hexagon:2611.00520v1

FLOW RECONFIGURATION WITH ONE MODULUS FOR ALL GRAPHS, AND CERECEDA'S CONJECTURE FOR BIPARTITE 2-DEGENERATE GRAPHS

DAVE GOEL

Version 2.30, October 23, 2026.

Abstract. This paper answers a question of Esperet, Hendrey, Lagoutte, Marseloo, Norin and Steiner on flow reconfiguration and makes inroads into Cereceda's recolouring conjecture for $d = 2$. Parts I and II are about flows. For a finite multigraph G and an integer $k \geq 2$, let $M_k(G)$ be the graph on the nowhere-zero $\mathbb{Z}_k$-flows of G, and $I_k(G)$ the graph on the integer flows F of G with $1 \leq |F(e)| \leq k-1$ on every edge; two flows are adjacent if the support of their difference is a cycle. We prove that $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G if $k = 40$, if $k = 48$, or if k is a multiple of 4 with $k \geq 52$. This answers their Problem 1.2. The same holds for $k = 66$, for every prime power p^N with $N \geq 6$, and for a set of moduli of natural density one; these results do not cover every $k \geq 40$. A master theorem for finite abelian groups gives the connectivity of the flow graph $F(G, \Gamma)$ for every abelian group Γ of order 40, and a binary form of it for every abelian group of exponent dividing 4 and order at least 32, in particular $\mathbb{F}_2^5$ (Esperet et al. proved $\mathbb{F}_2^8$). Part III is about colourings: two proper k-colourings of a graph G are adjacent if they differ at exactly one vertex. Cereceda conjectured that this graph has diameter $O(n^2)$ for every d-degenerate graph on n vertices and $k = d+2$; the case $d = 2$ is open. For four colours we prove the diameter bound $2n(n+1)$ for every bipartite graph of degeneracy at most 2, with the order n^2 sharp in this class, also at girth 6; the bound $5n(n+1)$ when an independent set of vertices of degree at most 2 is added; and, with a computer-verified certificate, $4n(n+1)$ for every graph of cone width at most 2, a class that contains all graphs of treewidth at most 2 and some graphs of treewidth 3 with triangles.

Summary of results

This paper has three parts. Parts I and II are about the reconfiguration of nowhere-zero flows; Part III is about the reconfiguration of proper colourings ($\mathcal{R}_k(G)$ is the graph on the proper k-colourings of G, two of them adjacent if they differ at exactly one vertex). On a plane graph G the two objects are dual to each other [Tu1]: a proper $\mathbb{Z}_k$-colouring of the dual graph G^* gives a nowhere-zero $\mathbb{Z}_k$-flow of G (on each edge, the colour of the face on its left minus the colour of the face on its right), every nowhere-zero $\mathbb{Z}_k$-flow of G comes in this way from exactly k colourings, and recolouring one vertex of G^* changes the flow along the boundary of one face of G, which is a cycle if G is 2-connected. Part III does not use Parts I and II, and Parts I and II do not use Part III. The new results are the following.

- *One modulus for all graphs* (Theorem 9.2). For every finite bridgeless loopless multigraph G, the flow graphs $M_k(G)$ and $I_k(G)$ are connected for $k = 40$, $k = 48$ and every multiple k of 4 with $k \geq 52$. This answers Problem 1.2 of [EHLMNS] positively.
- *Further moduli.* The same for $k = 66$ (Corollary 5.5), for every prime power p^N with $N \geq 6$ (Theorem 7.4), and for every k that is not a prime power and has at least

four prime factors, counted with multiplicity, except 24, 36 and 54 (Theorem 9.5); the set of all moduli k with this property has natural density 1 (Corollary 9.9); for one fixed graph, every k above a bound that depends on the graph, primes included (Section 10).

- *Groups* (Theorem 4.5). Let Γ be a finite abelian group that is not cyclic of prime-power order, and let $|\Gamma|$ have a divisor s with $4 \le s \le |\Gamma|/10$. Then the flow graph $F(G,\Gamma)$ is connected for every finite bridgeless multigraph G, loops allowed; for example for every abelian group of order 40. This answers the question of [EHLMNS] whether the order 2^8 of the group $\mathbb{F}_2^8$ in their positive answer to Problem 1.4 can be improved.
- *Binary groups* (Part II, Theorem 13.1 and Corollary 13.2). The same for every abelian group of exponent dividing 4 and order at least 32, in particular for $\mathbb{F}_2^5$.
- *Edge connectivity, odd vertices and cubic graphs* (Part II, Theorems 13.3, 22.1, 28.6 and 24.8). $F(G,\mathbb{F}_2^m)$ is connected for every 4-edge-connected multigraph G and every $m \ge 3$; $F(G,\mathbb{F}_2^3)$ and $F(G,\mathbb{F}_2^4)$ are connected if every component of the bridgeless multigraph G has at most 44, respectively 19022, vertices of odd degree; and $F(G,\mathbb{F}_2^4)$ is nonempty and connected for every finite simple connected bridgeless cubic graph G in which every cyclic edge cut has at least eleven edges.
- *Recolouring bipartite 2-degenerate graphs* (Theorem 32.2). Every bipartite graph G on n vertices of degeneracy at most 2 has $\operatorname{diam}\mathcal{R}_4(G) \le 2n(n+1)$. So the case $d=2$ of Cereceda's conjecture [Ce] (Conjecture 32.1) holds for bipartite graphs. The same bound holds when G has an order with at most two later neighbours per vertex in which every cone (Section 33) induces a bipartite graph; such graphs need not be bipartite (Corollary 34.8).
- *The order n^2 is sharp* (Proposition 32.3) in this class, also at girth 6.
- *An added independent set* (Theorem 32.5). $\operatorname{diam}\mathcal{R}_4(G) \le 5n(n+1)$ if G has an independent set Z of vertices of degree at most 2 such that $G-Z$ is bipartite and 2-degenerate; this allows triangles and many odd cycles.
- *Cone width two* (Theorem 32.4, computer-assisted). $\operatorname{diam}\mathcal{R}_4(G) \le 4n(n+1)$ for every graph G of cone width at most 2, a class that contains all graphs of treewidth at most 2 and some graphs of treewidth 3 with triangles, and is closed under subgraphs. (For treewidth at most 2 alone, the bound $2n(n+1)$ of [BB] is smaller.)

Part I. One modulus for all graphs

0. Introduction

The problem. Let G be a finite multigraph and $k \ge 2$ an integer. A *nowhere-zero $\mathbb{Z}_k$-flow* of G gives every edge a non-zero element of $\mathbb{Z}_k$, with conservation at every vertex; an *integer k-flow* gives every edge an integer $F(e)$ with $1 \le |F(e)| \le k-1$, with conservation at every vertex. The reconfiguration graphs $M_k(G)$ and $I_k(G)$ have these flows as their vertices; two flows are adjacent if their difference is $\lambda\chi_C$ for a cycle C of G and a non-zero λ, that is, if the support of the difference is a cycle. For a finite abelian group Γ the graph $F(G,\Gamma)$ is defined in the same way, and $M_k(G) = F(G,\mathbb{Z}_k)$. The exact definitions are in Section 1. A graph with a bridge has no nowhere-zero flow, so the graphs G of interest are the bridgeless ones.

