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Modular Kittaneh Numerical Radius Inequalities

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Abstract: We show that for adjointable morphisms on Hilbert C*-module over a commutative unital C*-algebra, a version of breakthrough numerical radius inequalities obtained by Kittaneh [*Stud. Math., 2005*] hold.
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1. Introduction

The **numerical range** or the **field of values** [11–16, 18, 19, 36] of a continuous linear operator T acting on a complex Hilbert space $\mathcal{H}$ (with the inner product linear in the first variable and conjugate linear in the second variable) is defined as

$$W_{\mathbb{C}}(T) \coloneqq \{\langle Th, h\rangle : h \in \mathcal{H}, \|h\| = 1\} \subseteq \mathbb{C}$$

and the **numerical radius** of T is defined as

$$w_{\mathbb{C}}(T) \coloneqq \sup_{h\in\mathcal{H}, \|h\|=1} |\langle Th, h\rangle| \in [0, \infty).$$

Using the Cauchy-Schwarz inequality and the generalized polarization identity [13], we get

$$\frac{\|T\|}{2} \leq w_{\mathbb{C}}(T) \leq \|T\|. \tag{1}$$

(**Classical Numerical Radius Inequalities**)

Let $\mathcal{B}(\mathcal{H})$ be the C*-algebra of all continuous linear operators acting on $\mathcal{H}$. Inequalities (1) say that the map $w_{\mathbb{C}}(\cdot) : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ is a norm which is equivalent to the operator norm. In 2005, Kittaneh made a far reaching generalization of Inequalities (1) [22].

Theorem 1.1. *[22] (**Kittaneh Numerical Radius Inequalities**) Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\sqrt{\|T^*T + TT^*\|} \leq w_{\mathbb{C}}(T) \leq \frac{1}{\sqrt{2}}\sqrt{\|T^*T + TT^*\|}. \tag{2}$$

Since

$$\frac{\|T\|}{2} \leq \frac{1}{2}\sqrt{\|T^*T + TT^*\|} \leq \frac{1}{\sqrt{2}}\sqrt{\|T^*T + TT^*\|} \leq \|T\|, \quad \forall T \in \mathcal{B}(\mathcal{H}),$$

Inequalities (2) improve Inequalities (1). Our fundamental motivation comes from the following question: What is the noncommutative analogue of Theorem 1.1? This is then naturally connected with the notion of Hilbert C*-modules which are first introduced by Kaplansky [21] for modules over commutative C*-algebras and later developed for modules over arbitrary C*-algebras by Paschke [27] and Rieffel [32].

Definition 1.2. *[21, 27, 32] Let $\mathcal{A}$ be a unital C*-algebra. A right module $\mathcal{E}$ over $\mathcal{A}$ is said to be a **semi-inner product C*-module** if there exists a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that the following hold.*

(i) $\langle x, x\rangle \geq 0$, $\forall x \in \mathcal{E}$.
(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, $\forall x, y, z \in \mathcal{E}$.
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$, $\forall x, y \in \mathcal{E}$, $\forall a \in \mathcal{A}$.
(iv) $\langle x, y\rangle = \langle y, x\rangle^*$, $\forall x, y \in \mathcal{E}$.

Semi-inner product C*-modules satisfy noncommutative Cauchy-Schwarz inequality.

Theorem 1.3. *[23] (**Cauchy-Schwarz-Kaplansky-Paschke-Rieffel Inequality**) Let $\mathcal{E}$ be a semi-inner product C*-module. Then*

$$\langle y, x\rangle\langle x, y\rangle \leq \|\langle x, x\rangle\|\langle y, y\rangle, \quad \forall x, y \in \mathcal{E}.$$

In particular,

$$\|\langle y, x\rangle\|^2 \leq \|\langle x, x\rangle\|\|\langle y, y\rangle\|, \quad \forall x, y \in \mathcal{E}.$$

Definition 1.4. *[23] A semi-inner product C*-module $\mathcal{E}$ is said to be **inner product C*-module** if $x \in \mathcal{E}$ satisfies $\langle x, x\rangle = 0$, then $x = 0$.*

Definition 1.5. *[23] An inner product C*-module $\mathcal{E}$ is said to be **Hilbert C*-module** if $\mathcal{E}$ is complete w.r.t. the norm $\|x\| := \sqrt{\|\langle x, x\rangle\|}$, $\forall x \in \mathcal{E}$.*

In this article, we derive noncommutative version of classical numerical radius inequalities and modular version of Kitttaneh numerical radius inequalities.

