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# THE LEE (2026) CONJECTURE ON THE DIAMETERS OF ORDER AND CHAIN POLYTOPES FAILS IN BOTH DIRECTIONS, AND OTHER MISCELLANEA

DAVE GOEL

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Abstract. For a finite poset $P$, Stanley's order polytope $\mathcal{O}(P)$ and chain polytope $\mathcal{C}(P)$ have the same number of vertices and, by a theorem of Hibi, Li, Sahara and Shikama, the same number of edges. J. Lee recently conjectured that for a *connected* poset the diameters of their graphs differ by at most 1. We show that this fails in both directions.[1] For every $k \geq 1$ we describe a connected poset $B_k$ with $3k$ elements on three consecutive antidiagonals of the grid $\mathbb{Z}^2$, for which the graph of $\mathcal{O}(B_k)$ has diameter exactly $k$, while the graph of $\mathcal{C}(B_k)$ has diameter exactly $\lfloor 2k/3 \rfloor + 1$. Their difference is therefore $\lceil k/3 \rceil - 1$, and their ratio tends to $3/2$. Hence no constant bounds $\operatorname{diam}\mathcal{O}(P) - \operatorname{diam}\mathcal{C}(P)$ on connected posets. In the other direction, for every $n \geq 4$ we construct a connected poset $P_n$ with $2n^2 + 5n + 1$ elements such that $\operatorname{diam}\mathcal{O}(P_n) = 2$ and $\operatorname{diam}\mathcal{C}(P_n) = 4$. We also show that $\operatorname{diam}\mathcal{O}(P) \leq 2$ implies $\operatorname{diam}\mathcal{C}(P) \leq 4$ for every finite poset $P$, so a difference of 2 is the largest possible at order diameter 2.

## 1. Introduction

Let $P$ be a finite poset. An *ideal* of $P$ is a subset that contains, with each of its elements, every smaller element; an *antichain* is a set of pairwise incomparable elements. Stanley [Sta86] attached two polytopes to $P$: the *order polytope* $\mathcal{O}(P)$, whose vertices are the indicator vectors of the ideals[2], and the *chain polytope* $\mathcal{C}(P)$, whose vertices are the indicator vectors of the antichains. The two polytopes have the same number of vertices, and Hibi, Li, Sahara and Shikama [HLSS17] proved that their graphs have the same number of edges. Lee [Lee26] refined this edge theorem and compared further invariants of the two graphs, among them the diameter; we write diam for the diameter of the graph (the 1-skeleton) of a polytope, and we call $\operatorname{diam}\mathcal{O}(P)$ and $\operatorname{diam}\mathcal{C}(P)$ the *order diameter* and the *chain diameter* of $P$. Both diameters are additive over disjoint unions of posets, since $\mathcal{O}(P \sqcup Q) = \mathcal{O}(P) \times \mathcal{O}(Q)$, $\mathcal{C}(P \sqcup Q) = \mathcal{C}(P) \times \mathcal{C}(Q)$ and the graph of a product of polytopes is the Cartesian product of their graphs. There are small connected posets on which the two diameters differ by 1, in either direction [Lee26, Example 5]; so they can differ by any amount on posets with many components. For connected posets Lee reports a computer search that found no difference larger than 1, and states the following conjecture, which we call the Lee (2026) conjecture.

**Conjecture 1.1** ([Lee26, Conjecture 29])**.** *For every connected finite poset $P$,*

$$|\operatorname{diam}\mathcal{O}(P) - \operatorname{diam}\mathcal{C}(P)| \leq 1.$$

Versions 2 and 3 of [Lee26] withdraw the conjecture, and version 3 cites an earlier version of this paper.

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[1] This is an expanded version of the following focused version [Goel26]: https://hexagonmath.org/2611.00523

[2] Stanley uses filters; passing to complements changes nothing in what follows.

We show that the conjecture fails, and that it cannot be repaired by a larger constant. For example, the band $B_7$ of Definition 1.2, with 21 elements, has order diameter 7 and chain diameter 5, and the poset $P_4$ of Section 9, with 53 elements, has order diameter 2 and chain diameter 4. The word *connected* matters: for posets with many components a large difference is immediate from additivity. For a connected poset $P$ that is not a chain, both diameters lie between 2 and the width $w(P)$ [Lee26, Corollary 25], so they differ by at most $w(P) - 2$. Our examples show that the difference can indeed grow linearly with the width.

**Results.** Our results are listed below, and Table 1 compares them with the best earlier results.

- The order diameter can exceed the chain diameter by any amount on connected posets: the bands $B_k$ of Theorem 1.3. This disproves Conjecture 1.1 in the direction $\operatorname{diam}\mathcal{O}(P) - \operatorname{diam}\mathcal{C}(P) \leq 1$, and no larger constant can replace 1; for disconnected posets the difference is unbounded, by additivity [Lee26].
- The chain diameter can exceed the order diameter by 2 on connected posets: the posets $P_n$ ($n \geq 4$) of Theorem 1.4. This disproves the other half of Conjecture 1.1, $\operatorname{diam}\mathcal{C}(P) - \operatorname{diam}\mathcal{O}(P) \leq 1$; the small examples of [Lee26, Example 5] reach a difference of 1 in either direction.
- Order diameter at most two forces chain diameter at most four, for every finite poset (Theorem 1.5). So $P_n$ ($n \geq 4$) has the largest possible difference among posets of order diameter two; unlike the bound of both diameters by the width [Lee26, Corollary 25], this bound does not depend on $w(P)$.
- The ratio $\operatorname{diam}\mathcal{O}(B_k)/\operatorname{diam}\mathcal{C}(B_k)$ tends to $3/2$ (Theorem 1.3); for a connected poset $P$ that is not a chain, the bounds of [Lee26, Corollary 25] give only $\operatorname{diam}\mathcal{O}(P)/\operatorname{diam}\mathcal{C}(P) \leq w(P)/2$. Whether ratios larger than $3/2$, or an unbounded difference $\operatorname{diam}\mathcal{C}(P) - \operatorname{diam}\mathcal{O}(P)$, occur on connected posets is open (Section 14).

TABLE 1. Results of this paper and the best results before it.

| New result | Best result before this paper |
|---|---|
| $\operatorname{diam}\mathcal{O}(P) - \operatorname{diam}\mathcal{C}(P)$ is unbounded on connected posets; $\operatorname{diam}\mathcal{O}(B_k) = k$ and $\operatorname{diam}\mathcal{C}(B_k) = \lfloor 2k/3 \rfloor + 1$ (Theorem 1.3) | no difference above 1 in Lee's computer search; at most $w(P) - 2$ if $P$ is not a chain (by [Lee26, Corollary 25]) |
| $\operatorname{diam}\mathcal{C}(P) - \operatorname{diam}\mathcal{O}(P) = 2$ for the connected posets $P = P_n$, $n \geq 4$ (Theorem 1.4) | difference 1 [Lee26, Example 5] |
| $\operatorname{diam}\mathcal{O}(P) \leq 2$ implies $\operatorname{diam}\mathcal{C}(P) \leq 4$ (Theorem 1.5) | $\operatorname{diam}\mathcal{C}(P) \leq w(P)$ [Lee26, Corollary 25] |

**Definition 1.2.** For $k \geq 1$ the *band* $B_k$ is the poset on the $3k$ elements $a_i, b_i, c_i$ ($1 \leq i \leq k$) whose cover relations are

$$a_i < b_i < c_i \quad (1 \leq i \leq k), \qquad a_i < b_{i+1} < c_i \quad (1 \leq i \leq k-1).$$

With the product order of $\mathbb{Z}^2$, the $3k$ points $a_i = (i, -i)$, $b_i = (i, 1-i)$, $c_i = (i+1, 1-i)$ ($1 \leq i \leq k$) form a copy of $B_k$: in $\mathbb{Z}^2$ we have $a_i < b_j$ exactly if $j \in \{i, i+1\}$, $b_i < c_j$ exactly if $j \in \{i-1, i\}$, and $a_i < c_j$ exactly if $|i-j| \leq 1$, and no other pair of these points is comparable; these are the comparable pairs of $B_k$ (Lemma 3.1). The points lie on three consecutive antidiagonals; see Figure 1. The band is connected, since $a_i < b_i < c_i$ for $1 \leq i \leq k$ and $a_i < b_{i+1}$ for $1 \leq i < k$, and it has width $k$ (Lemma 3.1).

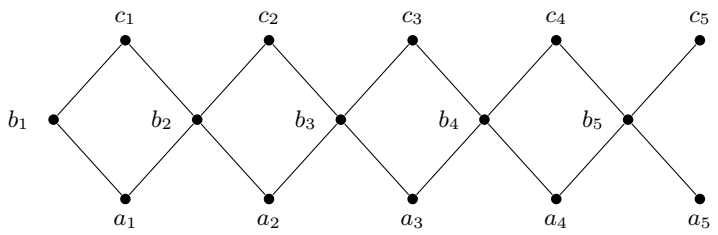



FIGURE 1. The Hasse diagram of the band $B_5$. Every $b_i$ with $i \geq 2$ is the center of a five-element subposet with two incomparable elements below it and two incomparable elements above it.

**Theorem 1.3.** *For every $k \geq 1$ the band $B_k$ is a connected poset with $3k$ elements and*

(i) $\operatorname{diam} \mathcal{O}(B_k) = k$;
(ii) $\operatorname{diam} \mathcal{C}(B_k) = \lfloor 2k/3 \rfloor + 1$.

*Consequently* $\operatorname{diam} \mathcal{O}(B_k) - \operatorname{diam} \mathcal{C}(B_k) = \lceil k/3 \rceil - 1$, *which is unbounded. In particular $B_7$, with* 21 *elements and diameters* 7 *and* 5, *is a counterexample to Conjecture 1.1. Moreover,*

$$\lim_{k \to \infty} \frac{\operatorname{diam} \mathcal{O}(B_k)}{\operatorname{diam} \mathcal{C}(B_k)} = \frac{3}{2}.$$

Part (i) rests on a potential, an alternating sum over the elements $b_1, \ldots, b_k$, that changes by at most one along every edge of the graph of $\mathcal{O}(B_k)$ (Section 4). For part (ii), we first reduce to disjoint antichains whose union is an antichain. Explicit walks handle each maximal set of consecutive columns $\{a_i, b_i, c_i\}$ that meet this union, and an induction across the columns that do not meet it reduces the rest to a few short cases whose walks can be joined (Section 5). A matching lower bound comes from a potential along walks that start at the empty antichain: it counts the columns visited so far and their maximal intervals, together with certain maximal blocks of at least two consecutive columns that each contain an element $b_i$ of the current antichain, and it shows that the empty antichain is at distance at least $\lfloor 2k/3 \rfloor + 1$ from a suitable antichain of elements $b_i$ (Section 6). Whether $B_7$ is a smallest connected counterexample is open.

For the other half of Conjecture 1.1, $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P) \leq 1$, we construct, in Section 9, a second family of connected posets.

**Theorem 1.4.** *For every $n \geq 4$, the poset $P_n$ of Definition 9.1 is connected, has $2n^2 + 5n + 1$ elements, and*

$$\operatorname{diam} \mathcal{O}(P_n) = 2, \qquad \operatorname{diam} \mathcal{C}(P_n) = 4.$$

*In particular $P_4$ is a connected poset with* 53 *elements whose chain diameter exceeds its order diameter by* 2.

The upper bound $\operatorname{diam} \mathcal{C}(P_n) \leq 4$ is an instance of a general fact.

**Theorem 1.5.** *Let $P$ be a finite poset with* $\operatorname{diam} \mathcal{O}(P) \leq 2$. *Then* $\operatorname{diam} \mathcal{C}(P) \leq 4$.

Theorems 1.4 and 1.5 together show that the largest value of $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P)$ over posets $P$ with $\operatorname{diam} \mathcal{O}(P) = 2$ is exactly 2. Whether $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P)$ is bounded on connected posets remains open (Question 14.1), and so does whether $P_4$ is a smallest connected poset with $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P) = 2$.

Section 2 recalls the adjacency criteria and a bound by components, and Sections 3–6 treat the band. Sections 7–12 concern the inequality $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P) \leq 1$: Section 7 collects the definitions they use, Section 8 proves Theorem 1.5, Section 9 defines $P_n$, Section 10 proves $\operatorname{diam} \mathcal{O}(P_n) = 2$, Section 11 shows, for two particular antichains $A$ and $B$ of $P_n$, that the antichains

equal or adjacent to $A$, and those equal or adjacent to $B$, have a few explicit forms, and deduces that $A$ and $B$ are at distance 4 in the graph of $\mathcal{C}(P_n)$ for $n \geq 4$; Section 12 contains remarks. Section 13 reports computer checks that are not used in the proofs, and Section 14 states open questions. A short version of this paper [Goel26] contains both families of counterexamples; for the bands it gives a shorter proof of the weaker bound $\operatorname{diam} \mathcal{C}(B_k) \leq \lfloor 3(k+1)/4 \rfloor$, which already gives a difference of 2 for $k = 8$.

## 2. Preliminaries

Two elements of a poset are *comparable* if one is smaller than the other. A subset $D$ of $P$ is *connected* if it is not empty and the comparability graph of $P$ induces a connected graph on $D$; it is *convex* if $x < y < z$ and $x, z \in D$ imply $y \in D$. The *components* of a set are its maximal connected subsets. The *width* $w(P)$ is the largest size of an antichain. We write $X \triangle Y$ for the symmetric difference. For integers $s, e$ we write $[s, e] = \{s, s+1, \ldots, e\}$; this set is empty if $e < s$.

**Theorem 2.1** (Ikebe and Tamura [IT95], Hibi and Li [HL19]; for antichains Chvátal [Chv75]; see [Lee26, Theorem 4])**.** *Two ideals $I \neq J$ of $P$ are adjacent in the graph of $\mathcal{O}(P)$ exactly if they are nested ($I \subseteq J$ or $J \subseteq I$) and $I \triangle J$ is connected. Two antichains $A \neq B$ of $P$ are adjacent in the graph of $\mathcal{C}(P)$ exactly if $A \triangle B$ is connected.*

A *move* is a step along an edge of one of these graphs; it exchanges the connected set $I \triangle J$, respectively $A \triangle B$.

**Proposition 2.2** ([Lee26, Proposition 24])**.** *The distance of two ideals $I, J$ in the graph of $\mathcal{O}(P)$ is at most the number of components of $I \triangle J$, and the distance of two antichains $A, B$ in the graph of $\mathcal{C}(P)$ is at most the number of components of $A \triangle B$.*

Such walks are obtained as follows. On the order side, remove the components of $I \setminus J$ and then add those of $J \setminus I$. On the chain side, the components can be exchanged one at a time in any order: every set $A \triangle Q$, with $Q$ a union of components of $A \triangle B$, is an antichain. By [Lee26, Corollary 25], both diameters are at most $w(P)$; for the chain polytope, this bound is also a special case of [ABBLS21, Proposition 16].

## 3. The comparable pairs of the band

**Lemma 3.1.** *In $B_k$ the comparable pairs are exactly*

$$a_i < b_i,\ b_{i+1}; \qquad b_i < c_{i-1},\ c_i; \qquad a_i < c_{i-1},\ c_i,\ c_{i+1},$$

*where only indices in $\{1, \ldots, k\}$ are kept. In particular two elements whose indices differ by two or more are incomparable, the sets $\{a_1, \ldots, a_k\}$, $\{b_1, \ldots, b_k\}$, $\{c_1, \ldots, c_k\}$ are antichains, and $w(B_k) = k$.*

*Proof.* The first two groups are the cover relations. The elements above $a_i$ are $b_i, b_{i+1}$ and what lies above them, namely $c_{i-1}, c_i$ and $c_i, c_{i+1}$. No element lies above a $c_j$ or below an $a_j$. The $k$ chains $a_i < b_i < c_i$ partition $B_k$, and the $a_i$ form an antichain. □

We call $\{a_i, b_i, c_i\}$ the $i$-th *column*; it is a chain. The antichains $\{a_1, \ldots, a_k\}$, $\{b_1, \ldots, b_k\}$ and $\{c_1, \ldots, c_k\}$ are the *rows* $a$, $b$ and $c$; the row $b$ is the *middle row*, and the rows $a$ and $c$ are the *outer rows*. By Lemma 3.1 the comparable pairs between two different columns $s < t$ exist only for $t = s + 1$, and they are the four pairs

$$a_s < b_{s+1}, \qquad a_s < c_{s+1}, \qquad a_{s+1} < c_s, \qquad b_{s+1} < c_s. \tag{1}$$

## 4. The order diameter of the band

**Lemma 4.1** (Interval lemma)**.** *Let $D$ be a connected convex subset of $B_k$. Then $\{i : b_i \in D\}$ is empty or an interval of integers.*

*Proof.* Let $b_l, b_r \in D$ and $l < j < r$, and suppose $b_j \notin D$. Split $B_k \setminus \{b_j\}$ into

$$L_j = \{a_i, b_i, c_i : i < j\}, \qquad R_j = \{a_i, c_i : i \geq j\} \cup \{b_i : i > j\}.$$

By Lemma 3.1 the only comparable pairs with one element in $L_j$ and one in $R_j$ are $a_{j-1} < c_j$ and $a_j < c_{j-1}$, and in both of them $b_j$ lies strictly between the two elements. A path in $D$ from $b_l \in L_j$ to $b_r \in R_j$ avoids $b_j$, so it uses one of these two pairs; then both of its elements are in $D$, and convexity puts $b_j$ into $D$, a contradiction. □

**Theorem 4.2.** *For every $k \geq 1$, $\operatorname{diam} \mathcal{O}(B_k) = k$.*

*Proof.* For an ideal $I$ put $\Phi(I) = \sum_{i=1}^{k} (-1)^i [\, b_i \in I \,]$, where $[\, b_i \in I \,]$ is 1 if $b_i \in I$ and 0 otherwise. Let $I \subseteq J$ be adjacent ideals, so that $D = J \setminus I$ is connected (Theorem 2.1). The set $D$ is convex: if $x < y < z$ with $x, z \in D$, then $y \in J$, and $y \in I$ would give $x \in I$. By Lemma 4.1, $\Phi(J) - \Phi(I)$ is an alternating sum over an interval of indices, so it is $-1$, 0 or 1. Hence the distance of two ideals is at least the absolute value of the difference of their values of $\Phi$. Let $I_0$ consist of all $a_i$ and the $b_i$ with $i$ even, and $I_1$ of all $a_i$ and the $b_i$ with $i$ odd. Both are ideals, because below $b_i$ there are only $a_{i-1}$ and $a_i$, and $\Phi(I_0) - \Phi(I_1) = \lfloor k/2 \rfloor + \lceil k/2 \rceil = k$. So $\operatorname{diam} \mathcal{O}(B_k) \geq k$. The reverse inequality follows from Proposition 2.2: one element from each component of $I \triangle J$ gives an antichain, so $I \triangle J$ has at most $w(B_k) = k$ components. □

## 5. An upper bound for the chain diameter of the band

**Theorem 5.1.** *For every $k \geq 1$, $\operatorname{diam} \mathcal{C}(B_k) \leq \lfloor 2k/3 \rfloor + 1$.*

Together with Corollary 6.4, Theorem 5.1 gives part (ii) of Theorem 1.3. This section contains the proof of Theorem 5.1. Throughout, $d(X, Y)$ is the distance of two antichains $X, Y$ in the graph of $\mathcal{C}(B_k)$, and

$$f(0) = 0, \qquad f(m) = \lfloor 2m/3 \rfloor + 1 \quad (m \geq 1).$$

*Outline of the proof.* Lemma 5.3 reduces the theorem to pairs of disjoint antichains whose union is an antichain $U$. Walks for such pairs are built from short walks on a few columns, which are performed one after the other or merged into common moves (Lemma 5.4). (Splits, walks, runs and empty columns are defined in Subsection 5.1.) Subsections 5.2–5.5 treat a set $U$ without empty columns; every split of such a set with $L \geq 2$ columns has a walk of length at most $\lceil L/2 \rceil + 1$ (Proposition 5.15). An induction across empty columns (Subsection 5.8) leaves one family of patterns (Proposition 5.21), in which runs of lengths 3, then 2 or 5, ..., and finally 3 are separated by single empty columns. For these, local walks are joined across each empty column, and each join merges two moves into one (Subsections 5.6 and 5.7).