These graphs are studied by Esperet, Hendrey, Lagoutte, Marseloo, Norin and Steiner [EHLMNS]. Their Problem 1.2 asks whether there is *one* integer k such that $I_k(G)$ is connected for every 2-edge-connected graph G; they prove that $I_k(G)$ is connected

for a modulus k that depends on G. For groups they prove that $F(G, \mathbb{F}_2^8)$ is connected for every 2-edge-connected graph G ($\mathbb{F}_2^m$ is the elementary abelian group of order 2^m), and that $F(G, \Gamma)$ is connected for every abelian group Γ whose order is at least an explicit bound of the order of 10^{694}; they also show that the modulus 5 fails. (In [EHLMNS] the graphs $I_k(G)$ and $M_k(G)$ are $\mathcal{F}(G, k)$ and $\mathcal{F}(G, \mathbb{Z}_k)$.) Their Problem 2.6 asks whether the connectivity of $M_k(G)$ implies the connectivity of $I_k(G)$; Cranston, Li, Su, Wang and Xu [CLSWX] conjecture that $I_k(G)$ is connected if and only if $M_k(G)$ is connected.

The results. All results below are for every finite bridgeless loopless multigraph G; G need not be connected. "k is reached" means: one of the results of this paper applies to k (in Section 9: k is *covered*, Definition 9.6); then $M_k(G)$ and $I_k(G)$ are connected for every such G.

(1) *The bases* (Theorem 9.2). $k = 40$, $k = 48$ and every multiple k of 4 with $k \geq 52$ are reached. In particular $k = 40$ gives a positive answer to Problem 1.2 of [EHLMNS].

(2) *Further moduli.* $k = 66$ (Corollary 5.5); every prime power $k = p^N$ with $N \geq 6$ (Theorem 7.4); every product $k = ab$ of two coprime *admitted factors*, that is integers a, b each of which is at least 43 or a product de with $d \geq 4$, $e \geq 7$ (Corollary 8.9): so every multiple of 1120, every pq with different primes $p, q \geq 43$, and every p^2q with different primes $p \geq 7$, $q \geq 43$; every k that is not a prime power and has at least four prime factors, counted with multiplicity, except 24, 36 and 54 (Theorem 9.5); every $k = 2p^3$ with a prime $p \geq 5$ (Proposition 9.4); and every k to which Criterion 5.11 applies.

(3) *Exactly which k these results reach* (Corollary 9.8, Proposition 9.7), by the number $\Omega(k)$ of prime factors with multiplicity: see Table 1. The reached moduli have natural density 1 (Corollary 9.9). Up to 100 they are 40, 48, 52, 56, 60, 64, 66, 68, 72, 76, 78, 80, 84, 88, 90, 92, 96, 100.

(4) *One graph* (Section 10). For a fixed G the results go from a base k_0 to every $k \geq k_0$, prime k included: see Table 2. For example, $M_k(G)$ and $I_k(G)$ are connected for every $k \geq 40$ if G has at most 149 vertices, or at most 189 edges, or no cycle with more than 39 edges, or if G is connected with diameter at most 3.

(5) *Groups* (Theorem 4.5). If Γ is not cyclic of prime-power order and $|\Gamma|$ has a divisor d with $4 \leq d \leq |\Gamma|/10$, then $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G, loops allowed; the least such orders are 40, 44, 48, 50. The final remarks of [EHLMNS] name two main questions left open there: Problem 1.2, and whether the order $2^8 = 256$ of the group $\mathbb{F}_2^8$ in the positive answer to their Problem 1.4 can be improved. Item (1) answers the first, and this theorem answers the second: every abelian group of order 40 works. By Corollary 13.2 of Part II, so does every abelian group of order 32 whose exponent divides 4.

(6) *Loops.* Every k of (1) and (2) is reached through Theorem 3.19 or through Theorem 8.8, and part (a) of these two theorems gives the connectivity of $M_k(G)$ also for every finite bridgeless multigraph G with loops.

Open cases. For all graphs, the moduli that are not reached include those from 6 to 39, those from 41 to 47, and the primes from 7 on (Remark 9.10); the modulus 5 fails [EHLMNS]. Table 1 gives exactly which moduli are reached. For the modular graph alone more is known: by the theorem of [EHLMNS] for large groups, $M_k(G)$ is connected for every k above a bound of the order of 10^{694}, primes included. For $k = 44$ the group theorem gives

the connectivity of $M_{44}(G)$ for every bridgeless G; the connectivity of $I_{44}(G)$ is a case of Problem 2.6 of [EHLMNS]. Section 11 explains where the method stops, and Section 12 lists open problems.

TABLE 1. The moduli k for which this paper proves that $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G (Corollary 9.8; "covered" in Definition 9.6).

k	reached exactly when
prime power p^e	$e \geq 6$ (no prime)
pq, $p \neq q$ primes	$p \geq 43$ and $q \geq 43$
$\Omega(k) = 3$, not a prime power	Proposition 9.7: p^2q by the rules (α), (β), (γ) there; every odd pqr except 105; every $2pq$ except 30, 42, 70, 110, 182, 286, 506, 646, 1406, 1798
$\Omega(k) \geq 4$, not a prime power	$k \notin \{24, 36, 54\}$

TABLE 2. One graph G (finite, bridgeless, loopless): the moduli k for which this paper proves that $M_k(G)$ and $I_k(G)$ are connected. $c(G)$ is the number of edges of a longest cycle, n the number of vertices.

condition on G	every k with	where
none	$k \geq \max(48, c(G) + 4)$	Cor. 10.6
$c(G) \leq 39$	$k \geq 40$	Cor. 10.6
at most 189 edges (at most 235 edges)	$k \geq 40$ ($k \geq 48$)	Cor. 10.12
at most 149 vertices (at most 187 vertices)	$k \geq 40$ ($k \geq 48$)	Cor. 10.16
$n \geq 150$ vertices	$k \geq 2n/\ln n + 4$	Cor. 10.16
every edge on a cycle with at most 8 edges	$k \geq 40$	Cor. 10.22
connected, diameter at most 3	$k \geq 40$	Cor. 10.23
all edges but at most 40 on cycles with at most 8 edges	$k \geq 40$	Cor. 10.24

The method. *From groups to integers.* A non-zero element z of $\mathbb{Z}_k$ is *liftable* if, for every loopless G and every move $f \to f + z\chi_C$ of $M_k(G)$, all integer lifts of the two flows lie in one component of $I_k(G)$. Section 1 shows that ± 1, ± 2 and $\pm q$ for every divisor $q < k$ of k are liftable, and that cz is liftable with z if the unit c is 1 modulo both factors of a factorization of k (so $\pm(1 + 2p)$ is liftable for $k = 4p$). Integer lifts exist, and the lifts of one flow are joined in $I_k(G)$. So if any two nowhere-zero $\mathbb{Z}_k$-flows are joined by moves whose steps are liftable, then $I_k(G)$ is connected (Theorem 3.19).