2. Modular Kittaneh Numerical Radius Inequalities

Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Similar to Hilbert space case, we define the **noncommutative numerical range** of an adjointable morphism $T : \mathcal{E} \to \mathcal{E}$ as

$$W_{\mathcal{A}}(T) := \{\langle x, Tx\rangle : x \in \mathcal{E}, \|x\| = 1\} \subseteq \mathcal{A}$$

and the **noncommutative numerical radius** of T is defined as

$$w_{\mathcal{A}}(T) \coloneqq \sup_{x\in\mathcal{E},\|x\|=1} \|\langle x, Tx\rangle\| \in [0,\infty).$$

Note that our definition of numerical range and radius on Hilbert C*-modules differ from the existing same notion on Hilbert C*-modules [26, 30, 39]. We give various examples. In all following examples, $\mathcal{A}$ is a unital C*-algebra.

Example 2.1.

$$W_{\mathcal{A}}(1) = \{a^*a : a \in \mathcal{A}, \|a\| = 1\}$$

and

$$w_{\mathcal{A}}(1) = 1.$$

Example 2.2. *Let $d \in \mathbb{N}$ and I_d be the identity matrix of size d by d. Then we have*

$$W_{\mathcal{A}}(I_d) = \{\langle x, x\rangle : x \in \mathcal{A}^d, \|x\| = 1\}$$

and

$$w_{\mathcal{A}}(I_d) = 1.$$

Example 2.3. *Define*

$$M \coloneqq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} W_{\mathcal{A}}(M) &= \{\langle x, Mx\rangle : x \in \mathcal{A}^2, \|x\| = 1\} \\ &= \{a^*b : a, b \in \mathcal{A}, \|a^*a + b^*b\| = 1\} \end{aligned}$$

and

$$\begin{aligned} w_{\mathcal{A}}(M) &= \sup\{\|a^*b\| : a, b \in \mathcal{A}, \|a^*a + b^*b\| = 1\} \\ &\leq 1. \end{aligned}$$

We note that

$$a^*bb^*a \leq \frac{a^*a + b^*b}{2}, \quad \forall a, b \in \mathcal{A}.$$

Therefore

$$\begin{aligned} w_{\mathcal{A}}(M) &= \sup\{\|a^*b\| : a, b \in \mathcal{A}, \|a^*a + b^*b\| = 1\} \\ &\leq \frac{1}{\sqrt{2}} \sup\{\sqrt{\|a^*a + b^*b\|} : a, b \in \mathcal{A}, \|a^*a + b^*b\| = 1\} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Example 2.4. *Let $d \in \mathbb{N}$ and $b_1, \ldots, b_d \in \mathcal{A}$. Define* $\operatorname{diag}((b_j)_{j=1}^d)$ *as the diagonal matrix with diagonal entries $b_1, \ldots, b_d$. Then*

$$W_{\mathcal{A}}(\operatorname{diag}((b_j)_{j=1}^d)) = \left\{ \sum_{j=1}^d a_j^* b_j a_j : (a_j)_{j=1}^d \in \mathcal{A}^d, \left\| \sum_{j=1}^d a_j^* a_j \right\| = 1 \right\}$$

and

$$w_{\mathcal{A}}(\text{diag}((b_j)_{j=1}^d)) = \sup\left\{\left\|\sum_{j=1}^d a_j^* b_j a_j\right\| : (a_j)_{j=1}^d \in \mathcal{A}^d, \left\|\sum_{j=1}^d a_j^* a_j\right\| = 1\right\}.$$

Example 2.5. *Let $R \in \mathbb{M}_d(\mathcal{A})$ be the right shift matrix defined by*

$$R \coloneqq \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Thus the action of R is

$$R(a_j)_{j=1}^d = (0, a_1, a_2, \ldots, a_{d-1}), \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d.$$

We then have

$$\langle (a_j)_{j=1}^d, R(a_j)_{j=1}^d \rangle = a_2^* a_1 + \cdots + a_d^* a_{d-1}, \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d.$$

