ is computed completely: $K_{B(5,2)}$ is a single copy of $(3, 1^{19})$ (Theorem 9.27),

so at $n = 5$ the deep types of Remark 9.26 carry nothing; $U_{B(5,2)} = 0$ itself is classical. To our knowledge, any bound below 21 would be new. Two routes are visible. First, $d(5) \le 20$ follows from the finite list of Theorem G for $k = 4$ at $n = 5$ (Remark 6.12; shapes of sizes 17 to 21). Second, $d(5) \le 19$ follows from $\mathrm{PIdeg}(M_{2,3}(\Lambda)) \ge 20$, by Theorem 8.9 with $n = 5$ and $(a, b) = (2, 3)$: $S = (5-2)(5-3) = 6$, the hypothesis is $\mathrm{PIdeg} \ge n^2 - S + 1 = 20$, and the conclusion is $d(5) \le n^2 - S = 19$; this is Problem 11.5(b) below.

$n = 6$. Here our bound is new: $21 \le d(6) \le 34$, while Razmyslov's bound is 36 and Higman's recursion gives 43. A positive answer to Problem 11.1 at $n = 6$ gives 33. By Higman's recursion $d(6) \le 2d(5) + 1$ (Subsection 1.2), a bound $d(5) \le 16$ would also give $d(6) \le 33$, and Kuzmin's conjecture for $n = 5$ would give $d(6) \le 31$; so the case $n = 5$ matters for $n = 6$.

11.2. **Open problems.**

Problem 11.1. Let $n \ge 6$ and $N = n^2 - 3$. Is $K_{B(n,2)}$ a single copy of $(3, 1^{N-3})$? Weaker, and enough: is $U_{B(n,2)} = 0$?

Either answer, if positive, gives $d(n) \le n^2 - 3$ (Corollary 9.25 and Theorem F), new for every $n \ge 6$; for example $d(6) \le 33$. The first question has a positive answer on every conjugation type of depth at most $n-1$ (Theorem 9.24), on every type with at most two columns (Corollary 9.42(b)), and for $n = 4$ and $n = 5$ on all types (Remark 9.10, Theorem 9.27), and for $n = 6$ on every type of depth at most 12 and every type κ with $\kappa'_3 \le 4$ (Theorem 11.9, computer-assisted; 33 types remain); for $n = 7$ on every type κ with $\kappa'_3 \le 1$ (Corollary 9.164, computer-assisted); and for $n = 8$ no type κ with $\kappa' = (61 - a, a - 1, 1)$, $a = 9, 11, 13$, occurs (Corollary 9.155, computer-assisted); and, for every $n \ge 4$, no type κ with $\kappa' = (n^2 - a - 3, a - 1, 1)$ and $a \ge 3$ occurs (Theorem 9.24 for $a \le n-1$, Corollary 9.174 for odd and Corollary 9.189 for even $a \ge n$); and, for every $n \ge 4$, no type κ with $\kappa'_3 = 2$ occurs (Corollary 9.258). The types of $K_{B(n,2)}$ have at most three columns (Corollary 7.6(a)), so their depth is at most $2N/3$; the window method reaches depth about $2n-3$ (Remark 9.26). The corners of $B(n, 2)$ are $(n-2, n)$, $(n-1, n-1)$ and $(n, n-2)$; so, by Corollary 8.3 and $M_{k,l}(\Lambda) \cong M_{l,k}(\Lambda)$, the second question asks whether $M_{n-3,n-1}(\Lambda) \oplus M_{n-2,n-2}(\Lambda)$ has no nonzero identity of degree at most $n^2 - 4$. A lower bound $\mathrm{PIdeg}(M_{n-2,n-2}(\Lambda)) \ge n^2 - 3$ for the second summand alone would give more, $d(n) \le n^2 - 4$ (Theorem 8.9 with $a = b = n - 2$).

In the language of Subsection 9.11, the first question of Problem 11.1 is the case $s = n$ of the following conjecture on rectangles (Lemma 9.32(iii) and Theorem 9.37); the rectangles with a side at most 3 are excluded, by Proposition 9.34 and Theorem 9.36.

Conjecture 11.2. *Let $n, s \ge 4$. For every $\tau \vdash ns$ with exactly three rows ($\tau_3 \ge 1$) other than $(ns - 4, 2, 2)$, $\mathrm{K}_s(n)_\tau = 0$.*

At $\tau = (ns - 4, 2, 2)$ the space is a line (Theorem 9.37). By Theorem 9.33 the conjecture is symmetric in n and s, and by computation it holds for $(n, s) = (4, 4)$, $(4, 5)$, $(5, 4)$ and $(5, 5)$ (Remark 9.38(a)), and at $(6, 6)$ for the types with $\tau_1 \ge 22$ or $\tau_3 \le 5$ (Theorem 11.9, computer-assisted), at $(n, s) = (7, 7)$ for every $\tau = (47 - a, a, 2)$ with $a \ge 3$ (Corollary 9.164, computer-assisted), at $(8, 8)$ for $\tau = (62 - a, a, 2)$ with $a = 9, 11, 13$ (Corollary 9.155, computer-assisted), and at every (n, n) with $n \ge 4$ for $\tau = (n^2 - a - 2, a, 2)$ with $n \le a \le (n^2 - 2)/2$ (Corollary 9.174 for odd a, Corollary 9.189 for even a), and for $\tau = (n^2 - a - 3, a, 3)$ with $3 \le a \le (n^2 - 3)/2$ (Corollary 9.258). The condition $\tau_3 \ge 1$ is needed: for $\tau_3 = 0$, $\mathrm{K}_s(n)_\tau = \mathrm{R}_s(n)_\tau$, which contains $\det(P_1)^s$ for $\tau = (ns)$. For $\tau_3 = 1$ the conjecture holds, and more: $\mathrm{K}_s(n)_\tau = 0$ for all $n \ge 3$, $s \ge 1$ and $\tau_3 = 1$ (Corollary 9.42(a)). For $\tau = (ns - 5, 3, 2)$ it holds as well (Corollary 9.49).

For $\tau_3 = 2$, Theorem 9.48 and the computations of Remark 9.50(4) suggest the following (for $b = 1$ the only possible eigenvalue is $(d-2)/n$, by Theorem 9.41).