The master theorem (Section 3). Let Γ be a finite abelian group, B a subgroup, $A = \Gamma/B$, and $\Sigma = -\Sigma$ a set of non-zero elements of Γ, the *steps*. Theorem 3.11 says: if the steps join the elements of A after the deletion of any 6 elements (R1), join the elements of B after the deletion of any 2 elements (R2), join the non-zero elements of Γ (R3*), if an averaging condition (AV″) holds, and if $|A| \geq 10$ and $|B| \geq 4$ (Corollary 3.15; more precisely, $|A| \geq 8$ and certain counts), then any two nowhere-zero Γ-flows of a bridgeless multigraph are joined by moves with steps in Σ. The proof is an induction on the number of vertices. Loops are handled by (R3*), and a cut with two edges by a contraction. In a 3-edge-connected graph, (AV″) leads to a flow whose A-coordinate has no zero on a *good subgraph* H: a cycle with at

most 5 edges, or a subdivision of a 3-edge-connected graph H_0 in which every edge becomes a path with at most 3 edges. Such an H exists in every loopless graph with $n \geq 3$ vertices and more than $(4/3)\,n - 2$ edges (Section 2). The graph G/H is smaller, and each of its moves is lifted to moves of G that keep the A-coordinate non-zero on H (Lemma 3.10). For this, every block of H_0 is expanded to a truncated cubic graph; their union is Q, and the well-colouring lemma of the appendix (Lemma A.5, one of the two facts of Jaeger, Linial, Payan and Tarsi [JLPT] that the appendix proves) gives a *frame* of H: a spanning forest of Q, with a tree in every component, whose edges come in pairs, and one *official cycle* for every pair. The A-flows of Q are combinations of the official cycles; the coefficients are chosen in reverse order to avoid lists of forbidden values (the reverse choice), and they are changed one step at a time inside the allowed flows (the walk lemma). A thread has at most 3 edges, so a list has at most 3 values; the largest count is $9 < |A|$, and in B it is $3 < |B|$. This is where $|A| \geq 10$ and $|B| \geq 4$ come from, hence the bound $k = ab \geq 40$.

This scheme of extraction and contraction follows [EHLMNS, Section 6, in particular Lemmas 6.1 and 6.14]: the averaging that finds the subgraph, and the preparation of the flow on it before a move of the contracted graph is lifted; their Section 6.2 also brings in the closure theory of [JLPT]. What we add is the quotient Γ/B with a prescribed set of steps, the counts for the avoidance and for the averaging, and the double step of Section 6; together with the liftable steps of Section 1 this gives the results on integer flows.

Averaging (Section 4). If $\Gamma = \Gamma_1 \times \Gamma_2$ with both factors non-trivial, and the steps generate both factors, then random changes of one coordinate on the support of the other make the value of every edge almost uniform. So some flow has at most $m/9$ edges where the A-coordinate is 0, if $(|B| - 1)/(|\Gamma| - 1) < 1/9$ (Lemma 4.1); with the count of the edges this gives a good subgraph.

The moduli. For $k = 40 = 10 \cdot 4$ the steps are ± 1, ± 2 and $\pm q$ for the divisors q; modulo 10 they miss only ± 3, which an elementary lemma allows; and $\mathbb{Z}_{40} = \mathbb{Z}_8 \times \mathbb{Z}_5$ (Section 5; 48 and 66 in the same way, and Criterion 5.11 in general). For $k = 4p$ the steps modulo p are only ± 1, ± 2, ± 4, and six deletions can isolate a value, so (R1) fails. The frame form of the master theorem (Theorem 3.12) needs (R1) only with five deletions, which holds by the theorem of Cauchy and Davenport (and Vosper's theorem shows that six deletions can only isolate one value), and replaces the walk lemma by a walk with *double steps* that pass an isolated value; one of four choices (two routes, and the two lifts ± 1, $\pm(1 + 2p)$ of the step ± 1) works, by a parity argument in $B = \mathbb{Z}_4$ (Section 6). A cyclic group of prime-power order is not a product, so for $k = p^N$ the averaging is replaced by a flattening of the p-adic valuations of the values (Section 7). For two coprime factors a, b, moves that change one residue on a cycle where the other residue has no zero lift to integers whatever their step (Section 8). Section 9 puts the moduli together.

One graph (Section 10). If $I_{k-1}(G)$ is connected and every flow of $I_k(G)$ is joined to a flow without the value $\pm(k-1)$, then $I_k(G)$ is connected. This step from $k-1$ to k is made under a bound on the longest cycle, on the number of edges, on the number of vertices, or on the lengths of short cycles, starting at a base.

The results of Part II. Part II is mainly about groups of exponent 2 or 4, with the moves and the graphs $F(G, \Gamma)$ of Part I. Its main theorem (Theorem 13.1) is a binary form of the master theorem: if a subgroup B of Γ and $A = \Gamma/B$ satisfy $2B = 0$, $2A = 0$, $|B| \geq 4$, $|A| \geq 8$ and $(|B|-1)/(|\Gamma|-1) \leq 1/9$, then $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G. Its proof uses the extraction and the frames of Part I (Section 2 and the appendix)

and the estimate of Remark 4.2 (from the proof of Lemma 4.1), which also covers the ratio 1/9. So $F(G,\Gamma)$ is connected for every finite abelian group Γ whose exponent divides 4 and whose order is at least 32 (Corollary 13.2); at order 32 these groups are $\mathbb{F}_2^5$, $\mathbb{Z}_4\times\mathbb{F}_2^3$ and $\mathbb{Z}_4^2\times\mathbb{Z}_2$, which item (5) does not reach. Part II also proves, independently of Part I and with the tree-packing theorem of Nash-Williams and Tutte, that $F(G,\mathbb{F}_2^m)$ is connected for every 4-edge-connected multigraph G and every $m\geq 3$ (Theorem 13.3); and that $F(G,\mathbb{F}_2^3)$ and $F(G,\mathbb{F}_2^4)$ are connected if every component of the finite bridgeless multigraph G has at most 44, respectively 19022, vertices of odd degree (Theorems 22.1 and 28.6). These two bounds use the unpublished manuscript [Go] only through its two-odd-vertex theorem [Go, Theorem 1.2(c)]; the local lemmas of [Go] that Part II needs are proved again there. No proof in Part I uses Part II.

Organization. Section 1: conventions, integer lifts, liftable steps. Section 2: good subgraphs, the truncated cubic reduction, threads, frames and official cycles. Section 3: step systems, the master theorem and its integer form. Section 4: averaging in a product, and the group theorem. Sections 5–9: the moduli. Section 10: one graph. Section 11: limits of the method. Section 12: open problems. Part II starts with its own statement of results and scope. Part III, on recolouring, starts with its own introduction (Section 32) and is independent of Parts I and II. The appendix, placed after Part III, proves the closure lemma used in Parts I and II. The results for $k=40$, 48 and 66, and so the answer to Problem 1.2 of [EHLMNS], use only Sections 1–5 and the appendix.

Theorems quoted from the literature. The proofs of Sections 1–10 use the following theorems from the literature, in the form stated here, and no other ones. Standard facts are used without a reference: the Chinese remainder theorem; a finite abelian group is a direct product of cyclic groups of prime-power order and has a subgroup of every order that divides its order; blocks of a graph. Some published results are proved again, in the form that is used: Tutte's lifting theorem (Corollary 1.3; [EHLMNS, Theorem 2.3]), the connected fibres (Lemma 1.4; [EHLMNS, Lemma 5.5]), the extraction lemma (Lemma 2.2; [EHLMNS, Lemma 6.1]), the closure lemma of [JLPT] (Lemma A.7), the signed cycle decomposition (Lemma 1.1 (b)), and the potential of a cost function without negative cycles (Lemma 10.13).

Theorem 0.1 (Seymour [Se]; Tutte [Tu1])**.** *Every bridgeless multigraph, loops allowed, has a nowhere-zero flow with values in any abelian group of order at least* 6.

Used in the proof of Theorem 3.11 (the target flow) and of Theorem 8.8.

