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DIAMETERS OF THE ORDER AND CHAIN POLYTOPES OF A CONNECTED POSET

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Abstract. For a finite poset P, Stanley's order polytope $\mathcal{O}(P)$ and chain polytope $\mathcal{C}(P)$ have the same number of vertices and, by a theorem of Hibi, Li, Sahara and Shikama, the same number of edges. On disconnected posets the diameters of their graphs can differ by any amount, since both diameters add over disjoint unions and already differ on small examples, but on connected posets a computer search of Lee found no difference larger than 1. We show that on connected posets, too, the two diameters can be far apart. For every $k \geq 1$ we describe a connected poset B_k with $3k$ elements whose order polytope has graph diameter k and whose chain polytope has graph diameter at most $\lfloor 3(k+1)/4 \rfloor$; hence the first diameter can exceed the second by any amount. For every $n \geq 4$ we construct a connected poset P_n with $2n^2+5n+1$ elements whose order polytope has graph diameter 2 and whose chain polytope has graph diameter 4. Finally, for every finite poset P, if the graph of $\mathcal{O}(P)$ has diameter at most 2, then the graph of $\mathcal{C}(P)$ has diameter at most 4; so 2 is the largest possible difference when the order polytope has graph diameter 2.

1. Introduction

Let P be a finite poset. An *ideal* of P is a subset that contains, with each of its elements, every smaller element; two distinct elements are *comparable* if one is smaller than the other, and *incomparable* otherwise; an *antichain* is a set of pairwise incomparable elements. The poset P is *connected* if it is nonempty and its comparability graph, in which two elements are joined when they are comparable, is connected. Stanley [Sta86] associated two polytopes with P: the *order polytope* $\mathcal{O}(P)$, whose vertices are the indicator vectors of the ideals[1], and the *chain polytope* $\mathcal{C}(P)$, whose vertices are the indicator vectors of the antichains. The *graph* of a polytope has the vertices of the polytope as its vertices and the edges of the polytope as its edges. We call the diameters $\operatorname{diam}\mathcal{O}(P)$ and $\operatorname{diam}\mathcal{C}(P)$ of the graphs of $\mathcal{O}(P)$ and $\mathcal{C}(P)$ the *order diameter* and the *chain diameter* of P. The two polytopes have the same number of vertices (an ideal is determined by the antichain of its maximal elements [Sta86]), and Hibi, Li, Sahara and Shikama [HLSS17] proved that their graphs have the same number of edges. Lee [Lee26] refined this edge theorem and compared further invariants of the two graphs, among them the diameter.

Both diameters are additive over disjoint unions of posets, since $\mathcal{O}(P \sqcup Q) = \mathcal{O}(P) \times \mathcal{O}(Q)$ and $\mathcal{C}(P \sqcup Q) = \mathcal{C}(P) \times \mathcal{C}(Q)$, and the graph of a product of polytopes is the Cartesian product of their graphs, in which distances add. Disjoint unions of copies of Lee's small connected posets on which the two diameters differ by 1, in either direction, therefore give posets with many components on which they differ by any amount [Lee26, Example 5]. Joining the components by a new least element does not keep such a difference: Lee showed that both diameters are at most 2 for every

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The Lean 4 proof of Theorems 1.3, 1.4 and 1.5, which takes the edge criteria of Theorem 2.1 as given: `https://github.com/david44hello/papers_supporting_materials/tree/lee-2026-conjecture-v1.50/lee-2026-conjecture`.

[1]Stanley uses filters (up-sets); the map $\xi \mapsto \mathbf{1} - \xi$ of $\mathbb{R}^P$, where $\mathbf{1}$ is the all-ones vector, carries Stanley's order polytope onto the one used here, so the two graphs are isomorphic.

poset with a least or a greatest element [Lee26, Proposition 26]. Indeed, if P' is P with a new least element $\hat{0}$ adjoined, then every nonempty ideal of P' contains $\hat{0}$, so $\mathcal{O}(P')$ is a pyramid over a copy of $\mathcal{O}(P)$, with apex the indicator vector 0 of the empty ideal; and $\{\hat{0}\}$ is the only antichain of P' that contains $\hat{0}$, so $\mathcal{C}(P')$ is a pyramid over a copy of $\mathcal{C}(P)$, with apex the indicator vector of $\{\hat{0}\}$. The apex of a pyramid is adjacent to every other vertex, so both diameters of P' are at most 2.

The question is therefore how far apart the two diameters can be on a connected poset. On connected posets Lee reports a computer search that found no difference larger than 1, and states the following conjecture.

Conjecture 1.1 (Lee [Lee26, Conjecture 29])**.** *For every connected finite poset P,*

$$|\operatorname{diam}\mathcal{O}(P) - \operatorname{diam}\mathcal{C}(P)| \leq 1.$$

Versions 2 and 3 of [Lee26] withdraw the conjecture; version 3 cites an earlier version of [Goel26], the expanded version of this paper.

We show that both inequalities $\operatorname{diam}\mathcal{O}(P) - \operatorname{diam}\mathcal{C}(P) \leq 1$ and $\operatorname{diam}\mathcal{C}(P) - \operatorname{diam}\mathcal{O}(P) \leq 1$ fail on connected posets, and that no constant can replace 1 in the first one. For a connected poset P that is not a chain, both diameters lie between 2 and the width $w(P)$ [Lee26, Corollary 25], where $w(P)$ is the largest size of an antichain; hence they differ by at most $w(P) - 2$. Our first family shows that the difference can grow linearly with the width.

Results (see also Table 1)**.**

- The diameter of the graph of $\mathcal{O}(P)$ can exceed that of $\mathcal{C}(P)$ by any amount on connected posets: the bands B_k of Definition 1.2 and Theorem 1.3. This disproves Conjecture 1.1.
- The diameter of the graph of $\mathcal{C}(P)$ can exceed that of $\mathcal{O}(P)$ by 2 on connected posets: the posets P_n, $n \geq 4$, of Definition 6.1 and Theorem 1.4. This disproves Conjecture 1.1 in the other direction; the examples of [Lee26, Example 5] reach a difference of 1.
- If $\operatorname{diam}\mathcal{O}(P) \leq 2$, then $\operatorname{diam}\mathcal{C}(P) \leq 4$, for every finite poset P (Theorem 1.5). Unlike the bound $\operatorname{diam}\mathcal{C}(P) \leq w(P)$ [Lee26, Proposition 24], this bound does not depend on the width.

TABLE 1. Results of this paper and the best results before it.

New result	Best result before this paper
$\operatorname{diam}\mathcal{O}(P) - \operatorname{diam}\mathcal{C}(P)$ is unbounded on connected posets (Theorem 1.3)	no difference above 1 in Lee's computer search; at most $w(P) - 2$ if P is not a chain (by [Lee26, Corollary 25])
$\operatorname{diam}\mathcal{C}(P) - \operatorname{diam}\mathcal{O}(P) = 2$ for the connected posets $P = P_n$, $n \geq 4$ (Theorem 1.4)	difference 1 [Lee26, Example 5]
$\operatorname{diam}\mathcal{O}(P) \leq 2$ implies $\operatorname{diam}\mathcal{C}(P) \leq 4$ (Theorem 1.5)	$\operatorname{diam}\mathcal{C}(P) \leq w(P)$ [Lee26, Proposition 24]

Definition 1.2. For $k \geq 1$ the *band* B_k is the poset on the $3k$ elements a_i, b_i, c_i $(1 \leq i \leq k)$ whose order is generated by the relations

$$a_i < b_i < c_i \quad (1 \leq i \leq k), \qquad a_i < b_{i+1} < c_i \quad (1 \leq i \leq k-1).$$

With the product order of $\mathbb{Z}^2$, the $3k$ points $a_i = (i, -i)$, $b_i = (i, 1-i)$, $c_i = (i+1, 1-i)$ $(1 \leq i \leq k)$ induce a subposet of $\mathbb{Z}^2$ isomorphic to B_k (compare the coordinates with Lemma 3.1). These points lie on three neighboring antidiagonals; Figure 1 shows this picture for $k = 5$, rotated by 45°. The band is connected and has width k (Lemma 3.1).

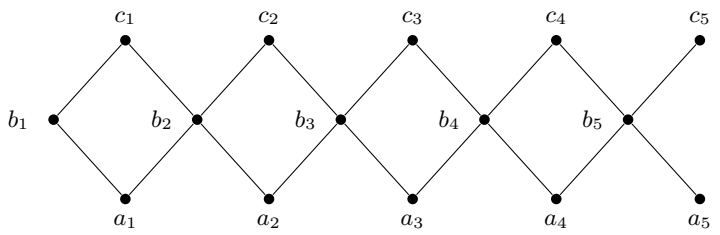


FIGURE 1. The Hasse diagram of the band B_5.

Theorem 1.3. *For every $k \geq 1$ the band B_k is a connected poset with $3k$ elements, and*

(i) $\operatorname{diam} \mathcal{O}(B_k) = k$;
(ii) $\operatorname{diam} \mathcal{C}(B_k) \leq \lfloor 3(k+1)/4 \rfloor$.

Consequently

$$\operatorname{diam} \mathcal{O}(B_k) - \operatorname{diam} \mathcal{C}(B_k) \geq \lceil (k-3)/4 \rceil,$$

which tends to infinity with k. In particular B_8, with 24 elements, is a counterexample to Conjecture 1.1.

Part (i) rests on the signed count of the elements b_i that an ideal contains, each b_i counted with the sign $(-1)^i$; this count changes by at most one along every edge of the graph of $\mathcal{O}(B_k)$ (Section 3). For part (ii) we reduce to pairs of disjoint antichains whose union is an antichain U, build explicit walks in the graph of $\mathcal{C}(B_k)$ when every column $\{a_i, b_i, c_i\}$ contains an element of U, and use induction on k, cutting B_k at a column that contains no element of U (Section 4). The weaker bound (ii) suffices here and has a shorter proof. The expanded version [Goel26] of this paper sharpens (ii) to $\operatorname{diam} \mathcal{C}(B_k) = \lfloor 2k/3 \rfloor + 1$; hence B_7, with diameters 7 and 5, is also a counterexample.

For the other direction we construct, in Section 6, a second family of connected posets.

Theorem 1.4. *For every $n \geq 4$, the poset P_n of Definition 6.1 is connected, has $2n^2 + 5n + 1$ elements, and*

$$\operatorname{diam} \mathcal{O}(P_n) = 2, \qquad \operatorname{diam} \mathcal{C}(P_n) = 4.$$

In particular P_4 is a connected poset with 53 elements and $\operatorname{diam} \mathcal{C}(P_4) - \operatorname{diam} \mathcal{O}(P_4) = 2$.