Conjecture 11.3. *Let $n \ge 4$ and $a \ge 3$. Every eigenvalue κ of the pencil $\kappa\mathcal{Z} - \mathcal{J}$ on $\mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$ satisfies $n\kappa - a \in \{0, 3\}$ or $\kappa \in \{2, -2\}$.*

For $a=3$ this is Theorem 9.48. For $a=4$ (all but finitely many n), $a=5$ ($n=11,12$) and $a=6$ ($n=13$) it holds by computation, which gives an upper bound on the spectrum (Remark 9.50(4)). For every $a\ge 3$ and every $n\ge a+6$ it holds, by hand (Theorem 9.150); at $s=n$ this range is depth at most $n-6$, inside Theorem 9.24, so it gives no new bound on $d(n)$. At $\kappa=s=n\ge 4$, $n\kappa-a\in\{0,3\}$ would give $\tau_1=n^2-a-2\in\{-2,1\}<\tau_2=a$, and the eigenvalue n is not ± 2; so, by Corollary 9.31(b) and Lemma 9.32(ii), (iii), Conjecture 11.3 implies $\mathrm{K}_n(n)_{(n^2-a-2,a,2)}=0$ for every $a\ge 3$, and then, with Theorem 9.37 and Corollary 9.42(b), every conjugation type κ of $K_{B(n,2)}$ with $\kappa'_3\le 1$ is $(3,1^{N-3})$. At the 18 pairs (n,a) of Theorem 9.154 (a odd, $4\le n\le 8$, $n\le a$) the statement "$\kappa=n$ is not an eigenvalue" is proved without the conjecture (computer-assisted), so $\mathrm{K}_n(n)_{(n^2-a-2,a,2)}=0$ there (Corollary 9.155); and for every $n\ge 4$ and every dominant $a\ge n$ that statement follows from step (2) of its proof alone (Corollary 9.160). Step (2) holds for every odd dominant a (Theorem 9.173), so for every $n\ge 4$ and every odd dominant $a\ge n$ the statement holds and $\mathrm{K}_n(n)_{(n^2-a-2,a,2)}=0$ (Corollary 9.174). For even dominant a step (2) fails, but one more row, a triangle of order-one rows on the stable line, takes its place (Theorems 9.180 and 9.188); so the statement holds and $\mathrm{K}_n(n)_{(n^2-a-2,a,2)}=0$ for every $n\ge 4$ and every dominant $a\ge n$ of either parity, and every type κ of $K_{B(n,2)}$ with $\kappa'_3\le 1$ is $(3,1^{N-3})$ (Corollary 9.189). So the consequence of Conjecture 11.3 noted above now holds at every $n\ge 4$ without the conjecture. At $n=7$ it was first shown with a computer, also in the pencil form for $3\le a\le 23$ (Theorem 9.163 and Corollary 9.164).

The value $n\kappa-a=3$ occurs for every $a\ge 3$ and $n\ge a+2$ (Corollary 9.63), and for $a=3$ and every $n\ge 4$ (Corollary 9.61), with an eigenvector obtained by lowering the line of Theorem 9.37. The value $n\kappa-a=0$ occurs for every $a\ge 3$ and $n\ge a+2$, and for $a=3$ and every $n\ge 4$ (Theorem 9.102, Corollary 9.100); Corollary 9.103 lists where each value is known to occur, and Corollary 9.81(b) bounds the number of eigenvalues by $\dim \mathcal{F}^{\mathcal{E}}_{(a,2)}(n)$. For $a\ge 3$ and $n\ge a+2$ the eigenspace of $n\kappa-a=0$ has dimension at least the number of partitions of a into parts ≥ 2 (Theorem 9.105). By Corollary 9.114, Conjecture 11.3 follows, by induction on a, from the principle (P1) of Subsection 9.21: every eigenvector at $\kappa\ne\pm 2$ lies in the $\mathfrak{gl}_3$-module generated by passing highest weight vectors of higher weight. (P1) is stronger than Conjecture 11.3; it implies that every eigenvalue has $n\kappa\in\mathbb{Z}$, as does Conjecture 11.3 (Remark 9.115).

Problem 11.4. Let $n\ge 6$. Do statements (ii) and (iii) in the proof of Theorem G hold?

With Problem 11.1 they give $d(n)\le n^2-4$ (Theorem G, $k=3$); for $n=5$ they are open too, and they would follow from $d(5)\le 20$ and $d(5)\le 19$, respectively. For (iii), $U_{\lambda(3,1,1)}=0$, Section 10 leaves only the conjugation types of depth at least $\max(n-2,6)$ (Remark 10.16) whose conjugate has at least three rows (Corollary 10.71; computer-assisted). Unlike for $B(n,2)$, every stable window $\mathcal{Y}_m((3,1,1))$ is zero (Theorem 10.15(a)), so no element of $K_{\lambda(3,1,1)}$ is known, and we ask whether the stronger statement $K_{\lambda(3,1,1)}=0$ holds for all $n\ge 5$. If the depths of the types of $K_{\lambda(3,1,1)}$ are bounded by a constant J, then Theorem 10.15(b) gives $K_{\lambda(3,1,1)}=0$ for $n\ge\max(5,J+3)$. Statement (ii) is a kernel statement for the conjugate pair $\lambda(3,1)$, $\lambda(2,1,1)$ (Lemma 6.11). Its tests are the live corners of $(2,1,1)$ (Theorem 10.24), and for $n\ge 6$ the types of six weights of the first layer do not occur in $K_{\lambda(3,1)}$ (Theorem 10.28), and for $n\ge 4$ neither do the two-row types $\kappa'=(N-j,j)$, $1\le j\le n-3$ (Corollary 10.36), nor, for every $n\ge 3$, any two-row type at any depth (Corollary 10.51), nor any type of depth at most 4 for $n\ge 7$, at most 5 for $n\ge 8$, at most 6 for $n\ge 9$ (Corollaries 10.51 and 10.57; computer-assisted); the rest of (ii) is open.

Problem 11.5. (a) Is $\mathrm{PIdeg}(M_{2,2}(\Lambda))=14$? (b) Is $\mathrm{PIdeg}(M_{2,3}(\Lambda))=20$?

Both are cases of the open lower half of Samoilov's conjecture: (a) is the first square after $M_{1,1}(\Lambda)$ (Proposition 8.5(b)), and squares govern Theorem H(a) (Corollary 8.11); (b) gives $d(5)\le 19$ (Subsection 11.1). In both cases Lemma 8.14 gives an identity of the conjectured degree (k_5 and k_6), so only the lower bounds are open; Proposition 8.5(d) gives 9 and 15. An identity of degree

m gives identities of every larger degree, so by Corollary 8.4 each question is one finite statement: (a) says $C_{14} \cap \bigoplus_{\mu\vdash 14,\ \mu\supseteq(3,3,3)} I_\mu = 0$, and (b) says $C_{20} \cap \bigoplus_{\mu\vdash 20,\ \mu\supseteq(4,4,4,3)} I_\mu = 0$ ($(3,3,3) = \Xi_{2,2}$ and $(4,4,4,3) = \Xi_{2,3}$). Through Theorem 8.9, (a) gives only $d(4) \le 12$ ($n = 4$, $a = b = 2$: $S = 4$, and $n^2 - S + 1 = 13 \le 14$; for $n = 5$ the pair $(2,2)$ would need PIdeg $\ge 17 > 14$). Theorem 11.11 (Subsection 11.3; computer-assisted) answers (a): the conjectured value holds at $(2,2)$.