Theorem 0.2 (Cauchy–Davenport [Ca, Da])**.** *Let p be a prime, and let U_1 and U_2 be non-empty subsets of $\mathbb{Z}_p$. Then*

$$|U_1+U_2| \;\geq\; \min\{\,p,\;|U_1|+|U_2|-1\,\},\qquad \text{where } U_1+U_2=\{\,u_1+u_2 : u_1\in U_1,\;u_2\in U_2\,\}.$$

Used in Lemma 5.8 and Lemma 6.1.

Theorem 0.3 (Vosper [Vo])**.** *Let p be a prime, and let U_1, U_2 be subsets of $\mathbb{Z}_p$ with $|U_1|\geq 2$, $|U_2|\geq 2$ and $|U_1+U_2|=|U_1|+|U_2|-1\leq p-2$. Then U_1 and U_2 are arithmetic progressions with the same difference.*

Used in Lemma 6.1.

Theorem 0.4 (Mader, Watkins [Ma, Wa])**.** *Let Δ be a connected vertex-transitive graph without loops and without multiple edges, in which every vertex has $h\geq 2$ neighbours. Then*

the vertex-connectivity of Δ is at least $2(h+1)/3$: the deletion of any set of fewer than $2(h+1)/3$ vertices leaves a connected graph. (*See also* [GR, Theorem 3.4.2, p. 40].)

Used in Lemma 5.9, hence in the last sentence of Criterion 5.11 and in its uses, and in Theorem 7.4 for an odd prime p (the check of (R1)).

Theorem 0.5 (Hoffman's circulation theorem [Ho])**.** *Let G be a multigraph with an orientation, and let $l \le u$ be integer functions on its arcs. There is an integer flow Y of G (in this orientation) with $l(e) \le Y(e) \le u(e)$ for every arc e if and only if, for every set X of vertices, the sum of l over the arcs that leave X is at most the sum of u over the arcs that enter X.*

Used three times in Section 10.

Theorem 0.6 (Euler; see [HW])**.** *The sum of $1/p$ over all primes p diverges.*

Theorem 0.7 (Chebyshev's inequality; see [Fe])**.** *Let X be a real random variable on a finite probability space, with mean μ and variance V. Then $\Pr[\,|X-\mu| \ge t\,] \le V/t^2$ for every real $t > 0$.*

Theorems 0.6 and 0.7 are used only in Corollary 9.9.

1. Conventions; lifting to integers; liftable steps

Graphs, flows and the reconfiguration graphs. Graphs are finite multigraphs. Loops are allowed, unless the graph is called loopless. A *cycle* is a connected subgraph in which every vertex has degree 2; a loop is a cycle of length 1, and two parallel edges form a cycle of length 2. An edge that lies on no cycle is a *bridge*; a graph is *bridgeless* if it has no bridge, that is, if every edge lies on a cycle. A graph without edges is bridgeless. A multigraph is *j-edge-connected* if, for every partition of its vertex set into two non-empty sets, at least j edges have one end in each set.

Every graph has a fixed reference orientation: every edge has a tail and a head (for a loop they are the same vertex). Let x be a map from $E(G)$ to an abelian group. Its *boundary* at a vertex v is

$$(\partial x)(v) \;=\; \sum_{e\,:\,v \text{ is the head of } e} x(e) \;-\; \sum_{e\,:\,v \text{ is the tail of } e} x(e).$$

A loop at v is in both sums, so it adds nothing. The map x is a *flow* if $\partial x = 0$ (conservation at every vertex). An *integer flow* is a flow with values in $\mathbb{Z}$. Groups are finite abelian groups, if nothing else is said. For a group Γ, a *Γ-flow* of G is a flow $E(G) \to \Gamma$; it is *nowhere zero* if no value is 0. The *support* $\mathrm{supp}(x)$ is the set of the edges e with $x(e) \ne 0$.

For a cycle C with a direction, χ_C is its signed indicator: $\chi_C(e) = 1$ if C passes the edge e from the tail to the head (*forward*), $\chi_C(e) = -1$ if C passes e from the head to the tail, and $\chi_C(e) = 0$ if e is not on C. So χ_C is an integer flow, and $\lambda\chi_C$ is a Γ-flow for every λ in Γ.

If the reference orientation of one edge is reversed and the value of every flow on this edge is replaced by its negative, then flows stay flows, and nothing else changes. No statement of this paper depends on the reference orientation, and we change it freely inside proofs.

$F(G,\Gamma)$ is the graph on the nowhere-zero Γ-flows of G; two flows are adjacent if their difference is $\lambda\chi_C$ for a cycle C of G and $\lambda \ne 0$ (a *move*). $M_k(G) = F(G,\mathbb{Z}_k)$. $I_k(G)$ is the graph on the integer flows F of G with $1 \le |F(e)| \le k-1$ for every edge e; two of them are adjacent if their difference is $\lambda\chi_C$ with an integer $\lambda \ne 0$. An *integer lift* of a $\mathbb{Z}_k$-flow f is a vertex F of $I_k(G)$ with $F = f$ modulo k. These are the three *reconfiguration graphs*.

In each of the three graphs, two different vertices are adjacent if and only if the support of their difference is the edge set of a cycle: a flow whose support is a cycle C has, by conservation at the vertices of C, the form $\lambda\chi_C$.

A graph G without edges has exactly one flow, the empty map; it is a vertex of each of the three reconfiguration graphs. A reconfiguration graph that has no vertex counts as connected.

In this section G is a finite multigraph, and loops are allowed. Only Definition 1.7, Lemma 1.12 and Corollary 1.13 speak of loopless graphs.

The cycle decomposition.

Lemma 1.1 (Cycle decomposition)**.** *Let G be a multigraph.*

(a) *Let A be an abelian group, finite or not, and let T_0 be a spanning forest of G (a spanning tree in every component of G). For an edge e outside T_0 let C_e be the fundamental cycle of e in T_0, directed along e. Then every A-flow h of G is*

$$h = \sum_{e\notin E(T_0)} h(e)\,\chi_{C_e},$$

and this is the only way to write h as a sum $\sum_{e\notin E(T_0)} x_e\chi_{C_e}$ with x_e in A. In particular every A-flow of G is a sum of flows $x\chi_C$ with x in A and C a cycle of G.

(b) (*The signed form.*) *Every integer flow h of G is a sum*

$$h = \chi_{O_1} + \cdots + \chi_{O_j} \qquad (j \geq 0)$$

of signed indicators of directed cycles $O_1, \ldots, O_j$ of G (a cycle can occur several times), such that every O_i passes each of its edges in the direction of the sign of h: $\chi_{O_i}(e)\,h(e) > 0$ for every edge e of O_i. Hence, for every edge e: exactly $|h(e)|$ of the cycles contain e; every sum of some of the numbers $\chi_{O_1}(e), \ldots, \chi_{O_j}(e)$ lies between 0 and $h(e)$; and all cycles lie in $\operatorname{supp}(h)$*. If all values of h are in $\{-1, 0, 1\}$, then $O_1, \ldots, O_j$ are pairwise edge-disjoint.*

Proof. (a) Put $h' = h - \sum_{e\notin E(T_0)} h(e)\chi_{C_e}$. This is an A-flow. The cycle C_e contains e and no other edge outside T_0, and $\chi_{C_e}(e) = 1$. (A loop is not in T_0, and it is its own fundamental cycle.) So h' is 0 on every edge outside T_0. Suppose that $h' \neq 0$. The edges with $h' \neq 0$ form a forest with at least one edge. It has a vertex that meets exactly one of these edges. Conservation at this vertex says that the value of h' on this edge is 0, a contradiction. So $h' = 0$. The sum $\sum x_e\chi_{C_e}$ has the value x_e on the edge e outside T_0; so the coefficients are determined by the flow.