5.1. **Words, splits and merged walks.** An antichain contains at most one element of each column. We describe an antichain $X$ by the *word* $\omega_1 \omega_2 \cdots \omega_k$ over the alphabet $\{a, b, c, -\}$, where $\omega_i$ is the letter of the element of $X$ in the $i$-th column, and $\omega_i = -$ if there is none; then the column is *empty*. By (1), a word over $\{a, b, c, -\}$ is the word of an antichain exactly if it has no two neighboring letters $\omega_i \omega_{i+1}$ equal to $ab$, $ac$, $ca$ or $cb$; a letter $b$ or $-$ may be followed by any letter. A *run* of a word is a maximal set of consecutive nonempty columns. Inside a run a letter $a$ can only be followed by $a$, and $c$ only by $c$; so the word of a run is $b^i x^j$ with $x \in \{a, c\}$ and $i, j \geq 0$.

The elements $a_i$ and $c_i$ are *outer* elements, and $b_i$ is a *middle* element. For $x \in \{a, c\}$ we write $x_i$ for the element of the row $x$ in column $i$, and $\bar{x}$ for the other outer letter. By Lemma 3.1, two distinct elements are comparable exactly in the following cases:

(P1) $b_j$ and one of $a_{j-1}, a_j, c_{j-1}, c_j$;
(P2) $a_s$ and $c_t$ with $|s-t| \le 1$.

In particular, two elements of one row are incomparable, and two elements of one column are comparable. Comparability depends only on the rows of the two elements and on the difference of their column indices. So a translation of the column indices maps antichains to antichains, connected sets to connected sets, and walks to walks, provided that the translated elements exist. We therefore describe local constructions on the columns $1, \dots, L$ or $0, \dots, L$ of a band, and use them on any interval of columns of $B_k$. For integers $s \le e$, the *band on the columns* $s, \dots, e$ is the poset of the elements $a_i, b_i, c_i$ $(s \le i \le e)$ with the comparable pairs of Lemma 3.1; it is a translated copy of $B_{e-s+1}$.

**Lemma 5.2.** *A nonempty set $F \subseteq B_k$ is connected exactly if the columns that meet $F$ form an interval and, for any two neighboring columns of this interval, some element of $F$ in one of them is comparable to some element of $F$ in the other.*

*Proof.* The elements of one column form a chain, and elements of columns that are not neighbors are incomparable. □

A *walk* of length $n = |W|$ from $X_0$ to $X_n$ is a sequence $W = (X_0, \dots, X_n)$ of antichains in which every *difference* $X_{t-1} \triangle X_t$ is connected; by Theorem 2.1 consecutive antichains are then adjacent. The antichains $X_0, \dots, X_n$ are the *states* of $W$, and $X_0$ and $X_n$ are its *initial* and *final sets*. The *move* $t$ exchanges this difference. We often write walks as sequences of words. The *reversal* $(X_n, \dots, X_0)$ of a walk is a walk. Let $\sigma$ exchange $a_i$ and $c_i$ for every $i$ and fix every $b_i$. By (P1) and (P2), $\sigma$ preserves comparability, so it maps walks to walks; on words it exchanges the letters $a$ and $c$. A sequence of sets *runs through* a walk $W$ if it becomes $W$ after consecutive repetitions are merged.

The *outer trace* of a walk $W = (X_0, \dots, X_n)$ at a column $e$ is the sequence of the outer elements of $X_0, \dots, X_n$ in column $e$, written $-$ when there is none, after consecutive repetitions are merged. For example, if column $e$ holds $b_e$, then $c_e$, then $a_e$, then nothing, the outer trace at $e$ is $(-, c_e, a_e, -)$.

**Lemma 5.3** (Splits)**.** *Let $X, Y$ be antichains of a finite poset $P$. There are disjoint antichains $A \subseteq X$ and $B \subseteq Y$ such that $A \cup B$ is an antichain and $d_{\mathcal{C}(P)}(X, Y) \le d_{\mathcal{C}(P)}(A, B)$.*

*Proof.* Write $d = d_{\mathcal{C}(P)}$. For a set $S$ of elements let $N(S)$ be its open neighborhood in the comparability graph, that is, the set of elements that are comparable to some element of $S$ (comparable elements are distinct). For antichains $R, S$ put $\pi_R(S) = S \cup (R \setminus N(S))$; this is an antichain. We first show that if $S, S'$ are adjacent antichains, then $\pi_R(S)$ and $\pi_R(S')$ are equal or adjacent. Let $F = S \triangle S'$, which is connected by Theorem 2.1, and $F' = \pi_R(S) \triangle \pi_R(S')$.

Let $x \in F' \setminus F$. Elements of $S \cap S'$ lie in both images, so $x \notin S \cup S'$. Then $x \in R$, and $x$ lies in exactly one of $N(S)$, $N(S')$, say in $N(S)$. So $x$ is comparable to some $z \in S$, and $z \notin S'$. Hence every element of $F' \setminus F$ is comparable to an element of $F$.

Let $x \in F$, say $x \in S \setminus S'$, so that $x \in \pi_R(S)$. If $|F| \ge 2$, the connected set $F$ contains an element $y$ comparable to $x$. Since $S$ is an antichain, $y \in S'$; so $x \in N(S')$ and $x \notin \pi_R(S')$. Thus $F \subseteq F'$, and $F'$ is connected. If $F = \{x\}$ and $x \notin R$, then again $x \notin \pi_R(S')$, and $F'$ is connected. If $F = \{x\}$ and $x \in R$, then $x \notin N(S')$, because $S' \cup \{x\} = S$ is an antichain; so $x \in \pi_R(S')$. Moreover no element of the antichain $R$ is comparable to $x$, so $F' \subseteq F = \{x\}$; as $x$ lies in both images, $F'$ is empty, that is, $\pi_R(S) = \pi_R(S')$. With Theorem 2.1 this proves the claim, and consequently $d(\pi_R(S), \pi_R(S')) \le d(S, S')$ for all antichains $S, S'$.

Now let $K = X \cap Y$, $X_1 = X \setminus K$, $Y_1 = Y \setminus K$, $R = X_1 \cap N(Y_1)$, $A = X_1 \setminus R$ and $B = Y_1$. Since $X$ and $Y$ are antichains, $\pi_K(X_1) = X$ and $\pi_K(Y_1) = Y$; hence $d(X, Y) \le d(X_1, Y_1)$. Since $R \subseteq X_1$ and $R \subseteq N(Y_1)$, we have $\pi_R(A) = X_1$ and $\pi_R(B) = B = Y_1$; hence $d(X_1, Y_1) \le d(A, B)$.

Finally $A \subseteq X$ and $B \subseteq Y$ are disjoint, and no element of $A$ is comparable to an element of $B$; so $A \cup B$ is an antichain. □

A pair $(A, B)$ of disjoint sets whose union $U$ is an antichain is a *split* of $U$. The elements of $A$ are *initial* and those of $B$ are *final*. A *walk for* the split $(A, B)$ is a walk from $A$ to $B$, and the split *has a walk* of length $n$ if there is such a walk of length $n$. By Lemma 5.3, $\operatorname{diam} \mathcal{C}(B_k)$ is the largest distance $d(A, B)$ over all splits of antichains of $B_k$. The reversal of a walk from $A$ to $B$ is a walk for the split $(B, A)$, in which initial and final elements are exchanged.

**Lemma 5.4** (Merging walks)**.** *Let $D_1, \dots, D_r$ be pairwise disjoint sets of elements of $B_k$, and for each $t$ let $W_t = (X_0^t, \dots, X_{n_t}^t)$ be a walk whose states are subsets of $D_t$. Group all moves of these walks into* steps *$S_1, \dots, S_N$, so that each step is a nonempty set of moves, at most one of each walk, every move lies in exactly one step, and, for each walk, a later move lies in a later step. Let $Z_\alpha$ be the union of the states that the walks have reached after the steps $S_1, \dots, S_\alpha$ $(0 \le \alpha \le N)$. Suppose that*

(i) *every $Z_\alpha$ is an antichain, and*
(ii) *for every step, the graph on its moves, in which two moves are joined when some element of the difference of one is comparable to some element of the difference of the other, is connected.*

*Then $(Z_0, \dots, Z_N)$ is a walk from $\bigcup_t X_0^t$ to $\bigcup_t X_{n_t}^t$ of length $N = \sum_t n_t - \sum_\alpha (|S_\alpha| - 1)$. For each $t$, the sets $Z_\alpha \cap D_t$ run through $W_t$; in particular, if all elements of a column that occur in states of the walks lie in $D_t$, the walk $(Z_\alpha)$ has the same outer trace at this column as $W_t$.*

*Proof.* Since the sets $D_t$ are disjoint, $Z_{\alpha-1} \triangle Z_\alpha$ is the disjoint union of the differences of the moves in $S_\alpha$. These are connected, and by (ii) their union is connected. □

We call $(Z_\alpha)$ a *merged walk*, and describe its steps in words: a move is performed *alone* or *together with* another move. If the walks are performed one after the other, all steps have one move and condition (ii) holds trivially; condition (i) then holds, for instance, when elements of different sets $D_t$ lie in columns at distance at least two.

**Lemma 5.5** (Adding an element)**.** *Let $W = (X_0, \dots, X_n)$ be a walk, and let $y$ be an element that lies in no state of $W$ and is comparable to an element of some state of $W$.*

(i) *If $X_0$ has no element comparable to $y$, let $\mu$ be the least index such that $X_\mu$ has one. Then $(X_0 \cup \{y\}, \dots, X_{\mu-1} \cup \{y\}, X_\mu, \dots, X_n)$ is a walk of length $n$.*
(ii) *If $X_n$ has no element comparable to $y$, let $\nu$ be the largest index such that $X_{\nu-1}$ has one. Then $(X_0, \dots, X_{\nu-1}, X_\nu \cup \{y\}, \dots, X_n \cup \{y\})$ is a walk of length $n$.*

*Proof.* (i) Merge $W$ with the walk $(\{y\}, \varnothing)$, performing its move together with the move $\mu$ of $W$. The sets $X_t \cup \{y\}$ with $t < \mu$ are antichains by the choice of $\mu$. The state $X_\mu$ contains an element comparable to $y$, which is not in $X_{\mu-1}$; it lies in the difference of the move $\mu$. Lemma 5.4 applies. (ii) follows from (i) by reversal. □

**Lemma 5.6** (Reflection)**.** *Let $L \ge 1$, and let $\Pi_L$ consist of the elements $a_t, c_t$ $(0 \le t \le L)$ and $b_t$ $(1 \le t \le L)$ of the band on the columns $0, \dots, L$. The involution $\rho$ of $\Pi_L$ given by*

$$\rho(b_t) = b_{L+1-t}, \qquad \rho(a_t) = a_{L-t}, \qquad \rho(c_t) = c_{L-t}$$

*preserves comparability and incomparability. Hence it maps walks with states in $\Pi_L$ to walks with states in $\Pi_L$.*

*Proof.* By (P1), the elements comparable to $b_t$ are $a_{t-1}, a_t, c_{t-1}, c_t$; they lie in $\Pi_L$, and $\rho$ maps them to $a_{L+1-t}, a_{L-t}, c_{L+1-t}, c_{L-t}$, which are the elements comparable to $b_{L+1-t}$. By (P2), $a_s$ and $c_t$ are comparable exactly if $|(L-s) - (L-t)| \le 1$. The map $\rho$ preserves rows. □

The map $\rho$ sends the outer elements of column $t$ to column $L-t$, in particular those of column $L$ to column 0, and the middle element of column $t$ to column $L+1-t$.

### 5.2. **Walks along one outer row.**

**Lemma 5.7** (Rows of helpers)**.** *Let $s \le e$ be columns, $j = e-s+1$ and $U = \{a_s, \dots, a_e\}$. Let $T$ be either the set $\{s+1, s+3, \dots\} \cap [s,e]$ of* even helper positions *(the second, fourth, ... columns of the interval), provided that $j \ge 2$, or the set $\{s, s+2, \dots\} \cap [s,e]$ of* odd helper positions *(the first, third, ... columns). Put $C = \{c_t : t \in T\}$, whose elements are the* helpers*, and $m = |C|$. For every split $(A,B)$ of $U$ there is a walk from $A$ to $B$ of length $m+1$, with states in $U \cup C$, of the following form. Its first phase exchanges the components of $A \cup C$ one at a time, in any order, and ends at $C$; its second phase exchanges the components of $C \cup B$ one at a time, in any order. Thus every $c_t \in C$ is present exactly from the move that exchanges its component in the first phase to the move that exchanges its component in the second phase. If $c_e \in C$, then the outer trace at $e$ is $(a_e, c_e, -)$ if $a_e \in A$, and $(-, c_e, a_e)$ if $a_e \in B$.*

For even helper positions $m = \lfloor j/2 \rfloor$, and the length is $\lfloor j/2 \rfloor + 1$; for odd ones $m = \lceil j/2 \rceil$, and the length is $\lceil j/2 \rceil + 1$.

*Proof.* The helpers $c_t \in C$ are pairwise incomparable, and by (P2) the element $a_u$ is comparable to $c_t$ exactly if $|u-t| \le 1$. Consecutive elements of $T$ differ by two, the least one is $s$ or $s+1$, and the largest is $e$ or $e-1$. Hence every element $a_u$ of $U$ is comparable to one or two helpers, and to two exactly if $u = t+1$ with $t, t+2 \in T$; we call such an element $a_u$ a *glue* element. There are $m-1$ glue elements. So every component of $A \cup C$ contains a helper, and two consecutive helpers $c_t$, $c_{t+2}$ lie in the same component exactly if $a_{t+1} \in A$. Therefore $A \cup C$ has $m - l_A$ components, where $l_A$ is the number of glue elements in $A$, and $C \cup B$ has $m - l_B$ components. By Proposition 2.2, applied to the disjoint antichains $A$ and $C$ and to $C$ and $B$, exchanging these components one at a time, in any order, gives walks from $A$ to $C$ and from $C$ to $B$. Their total length is $2m - l_A - l_B = m+1$, because every glue element lies in exactly one of $A$, $B$. If $c_e \in C$, then $c_{e-1} \notin C$, so $a_e$ lies in the component of $c_e$ in its phase. Hence $a_e \in A$ is removed in the move that adds $c_e$, and $a_e \in B$ is added in the move that removes $c_e$. □

**Corollary 5.8.** *Let $j = e-s+1 \ge 1$. Every split of $\{a_s, \dots, a_e\}$ has a walk of length $\lfloor j/2 \rfloor + 1$ with states in the columns $s, \dots, e$; no state contains a middle element, and $a_s$ is the only element of column $s$ that occurs in a state. The same holds for $\{c_s, \dots, c_e\}$, with $c_s$.*

*Proof.* For $j = 1$, exchange $a_s$ alone; for $j \ge 2$, use Lemma 5.7 with even helper positions, none of which is $s$. Apply $\sigma$ for the second statement. □

### 5.3. **Tame walks.**

**Definition 5.9.** Let $i \ge 1$. A walk $W = (X_0, \dots, X_n)$ is *tame at column $i$* if all its states consist of elements of the columns $\le i$, and its outer trace at column $i$ is $(-, h, -)$ for an outer element $h$ of column $i$. Then $h$ lies exactly in the states $X_\mu, \dots, X_{\nu-1}$ for some $1 \le \mu < \nu \le n$; it *arrives* with the move $\mu$ and *departs* with the move $\nu$.

The reversal of a tame walk is tame. The image of a tame walk under $\sigma$ is tame, and $h$ changes its row. If the initial and final sets of $W$ consist of middle elements, $\sigma(W)$ has the same initial and final sets; then the row of $h$ can be prescribed.

**Lemma 5.10** (Tame bases)**.** *For each split of $\{b_1\}$ there is a walk of length 2 that is tame at column 1, with states in column 1, and a walk of length 3 that is tame at column 1, with states in the columns $0, 1$, which contains $a_0$ and $c_0$ in some states but never $b_0$. For each split of $\{b_1, b_2\}$ there is a walk of length 3 that is tame at column 2, with states in the columns $1, 2$.*

TABLE 2. Tame bases. The words of entry 1 are on column 1, those of entry 2 on the columns $0, 1$, and those of entries 3 and 4 on the columns $1, 2$. In this table and in Table 3, the first word is the initial set and the last word the final set. For each move, the table lists one comparable pair of elements of the difference for every two neighboring columns of the difference; '—' marks a difference inside one column; in the words, $-$ is the letter of an empty column. By Lemma 5.2, every difference is connected.

| no. | walk | move 1 | move 2 | move 3 |
|---|---|---|---|---|
| 1 | `b` $\to$ `c` $\to$ `-` | — | — | |
| 2 | `-b` $\to$ `aa` $\to$ `c-` $\to$ `--` | $a_0 < b_1$ | $a_1 < c_0$ | — |
| 3 | `--` $\to$ `a-` $\to$ `cc` $\to$ `bb` | — | $a_1 < c_2$ | $b_2 < c_1$ |
| 4 | `b-` $\to$ `a-` $\to$ `cc` $\to$ `-b` | — | $a_1 < c_2$ | $b_2 < c_1$ |

*Proof.* Entries 1 and 2 of Table 2 give the first two walks when $b_1$ is initial; their outer traces at column 1 are $(-, c_1, -)$ and $(-, a_1, -)$. Entries 3 and 4 give the walks for the splits $(\varnothing, \{b_1, b_2\})$ and $(\{b_1\}, \{b_2\})$; their outer traces at column 2 are $(-, c_2, -)$. All words are antichains, and all differences are connected. The reversals of these walks cover the remaining splits. □

**Lemma 5.11** (Tame extension)**.** *Let $i \geq 1$, and let $W$ be a walk from $A'$ to $B'$ that is tame at column $i$, where $A' \cup B'$ consists of middle elements. Then for every split $(A'', B'')$ of $\{b_{i+1}, b_{i+2}\}$ there is a walk from $A' \cup A''$ to $B' \cup B''$ of length $|W| + 1$ that is tame at column $i + 2$, and whose restrictions to the columns $\leq i$ run through $W$.*

*Proof.* By reversal we may assume that $b_{i+1}$ is final. Since $\sigma$ fixes $A' \cup A''$ and $B' \cup B''$, we may apply $\sigma$ to $W$, carry out the construction, and apply $\sigma$ to the result; so we may assume that the outer element of column $i$ in $W$ is $c_i$, with arrival $\mu$ and departure $\nu$. Let $V$ be the following walk on the columns $i + 1, i + 2$:

`--` $\to$ `a-` $\to$ `cc` $\to$ `bb` if $b_{i+2}$ is final, `-b` $\to$ `aa` $\to$ `c-` $\to$ `b-` if $b_{i+2}$ is initial.

Its differences are connected: $\{a_{i+1}\}$; $a_{i+1} < c_{i+1}, c_{i+2}$; $b_{i+1} < c_{i+1} > b_{i+2} < c_{i+2}$ in the first case, and $a_{i+1} < b_{i+2} > a_{i+2}$; $a_{i+1} < c_{i+1} > a_{i+2}$; $b_{i+1} < c_{i+1}$ in the second. Merge $W$ and $V$: perform move 1 of $V$ alone just before move $\mu$ of $W$, move 2 of $V$ together with move $\mu$, and move 3 of $V$ together with move $\nu$.