The upper bound $\operatorname{diam} \mathcal{C}(P_n) \leq 4$ is a special case of the following theorem.

Theorem 1.5. *Let P be a finite poset with $\operatorname{diam} \mathcal{O}(P) \leq 2$. Then $\operatorname{diam} \mathcal{C}(P) \leq 4$.*

Theorems 1.4 and 1.5 together show that the largest value of $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P)$ over posets P with $\operatorname{diam} \mathcal{O}(P) = 2$ is exactly 2.

Section 2 collects the tools used for both families. Sections 3 and 4 prove Theorem 1.3. Section 5 proves Theorem 1.5, and Section 6 defines P_n and proves Theorem 1.4. Section 7 states two open questions. An expanded version of this paper, with further results, is posted on Zenodo [Goel26].

2. Preliminaries

A subset D of P is *connected* if it is not empty and the comparability graph of P induces a connected graph on D; it is *convex* if $x < y < z$ and $x, z \in D$ imply $y \in D$. The *components* of a set are its maximal connected subsets. We identify each vertex of $\mathcal{O}(P)$ with the corresponding ideal and each vertex of $\mathcal{C}(P)$ with the corresponding antichain. We write $X \subset Y$ for proper inclusion, $X \triangle Y$ for the symmetric difference, and $d_{\mathcal{O}(P)}$ and $d_{\mathcal{C}(P)}$, or briefly $d_{\mathcal{O}}$ and $d_{\mathcal{C}}$, for the distances in the graphs of $\mathcal{O}(P)$ and $\mathcal{C}(P)$. We write $\downarrow S$ and $\uparrow S$ for the sets of elements below or equal to, respectively above or equal to, some element of a set $S \subseteq P$, and $\downarrow p = \downarrow\{p\}$, $\uparrow p = \uparrow\{p\}$.

Theorem 2.1 (Ikebe and Tamura [IT95], Hibi and Li [HL19]; for antichains Chvátal [Chv75]; see [Lee26, Theorem 4])**.** *Two ideals $I \neq J$ of P are adjacent in the graph of $\mathcal{O}(P)$ if and only if they are nested ($I \subset J$ or $J \subset I$) and $I \triangle J$ is connected. Two antichains $A \neq B$ of P are adjacent in the graph of $\mathcal{C}(P)$ if and only if $A \triangle B$ is connected.*

The polytope $\mathcal{C}(P)$ is the stable set polytope of the comparability graph of P [Sta86]; so the second part of Theorem 2.1 is Chvátal's adjacency criterion for stable set polytopes.

A *walk* from X_0 to X_n is a sequence $W = (X_0, \dots, X_n)$ of antichains of P, its *states*, in which every *difference* $X_{t-1} \triangle X_t$ is connected; its *length* is $\ell(W) = n$. The t-th *move* of W replaces X_{t-1} by X_t; its difference is $X_{t-1} \triangle X_t$, and we say that the move *exchanges* this set. By Theorem 2.1, consecutive states of a walk are adjacent in the graph of $\mathcal{C}(P)$, so $d_{\mathcal{C}}(X_0, X_n) \leq \ell(W)$. The *reversal* $(X_n, \dots, X_0)$ of a walk is a walk.

Lemma 2.2 (Components)**.** *Let $J \subseteq I$ be ideals of P, and let C be a union of components of $I \setminus J$. Then $J \cup C$ and $I \setminus C$ are ideals.*

Proof. Let $z \in C$ and $z' < z$. Then $z' \in I$. If $z' \notin J$, then $z' \in I \setminus J$ is comparable to z, so it lies in the component of z, and $z' \in C$. Hence $J \cup C$ is an ideal. Now let $z \in I \setminus C$ and $z' < z$, and suppose $z' \in C$. Then $z' \notin J$, so $z \notin J$; thus $z \in I \setminus J$ is comparable to z', so z lies in the component of z', which is contained in C; hence $z \in C$, a contradiction. Hence $I \setminus C$ is an ideal. □

Proposition 2.3 ([Lee26, Proposition 24])**.** *The distance between two ideals I, J in the graph of $\mathcal{O}(P)$ is at most the number of components of $I \triangle J$. For two antichains A, B, exchanging the components of $A \triangle B$ one at a time, in any order, gives a walk from A to B; hence their distance in the graph of $\mathcal{C}(P)$ is at most the number of components of $A \triangle B$. Consequently both diameters are at most $w(P)$.*

Proof. No element of $I \setminus J$ is comparable to an element of $J \setminus I$, since I and J are ideals; so the components of $I \triangle J$ are those of $I \setminus J$ and those of $J \setminus I$. Removing the components of $I \setminus J$ from I one at a time, and then adding those of $J \setminus I$ one at a time, gives ideals by Lemma 2.2 (applied to $I \cap J \subseteq I$ and to $I \cap J \subseteq J$), and consecutive ideals are nested with a connected difference. For the chain polytope, let Q be a union of components of $A \triangle B$. Let $x \in A \setminus Q$ and $y \in B \cap Q$. If $x \in B$, then x and y lie in the antichain B; otherwise x and y lie in different components of $A \triangle B$. In both cases they are incomparable. Hence $A \triangle Q = (A \setminus Q) \cup (B \cap Q)$ is an antichain. When the components are exchanged one at a time, Q grows by one component at each step, so consecutive antichains $A \triangle Q$ differ by one component, which is connected. Finally, elements of different components are incomparable, so one element from each component gives an antichain, and the number of components is at most $w(P)$. □

For the chain polytope, the bound $\operatorname{diam} \mathcal{C}(P) \leq w(P)$ is also a special case of [ABBLS21, Proposition 16].

A pair (A, B) of disjoint sets whose union U is an antichain is a *split* of U. The following lemma shows that the chain diameter is attained at a split.

Lemma 2.4 (Splits)**.** *Let X, Y be antichains of a finite poset P. There are disjoint antichains $A \subseteq X$ and $B \subseteq Y$ such that $A \cup B$ is an antichain and $d_{\mathcal{C}(P)}(X, Y) \leq d_{\mathcal{C}(P)}(A, B)$. Consequently $\operatorname{diam} \mathcal{C}(P)$ is the largest distance $d_{\mathcal{C}(P)}(A, B)$ over all splits (A, B) of antichains of P.*

Proof. Write $d = d_{\mathcal{C}(P)}$. For a set S of elements let $N(S)$ be its open neighborhood in the comparability graph, that is, the set of elements that are comparable to some element of S (comparable elements are distinct). For antichains R, S put $\pi_R(S) = S \cup (R \setminus N(S))$; this is an antichain. We first show that if S, S' are adjacent antichains, then $\pi_R(S)$ and $\pi_R(S')$ are equal or adjacent. Let $F = S \triangle S'$, which is connected by Theorem 2.1, and $F' = \pi_R(S) \triangle \pi_R(S')$.

Let $x \in F' \setminus F$. Since $x \notin F = S \triangle S'$, either $x \in S \cap S'$ or $x \notin S \cup S'$; elements of $S \cap S'$ lie in both images, so $x \notin S \cup S'$. Then x lies in $\pi_R(S)$ exactly when $x \in R \setminus N(S)$, and likewise for S'; as x lies in exactly one of the two images, $x \in R$, and x lies in exactly one of $N(S)$, $N(S')$, say in $N(S)$. So x is comparable to some $z \in S$, and $z \notin S'$, since $x \notin N(S')$. Hence every element of $F' \setminus F$ is comparable to an element of F.

Let $x \in F$, say $x \in S \setminus S'$, so that $x \in \pi_R(S)$. If $|F| \geq 2$, the connected set F contains an element y comparable to x. Since S is an antichain, $y \in S' \setminus S$; hence $x \in N(S')$ and $x \notin \pi_R(S')$. Thus $F \subseteq F'$, and F' is connected. If $F = \{x\}$ and $x \notin R$, then again $x \notin \pi_R(S')$, so $F \subseteq F'$ and F' is connected. If $F = \{x\}$ and $x \in R$, then $x \notin N(S')$, because $S' \cup \{x\} = S$ is an antichain; so $x \in \pi_R(S')$. Moreover no element of the antichain R is comparable to x, so $F' \setminus F$ is empty and $\pi_R(S) = \pi_R(S')$. With Theorem 2.1 this proves the claim, and consequently $d(\pi_R(S), \pi_R(S')) \leq d(S, S')$ for all antichains S, S'.

Now let $K = X \cap Y$, $X_1 = X \setminus K$, $Y_1 = Y \setminus K$, $R = X_1 \cap N(Y_1)$, $A = X_1 \setminus R$ and $B = Y_1$. Since X and Y are antichains, $\pi_K(X_1) = X$ and $\pi_K(Y_1) = Y$; hence $d(X, Y) \leq d(X_1, Y_1)$. Since $R \subseteq X_1$ and $R \subseteq N(Y_1)$, we have $\pi_R(A) = X_1$ and $\pi_R(B) = B = Y_1$; hence $d(X_1, Y_1) \leq d(A, B)$. Finally $A \subseteq X$ and $B \subseteq Y$ are disjoint, and no element of A is comparable to an element of B; hence $A \cup B$ is an antichain. □

3. The band and its order polytope

The order diameter of the band is found with a potential function. Two elements of B_k whose columns are two or more apart are incomparable (Lemma 3.1), so a connected convex set meets the middle elements b_i in an interval of indices (Lemma 3.2). Along an edge of the graph of $\mathcal{O}(B_k)$ an ideal gains or loses such a set (Theorem 2.1; the difference of two nested ideals is convex), so the alternating count of the elements b_i in an ideal changes by at most one per step, and two ideals whose counts differ by k are at least k steps apart; the width of B_k gives the matching upper bound.

Lemma 3.1. *In B_k the comparable pairs are exactly*

$$a_i < b_i,\ b_{i+1}; \qquad b_i < c_{i-1},\ c_i; \qquad a_i < c_{i-1},\ c_i,\ c_{i+1},$$

where terms with indices outside $\{1, \ldots, k\}$ are omitted. In particular two elements whose indices differ by two or more are incomparable, the sets $\{a_1, \ldots, a_k\}$, $\{b_1, \ldots, b_k\}$, $\{c_1, \ldots, c_k\}$ are antichains, B_k is connected, and $w(B_k) = k$.