(b) Induction on $\sum_e |h(e)|$. If this sum is 0, take $j = 0$. Otherwise reverse the reference orientation of every edge e with $h(e) < 0$; now $h \geq 0$. If a loop has a positive value, let O be this loop, passed forward. Otherwise use conservation: at every vertex, the sum of h over the edges that enter is equal to the sum over the edges that leave (loops are left out on both sides). So the head of an edge with a positive value is the tail of an edge with a positive value. Start with an edge with a positive value and go on in this way. The walk does not end, so it comes back to a vertex that it has seen. This gives a directed cycle O that passes all its edges forward, and $h > 0$ on all its edges. Now $h - \chi_O$ is an integer flow with $h - \chi_O \geq 0$ and with a smaller sum. By induction it is a sum of signed indicators of directed cycles that pass their edges forward. Add χ_O. In the first reference orientation, "forward" is the direction of the sign of $h(e)$.

For a fixed edge e, all non-zero numbers $\chi_{O_i}(e)$ have the sign of $h(e)$, and their sum is $h(e)$. This gives the three consequences. If $|h(e)| \le 1$ for all e, every edge is on at most one of the cycles. □

Rounding, integer lifts, and the fibres.

Lemma 1.2 (Rounding)**.** *Let x be a map from $E(G)$ to the real interval $[0,1]$ such that $(\partial x)(v)$ is an integer for every vertex v. Then there is a map $z\colon E(G) \to \{0,1\}$ with $\partial z = \partial x$ and with $z(e) = x(e)$ for every edge e with $x(e)$ in $\{0,1\}$.*

Proof. Call an edge e fractional if $0 < x(e) < 1$. If there is no fractional edge, take $z = x$. Otherwise the fractional edges contain a cycle C. Indeed, a fractional loop is such a cycle. If no loop is fractional, then no vertex v meets exactly one fractional edge: the other edges at v have integer values, $(\partial x)(v)$ is an integer, and so the one fractional value would be an integer. A loopless graph with at least one edge in which no vertex has degree 1 contains a cycle.

Give C a direction. Let $\delta > 0$ be the least number such that $x + \delta\chi_C$ has the value 0 or 1 on some edge of C; it exists, because all values of x on C are strictly between 0 and 1. The map $x + \delta\chi_C$ has values in $[0,1]$ and the same boundary as x, because $\partial\chi_C = 0$. The values on the edges that are not fractional have not changed, and the number of fractional edges is smaller. Repeat this until no edge is fractional. □

Corollary 1.3 (Integer lifts)**.** *Let $k \ge 2$ and let f be a $\mathbb{Z}_k$-flow of G; zeros are allowed. Then there is an integer flow F of G with $F = f$ modulo k, with $|F(e)| \le k-1$ for every edge e, and with $F(e) = 0$ exactly for the edges e with $f(e) = 0$. In particular every nowhere-zero $\mathbb{Z}_k$-flow of G has an integer lift.*

Proof. For every edge e let $r(e)$ in $\{0,\dots,k-1\}$ be the residue of $f(e)$. Then $\partial r = \partial f = 0$ modulo k, so the map $x = r/k$ has values in $[0,1)$ and an integer boundary. Lemma 1.2 gives $z\colon E(G) \to \{0,1\}$ with $\partial z = \partial x$ and with $z(e) = 0$ if $r(e) = 0$. Put $F = r - kz$. Then $\partial F = \partial r - k\,\partial x = 0$, and $F = f$ modulo k. If $r(e) = 0$, then $F(e) = 0$. If $r(e) \ne 0$, then $F(e)$ is $r(e)$ or $r(e) - k$; the first number is in $[1, k-1]$ and the second one is in $[-(k-1), -1]$. □

Corollary 1.3 is Tutte's lifting theorem in the form of [EHLMNS, Theorem 2.3].

Lemma 1.4 (Connected fibres)**.** *Let $k \ge 2$ and let f be a nowhere-zero $\mathbb{Z}_k$-flow of G. Then the integer lifts of f induce a connected subgraph of $I_k(G)$.*

Proof. Let F and F' be integer lifts of f. On every edge e, both $F(e)$ and $F'(e)$ are in $\{r(e),\, r(e) - k\}$, where $r(e)$ in $\{1,\dots,k-1\}$ is the residue of $f(e)$. So $h = (F' - F)/k$ is an integer flow with values in $\{-1, 0, 1\}$. By Lemma 1.1 (b), $h = \chi_{O_1} + \cdots + \chi_{O_j}$ with pairwise edge-disjoint directed cycles. Put $F_i = F + k\,(\chi_{O_1} + \cdots + \chi_{O_i})$. On every edge, F_i has the value of F or the value of F'. So every F_i is an integer lift of f, $F_0 = F$, $F_j = F'$, and $F_i - F_{i-1} = k\chi_{O_i}$. This is a path from F to F' in $I_k(G)$ through integer lifts of f. □

Lemma 1.4 is [EHLMNS, Lemma 5.5].

The divisor extension. Let $k = mq$ with integers $m, q \ge 2$. Reduction modulo q is a homomorphism from $\mathbb{Z}_k$ onto $\mathbb{Z}_q$; so the *reduction modulo q* of a $\mathbb{Z}_k$-flow, taken edge by edge, is a $\mathbb{Z}_q$-flow. The multiples of q in $\mathbb{Z}_k$ form the subgroup $q\mathbb{Z}_k$, and $qj \mapsto j$ modulo m is an isomorphism from $q\mathbb{Z}_k$ onto $\mathbb{Z}_m$. For a $\mathbb{Z}_k$-flow y with all values in $q\mathbb{Z}_k$ we write y/q for its image, a $\mathbb{Z}_m$-flow.

The next two proofs use the same setting.

Setting. w is a $\mathbb{Z}_q$-flow of G, and E_0 is the set of the edges e with $w(e) = 0$. By Corollary 1.3 there is an integer flow W of G with $W = w$ modulo q, with $|W(e)| \le q-1$, and with $W(e) = 0$ exactly on E_0. Reverse the reference orientation of every edge with $W(e) < 0$. So from now on

$$1 \le W(e) \le q-1 \quad \text{for every edge } e \notin E_0, \qquad W(e) = 0 \quad \text{for every edge } e \in E_0.$$

A *state* is a nowhere-zero $\mathbb{Z}_k$-flow of G whose reduction modulo q is w.

Claim 1 (the standard lift of a state). *For every state h there is an integer flow L of G with $|L(e)| \le m-1$ for every edge e, such that $F = W + qL$ is an integer lift of h. For every edge e in E_0: $L(e) \ne 0$, and $h(e) = qL(e)$ modulo k.*

Proof. The $\mathbb{Z}_k$-flow $h - W$ has the reduction 0 modulo q, so all its values are in $q\mathbb{Z}_k$, and $y = (h-W)/q$ is a $\mathbb{Z}_m$-flow. By Corollary 1.3 there is an integer flow L with $L = y$ modulo m, with $|L(e)| \le m-1$, and with $L(e) = 0$ exactly where $y(e) = 0$. Put $F = W + qL$. This is an integer flow. The residue of $qL(e)$ modulo k depends only on the residue of $L(e)$ modulo m; so $qL = h - W$ modulo k, and $F = h$ modulo k.

Let $e \notin E_0$. Then $F(e) = W(e)$ modulo q, and this is not 0; so $F(e) \ne 0$. And $1 - q(m-1) \le F(e) \le q - 1 + q(m-1)$, so $|F(e)| \le k-1$.