Condition (i) of Lemma 5.4. An element of a column $\leq i - 1$, and $b_i$, is incomparable to every element of the columns $i + 1, i + 2$. Column $i$ contains no outer element of $W$ other than $c_i$, and $c_i$ is comparable in these columns only to $a_{i+1}$ and $b_{i+1}$. Now $c_i$ is present exactly while $V$ is in its state `cc` or `c-`, which contains neither. Condition (ii): the move $\mu$ of $W$ adds $c_i$, and the move 2 of $V$ removes $a_{i+1}$; the move $\nu$ removes $c_i$, and the move 3 adds $b_{i+1}$; and $a_{i+1}, b_{i+1} < c_i$.

The merged walk has length $|W| + 3 - 2$. If $b_{i+2}$ is final, the only outer element of column $i + 2$ in the merged walk is $c_{i+2}$, present exactly in the consecutive states between the moves $\mu$ and $\nu$ of $W$; if $b_{i+2}$ is initial, it is $a_{i+2}$, present exactly in the one state after the move 1 of $V$. So it is tame at column $i + 2$. □

**Corollary 5.12.** *Let $\tau(i) = \lfloor i/2 \rfloor + 2$ and $i \geq 1$. For every split of $\{b_1, \ldots, b_i\}$ there is a walk of length $\tau(i)$ that is tame at column $i$, with states in the columns $1, \ldots, i$. If $i$ is odd, there is also a walk of length $\tau(i) + 1$ that is tame at column $i$, with states in the columns $0, \ldots, i$, which never contains $b_0$ and contains an outer element of column $0$ in some state. In both cases the row of the outer element of column $i$ can be prescribed.*

*Proof.* Start from Lemma 5.10 with $i = 1$ or $i = 2$ and apply Lemma 5.11 repeatedly; each application adds two columns and one move, and $\tau(i+2) = \tau(i) + 1$. The added columns are $\geq 2$,

and the restrictions to the old columns run through the old walk; so the walk never contains $b_0$, and it keeps a state with an outer element of column 0. The row is prescribed by $\sigma$. □

The second kind of walk is called *buffered*; column 0 is its buffer.

5.4. **Extending a walk along an outer row.** Both lemmas of this subsection extend a walk whose initial and final sets consist of middle elements by the elements $a_{i+1}, \dots, a_L$ (Lemma 5.14 also by $b_i$); by $\sigma$, they also extend it by $c_{i+1}, \dots, c_L$.

**Lemma 5.13** (Junction)**.** *Let $i, j \geq 1$ and $L = i + j$. Let $W$ be a walk from $A'$ to $B'$, where $A' \cup B'$ consists of middle elements, whose states consist of elements of the columns $\leq i$, and some state of which contains $c_i$. Then for every split $(A'', B'')$ of $\{a_{i+1}, \dots, a_L\}$ there is a walk from $A' \cup A''$ to $B' \cup B''$ of length $|W| + \lfloor j/2 \rfloor$, with states in the columns $\leq L$, whose restrictions to the columns $\leq i$ run through $W$. If $j$ is even, its outer trace at $L$ is $(a_L, c_L, -)$ if $a_L \in A''$, and $(-, c_L, a_L)$ if $a_L \in B''$.*

*Proof.* By reversal we may assume that $a_{i+1} \in A''$. Let $Q$ be the walk of Corollary 5.8 for the split $(A'', B'')$, on the columns $i + 1, \dots, L$; its length is $\lfloor j/2 \rfloor + 1$, and $a_{i+1}$ is the only element of column $i + 1$ in its states. If $j \geq 2$, $Q$ is the walk of Lemma 5.7 with even helper positions $i + 2, i + 4, \dots$; let $K$ be the move of its first phase that exchanges the component of $a_{i+1}$, and order the first phase so that $K$ comes last. If $j = 1$, let $K$ be the only move of $Q$; then the merge below has no other move of the first phase and no second phase. Merge: perform the moves of the first phase of $Q$ before $K$; then the moves of $W$, where $K$ is performed together with the move of $W$ at which $c_i$ first appears; then the second phase of $Q$.

By (P1) and (P2), the only comparable pair formed by an element of a state of $W$ and an element of a state of $Q$ is $a_{i+1} < c_i$. The element $a_{i+1}$ is present exactly before $K$, and $c_i$ is absent then; this is condition (i) of Lemma 5.4. In the joint step, $c_i$ is added and $a_{i+1}$ is removed; this is condition (ii). The length is $|W| + \lfloor j/2 \rfloor + 1 - 1$. If $j$ is even, then $c_L$ is a helper of $Q$, and the outer trace follows from Lemma 5.7 and Lemma 5.4. □

**Lemma 5.14** (Half junction)**.** *Let $i \geq 2$, $j \geq 1$ and $L = i + j$. Let $W$ be a walk from $A'$ to $B'$ that is tame at column $i - 1$, where $A' \cup B'$ consists of middle elements. Then for every split $(A'', B'')$ of $\{b_i, a_{i+1}, \dots, a_L\}$ there is a walk from $A' \cup A''$ to $B' \cup B''$ of length $|W| + \lceil j/2 \rceil$, with states in the columns $\leq L$, whose restrictions to the columns $\leq i - 1$ run through $W$ or through $\sigma(W)$. If $j$ is odd, its outer trace at $L$ is $(a_L, c_L, -)$ if $a_L \in A''$, and $(-, c_L, a_L)$ if $a_L \in B''$.*

*Proof.* By reversal we may assume that $a_{i+1} \in A''$. Let $h$, $\mu$ and $\nu$ be the outer element of column $i - 1$ in $W$ and its arrival and departure.

*Step 1: a walk $W'$ on the columns $\leq i$.* We may replace $W$ by $\sigma(W)$, which is tame at column $i - 1$ and has the same initial and final sets. If $b_i$ is initial, choose $h = c_{i-1}$ in this way, and merge $W$ with the walk $\mathtt{b} \to \mathtt{c} \to \mathtt{a} \to \mathtt{-}$ on column $i$: its move 1 together with the move $\mu$ of $W$, its move 2 together with the move $\nu$, and its move 3 alone just after the move $\nu$. If $b_i$ is final, choose $h = a_{i-1}$, and merge $W$ with $\mathtt{-} \to \mathtt{c} \to \mathtt{a} \to \mathtt{b}$: its move 1 alone just before the move $\mu$, its move 2 together with $\mu$, and its move 3 together with $\nu$. Each difference of these walks lies in column $i$ and is connected. Elements of the columns $\leq i - 2$, and $b_{i-1}$, are incomparable to column $i$, and the only outer element of column $i - 1$ in $W$ is $h$, present exactly between the moves $\mu$ and $\nu$. In the first case column $i$ holds $c_i$ during this time, and $b_i$ or $a_i$ only when $h$ is absent; in the second case it holds $a_i$ during this time, and $c_i$ or $b_i$ only when $h$ is absent. By (P1) and (P2) this gives condition (i) of Lemma 5.4. Condition (ii) holds because $b_i < c_{i-1}$ and $a_i < c_{i-1}$ in the first case, and $a_{i-1} < c_i$ and $a_{i-1} < b_i$ in the second. The merged walk $W'$ has length $|W| + 1$. Its states consist of elements of the columns $\leq i$, its restrictions to the columns $\leq i - 1$ run through $W$ or $\sigma(W)$, and its outer trace at column $i$ is $(-, c_i, a_i, -)$ in both cases. Let $\mu'$ be the move of $W'$ at which $c_i$ arrives, and $\nu' > \mu'$ the move at which $a_i$ arrives.

*Step 2: two merges.* Let $S = \{a_{i+1}, \dots, a_L\}$, and let $Q$ be the walk of Lemma 5.7 for the split $(A'' \cap S, B'' \cap S)$ of $S$, with odd helper positions $i+1, i+3, \dots$; its length is $\lceil j/2 \rceil + 1$. Its only helper comparable to $a_{i+1}$ is $c_{i+1}$. Let $K_1$ be the move of its first phase that exchanges the component of $c_{i+1}$, which contains $a_{i+1}$, and let $K_2$ be the move of its second phase that exchanges the component of $c_{i+1}$. Order the phases so that $K_1$ is the last move of the first phase and $K_2$ the first move of the second. Merge $W'$ and $Q$: perform the first phase before $K_1$; then the moves of $W'$, with $K_1$ together with $\mu'$ and $K_2$ together with $\nu'$; then the rest of the second phase.

The states of $W'$ lie in the columns $\leq i$, and those of $Q$ in the columns $\geq i+1$, where column $i+1$ holds only $a_{i+1}$ or $c_{i+1}$. Since $b_i$ is incomparable to column $i+1$, the only comparable pairs between the two walks are $a_{i+1} < c_i$ and $a_i < c_{i+1}$. The element $a_{i+1}$ is present exactly before $K_1$, that is, before $c_i$ first arrives. The element $c_{i+1}$ is present exactly between $K_1$ and $K_2$, that is, between the arrivals of $c_i$ and of $a_i$, when $a_i$ is absent. This is condition (i). Condition (ii) holds because the joint steps add $c_i$ and remove $a_{i+1}$, and add $a_i$ and remove $c_{i+1}$. The length is $|W| + 1 + \lceil j/2 \rceil + 1 - 2$. If $j$ is odd, $c_L$ is a helper of $Q$, and the outer trace follows from Lemmas 5.7 and 5.4. □

5.5. **Runs without empty columns.**

**Proposition 5.15.** *Let $L \geq 2$, and let $U$ be an antichain without empty columns of the band on the columns $1, \dots, L$, with the word $b^i x^j$. Every split of $U$ has a walk of length at most $\lceil L/2 \rceil + 1$ if $i \geq 1$, and at most $\lfloor L/2 \rfloor + 1$ if $i = 0$.*

The walks lie in this band, that is, in the columns of $U$. Transported to an interval of columns of $B_k$ by translation, they use no element outside these columns.

*Proof.* By $\sigma$ we may assume $x = a$. Let $(A, B)$ be a split of $U$.

If $i = 0$, use Corollary 5.8.

If $i$ is odd, take a walk $W$ for the split of $\{b_1, \dots, b_i\}$ that is tame at column $i$, of length $\tau(i) = (i+3)/2$, whose outer element of column $i$ is $c_i$ (Corollary 5.12). If $j = 0$, this is the walk. If $j \geq 1$, Lemma 5.13 gives a walk of length $(i+3)/2 + \lfloor j/2 \rfloor$, which is $\lceil L/2 \rceil + 1$.

If $i \geq 2$ is even, take a walk $W$ for the split of $\{b_1, \dots, b_{i-1}\}$ that is tame at column $i-1$, of length $\tau(i-1) = i/2+1$. If $j \geq 2$ is even, Lemma 5.14 gives a walk of length $i/2+1+j/2 = L/2+1$. Otherwise $b_i$ is comparable to the outer element of column $i-1$ in $W$, and to no element of the first and last states of $W$, which are middle elements. Lemma 5.5 gives a walk $W_1$ of length $i/2+1$ for the split of $\{b_1, \dots, b_i\}$, with states in the columns $\leq i$, whose column $i$ contains no element but $b_i$. If $j = 0$, this is the walk, of length $L/2+1$. If $j$ is odd, perform $W_1$ and then the walk of Corollary 5.8 on the columns $i+1, \dots, L$, of length $(j+1)/2$, whose column $i+1$ contains only $a_{i+1}$. The element $b_i$ is incomparable to column $i+1$, and other columns of the two walks are at distance at least two; so this is a walk, of length $i/2 + 1 + (j+1)/2 = \lceil L/2 \rceil + 1$. □

5.6. **Joining walks across empty columns.** In this subsection and the next, $U$ is an antichain of $B_k$ whose runs $R_1, \dots, R_r$ ($r \geq 2$), from left to right, are separated by single empty columns, with columns 1 and $k$ nonempty, and $(A, B)$ is a split of $U$. Let $e_t$ be the last column of $R_t$, and for $t \geq 2$ let $g_t = e_{t-1} + 1$ be the empty column before $R_t$, its *gap*. The *domain* $D_1$ consists of all elements of the columns of $R_1$. For $t \geq 2$, the domain $D_t$ consists of all elements of the columns of $R_t$ and the two outer elements $a_{g_t}$, $c_{g_t}$ of its gap. A *local walk* for $R_t$ is a walk from $A \cap D_t$ to $B \cap D_t$ with states in $D_t$; it never contains the middle element of a gap.

The domains are pairwise disjoint. Elements of different domains are comparable only in the pairs $(a_{e_t}, c_{g_{t+1}})$ and $(c_{e_t}, a_{g_{t+1}})$. Indeed, elements of different domains lie in columns at distance at least two, unless they lie in the columns $e_t$ and $g_{t+1} = e_t + 1$ for some $t$. Column $g_{t+1}$ contributes

only $a_{g_{t+1}}$ and $c_{g_{t+1}}$; the element $b_{e_t}$ is comparable to neither, and two outer elements of one row are incomparable.

**Definition 5.16** (Ports)**.** Let $W$ be a local walk for $R_t$, let $e = e_t$, and let $x \in \{a, c\}$.

(i) $W$ has a *forward exit in row $x$* if its outer trace at $e$ ends with $x_e, -$. The exit is the move of this last change; after it, column $e$ contains no outer element.

(ii) $W$ has a *backward exit in row $x$* if its outer trace at $e$ begins with $-, x_e$. The exit is the move of this first change; before it, column $e$ contains no outer element.

(iii) Let $t \geq 2$ and $g = g_t$. The local walk $W$ *visits its gap* if some state of $W$ contains $a_g$ or $c_g$. Since the gap is empty in $U$, the outer trace at $g$ then begins with $-, y_g$ for some $y$; the move of this first change is a *forward entrance in row $y$*, and before it the gap is empty. Likewise the trace ends with $z_g, -$ for some $z$; the move of this last change is a *backward entrance in row $z$*, and after it the gap is empty.

An exit of a local walk for $R_t$ and an entrance of a local walk for $R_{t+1}$ *match* if both are forward or both are backward, and their rows are different. Exits and entrances are called *ports*.

The outer element that gives a port its row may belong to $U$. For instance, if $a_e$ is initial and no outer element enters column $e$ after its removal, then this removal is a forward exit in row $a$. If the last letter of $R_t$ is outer, then the outer trace at $e_t$ begins with it if it is initial, and ends with it if it is final. So local walks for $R_t$ have only forward exits if this letter is outer and initial, and only backward exits if it is outer and final; the direction of these exits is the *admissible* direction of $R_t$ (with respect to the split $(A, B)$). If the last letter is $b$, both directions are admissible.

**Lemma 5.17** (Joining)**.** *Let $W_t$ be a local walk for $R_t$ $(1 \leq t \leq r)$, and suppose that for each $t < r$ some exit of $W_t$ matches some entrance of $W_{t+1}$. Then $d(A, B) \leq \sum_t |W_t| - (r-1)$.*

*Proof.* For each $t < r$, identify the chosen exit of $W_t$ with the chosen entrance of $W_{t+1}$. Consider the graph whose vertices are the classes of moves under these identifications, with an arc from each move of each $W_t$ to the next move of $W_t$. The moves of one walk form a path. The union of the paths of $W_1, \ldots, W_{t+1}$ arises from the union of the paths of $W_1, \ldots, W_t$ by attaching the path of $W_{t+1}$ at one vertex, its entrance; this holds also when the entrance of $W_{t+1}$ is its exit. So the underlying undirected graph is a tree, there is no directed cycle, and some linear order of the classes extends all arcs. Each identification links a move of $W_t$ to a move of $W_{t+1}$, and a move is linked to $W_{t+1}$ only as the exit of $W_t$ and to $W_{t-1}$ only as the entrance of $W_t$. So a class consists of moves of consecutive walks $W_t, \ldots, W_{t'}$, one of each, where every move but the last is the chosen exit of its walk and every move but the first is the chosen entrance of its walk. Performing the classes in this order as steps defines a merged walk as in Lemma 5.4, with $\sum_t |W_t| - (r-1)$ steps. It runs from $A$ to $B$, since the domains contain $U$ and are disjoint.

Condition (i). Consider a state of this sequence, and let $X$ and $X'$ be the states reached by $W_t$ and $W_{t+1}$. By the paragraph before Definition 5.16, it suffices to show that $X$ has no outer element in column $e_t$ or $X'$ has an empty gap. For a forward exit and entrance: if the joint step has been performed, $X$ is after the exit, so column $e_t$ of $X$ has no outer element; otherwise $X'$ is before the entrance, and its gap is empty. For a backward exit and entrance: if the joint step has been performed, $X'$ is after the entrance, and its gap is empty; otherwise $X$ is before the exit, and column $e_t$ of $X$ has no outer element.

Condition (ii). A step with several moves consists of moves of consecutive walks $W_t, \ldots, W_{t'}$, identified at the boundaries between them. At each such boundary, the exit of $W_s$ adds or removes an element $x_{e_s}$, and the matching entrance of $W_{s+1}$ adds or removes $\bar{x}_{g_{s+1}}$, where $g_{s+1} = e_s + 1$. These two elements are comparable by (P2). □

5.7. **Local walks for short runs.** We keep the notation of Subsection 5.6 and describe local walks in local coordinates: the columns of the run are $1, \ldots, L$ and its gap is column 0. Words

are written on the columns $1, \dots, L$ for the first run and on the columns $0, \dots, L$ otherwise. The statements of the three lemmas below do not change under $\sigma$, which exchanges the rows of all ports, nor under reversal, which exchanges initial and final elements and forward and backward ports and keeps rows and visits to the gap. We use both symmetries freely; in particular, by $\sigma$ we may assume that the word of the run is $b^i a^j$.