Proof. The first two groups are the relations of Definition 1.2. The elements above a_i are b_i, b_{i+1} and what lies above them, namely c_{i-1}, c_i and c_i, c_{i+1}. No element lies above a c_j or below an a_j. The comparabilities $a_i < b_i < c_i$ and $a_i < c_{i+1}$ connect all elements. The k chains $a_i < b_i < c_i$ cover B_k, and the a_i form an antichain. □

We call $\{a_i, b_i, c_i\}$ the i-th *column*; it is a chain. By Lemma 3.1 the comparable pairs between two different columns $s < t$ exist only for $t = s + 1$, and they are the four pairs

$$a_s < b_{s+1}, \qquad a_s < c_{s+1}, \qquad a_{s+1} < c_s, \qquad b_{s+1} < c_s. \tag{1}$$

Lemma 3.2 (Interval lemma)**.** *Let D be a connected convex subset of B_k. Then $\{i : b_i \in D\}$ is empty or an interval of integers.*

Proof. Let $b_l, b_r \in D$ and $l < j < r$, and suppose $b_j \notin D$. Partition $B_k \setminus \{b_j\}$ into

$$L_j = \{a_i, b_i, c_i : i < j\}, \qquad R_j = \{a_i, c_i : i \geq j\} \cup \{b_i : i > j\}.$$

By Lemma 3.1 the only comparable pairs with one element in L_j and one in R_j are $a_{j-1} < c_j$ and $a_j < c_{j-1}$, and in both of them b_j lies strictly between the two elements. A path in D from

$b_l \in L_j$ to $b_r \in R_j$ avoids b_j, so it uses one of these two pairs; then both of its elements are in D, and convexity puts b_j into D, a contradiction. □

Theorem 3.3. *For every $k \geq 1$, $\operatorname{diam} \mathcal{O}(B_k) = k$.*

Proof. For an ideal I put $\Phi(I) = \sum_{i=1}^{k} (-1)^i [\, b_i \in I \,]$, where $[\, b_i \in I \,]$ is 1 if $b_i \in I$ and 0 otherwise. Let $I \subset J$ be adjacent ideals, so that $D = J \setminus I$ is connected (Theorem 2.1). The set D is convex: if $x < y < z$ with $x, z \in D$, then $y \in J$, and $y \in I$ would give $x \in I$. By Lemma 3.2, $\Phi(J) - \Phi(I)$ is an alternating sum over an interval of indices, so it is -1, 0 or 1. Hence $d_{\mathcal{O}}(I, J) \geq |\Phi(I) - \Phi(J)|$ for all ideals I, J. Let I_0 consist of all a_i and the b_i with i even, and I_1 of all a_i and the b_i with i odd. Both are ideals, because below b_i there are only a_{i-1} and a_i, and $\Phi(I_0) - \Phi(I_1) = \lfloor k/2 \rfloor + \lceil k/2 \rceil = k$. Hence $\operatorname{diam} \mathcal{O}(B_k) \geq k$. The reverse inequality is Proposition 2.3, since $w(B_k) = k$. □

4. The chain polytope of the band

Theorem 4.1. *For every $k \geq 1$, $\operatorname{diam} \mathcal{C}(B_k) \leq \lfloor 3(k+1)/4 \rfloor$.*

Throughout this section, $d(X, Y)$ is the distance between two antichains X, Y in the graph of $\mathcal{C}(B_k)$. By Lemma 2.4 it suffices to bound $d(A, B)$ for every split (A, B) of every antichain of B_k. Subsections 4.1–4.4 construct walks by combining walks on a small number of columns, either one after the other or merged into common moves. Subsection 4.5 combines them: if $k \geq 2$ and every column of B_k contains an element of an antichain U, then for every split (A, B) of U there is a walk from A to B of length at most $\lceil k/2 \rceil + 1$ (Proposition 4.13). An induction on k, which cuts B_k at a column that contains no element of U, completes the proof (Subsection 4.6).

4.1. **Words and merged walks.** An antichain contains at most one element of each column. We describe an antichain X by the *word* $x_1 x_2 \cdots x_k$ over the alphabet $\{a, b, c, \text{-}\}$, where x_i is the letter of the element of X in the i-th column (the letter of a_i, b_i, c_i is a, b, c), and $x_i = \text{-}$ if there is none; we then call the i-th column *empty* in X. By (1), a word describes an antichain if and only if it has no two neighboring letters $x_i x_{i+1}$ equal to ab, ac, ca or cb; a letter b or - may be followed by any letter. A *run* of a word is a maximal set of consecutive nonempty columns. Inside a run a letter a can only be followed by a, and c only by c; hence the word of a run is $b^i x^j$ with $x \in \{a, c\}$ and $i, j \geq 0$. In tables and displayed walks, words are set in typewriter type.

The elements $a_1, \ldots, a_k$ form the *row* a, and similarly for the rows b and c. The elements a_i and c_i are *outer* elements, and b_i is a *middle* element. By Lemma 3.1, two distinct elements are comparable exactly in the following cases:

(P1) b_j and one of $a_{j-1}, a_j, c_{j-1}, c_j$;
(P2) a_s and c_t with $|s - t| \leq 1$.

In particular, two elements of one row are incomparable, and two elements of one column are comparable. Comparability depends only on the rows of the two elements and on the difference of their column indices. Hence a translation of the column indices maps antichains to antichains, connected sets to connected sets, and walks to walks, provided that the translated elements exist. In Subsection 4.6 we use this to regard walks in a smaller band B_p as walks on any p consecutive columns of B_k.

Lemma 4.2. *A nonempty set $F \subseteq B_k$ is connected if and only if its columns form an interval and, for any two neighboring columns of this interval, some element of F in one of them is comparable to some element of F in the other.*

Proof. The elements of one column form a chain, and elements of columns that are not neighbors are incomparable. □

We often write walks as sequences of words. Let σ exchange a_i and c_i for every i and fix every b_i. By (P1) and (P2), σ preserves comparability (by Lemma 3.1 it even reverses the order), so it maps walks to walks; on words it exchanges the letters a and c.

The *outer trace* of a walk $W = (X_0, \dots, X_n)$ at a column e is the sequence of the outer elements of $X_0, \dots, X_n$ in column e, written - when there is none, after consecutive repetitions are merged. For example, if column e holds b_e, then c_e, then a_e, then nothing, the outer trace at e is $(-, c_e, a_e, -)$.

A *walk for* a split (A, B) is a walk from A to B. For a set S, the split *induced* by (A, B) on S is $(A \cap S, B \cap S)$. The elements of A are *initial* and those of B are *final*. The reversal of a walk from A to B is a walk for the split (B, A), in which initial and final elements are exchanged.

Lemma 4.3 (Merging walks)**.** *Let $D_1, \dots, D_r$ be pairwise disjoint sets of elements of B_k, and for each q let $W_q = (X_0^q, \dots, X_{n_q}^q)$ be a walk whose states are subsets of D_q. Suppose that the moves of all these walks are partitioned into nonempty* steps *$S_1, \dots, S_M$ such that each step contains at most one move of each walk and, for each q, the moves of W_q lie in steps of increasing index. Let Z_α be the union of the states that the walks have reached after the steps $S_1, \dots, S_\alpha$ $(0 \le \alpha \le M)$. Assume moreover that*

(i) *every Z_α is an antichain, and*
(ii) *for every step, the graph on its moves, in which two moves are joined when some element of the difference of one is comparable to some element of the difference of the other, is connected.*

Then $(Z_0, \dots, Z_M)$ is a walk from $\bigcup_q X_0^q$ to $\bigcup_q X_{n_q}^q$ of length $M = \sum_q n_q - \sum_\alpha (|S_\alpha| - 1)$. If all elements of a column that occur in states of the walks lie in D_q, then (Z_α) has the same outer trace at this column as W_q.

Proof. Since the sets D_q are disjoint, $Z_{\alpha-1} \mathbin{\triangle} Z_\alpha$ is the disjoint union of the differences of the moves in S_α. These are connected, and by (ii) their union is connected. Each $Z_\alpha \cap D_q$ is the state that W_q has reached after the steps $S_1, \dots, S_\alpha$, which gives the statement on outer traces. □

We call (Z_α) a *merged walk*. A move K is performed *alone* if its step is $\{K\}$, and *together with* a move K' if both lie in one step; K is performed *just before* K', and K' *just after* K, if the step of K' immediately follows the step of K. If the walks are performed one after the other, all steps have one move and condition (ii) holds trivially; condition (i) then holds, for instance, when elements of different sets D_q lie in columns at distance at least two.

Lemma 4.4 (Adding an element)**.** *Let $W = (X_0, \dots, X_n)$ be a walk, and let y be an element that lies in no state of W and is comparable to an element of some state of W.*

(i) *If X_0 has no element comparable to y, let μ be the least index such that X_μ has one. Then $(X_0 \cup \{y\}, \dots, X_{\mu-1} \cup \{y\}, X_\mu, \dots, X_n)$ is a walk of length n.*
(ii) *If X_n has no element comparable to y, let ν be the largest index such that $X_{\nu-1}$ has one. Then $(X_0, \dots, X_{\nu-1}, X_\nu \cup \{y\}, \dots, X_n \cup \{y\})$ is a walk of length n.*

Proof. (i) Merge W with the walk $(\{y\}, \varnothing)$, performing its move together with move μ of W. The sets $X_t \cup \{y\}$ with $t < \mu$ are antichains by the choice of μ. The state X_μ contains an element comparable to y, which is not in $X_{\mu-1}$; it lies in the difference of move μ. Lemma 4.3 applies. (ii) follows from (i) by reversal. □

4.2. **Splits of one outer row.** The next lemma joins the two parts A and B of a split of a row of outer elements $a_s, \dots, a_e$ through helpers: elements c_t of every second column, which are pairwise incomparable. The walk first exchanges A for the set of helpers, one component at a time, and then exchanges the helpers for B.

Lemma 4.5. *Let $s \le e$ be columns, $[s,e] = \{s, s+1, \dots, e\}$, $j = e - s + 1$ and $U = \{a_s, \dots, a_e\}$. Let T be either the set $\{s+1, s+3, \dots\} \cap [s,e]$ of* even helper positions *(the second, fourth, ... columns of the interval), provided that $j \ge 2$, or the set $\{s, s+2, \dots\} \cap [s,e]$ of* odd helper positions *(the first, third, ... columns). Put $C = \{c_t : t \in T\}$, whose elements are the* helpers*, and $m = |C|$. For every split (A,B) of U there is a walk from A to B of length $m+1$, with states in $U \cup C$, of the following form. Its first phase exchanges the components of $A \cup C$ one at a time, in any order, and ends at C; its second phase exchanges the components of $C \cup B$ one at a time, in any order. Thus every $c_t \in C$ lies exactly in the states after the move that exchanges its component in the first phase and before the move that exchanges its component in the second phase.*

For even helper positions $m = \lfloor j/2 \rfloor$, and the length is $\lfloor j/2 \rfloor + 1$; for odd ones $m = \lceil j/2 \rceil$, and the length is $\lceil j/2 \rceil + 1$.