Hence

$$\begin{aligned} \|\langle (a_j)_{j=1}^d, R(a_j)_{j=1}^d \rangle\| &= \|a_2^* a_1 + \cdots + a_d^* a_{d-1}\| \\ &\leq \left\|\sum_{j=2}^d a_j^* a_j\right\|^{\frac{1}{2}} \left\|\sum_{k=1}^{d-1} a_k^* a_k\right\|^{\frac{1}{2}} \\ &\leq \left\|\sum_{j=1}^d a_j^* a_j\right\|^{\frac{1}{2}} \left\|\sum_{k=1}^{d} a_k^* a_k\right\|^{\frac{1}{2}} \\ &= \left\|\sum_{j=1}^d a_j^* a_j\right\|, \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d. \end{aligned}$$

Therefore

$$W_{\mathcal{A}}(R) = \left\{a_2^* a_1 + \cdots + a_d^* a_{d-1} : (a_j)_{j=1}^d \in \mathcal{A}^d, \left\|\sum_{j=1}^d a_j^* a_j\right\| = 1\right\}$$

and

$$\begin{aligned} w_{\mathcal{A}}(R) &= \sup\left\{\|a_2^* a_1 + \cdots + a_d^* a_{d-1}\| : (a_j)_{j=1}^d \in \mathcal{A}^d, \left\|\sum_{j=1}^d a_j^* a_j\right\| = 1\right\} \\ &\leq 1. \end{aligned}$$

Example 2.6. *Let $\mathcal{A}$ be a unital C*-algebra and $\ell^2(\mathcal{A})$ be the standard Hilbert C*-module defined by*

$$\ell^2(\mathcal{A}) \coloneqq \left\{\{a_n\}_{n=1}^\infty : a_n \in \mathcal{A}, \forall n \in \mathbb{N}, \sum_{n=1}^\infty a_n^* a_n < \infty\right\}$$

equipped with the inner product

$$\langle \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \rangle \coloneqq \sum_{n=1}^{\infty} a_n^* b_n, \quad \forall \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \ell^2(\mathcal{A})$$

and the norm

$$\|\{a_n\}_{n=1}^{\infty}\| = \left\| \sum_{n=1}^{\infty} a_n^* a_n \right\|^{\frac{1}{2}}, \quad \forall \{a_n\}_{n=1}^{\infty} \in \ell^2(\mathcal{A}).$$

Let L be the left shift morphism defined by

$$L : \ell^2(\mathcal{A}) \ni \{a_n\}_{n=1}^{\infty} \mapsto L\{a_n\}_{n=1}^{\infty} \coloneqq \{a_{n+1}\}_{n=1}^{\infty} \in \ell^2(\mathcal{A}).$$

Then

$$W_{\mathcal{A}}(L) = \left\{ \sum_{n=1}^{\infty} a_n^* a_{n+1} : \{a_n\}_{n=1}^{\infty} \in \ell^2(\mathcal{A}), \left\| \sum_{n=1}^{\infty} a_n^* a_n \right\| = 1 \right\}$$

and

$$\begin{aligned} w_{\mathcal{A}}(L) &= \sup \left\{ \left\| \sum_{n=1}^{\infty} a_n^* a_{n+1} \right\| : \{a_n\}_{n=1}^{\infty} \in \ell^2(\mathcal{A}), \left\| \sum_{n=1}^{\infty} a_n^* a_n \right\| = 1 \right\} \\ &\leq 1. \end{aligned}$$

Example 2.7. *Define*

$$A \coloneqq \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$W_{\mathcal{A}}(A) = \{x^*ax + x^*by + y^*cx + y^*dy : x, y \in \mathcal{A}, \|x^*x + y^*y\| = 1\}$$

and

$$\begin{aligned} w_{\mathcal{A}}(A) &= \sup\{\|x^*ax + x^*by + y^*cx + y^*dy\| : x, y \in \mathcal{A}, \|x^*x + y^*y\| = 1\} \\ &\leq \|a\| + \|b\| + \|c\| + \|d\|. \end{aligned}$$

Given $r > 0$ and $v \in \mathcal{A}$, the closed disc centered at $v \in \mathcal{A}$ of radius r is defined as

$$\mathbb{D}_{\mathcal{A}}[v, r] \coloneqq \{z \in \mathcal{A} : \|z - v\| \leq r\}.$$

Following are basic properties of noncommutative numerical range and radius.