TABLE 3. Local walks for short runs; the conventions are those of Table 2. In every word the gap, if present, is the first column. Entries 1–8 have the outer trace $(-, c_3, a_3)$ at column 3. Entries 9–16 visit the gap. Entries 17–20 visit the gap, and their outer traces at column 2 are $(-, a_2, -)$, $(-, a_2, -)$, $(-, c_2, -)$, $(-, a_2, -)$. Entries 21–24 visit the gap and have the outer trace $(-, c_2, a_2)$ at column 2.

| no. | walk | move 1 | move 2 | move 3 |
|---|---|---|---|---|
| *First runs bba, columns* $1, 2, 3$ | | | | |
| 1 | `---` → `-a-` → `ccc` → `bba` | — | $a_2 < c_1$, $a_2 < c_3$ | $b_2 < c_1$, $a_3 < c_2$ |
| 2 | `b--` → `ba-` → `ccc` → `-ba` | — | $a_2 < c_1$, $a_2 < c_3$ | $b_2 < c_1$, $a_3 < c_2$ |
| 3 | `-b-` → `aa-` → `bcc` → `b-a` | $a_1 < b_2$ | $a_1 < c_2$, $a_2 < c_3$ | $a_3 < c_2$ |
| 4 | `bb-` → `aa-` → `-cc` → `--a` | $a_1 < b_2$ | $a_1 < c_2$, $a_2 < c_3$ | $a_3 < c_2$ |
| *First runs baa, columns* $1, 2, 3$ | | | | |
| 5 | `---` → `-a-` → `ccc` → `baa` | — | $a_2 < c_1$, $a_2 < c_3$ | $a_2 < c_1$, $a_2 < c_3$ |
| 6 | `b--` → `ba-` → `ccc` → `-aa` | — | $a_2 < c_1$, $a_2 < c_3$ | $a_2 < c_1$, $a_2 < c_3$ |
| 7 | `-a-` → `aa-` → `bcc` → `b-a` | — | $a_1 < c_2$, $a_2 < c_3$ | $a_3 < c_2$ |
| 8 | `ba-` → `aa-` → `-cc` → `--a` | — | $a_1 < c_2$, $a_2 < c_3$ | $a_3 < c_2$ |
| *Last runs bba, columns* $0, 1, 2, 3$ | | | | |
| 9 | `----` → `-a--` → `ccc-` → `-bba` | — | $a_1 < c_0$, $a_1 < c_2$ | $b_1 < c_0$, $b_2 < c_1$, $a_3 < c_2$ |
| 10 | `-b--` → `aa--` → `-cc-` → `--ba` | $a_0 < b_1$ | $a_0 < c_1$, $a_1 < c_2$ | $b_2 < c_1$, $a_3 < c_2$ |
| 11 | `--b-` → `-cc-` → `aa-a` → `-b-a` | $b_2 < c_1$ | $a_0 < c_1$, $a_1 < c_2$, $a_3 < c_2$ | $a_0 < b_1$ |
| 12 | `-bb-` → `ccc-` → `-a-a` → `---a` | $b_1 < c_0$, $b_2 < c_1$ | $a_1 < c_0$, $a_1 < c_2$, $a_3 < c_2$ | — |
| *Last runs baa, columns* $0, 1, 2, 3$ | | | | |
| 13 | `----` → `-a--` → `ccc-` → `-baa` | — | $a_1 < c_0$, $a_1 < c_2$ | $b_1 < c_0$, $a_2 < c_1$, $a_3 < c_2$ |
| 14 | `-b--` → `aa--` → `-cc-` → `--aa` | $a_0 < b_1$ | $a_0 < c_1$, $a_1 < c_2$ | $a_2 < c_1$, $a_3 < c_2$ |
| 15 | `--a-` → `-ccc` → `aa-a` → `-b-a` | $a_2 < c_1$, $a_2 < c_3$ | $a_0 < c_1$, $a_1 < c_2$, $a_3 < c_2$ | $a_0 < b_1$ |
| 16 | `-ba-` → `cccc` → `-a-a` → `---a` | $b_1 < c_0$, $a_2 < c_1$, $a_2 < c_3$ | $a_1 < c_0$, $a_1 < c_2$, $a_3 < c_2$ | — |
| *Interior runs of length* $2$*, columns* $0, 1, 2$ | | | | |
| 17 | `---` → `-c-` → `aaa` → `-bb` | — | $a_0 < c_1$, $a_2 < c_1$ | $a_0 < b_1$, $a_1 < b_2$ |
| 18 | `-bb` → `aaa` → `-c-` → `---` | $a_0 < b_1$, $a_1 < b_2$ | $a_0 < c_1$, $a_2 < c_1$ | — |
| 19 | `-b-` → `aa-` → `-cc` → `--b` | $a_0 < b_1$ | $a_0 < c_1$, $a_1 < c_2$ | $b_2 < c_1$ |
| 20 | `--b` → `-aa` → `cc-` → `-b-` | $a_1 < b_2$ | $a_1 < c_0$, $a_2 < c_1$ | $b_1 < c_0$ |
| 21 | `---` → `-a-` → `ccc` → `-ba` | — | $a_1 < c_0$, $a_1 < c_2$ | $b_1 < c_0$, $a_2 < c_1$ |
| 22 | `-b-` → `aa-` → `-cc` → `--a` | $a_0 < b_1$ | $a_0 < c_1$, $a_1 < c_2$ | $a_2 < c_1$ |
| 23 | `---` → `c--` → `c-c` → `-aa` | — | — | $a_1 < c_0$, $a_1 < c_2$ |
| 24 | `-a-` → `aa-` → `-cc` → `--a` | — | $a_0 < c_1$, $a_1 < c_2$ | $a_2 < c_1$ |

**Lemma 5.18** (First runs)**.** *Let $R_1$ have length* 3 *and a word that begins with $b$. For the admissible direction of $R_1$, or for either direction if both are admissible, and for each row, there is a local walk for $R_1$ of length at most* 3 *with an exit of this direction in this row.*

*Proof.* By $\sigma$, the word is *bbb*, *bba* or *baa*. For *bbb*, take a walk of length $\tau(3) = 3$ that is tame at column 3 (Corollary 5.12). Its outer trace at column 3 is $(-, h, -)$, so it has a forward and a backward exit in the row of $h$, which can be prescribed.

Now let the word be *bva* with $v \in \{a, b\}$. For row $a$, exchange the three elements of $U \cap D_1$ one at a time; all states are subsets of $U$, and the outer trace at column 3 is $(a_3, -)$ or $(-, a_3)$, which gives an exit in row $a$ in the admissible direction. For row $c$ we may assume by reversal

that $a_3$ is final, so the admissible direction is backward. Entries 1–4 (for $v = b$) and 5–8 (for $v = a$) of Table 3 give a walk for each of the four splits of $\{b_1, v_2\}$. All words are antichains and all differences are connected, and the outer trace at column 3 is $(-, c_3, a_3)$; this is a backward exit in row $c$. □

**Lemma 5.19** (Last runs)**.** *Let $R_r$ have length 3 and a word that begins with $b$. There is a local walk for $R_r$ of length at most 3 that visits its gap.*

*Proof.* By $\sigma$, the word is *bbb*, *bba* or *baa*. First let it be *bbb*, and let $(A_r, B_r) = (A \cap D_r, B \cap D_r)$. Use the reflection $\rho$ of Lemma 5.6 with $L = 3$, so that $\rho(b_t) = b_{4-t}$, $\rho(a_t) = a_{3-t}$ and $\rho(c_t) = c_{3-t}$. By Corollary 5.12 there is a walk $W$ from $\rho(A_r)$ to $\rho(B_r)$ of length 3 that is tame at column 3, with states in the columns $1, 2, 3$. Since $\rho(b_t) = b_{4-t}$, the element $b_t$ is initial for $W$ exactly if $b_{4-t} \in A_r$. By Lemma 5.6, $\rho(W)$ is a walk from $A_r$ to $B_r$ with states in $\Pi_3 \subseteq D_r$. If $h$ is the outer element of column 3 in $W$, then $\rho(h)$ is an outer element of column 0, so $\rho(W)$ visits its gap.

Now let the word be *bva* with $v \in \{a, b\}$; by reversal we may assume that $a_3$ is final. Entries 9–12 (for $v = b$) and 13–16 (for $v = a$) of Table 3 give a walk for each of the four splits of $\{b_1, v_2\}$, and each of them visits the gap. □

**Lemma 5.20** (Interior runs)**.** *Let $1 < t < r$, let $R_t$ have length $L \in \{2, 5\}$ and the word $b^i x^j$, and let $i \geq 1$ if $L = 5$. For the admissible direction of $R_t$, or for either direction if both are admissible, and for each row, there is a local walk for $R_t$ that visits its gap, has an exit of this direction in this row, and has length at most 3 if $L = 2$ and at most 5 if $L = 5$.*

*Proof.* By $\sigma$ we may assume $x = a$.

*Case $j = 0$.* For $L = 2$ take entries 17–20 of Table 3, which cover the four splits of $\{b_1, b_2\}$. For $L = 5$ take a buffered walk of length $\tau(5) + 1 = 5$ that is tame at column 5 (Corollary 5.12), with the buffer as gap. In both cases the gap is visited, the outer trace at column $L$ is $(-, h, -)$ for an outer element $h$, and $\sigma$ prescribes the row of $h$. So there are exits of both directions in each row.

In the remaining cases $j \geq 1$, the last letter is $a_L$, and an outer trace $(a_L, -)$ or $(-, a_L)$ at column $L$ gives an exit in row $a$ in the admissible direction, while $(a_L, c_L, -)$ or $(-, c_L, a_L)$ gives an exit in row $c$ in the admissible direction.

*Case $j \geq 1$, exit in row $a$.* If $i = 0$, then $L = 2$ and the word is *aa*. Add $a_0$, and then use Lemma 5.7 for the split $((A \cap D_t) \cup \{a_0\}, B \cap D_t)$ of $\{a_0, a_1, a_2\}$, with the even helper position 1. This walk has length $1 + 2$, visits the gap, and has the outer trace $(a_2, -)$ or $(-, a_2)$ at column 2.

If $i \geq 1$, use the reflection $\rho$ of Lemma 5.6. It maps $U \cap D_t$, which has $b_1, \ldots, b_i$ and $a_{i+1}, \ldots, a_L$, to the set with $a_0, \ldots, a_{j-1}$ and $b_{j+1}, \ldots, b_L$; column $j$ is empty. Let $W_1$ be the walk of Corollary 5.8 for the induced split of $\{a_0, \ldots, a_{j-1}\}$, of length $\lfloor j/2 \rfloor + 1$, in which $a_0$ is the only outer element of column 0. Let $W_2$ be a walk for the induced split of $\{b_{j+1}, \ldots, b_L\}$ that is tame at column $L$, with states in the columns $j + 1, \ldots, L$, of length $\tau(i)$ (Corollary 5.12, translated). The columns of $W_1$ and $W_2$ are at distance at least two, so performing $W_1$ and then $W_2$ gives a walk $W^*$ from $\rho(A \cap D_t)$ to $\rho(B \cap D_t)$ with states in $\Pi_L$. Its length is $\lfloor j/2 \rfloor + \lfloor i/2 \rfloor + 3$, which is 3 for $(i, j) = (1, 1)$, and 5 for $L = 5$ and $i, j \geq 1$, because then exactly one of $i, j$ is odd and $\lfloor i/2 \rfloor + \lfloor j/2 \rfloor = (i + j - 1)/2 = 2$. By Lemma 5.6, $\rho(W^*)$ is a local walk for $R_t$. Its column 0 contains the image of the outer element of column $L$ of $W_2$, so it visits the gap. The outer elements of column $L$ of $\rho(W^*)$ are the images of the outer elements of column 0 of $W^*$, and $a_0$ is the only one of these; so the outer trace of $\rho(W^*)$ at column $L$ is $(a_L, -)$ or $(-, a_L)$.

*Case $j \geq 1$, exit in row $c$, $L = 2$.* By reversal we may assume that $a_2$ is final. For the word *ba*, entries 21 and 22 of Table 3 cover the two splits of $\{b_1\}$; for the word *aa*, entries 23 and 24 cover the two splits of $\{a_1\}$. They visit the gap and have the outer trace $(-, c_2, a_2)$ at column 2.

*Case $j \geq 1$, exit in row $c$, $L = 5$, $i$ odd.* Then $j$ is even. Take a buffered walk $W$ for the split of $\{b_1, \ldots, b_i\}$ that is tame at column $i$, with the outer element $c_i$ there, of length $\tau(i) + 1 = (i + 5)/2$.

Lemma 5.13 gives a walk of length $(i+5)/2 + j/2 = 5$. Its restrictions to the columns $\le i$ run through $W$, so it visits the gap. Its outer trace at column 5 is $(a_5, c_5, -)$ or $(-, c_5, a_5)$.

*Case $j \ge 1$, exit in row $c$, $L = 5$, $i$ even.* Then $i \ge 2$ and $j$ is odd. Take a buffered walk $W$ for the split of $\{b_1, \dots, b_{i-1}\}$ that is tame at column $i-1$, of length $\tau(i-1)+1 = i/2+2$. Lemma 5.14 gives a walk of length $i/2 + 2 + (j+1)/2 = 5$. Its restrictions to the columns $\le i-1$ run through $W$ or $\sigma(W)$; each of $W$ and $\sigma(W)$ has a state that contains an outer element of column 0, so the walk visits the gap. Its outer trace at column 5 is $(a_5, c_5, -)$ or $(-, c_5, a_5)$.

All walks constructed here have states in the columns $0, \dots, L$ and never contain $b_0$; so they are local walks. □

**Proposition 5.21.** *Let $U$ be an antichain of $B_k$ whose runs $R_1, \dots, R_r$ ($r \ge 2$) are separated by single empty columns, with columns 1 and $k$ nonempty. Suppose that $R_1$ and $R_r$ have length 3 and words beginning with $b$, and that every other run has length 2, or length 5 and a word containing $b$. Then $d(A, B) \le f(k)$ for every split $(A, B)$ of $U$.*

*Proof.* Let $n_2$ and $n_5$ be the numbers of runs of lengths 2 and 5. Then $r = n_2 + n_5 + 2$, $k = 6 + 2n_2 + 5n_5 + (r-1) = 7 + 3n_2 + 6n_5$, and $f(k) = 5 + 2n_2 + 4n_5$.

Choose local walks from right to left. Let $W_r$ be a local walk for $R_r$ of length at most 3 that visits its gap (Lemma 5.19). Let $t < r$, and suppose that $W_{t+1}$ has been chosen and visits its gap. Choose an admissible direction of $R_t$. The walk $W_{t+1}$ has an entrance of this direction, in some row $y$. By Lemma 5.20 if $t \ge 2$, and by Lemma 5.18 if $t = 1$, there is a local walk $W_t$ for $R_t$ with an exit of this direction in row $\bar{y}$, of length at most 3, 3 or 5 according as $|R_t|$ is 3, 2 or 5; if $t \ge 2$, it visits its gap. The exit of $W_t$ matches the entrance of $W_{t+1}$. By Lemma 5.17,

$$d(A, B) \le 3 + 3 + 3n_2 + 5n_5 - (r-1) = 5 + 2n_2 + 4n_5 = f(k).$$ □

5.8. **Proof of Theorem 5.1.**

**Lemma 5.22.** *The function $f$ has the following properties.*

(i) *For integers $p, q \ge 0$, $f(p) + f(q) \le f(p+q+1)$, unless $p$ and $q$ are positive multiples of 3; in that case $f(p) + f(q) = f(p+q+1) + 1$.*
(ii) *For $k \ge 2$, $\lceil k/2 \rceil + 1 \le f(k)$.*
(iii) *For $m \ge 2$, $\lceil 3m/2 \rceil + 1 \le f(3m) - 1$ and $\lceil (3m+2)/2 \rceil + 1 \le f(3m+2) - 1$.*

*Proof.* (i) The function $f$ is nondecreasing, which settles $p = 0$ or $q = 0$. For $p, q \ge 1$ write $2p = 3\lfloor 2p/3 \rfloor + r_p$ with $r_p \in \{0, 1, 2\}$, so that $r_p = 0$ exactly if 3 divides $p$, and similarly for $q$. Then

$$f(p+q+1) - f(p) - f(q) = \lfloor (r_p + r_q + 2)/3 \rfloor - 1,$$

which is $-1$ if $r_p = r_q = 0$ and nonnegative otherwise. (ii) and (iii) use that $\lceil y \rceil \le n$ holds for an integer $n$ exactly if $y \le n$. (ii) Let $k \ge 2$, and write $k = 3m$, $3m+1$ or $3m+2$. Then $f(k) - 1 = 2m$, $2m$, $2m+1$, respectively, and $k/2$ is at most this value, since $m \ge 1$ in the first two cases. (iii) We have $f(3m) - 2 = 2m - 1 \ge 3m/2$ and $f(3m+2) - 2 = 2m \ge (3m+2)/2$ for $m \ge 2$. □

*Proof of Theorem 5.1.* By Lemma 5.3, it suffices to show that $d(A, B) \le f(k)$ for every split $(A, B)$ of every antichain $U$ of $B_k$. We use induction on $k$. For $k = 1$, $|U| \le 1$ and $d(A, B) \le 1 = f(1)$. Let $k \ge 2$.

*Case 1: $U$ has no empty column.* Its word is $b^i x^j$. By Proposition 5.15 and Lemma 5.22(ii), $d(A, B) \le \lceil k/2 \rceil + 1 \le f(k)$.

*Case 2: $U$ has an empty column $g$ such that $g-1$ and $k-g$ are not both positive multiples of 3.* The columns $1, \dots, g-1$ and $g+1, \dots, k$ induce copies of $B_{g-1}$ and $B_{k-g}$; a walk in one of them is a walk in $B_k$, by translation. By induction, the splits induced by $(A, B)$ on these copies have walks of lengths at most $f(g-1)$ and $f(k-g)$, where $f(0) = 0$ covers an empty side. Their

columns are at distance at least two, so performing one after the other gives a walk from $A$ to $B$. By Lemma 5.22(i), its length is at most $f(g-1)+f(k-g) \le f(k)$.

*Case 3: for every empty column $g$, both $g-1$ and $k-g$ are positive multiples of 3.* There is such a column, so $k-1=(g-1)+(k-g)$ is divisible by 3, every empty column $g$ satisfies $g \equiv 1 \pmod 3$ and $4 \le g \le k-3$, and no two empty columns are neighbors. So the runs $R_1, \dots, R_r$ ($r \ge 2$) are separated by single empty columns, columns 1 and $k$ are nonempty, the lengths of $R_1$ and $R_r$ are positive multiples of 3, and the lengths of the other runs are $\equiv 2 \pmod 3$. The $r-1$ empty columns lie between the runs, so $\sum_s |R_s| = k-r+1$. Since $k \equiv 1 \pmod 3$,

$$\sum_{s=1}^{r} f(|R_s|) = \frac{2(k-r+1)}{3} + 2 + \frac{2(r-2)}{3} = \frac{2k+4}{3} = f(k)+1,$$

because $f(n)=(2n+3)/3$ for $n \equiv 0$, $f(n)=(2n+2)/3$ for $n \equiv 2$, and $f(k)=(2k+1)/3$.

Each run $R_s$ has length at least 2, and its word is $b^i x^j$. Proposition 5.15 gives a walk for the split induced on $R_s$, inside the columns of $R_s$, of length $\ell_s \le \lceil |R_s|/2 \rceil + 1 \le f(|R_s|)$ (Lemma 5.22(ii)), and $\ell_s \le \lfloor |R_s|/2 \rfloor + 1$ if the word of $R_s$ has no $b$. The runs are at distance at least two from each other, so performing these walks one after the other gives $d(A,B) \le \sum_s \ell_s$. This is at most $f(k)$ if some run *saves*, that is, $\ell_s \le f(|R_s|)-1$. By Lemma 5.22(iii) every run of length at least 6 saves, since $|R_s| = 3m$ or $3m+2$ with $m \ge 2$. A run of length 3 or 5 whose word has no $b$ saves, since $\lfloor 3/2 \rfloor + 1 = f(3)-1$ and $\lfloor 5/2 \rfloor + 1 = f(5)-1$. If no run saves, then $R_1$ and $R_r$ have length 3 and words beginning with $b$, and every other run has length 2, or length 5 and a word containing $b$. Then Proposition 5.21 gives $d(A,B) \le f(k)$. □

## 6. A lower bound and the exact diameters

For a set $T \subseteq \{1,\dots,k\}$, write $b_T = \{b_i : i \in T\}$. Let $r(T)$ be the number of maximal intervals of integers contained in $T$, and let $s(T)$ be the number of those with at least two elements; both counts are zero for $T = \varnothing$.

**Theorem 6.1.** *For every $T \subseteq \{1,\dots,k\}$,*

$$d_{\mathcal{C}(B_k)}(\varnothing, b_T) \ge \left\lceil \frac{|T| + r(T) + s(T)}{2} \right\rceil.$$

The proof uses a potential $H(V,S)$ that depends on the current antichain $S$ and on the set $V$ of columns visited so far.

6.1. **Visited columns.** For an antichain $X$, let $\operatorname{supp}(X)$ be the set of indices of the columns that contain an element of $X$. For a walk $\varnothing = X_0, X_1, \dots, X_t$, the *visited set* after $i$ moves is $V_i = \operatorname{supp}(X_0) \cup \dots \cup \operatorname{supp}(X_i)$. For a set $Z$ of column indices put $p(Z) = |Z| + r(Z)$; all sets of column indices are subsets of $\{1,\dots,k\}$. We write $[\,\cdot\,]$ for the Iverson bracket: it is 1 if the statement inside holds and 0 otherwise.