The last problem is a conjecture about single blocks. For a self-conjugate λ let $h(\lambda) = (h_{11}, \dots, h_{dd})$ be its diagonal hook lengths, as in Frobenius' theorem in Section 4; here d is the size of the Durfee square, the h_{ii} are distinct odd numbers, and $h_{ii} = 2(\lambda_i - i) + 1$.

Conjecture 11.6. *Let $\lambda = \lambda'$. Then $U_\lambda \ne 0$ if and only if $h_{ii} - h_{i+1,i+1} \ge 4$ for $1 \le i < d$; equivalently, if and only if $\lambda_1 > \lambda_2 > \cdots > \lambda_d$.*

The two forms agree since $h_{ii} - h_{i+1,i+1} = 2(\lambda_i - \lambda_{i+1}) + 2$. Call such a λ a *gap shape*.

Remark 11.7 (what Conjecture 11.6 would give). A gap shape λ inside the $n \times n$ box has $|\lambda| \le n(n+1)/2$, with equality only for St_n. Indeed $h_{11} \le 2n-1$, so $h_{ii} \le 2n+3-4i$; the diagonal hooks partition the cells of λ, so $|\lambda| = \sum_i h_{ii}$ is at most the sum of the positive numbers $2n+3-4i$, which are the diagonal hook lengths of St_n; and equality forces $h(\lambda) = h(\mathrm{St}_n)$, hence $\lambda = \mathrm{St}_n$. So Conjecture 11.6 implies $U_\lambda = 0$ for every shape λ inside the box with $|\lambda| > n(n+1)/2$ (for $\lambda \ne \lambda'$ this holds anyway, by Proposition 5.1): Kuzmin's conjecture block by block. It does not give $d(n)$, since Formanek's criterion is joint (Proposition 2.1); but it implies the hypothesis of Theorem F for $n \ge 4$ and statement (iii) in the proof of Theorem G for $n \ge 4$ (neither shape is a gap shape, as each has two equal rows inside its Durfee square, and both have more than $n(n+1)/2$ cells), though not the kernel statement (ii).

It agrees with every block result of this paper. A self-conjugate rectangle is a square (m^m); for $m \ge 2$ it has equal rows and $U = 0$ (Remark 5.2), and $U_{(1)} = C_1 \ne 0$. A self-conjugate shape with two corners is a hook ($d = 1$) or $((u+v)^u, u^v)$ with $u \ge 2$, which has $\lambda_1 = \lambda_2$ (Theorem D). The shapes A_k have $h = (2k+1, 1)$ (Theorem E), and the staircases have all gaps equal to 4 (Theorem 8.15): they are the extreme gap shapes. Of the 27 self-conjugate partitions of $N \le 15$, these results and $d(4) = 10$ decide 22, all as the conjecture predicts; the other five are open: $(5,3,2,1,1)$, $(6,3,2,1^3)$ and $(5,4,2,2,1)$, predicted $U \ne 0$, and $(5,3,3,1,1)$ and $(6,3,3,1^3)$, predicted $U = 0$.

In the language of Section 8: for $a \le b$, the smallest gap shape containing $\Xi_{a,b}$ has size $(a+1)(2b+1)$. Indeed a gap shape $\mu \supseteq \Xi_{a,b}$ has $d \ge a+1$ and $\mu_{a+1} \ge b+1$, so $\mu_i \ge a+b+2-i$ and $h_{ii} \ge 2(a+b+2-2i)+1$ for $i \le a+1$, and these $a+1$ hooks sum to $(a+1)(2b+1)$; the shape with exactly these diagonal hooks contains $\Xi_{a,b}$ (it is St_{a+b+1} when $b - a \le 1$). So, by Corollary 8.4, Conjecture 11.6 predicts an identity of $M_{a,b}(\Lambda)$ of degree $(a+1)(2b+1) - 1$, which is Samoilov's theorem. The lower half of Samoilov's conjecture is a joint statement over all $\mu \supseteq \Xi_{a,b}$ and does not follow from Conjecture 11.6.

For each fixed k, the reduction of Theorem G (Corollary 6.8 for $k > 5$) leads to finitely many patterns, but their number grows fast with k (Remark 6.10), and no one-shape argument passes $k = S(n) - 1$ (Corollary 8.17). A saving that grows with n therefore needs statements that are uniform in the shape. Conjecture 11.6 is one such statement for single blocks, and Samoilov's conjecture, through Theorem H, is another for one algebra at a time.

11.3. **Two computed cases.** This subsection proves Conjecture 11.2 at $(n,s) = (6,6)$ for 74 of its 107 types (Theorem 11.9) and answers Problem 11.5(a) (Theorem 11.11); both proofs are computer-assisted.

The method of Theorem 11.9 treats one GL_3-weight space at a time. Let $\mathrm{Inv}_s(n)$ be the space of the $\mathrm{SL}_n \times \mathrm{SL}_n$-invariant polynomials on M_n^3 of total degree ns (with GL_3 acting as in Subsection 9.7), $\mathrm{Inv}_s^{\mathrm{S}}(n)$ its subspace cut out by (S1) and (S2), and, for partitions τ, τ_0 with at most three rows,

$\mathrm{Kos}(\tau,\tau_0)$ the Kostka number, the multiplicity of the weight τ_0 in the GL_3-module V_τ; it is positive if and only if τ dominates τ_0, $\tau \trianglerighteq \tau_0$.

Lemma 11.8 (weight spaces)**.** $\mathrm{Inv}_s^{\mathrm{S}}(n)$ *is a* GL_3*-submodule of* $\mathrm{Inv}_s(n)$*, and for every weight* τ_0*,*

$$\dim \mathrm{Inv}_s^{\mathrm{S}}(n)_{\tau_0} = \sum_{\tau} \dim \mathrm{K}_s(n)_\tau \cdot \mathrm{Kos}(\tau,\tau_0),$$

the sum over the partitions τ *of* ns *with at most three rows.*

Proof. (a) $\Omega_u f(P) = \sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)\, D^3 f(P)[uy_{\sigma(1)}^T \otimes e_1, uy_{\sigma(2)}^T \otimes e_2, uy_{\sigma(3)}^T \otimes e_3]$, where $D^3 f(P)$ is the third derivative (a symmetric trilinear form) and $X \otimes e_a$ puts X in slot a. (b) Put $F(y_1,y_2,y_3;v_1,v_2,v_3) := D^3 f(P)[uy_1^T \otimes v_1, uy_2^T \otimes v_2, uy_3^T \otimes v_3]$ for row vectors $v_a \in \mathbb{C}^3$; here, for $v = (\nu_1,\nu_2,\nu_3)$, $X \otimes v := (\nu_1 X, \nu_2 X, \nu_3 X)$. F is trilinear in (v_1,v_2,v_3) and invariant under permuting the pairs (y_a,v_a), as $D^3 f(P)$ is symmetric. So $\mathrm{Alt}_y F(y;v) := \sum_\sigma \mathrm{sgn}(\sigma) F(y_{\sigma(1)},y_{\sigma(2)},y_{\sigma(3)};v_1,v_2,v_3)$ satisfies $\mathrm{Alt}_y F(y;v_{\pi(1)},v_{\pi(2)},v_{\pi(3)}) = \mathrm{sgn}(\pi)\,\mathrm{Alt}_y F(y;v)$: it is an alternating trilinear form in (v_1,v_2,v_3), so $\mathrm{Alt}_y F(y;e_1g,e_2g,e_3g) = \det(g)\,\mathrm{Alt}_y F(y;e_1,e_2,e_3)$. (c) $\mathrm{Inv}_s(n)$ is GL_3-stable (the action commutes with $\mathrm{SL}_n \times \mathrm{SL}_n$ and keeps the total degree), and