Let $e \in E_0$. Then $W(e) = 0$ and $h(e) \ne 0$, so $y(e) \ne 0$ and $L(e) \ne 0$. So $F(e) = qL(e)$ is not 0, and $|F(e)| \le q(m-1) < k$. □

Claim 2 (one step of size q). *Let O be a cycle of G with a direction that passes each of its edges outside E_0 forward. Let h and $h - q\chi_O$ be states. Then some integer lift of h is adjacent in $I_k(G)$ to some integer lift of $h - q\chi_O$.*

Proof. Take L and $F = W + qL$ from Claim 1. We show that $F - q\chi_O$ is a vertex of $I_k(G)$. It is an integer flow, it is equal to $h - q\chi_O$ modulo k, and outside O it is equal to F.

Let e be an edge of O outside E_0. Then $\chi_O(e) = 1$, and the new value is $W(e) + q\,(L(e) - 1)$. It is equal to $W(e)$ modulo q, so it is not 0. It is at least $W(e) - qm \ge 1 - k$ and at most $W(e) + q(m-2) \le k - q - 1$.

Let e be an edge of O in E_0. Choose its reference orientation such that O passes e forward (on E_0 the orientation was free). By Claim 1, $h(e) = qi$ modulo k with $1 \le i \le m-1$, and $L(e) = i$ modulo m; so $L(e)$ is i or $i - m$. The state $h - q\chi_O$ has the value $q(i-1)$ on e, which is not 0; so $i \ge 2$. The new integer value on e is $q(i-1)$ or $q(i-m-1)$. Here $1 \le i - 1 \le m - 2$ and $1 - m \le i - m - 1 \le -2$. So the new value is not 0, and its absolute value is at most $q(m-1) < k$.

So $F - q\chi_O$ is an integer lift of $h - q\chi_O$. It is adjacent to F, because $q \ne 0$. □

Theorem 1.5 (Divisor extension)**.** *Let $k = mq$ with $m, q \ge 2$. Let f and g be nowhere-zero $\mathbb{Z}_k$-flows of G that have the same reduction w modulo q, and let $f(e) = g(e)$ for every edge e with $w(e) = 0$. Then all integer lifts of f and all integer lifts of g lie in one component of $I_k(G)$.*

Proof. Use the setting above for this w. Then f and g are states. Let G' be the graph with the vertex set $V(G)$ and the edge set $E(G) \setminus E_0$. The $\mathbb{Z}_k$-flow $g - f$ has all values in $q\mathbb{Z}_k$, and it is 0 on E_0. So $y_0 = (g-f)/q$ is a $\mathbb{Z}_m$-flow of G'. By Corollary 1.3, used for G', there is an integer flow L_0 of G' with $L_0 = y_0$ modulo m and $|L_0(e)| \le m-1$.

The integer flow $mW + L_0$ of G' is positive on every edge of G': $|L_0(e)| \le m - 1 < m \le mW(e)$. By Lemma 1.1 (b),

$$mW + L_0 = \chi_{O_1} + \cdots + \chi_{O_j}$$

with directed cycles $O_1, \dots, O_j$ of G' that pass all their edges forward. Modulo m this gives $y_0 = \chi_{O_1} + \cdots + \chi_{O_j}$, and so $g - f = q\,(\chi_{O_1} + \cdots + \chi_{O_j})$ modulo k.

Put $h_0 = g$ and $h_i = h_{i-1} - q\chi_{O_i}$ for $i = 1, \dots, j$. Then $h_j = f$. Every h_i is a state: its reduction modulo q is w, so it is not 0 outside E_0; and on E_0 it is equal to g, because the cycles O_i have no edge in E_0. By Claim 2, some integer lift of h_{i-1} is adjacent to some integer lift of h_i. By Lemma 1.4 all integer lifts of one h_i lie in one component of $I_k(G)$. So all integer lifts of $g = h_0$ and of $f = h_j$ lie in one component. (If $j = 0$, then $f = g$, and this is Lemma 1.4.) □

Remark 1.6. For $m = 2$ the two flows of Theorem 1.5 are always equal on the edges with $w(e) = 0$: the only element of $q\mathbb{Z}_k$ other than 0 is q. For $m \geq 3$ the theorem says nothing about two flows that are different on such an edge. Lemma 1.10 below treats one such case, a single move of size q.

Liftable steps.

Definition 1.7 (Liftable step)**.** Let $k \geq 3$. An element $z \neq 0$ of $\mathbb{Z}_k$ is *liftable* if for every finite loopless multigraph G and every move $f \to f + z\chi_C$ of $M_k(G)$, all integer lifts of f and all integer lifts of $f + z\chi_C$ lie in one component of $I_k(G)$.

In the next three lemmas C is a cycle of G with a direction. In the first two of them we choose the reference orientation such that C passes all its edges forward. So χ_C is 1 on the edges of C and 0 on all other edges.

Lemma 1.8 (A step of size 1)**.** *Let $k \geq 2$, and let f and $f + \chi_C$ be nowhere-zero $\mathbb{Z}_k$-flows of G. Then for every integer lift F of f, the flow $F + \chi_C$ is an integer lift of $f + \chi_C$.*

Proof. $F + \chi_C$ is an integer flow, and it is equal to $f + \chi_C$ modulo k. Let e be an edge of C and $r(e)$ in $\{1, \dots, k-1\}$ the residue of $f(e)$. Then $r(e) \neq k - 1$, because $f + \chi_C$ is not 0 on e. The value $F(e)$ is $r(e)$ or $r(e) - k$. So $F(e) + 1$ is in $[2, k-1]$ or in $[2-k, -1]$. □

Lemma 1.9 (A step of size 2)**.** *Let $k \geq 3$, and let f and $f + 2\chi_C$ be nowhere-zero $\mathbb{Z}_k$-flows of G. Then there is an integer lift F of f such that $F + 2\chi_C$ is an integer lift of $f + 2\chi_C$.*

The flow $f + \chi_C$ can have zeros, so Lemma 1.8 cannot be used twice.

Proof. For every edge e let $r(e)$ in $\{1, \dots, k-1\}$ be the residue of $f(e)$. No edge of C has $r(e) = k - 2$, because $f + 2\chi_C$ is nowhere zero. Let E_1 be the set of the edges e of C with $r(e) = k - 1$. Put

$$x = (r + \chi_C)/k.$$

All values of x are in $[0, 1]$, and $x(e) = 1$ exactly for the edges e in E_1. The boundary $\partial x = (\partial r)/k$ is an integer at every vertex, because $\partial r = \partial f = 0$ modulo k. By Lemma 1.2 there is $z\colon E(G) \to \{0, 1\}$ with $\partial z = \partial x$ and with $z(e) = 1$ for every e in E_1. Put $F = r - kz$. As in the proof of Corollary 1.3, F is an integer lift of f.