Proposition 2.8. *Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $T, S : \mathcal{E} \to \mathcal{E}$ be adjointable morphisms and $a, b \in \mathcal{A}$. Let I be the identity morphism on $\mathcal{E}$.*

(i) $W_{\mathcal{A}}(T) \subseteq \mathbb{D}_{\mathcal{A}}[0, \|T\|]$.
(ii) $0 \leq w_{\mathcal{A}}(T) \leq \|T\|$.
(iii) $W_{\mathcal{A}}(T + S) \subseteq W_{\mathcal{A}}(T) + W_{\mathcal{A}}(S)$.
(iv) $w_{\mathcal{A}}(S + T) \leq w_{\mathcal{A}}(S) + w_{\mathcal{A}}(T)$.
(v) $W_{\mathcal{A}}(Ta) = W_{\mathcal{A}}(T)a$.
(vi) $w_{\mathcal{A}}(Ta) \leq \|a\| w_{\mathcal{A}}(T)$.
(vii) $W_{\mathcal{A}}(T^*) = (W_{\mathcal{A}}(T))^*$.
(viii) $w_{\mathcal{A}}(T^*) = w_{\mathcal{A}}(T)$.

(ix) $W_{\mathcal{A}}(Ta+b)=W_{\mathcal{A}}(T)a+b$.

(x) $W_{\mathcal{A}}(\mathrm{Re}(T))=\mathrm{Re}(W_{\mathcal{A}}(T))$ *and* $W_{\mathcal{A}}(\mathrm{Im}(T))=\mathrm{Im}(W_{\mathcal{A}}(T))$.

(xi) *If* $T\geq 0$, *then* $W_{\mathcal{A}}(T)\subseteq\mathcal{A}^+$.

(xii) *Let* $\mathcal{E}_0$ *be a Hilbert C*-module over a unital C*-algebra* $\mathcal{A}$ *such that* $\mathcal{E}$ *is orthogonally complementable in* $\mathcal{E}_0$. *If an adjointable morphism* $V:\mathcal{E}_0\to\mathcal{E}_0$ *is a dilation of* T, *then* $W_{\mathcal{A}}(T)\subseteq W_{\mathcal{A}}(V)$.

(xiii) *Let* $\mathcal{E}_1$ *be an orthogonally complementable closed submodule in* $\mathcal{E}$. *Let* $P:\mathcal{E}\to\mathcal{E}_1$ *be an onto orthogonal projection. Then* $W_{\mathcal{A}}(PT_{|\mathcal{E}_1})\subseteq W_{\mathcal{A}}(T)$.

(xiv) $W_{\mathcal{A}}(U^*TU)=W_{\mathcal{A}}(T)$ *for every unitary morphism* $U:\mathcal{E}\to\mathcal{E}$.

(xv) *Let* $a\in\mathcal{A}$ *be such that there exists a* $x\in\mathcal{E}$ *with* $Tx=xa$ *and* $\langle x,x\rangle=1$. *Then* $a\in W_{\mathcal{A}}(T)$. *In other words,*

$$\sigma_1(T):=\{a\in\mathcal{A}:\exists x\in\mathcal{E},Tx=xa,\langle x,x\rangle=1\}\subseteq W_{\mathcal{A}}(T)$$

and

$$\|a\|\leq w_{\mathcal{A}}(T),\quad \forall a\in\sigma_1(T).$$

(xvi)

$$w_{\mathcal{A}}(ST+TS)\leq\frac{w_{\mathcal{A}}((S+T)^2)+w_{\mathcal{A}}((S-T)^2)}{2}.$$

In particular, if $ST=TS$, *then*

$$w_{\mathcal{A}}(ST)\leq\frac{w_{\mathcal{A}}((S+T)^2)+w_{\mathcal{A}}((S-T)^2)}{4}.$$

(xvii) *If a sequence* $\{T_n\}_{n=1}^{\infty}$ *of adjointable morphisms on* $\mathcal{E}$ *converges to an adjointable morphism* T *on* $\mathcal{E}$ *in the morphism norm, then the sequence* $\{w_{\mathcal{A}}(T_n)\}_{n=1}^{\infty}$ *converges to* $w_{\mathcal{A}}(T)$ *in the norm.*

We next derive a noncommutative analogue of Inequalities (1).