$$D^3(g.f)(P)[X_a \otimes e_a] = D^3 f(Pg)[X_a \otimes e_a g];$$

by (b), $\Omega_u(g.f)(P) = \det(g)\,\Omega_u f(Pg)$, and the same holds for (S2) with the updates $y_a u^T$ (only the trilinearity and the pair symmetry are used). So $\mathrm{Inv}_s^{\mathrm{S}}(n)$ is a GL_3-submodule; this is the classical identity behind the Cayley Ω-process (compare Lemma 9.111(c), its Lie algebra form for (S1) on $\mathcal{V}_s$). (d) GL_3 is reductive, so $\mathrm{Inv}_s^{\mathrm{S}}(n) = \bigoplus_\tau M_\tau \otimes V_\tau$ with M_τ the highest weight vectors of weight τ in $\mathrm{Inv}_s^{\mathrm{S}}(n)$, that is $\mathrm{K}_s(n)_\tau$; take the weight space τ_0. (The tests are linear conditions with rational coefficients, so the dimensions over $\mathbb{Q}$ and $\mathbb{C}$ agree.) □

If $\tau_3 = 0$, then $\mathrm{K}_s(n)_\tau$ is all of $\mathrm{R}_s(n)_\tau$ by Lemma 9.32(iii), and its dimension is $g((s^n),(s^n),\tau)$ by Lemma 9.32(i). Also $\dim \mathrm{K}_s(n)_{(ns-4,2,2)} = 1$ for $n \ge 3$ and $s \ge 4$ (Theorem 9.37), and $\mathrm{Kos}((ns-4,2,2),\tau_0)$ is 1 if $\tau_0 \trianglelefteq (ns-4,2,2)$ and 0 otherwise (in a semistandard tableau of this shape row 3 is $(3,3)$ and row 2 is $(2,2)$). So, with $T(\tau_0) := \sum_{\tau_3=0} g((s^n),(s^n),\tau)\,\mathrm{Kos}(\tau,\tau_0)$ (the two-row part),

$$\dim \mathrm{Inv}_s^{\mathrm{S}}(n)_{\tau_0} \ge T(\tau_0) + \mathrm{Kos}((ns-4,2,2),\tau_0), \tag{14}$$

with equality if and only if $\mathrm{K}_s(n)_\tau = 0$ for every three-row $\tau \trianglerighteq \tau_0$ other than $(ns-4,2,2)$.

Theorem 11.9 (computer-assisted)**.** *Let* $n = s = 6$*. For every* $\tau \vdash 36$ *with exactly three rows,* $\tau \ne (32,2,2)$*, and* $\tau_1 \ge 22$ *or* $\tau_3 \le 5$*,* $\mathrm{K}_6(6)_\tau = 0$ *(74 types; at* $(32,2,2)$ *the space is a line, Theorem 9.37). Equivalently (Lemma 9.32(iii), Theorem 9.18), among the conjugation types* κ *of* $K_{B(6,2)}$ *with* $\tau = \kappa' + (1,1,1)$ *satisfying* $\tau_1 \ge 22$ *(depth at most* 12*) or* $\tau_3 \le 5$ *(*$\kappa_3' \le 4$*), only* $(3,1^{30})$ *occurs, once.*

Proof. For three-row $\tau \vdash 36$: $\tau \trianglerighteq (22,7,7)$ if and only if $\tau_1 \ge 22$ (then $\tau_3 \le 7$, so $\tau_1+\tau_2 \ge 29$), and $\tau \trianglerighteq (16,15,5)$ if and only if $\tau_3 \le 5$ ($\tau_1+\tau_2 \ge 31$ forces $\tau_1 \ge 16$). At these two weights the computed upper bound below equals the lower bound (14), so Lemma 11.8 gives the claim:

τ_0	$\dim \mathrm{Inv}_6(6)_{\tau_0}$	$T(\tau_0)$	$\mathrm{Kos}((32,2,2),\tau_0)$	lower bound	computed
$(22,7,7)$	4796	212	1	213	213
$(16,16,4)$	4320	271	1	272	272
$(16,15,5)$	9209	317	1	318	318

(the weight $(16,16,4)$, which settles $\tau_3 \le 4$, is a second certificate for the 20 types with $\tau_1 \le 21$, $\tau_3 \le 4$). *The computed upper bound.* The program draws $nb = D+12$ determinantal invariants (D the rank below) $g_T(P) = [z^{\tau_0}]\det(\sum_a z_a T_a \otimes P_a)$, $T_a \in M_6(\mathbb{Z})$, $T_1 = I$; each g_T has integer coefficients and lies in $\mathrm{Inv}_6(6)_{\tau_0}$. Their evaluation matrix at random points has rank D modulo $p = 999983$, so $\dim \mathrm{Inv}_6(6)_{\tau_0} \ge D$; D equals $\dim \mathrm{Inv}_6(6)_{\tau_0}$ as computed in two independent ways

($\sum_\tau g$ Kos with Kronecker coefficients by characters, and a character-free count by Littlewood–Richardson coefficients), so the g_T span the weight space over $\mathbb{Q}$ and their relation space has dimension $nb-D=12$. The rows are the tests (S1), (S2) at integer base points $P=(I,A,B)$ with integer u, y; the program uses $f(P)=\det(P_1)^s f(I,P_1^{-1}P_2,P_1^{-1}P_3)$, true for every $f\in\mathrm{Inv}_s(n)$, and an exp-trace formula whose denominators are prime to p, so each row is the reduction modulo p of a true integer linear condition on the coefficients c whenever $\sum_i c_i g_{T_i}\in \mathrm{Inv}_6^{\mathrm{S}}(6)$. So the nullity of the row matrix modulo p is at least $\dim \mathrm{Inv}_6^{\mathrm{S}}(6)_{\tau_0}+nb-D$, and the computed value, the nullity minus $nb-D$, is an upper bound. One prime suffices, as the two bounds are equal. $T(\tau_0)$ is computed by Littlewood–Richardson without characters: $g((6^6),(6^6),(36-k,k))$ is given by Proposition 7.8, and the two-row Kostka numbers are counted by hand. □