Proof. The helpers $c_t \in C$ are pairwise incomparable, and by (P2) the element a_u is comparable to c_t if and only if $|u - t| \le 1$. Consecutive elements of T differ by two, the least one is s or $s+1$, and the largest is e or $e-1$. Hence every element a_u of U is comparable to one or two helpers, and to two if and only if $u = t+1$ with $t, t+2 \in T$. There are $m-1$ such *glue* elements. So every component of $A \cup C$ contains a helper, and two consecutive helpers c_t, c_{t+2} lie in the same component if and only if $a_{t+1} \in A$. Therefore $A \cup C$ has $m - \gamma_A$ components, where γ_A is the number of glue elements in A, and $C \cup B$ has $m - \gamma_B$ components. By Proposition 2.3, applied to the disjoint antichains A and C and to C and B, exchanging these components one at a time, in any order, gives walks from A to C and from C to B. Their total length is $2m - \gamma_A - \gamma_B = m+1$, because every glue element lies in exactly one of A, B. □

Corollary 4.6. *Let $j = e - s + 1 \ge 1$. Every split of $\{a_s, \dots, a_e\}$ has a walk of length $\lfloor j/2 \rfloor + 1$ whose states contain no middle element and lie in the columns $s, \dots, e$, and in which a_s is the only element of column s that occurs. The same holds for $\{c_s, \dots, c_e\}$, with c_s.*

Proof. For $j = 1$, exchange a_s alone; for $j \ge 2$, use Lemma 4.5 with even helper positions, none of which is s. In both cases the states lie in $U \cup C$, which contains no middle element. Apply σ for the second statement. □

4.3. **Tame walks.** From now on all walks are walks in B_k, so every column index that occurs is at most k.

Definition 4.7. Let $i \ge 1$. A walk $W = (X_0, \dots, X_n)$ is *tame at column i* if all its states consist of elements of the columns $\le i$, and its outer trace at column i is $(\text{-}, h, \text{-})$ for an outer element h of column i. Then h lies exactly in the states $X_\mu, \dots, X_{\nu-1}$ for some $1 \le \mu < \nu \le n$; it *arrives* with move μ and *departs* with move ν.

Tame walks are designed for Lemma 4.9: a walk that is tame at column i extends to the middle elements of the next two columns at the cost of a single move. The reversal of a tame walk is tame. The image of a tame walk under σ is tame, and $\sigma(h)$ is the outer element of column i in the other row. If the initial and final sets of W consist of middle elements, $\sigma(W)$ has the same initial and final sets; then the row of h can be prescribed.

Lemma 4.8 (Tame bases)**.** *For each split of $\{b_1\}$ there is a walk of length 2 that is tame at column 1, with states in column 1. For each split of $\{b_1, b_2\}$ there is a walk of length 3 that is tame at column 2, with states in the columns $1, 2$.*

Proof. Line 1 of Table 2 gives the first walk when b_1 is initial; its outer trace at column 1 is $(\text{-}, c_1, \text{-})$. Lines 2 and 3 give the walks for the splits $(\varnothing, \{b_1, b_2\})$ and $(\{b_1\}, \{b_2\})$; their outer traces at column 2 are $(\text{-}, c_2, \text{-})$. All words describe antichains, and all differences are connected. The reversals of these walks cover the remaining splits. □

TABLE 2. Tame bases. The words of line 1 are on column 1, and those of lines 2 and 3 on the columns $1, 2$. In each line, the first word is the initial set and the last word the final set. Every move of line 1 and every first move has its difference in one column. For moves 2 and 3 of lines 2 and 3, the difference meets both columns, and the table lists a comparable pair of its elements, one in each column. By Lemma 4.2, every difference is connected.

no.	walk	move 2	move 3
1	`b` $\to$ `c` $\to$ `-`		
2	`--` $\to$ `a-` $\to$ `cc` $\to$ `bb`	$a_1 < c_2$	$b_2 < c_1$
3	`b-` $\to$ `a-` $\to$ `cc` $\to$ `-b`	$a_1 < c_2$	$b_2 < c_1$

Lemma 4.9 (Tame extension)**.** *Let $i \geq 1$, and let W be a walk from A' to B' that is tame at column i, where $A' \cup B'$ consists of middle elements. Then for every split (A'', B'') of $\{b_{i+1}, b_{i+2}\}$ there is a walk from $A' \cup A''$ to $B' \cup B''$ of length $\ell(W) + 1$ that is tame at column $i + 2$.*

Proof. By reversal we may assume that b_{i+1} is final. Since σ fixes $A' \cup A''$ and $B' \cup B''$, we may apply σ to W, carry out the construction, and apply σ to the result; so we may assume that the outer element of column i in W is c_i, with arrival μ and departure ν. Let V be the following walk on the columns $i + 1, i + 2$:

`--` $\to$ `a-` $\to$ `cc` $\to$ `bb` if b_{i+2} is final, `-b` $\to$ `aa` $\to$ `c-` $\to$ `b-` if b_{i+2} is initial.

Its differences are connected: $\{a_{i+1}\}$; $a_{i+1} < c_{i+1}, c_{i+2}$; $b_{i+1}, b_{i+2} < c_{i+1}$ and $b_{i+2} < c_{i+2}$ in the first case, and $a_{i+1}, a_{i+2} < b_{i+2}$; $a_{i+1}, a_{i+2} < c_{i+1}$; $b_{i+1} < c_{i+1}$ in the second. We merge W and V: perform move 1 of V alone just before move μ of W, move 2 of V together with move μ, and move 3 of V together with move ν.

We verify condition (i) of Lemma 4.3. Every element of the columns $\leq i - 1$, and also b_i, is incomparable to every element of the columns $i + 1, i + 2$. Column i contains no outer element of W other than c_i, and c_i is comparable in these columns only to a_{i+1} and b_{i+1}. Now c_i is present exactly while V is in its state `cc` or `c-`, which contains neither. Condition (ii): move μ of W adds c_i, and move 2 of V removes a_{i+1}; move ν removes c_i, and move 3 adds b_{i+1}; and $a_{i+1}, b_{i+1} < c_i$.

The merged walk has length $\ell(W) + 3 - 2 = \ell(W) + 1$. If b_{i+2} is final, the only outer element of column $i + 2$ in the merged walk is c_{i+2}, present exactly in the states between moves μ and ν of W; if b_{i+2} is initial, it is a_{i+2}, present only in the state after move 1 of V. Hence the merged walk is tame at column $i + 2$. □

Corollary 4.10. *Let $i \geq 1$ and $\tau(i) = \lfloor i/2 \rfloor + 2$. For every split of $\{b_1, \ldots, b_i\}$ there is a walk of length $\tau(i)$ that is tame at column i, with states in the columns $1, \ldots, i$, and the row of its outer element of column i can be prescribed.*

Proof. Start from Lemma 4.8 with $i = 1$ or $i = 2$ and apply Lemma 4.9 repeatedly; each application adds two columns and one move, and $\tau(i + 2) = \tau(i) + 1$. Applying σ if necessary, we can prescribe the row. □

4.4. **Junctions.** For columns $i < L$, Lemma 4.11 extends a walk on the columns $\leq i$ by $a_{i+1}, \ldots, a_L$, and Lemma 4.12 extends a walk on the columns $\leq i - 1$ by $b_i, a_{i+1}, \ldots, a_L$. In both lemmas, the initial and final sets of the given walk consist of middle elements.

Lemma 4.11 (Junction)**.** *Let $i, j \geq 1$ and $L = i + j$. Let W be a walk from A' to B' such that $A' \cup B'$ consists of middle elements, every state of W lies in the columns $\leq i$, and some state of W contains c_i. Then for every split (A'', B'') of $\{a_{i+1}, \ldots, a_L\}$ there is a walk from $A' \cup A''$ to $B' \cup B''$ of length $\ell(W) + \lfloor j/2 \rfloor$, with states in the columns $\leq L$.*

Proof. By reversal we may assume that $a_{i+1} \in A''$. Let Q be the walk of Corollary 4.6 for the split (A'', B''), on the columns $i+1, \dots, L$; its length is $\lfloor j/2 \rfloor + 1$, its states contain no middle element, and a_{i+1} is the only element of column $i+1$ that occurs. If $j \geq 2$, Q is the walk of Lemma 4.5 with even helper positions $i+2, i+4, \dots$; let K be the move of its first phase that exchanges the component of a_{i+1}, and order the first phase so that K comes last. If $j = 1$, let K be the only move of Q; then the first phase consists of K, and the second phase has no moves. We merge the walks as follows: perform the moves of the first phase of Q before K; then the moves of W, where K is performed together with the move of W at which c_i first appears; then the second phase of Q.

By (P1) and (P2), the only comparable pair formed by an element of a state of W and an element of a state of Q is $a_{i+1} < c_i$. The element a_{i+1} is present exactly before K, and c_i is absent then; this is condition (i) of Lemma 4.3. In the step that performs K together with that move of W, c_i is added and a_{i+1} is removed; this is condition (ii). The length is $\ell(W) + \lfloor j/2 \rfloor + 1 - 1 = \ell(W) + \lfloor j/2 \rfloor$. □

Lemma 4.12 (Half junction)**.** *Let $i \geq 2$, $j \geq 1$ and $L = i + j$. Let W be a walk from A' to B' that is tame at column $i-1$, where $A' \cup B'$ consists of middle elements. Then for every split (A'', B'') of $\{b_i, a_{i+1}, \dots, a_L\}$ there is a walk from $A' \cup A''$ to $B' \cup B''$ of length $\ell(W) + \lceil j/2 \rceil$, with states in the columns $\leq L$.*

Proof. By reversal we may assume that $a_{i+1} \in A''$. Let h, μ and ν be the outer element of column $i-1$ in W and its arrival and departure.

Part 1: a walk W' on the columns $\leq i$. Since $\sigma(W)$ is tame at column $i-1$ and has the same initial and final sets, we may replace W by $\sigma(W)$ and so assume that $h = c_{i-1}$ if b_i is initial and $h = a_{i-1}$ if b_i is final. If b_i is initial, merge W with the walk $\mathtt{b} \to \mathtt{c} \to \mathtt{a} \to \mathtt{-}$ on column i: its move 1 together with move μ of W, its move 2 together with move ν, and its move 3 alone just after move ν. If b_i is final, merge W with $\mathtt{-} \to \mathtt{c} \to \mathtt{a} \to \mathtt{b}$: its move 1 alone just before move μ, its move 2 together with μ, and its move 3 together with ν. Each difference of these walks lies in column i and is connected. Elements of the columns $\leq i-2$, and b_{i-1}, are incomparable to column i, and the only outer element of column $i-1$ in W is h, present exactly between the moves μ and ν. In the first case column i holds c_i during this time, and b_i or a_i only when h is absent; in the second case it holds a_i during this time, and c_i or b_i only when h is absent. By (P1) and (P2) this gives condition (i) of Lemma 4.3. Condition (ii) holds because $b_i < c_{i-1}$ and $a_i < c_{i-1}$ in the first case, and $a_{i-1} < c_i$ and $a_{i-1} < b_i$ in the second. The merged walk W' has length $\ell(W) + 1$. Its states consist of elements of the columns $\leq i$, and its outer trace at column i is $(\mathtt{-}, c_i, a_i, \mathtt{-})$ in both cases. Let μ' be the move of W' at which c_i is added, and $\nu' > \mu'$ the move at which a_i is added.