Let $S \to S'$ be a move, and let $V$ be a set of column indices with $\operatorname{supp}(S) \subseteq V$. The set $F = S \triangle S'$ is connected by Theorem 2.1, so its columns form an interval $[L,R]$ by Lemma 5.2; in this section $L$ and $R$ always denote these ends. Put $V' = V \cup \operatorname{supp}(S')$. A column is *new* if its index lies in $V' \setminus V$. Put $\varepsilon_L = [\,L \text{ is new and } L-1 \notin V\,]$ and $\varepsilon_R = [\,R \text{ is new and } R+1 \notin V\,]$.

**Lemma 6.2.** *Let $S \to S'$ be a move, and let $V$, $V'$, $[L,R]$, $\varepsilon_L$ and $\varepsilon_R$ be as above.*

(a) *Every new column lies in $[L,R]$, and no two new columns are neighbors.*
(b) *$p(V') - p(V) = \varepsilon_L + \varepsilon_R \le 2$.*
(c) *If $Z \subseteq Z'$ are sets of column indices, then $p(Z) \le p(Z')$.*

*Proof.* For a set $Z$ of column indices and an index $j \notin Z$, let $n_Z(j) = |\{j-1, j+1\} \cap Z|$ be the number of neighbors of $j$ in $Z$.

*Step 1.* If $j \notin Z$, then $p(Z\cup\{j\})-p(Z) = 2-n_Z(j)$. Indeed, $|Z|$ increases by one. If $n_Z(j)=0$, then $\{j\}$ is a new maximal interval of $Z\cup\{j\}$; if $n_Z(j)=1$, then $j$ extends one maximal interval of $Z$; if $n_Z(j)=2$, then $j$ joins two maximal intervals of $Z$ into one. So $r(Z)$ changes by $1-n_Z(j)$ in each of the three cases.

*Step 2: proof of* (c). Add the indices of $Z'\setminus Z$ to $Z$ one at a time. By Step 1, adding an index $j$ to the current set $Z_0$ changes $p$ by $2-n_{Z_0}(j)\geq 0$. Hence $p(Z)\leq p(Z')$.

*Step 3: first part of* (a). If $j$ is new, then $S$ has no element in column $j$, because $\operatorname{supp}(S)\subseteq V$, while $S'$ has one. That element lies in $F$, so $j\in[L,R]$.

*Step 4: second part of* (a). Suppose that two neighboring columns $j, j+1$ are both new. By Step 3 both lie in $[L,R]$. Hence, by Lemma 5.2, some $x\in F$ in one of them is comparable to some $y\in F$ in the other. Neither column contains an element of $S$, so $x,y\in S'$. This contradicts the fact that $S'$ is an antichain.

*Step 5: proof of* (b). Every column of $[L,R]$ contains an element of $F\subseteq S\cup S'$, so its index lies in $V$ or in $\operatorname{supp}(S')$, hence in $V'$. Therefore a column of $[L,R]$ that is not new lies in $V$. Now add the new columns to $V$ one at a time. By Step 4 no neighbor of a new column is new, so its neighbors in the current set are its neighbors in $V$. Therefore, by Step 1, adding a new column $j$ increases $p$ by $2-n_V(j)$. Let $j$ be new. If $L<j<R$, then $j-1$ and $j+1$ are columns of $[L,R]$ that are not new (Step 4), so they lie in $V$; thus $n_V(j)=2$, and $j$ contributes 0. If $j=L<R$, then in the same way $L+1\in V$, so $n_V(L)=1+[L-1\in V]$, and $L$ contributes $1-[L-1\in V]=\varepsilon_L$. Symmetrically, if $j=R>L$, then $R$ contributes $\varepsilon_R$. If $j=L=R$, then $j$ contributes $2-[L-1\in V]-[R+1\in V]=\varepsilon_L+\varepsilon_R$. Finally, $\varepsilon_L=0$ if $L$ is not new, and $\varepsilon_R=0$ if $R$ is not new. Summing the contributions gives $p(V')-p(V)=\varepsilon_L+\varepsilon_R$, and $\varepsilon_L,\varepsilon_R\leq 1$. □

6.2. **A potential for long runs in the middle row.** We use the word representation from Section 5: the word of an antichain $S$ is $\omega_1\omega_2\cdots\omega_k$, where $\omega_j\in\{a,b,c,-\}$ is the letter of the element of $S$ in column $j$, and $\omega_j=-$ if column $j$ contains no element of $S$. We also put $\omega_0=\omega_{k+1}=-$. As in Subsection 5.1, the letters $a$ and $c$ are the *outer letters*. Throughout, $V$ is a set of column indices with $\operatorname{supp}(S)\subseteq V$.

*Components.* A *component* of a set $Z$ of column indices is a maximal interval $[s,e]$ contained in $Z$ (a different notion from the components of a set of elements in Section 2). Thus $Z$ has exactly $r(Z)$ components.

*Long $b$-runs.* A *long $b$-run* of $S$ is an interval $[s,e]$ with $s<e$ such that $\omega_j=b$ for $s\leq j\leq e$, while $\omega_{s-1}\neq b$ and $\omega_{e+1}\neq b$. Its *start* is $s$ and its *end* is $e$. A long $b$-run is an interval contained in $\operatorname{supp}(S)\subseteq V$, so it lies in a single component of $V$. Two distinct long $b$-runs are disjoint, and some column between them has a letter other than $b$, by maximality.

*Marks.* Let $D$ be a component of $V$. A *mark* of $(V,S)$ in $D$ is either an *outer mark*, that is, a set $\{j\}$ with $j\in D$ and $\omega_j\in\{a,c\}$, or a *run mark*, that is, a long $b$-run of $S$ contained in $D$. Distinct marks are disjoint intervals: $\omega$ takes the value $a$ or $c$ on an outer mark and the value $b$ on a run mark, and distinct long $b$-runs are disjoint. A mark $\xi$ lies *to the left of* a mark $\zeta$ if $\max\xi<\min\zeta$; of two distinct marks, exactly one lies to the left of the other, because they are disjoint intervals. The *left neighbor* of a mark $\zeta$ in $D$ is the rightmost mark in $D$ that lies to the left of $\zeta$, if such a mark exists.

*Credited runs.* A long $b$-run $\beta$ of $S$ in the component $D$ of $V$ is *credited* if it has no left neighbor in $D$, or if its left neighbor is a run mark. It is *blocked* if its left neighbor is an outer mark. Let $q_D(V,S)$ be the number of credited long $b$-runs in $D$, and let $q(V,S)$ be the sum of $q_D(V,S)$ over all components $D$ of $V$.

Equivalently, a long $b$-run in $D$ is credited exactly when no outer letter lies in $D$ between it and the preceding long $b$-run in $D$; if there is no preceding long $b$-run in $D$, the condition is that no outer letter lies in $D$ to the left of it. The indices $\min D-1$ and $\max D+1$ are not in $V$, so

$\omega$ is not $b$ at these indices. Therefore the run marks in $D$ are the maximal intervals of length at least two in $D$ on which $\omega$ equals $b$. Consequently, we have the following locality property:

(2) the marks in $D$, and the number $q_D(V,S)$, depend only on $D$ and on the letters $\omega_j$ with $j \in D$.

Define

$$H(V,S) = p(V) + q(V,S).$$

**Lemma 6.3.** *Let $S \to S'$ be a move, let $V$ be a set of column indices with $\operatorname{supp}(S) \subseteq V$, and put $V' = V \cup \operatorname{supp}(S')$. Then*

$$H(V',S') - H(V,S) \le 2.$$

We prove Lemma 6.3 through six claims. Claims 1 and 2 describe the move and the component $C'$ of $V'$ that contains its columns $[L,R]$; Claim 3 shows that a long $b$-run of $S$ and one of $S'$ with the same start are either both credited or both not credited when the two words agree on a suitable interval to their left; Claims 4–6 compare the credited runs before and after the move.

*Notation for the proof.* Fix $V$ and the move $S \to S'$, and let $\omega$ and $\omega'$ be the words of $S$ and $S'$. Let $F = S \triangle S'$, $[L,R]$, $V'$, the new columns and $\varepsilon_L$, $\varepsilon_R$ be as in Subsection 6.1. An element of $S$ or $S'$ outside the columns of $[L,R]$ is not in $F$, so it lies in both $S$ and $S'$. Hence

$$\omega_j = \omega'_j \qquad \text{for all } j \notin [L,R].$$

**Claim 1.** *Let $L \le j < R$.*

(a) *If $\omega'_j \in \{b,-\}$, then $\omega_j \in \{a,c\}$ and $\omega'_{j+1} \ne -$.*
(b) *If $\omega_j = -$, then $\omega'_j \in \{a,c\}$.*

*Proof of Claim 1. Step 1.* The columns $j$ and $j+1$ are neighboring columns of $[L,R]$, and $F$ is connected. So, by Lemma 5.2, there are comparable elements $x \in F$ in column $j$ and $y \in F$ in column $j+1$.

*Step 2.* The element $x$ is $a_j$ or $c_j$. Indeed, by (1) the comparable pairs between the columns $j$ and $j+1$ are $a_j < b_{j+1}$, $a_j < c_{j+1}$, $a_{j+1} < c_j$ and $b_{j+1} < c_j$, and their elements in column $j$ are $a_j$ and $c_j$.

*Step 3.* One of $x, y$ lies in $S \setminus S'$ and the other in $S' \setminus S$. Indeed, $F = (S \setminus S') \cup (S' \setminus S)$, and the comparable elements $x, y$ cannot both lie in the antichain $S$, nor both in the antichain $S'$.

*Step 4: proof of* (a). Let $\omega'_j \in \{b,-\}$. Then the element of $S'$ in column $j$, if there is one, is $b_j$, which is neither $a_j$ nor $c_j$. By Step 2, $x \notin S'$. So $x$ is the element of $S$ in column $j$, and $\omega_j \in \{a,c\}$ by Step 2. By Step 3, $y \in S'$; thus $S'$ has an element in column $j+1$, that is, $\omega'_{j+1} \ne -$.

*Step 5: proof of* (b). Let $\omega_j = -$. Then $S$ has no element in column $j$, so $x \in S'$. Thus $\omega'_j$ is the letter of $x$, which is $a$ or $c$ by Step 2. □

**Claim 2** (The component of the move)**.**

(a) *The interval $[L,R]$ is contained in $V'$, hence in a single component of $V'$. We denote this component by $C' = [u,v]$.*
(b) *Every new column lies in $[L,R]$. If $u \le L-1$, then $[u,L-1]$ is contained in a component $D_{\text{left}}$ of $V$ with $\min D_{\text{left}} = u$. If $R+1 \le v$, then $[R+1,v]$ is contained in a component $D_{\text{right}}$ of $V$ with $\max D_{\text{right}} = v$.*
(c) *Every component of $V$ is contained in $C'$ or disjoint from $C'$. The components of $V$ that are disjoint from $C'$ are exactly the components of $V'$ other than $C'$, and $\omega'$ agrees with $\omega$ on each of them.*
(d) *$p(V') - p(V) = \varepsilon_L + \varepsilon_R$.*
(e) *If $\varepsilon_L = 1$, then $L \notin V$ and $u = L$. If $\varepsilon_R = 1$, then $R \notin V$ and $v = R$.*

*Proof of Claim 2. Step 1: proof of* (a). Every column of $[L, R]$ contains an element of $F \subseteq S \cup S'$, so its index lies in $\operatorname{supp}(S) \subseteq V$ or in $\operatorname{supp}(S')$, hence in $V'$. An interval contained in $V'$ lies in one component of $V'$, by the definition of components.

*Step 2: proof of* (b). By Lemma 6.2(a), every new column lies in $[L, R]$. Hence the columns of $[u, L-1]$ and of $[R+1, v]$ are not new; as they lie in $C' \subseteq V'$, they lie in $V$. The index $u-1$ is not in $V'$, because $C'$ is a component of $V'$; so $u-1 \notin V$. Therefore the component of $V$ that contains the interval $[u, L-1]$ begins at $u$. In the same way, $v+1 \notin V$, and the component of $V$ that contains $[R+1, v]$ ends at $v$.

*Step 3: proof of* (c). A component $D$ of $V$ is an interval contained in $V \subseteq V'$, so it lies in one component of $V'$. Hence $D \subseteq C'$ or $D \cap C' = \varnothing$. Now let $D' \neq C'$ be a component of $V'$. Then $D' \cap C' = \varnothing$, so $D'$ contains no column of $[L, R]$, and by (b) no new column. Hence $D' \subseteq V$. An interval in $V$ that contains $D'$ is also an interval in $V'$, so it equals $D'$ by the maximality of $D'$. Thus $D'$ is a component of $V$, and it is disjoint from $C'$. Conversely, let $D$ be a component of $V$ disjoint from $C'$. It lies in a component $D'$ of $V'$, and $D' \neq C'$ because $D \neq \varnothing$ and $D \cap C' = \varnothing$. As just shown, $D'$ is a component of $V$; it contains $D$, so $D' = D$ by the maximality of $D$. Finally, a component of $V$ disjoint from $C'$ is disjoint from $[L, R]$, so the words $\omega$ and $\omega'$ agree on it.

*Step 4: proof of* (d). This is Lemma 6.2(b).

*Step 5: proof of* (e). Let $\varepsilon_L = 1$. Then $L$ is new, so $L \notin V$, and $L-1 \notin V$. The column $L-1$ is not in $[L, R]$, so it is not new, by (b); hence $L-1 \notin V'$. Thus the component $C'$ of $V'$, which contains $L$, begins at $L$, that is, $u = L$. The statement for $\varepsilon_R = 1$ follows in the same way. □

*Left, central and right runs.* In the rest of this proof, a *run* of $S'$ means a long $b$-run of $S'$. Let $\beta = [s, e]$ be a long $b$-run of $S'$ contained in $C'$. We call $\beta$ a *left run* if $s+1 \leq L-1$, a *central run* if $L \leq s+1 \leq R+1$, and a *right run* if $s+1 \geq R+2$. Each long $b$-run of $S'$ in $C'$ is of exactly one of these kinds. If there are right runs, we denote the leftmost one by $\psi$. Thus a run that starts at $L-2$ or earlier is a left run, even if it continues into $[L, R]$; a run that starts in $[L-1, R]$ is a central run, even if it continues beyond $R$; and a run that starts at $R+1$ or later is a right run, even if $S$ has the letter $b$ in column $R$. Claim 4 treats the central runs, and Claim 5 the left and right runs.

**Claim 3** (Locality)**.** *Let $\beta$ be a long $b$-run of $S$ in a component $D$ of $V$, and let $\beta'$ be a long $b$-run of $S'$ in a component $D'$ of $V'$, both with the same start $s$. Let $t \leq s$ be an index such that*

(i) *$[t, s] \subseteq D \cap D'$, and $\omega_j = \omega'_j$ for $t \leq j \leq s-1$; and*
(ii) *either* (A) *$t = \min D = \min D'$, or* (B) *$t \leq s-1$, and $t$ is the end of a long $b$-run $\eta$ of $S$ and the end of a long $b$-run $\eta'$ of $S'$.*

*Then $\beta$ is credited in $(V, S)$ exactly if $\beta'$ is credited in $(V', S')$.*

*Proof of Claim 3.* Put $t^* = t$ in case (A) and $t^* = t+1$ in case (B), and call $[t^*, s-1]$ the *window*. It is contained in $[t, s-1]$, so by (i) it lies in $D \cap D'$ and $\omega = \omega'$ on it. The hypotheses are symmetric in $(V, S, D, \beta, \eta)$ and $(V', S', D', \beta', \eta')$.

*Step 1.* $\omega_{s-1} \neq b$ and $\omega'_{s-1} \neq b$, because $\beta$ and $\beta'$ are long $b$-runs that start at $s$.

*Step 2.* Every mark $\xi$ of $(V, S)$ in $D$ that lies to the left of $\beta$ is contained in the window, or, in case (B), equals $\eta$ or lies to the left of $\eta$. Indeed, $\xi \subseteq D$ and $\max \xi \leq s-1$. In case (A), $\min \xi \geq \min D = t$, so $\xi \subseteq [t, s-1]$, which is the window. In case (B), the run $\eta$ is an interval of $\operatorname{supp}(S) \subseteq V$ that contains $t \in D$, so $\eta \subseteq D$ and $\eta$ is a run mark in $D$. If $\xi \neq \eta$, then $\xi$ is disjoint from $\eta$, so $t \notin \xi$. Since $\xi$ is an interval with $\max \xi \leq s-1$, either $\xi \subseteq [t+1, s-1]$, which is the window, or $\max \xi < t$; in the latter case $\xi$ lies to the left of $\eta$, because $\eta$ is an interval ending at $t$ and disjoint from $\xi$. By symmetry, the same holds for the marks of $(V', S')$ in $D'$ that lie to the left of $\beta'$, with $\eta'$ in place of $\eta$.

*Step 3.* The marks of $(V, S)$ in $D$ that are contained in the window are exactly the marks of $(V', S')$ in $D'$ that are contained in the window, with the same kinds. For outer marks this holds because $\omega = \omega'$ on the window. Let $\kappa = [i_1, i_2]$ be a long $b$-run of $S$ contained in the window. Then $\omega'_j = \omega_j = b$ for $i_1 \le j \le i_2$. Since $\omega_{i_2} = b$ and $\omega_{s-1} \neq b$ (Step 1), we have $i_2 \le s-2$; so $i_2 + 1$ lies in the window, and $\omega'_{i_2+1} = \omega_{i_2+1} \neq b$. For the left side, consider $i_1 - 1 \ge t^* - 1$. In case (A), if $i_1 = t$, then $i_1 - 1 = \min D' - 1 \notin V'$, so $\omega'_{i_1-1} = -$. In case (B), $i_1 \neq t+1$, because $\omega_t = b$ (as $t \in \eta$) would otherwise extend $\kappa$ to the left; so $i_1 \ge t+2$. In all remaining cases $i_1 - 1$ lies in the window, and $\omega'_{i_1-1} = \omega_{i_1-1} \neq b$. Hence $\kappa$ is also a long $b$-run of $S'$, and it is a run mark of $(V', S')$ in $D'$ because the window lies in $D'$. By symmetry, every long $b$-run of $S'$ contained in the window is a long $b$-run of $S$.

*Step 4.* Suppose that the window contains a mark of $(V, S)$ in $D$. By Step 2, the left neighbor of $\beta$ is then the rightmost mark contained in the window. By Step 3 and by Step 2 for $(V', S')$, the left neighbor of $\beta'$ is the same set, of the same kind. So $\beta$ and $\beta'$ are both credited or both blocked.

*Step 5.* Suppose that the window contains no mark of $(V, S)$ in $D$; by Step 3 it then contains no mark of $(V', S')$ in $D'$ either. In case (A), Step 2 shows that $\beta$ and $\beta'$ have no left neighbors, so both are credited. In case (B), Step 2 shows that the left neighbors of $\beta$ and $\beta'$ are the run marks $\eta$ and $\eta'$, so both are credited. □

**Claim 4** (Central runs)**.**

(a) *At most one central run of $S'$ is credited, and a credited central run is the leftmost central run.*

(b) *If $L < R$ and $\varepsilon_L = 1$, then no central run of $S'$ is credited.*

*Proof of Claim 4. Step 1.* Let $\beta_1 = [s_1, e_1]$ and $\beta_2 = [s_2, e_2]$ be central runs with $s_1 < s_2$, such that no central run starts strictly between $s_1$ and $s_2$. Then $L \le e_1 \le R - 2$. Indeed, $e_1 \ge s_1 + 1 \ge L$ because $\beta_1$ is a central run. Distinct long $b$-runs are separated by a column without $b$, so $s_2 \ge e_1 + 2$; and $s_2 \le R$ because $\beta_2$ is a central run. Hence $e_1 \le R - 2$.