Remark 11.10. (1) The paper had 23 of these 75 types before: depth at most $n-1=5$ ($\tau_1\ge 29$, 12 types, Theorem 9.24) and $\tau_3=1$ (17 types, Corollary 9.42), 6 in both; 52 are new. The method comes from the certificate at $n=5$ (one weight $(9,8,8)$, the tests at points with $P_1=I+t_1uy^T$; Remark 9.29(c), Remark 9.38(a)); new are Lemma 11.8 as a written proof of the GL_3-stability and the use of non-balanced weights with the lower bound (14). (2) Controls. In every $n=6$ run the computed value equals the proved lower bound, consistent with the rows vanishing on the known part of the weight space; at $n=s=3$, $\tau_0=(3,3,3)$, the method gives $10=8+1+1$, above the value $9=8+1$ of the two-row part and the type $(3,1,1,1)$ alone; 10 is the known answer (the types $(3,1,1,1)$ and $(3,3)$, each once: Remark 9.10, Theorem 9.36); at $n=4$ and $n=5$ it gives the known answers $30=29+1$ and $122=121+1$. A second program (the tests (S1) from the definition at $s=n=6$, exact modulo a prime, with spanning certified by the dimension by characters) settles 68 of the 75 types independently; the types $(16,16,4)$, $(17,15,4)$, $(18,14,4)$, $(16,15,5)$, $(17,14,5)$, $(18,13,5)$, $(19,12,5)$ rest on the first program alone. (3) Open: the 33 types with $\tau_1\le 21$ and $\tau_3\ge 6$, all with $g((6^6),(6^6),\tau)>0$ (from $(12,12,12)$, $g=58$, to $(16,12,8)$, $g=985$). On the route of Problem 11.1, $d(6)\le 33$ follows once they are excluded (or once $U_{B(6,2)}=0$); Theorem 11.9 gives no bound on $d(6)$ by itself.

Theorem 11.11 (computer-assisted)**.** *Let F be a field of characteristic* 0. *Then* $\mathrm{PIdeg}(M_{2,2}(\Lambda))=14$.

Proof. ($\le$) Lemma 8.14 with $n=5$ ($k+l=4\le n-1$): $k_5(x,y)=\sum_\pi \mathrm{sgn}(\pi)\,x^{\pi(0)}yx^{\pi(1)}y\cdots yx^{\pi(4)}$ is an identity of $M_{2,2}(\Lambda)$ of degree $0+1+2+3+4=10$ in x and 4 in y; it is nonzero, as distinct π give distinct words. ($\ge$) Corollary 8.4 with the single pair $(2,2)$, $\Xi_{2,2}=(3,3,3)$, gives $\mathrm{PIdeg}(M_{2,2}(\Lambda))=\min\{N: C_N\cap\bigoplus_{\mu\vdash N,\ \mu\supseteq(3,3,3)} I_\mu\neq 0\}-1$. If this intersection were nonzero for some $N\le 14$, there would be an identity of degree $N-1\le 13$, and multiplying by new letters one of degree exactly 13 (as in the proof of Corollary 8.4); so the intersection would be nonzero at $N=14$, against Proposition 11.12. C_N and the I_μ are defined over $\mathbb{Q}$ (the central idempotents have rational coefficients), so the dimension of the intersection is the same for every field of characteristic 0. □

Proposition 11.12 (computer-assisted)**.** *Over* $\mathbb{Q}$, $C_{14}\cap\bigoplus_{\mu\vdash 14,\ \mu\supseteq(3,3,3)} I_\mu=0$.

Proof. We work over $\mathbb{Q}\subset\mathbb{R}$. Put $N=14$, $m=13$, let $S_m=\mathrm{Sym}([13])$ (it fixes the extra letter 14) and fix the N-cycle $c=(14\ 1\ 2\ \cdots\ 13)$. S_m acts on the N-cycles by conjugation; the centralizer of c in S_N is $\langle c\rangle$ and $\langle c\rangle\cap S_m=1$, so the action is free, and as 13! is the number of 14-cycles it is transitive. So $\sigma\mapsto\sigma c\sigma^{-1}$ identifies $\mathbb{Q}[S_m]$ with C_N. For $S:=\{\mu\vdash 14:\mu\supseteq(3,3,3)\}$ the central idempotent $e_S=\sum_{\mu\in S}(f^\mu/N!)\chi^\mu$ is real, so multiplication by e_S is the orthogonal projection P_S onto $\bigoplus_S I_\mu$ (the permutations orthonormal). Let Q be the orthogonal projection onto C_N. For $v\in C_N$, $v\in\mathrm{Im}\,P_S$ if and only if $QP_SQv=v$ (if $QP_SQv=v$, then $\langle P_Sv,v\rangle=|v|^2$ and $|P_Sv|\le|v|$ force $P_Sv=v$). Under the identification, QP_SQ is right multiplication by $h=\sum_\pi h(\pi)\pi$, $h(\pi):=e_S(c^{-1}\pi c\pi^{-1})$ ($\pi\in S_m$), and $h(\pi^{-1})=h(\pi)$, as e_S is a class function with $e_S(g)=e_S(g^{-1})$.

By the Fourier transform on S_m,

$$\dim\Big(C_N \cap \bigoplus_{\mu\in S} I_\mu\Big) = \sum_{\alpha\vdash 13} f^\alpha \dim\ker(\rho_\alpha(h) - I).$$

$x := N!\,h$ has the integer values $x(\pi) = \sum_{\mu\in S} f^\mu \chi^\mu(c^{-1}\pi c\pi^{-1})$, and $\ker(\rho_\alpha(h)-I) = \ker(\rho_\alpha(x) - 14!\cdot I)$. So the proposition says that the 101 matrices $\rho_\alpha(x) - 14!\cdot I$, $\alpha\vdash 13$ (of size $f^\alpha \le 21450$), are nonsingular. This is an exact computation modulo the prime $p = 999983$: the Fourier transform of x on S_{13} by the coset recursion $S_k = \bigsqcup_i c_i S_{k-1}$ in Young's seminormal form, and the rank of each matrix by exact elimination; all 101 nullities are 0. In Young's seminormal form the entries are $1/r$, 1 and $1-1/r^2$ with $|r| \le 12 < p$ (axial distances), so every $\rho_\alpha(\pi)$ has entries in $\mathbb{Z}[1/12!] \subseteq \mathbb{Z}_{(p)}$; x is integral, so $\det(\rho_\alpha(x) - 14!\cdot I) \in \mathbb{Z}_{(p)}$ reduces to the determinant modulo p, and a nonzero determinant modulo p is nonzero over $\mathbb{Q}$. The arithmetic is exact: every intermediate value is an integer below 2^{53}. □

Remark 11.13. (1) Two checks by hand. For $\alpha = (13)$, $\pi \mapsto \pi c\pi^{-1}$ is a bijection onto the 14-cycles and their class sum is central, so $\sum_\pi x(\pi) = 13!\sum_{\mu\in S}\chi^\mu(c)^2 = 0$, as χ^μ vanishes on an N-cycle unless μ is a hook; so $\rho_{(13)}(x) = 0 \ne 14!$. Likewise, on each coset $\{\sigma\in S_{13} : \sigma(13) = j\}$, $j\in[13]$, the sum of x is 0 (with c as in the proof of Proposition 11.12): there $\sigma c\sigma^{-1} = (j\,N)e$ with e running over the 13-cycles of $\mathrm{Sym}([14]\setminus\{j\})$, whose class sum acts on each S^ν, $\nu = \mu$ minus a corner $\supseteq (3,3,2)$, by 0 (no such ν is a hook). The stored data of the computation have these 13 values.