Part 2: merging W' with a walk on the columns $> i$. Let $S = \{a_{i+1}, \dots, a_L\}$, and let Q be the walk of Lemma 4.5 for the split $(A'' \cap S, B'' \cap S)$ of S, with odd helper positions $i+1, i+3, \dots$; its length is $\lceil j/2 \rceil + 1$. Its only helper comparable to a_{i+1} is c_{i+1}. Let K_1 be the move of its first phase that exchanges the component of c_{i+1}, which contains a_{i+1}, and let K_2 be the move of its second phase that exchanges the component of c_{i+1}. Order the phases so that K_1 is the last move of the first phase and K_2 the first move of the second. We merge W' and Q: perform the first phase before K_1; then the moves of W', with K_1 together with μ' and K_2 together with ν'; then the rest of the second phase.

The states of W' lie in the columns $\leq i$, and those of Q in the columns $\geq i+1$, where column $i+1$ holds only a_{i+1} or c_{i+1}. Since b_i is incomparable to column $i+1$, the only comparable pairs between the two walks are $a_{i+1} < c_i$ and $a_i < c_{i+1}$. The element a_{i+1} is present exactly before K_1, that is, before c_i first arrives. The element c_{i+1} is present exactly between K_1 and K_2, that is, between the arrivals of c_i and of a_i, when a_i is absent. This is condition (i). Condition (ii)

holds because the joint steps add c_i and remove a_{i+1}, and add a_i and remove c_{i+1}. The length is $\ell(W) + 1 + \lceil j/2 \rceil + 1 - 2 = \ell(W) + \lceil j/2 \rceil$. □

4.5. **Antichains without empty columns.**

Proposition 4.13. *Let $L \geq 2$, and let U be an antichain of B_L without empty columns; its word is $b^i x^j$ with $x \in \{a, c\}$, $i, j \geq 0$ and $i + j = L$ (Subsection 4.1). Every split of U has a walk of length at most $\lceil L/2 \rceil + 1$ if $i \geq 1$, and at most $\lfloor L/2 \rfloor + 1$ if $i = 0$.*

Proof. By σ we may assume $x = a$. Let (A, B) be a split of U.

If $i = 0$, use Corollary 4.6.

If i is odd, take a walk W for the split induced by (A, B) on $\{b_1, \ldots, b_i\}$ that is tame at column i, of length $\tau(i) = (i + 3)/2$, whose outer element of column i is c_i (Corollary 4.10). If $j = 0$, W is the required walk. If $j \geq 1$, Lemma 4.11 gives a walk of length $(i + 3)/2 + \lfloor j/2 \rfloor$, which is $\lceil L/2 \rceil + 1$.

If $i \geq 2$ is even, take a walk W for the split induced by (A, B) on $\{b_1, \ldots, b_{i-1}\}$ that is tame at column $i - 1$, of length $\tau(i - 1) = i/2 + 1$. If $j \geq 1$, Lemma 4.12 gives a walk of length $i/2 + 1 + \lceil j/2 \rceil = \lceil L/2 \rceil + 1$. If $j = 0$, then b_i is comparable to the outer element of column $i - 1$ in W, and to no element of the first and last states of W, which consist of middle elements; hence Lemma 4.4, part (i) if $b_i \in A$ and part (ii) if $b_i \in B$, adds b_i to W and gives the required walk, of length $i/2 + 1 = L/2 + 1$. □

4.6. **Proofs of Theorems 4.1 and 1.3.** Put $g(k) = \lfloor 3(k + 1)/4 \rfloor$ for $k \geq 0$; thus $g(0) = 0$ and $g(1) = 1$. Let B_0 be the empty poset. The factor $3/4$ is what the induction loses, as the following example shows. Let $k = 4m + 3$, and let U consist of the middle elements b_i with $i \not\equiv 0 \pmod 4$. The induction cuts B_k at the m empty columns into $m + 1$ blocks of three full columns, and Proposition 4.13 bounds each block by $\lceil 3/2 \rceil + 1 = 3$ moves, so the bound obtained is $3(m + 1) = g(k)$. The proof of the exact value $\operatorname{diam} \mathcal{C}(B_k) = \lfloor 2k/3 \rfloor + 1$ in [Goel26] avoids this loss.

Lemma 4.14. *The function g has the following properties.*

(i) *For all integers $p, q \geq 0$, $g(p) + g(q) \leq g(p + q + 1)$.*
(ii) *For every $k \geq 2$, $\lceil k/2 \rceil + 1 \leq g(k)$.*

Proof. (i) Since $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$, we have $\lfloor 3(p + 1)/4 \rfloor + \lfloor 3(q + 1)/4 \rfloor \leq \lfloor 3(p + q + 2)/4 \rfloor$. (ii) For $k = 2, 3, 4, 5$ both sides equal $2, 3, 3, 4$. For $k \geq 6$, $g(k) \geq 3k/4 \geq (k + 3)/2 \geq \lceil k/2 \rceil + 1$. □

Proof of Theorem 4.1. We show by strong induction on $k \geq 0$ that every split (A, B) of every antichain U of B_k has a walk of length at most $g(k)$; by Lemma 2.4 and Theorem 2.1, this proves the theorem. For $k \leq 1$ we have $|U| \leq 1$: if $U = \varnothing$, the walk $(\varnothing)$ has length 0, and if $U = \{u\}$, then $k = 1$ and exchanging u gives a walk of length $1 = g(1)$. Let $k \geq 2$.

If U has no empty column, its word is $b^i x^j$. By Proposition 4.13 and Lemma 4.14(ii), there is a walk of length at most $\lceil k/2 \rceil + 1 \leq g(k)$.

Otherwise let e be an empty column of U. The columns $1, \ldots, e - 1$ and $e + 1, \ldots, k$ induce copies of B_{e-1} and B_{k-e}; by translation, a walk in one of them is a walk in B_k. By induction, the splits induced by (A, B) on these copies have walks of lengths at most $g(e - 1)$ and $g(k - e)$. Elements of the two copies lie in columns at distance at least two, so they are incomparable, and performing one walk after the other gives a walk for (A, B). By Lemma 4.14(i), its length is at most $g(e - 1) + g(k - e) \leq g(k)$. □

Proof of Theorem 1.3. The band has $3k$ elements and is connected by Lemma 3.1. Theorems 3.3 and 4.1 give (i) and (ii). Hence $\operatorname{diam} \mathcal{O}(B_k) - \operatorname{diam} \mathcal{C}(B_k) \geq k - \lfloor 3(k + 1)/4 \rfloor = \lceil (k - 3)/4 \rceil$, since $k - \lfloor y \rfloor = \lceil k - y \rceil$ for every integer k and real y. For $k = 8$ this is 2. □

5. Order diameter at most two implies chain diameter at most four

In this section P is an arbitrary finite poset. The proof has two steps. First, the hypothesis $\operatorname{diam}\mathcal{O}(P) \leq 2$ forces a local structure: two incomparable elements that have a third element incomparable to both have a common lower bound or a common upper bound (Lemma 5.1), since otherwise two particular ideals would have no common neighbor. Then, for a split (A, B) of an antichain U, one step at each end replaces the elements of U below a maximal element t by t itself, and the common lower bounds given by Lemma 5.1 join the two new antichains in at most two steps; so $d_{\mathcal{C}}(A, B) \leq 1 + 2 + 1 = 4$.

A *common lower bound* of p and q is an element z with $z \leq p$ and $z \leq q$; common upper bounds are defined dually.

Lemma 5.1 (Pair lemma)**.** *Suppose that* $\operatorname{diam}\mathcal{O}(P) \leq 2$*. Let* p, q *be incomparable elements of* P*, and suppose that some third element* $r \notin \{p, q\}$ *is incomparable to both. Then* p *and* q *have a common lower bound or a common upper bound.*

Proof. The sets $I = {\downarrow}p \cup {\downarrow}q$ and $J = P \setminus ({\uparrow}p \cup {\uparrow}q)$ are ideals. An element $x \in I \setminus J$ satisfies $x \leq p$ or $x \leq q$, and $x \geq p$ or $x \geq q$. As p, q are incomparable, this forces $x \in \{p, q\}$; hence $I \setminus J = \{p, q\}$. Also $r \in J \setminus I$. Thus I and J are not nested, and therefore not adjacent, and since $\operatorname{diam}\mathcal{O}(P) \leq 2$ there is an ideal K adjacent to both. By Theorem 2.1, K is nested with I and with J. The cases $J \subseteq K \subseteq I$ and $I \subseteq K \subseteq J$ are impossible, since $r \in J \setminus I$ and $p \in I \setminus J$. So $K \subseteq I \cap J$ or $K \supseteq I \cup J$.

Suppose that p, q have no common lower bound and no common upper bound. Then ${\downarrow}p$ and ${\downarrow}q$ are disjoint, and no element of ${\downarrow}p$ is comparable to an element of ${\downarrow}q$: if $x \leq p$, $y \leq q$ and $x \leq y$, then x is a common lower bound, and similarly if $y \leq x$. Dually for ${\uparrow}p$ and ${\uparrow}q$. If $K \subseteq I \cap J$, then $I \setminus K$ contains p and q and lies in ${\downarrow}p \cup {\downarrow}q$, so it is not connected. If $K \supseteq I \cup J$, then $K \setminus J$ contains $I \setminus J = \{p, q\}$ and lies in $P \setminus J = {\uparrow}p \cup {\uparrow}q$, so it is not connected. Both contradict Theorem 2.1. □

Proof of Theorem 1.5. By Lemma 2.4 it suffices to show $d_{\mathcal{C}}(A, B) \leq 4$ for every split (A, B) of an antichain U. We may assume $U \neq \varnothing$. Pick $u \in U$ and a maximal element t of P with $t \geq u$. Let $U_t = U \cap {\downarrow}t$; it contains u. If $t \in U$ then $t = u$ and $U_t = \{u\}$. Put

$$U' = (U \setminus U_t) \cup \{t\}, \qquad A_t = (A \setminus U_t) \cup \{t\}, \qquad B_t = (B \setminus U_t) \cup \{t\}.$$

The set U' is an antichain: t is maximal and lies above no element of $U \setminus U_t$. The difference $A \mathbin{\triangle} A_t$ is empty if $A \cap U_t = \{t\}$, and otherwise it is $(A \cap U_t) \cup \{t\}$, which is connected because every element of $A \cap U_t$ lies below t; so A and A_t are equal or adjacent. In the same way B_t and B are equal or adjacent. Moreover $A_t \mathbin{\triangle} B_t = U \setminus U_t$ and $A_t \cup B_t = U'$.