$F + 2\chi_C$ is an integer flow, it is equal to $f + 2\chi_C$ modulo k, and outside C it is equal to F. For e in E_1: $F(e) = -1$, and the new value is 1. For an edge e of C outside E_1: $r(e) \leq k - 3$. If $F(e) = r(e)$, then $3 \leq F(e) + 2 \leq k - 1$. If $F(e) = r(e) - k$, then $3 - k \leq F(e) + 2 \leq -1$. So $F + 2\chi_C$ is a vertex of $I_k(G)$. □

Lemma 1.10 (Divisor move)**.** *Let q be a divisor of k with $1 \leq q < k$, and let f and $f + q\chi_C$ be nowhere-zero $\mathbb{Z}_k$-flows of G. Then there is a path in $I_k(G)$ from an integer lift of f to an integer lift of $f + q\chi_C$.*

Proof. For $q = 1$ this is Lemma 1.8. Let $q \geq 2$ and $k = mq$; then $m \geq 2$. Let w be the reduction of f modulo q, and use the setting of this section for w, with E_0, W and the states. The reference orientation is now the one of the setting, and χ_C is read in it. The flows f and $f + q\chi_C$ are states.

The integer flow $mW - \chi_C$ has a value of at least $m - 1 \geq 1$ on every edge outside E_0, and the value $-\chi_C(e)$ on every edge e in E_0. By Lemma 1.1 (b),

$$mW - \chi_C = \chi_{O_1} + \cdots + \chi_{O_j}$$

with directed cycles that pass their edges in the direction of the sign of $mW - \chi_C$. So every O_i passes its edges outside E_0 forward. An edge of E_0 that is not on C is on no O_i. An edge e of E_0 on C is on exactly one O_i, and $\chi_{O_i}(e) = -\chi_C(e)$.

Put $h_0 = f$ and $h_i = h_{i-1} - q\chi_{O_i}$ for $i = 1, \ldots, j$. Then $h_j = f - q\,(mW - \chi_C) = f - kW + q\chi_C = f + q\chi_C$ modulo k. Every h_i is a state. Indeed, its reduction modulo q is w, so it is not 0 outside E_0. On an edge of E_0 outside C it is equal to f. On an edge e of E_0 on C it has the value $f(e)$, or the value $f(e) + q\chi_C(e)$ of the state $f + q\chi_C$; both are not 0.

By Claim 2, for every i some integer lift of h_{i-1} is adjacent to some integer lift of h_i. By Lemma 1.4, all integer lifts of one h_i are joined in $I_k(G)$. Together this is a path from an integer lift of f to an integer lift of $f + q\chi_C$. □

Lemma 1.11 (Multiplier)**.** *Let $k = mq$ with $m, q \geq 2$, and let c be an integer that is invertible modulo k, with $c = 1$ modulo m and modulo q. Then for every nowhere-zero $\mathbb{Z}_k$-flow f of G, the map cf is a nowhere-zero $\mathbb{Z}_k$-flow of G, and all integer lifts of f and all integer lifts of cf lie in one component of $I_k(G)$.*

Proof. Multiplication by c is an automorphism of $\mathbb{Z}_k$, so cf is a nowhere-zero $\mathbb{Z}_k$-flow. Because q divides $c - 1$, the flows f and cf have the same reduction w modulo q. Let $w(e) = 0$. Then $f(e) = qi$ for an integer i, and $cf(e) - f(e) = (c - 1)\,qi$ is a multiple of $mq = k$, because m divides $c - 1$. So $cf(e) = f(e)$. Theorem 1.5 gives the result. □

Lemma 1.12 (Liftable steps)**.** *Let $k \geq 3$.*

(a) *Every nowhere-zero $\mathbb{Z}_k$-flow of a finite loopless multigraph G has an integer lift, and all its integer lifts lie in one component of $I_k(G)$.*

(b) *1 and -1 are liftable.*

(c) *2 and -2 are liftable.*

(d) *q and $-q$ are liftable for every divisor q of k with $1 \leq q < k$.*

(e) *Let $k = mq$ with $m, q \geq 2$, and let c be invertible modulo k with $c = 1$ modulo m and modulo q. If z is liftable, then cz is liftable. In particular, for $k = 4p$ with an odd prime p, the elements $1 + 2p$ and $-(1 + 2p)$ are liftable.*

Proof. (a) Corollary 1.3 and Lemma 1.4.

(b), (c), (d) Let z be 1, or 2, or a divisor q of k with $1 \leq q < k$, and let $f \to f + z\chi_C$ be a move of $M_k(G)$. By Lemma 1.8, Lemma 1.9 or Lemma 1.10, some integer lift of f is joined in $I_k(G)$ to some integer lift of $f + z\chi_C$. With (a) for f and for $f + z\chi_C$, all integer lifts of the two flows lie in one component. A move $f \to f - z\chi_C$ is the move $f \to f + z\chi_{C'}$ for the cycle C' that is C with the other direction. So $-z$ is liftable, too.

(e) Let c' be an integer with $cc' = 1$ modulo k. Then c' is invertible modulo k, and $c' = 1$ modulo m and modulo q. Let $f \to g = f + cz\chi_C$ be a move of $M_k(G)$. Then $c'f$ and $c'g = c'f + z\chi_C$ are nowhere-zero $\mathbb{Z}_k$-flows, and $c'f \to c'g$ is a move of $M_k(G)$. Because z

is liftable, all integer lifts of $c'f$ and of $c'g$ lie in one component of $I_k(G)$. By Lemma 1.11, used for the flow $c'f$ and for the flow $c'g$, all integer lifts of $c'f$ and of $f = c\,(c'f)$ lie in one component, and all integer lifts of $c'g$ and of $g = c\,(c'g)$ lie in one component. So all integer lifts of f and of g lie in one component.

Let $k = 4p$ with an odd prime p. Take $m = 2$, $q = 2p$ and $c = 1 + 2p$. Then $c = 1$ modulo 2 and modulo $2p$. The number c is odd, and it is 1 modulo p; so it is invertible modulo $4p$. By (b), $z = 1$ and $z = -1$ are liftable. So $1 + 2p$ and $-(1 + 2p)$ are liftable. □

Corollary 1.13 (The steps for $k = 4p$)**.** *Let $k = 4p$ with an odd prime p. Then $1 + 2p$ and $-(1 + 2p)$ are liftable, and every element of the set*

$$\{\pm 1,\ \pm(1 + 2p),\ \pm 2,\ \pm 4,\ \pm p,\ 2p\}$$

of elements of $\mathbb{Z}_{4p}$ is liftable.

Proof. $\pm(1 + 2p)$: Lemma 1.12 (e). ± 1: (b). ± 2: (c). The numbers 4, p and $2p$ are divisors of $4p$ that are smaller than $4p$, and $-2p = 2p$ in $\mathbb{Z}_{4p}$; so (d) gives ± 4, $\pm p$ and $2p$. □

2. Extraction; the truncated cubic reduction; threads; official cycles

Good subgraphs and the extraction lemma.

Definition 2.1 (Good subgraph)**.** A *good subgraph* of a multigraph is a subgraph H of one of two kinds:

(CYC) a cycle with at most 5 edges and at least 2 vertices;

(SUB) a subdivision of a loopless 3-edge-connected multigraph H_0 on at least two vertices, in which every edge of H_0 is replaced by a path with at most 3 edges.

In (SUB) the paths are internally disjoint, and their interior vertices are the vertices of H that are not vertices of H_0; these have degree 2 in H.