(2) Computations, not used in the proof. A full rerun at the second prime 999979, with the same program and a new store, gives all 101 nullities 0. The traces $\operatorname{tr}\rho_\alpha(x)$ of all 101 components built from the stored data of the main run agree modulo 999983 with exact traces computed independently from the class sums of x and the characters (97 of them are nonzero); $\rho_\alpha(h)$ is positive semidefinite with spectrum in $[0,1]$ in the orthogonal form, and every exact trace lies in $[0, f^\alpha\cdot 14!]$. The program reproduces the known nonzero cases at smaller N: 315 for $\mu\supseteq(2,2)$, $N=7$ ($M_{1,1}(\Lambda)$, PI degree 5) and 4614 for $\mu\supseteq(3,3,2)$, $N=10$ (the dimension of the degree-9 identities of $M_{1,2}(\Lambda)$ in [Vi11]); earlier programs give 0 for $\mu\supseteq(3,3,3)$ at $N = 9,\dots,13$.

(3) Before Theorem 11.11, Proposition 8.5(d) gave $\mathrm{PIdeg}(M_{2,2}(\Lambda)) \ge 9$, which already follows from Vishne's theorem that $M_{1,2}(\Lambda)$, a corner subalgebra of $M_{2,2}(\Lambda)$, has no identity of degree 8 [Vi02]; Popov's unpublished step $\mathrm{PIdeg}(M_{k+1,l}(\Lambda)) \ge \mathrm{PIdeg}(M_{k,l}(\Lambda)) + 3$, reported in [Vi11], would give 12. The other known cases of the lower half of Samoilov's conjecture with $1\le k\le l$ are $(1,1)$ and $(1,2)$ (the true values 5 and 9 after Proposition 8.5). Not claimed: any other (k,l) (for example $(1,3)$, value 13, stays open), the general lower half, and any new bound on $d(n)$: through Theorem 8.9 the theorem gives only $d(4)\le 12$, and $d(4) = 10$ is known.

Appendix A. The re-check of $d(4)\le 10$

This appendix describes the computation quoted in the introduction. By (F0) and (F1), $d(n)\le m$ if and only if the elements $a\cdot L(v_1,\dots,v_n)\cdot b$ of (F1) span V_m.

Identify V_m with $F[S_m]$: the word $x_{\sigma(1)}\cdots x_{\sigma(m)}$ is identified with σ. Then renaming the variables by τ ($x_i\mapsto x_{\tau(i)}$) is left multiplication by τ, and $W_{n,m}$ is a left ideal. Every spanning element of (F1) is a renaming of one in which the word $a\,v_1\cdots v_n\,b$ is $x_1x_2\cdots x_m$. For a sequence $c = (c_0,c_1,\dots,c_{n+1})$ of integers with sum m, $c_0,c_{n+1}\ge 0$ and $c_1,\dots,c_n\ge 1$, let $g_c\in F[S_m]$ be the element $a\cdot L(v_1,\dots,v_n)\cdot b$ in which $a, v_1,\dots,v_n, b$ are the consecutive pieces of $x_1x_2\cdots x_m$ of lengths $c_0,c_1,\dots,c_{n+1}$. Hence

$$W_{n,m} = \sum_c F[S_m]\,g_c,$$

a sum over $\binom{m+1}{n+1}$ sequences c (462 for $(n,m) = (4,10)$). For $\mu\vdash m$ the image $\rho_\mu(F[S_m]\,g_c) = \mathrm{End}(S^\mu)\,\rho_\mu(g_c)$ is the set of matrices whose rows lie in the row space of $\rho_\mu(g_c)$. So, by (F2),

$W_{n,m} = V_m$ if and only if for every $\mu \vdash m$ the matrices $\rho_\mu(g_c)$, stacked on top of each other, have rank f^μ.

The matrices $\rho_\mu(\sigma)$ are taken in Young's seminormal form. Its entries are rational, and their denominators are products of axial distances, which are smaller than m. So for a prime $p > m$ the entries are p-integral, and the rank modulo p of a p-integral matrix is at most its rank over $\mathbb{Q}$. Hence full rank modulo p gives full rank over $\mathbb{Q}$, and so over every field of characteristic 0.

For $(n, m) = (4, 10)$ the stacked matrices have full rank f^μ modulo $p = 2147483629$ and modulo $p = 2147483587$, for all 42 partitions $\mu \vdash 10$. (One prime suffices; the second one checks the program.) Few of the g_c are needed: the program adds the matrices $\rho_\mu(g_c)$ in a random order and stops at full rank, and for every μ it stopped after at most 8 of the 462 elements. Hence $W_{4,10} = V_{10}$, that is $d(4) \leq 10$, and $d(4) = 10$ by Kuzmin's bound. For every $\mu \vdash 10$ the program's matrices $\rho_\mu(s_k)$ satisfy the Coxeter relations of S_{10} modulo $p = 2147483629$, and the trace of $\rho_\mu(\sigma)$ equals $\chi_\mu(\sigma)$ modulo p for one σ in each conjugacy class (an independent program). The programs and logs are in the ancillary files.

Acknowledgments

This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Google's Gemini, Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

References

[ABW82] K. Akin, D. A. Buchsbaum and J. Weyman, Schur functors and Schur complexes, *Adv. Math.* **44** (1982), 207–278.

[AL50] S. A. Amitsur and J. Levitzki, Minimal identities for algebras, *Proc. Amer. Math. Soc.* **1** (1950), 449–463.

[Be88] A. Berele, Trace identities and $\mathbb{Z}/2\mathbb{Z}$-graded invariants, *Trans. Amer. Math. Soc.* **309** (1988), 581–589.

[BR87] A. Berele and A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Adv. Math.* **64** (1987), 118–175.

[BGG71] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Structure of representations generated by vectors of highest weight, *Funct. Anal. Appl.* **5** (1971), 1–8.

[BWZ10] A. A. H. Brown, S. van Willigenburg and M. Zabrocki, Expressions for Catalan Kronecker products, *Pacific J. Math.* **248** (2010), 31–48.

[CE48] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[CLR10] P. Caputo, T. M. Liggett and T. Richthammer, Proof of Aldous' spectral gap conjecture, *J. Amer. Math. Soc.* **23** (2010), 831–851.