If $|U \setminus U_t| \leq 2$, then $d_{\mathcal{C}}(A_t, B_t) \leq |A_t \mathbin{\triangle} B_t| \leq 2$ by Proposition 2.3. Otherwise let $v \in U \setminus U_t$ be arbitrary. Then v and t are incomparable, since $v \notin {\downarrow}t$ and t is maximal, and any $r \in U \setminus (U_t \cup \{v\})$ is a third element incomparable to both. By Lemma 5.1, v and t have a common lower bound or a common upper bound. Since t is maximal, its only upper bound is t itself, and $v \not\leq t$; hence v and t have no common upper bound, and there is a common lower bound z_v, with $z_v < v$ and $z_v < t$. Let M_t be the set of minimal elements m of P with $m \leq t$. It is a nonempty antichain, and $t \notin M_t$ because $z_v < t$. No element of $U \setminus U_t$ lies below t, so M_t is disjoint from U'. Every $v \in U \setminus U_t$ lies above an element of M_t, namely any minimal element m of P with $m \leq z_v$ (as $z_v < v$ and $z_v < t$). Hence $A_t \mathbin{\triangle} M_t = (A \setminus U_t) \cup \{t\} \cup M_t$ is connected through t, which lies above M_t, and in the same way $M_t \mathbin{\triangle} B_t = M_t \cup (B \setminus U_t) \cup \{t\}$ is connected. Hence (A_t, M_t, B_t) is a walk.

In both cases $d_{\mathcal{C}}(A, B) \leq 1 + 2 + 1 = 4$. □

6. Connected posets with order diameter two and chain diameter four

In this section the letters a_i, b_j, h, σ, A, B, U, Q, T, W and Z_A, Z_B denote elements, maps and sets of the poset P_n defined below, not of the band; in particular A, B and U are the specific antichains of Definition 6.1.

The poset P_n is built so that the two antichains A and B of targets are far apart in the graph of $\mathcal{C}(P_n)$, while all ideals stay close in the graph of $\mathcal{O}(P_n)$. Subsection 6.2 shows that any two ideals are at distance at most 2 in the graph of $\mathcal{O}(P_n)$. Subsection 6.3 lists the antichains that are equal or adjacent to A, and by symmetry those for B, and shows that no walk of length at most 3 joins A to B. The reason is that an antichain equal or adjacent to A either lies in $\downarrow A$ or still contains at least $n-1$ targets of A (Lemma 6.8), the same holds for B, and no antichain of either kind for A is equal or adjacent to one for B (Proposition 6.11). The upper bound $\operatorname{diam}\mathcal{C}(P_n) \le 4$ is Theorem 1.5.

6.1. **The construction.**

Definition 6.1. Let $n \ge 2$ and $[n] = \{1, \dots, n\}$. The poset P_n has the following elements:

- the *targets* a_i and b_j $(i, j \in [n])$; put $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ and $U = A \cup B$;
- for each 2-element subset $E \subseteq [n]$, two *pair minima* m_E and m'_E; we also write $m_{ik} = m_{ki} = m_{\{i,k\}}$ and $m'_{jl} = m'_{lj} = m'_{\{j,l\}}$; put $M_A = \{m_E\}$ and $M_B = \{m'_E\}$, where E runs over the 2-element subsets of $[n]$;
- the *lower connectors* y_i and y'_j $(i, j \in [n])$; put $Y_A = \{y_1, \dots, y_n\}$ and $Y_B = \{y'_1, \dots, y'_n\}$;
- the *hub* h;
- the *upper connectors* w_i and w'_j $(i, j \in [n])$; put $W_A = \{w_1, \dots, w_n\}$, $W_B = \{w'_1, \dots, w'_n\}$ and $W = W_A \cup W_B$;
- the *tops* t_{ij} $(i, j \in [n])$; put $T = \{t_{ij}\}$.

The order is the reflexive and transitive closure of the relations

$$m_E < y_i \ (i \in E), \qquad y_i < a_i, \qquad y_i < h, \qquad m'_E < y'_j \ (j \in E), \qquad y'_j < b_j, \qquad y'_j < h,$$
$$a_i < w_i, \qquad h < w_i, \qquad b_j < w'_j, \qquad h < w'_j, \qquad w_i < t_{ij}, \qquad w'_j < t_{ij}.$$

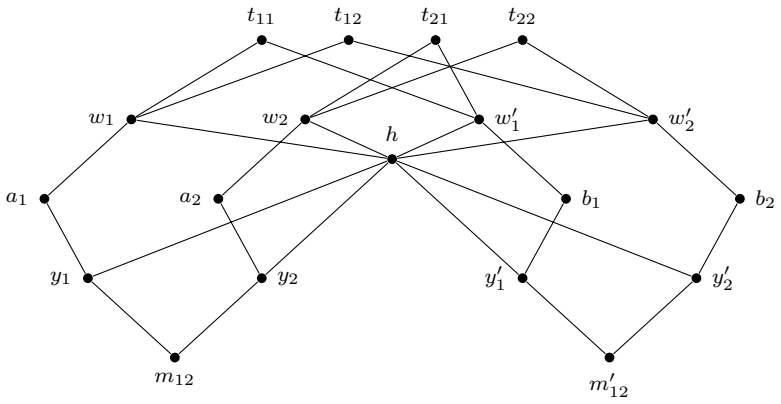


FIGURE 2. The Hasse diagram of P_2. Theorem 1.4 requires $n \ge 4$; for every n there are pair minima m_E and m'_E for each 2-element subset E of $[n]$, and one top t_{ij} for each pair (i, j).

Figure 2 shows P_2. In P_n, y_i lies above the $n-1$ pair minima m_{ik} $(k \ne i)$, each pair minimum m_{ik} lies below a_i and a_k, and t_{ij} lies above a_i and b_j. The number of elements is $2n + n(n-1) + 2n + 1 + 2n + n^2 = 2n^2 + 5n + 1$, which is 53 for $n = 4$. The function ρ with values 0 on $M_A \cup M_B$,

1 on $Y_A \cup Y_B$, 2 on $U \cup \{h\}$, 3 on W and 4 on T, which we call the *rank*, increases by one along each listed relation; hence the reflexive and transitive closure of these relations is antisymmetric, P_n is a poset, and $p < q$ implies $\rho(p) < \rho(q)$. We also use

$$Q = M_A \cup M_B \cup Y_A \cup Y_B, \qquad \Lambda_A = M_A \cup Y_A \cup A, \qquad \Lambda_B = M_B \cup Y_B \cup B,$$

so that P_n is the disjoint union of Λ_A, Λ_B, $\{h\}$, W and T. For a target u we write u^- for its lower connector and u^+ for its upper connector: $a_i^- = y_i$, $a_i^+ = w_i$, $b_j^- = y'_j$, $b_j^+ = w'_j$.

Let σ be the involution of P_n that exchanges a_i and b_i, m_E and m'_E, y_i and y'_i, w_i and w'_i, and t_{ij} and t_{ji}, and fixes h. It maps the listed relations onto the listed relations, so it is an automorphism of P_n that exchanges A and B and fixes h; it induces automorphisms of the graphs of $\mathcal{O}(P_n)$ and $\mathcal{C}(P_n)$.

Lemma 6.2. *In P_n:*

(i) *The elements below a_i are y_i and the m_{ik} ($k \neq i$); they all lie in $\downarrow y_i$. The elements above a_i are w_i and $t_{i1}, \dots, t_{in}$; they all lie in $\uparrow w_i$. Applying σ gives the same statement for b_j, with y'_j, the m'_{jl}, w'_j and $t_{1j}, \dots, t_{nj}$.*

(ii) *$\downarrow h = Q \cup \{h\}$ and $\uparrow h = \{h\} \cup W \cup T$. So h is comparable to every element outside $U \cup \{h\}$ and to no target, and every element of Q lies below every element of $W \cup T$.*

(iii) *No element of Λ_A is comparable to an element of Λ_B.*

(iv) *Every element of $W_A \cup T$ lies above exactly one target of A (w_i and t_{ij} above a_i), and no element of W_A lies above a target of B. The same holds for $W_B \cup T$ and B (t_{ij} lies above b_j). The sets W and T are antichains.*

(v) *The comparability graph of P_n is connected.*

Proof. The proof is a direct check of the listed relations. The only relation with a_i on the right is $y_i < a_i$, and the only ones with y_i on the right are $m_{ik} < y_i$; this gives the elements below a_i. The only relation with a_i on the left is $a_i < w_i$, and above w_i there are only the t_{ij}; this gives the elements above a_i and proves (i). Below h are the lower connectors and, below them, all pair minima (each 2-element subset E contains some i, and $m_E < y_i$); above h are all upper connectors and, above them, all tops. This proves (ii). For (iii): the elements above an element of Λ_A lie in $\Lambda_A \cup \{h\} \cup W \cup T$ (by (i) and (ii), as $\uparrow y_i = \{y_i, a_i\} \cup \uparrow h$), and symmetrically for Λ_B; so no element of Λ_B lies above an element of Λ_A, and conversely. Item (iv) follows from $\downarrow w_i = \{w_i, a_i, h\} \cup Q$ and $\downarrow t_{ij} = \{t_{ij}, w_i, w'_j, a_i, b_j, h\} \cup Q$; elements of equal rank are incomparable, since $p < q$ implies $\rho(p) < \rho(q)$. Finally every element outside U is comparable to h or equal to it, and every target u is comparable to u^-; this gives (v). □

6.2. **The order diameter of P_n.** Throughout this subsection $n \geq 2$ and $P = P_n$.

Lemma 6.3. *In P_n:*

(i) *Every ideal I with $h \in I$ is connected.*

(ii) *For every ideal J with $h \notin J$, the set $P \setminus J$ is connected and contains $\uparrow h$.*

Proof. (i) I contains $\downarrow h = Q \cup \{h\}$. Every element of I outside U is h or comparable to h (Lemma 6.2(ii)), and every target $u \in I$ is comparable to $u^- \in Q \subseteq I$. (ii) $P \setminus J$ is an up-set (filter) that contains h, so it contains $\uparrow h = \{h\} \cup W \cup T$. Every element of $P \setminus J$ outside U is h or comparable to h, and every target $u \in P \setminus J$ is comparable to $u^+ \in W \subseteq P \setminus J$. □

Now let I, J be ideals with $h \in I$ and $h \notin J$; we call (I, J) a *mixed pair*. Let Γ_h be the component of $I \setminus J$ that contains h, and let $\mathrm{Iso}(I, J)$ be the set of targets $u \in I \setminus J$ with $u^- \in J$ and $u^+ \notin I$.