Theorem 2.9. *Let* $\mathcal{E}$ *be a Hilbert C*-module over a unital C*-algebra* $\mathcal{A}$. *Let* $T:\mathcal{E}\to\mathcal{E}$ *be an adjointable morphism. Then*

$$\frac{\|T\|}{4}\leq w_{\mathcal{A}}(T)\leq\|T\|. \tag{3}$$

Proof. For $x,y\in\mathcal{E}$. Then

$$4\langle y,Tx\rangle=\langle x+y,T(x+y)\rangle-\langle x-y,T(x-y)\rangle+i\langle x+iy,T(x+iy)\rangle-i\langle x-iy,T(x-iy)\rangle.$$

Therefore

$$\begin{aligned}
4\|T\|&=4\sup_{x,y\in\mathcal{E},\|x\|=\|y\|=1}\|\langle y,Tx\rangle\|\\
&=\sup_{x,y\in\mathcal{E},\|x\|=\|y\|=1}\|\langle x+y,T(x+y)\rangle-\langle x-y,T(x-y)\rangle+i\langle x+iy,T(x+iy)\rangle-i\langle x-iy,T(x-iy)\rangle\|\\
&\leq w_{\mathcal{A}}(T)\sup_{x,y\in\mathcal{E},\|x\|=\|y\|=1}(\|x+y\|^2+\|x-y\|^2+\|x+iy\|^2+\|x-iy\|^2)\\
&\leq w_{\mathcal{A}}(T)\sup_{x,y\in\mathcal{E},\|x\|=\|y\|=1}((\|x\|+\|y\|)^2+(\|x\|+\|y\|)^2+(\|x\|+\|y\|)^2+(\|x\|+\|y\|)^2)\\
&=16w_{\mathcal{A}}(T).
\end{aligned}$$

□

We next derive modular version of Kittaneh numerical radius inequalities with the help of the following lemma.

Lemma 2.10. *Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $T : \mathcal{E} \to \mathcal{E}$ be an adjointable self-adjoint morphism. Then*

$$\sup_{x\in\mathcal{E},\|x\|=1} \|\langle x, Tx\rangle\| \leq \|T\| \leq 2 \sup_{x\in\mathcal{E},\|x\|=1} \|\langle x, Tx\rangle\|.$$

In other words,

$$\frac{\|T\|}{2} \leq w_{\mathcal{A}}(T) \leq \|T\|.$$

Proof. We clearly have the first inequality. Define

$$M \coloneqq \sup_{x\in\mathcal{E},\|x\|=1} \|\langle x, Tx\rangle\|.$$

Now let $x, y \in \mathcal{E}$. Then

$$\langle x+y, T(x+y)\rangle - \langle x-y, T(x-y)\rangle = 2(\langle x, Ty\rangle + \langle y, Tx\rangle).$$

Therefore

$$\begin{aligned}\|\langle x, Ty\rangle + \langle y, Tx\rangle\| &\leq \frac{1}{2}(\|\langle x+y, T(x+y)\rangle\| + \|\langle x-y, T(x-y)\rangle\|)\\ &\leq \frac{1}{2}(M\|x+y\|^2 + M\|x-y\|^2)\\ &\leq M(\|x\|^2 + \|y\|^2 + 2\|x\|\|y\|).\end{aligned}$$

Since, T is self-adjoint,

$$\|\langle x, Ty\rangle + \langle Ty, x\rangle\| \leq M(\|x\|^2 + \|y\|^2 + 2\|x\|\|y\|), \quad \forall x, y \in \mathcal{E}. \tag{4}$$