[CST10] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, *Representation Theory of the Symmetric Groups: The Okounkov–Vershik Approach, Character Formulas, and Partition Algebras*, Cambridge Stud. Adv. Math. **121**, Cambridge Univ. Press, 2010.

[CV90] O. A. Chalykh and A. P. Veselov, Commutative rings of partial differential operators and Lie algebras, *Comm. Math. Phys.* **126** (1990), 597–611.

[CW12] S.-J. Cheng and W. Wang, *Dualities and Representations of Lie Superalgebras*, Grad. Stud. Math. **144**, Amer. Math. Soc., 2012.

[Cv08] P. Cvitanović, *Group Theory: Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press, 2008.

[DEP80] C. De Concini, D. Eisenbud and C. Procesi, Young diagrams and determinantal varieties, *Invent. Math.* **56** (1980), 129–165.

[DI43] Ya. S. Dubnov and V. K. Ivanov, Sur l'abaissement du degré des polynômes en affineurs, *C. R. (Doklady) Acad. Sci. URSS* **41** (1943), 95–98.

[DJO94] C. F. Dunkl, M. F. E. de Jeu and E. M. Opdam, Singular polynomials for finite reflection groups, *Trans. Amer. Math. Soc.* **346** (1994), 237–256.

[DML09] V. Drensky and L. Makar-Limanov, The conjecture of Nowicki on Weitzenböck derivations of polynomial algebras, *J. Algebra Appl.* **8** (2009), 41–51.

[Do18] M. Domokos, Polynomial bound for the nilpotency index of finitely generated nil algebras, *Algebra Number Theory* **12** (2018), 1233–1242.

[Dr95] V. Drensky, Commutative and noncommutative invariant theory, in: *Math. and Education in Math.* (Proc. 24th Spring Conf. Union Bulg. Math., Svishtov, 1995), 14–50; arXiv:2302.10052.

[Du04] C. F. Dunkl, Singular polynomials for the symmetric groups, *Int. Math. Res. Not.* **2004**, no. 67, 3607–3635.
[Du05] C. F. Dunkl, Singular polynomials and modules for the symmetric groups, *Int. Math. Res. Not.* **2005**, no. 39, 2409–2436.
[Du35] J. Dubnov, Sur une généralisation de l'équation de Hamilton–Cayley et sur les invariants simultanés de plusieurs affineurs, *Proc. Seminar Vector Tensor Analysis, Moscow State Univ.* **2/3** (1935), 351–367.
[dR54] G. de Rham, Sur la division de formes et de courants par une forme linéaire, *Comment. Math. Helv.* **28** (1954), 346–352.
[EF01] D. Eisenbud and E. Frenkel, Appendix to [Mu01].
[EFP11] D. Ellis, E. Friedgut and H. Pilpel, Intersecting families of permutations, *J. Amer. Math. Soc.* **24** (2011), 649–682.
[EG02] P. Etingof and V. Ginzburg, On m-quasi-invariants of a Coxeter group, *Mosc. Math. J.* **2** (2002), 555–566.
[FV03] G. Felder and A. P. Veselov, Action of Coxeter groups on m-harmonic polynomials and Knizhnik–Zamolodchikov equations, *Mosc. Math. J.* **3** (2003), 1269–1291.
[Fo86] E. Formanek, Generating the ring of matrix invariants, in: *Ring Theory (Antwerp, 1985)*, Lecture Notes in Math. **1197**, Springer, 1986, 73–82.
[Fo87] E. Formanek, The invariants of $n \times n$ matrices, in: *Invariant Theory*, Lecture Notes in Math. **1278**, Springer, 1987, 18–43.
[Fo90] E. Formanek, The Nagata–Higman theorem, *Acta Appl. Math.* **21** (1990), 185–192.
[Fo91] E. Formanek, *The Polynomial Identities and Invariants of $n \times n$ Matrices*, CBMS Regional Conf. Ser. Math. **78**, Amer. Math. Soc., 1991.
[GWXZ09] A. Garsia, N. Wallach, G. Xin and M. Zabrocki, Hilbert series of invariants, constant terms and Kostka–Foulkes polynomials, *Discrete Math.* **309** (2009), 5206–5230.
[Ga98] F. R. Gantmacher, *The Theory of Matrices*, Vol. 2, translated from the Russian by K. A. Hirsch, AMS Chelsea Publishing, Providence, RI, 1998.
[Ge94] F. Geoffriau, Sur le centre de l'algèbre enveloppante d'une algèbre de Takiff, *Ann. Math. Blaise Pascal* **1** (1994), no. 2, 15–31.
[Go26] D. Goel, Improving Razmyslov's (1974) bound: the Nagata–Higman index is at most $n^2 - 2$ for $n \geq 3$, Zenodo, 2026, https://doi.org/10.5281/zenodo.23627984
[He74] H. Helling, Eine Kennzeichnung von Charakteren auf Gruppen und assoziativen Algebren, *Comm. Algebra* **1** (1974), 491–501.
[Hi56] G. Higman, On a conjecture of Nagata, *Proc. Cambridge Philos. Soc.* **52** (1956), 1–4.
[HU91] R. Howe and T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, *Math. Ann.* **290** (1991), 565–619.
[Hu72] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Grad. Texts in Math. **9**, Springer, 1972.
[Hu08] J. E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*, Grad. Stud. Math. **94**, Amer. Math. Soc., 2008.
[Ja77] J. C. Jantzen, Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren, *Math. Ann.* **226** (1977), 53–65.
[Ja78] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math. **682**, Springer, 1978.
[JK81] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia Math. Appl. **16**, Addison-Wesley, 1981.
[Ju74] A.-A. A. Jucys, Symmetric polynomials and the center of the symmetric group ring, *Rep. Math. Phys.* **5** (1974), 107–112.
[KT99] I. Kantor and I. Trishin, On the Cayley–Hamilton equation in the supercase, *Comm. Algebra* **27** (1999), 233–259.
[Ke91] A. R. Kemer, *Ideals of Identities of Associative Algebras*, Transl. Math. Monogr. **87**, Amer. Math. Soc., 1991.
[KV05] H. M. Khudaverdian and Th. Th. Voronov, Berezinians, exterior powers and recurrent sequences, *Lett. Math. Phys.* **74** (2005), 201–228.
[Kh08] J. Khoury, A Groebner basis approach to solve a conjecture of Nowicki, *J. Symbolic Comput.* **43** (2008), 908–922.
[Ki71] R. C. King, The dimensions of irreducible tensor representations of the orthogonal and symplectic groups, *Canad. J. Math.* **23** (1971), 176–188.
[Kl00] A. A. Klein, Bounds for indices of nilpotency and nility, *Arch. Math. (Basel)* **74** (2000), 6–10.
[Ko58] B. Kostant, A theorem of Frobenius, a theorem of Amitsur–Levitzki and cohomology theory, *J. Math. Mech.* **7** (1958), 237–264.
[Ko63] B. Kostant, Lie group representations on polynomial rings, *Amer. J. Math.* **85** (1963), 327–404.