Lemma 6.4 (Mixed pairs)**.** *Let (I, J) be a mixed pair.*

(i) *$Q \subseteq I$, $J \subseteq Q \cup U$, and $J \setminus I \subseteq U$.*
(ii) *If $J \cap (A \cup Y_A)$ is nonempty, its elements lie in one component of J, and this component is contained in Λ_A. The same holds for $J \cap (B \cup Y_B)$ and Λ_B (apply σ).*
(iii) *$I \setminus J = \Gamma_h \cup \mathrm{Iso}(I, J)$, and every $u \in \mathrm{Iso}(I, J)$ forms a component $\{u\}$ of $I \setminus J$.*

Proof. (i) $Q \subseteq {\downarrow}h \subseteq I$. Since J is an ideal without h, it contains no element of ${\uparrow}h = \{h\} \cup W \cup T$; hence $J \subseteq Q \cup U \subseteq I \cup U$.

(ii) Each element of $A \cup Y_A$ has an index: a_k and y_k have index k, and both lie above every m_{kl} ($l \neq k$). Let $z, z' \in J \cap (A \cup Y_A)$ have indices k, l. If $k = l$, then z, z' are equal or comparable. If $k \neq l$, then $m_{kl} \in J$ lies below both. Hence z and z' lie in one component of J. By Lemma 6.2(iii) and $J \subseteq Q \cup U = \Lambda_A \cup \Lambda_B$, every component of J lies in Λ_A or in Λ_B. The component that contains $J \cap (A \cup Y_A)$ meets $A \cup Y_A \subseteq \Lambda_A$, so it lies in Λ_A.

(iii) Every element of $I \setminus J$ outside U is h or comparable to $h \in I \setminus J$, so it lies in Γ_h. Let $u \in I \setminus J$ be a target. If $u^- \notin J$, then $u^- \in Q \setminus J \subseteq I \setminus J$ joins u to h. If $u^+ \in I$, then $u^+ \in I \setminus J$ by (i), and it joins u to h. Otherwise $u \in \mathrm{Iso}(I, J)$: the elements below u lie in ${\downarrow}u^- \subseteq J$, and the elements above u lie in ${\uparrow}u^+$, which is disjoint from I (Lemma 6.2(i)). So no element of $I \setminus J$ is comparable to u. □

Lemma 6.5 (A common neighbor of I and J)**.** *Let (I, J) be a mixed pair, and suppose that $w_i \notin I$ and $w'_j \notin I$ for some $i, j \in [n]$. Let Γ be the component of $P \setminus I$ that contains w_i, and $K = I \cup \Gamma$. Then:*

(i) *Γ contains every element of $W \setminus I$ and every element of $U \setminus I$;*
(ii) *K is an ideal adjacent to I, and $J \subset K$;*
(iii) *$K \setminus J$ is connected, so K is adjacent to J, and $d_{\mathcal{O}}(I, J) \leq 2$.*

Proof. (i) Since I is an ideal, a top above an element outside I is outside I. Let $w_k \notin I$. Then $t_{kj}, t_{ij} \notin I$, and $(w_k, t_{kj}, w'_j, t_{ij}, w_i)$ is a path in the comparability graph of $P \setminus I$; so $w_k \in \Gamma$. Let $w'_l \notin I$. Then $t_{il} \notin I$, and (w'_l, t_{il}, w_i) is such a path; hence $w'_l \in \Gamma$. A target $u \notin I$ has $u^+ \notin I$, so u is joined to $u^+ \in \Gamma$.

(ii) K is an ideal by Lemma 2.2 (with the ideals $I \subseteq P$). $K \setminus I = \Gamma$ is connected, so K is adjacent to I. By Lemma 6.4(i), $J \subseteq Q \cup U$; here $Q \subseteq I$, and every target of J outside I lies in Γ by (i). Hence $J \subseteq K$, and $J \neq K$ since $h \in K \setminus J$.

(iii) By (i), K contains all of W, and $W \cap J = \varnothing$. Every element of $K \setminus J$ outside U is h or comparable to h; every target $u \in K \setminus J$ is comparable to $u^+ \in K \setminus J$, which is comparable to h. So $K \setminus J$ is connected, and (I, K, J) is a path in the graph of $\mathcal{O}(P)$. □

Proposition 6.6. *For every mixed pair (I, J), $d_{\mathcal{O}}(I, J) \leq 2$.*

Proof. Let $\mathrm{Iso} = \mathrm{Iso}(I, J)$ and $\Theta = \mathrm{Iso} \cup (J \setminus I)$, a set of targets. For $u \in \mathrm{Iso}$ we have $u^+ \notin I$ by definition; for $u \in J \setminus I$ we have $u \notin I$, hence $u^+ \notin I$. Hence if Θ meets both A and B, Lemma 6.5 applies; otherwise $\Theta \subseteq A$ or $\Theta \subseteq B$, since every target lies in A or in B. The automorphism σ fixes h, so it maps mixed pairs to mixed pairs; it exchanges A and B, and $\sigma(u^-) = \sigma(u)^-$ and $\sigma(u^+) = \sigma(u)^+$ for every target u. Hence $\mathrm{Iso}(\sigma I, \sigma J) = \sigma(\mathrm{Iso}(I, J))$ and $\sigma J \setminus \sigma I = \sigma(J \setminus I)$, so σ carries Θ to the corresponding set of $(\sigma I, \sigma J)$. As σ preserves distances, we may assume from now on that $\Theta \subseteq A$.

If $\Theta = \varnothing$, then $J \subseteq I$ and $I \setminus J = \Gamma_h$ is connected (Lemma 6.4(iii)), so I and J are adjacent.

Let $\Theta \neq \varnothing$. Then $J \cap (A \cup Y_A) \neq \varnothing$: a target of $J \setminus I$ lies in $J \cap A$, and $u \in \mathrm{Iso}$ has $u^- \in J \cap Y_A$. Let J_1 be the component of J that contains $J \cap (A \cup Y_A)$ (Lemma 6.4(ii)); then $J_1 \subseteq \Lambda_A$. Put $K = J \setminus J_1$. By Lemma 2.2 (with the ideals $\varnothing \subseteq J$), K is an ideal, and $J \setminus K = J_1$ is connected; so K is adjacent to J. Since $J \setminus I \subseteq J \cap A \subseteq J_1$, we have $K \subseteq I$, and

$$I \setminus K = (I \setminus J) \cup (I \cap J_1) = \Gamma_h \cup \mathrm{Iso} \cup (I \cap J_1).$$

We show that every element of this set is joined to h inside it; then $I \setminus K$ is connected, and K is adjacent to I (note $h \in I \setminus K$).

- Elements of Γ_h: by definition.
- Elements of $I \cap J_1$ in Q: they are comparable to h.
- Targets in $I \cap J_1$: such a target is some a_k, as $J_1 \subseteq \Lambda_A$. Then $y_k \in J$ lies in J_1 by the choice of J_1, and in $Q \subseteq I$; hence $y_k \in I \cap J_1$ is comparable to a_k and to h.
- Elements of Iso: they are targets of A, since $\mathrm{Iso} \subseteq \Theta \subseteq A$. For $a_k \in \mathrm{Iso}$, $y_k = a_k^- \in J \cap Y_A \subseteq J_1$ and $y_k \in Q \subseteq I$; so $y_k \in I \cap J_1$ is comparable to a_k and to h.

Hence (I, K, J) is a path, and $d_{\mathcal{O}}(I, J) \leq 2$. □

Theorem 6.7. *For every $n \geq 2$, $\operatorname{diam} \mathcal{O}(P_n) = 2$.*

Proof. Let I, J be ideals. If both contain h, both are connected by Lemma 6.3, so both are adjacent to $\varnothing$. If neither contains h, then $P \setminus I$ and $P \setminus J$ are connected, so both are adjacent to P. Otherwise one of them contains h and the other does not, and Proposition 6.6 applies. Hence $\operatorname{diam} \mathcal{O}(P_n) \leq 2$. The ideals $\{m_{12}\}$ and $\{m'_{12}\}$ are not nested, so they are not adjacent, and $\operatorname{diam} \mathcal{O}(P_n) \geq 2$. □

6.3. **The chain diameter of P_n.** Throughout this subsection $n \geq 2$ and $P = P_n$. For an antichain X let $\mathcal{N}[X]$ be the set of antichains that are equal or adjacent to X. We list the forms that a member of $\mathcal{N}[A]$ can take; by the automorphism σ this also describes $\mathcal{N}[B]$.

Let Z_A be the set of elements outside A that are incomparable to every target of A. By Lemma 6.2, $\downarrow A = \Lambda_A$ and $\uparrow A = A \cup W_A \cup T$, so

$$Z_A = \Lambda_B \cup \{h\} \cup W_B, \qquad Z_B = \Lambda_A \cup \{h\} \cup W_A.$$

In the next lemma, the type labels (D), (E), (L) and (H) stand for deletion, extension, lower and higher.

Lemma 6.8 (Neighbors of A)**.** *Every antichain $X \in \mathcal{N}[A]$ with $X \neq A$ has one of the following forms.*

(D) $X = A \setminus \{a_i\}$ *for some i.*
(E) $X = A \cup \{z\}$ *for some $z \in Z_A$.*
(L) $X = (A \setminus R) \cup S$*, where $\varnothing \neq R \subseteq A$, $\varnothing \neq S \subseteq M_A \cup Y_A$, and every element of S lies below some element of R and below no element of $A \setminus R$.*
(H) $X = (A \setminus \{a_i\}) \cup S$ *for some i, where $S = \{w_i\}$ or $\varnothing \neq S \subseteq \{t_{i1}, \ldots, t_{in}\}$.*

Proof. Let $R = A \setminus X$ and $S = X \setminus A$. By Theorem 2.1, $A \triangle X = R \cup S$ is connected. If $S = \varnothing$, then R is a connected subset of an antichain, so $|R| = 1$: type (D). If $R = \varnothing$, then S is a connected subset of the antichain X, so $S = \{z\}$; as $A \cup \{z\}$ is an antichain, $z \in Z_A$: type (E).