For a given $z \in \mathcal{E}$ with $\|z\| = 1$ and $\|Tz\| \neq 0$, we take

$$x = \frac{Tz}{\|Tz\|}, \quad y = z$$

in Inequality (4). Then we have

$$2\|Tz\| \leq 4M.$$

By varying $z \in \mathcal{E}$ with $\|z\| = 1$ we get $\|T\| \leq 2M$. □

Theorem 2.11. *Let $\mathcal{E}$ be a Hilbert C*-module over a unital commutative C*-algebra $\mathcal{A}$. Let $T : \mathcal{E} \to \mathcal{E}$ be an adjointable morphism. Then*

$$\frac{\sqrt{\|T^*T + TT^*\|}}{4} \leq w_{\mathcal{A}}(T) \leq \frac{1}{\sqrt{2}}\sqrt{\|T^*T + TT^*\|}. \tag{5}$$

Proof. Define

$$A \coloneqq \frac{T+T^*}{2}, \quad B \coloneqq \frac{T-T^*}{2i}.$$

Then A and B are self-adjoint, $T = A + iB$, $T^* = A - iB$ and

$$2A^2 + 2B^2 = T^*T + TT^*.$$

Let $x \in \mathcal{E}$. Using Theorem 1.3 and since $\mathcal{A}$ is commutative, we get

$$\begin{aligned}\langle x, Tx\rangle\langle Tx, x\rangle &= \langle x, (A+iB)x\rangle\langle (A+iB)x, x\rangle\\ &= (\langle x, Ax\rangle + i\langle x, Bx\rangle)(\langle Ax, x\rangle - i\langle Bx, x\rangle)\\ &= \langle x, Ax\rangle\langle Ax, x\rangle + \langle x, Bx\rangle\langle Bx, x\rangle\\ &\geq \frac{1}{2}(\langle x, Ax\rangle \pm \langle x, Bx\rangle)(\langle Ax, x\rangle \pm \langle Bx, x\rangle)\\ &= \frac{1}{2}\langle x, (A\pm B)x\rangle\langle (A\pm B)x, x\rangle.\end{aligned}$$

Hence by using Lemma 2.10,

$$\begin{aligned}w_{\mathcal{A}}(T)^2 &= \sup_{x\in\mathcal{E}, \|x\|=1} \|\langle x, Tx\rangle\|^2\\ &= \sup_{x\in\mathcal{E}, \|x\|=1} \|\langle x, Tx\rangle\langle Tx, x\rangle\|\\ &\geq \frac{1}{2}\sup_{x\in\mathcal{E}, \|x\|=1} \|\langle x, (A\pm B)x\rangle\langle (A\pm B)x, x\rangle\|\\ &= \frac{1}{2}\sup_{x\in\mathcal{E}, \|x\|=1} \|\langle x, (A\pm B)x\rangle\|^2\\ &\geq \frac{1}{2}\frac{1}{4}\|A\pm B\|^2\\ &\geq \frac{1}{8}\|A\pm B\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}2w_{\mathcal{A}}(T)^2 &\geq \frac{1}{8}\|A+B\|^2 + \frac{1}{8}\|A-B\|^2\\ &= \frac{1}{8}\|(A+B)^2\| + \frac{1}{8}\|(A-B)^2\|\\ &\geq \frac{1}{8}\|(A+B)^2 + (A-B)^2\|\\ &= \frac{1}{8}\|2A^2 + 2B^2\|\\ &= \frac{1}{8}\|T^*T + TT^*\|.\end{aligned}$$

Now again using commutativity of C*-algebra and Theorem 1.3,

$$\begin{aligned}\langle x, Tx\rangle\langle Tx, x\rangle &= \langle x, (A+iB)x\rangle\langle (A+iB)x, x\rangle\\ &= (\langle x, Ax\rangle + i\langle x, Bx\rangle)(\langle Ax, x\rangle - i\langle Bx, x\rangle)\\ &= \langle x, Ax\rangle\langle Ax, x\rangle + \langle x, Bx\rangle\langle Bx, x\rangle\\ &\leq \|x\|^2\langle Ax, Ax\rangle + \|x\|^2\langle Bx, Bx\rangle\\ &= \|x\|^2\langle x, A^2x\rangle + \|x\|^2\langle x, B^2x\rangle\\ &= \|x\|^2\langle x, (A^2+B^2)x\rangle.\end{aligned}$$