[Kos50] J.-L. Koszul, Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. France* **78** (1950), 65–127.
[KOV93] S. Kerov, G. Olshanski and A. Vershik, Harmonic analysis on the infinite symmetric group. A deformation of the regular representation, *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), 773–778.
[KOV04] S. Kerov, G. Olshanski and A. Vershik, Harmonic analysis on the infinite symmetric group, *Invent. Math.* **158** (2004), 551–642.
[Ku75] E. N. Kuz'min, On the Nagata–Higman theorem (Russian), in: *Mathematical Structures, Computational Mathematics, Mathematical Modelling* (Proc. dedicated to the 60th birthday of Acad. L. Iliev), Sofia, 1975, 101–107.
[Le73] U. Leron, Multilinear identities of the matrix ring, *Trans. Amer. Math. Soc.* **183** (1973), 175–202.
[Lo12] A. A. Lopatin, On the nilpotency degree of the algebra with identity $x^n = 0$, *J. Algebra* **371** (2012), 350–366.
[LO13] J. M. Landsberg and G. Ottaviani, Equations for secant varieties of Veronese and other varieties, *Ann. Mat. Pura Appl.* **192** (2013), 569–606.
[LS01] A. S. Lewis and H. S. Sendov, Twice differentiable spectral functions, *SIAM J. Matrix Anal. Appl.* **23** (2001), 368–386.
[Ma95] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford Univ. Press, 1995.
[Man11] L. Manivel, On rectangular Kronecker coefficients, *J. Algebraic Combin.* **33** (2011), 153–162.
[Mat86] H. Matsumura, *Commutative Ring Theory*, Cambridge Stud. Adv. Math. 8, Cambridge University Press, 1986.
[MFF86] F. G. Malikov, B. L. Feigin and D. B. Fuchs, Singular vectors in Verma modules over Kac–Moody algebras, *Funct. Anal. Appl.* **20** (1986), no. 2, 103–113.
[MV11] R. L. Mkrtchyan and A. P. Veselov, On duality and negative dimensions in the theory of Lie groups and symmetric spaces, *J. Math. Phys.* **52** (2011), 083514.
[Mu01] M. Mustaţă, Jet schemes of locally complete intersection canonical singularities, *Invent. Math.* **145** (2001), 397–424.
[Mu81] G. E. Murphy, A new construction of Young's seminormal representation of the symmetric groups, *J. Algebra* **69** (1981), 287–297.
[Na52] M. Nagata, On the nilpotency of nil-algebras, *J. Math. Soc. Japan* **4** (1952), 296–301.
[Now94] A. Nowicki, *Polynomial Derivations and their Rings of Constants*, Uniwersytet Mikołaja Kopernika, Toruń, 1994.
[OV96] A. Okounkov and A. Vershik, A new approach to representation theory of symmetric groups, *Selecta Math. (N.S.)* **2** (1996), 581–605.
[Ol99] P. J. Olver, *Classical Invariant Theory*, London Math. Soc. Student Texts **44**, Cambridge University Press, 1999.
[PP13] I. Pak and G. Panova, Strict unimodality of q-binomial coefficients, *C. R. Math. Acad. Sci. Paris* **351** (2013), 415–418.
[PPV16] I. Pak, G. Panova and E. Vallejo, Kronecker products, characters, partitions, and the tensor square conjectures, *Adv. Math.* **288** (2016), 702–731.
[Po82] A. P. Popov, Identities of the tensor square of the Grassmann algebra, *Algebra and Logic* **21** (1982), 296–316.
[Pr76] C. Procesi, The invariant theory of $n \times n$ matrices, *Adv. Math.* **19** (1976), 306–381.
[Pr15] C. Procesi, On the theorem of Amitsur–Levitzki, *Israel J. Math.* **207** (2015), 151–154.
[Pr26] C. Procesi, Fermionic matrices and super Cayley–Hamilton algebras, arXiv:2605.22745 (2026).
[PY12] D. Panyushev and O. Yakimova, A remarkable contraction of semisimple Lie algebras, *Ann. Inst. Fourier* **62** (2012), 2053–2068.
[Ra74] Yu. P. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 723–756; English transl.: *Math. USSR Izv.* **8** (1974), 727–760.
[Ra87] Yu. P. Razmyslov, Trace identities and central polynomials in the matrix superalgebras $M_{n,k}$, *Math. USSR Sb.* **56** (1987), 187–206 (Russian original: *Mat. Sb.* **128(170)** (1985), 194–215).
[Rai88] M. Raïs, Champs de vecteurs invariants sur une algèbre de Lie réductive complexe, *J. Math. Soc. Japan* **40** (1988), 615–628.
[Ro22] A. Romano, On the T-ideal generated by the identity $f = x^n$, arXiv:2212.05994 (2022).
[Ros76] S. Rosset, A new proof of the Amitsur–Levitski identity, *Israel J. Math.* **23** (1976), 187–188.
[RT92] M. Raïs and P. Tauvel, Indice et polynômes invariants pour certaines algèbres de Lie, *J. Reine Angew. Math.* **425** (1992), 123–140.
[Sa10] L. M. Samoilov, An analog of the Amitsur–Levitzki theorem for matrix superalgebras, *Sib. Math. J.* **51** (2010), 491–495.
[Se85] A. N. Sergeev, The tensor algebra of the identity representation as a module over the Lie superalgebras $\mathfrak{gl}(n,m)$ and $Q(n)$, *Math. USSR Sb.* **51** (1985), 419–427.

[SZ04] I. P. Shestakov and N. Zhukavets, On associative algebras satisfying the identity $x^5 = 0$, *Algebra Discrete Math.* (2004), no. 1, 112–120.
[Str83] V. Strassen, Rank and optimal computation of generic tensors, *Linear Algebra Appl.* **52/53** (1983), 645–685.
[St89] E. Strickland, On algebras satisfying the identity $x^N = 0$, *Comm. Algebra* **17** (1989), 2795–2799.
[Te86] Y. Teranishi, The ring of invariants of matrices, *Nagoya Math. J.* **104** (1986), 149–161.
[VL93] M. Vaughan-Lee, An algorithm for computing graded algebras, *J. Symbolic Comput.* **16** (1993), 345–354.
[VL98] M. Vaughan-Lee, Superalgebras and dimensions of algebras, *Internat. J. Algebra Comput.* **8** (1998), 97–125.
[Ve68] D.-N. Verma, Structure of certain induced representations of complex semisimple Lie algebras, *Bull. Amer. Math. Soc.* **74** (1968), 160–166.
[Vi02] U. Vishne, Polynomial identities of $M_2(G)$, *Comm. Algebra* **30** (2002), 443–454.
[Vi11] U. Vishne, Polynomial identities of $M_{2,1}(G)$, *Comm. Algebra* **39** (2011), 2044–2050.
[Vu74] Th. Vust, *Covariants de groupes algébriques réductifs*, Thèse no. 1671, Université de Genève, 1974.