Let $R, S \neq \varnothing$. Since R and S are antichains and $R \cup S$ is connected with at least two elements, every $s \in S$ is comparable to some element of R; and since X is an antichain, s is comparable to no element of $A \setminus R$. Thus $s \in (\downarrow A \cup \uparrow A) \setminus A = M_A \cup Y_A \cup W_A \cup T$. Every element of $M_A \cup Y_A \subseteq Q$ lies below every element of $W_A \cup T$ (Lemma 6.2(ii)), so the antichain S lies in $M_A \cup Y_A$ or in $W_A \cup T$. In the first case X has type (L). In the second case every $s \in S$ lies above exactly one target of A (Lemma 6.2(iv)), which belongs to R. In the connected bipartite comparability graph on $R \cup S$, every element of S then has exactly one neighbor; a path between two elements of R would pass through an element of S with two neighbors in R. So $R = \{a_i\}$ for some i, and S is an antichain in $\uparrow a_i \setminus \{a_i\} = \{w_i, t_{i1}, \ldots, t_{in}\}$. As w_i lies below every t_{ij}, either $S = \{w_i\}$ or $S \subseteq \{t_{i1}, \ldots, t_{in}\}$: type (H). □

Applying σ, we see that every $Y \in \mathcal{N}[B]$ with $Y \neq B$ has one of the forms (D′) $B \setminus \{b_j\}$; (E′) $B \cup \{z\}$ with $z \in Z_B$; (L′) $(B \setminus R) \cup S$ with $\varnothing \neq R \subseteq B$, $\varnothing \neq S \subseteq M_B \cup Y_B$, where every element of

S lies below some element of R and below no element of $B \setminus R$; (H′) $(B \setminus \{b_j\}) \cup S$ with $S = \{w'_j\}$ or $\varnothing \neq S \subseteq \{t_{1j}, \dots, t_{nj}\}$.

Lemma 6.9 (Isolated elements)**.** *Let X, Y be antichains and $G = X \triangle Y$. If $x \in X \setminus Y$ is comparable to no element of Y, then no element of G is comparable to x. Consequently, if moreover $|G| \geq 2$, then G is not connected. The same holds with X and Y exchanged.*

Proof. An element of G comparable to x would lie in X or in Y. It is not in the antichain X, which contains x, and not in Y by hypothesis. □

We record which elements of a neighbor of B are equal or comparable to targets of A. Call an element z *attached* to a target u if $z = u$ or z is comparable to u.

Lemma 6.10. *Let $Y \in \mathcal{N}[B]$.*

(i) *If $Y = B$ or Y has type* (D′) *or* (L′)*, then $Y \subseteq \Lambda_B$, and no element of Y is attached to a target of A.*
(ii) *If $Y = B \cup \{z\}$ has type* (E′)*, then z is the only element of Y that can be attached to a target of A, and it is attached to at most two of them.*
(iii) *If $Y = (B \setminus \{b_j\}) \cup S$ has type* (H′)*, then $Y \cap A = \varnothing$, the only elements of Y comparable to targets of A are the tops $t_{pj} \in S$, and t_{pj} is comparable to a_p only.*

The same statements hold with A and B exchanged, for $X \in \mathcal{N}[A]$.

Proof. (i) follows from Lemma 6.2(iii). For (ii): $B \subseteq \Lambda_B$, and $z \in Z_B = \Lambda_A \cup \{h\} \cup W_A$. If $z = a_k$, it is attached to a_k only; if $z = m_{ik}$, to a_i and a_k; if $z = y_i$, to a_i only; if $z = h$, to none; if $z = w_i$, to a_i only (Lemma 6.2). For (iii): the elements of $B \setminus \{b_j\} \subseteq \Lambda_B$ are comparable to no target of A, by Lemma 6.2(iii), as $A \subseteq \Lambda_A$; w'_j is comparable to no target of A, and t_{pj} is comparable to a_p only, by Lemma 6.2(i). □

Proposition 6.11. *Let $n \geq 4$. For all $X \in \mathcal{N}[A]$ and $Y \in \mathcal{N}[B]$, the set $X \triangle Y$ is nonempty and disconnected. In particular $X \neq Y$, and X and Y are not adjacent.*

Proof. Let $G = X \triangle Y$. We call X *low* if $X = A$ or X has type (D) or (L), so that $X \subseteq \Lambda_A$ is nonempty; similarly Y is low if $Y = B$ or Y has type (D′) or (L′). Otherwise X has type (E) or (H) and contains at least $n - 1$ targets of A, and similarly for Y.

Case 1: X and Y are low. Then $X \subseteq \Lambda_A$ and $Y \subseteq \Lambda_B$ are nonempty and disjoint, so $G = X \cup Y$, and no element of X is comparable to an element of Y (Lemma 6.2(iii)). Hence G is not connected.

Case 2: X is low and Y is not. Y contains at least $n - 1 \geq 2$ targets of B. No element of $X \subseteq \Lambda_A$ is attached to a target of B (Lemma 6.2(iii)). So these targets lie in $Y \setminus X \subseteq G$ and are comparable to no element of X; as there are at least two of them, Lemma 6.9 applies. The case where Y is low and X is not is symmetric.

Case 3: $X = A \cup \{z\}$ and $Y = B \cup \{z'\}$, of types (E) *and* (E′). By Lemma 6.10(ii), at most two targets of A are attached to elements of Y. Hence at least $n - 2 \geq 1$ targets of A lie in $X \setminus Y$ and are comparable to no element of Y. By the same lemma with A and B exchanged, at least $n - 2 \geq 1$ targets of B lie in $Y \setminus X$; hence $|G| \geq 2$, and Lemma 6.9 applies.

Case 4: $X = A \cup \{z\}$ of type (E) *and $Y = (B \setminus \{b_j\}) \cup S$ of type* (H′). By the version of Lemma 6.10(ii) with A and B exchanged, z is the only element of X that can be attached to a target of B, and it is attached to at most two of them. The set Y contains the $n - 1 \geq 3$ targets of $B \setminus \{b_j\}$. So some $b \in Y \cap B$ is not attached to z; then $b \in Y \setminus X$, and b is comparable to no element of X. Also $A \subseteq X \setminus Y$, as $Y \cap A = \varnothing$; so $|G| \geq 2$, and Lemma 6.9 applies. The case of types (H) and (E′) is symmetric.

Case 5: $X = (A \setminus \{a_i\}) \cup S$ of type (H) *and $Y = (B \setminus \{b_j\}) \cup S'$ of type* (H′). Fix $p \neq i$. Then $a_p \in X \setminus Y$, so $a_p \in G$. The elements comparable to a_p are y_p, the m_{pk}, w_p and the t_{pq}. None of them lies in X, since X is an antichain that contains a_p. The only one that can lie in Y is t_{pj},

if $t_{pj} \in S'$. In that case $t_{pj} \notin X$, so $t_{pj} \in G$. The elements comparable to t_{pj} are the elements of $\downarrow t_{pj} \setminus \{t_{pj}\} = \{w_p, w'_j, a_p, b_j, h\} \cup Q$. Of these, only a_p lies in $X \cup Y$: $w_p \notin X$ since $p \neq i$, and $w_p \notin Y$; $w'_j \notin Y$ since $S' \subseteq T$ when $t_{pj} \in S'$, and $w'_j \notin X$; b_j lies in neither; and h and Q meet no set of type (H) or (H′). Hence the component of a_p in G is contained in $\{a_p, t_{pj}\}$. But G also contains the targets of $B \setminus \{b_j\}$, which lie outside $\{a_p, t_{pj}\}$; hence G is not connected.

The cases cover all pairs: by the remark on low sets, a non-low X has type (E) or (H), and a non-low Y has type (E′) or (H′). Each case exhibits elements of G, so G is nonempty and $X \neq Y$; and since G is disconnected, X and Y are not adjacent, by Theorem 2.1. □

Remark 6.12. Cases 1, 2, 3 and 5 of the proof of Proposition 6.11 use only $n \geq 3$; Case 4 is the only place where $n \geq 4$ is needed. For $n \in \{2, 3\}$, the sequence $A \to (A \setminus \{a_1\}) \cup \{t_{11}, \ldots, t_{1n}\} \to B \cup \{m_E\} \to B$, with $E = \{2, 3\}$ if $n = 3$ and $E = \{1, 2\}$ if $n = 2$, is a walk of length 3 whose middle states have types (H) and (E′). Its first difference is connected through a_1, and its second through m_E, which lies below the retained targets of A and below every top, while $t_{1j} > b_j$. (We do not determine $\operatorname{diam} \mathcal{C}(P_n)$ for $n \leq 3$.)

Theorem 6.13. *For every $n \geq 4$, $d_{\mathcal{C}(P_n)}(A, B) = 4$. Hence* $\operatorname{diam} \mathcal{C}(P_n) = 4$.

Proof. Upper bound. The sequence

$$A \;\to\; M_A \;\to\; \{h\} \;\to\; M_B \;\to\; B$$

is a walk. The states are antichains. $A \triangle M_A = A \cup M_A$ is connected, since m_{ik} lies below a_i and a_k and the complete graph on $[n]$ is connected. $M_A \triangle \{h\} = M_A \cup \{h\}$ is connected, since every element of M_A lies below h. The last two moves are symmetric.

Lower bound. Suppose $d_{\mathcal{C}}(A, B) \leq 3$, and let $A = V_0, V_1, \ldots, V_r = B$ be a shortest path, $1 \leq r \leq 3$. Put $X = V_1$ and $Y = V_{r-1}$. Then $X \in \mathcal{N}[A]$, $Y \in \mathcal{N}[B]$, and $X = Y$ (if $r = 2$) or X and Y are adjacent (if $r \in \{1, 3\}$; for $r = 1$, $X = B$ and $Y = A$). This contradicts Proposition 6.11.

So $\operatorname{diam} \mathcal{C}(P_n) \geq 4$. By Theorems 6.7 and 1.5, $\operatorname{diam} \mathcal{C}(P_n) \leq 4$. □

Proof of Theorem 1.4. P_n is connected by Lemma 6.2(v) and has $2n^2 + 5n + 1$ elements. Theorems 6.7 and 6.13 give the diameters. □

7. Questions

Question 7.1. Is $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P)$ bounded on connected finite posets P?

A connected poset P with $\operatorname{diam} \mathcal{C}(P) - \operatorname{diam} \mathcal{O}(P) \geq 3$ must have $\operatorname{diam} \mathcal{O}(P) \geq 3$. Indeed, if $\operatorname{diam} \mathcal{O}(P) \leq 1$, then any two ideals are nested; two incomparable elements p, q would give the non-nested ideals $\downarrow p$ and $\downarrow q$, so P is a chain, its antichains are $\varnothing$ and the singletons, and $\operatorname{diam} \mathcal{C}(P) = 1$. If $\operatorname{diam} \mathcal{O}(P) = 2$, Theorem 1.5 applies.

Question 7.2. Is the supremum of $\operatorname{diam} \mathcal{O}(P)/\operatorname{diam} \mathcal{C}(P)$ over connected finite posets P finite? Theorem 1.3 shows that it is at least $4/3$, and [Goel26] that it is at least $3/2$.

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