Therefore

$$\begin{aligned} w_{\mathcal{A}}(T)^2 &= \sup_{x\in\mathcal{E},\|x\|=1} \|\langle x, Tx\rangle\|^2 \\ &\leq \sup_{x\in\mathcal{E},\|x\|=1} \|x\|^2\|\langle x, (A^2+B^2)x\rangle\| \\ &\leq \sup_{x\in\mathcal{E},\|x\|=1} \|x\|\|A^2+B^2\|\|x\| \\ &= \|A^2+B^2\| \\ &= \frac{1}{2}\|T^*T+TT^*\|. \end{aligned}$$

□

We note that for any adjointable morphism $T : \mathcal{E} \to \mathcal{E}$,

$$\frac{\|T\|}{4} \leq \frac{\sqrt{\|T^*T+TT^*\|}}{4} \leq \sqrt{\|T^*T+TT^*\|} \leq \|T\|.$$

Therefore Inequalities (5) improve Inequalities (3) over commutative C*-algebras. We wish to note that Abu-Omar and Kittaneh [1] and Bhunia, Bag and Paul [3] improved Theorem 1.1 but the techniques used by them do not seem to carry over to Hilbert C*-modules.

3. Noncommutative Toeplitz-Hausdorff Problem

The celebrated Toeplitz-Hausdorff theorem says that the numerical range is always a convex subset of $\mathbb{C}$ [11, 13, 15, 16, 18, 25, 36]. We formulate the following problem.

Problem 3.1. *What is noncommutative version of Toeplitz-Hausdorff Theorem (fundamental theorem of the numerical range)?*

4. Noncommutative Crouzeix and von Neumann Inequality Problems

Breakthrough Crouzeix theorem says following.

Theorem 4.1. *[4, 6–8, 31] (**Crouzeix Theorem**) For every $d \in \mathbb{N}$ and for every matrix $M \in \mathbb{M}_d(\mathbb{C})$, we have*

(***Crouzeix-Palencia Inequality***) $$\|p(M)\| \leq (1+\sqrt{2})\sup\{|p(z)| : z \in W_{\mathbb{C}}(M)\}, \quad \forall p \in \mathbb{C}[z].$$

We formulate following problem.

Problem 4.2. *(**Noncommutative Crouzeix Problem**) Let $\mathcal{A}$ be a unital C*-algebra. Whether there is a universal constant $R_{\mathcal{A}}$ (which may depend upon $\mathcal{A}$) satisfying following: For every $d \in \mathbb{N}$ and for every matrix $M \in \mathbb{M}_d(\mathcal{A})$, we have*

$$\|p(M)\| \leq R_{\mathcal{A}}\sup\{\|p(z)\| : z \in W_{\mathcal{A}}(M)\}, \quad \forall p(z) = a_0 + za_1 + z^2a_2 + \cdots + z^na_n \in \mathcal{A}[z].$$

We conjecture that $R_{\mathcal{A}}$ exists and $R_{\mathcal{A}} = 4$.

An inequality which is very close to Crouzeix inequality is the von Neumann inequality.

Theorem 4.3. *[9, 34, 35, 38] (**von Neumann Theorem**) For every $d \in \mathbb{N}$ and for every matrix $M \in \mathbb{M}_d(\mathbb{C})$ with $\|M\| \leq 1$, we have*

(***von Neumann Inequality***) $$\|p(M)\| \leq \sup\{|p(z)| : z \in \mathbb{C}, |z| \leq 1\}, \quad \forall p \in \mathbb{C}[z].$$

Based on Theorem 4.3 we formulate following problem.

Problem 4.4. *(**Noncommutative von Neumann Inequality Problem**) Let $\mathcal{A}$ be a unital C*-algebra. Whether there is a universal constant $R_{\mathcal{A}}$ (which may depend upon $\mathcal{A}$) satisfying the following: For every $d \in \mathbb{N}$ and for every $A \in \mathbb{M}_d(\mathcal{A})$ with $\|A\| \leq 1$, we have*

$$\|p(A)\| \leq R_{\mathcal{A}} \sup\{\|p(z)\| : z \in \mathcal{A}, \|z\| \leq 1\}, \quad \forall p(z) = a_0 + za_1 + z^2a_2 + \cdots + z^n a_n \in \mathcal{A}[z].$$

Theorem 4.3 has been extended by Ando for two variables [2, 35]. Note that Ando theorem cannot be extended to more than two commuting matrices [5,37]. We observe that the Halmos dilation [14], Egervary dilation [10,24] and Sz.-Nagy dilation (Schaffer construction) [33] carry over to adjointable morphisms on Hilbert C*-modules [29] but this result will not give noncommutative von Neumann inequality. We also observe that the proof of Ando dilation [2] for commuting operators on Hilbert space will not carry over to commuting adjointable morphisms on Hilbert C*-modules (mainly because closed submodules need not be orthogonally complementable and C*-algebras need not have the invariant basis number property).