*Step 2.* $\omega'_{e_1+1} \in \{a, c\}$. Indeed, $L \le e_1 < R$ by Step 1 and $\omega'_{e_1} = b$, so Claim 1(a) with $j = e_1$ gives $\omega'_{e_1+1} \neq -$. Moreover $\omega'_{e_1+1} \neq b$, because $\beta_1$ ends at $e_1$.

*Step 3.* $\beta_2$ is blocked. The column $e_1 + 1$ lies between $\beta_1$ and $\beta_2$, hence in the interval $C'$ that contains both; by Step 2, $\{e_1 + 1\}$ is an outer mark in $C'$ to the left of $\beta_2$. A run mark of $(V', S')$ in $C'$ that lies between $\beta_1$ and $\beta_2$ would start at some $s$ with $s_1 < s < s_2$; then $L \le s+1 \le R+1$, so it would be a central run starting strictly between $s_1$ and $s_2$, which is excluded. So every mark strictly between $\beta_1$ and $\beta_2$ is an outer mark, and there is at least one. Hence the left neighbor of $\beta_2$ is an outer mark.

*Step 4: proof of* (a). Let $\beta_2$ be a central run other than the leftmost one, and let $\beta_1$ be the central run that precedes it among the central runs. By Step 3, $\beta_2$ is blocked.

*Step 5: proof of* (b). Let $L < R$ and $\varepsilon_L = 1$. By Claim 2(e), $L \notin V$ and $u = L$. Hence $\omega_L = -$, and Claim 1(b) with $j = L$ gives $\omega'_L \in \{a, c\}$; so $\{L\}$ is an outer mark in $C'$. Let $\beta = [s, e]$ be the leftmost central run, if there is one. Then $s \ge u = L$, and $s \neq L$ because $\omega'_L \neq b$; so the mark $\{L\}$ lies to the left of $\beta$. A run mark in $C'$ to the left of $\beta$ would start at some $s_0$ with $L \le s_0 < s$; then $L \le s_0 + 1 \le s \le R$, so it would be a central run to the left of $\beta$, which is excluded. So all marks in $C'$ to the left of $\beta$ are outer marks, and there is at least one. Hence $\beta$ is blocked, and by (a) no central run is credited. □

**Claim 5** (Left and right runs)**.**

(a) *Let $\beta'$ be a left run, with start $s$. Then $S$ has a long $b$-run $\beta$ with start $s$. It lies in the component $D_{\text{left}}$ of Claim* 2(b)*, and $D_{\text{left}} \subseteq C'$. Moreover, $\beta$ is credited in $(V, S)$ exactly if $\beta'$ is credited in $(V', S')$.*

(b) *Let $\beta'$ be a right run other than $\psi$. Then $\beta'$ is also a long $b$-run of $S$. It lies in the component $D_{\text{right}}$ of Claim* 2(b)*, and $D_{\text{right}} \subseteq C'$. Moreover, $\beta'$ is credited in $(V, S)$ exactly if it is credited in $(V', S')$.*

(c) *The long $b$-runs of $S$ assigned in* (a) *and* (b) *to different runs $\beta'$ are different.*

*Proof of Claim 5. Step 1: setting of* (a). Let $\beta' = [s, e']$. Then $s \geq u$ because $\beta' \subseteq C'$, and $s+1 \leq L-1$ because $\beta'$ is a left run. So $[u, s+1] \subseteq [u, L-1]$. By Claim 2(b) this interval lies in $D_{\text{left}}$, and $\min D_{\text{left}} = u$. By Claim 2(c), $D_{\text{left}} \subseteq C'$, since $D_{\text{left}}$ meets $C'$. The columns of $[u, s+1]$ are outside $[L, R]$, so $\omega = \omega'$ there.

*Step 2: the run $\beta$.* By Step 1, $\omega_s = \omega'_s = b$ and $\omega_{s+1} = \omega'_{s+1} = b$. If $s - 1 \geq u$, then $\omega_{s-1} = \omega'_{s-1} \neq b$, because $\beta'$ starts at $s$. If $s - 1 = u - 1$, then $s - 1 \notin V$ by Claim 2(b), so $\omega_{s-1} = -$. Hence the maximal interval containing $s$ on which $\omega$ equals $b$ starts at $s$ and contains $s+1$: it is a long $b$-run $\beta$ of $S$ with start $s$. It lies in the component of $V$ that contains $s$, which is $D_{\text{left}}$.

*Step 3: credited runs in* (a). Apply Claim 3 with $D = D_{\text{left}}$, $D' = C'$ and $t = u$, in case (A). Hypothesis (i) holds by Step 1, and $\min D_{\text{left}} = u = \min C'$. So $\beta$ is credited exactly if $\beta'$ is credited.

*Step 4: setting of* (b). Let $\psi = [s_1, e_1]$ and $\beta' = [s, e]$. Then $s_1 + 1 \geq R + 2$, so $s_1 \geq R + 1$. Since $\psi$ is the leftmost right run and $\beta' \neq \psi$, we have $s > s_1$, so $s \geq e_1 + 2$. All columns of $[s_1, v+1]$ lie outside $[L, R]$, so $\omega = \omega'$ there. Moreover $[s_1, v] \subseteq [R+1, v] \subseteq D_{\text{right}}$ by Claim 2(b), and $D_{\text{right}} \subseteq C'$ by Claim 2(c), since $D_{\text{right}}$ meets $C'$.

*Step 5: the run $\beta'$ in $S$.* The letters of $\omega$ on $\beta'$ are $b$. The neighboring indices $s-1$ and $e+1$ lie in $[s_1, v+1]$, so by Step 4, $\omega_{s-1} = \omega'_{s-1} \neq b$ and $\omega_{e+1} = \omega'_{e+1} \neq b$. Hence $\beta'$ is a long $b$-run of $S$. It lies in $[s_1, v] \subseteq D_{\text{right}}$.

*Step 6: credited runs in* (b). The index $e_1$ is the end of the long $b$-run $\psi$ of $S'$. It is also the end of a long $b$-run of $S$: by Step 4, $\omega_j = b$ for $s_1 \leq j \leq e_1$ and $\omega_{e_1+1} = \omega'_{e_1+1} \neq b$, so the maximal interval on which $\omega$ equals $b$ and which contains $[s_1, e_1]$ ends at $e_1$ and has at least two columns. Apply Claim 3 with $D = D_{\text{right}}$, $D' = C'$ and $t = e_1$, in case (B). We have $t \leq s - 2$. Hypothesis (i) holds, because $[e_1, s] \subseteq [s_1, v]$ lies in $D_{\text{right}} \cap C'$ and $\omega = \omega'$ there, by Step 4. So $\beta'$ is credited in $(V, S)$ exactly if it is credited in $(V', S')$.

*Step 7: proof of* (c). The runs assigned in (a) start at $s \leq L - 2$, and those assigned in (b) start at $s \geq R + 2$; so no run is assigned in both parts. In (a), different left runs have different starts, and the assigned runs have the same starts. In (b), each run is assigned to itself. □

**Claim 6** (A new column at the left end of its component)**.** *Let $L = R$, $\varepsilon_L = 1$ and $\varepsilon_R = 0$. Then $q(V', S') - q(V, S) \leq 1$.*

*Proof of Claim 6. Step 1.* Put $j = L = R$. By Claim 2(e), $j \notin V$ and $u = j$. Since $j$ is new and $\varepsilon_R = 0$, we have $j + 1 \in V$. As $j \in C'$ and $j + 1 \in V \subseteq V'$, the component $C'$ contains $j + 1$; so $v \geq j + 1$, and $D = [j + 1, v]$ is contained in $V$ by Claim 2(b). It is a component of $V$, because $j \notin V$ and $v + 1 \notin V$. It is the only component of $V$ contained in $C' = [j, v]$, because $V \cap C' = D$. By Claim 2(c), the other components of $V$ are the components of $V'$ other than $C'$, with the same letters; by (2), each of them contributes equally to $q(V, S)$ and $q(V', S')$. Hence

$$q(V', S') - q(V, S) = q_{C'}(V', S') - q_D(V, S).$$

*Step 2.* We have $\omega_j = -$ because $j \notin V$, and $\omega'_j \neq -$ because $j \in V' \setminus V$ forces $j \in \operatorname{supp}(S')$. Also $\omega'_i = \omega_i$ for $i \neq j$. Moreover $\omega'_{j-1} = -$, because $j-1 \notin V'$ (Claim 2(e) gives $u = j$).

*Step 3: $\omega'_j \in \{a, c\}$.* No long $b$-run of $S'$ contains $j$. A long $b$-run of $S'$ contained in $D$ is a long $b$-run of $S$, and conversely: the letters on $D$ agree, and at the index $j$ neither word has the letter $b$. So the marks of $(V', S')$ in $C'$ are the outer mark $\{j\}$, followed by the marks of $(V, S)$ in $D$. Every run mark in $D$ other than the leftmost mark of $D$ has the same left neighbor in both cases. If the leftmost mark of $D$ is a run mark, it is credited in $(V, S)$, since it has no left neighbor, and it is blocked in $(V', S')$, since its left neighbor is $\{j\}$. Hence $q_{C'}(V', S') \leq q_D(V, S)$.

*Step 4: $\omega'_j = b$ and $\omega_{j+1} \neq b$.* Then $\omega'_{j-1} = -$ (Step 2) and $\omega'_{j+1} = \omega_{j+1} \neq b$, so the letter $b$ in column $j$ is not part of a long $b$-run of $S'$, and it is not an outer letter. Neither word has the letter $b$ in column $j+1$, so every long $b$-run of $S$ or of $S'$ in $C'$ lies in $[j+2, v]$; there, and at the neighboring indices $j+1$ and $v+1$, the two words agree. So the marks of $(V', S')$ in $C'$ are exactly the marks of $(V, S)$ in $D$. Hence $q_{C'}(V', S') = q_D(V, S)$.

*Step 5: $\omega'_j = b$ and $\omega_{j+1} = b$.* Let $[j+1, e]$ be the maximal interval containing $j+1$ on which $\omega$ equals $b$; it starts at $j+1$, because $\omega_j = -$. Since $\omega'_{j-1} = -$ and $\omega' = \omega$ on $D$, the interval $[j, e]$ is a long $b$-run of $S'$. It is the leftmost mark in $C'$, because $j = \min C'$; so it is credited. If $e \geq j+2$, then $[j+1, e]$ is a long $b$-run of $S$; it is the leftmost mark in $D$ because $j+1 = \min D$, hence it is credited. All other marks of $(V, S)$ in $D$ and of $(V', S')$ in $C'$ lie in $[e+1, v]$, where the two words agree, and they are the same. So the marks in $C'$ are those in $D$, with $[j+1, e]$ replaced by $[j, e]$. Hence every other run mark has a left neighbor of the same kind in both cases, and $q_{C'}(V', S') = q_D(V, S)$. If $e = j+1$, then $\{j+1\}$ is a maximal interval on which $\omega$ equals $b$; it has one column, so it is not a mark. Since $\omega'_{j+2} = \omega_{j+2} \neq b$ (as $e = j+1$), all other marks of $(V, S)$ in $D$ and of $(V', S')$ in $C'$ lie in $[j+2, v]$, where the two words agree, and they are the same. So the marks in $C'$ are the run mark $[j, j+1]$, followed by the marks of $(V, S)$ in $D$, which all lie in $[j+2, v]$. The leftmost mark of $D$, if it is a run mark, is credited in both cases: in $(V, S)$ it has no left neighbor, and in $(V', S')$ its left neighbor is the run mark $[j, j+1]$. Every other run mark in $D$ has the same left neighbor in both cases. Hence $q_{C'}(V', S') = q_D(V, S) + 1$.

Steps 3–5 cover all letters $\omega'_j \in \{a, b, c\}$, and in each case $q_{C'}(V', S') - q_D(V, S) \leq 1$. □

*Proof of Lemma 6.3. Step 1.* By Claim 2(c), the components of $V'$ other than $C'$ are exactly the components of $V$ disjoint from $C'$, and $\omega' = \omega$ on each of them; by (2), each of them contributes the same number to $q(V', S')$ and to $q(V, S)$. Every other component of $V$ is contained in $C'$, by Claim 2(c). Hence

$$q(V', S') - q(V, S) = q_{C'}(V', S') - \sum_D q_D(V, S),$$

where $D$ runs over the components of $V$ contained in $C'$.

*Step 2.* Let $N'$ be the set of credited long $b$-runs of $S'$ in $C'$, and let $N'_{\text{left}}$, $N'_{\text{central}}$ and $N'_{\text{right}}$ be its left, central and right runs. By Claim 5, the runs in $N'_{\text{left}}$, and the runs in $N'_{\text{right}}$ other than $\psi$, are assigned different credited long $b$-runs of $S$, all lying in components of $V$ contained in $C'$. Hence the sum in Step 1 is at least $|N'_{\text{left}}| + |N'_{\text{right}} \setminus \{\psi\}|$, and

$$q(V', S') - q(V, S) \leq |N'_{\text{central}}| + [\,\psi \in N'_{\text{right}}\,].$$

*Step 3.* We show that $q(V', S') - q(V, S) \leq 2 - \varepsilon_L - \varepsilon_R$. If $L = R$, $\varepsilon_L = 1$ and $\varepsilon_R = 0$, Claim 6 gives this bound. Otherwise we bound the two terms of Step 2. First, $|N'_{\text{central}}| \leq 1 - \varepsilon_L$. By Claim 4(a), $|N'_{\text{central}}| \leq 1$. If $\varepsilon_L = 1$, then either $L < R$, and Claim 4(b) gives $N'_{\text{central}} = \varnothing$; or $L = R$ and $\varepsilon_R = 1$, and then $C' = [L, L]$ by Claim 2(e), which contains no long $b$-run. Second, $[\psi \in N'_{\text{right}}] \leq 1 - \varepsilon_R$. If $\varepsilon_R = 1$, then $v = R$ by Claim 2(e), while a right run starts at an index $s \geq R + 1 > v$; so there is no right run in $C'$.

*Step 4.* By Claim 2(d) and Step 3,

$$H(V', S') - H(V, S) = (\varepsilon_L + \varepsilon_R) + \big(q(V', S') - q(V, S)\big) \le 2. \qquad \square$$

*Proof of Theorem 6.1.* Let $\varnothing = X_0, X_1, \dots, X_t = b_T$ be a walk of length $t = d_{\mathcal{C}(B_k)}(\varnothing, b_T)$, with visited sets $V_i$ (Subsection 6.1). Then $\operatorname{supp}(X_{i-1}) \subseteq V_{i-1}$ and $V_i = V_{i-1} \cup \operatorname{supp}(X_i)$, so Lemma 6.3 gives $H(V_i, X_i) - H(V_{i-1}, X_{i-1}) \le 2$. Initially $H(\varnothing, \varnothing) = 0$. The word of $b_T$ has no outer letter, so $(V_t, b_T)$ has no outer mark, and every long $b$-run of $b_T$ is credited, whatever the visited set $V_t$ is. These long $b$-runs are the $s(T)$ maximal intervals in $T$ with at least two elements. Therefore

$$2t \ge H(V_t, b_T) = p(V_t) + s(T) \ge |T| + r(T) + s(T),$$

using Lemma 6.2(c) and $T \subseteq V_t$. $\square$

**Corollary 6.4.** *For every $k \ge 1$,*

$$\operatorname{diam} \mathcal{C}(B_k) \ge \lfloor 2k/3 \rfloor + 1.$$

*Proof.* Let $m \ge 1$ and $\ell \in \{2, 3, 4\}$, put $k = 3m + \ell - 3$, and let

$$T = [1, \ell] \cup \bigcup_{i=1}^{m-1} [\ell + 3i - 1, \ell + 3i] \subseteq \{1, \dots, k\}.$$

Its maximal intervals are $[1, \ell]$ and $m-1$ intervals of length two, separated by single missing indices. So $|T| = 2m+\ell-2$ and $r(T) = s(T) = m$, and Theorem 6.1 gives $d_{\mathcal{C}(B_k)}(\varnothing, b_T) \ge \lceil (4m+\ell-2)/2 \rceil$. This is $2m$, $2m+1$ and $2m+1$ for $\ell = 2, 3, 4$, that is, $\lfloor 2k/3 \rfloor + 1$ for $k = 3m-1$, $3m$ and $3m+1$. Every $k \ge 2$ has one of these forms. For $k = 1$, take $T = \{1\}$. $\square$

6.3. **Exact diameters.**

*Proof of Theorem 1.3.* The connectedness and the number of elements follow from Definition 1.2. Theorem 4.2 gives the order diameter. The upper bound in Theorem 5.1 and the lower bound in Corollary 6.4 give

$$\operatorname{diam} \mathcal{C}(B_k) = \lfloor 2k/3 \rfloor + 1.$$

Since $k - \lfloor 2k/3 \rfloor = \lceil k/3 \rceil$, the difference of the diameters is $\lceil k/3 \rceil - 1$. At $k = 7$ the diameters are 7 and 5. Since $(\lfloor 2k/3 \rfloor + 1)/k \to 2/3$, we also have

$$\frac{\operatorname{diam} \mathcal{O}(B_k)}{\operatorname{diam} \mathcal{C}(B_k)} = \frac{k}{\lfloor 2k/3 \rfloor + 1} \longrightarrow \frac{3}{2}. \qquad \square$$

**Corollary 6.5.** *For every $k \ge 1$, the eccentricity of the empty antichain in the graph of $\mathcal{C}(B_k)$ equals the diameter of that graph.*

*Proof.* The antichains $b_T$ in the proof of Corollary 6.4 are at distance at least $\lfloor 2k/3 \rfloor + 1$ from the empty antichain. No distance exceeds this value by Theorem 1.3. $\square$

## 7. Further preliminaries

In Sections 7–12, $P$ is an arbitrary finite poset unless stated otherwise, and a walk may also repeat a state (see below). From Section 9 on, the letters $a_i$, $b_j$, $h$, $x_j$, $y_i$, $w_i$, $u^-$, $u^+$, $\sigma$ and $A$, $B$, $U$, $L$, $T$, $W$ denote elements, maps and sets of the poset $P_n$ defined there, not of the band.

Two distinct elements of a poset are *comparable* if one is smaller than the other, and *incomparable* otherwise; an element is neither comparable nor incomparable to itself. A subset $D$ of $P$ is *connected* if it is not empty and the comparability graph of $P$ induces a connected graph on $D$. The *components* of a set are its maximal connected subsets. A set $D$ is a *star* at $x$ if $x \in D$ and $x$ is comparable to every other element of $D$; a star is connected. We write $X \triangle Y$ for the symmetric difference, $\downarrow S$ and $\uparrow S$ for the sets of elements below or equal to, respectively above or

equal to, some element of $S$, and $\downarrow p = \downarrow\{p\}$, $\uparrow p = \uparrow\{p\}$. A *common lower bound* of $p$ and $q$ is an element $c$ with $c \le p$ and $c \le q$; common upper bounds are defined dually.

We use the adjacency criteria of Theorem 2.1: two antichains $X \neq Y$ of $P$ are adjacent in the graph of $\mathcal{C}(P)$ exactly if $X \triangle Y$ is connected, and two ideals are adjacent exactly if they are nested and their difference is connected.

We write $d_{\mathcal{O}}$ and $d_{\mathcal{C}}$ for the graph distances. A *walk* of length $m$ from $X_0$ to $X_m$ is a sequence $(X_0, \dots, X_m)$ of antichains, its *states*, in which any two consecutive states are equal or have a connected symmetric difference. By Theorem 2.1, consecutive states are equal or adjacent, so $d_{\mathcal{C}}(X_0, X_m) \le m$.