5. Noncommutative Power Inequality Problem

Surprising power inequality for numerical radius says the following.

Theorem 5.1. *[28] For every bounded linear operator T on a Hilbert space,*

$$(\textbf{\textit{Power Inequality}}) \qquad w_{\mathbb{C}}(T^n) \leq (w_{\mathbb{C}}(T))^n, \quad \forall n \in \mathbb{N}.$$

We formulate the following problem.

Problem 5.2. *(**Noncommutative Power Inequality Problem**) Let $T : \mathcal{E} \to \mathcal{E}$ be an adjointable morphism. Whether*

$$w_{\mathcal{A}}(T^n) \leq (w_{\mathcal{A}}(T))^n, \quad \forall n \in \mathbb{N}?$$

6. Noncommutative Berger Power Dilation Problem

Berger obtained the following result connecting numerical radius and dilation theory.

Theorem 6.1. *[13, 35] For a continuous linear operator $T : \mathcal{H} \to \mathcal{H}$, the following are equivalent.*

(1) $w_{\mathbb{C}}(T) \leq 1$.

(2) There exist a Hilbert space $\mathcal{H}_0$ which contains $\mathcal{H}$ isometrically, $P_{\mathcal{H}} : \mathcal{H}_0 \to \mathcal{H}$ an onto orthogonal projection and unitary operator $U : \mathcal{H}_0 \to \mathcal{H}_0$ such that

$$T^n h = 2P_{\mathcal{H}} U^n h, \quad \forall n \in \mathbb{N}, \forall h \in \mathcal{H}.$$

We ask for the following.

Problem 6.2. *What is noncommutative version of Berger power dilation?*

7. Noncommutative Johnson Numerical Range Problem

In 1974, Johnson derived following important result.

Theorem 7.1. *[20] (**Johnson Numerical Range Theorem**) For every $A = [a_{j,k}]_{1 \leq j \leq d, 1 \leq k \leq d} \in \mathbb{M}_d(\mathbb{C})$,*

$$W_{\mathbb{C}}(A) \subseteq \mathbb{D}\left[0, \frac{1}{2} \max_{1 \leq j \leq d} \left\{\sum_{k=1}^{d} (|a_{j,k}| + |a_{k,j}|)\right\}\right].$$

In particular,

$$(\textbf{\textit{Johnson Numerical Radius Inequality}}) \quad w_{\mathbb{C}}(A) \leq \frac{1}{2} \max_{1\leq j\leq d} \left\{ \sum_{k=1}^{n} (|a_{j,k}| + |a_{k,j}|) \right\}.$$

We ask following problem.

Problem 7.2. *(**Noncommutative Johnson Numerical Range Problem**) What is noncommutative version of Johnson numerical range theorem?*

8. Noncommutative Holbrook Numerical Radius Inequality Problem

In 1969, Holbrook, derived following result for commuting product.

Theorem 8.1. *[17] Let A, B be continuous linear operators on the same Hilbert space such that $AB = BA$. Then*

$$(\textbf{\textit{Holbrook Numerical Radius Inequality for Commuting Product}}) \quad w_{\mathbb{C}}(AB) \leq 2w_{\mathbb{C}}(A)w_{\mathbb{C}}(B).$$

For arbitrary operators, Holbrook derived the following result.

Theorem 8.2. *[17] Let A, B be continuous linear operators on the same Hilbert space. Then*

$$(\textbf{\textit{Holbrook Numerical Radius Inequality for Product}}) \quad w_{\mathbb{C}}(AB + BA) \leq 4w_{\mathbb{C}}(A)w_{\mathbb{C}}(B).$$

We formulate the following problem.

Problem 8.3. *What is noncommutative version of Holbrook numerical radius inequality for product?*

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