**Lemma 7.1** (Components)**.** *Let $J \subseteq I$ be ideals of $P$, and let $C$ be a union of components of $I \setminus J$. Then $J \cup C$ and $I \setminus C$ are ideals.*

*Proof.* Let $z \in C$ and $z' < z$. Then $z' \in I$. If $z' \notin J$, then $z' \in I \setminus J$ is comparable to $z$, so it lies in the component of $z$, and $z' \in C$. Hence $J \cup C$ is an ideal. Now let $z \in I \setminus C$ and $z' < z$, and suppose $z' \in C$. Then $z' \notin J$, so $z \notin J$; thus $z \in I \setminus J$ is comparable to $z'$, and $z \in C$, a contradiction. Hence $I \setminus C$ is an ideal. □

We use the lemma with $J = \varnothing$ (removing a component of an ideal) and with $I = P$ (adding to an ideal $J$ a component of $P \setminus J$).

**Lemma 7.2.** *If $X, Y$ are antichains and $X \cup Y$ is an antichain, then $d_{\mathcal{C}}(X, Y) \le |X \triangle Y|$.*

*Proof.* Remove the elements of $X \setminus Y$ one at a time, then add those of $Y \setminus X$ one at a time. Every state is a subset of $X \cup Y$, hence an antichain, and every difference is a single element. □

We use the split lemma, Lemma 5.3, which holds for every finite poset $P$: for antichains $X, Y$ there are disjoint antichains $A \subseteq X$ and $B \subseteq Y$ such that $A \cup B$ is an antichain and $d_{\mathcal{C}}(X, Y) \le d_{\mathcal{C}}(A, B)$. A pair $(A, B)$ of disjoint sets whose union $U$ is an antichain is a *split* of $U$. By Lemma 5.3, $\operatorname{diam} \mathcal{C}(P)$ is the largest distance $d_{\mathcal{C}}(A, B)$ over all splits of antichains of $P$.

## 8. Order diameter at most two implies chain diameter at most four

**Lemma 8.1** (Pair lemma)**.** *Suppose that* $\operatorname{diam} \mathcal{O}(P) \le 2$*. Let $p, q$ be incomparable elements of $P$, and suppose that some third element $r \notin \{p, q\}$ is incomparable to both. Then $p$ and $q$ have a common lower bound or a common upper bound.*

*Proof.* The sets $I = \downarrow p \cup \downarrow q$ and $J = P \setminus (\uparrow p \cup \uparrow q)$ are ideals. An element $x \in I \setminus J$ satisfies $x \le p$ or $x \le q$, and $x \ge p$ or $x \ge q$. As $p, q$ are incomparable, this forces $x \in \{p, q\}$; so $I \setminus J = \{p, q\}$. Also $r \in J \setminus I$. So $I$ and $J$ are not nested, hence not adjacent, and since $\operatorname{diam} \mathcal{O}(P) \le 2$ there is an ideal $K$ adjacent to both. By Theorem 2.1, $K$ is nested with $I$ and with $J$. The cases $J \subseteq K \subseteq I$ and $I \subseteq K \subseteq J$ are impossible, since $r \in J \setminus I$ and $p \in I \setminus J$. So $K \subseteq I \cap J$ or $K \supseteq I \cup J$.

Suppose that $p, q$ have no common lower bound and no common upper bound. Then $\downarrow p$ and $\downarrow q$ are disjoint, and no element of $\downarrow p$ is comparable to an element of $\downarrow q$: if $x \le p$, $y \le q$ and $x \le y$, then $x$ is a common lower bound, and similarly if $y \le x$. Dually for $\uparrow p$ and $\uparrow q$. If $K \subseteq I \cap J$, then $I \setminus K$ contains $p$ and $q$ and lies in $\downarrow p \cup \downarrow q$, so it is not connected. If $K \supseteq I \cup J$, then $K \setminus J$ contains $I \setminus J = \{p, q\}$ and lies in $P \setminus J = \uparrow p \cup \uparrow q$, so it is not connected. Both contradict Theorem 2.1. □

*Proof of Theorem 1.5.* By Lemma 5.3 it suffices to show $d_{\mathcal{C}}(A, B) \le 4$ for every split $(A, B)$ of an antichain $U$. We may assume $U \neq \varnothing$. Pick $u \in U$ and a maximal element $t$ of $P$ with $t \ge u$. Let $U_t = U \cap \downarrow t$; it contains $u$. If $t \in U$ then $t = u$ and $U_t = \{u\}$. Put

$$U' = (U \setminus U_t) \cup \{t\}, \qquad A_t = (A \setminus U_t) \cup \{t\}, \qquad B_t = (B \setminus U_t) \cup \{t\}.$$

The set $U'$ is an antichain: $t$ is maximal and lies above no element of $U \setminus U_t$. The difference $A \triangle A_t$ is empty if $A \cap U_t = \{t\}$, and otherwise it is $(A \cap U_t) \cup \{t\}$, a star at $t$; so $A$ and $A_t$ are

equal or adjacent. In the same way $B_t$ and $B$ are equal or adjacent. Moreover $A_t \triangle B_t = U \setminus U_t$ and $A_t \cup B_t = U'$.

If $|U \setminus U_t| \le 2$, then $d_{\mathcal{C}}(A_t, B_t) \le 2$ by Lemma 7.2. Otherwise let $v \in U \setminus U_t$. Then $v$ and $t$ are incomparable, and any $r \in U \setminus (U_t \cup \{v\})$ is a third element incomparable to both. By Lemma 8.1, $v$ and $t$ have a common lower bound or a common upper bound. A common upper bound of $t$ is $t$ itself, and $v \not\le t$; so there is a common lower bound $z_v$, with $z_v < v$ and $z_v < t$. Let $M_t$ be the set of minimal elements of $P$ below $t$. It is a nonempty antichain, and $t \notin M_t$ because $z_v < t$. No element of $U \setminus U_t$ lies below $t$, so $M_t$ is disjoint from $U'$. Every $v \in U \setminus U_t$ lies above an element of $M_t$, namely any minimal element below $z_v$. Hence $A_t \triangle M_t = (A \setminus U_t) \cup \{t\} \cup M_t$ is connected through $t$, which lies above $M_t$, and in the same way $M_t \triangle B_t = M_t \cup (B \setminus U_t) \cup \{t\}$ is connected. So $(A_t, M_t, B_t)$ is a walk.

In both cases $d_{\mathcal{C}}(A, B) \le 1 + 2 + 1 = 4$. □

**Remark 8.2.** The proof does not use that $P$ is connected. When more than two elements of $U$ lie outside $\downarrow t$, its walk is $A \to A_t \to M_t \to B_t \to B$; otherwise its middle part consists of at most two moves, each exchanging one element.

## 9. The construction of $P_n$

From now on, $A$, $B$ and $U$ denote the specific antichains of the following construction.

**Definition 9.1.** Let $n \ge 2$ and $[n] = \{1, \dots, n\}$. The poset $P_n$ has the following elements:

- the *targets* $a_i$ and $b_j$ $(i, j \in [n])$; put $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ and $U = A \cup B$;
- the *pair minima* $m_E$ and $m'_E$, one of each for every 2-subset $E$ of $[n]$; we also write $m_{ik} = m_{ki} = m_{\{i,k\}}$ and $m'_{jl} = m'_{lj} = m'_{\{j,l\}}$; put $M_A = \{m_E : E \subseteq [n],\ |E| = 2\}$ and $M_B = \{m'_E : E \subseteq [n],\ |E| = 2\}$;
- the *lower connectors* $y_i$ and $x_j$ $(i, j \in [n])$; put $Y_A = \{y_1, \dots, y_n\}$ and $Y_B = \{x_1, \dots, x_n\}$;
- the *hub* $h$;
- the *upper connectors* $w_i$ and $w'_j$ $(i, j \in [n])$; put $W_A = \{w_1, \dots, w_n\}$, $W_B = \{w'_1, \dots, w'_n\}$ and $W = W_A \cup W_B$;
- the *tops* $t_{ij}$ $(i, j \in [n])$; put $T = \{t_{ij}\}$.

The order is the reflexive and transitive closure of the relations

$$m_E < y_i\ (i \in E), \qquad y_i < a_i, \qquad y_i < h, \qquad m'_E < x_j\ (j \in E), \qquad x_j < b_j, \qquad x_j < h,$$
$$a_i < w_i, \qquad h < w_i, \qquad b_j < w'_j, \qquad h < w'_j, \qquad w_i < t_{ij}, \qquad w'_j < t_{ij}.$$

So $y_i$ lies above the $n-1$ pair minima $m_{ik}$ $(k \ne i)$, each pair minimum $m_{ik}$ lies below $a_i$ and $a_k$, and $t_{ij}$ lies above $a_i$ and $b_j$. The number of elements is $2n+n(n-1)+2n+1+2n+n^2 = 2n^2+5n+1$, which is 53 for $n = 4$. The rank function with values 0 on $M_A \cup M_B$, 1 on $Y_A \cup Y_B$, 2 on $U \cup \{h\}$, 3 on $W$ and 4 on $T$ increases by one along each listed relation, so the relation generated is a partial order. We also use

$$L = M_A \cup M_B \cup Y_A \cup Y_B, \qquad \Lambda_A = M_A \cup Y_A \cup A, \qquad \Lambda_B = M_B \cup Y_B \cup B,$$

so that $P_n$ is the disjoint union of $\Lambda_A$, $\Lambda_B$, $\{h\}$, $W$ and $T$. For a target $u$ we write $u^-$ for its lower connector and $u^+$ for its upper connector: $a_i^- = y_i$, $a_i^+ = w_i$, $b_j^- = x_j$, $b_j^+ = w'_j$.

The map $\sigma$ with $\sigma(a_i) = b_i$, $\sigma(m_E) = m'_E$, $\sigma(y_i) = x_i$, $\sigma(w_i) = w'_i$, $\sigma(t_{ij}) = t_{ji}$, $\sigma(h) = h$, and $\sigma^2 = \mathrm{id}$, maps the listed relations onto the listed relations. So $\sigma$ is an automorphism of $P_n$ that exchanges $A$ and $B$ and fixes $h$; it induces automorphisms of the graphs of $\mathcal{O}(P_n)$ and $\mathcal{C}(P_n)$.

**Lemma 9.2.** *In $P_n$:*

(i) *The elements below $a_i$ are $y_i$ and the $m_{ik}$ ($k \neq i$); they all lie in $\downarrow y_i$. The elements above $a_i$ are $w_i$ and $t_{i1}, \dots, t_{in}$; they all lie in $\uparrow w_i$. Symmetrically (apply $\sigma$) for $b_j$, with $x_j$, the $m'_{jl}$, $w'_j$ and $t_{1j}, \dots, t_{nj}$.*

(ii) *$\downarrow h = L \cup \{h\}$ and $\uparrow h = \{h\} \cup W \cup T$. So $h$ is comparable to every element outside $U \cup \{h\}$ and to no target, and every element of $L$ lies below every element of $W \cup T$.*

(iii) *No element of $\Lambda_A$ is comparable to an element of $\Lambda_B$.*

(iv) *Every element of $W_A \cup T$ lies above exactly one target of $A$ ($w_i$ and $t_{ij}$ above $a_i$), and no element of $W_A$ lies above a target of $B$. Symmetrically for $W_B \cup T$ and $B$ ($t_{ij}$ lies above $b_j$). The sets $W$ and $T$ are antichains.*

(v) *The comparability graph of $P_n$ is connected.*

*Proof.* Follow the listed relations. The only relation with $a_i$ on the right is $y_i < a_i$, and the only ones with $y_i$ on the right are $m_{ik} < y_i$; this gives the elements below $a_i$. The only relation with $a_i$ on the left is $a_i < w_i$, and above $w_i$ there are only the $t_{ij}$; this gives the elements above $a_i$ and proves (i). Below $h$ are the lower connectors and, below them, all pair minima (each 2-subset $E$ contains some $i$, and $m_E < y_i$); above $h$ are all upper connectors and, above them, all tops. This proves (ii). For (iii): the elements above an element of $\Lambda_A$ lie in $\Lambda_A \cup \{h\} \cup W \cup T$ (by (i) and (ii), as $\uparrow y_i = \{y_i, a_i\} \cup \uparrow h$), and symmetrically for $\Lambda_B$; so no element of $\Lambda_B$ lies above an element of $\Lambda_A$, and conversely. Item (iv) follows from $\downarrow w_i = \{w_i, a_i, h\} \cup L$ and $\downarrow t_{ij} = \{t_{ij}, w_i, w'_j, a_i, b_j, h\} \cup L$; elements of equal rank are incomparable. Finally every element outside $U$ is comparable to $h$ or equal to it, and every target $u$ is comparable to $u^-$; this gives (v). □

**Remark 9.3.** The hub $h$ is comparable to every element outside $U \cup \{h\}$ and to no target (Lemma 9.2). This makes many ideals, and many complements of ideals, connected (Lemma 10.1), which keeps the order diameter at 2; for $n \geq 4$ the antichains $A$ and $B$ are nevertheless at distance 4 in the graph of $\mathcal{C}(P_n)$ (Theorem 11.5).

## 10. The order diameter of $P_n$

Throughout this section $n \geq 2$ and $P = P_n$.

**Lemma 10.1** (Connected ideals and their complements)**.** *In $P_n$:*

(i) *Every ideal $I$ with $h \in I$ is connected.*

(ii) *For every ideal $J$ with $h \notin J$, the set $P \setminus J$ is connected and contains $\uparrow h$.*

*Proof.* (i) $I$ contains $\downarrow h = L \cup \{h\}$. Every element of $I$ outside $U$ is $h$ or comparable to $h$ (Lemma 9.2(ii)), and every target $u \in I$ is comparable to $u^- \in L \subseteq I$. (ii) $P \setminus J$ is a filter that contains $h$, so it contains $\uparrow h = \{h\} \cup W \cup T$. Every element of $P \setminus J$ outside $U$ is $h$ or comparable to $h$, and every target $u \in P \setminus J$ is comparable to $u^+ \in W \subseteq P \setminus J$. □

Now let $I, J$ be ideals with $h \in I$ and $h \notin J$; we call $(I, J)$ a *mixed pair*. Let $\Gamma_h$ be the component of $I \setminus J$ that contains $h$, and let $\mathrm{Iso}(I, J)$ be the set of targets $u \in I \setminus J$ with $u^- \in J$ and $u^+ \notin I$.

**Lemma 10.2** (Mixed pairs)**.** *Let $(I, J)$ be a mixed pair.*

(i) *$L \subseteq I$, $J \subseteq L \cup U$, and $J \setminus I \subseteq U$.*

(ii) *If $J \cap (A \cup Y_A)$ is nonempty, its elements lie in one component of $J$, and this component is contained in $\Lambda_A$. Symmetrically for $J \cap (B \cup Y_B)$ and $\Lambda_B$.*

(iii) *$I \setminus J = \Gamma_h \cup \mathrm{Iso}(I, J)$, and every $u \in \mathrm{Iso}(I, J)$ forms a component $\{u\}$ of $I \setminus J$.*

*Proof.* (i) $L \subseteq \downarrow h \subseteq I$. Since $J$ is an ideal without $h$, it contains no element of $\uparrow h = \{h\} \cup W \cup T$; so $J \subseteq L \cup U \subseteq I \cup U$.

(ii) Each element of $A \cup Y_A$ has an index: $a_k$ and $y_k$ have index $k$, and both lie above every $m_{kl}$ ($l \neq k$). Let $z, z' \in J \cap (A \cup Y_A)$ have indices $k, l$. If $k = l$, then $z, z'$ are equal or comparable. If $k \neq l$, then $m_{kl} \in J$ lies below both. So $z$ and $z'$ lie in one component of $J$. By Lemma 9.2(iii) and $J \subseteq L \cup U = \Lambda_A \cup \Lambda_B$, every component of $J$ lies in $\Lambda_A$ or in $\Lambda_B$.

(iii) Every element of $I \setminus J$ outside $U$ is $h$ or comparable to $h \in I \setminus J$, so it lies in $\Gamma_h$. Let $u \in I \setminus J$ be a target. If $u^- \notin J$, then $u^- \in L \setminus J \subseteq I \setminus J$ joins $u$ to $h$. If $u^+ \in I$, then $u^+ \in I \setminus J$ by (i), and it joins $u$ to $h$. Otherwise $u \in \mathrm{Iso}(I, J)$: the elements below $u$ lie in $\downarrow u^- \subseteq J$, and the elements above $u$ lie in $\uparrow u^+$, which is disjoint from $I$ (Lemma 9.2(i)). So no element of $I \setminus J$ is comparable to $u$. □

**Lemma 10.3** (A common neighbor of $I$ and $J$)**.** *Let $(I, J)$ be a mixed pair, and suppose that $w_i \notin I$ and $w'_j \notin I$ for some $i, j \in [n]$. Let $\Gamma$ be the component of $P \setminus I$ that contains $w_i$, and $K = I \cup \Gamma$. Then:*

(i) *$\Gamma$ contains every element of $W \setminus I$ and every element of $U \setminus I$;*
(ii) *$K$ is an ideal adjacent to $I$, and $J \subsetneq K$;*
(iii) *$K \setminus J$ is connected, so $K$ is adjacent to $J$, and $d_{\mathcal{O}}(I, J) \leq 2$.*

*Proof.* (i) Since $I$ is an ideal, a top above an element outside $I$ is outside $I$. Let $w_k \notin I$. Then $t_{kj}, t_{ij} \notin I$, and $w_k < t_{kj} > w'_j < t_{ij} > w_i$ is a path in $P \setminus I$; so $w_k \in \Gamma$. Let $w'_l \notin I$. Then $t_{il} \notin I$, and $w'_l < t_{il} > w_i$; so $w'_l \in \Gamma$. A target $u \notin I$ has $u^+ \notin I$, so $u$ is joined to $u^+ \in \Gamma$.

(ii) $K$ is an ideal by Lemma 7.1 (with the ideals $I \subseteq P$). $K \setminus I = \Gamma$ is connected, so $K$ is adjacent to $I$. By Lemma 10.2(i), $J \subseteq L \cup U$; here $L \subseteq I$, and every target of $J$ outside $I$ lies in $\Gamma$ by (i). So $J \subseteq K$, and $J \neq K$ since $h \in K \setminus J$.

(iii) By (i), $K$ contains all of $W$, and $W \cap J = \varnothing$. Every element of $K \setminus J$ outside $U$ is $h$ or comparable to $h$; every target $u \in K \setminus J$ is comparable to $u^+ \in K \setminus J$, which is comparable to $h$. So $K \setminus J$ is connected, and $(I, K, J)$ is a path in the graph of $\mathcal{O}(P)$. □

**Proposition 10.4.** *For every mixed pair $(I, J)$, $d_{\mathcal{O}}(I, J) \leq 2$.*

*Proof.* Let $\mathrm{Iso} = \mathrm{Iso}(I, J)$ and $\Delta = \mathrm{Iso} \cup (J \setminus I)$, a set of targets. For $u \in \mathrm{Iso}$ we have $u^+ \notin I$ by definition; for $u \in J \setminus I$ we have $u \notin I$, hence $u^+ \notin I$. So if $\Delta$ meets both $A$ and $B$, Lemma 10.3 applies. The automorphism $\sigma$ fixes $h$, so it maps mixed pairs to mixed pairs; it exchanges $A$ and $B$, and $\sigma(u^-) = \sigma(u)^-$ and $\sigma(u^+) = \sigma(u)^+$ for every target $u$. Hence $\mathrm{Iso}(\sigma I, \sigma J) = \sigma(\mathrm{Iso}(I, J))$ and $\sigma J \setminus \sigma I = \sigma(J \setminus I)$, so $\sigma$ carries $\Delta$ to the corresponding set of $(\sigma I, \sigma J)$. As $\sigma$ preserves distances, we may assume from now on that $\Delta \subseteq A$.

If $\Delta = \varnothing$, then $J \subseteq I$ and $I \setminus J = \Gamma_h$ is connected (Lemma 10.2(iii)), so $I$ and $J$ are adjacent.

Let $\Delta \neq \varnothing$. Then $J \cap (A \cup Y_A) \neq \varnothing$: a target of $J \setminus I$ lies in $J \cap A$, and $u \in \mathrm{Iso}$ has $u^- \in J \cap Y_A$. Let $J_1$ be the component of $J$ that contains $J \cap (A \cup Y_A)$ (Lemma 10.2(ii)); then $J_1 \subseteq \Lambda_A$. Put $K = J \setminus J_1$. By Lemma 7.1 (with the ideals $\varnothing \subseteq J$), $K$ is an ideal, and $J \setminus K = J_1$ is connected; so $K$ is adjacent to $J$. Since $J \setminus I \subseteq J \cap A \subseteq J_1$, we have $K \subseteq I$, and

$$I \setminus K = (I \setminus J) \cup (I \cap J_1) = \Gamma_h \cup \mathrm{Iso} \cup (I \cap J_1).$$

We show that every element of this set is joined to $h$ inside it; then $I \setminus K$ is connected, and $K$ is adjacent to $I$ (note $h \in I \setminus K$).

- Elements of $\Gamma_h$: by definition.
- Elements of $I \cap J_1$ in $L$: they are comparable to $h$.
- Targets in $I \cap J_1$: such a target is some $a_k$, as $J_1 \subseteq \Lambda_A$. Then $y_k \in J$ lies in $J_1$ by the choice of $J_1$, and in $L \subseteq I$; so $y_k \in I \cap J_1$ is comparable to $a_k$ and to $h$.
- Elements of Iso: they are targets of $A$, since $\mathrm{Iso} \subseteq \Delta \subseteq A$. For $a_k \in \mathrm{Iso}$, $y_k = a_k^- \in J \cap Y_A \subseteq J_1$ and $y_k \in L \subseteq I$; so $y_k \in I \cap J_1$ is comparable to $a_k$ and to $h$.

Hence $(I, K, J)$ is a path, and $d_{\mathcal{O}}(I, J) \leq 2$. □

**Theorem 10.5.** *For every $n \geq 2$,* $\operatorname{diam} \mathcal{O}(P_n) = 2$.

*Proof.* Let $I, J$ be ideals. If both contain $h$, both are connected by Lemma 10.1, so both are adjacent to $\varnothing$. If neither contains $h$, then $P \setminus I$ and $P \setminus J$ are connected, so both are adjacent to $P$. Otherwise one of them contains $h$ and the other does not, and Proposition 10.4 applies. Hence $\operatorname{diam} \mathcal{O}(P_n) \leq 2$. The ideals $\{m_{12}\}$ and $\{m'_{12}\}$ are not nested, so they are not adjacent, and $\operatorname{diam} \mathcal{O}(P_n) \geq 2$. □

## 11. The chain distance of the two target sets of $P_n$

Throughout this section $P = P_n$. For an antichain $X$ let $\mathcal{N}[X]$ be the set of antichains that are equal or adjacent to $X$. We list the possible forms of the members of $\mathcal{N}[A]$; by the automorphism $\sigma$ this also describes $\mathcal{N}[B]$.

Let $Z_A$ be the set of elements outside $A$ that are incomparable to every target of $A$, and define $Z_B$ in the same way for $B$. By Lemma 9.2, $\downarrow A = \Lambda_A$ and $\uparrow A = A \cup W_A \cup T$, so

$$Z_A = \Lambda_B \cup \{h\} \cup W_B, \qquad Z_B = \Lambda_A \cup \{h\} \cup W_A.$$

**Lemma 11.1** (Neighbors of $A$)**.** *Every antichain $X \in \mathcal{N}[A]$ with $X \neq A$ has one of the following forms.*

(D) $X = A \setminus \{a_i\}$ *for some $i$.*
(E) $X = A \cup \{z\}$ *for some $z \in Z_A$.*
(L) $X = (A \setminus R) \cup S$*, where $\varnothing \neq R \subseteq A$, $\varnothing \neq S \subseteq M_A \cup Y_A$, and every element of $S$ lies below some element of $R$ and below no element of $A \setminus R$.*
(H) $X = (A \setminus \{a_i\}) \cup S$ *for some $i$, where $S = \{w_i\}$ or $\varnothing \neq S \subseteq \{t_{i1}, \ldots, t_{in}\}$.*

*The letters stand for delete, extend, lower and higher.*

*Proof.* Let $R = A \setminus X$ and $S = X \setminus A$. By Theorem 2.1, $A \triangle X = R \cup S$ is connected. If $S = \varnothing$, then $R$ is a connected subset of an antichain, so $|R| = 1$: type (D). If $R = \varnothing$, then $S$ is a connected subset of the antichain $X$, so $S = \{z\}$; as $A \cup \{z\}$ is an antichain, $z \in Z_A$: type (E).

Let $R, S \neq \varnothing$. Since $R$ and $S$ are antichains and $R \cup S$ is connected with at least two elements, every $s \in S$ is comparable to some element of $R$; and since $X$ is an antichain, $s$ is comparable to no element of $A \setminus R$. Thus $s \in (\downarrow A \cup \uparrow A) \setminus A = M_A \cup Y_A \cup W_A \cup T$. Every element of $M_A \cup Y_A \subseteq L$ lies below every element of $W_A \cup T$ (Lemma 9.2(ii)), so the antichain $S$ lies in $M_A \cup Y_A$ or in $W_A \cup T$. In the first case $X$ has type (L). In the second case every $s \in S$ lies above exactly one target of $A$ (Lemma 9.2(iv)), which belongs to $R$. In the connected bipartite comparability graph on $R \cup S$, every element of $S$ then has exactly one neighbor; a path between two elements of $R$ would pass through an element of $S$ with two neighbors in $R$. So $R = \{a_i\}$ for some $i$, and $S$ is an antichain in $\uparrow a_i \setminus \{a_i\} = \{w_i, t_{i1}, \ldots, t_{in}\}$. As $w_i$ lies below every $t_{ij}$, either $S = \{w_i\}$ or $S \subseteq \{t_{i1}, \ldots, t_{in}\}$: type (H). □

Applying $\sigma$, every $Y \in \mathcal{N}[B]$ with $Y \neq B$ has one of the forms (D′) $B \setminus \{b_j\}$; (E′) $B \cup \{z\}$ with $z \in Z_B$; (L′) $(B \setminus R) \cup S$ with $\varnothing \neq R \subseteq B$, $\varnothing \neq S \subseteq M_B \cup Y_B$, and every element of $S$ below some element of $R$ and below no element of $B \setminus R$; (H′) $(B \setminus \{b_j\}) \cup S$ with $S = \{w'_j\}$ or $\varnothing \neq S \subseteq \{t_{1j}, \ldots, t_{nj}\}$.

**Lemma 11.2** (An isolated element)**.** *Let $X, Y$ be antichains and $G = X \triangle Y$. If $u \in X \setminus Y$ is comparable to no element of $Y$, then no element of $G$ is comparable to $u$. Consequently, if moreover $|G| \geq 2$, then $G$ is not connected. The same holds with $X$ and $Y$ exchanged.*

*Proof.* An element of $G$ comparable to $u$ would lie in $X$ or in $Y$. It is not in the antichain $X$, which contains $u$, and not in $Y$ by hypothesis. □

We record which elements of a neighbor of $B$ are equal or comparable to targets of $A$. Call an element $z$ *attached* to a target $u$ if $z = u$ or $z$ is comparable to $u$.

**Lemma 11.3.** *Let $Y \in \mathcal{N}[B]$.*

(i) *If $Y = B$ or $Y$ has type (D′) or (L′), then $Y \subseteq \Lambda_B$, and no element of $Y$ is attached to a target of $A$.*

(ii) *If $Y = B \cup \{z\}$ has type (E′), then $z$ is the only element of $Y$ that can be attached to a target of $A$, and it is attached to at most two of them.*

(iii) *If $Y = (B \setminus \{b_j\}) \cup S$ has type (H′), then $Y \cap A = \varnothing$, the only elements of $Y$ comparable to targets of $A$ are the tops $t_{pj} \in S$, and $t_{pj}$ is comparable to $a_p$ only.*

*The same statements hold with $A$ and $B$ exchanged, for $X \in \mathcal{N}[A]$.*

*Proof.* (i) follows from Lemma 9.2(iii). For (ii): $B \subseteq \Lambda_B$, and $z \in Z_B = \Lambda_A \cup \{h\} \cup W_A$. If $z = a_k$, it is attached to $a_k$ only; if $z = m_{ik}$, to $a_i$ and $a_k$; if $z = y_i$, to $a_i$ only; if $z = h$, to none; if $z = w_i$, to $a_i$ only (Lemma 9.2). For (iii): $S \subseteq \{w'_j, t_{1j}, \ldots, t_{nj}\}$ and $B \setminus \{b_j\} \subseteq \Lambda_B$, so $Y \cap A = \varnothing$, and the elements of $B \setminus \{b_j\}$ are comparable to no target of $A$ (Lemma 9.2(iii)); $w'_j$ is comparable to no target of $A$ (Lemma 9.2(i)), and $t_{pj}$ is comparable to $a_p$ only (Lemma 9.2(iv)). □

**Proposition 11.4.** *Let $n \geq 4$. For all $X \in \mathcal{N}[A]$ and $Y \in \mathcal{N}[B]$, the set $X \triangle Y$ is nonempty and disconnected. In particular $X \neq Y$, and $X$ and $Y$ are not adjacent.*

*Proof.* Let $G = X \triangle Y$. We call $X$ *low* if $X = A$ or $X$ has type (D) or (L), so that $X \subseteq \Lambda_A$ is nonempty; similarly $Y$ is low if $Y = B$ or $Y$ has type (D′) or (L′). Otherwise $X$ has type (E) or (H) and contains at least $n - 1$ targets of $A$, and similarly for $Y$. Cases 1, 2, 3 and 5 below use only $n \geq 3$; Case 4 uses $n \geq 4$.

*Case 1: $X$ and $Y$ are low.* Then $X \subseteq \Lambda_A$ and $Y \subseteq \Lambda_B$ are nonempty and disjoint, so $G = X \cup Y$, and no element of $X$ is comparable to an element of $Y$ (Lemma 9.2(iii)). So $G$ is not connected.

*Case 2: $X$ is low and $Y$ is not.* $Y$ contains at least $n - 1 \geq 2$ targets of $B$. No element of $X \subseteq \Lambda_A$ is attached to a target of $B$. So these targets lie in $Y \setminus X \subseteq G$ and are comparable to no element of $X$; as there are at least two of them, Lemma 11.2 applies. The case where $Y$ is low and $X$ is not is symmetric.

*Case 3: $X = A \cup \{z\}$ and $Y = B \cup \{z'\}$, of types* (E) *and* (E′). By Lemma 11.3(ii), at most two targets of $A$ are attached to elements of $Y$. So at least $n - 2 \geq 1$ target of $A$ lies in $X \setminus Y$ and is comparable to no element of $Y$. By the same lemma with $A$ and $B$ exchanged, at least $n - 2 \geq 1$ target of $B$ lies in $Y \setminus X$; so $|G| \geq 2$, and Lemma 11.2 applies.

*Case 4: $X = A \cup \{z\}$ of type* (E) *and $Y = (B \setminus \{b_j\}) \cup S$ of type* (H′). By the version of Lemma 11.3(ii) with $A$ and $B$ exchanged, $z$ is the only element of $X$ that can be attached to a target of $B$, and it is attached to at most two of them. The set $Y$ contains the $n - 1 \geq 3$ targets of $B \setminus \{b_j\}$. So some $b \in Y \cap B$ is not attached to $z$; then $b \in Y \setminus X$, and $b$ is comparable to no element of $X$. Also $A \subseteq X \setminus Y$, as $Y \cap A = \varnothing$; so $|G| \geq 2$, and Lemma 11.2 applies. See Remark 12.1. The case of types (H) and (E′) is symmetric.

*Case 5: $X = (A \setminus \{a_i\}) \cup S$ of type* (H) *and $Y = (B \setminus \{b_j\}) \cup S'$ of type* (H′). Fix $p \neq i$. Then $a_p \in X \setminus Y$, so $a_p \in G$. The elements comparable to $a_p$ are $y_p$, the $m_{pk}$, $w_p$ and the $t_{pq}$. None of them lies in $X$: they are not targets, and none of them lies in $\uparrow a_i$, which contains $S$ (the elements below $a_p$ are not above $a_i$ by Lemma 9.2(i), and by Lemma 9.2(iv) no element of $\uparrow a_i \setminus \{a_i\}$ lies above $a_p$). The only one that can lie in $Y$ is $t_{pj}$, if $t_{pj} \in S'$. In that case $t_{pj} \notin X$, so $t_{pj} \in G$. The elements comparable to $t_{pj}$ are the elements of $\downarrow t_{pj} \setminus \{t_{pj}\} = \{w_p, w'_j, a_p, b_j, h\} \cup L$. Of these, only $a_p$ lies in $X \cup Y$: $w_p \notin X$ since $p \neq i$, and $w_p \notin Y$; $w'_j \notin Y$ since $S' \subseteq T$ when $t_{pj} \in S'$, and $w'_j \notin X$; $b_j$ lies in neither; and no element of $L \cup \{h\}$ lies in $X \cup Y$, because sets of types (H) and (H′) contain only targets, upper connectors and tops. Hence the component of $a_p$ in $G$ is contained in $\{a_p, t_{pj}\}$. But $G$ also contains the targets of $B \setminus \{b_j\}$, which lie outside $\{a_p, t_{pj}\}$; so $G$ is not connected.

The cases cover all pairs: by the remark on low sets, a non-low $X$ has type (E) or (H), and a non-low $Y$ has type (E′) or (H′). Each case exhibits elements of $G$, so $G$ is nonempty and $X \neq Y$; and since $G$ is disconnected, $X$ and $Y$ are not adjacent, by Theorem 2.1. □

**Theorem 11.5.** *For every $n \geq 4$, $d_{\mathcal{C}(P_n)}(A, B) = 4$. Hence* $\operatorname{diam} \mathcal{C}(P_n) = 4$.

*Proof. Upper bound.* The sequence

$$A \;\to\; M_A \;\to\; \{h\} \;\to\; M_B \;\to\; B$$

is a walk. The states are antichains. $A \triangle M_A = A \cup M_A$ is connected, since $m_{ik}$ lies below $a_i$ and $a_k$ and the complete graph on $[n]$ is connected. $M_A \triangle \{h\} = M_A \cup \{h\}$ is a star at $h$. The last two moves are symmetric.

*Lower bound.* Suppose $d_{\mathcal{C}}(A, B) \leq 3$, and let $A = Q_0, Q_1, \ldots, Q_d = B$ be a shortest path, $1 \leq d \leq 3$. Put $X = Q_1$ and $Y = Q_{d-1}$. Then $X \in \mathcal{N}[A]$, $Y \in \mathcal{N}[B]$, and $X = Y$ (if $d = 2$) or $X$ and $Y$ are adjacent (if $d \in \{1, 3\}$; for $d = 1$, $X = B$ and $Y = A$). This contradicts Proposition 11.4.

So $\operatorname{diam} \mathcal{C}(P_n) \geq 4$. By Theorems 10.5 and 1.5, $\operatorname{diam} \mathcal{C}(P_n) \leq 4$. □

*Proof of Theorem 1.4.* $P_n$ is connected by Lemma 9.2(v) and has $2n^2 + 5n + 1$ elements. Theorems 10.5 and 11.5 give the diameters. □

## 12. Remarks

**Remark 12.1** (Small $n$)**.** For $n \in \{2, 3\}$ the antichains $A$ and $B$ of $P_n$ are at distance at most 3. With $F = [n] \setminus \{1\}$ if $n = 3$, and $F = \{1, 2\}$ if $n = 2$, the sequence

$$A \;\to\; (A \setminus \{a_1\}) \cup \{t_{11}, \ldots, t_{1n}\} \;\to\; B \cup \{m_F\} \;\to\; B$$

is a walk: the first move is a star at $a_1$; in the middle difference, $m_F$ lies below the retained targets of $A$ and below every top, and $t_{1j}$ lies above $b_j$; the last move removes $m_F$. This is the case of types (H) and (E′), symmetric to Case 4 of the proof of Proposition 11.4, which $n \geq 4$ excludes: for $n \geq 4$, three targets of $A$ are retained, and the added pair minimum lies below only two of them. We do not determine $\operatorname{diam} \mathcal{C}(P_2)$ and $\operatorname{diam} \mathcal{C}(P_3)$; by Theorems 10.5 and 1.5, both are at most 4.

**Remark 12.2** (Computations)**.** The following checks were made before and after the hand proofs; they are not inputs to any proof. For $P_4$, three independently written programs find $|\mathcal{N}[A]| = |\mathcal{N}[B]| = 152$ (the antichain $A$ itself, 4 of type (D), 19 of type (E), 64 of type (H) and 64 of type (L)), and confirm that no pair in $\mathcal{N}[A] \times \mathcal{N}[B]$ is equal or adjacent; one of them also finds $|\mathcal{N}[A]| = |\mathcal{N}[B]| = 1165$ for $P_5$ (again counting $A$ and $B$ themselves), and again no such pair. The poset $P_4$ has 275744 ideals (and antichains); Lemma 10.1 holds for all of them. For mixed pairs, an ideal adjacent to both ideals of the pair, of the kinds used in Section 10, was found for every pair when $n \in \{2, 3\}$, for 220000 random pairs with $n = 4$ and, in a second program, for 20076 further random pairs with $4 \leq n \leq 7$. A further program searched all 275744 ideals of $P_4$ for a common neighbor of 12200 sampled mixed pairs and found one in every case.

**Remark 12.3** (Larger differences)**.** Whether $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P)$ is bounded on connected finite posets is Question 14.1. An example with difference at least 3 must have $\operatorname{diam} \mathcal{O}(P) \geq 3$. Indeed, the empty poset has both diameters 0. If $P$ is nonempty and $\operatorname{diam} \mathcal{O}(P) \leq 1$, then any two ideals are nested; incomparable elements $p, q$ would give the non-nested ideals $\downarrow p$ and $\downarrow q$, so $P$ is a chain. Then the antichains are $\varnothing$ and the singletons; any two of them are adjacent, and $\operatorname{diam} \mathcal{C}(P) = 1$. If $\operatorname{diam} \mathcal{O}(P) = 2$, Theorem 1.5 applies.

## 13. Supplementary computations

For the posets $P_n$ see Remark 12.2. Exact graph searches confirmed the formulas of Theorem 1.3 for $1 \le k \le 10$ before the general proof was obtained. At least two independently written programs agree for $k \le 8$. For $k = 9, 10$, the values come from one graph implementation. All these values were checked by breadth-first search from every vertex; for $k = 9, 10$ this also confirmed an earlier computation using exact eccentricity bounds on the same graphs. The programs use the adjacency criteria of Theorem 2.1. These finite checks are not inputs to any proof in this paper.

## 14. Questions

Theorem 1.4 gives connected posets whose order and chain diameters are 2 and 4, respectively. Thus the inequality $\operatorname{diam}\mathcal{C}(P) - \operatorname{diam}\mathcal{O}(P) \le 1$ also fails. We ask:

**Question 14.1.** Is $\operatorname{diam}\mathcal{C}(P) - \operatorname{diam}\mathcal{O}(P)$ unbounded on connected finite posets $P$? By Remark 12.3, an example with difference at least 3 must have order diameter at least 3.

**Question 14.2.** What is the supremum of $\operatorname{diam}\mathcal{O}(P)/\operatorname{diam}\mathcal{C}(P)$ over connected posets $P$? Theorem 1.3 shows that it is at least $3/2$. The ratio is strictly less than $3/2$ for every member of the band family, so a larger supremum would require other posets. Is it finite?

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