Lemma 2.2 (Extraction)**.** *Let G be a loopless multigraph with $n \geq 3$ vertices and more than $(4/3)\,n - 2$ edges. Call a subgraph of G* dense *if it has $n' \geq 3$ vertices and more than $(4/3)\,n' - 2$ edges.*

(a) *G contains a good subgraph.*

(b) *Let G contain no cycle with at most 5 edges, and let H be a dense subgraph of G with the least number of vertices (a* vertex-minimal *dense subgraph). Then H is a good subgraph of the kind* (SUB)*, and:*

- (X1) *The vertices of H_0 are the vertices of H of degree at least 3. H_0 is loopless and 3-edge-connected, it has at least two vertices, and every edge of H_0 is replaced by a path with at most 3 edges.*
- (X2) *H contains no cycle with at most 5 edges. So H is a simple graph, and the paths that replace the edges of a cycle of H_0 have together at least 6 edges. In particular, if two edges of H_0 are parallel, both are replaced by paths with 3 edges.*
- (X3) *$3\,|E(H)| \geq 4\,|V(H)| - 5$. So H has at most $3m_0 - 4n_0 + 5$ vertices of degree 2, where n_0 and m_0 are the numbers of vertices and of edges of H_0.*
- (X4) *For every set V' of vertices of H with $1 \leq |V'| \leq |V(H)| - 3$, more than $(4/3)\,|V'|$ edges of H have an end in V'.*

Proof. G is a dense subgraph of G, so there is a dense subgraph H with the least number of vertices. If G contains a cycle with at most 5 edges, this cycle is a good subgraph of the kind (CYC): it has at least 2 vertices, because G is loopless. So (a) follows from (b). For the proof of (b) let G contain no cycle with at most 5 edges. Then H contains no such cycle; in particular H has no parallel edges. Put $n_H = |V(H)|$ and $m_H = |E(H)|$.

(X3)*, first part.* H is dense, so $3m_H > 4n_H - 6$, that is $3m_H \geq 4n_H - 5$.

(X4). The graph $H - V'$ has $n_H - |V'| \geq 3$ vertices, fewer than H. So it is not dense: $3\,|E(H - V')| \leq 4\,(n_H - |V'|) - 6$. With (X3), the number of the edges of H with an end in V' is $m_H - |E(H - V')| \geq (4\,|V'| + 1)/3$.

Step 1: $n_H \geq 6$, $m_H > n_H$, and every vertex of H has degree at least 2. We have $m_H > (4/3)\,n_H - 2 \geq n_H - 1$, so $m_H \geq n_H$, and H contains a cycle. It has at least 6 edges, so $n_H \geq 6$. Then $(4/3)\,n_H - 2 \geq n_H$, so $m_H > n_H$. By (X4) for a set V' with one vertex, more than 4/3 edges meet this vertex.

Step 2: H is connected. Suppose that H is the disjoint union of two graphs H_1, H_2 with n_1, n_2 vertices and m_1, m_2 edges. By Step 1 every vertex has degree at least 2, so both graphs contain a cycle, and $n_1, n_2 \geq 6$. Both have fewer vertices than H, so they are not dense: $m_i \leq (4/3)\,n_i - 2$. Then $m_H \leq (4/3)\,n_H - 4$, a contradiction.

Step 3: the partitions with at most two edges. Let $V(H)$ be the disjoint union of two non-empty sets V_1, V_2 such that at most two edges of H join V_1 and V_2. Let H_i be the subgraph of H induced on V_i, with n_i vertices and m_i edges. If $m_i \leq (4/3)\,n_i - 2$ for $i = 1$ and for $i = 2$, then $m_H \leq (4/3)\,n_H - 4 + 2$, a contradiction. So $m_i > (4/3)\,n_i - 2$ for one i. This H_i has fewer vertices than H, so it is not dense; hence $n_i \leq 2$. A graph with at most 2 vertices and without parallel edges has no cycle. So $n_i = 1$ and $m_i = 0$, or $n_i = 2$ and $m_i = 1$. Every vertex of V_i has degree at least 2 in H, and at most two edges leave V_i. So exactly two edges leave V_i, and every vertex of V_i has degree exactly 2 in H. Result: *exactly two edges join V_1 and V_2, and one of the two sets has at most 2 vertices, all of degree 2 in H.*

Step 4: the threads. Let V_0 be the set of the vertices of H of degree at least 3. It is not empty: H is connected, every degree is at least 2, and $m_H > n_H$. Because H is connected and V_0 is not empty, H is the union of paths and cycles of the following kind (we call them *threads* here): the two ends of a thread are in V_0 (they are equal if the thread is a cycle), its interior vertices have degree 2 in H, every edge of H is on exactly one thread, and every vertex of degree 2 is an interior vertex of exactly one thread. Let I be the set of the interior vertices of a thread, and let I be not empty. The first and the last edge of the thread are the only edges between I and $V(H) \setminus I$. The set $V(H) \setminus I$ contains a vertex of degree at least 3. So by Step 3, $|I| \leq 2$. Hence every thread has at most 3 edges. A thread that is a cycle would be a cycle with at most 3 edges; there is none. So every thread is a path with two different ends in V_0. Let H_0 be the multigraph with the vertex set V_0 in which every thread is replaced by one edge between its ends. H_0 is loopless, and H is a subdivision of H_0 in which every edge of H_0 is replaced by a path with at most 3 edges.

Step 5: H_0 is 3-edge-connected and has at least two vertices. A thread has two different ends, so $|V_0| \geq 2$. Let V_0 be the disjoint union of two non-empty sets V_0', V_0'' such that at most two edges of H_0 join V_0' and V_0'' (possibly none). Put the interior vertices of every thread on the side of the first end of the thread. This is a partition of $V(H)$ into two sets; at most two edges of H join them (for every thread between V_0' and V_0'' its last edge); and both sets contain a vertex of degree at least 3. This contradicts Step 3.

Steps 4 and 5 give (X1). (X2): a cycle of H_0 becomes a cycle of H when its edges are replaced by their paths; it has at least 6 edges; two parallel edges of H_0 are a cycle of H_0, and each of the two paths has at most 3 edges. (X3), second part: let j be the number of the vertices of degree 2 of H. Then $n_H = n_0 + j$ and $m_H = m_0 + j$, and $3\,(m_0 + j) \geq 4\,(n_0 + j) - 5$ gives $j \leq 3m_0 - 4n_0 + 5$. □

Part (a) of Lemma 2.2 is the extraction lemma of [EHLMNS, Lemma 6.1]. The proof above treats the case of a cycle that is a thread explicitly.

Remark 2.3 (What the later sections use)**.** The proofs of the later sections use of a good subgraph H only the two lines of Definition 2.1. In the case (SUB) this is (X1): H_0 is loopless and 3-edge-connected, it has at least two vertices, and every edge of H_0 is replaced by a path with at most 3 edges. So every good subgraph serves, and Lemma 2.2 (a) is all that the proofs need. The equivalent form of (a): a loopless multigraph with $n \geq 3$ vertices that contains no good subgraph has at most $(4/3)\,n - 2$ edges.

Part (b) says more. Whenever the lemma is used for a graph G without a cycle with at most 5 edges, the good subgraph can be chosen as a dense subgraph with the least number of vertices, and then it has the properties (X2), (X3) and (X4) in addition. These three properties are not used in the proofs. They limit the edges of H_0 that are replaced by paths with 3 edges, and the conditions (S) and (NR) of Definition 2.15 speak of these edges. The bound in (X3) is attained. Let G consist of two vertices joined by three paths with 3 edges each. It has 8 vertices and $9 > (4/3) \cdot 8 - 2$ edges, and no cycle with fewer than 6 edges. A subgraph with $j \leq 7$ vertices has at most j edges if $j \geq 6$, and at most $j - 1$ edges if $j \leq 5$; so it is not dense. Hence $H = G$ in Lemma 2.2 (b), with $n_0 = 2$, $m_0 = 3$ and $6 = 3m_0 - 4n_0 + 5$ vertices of degree 2.

The truncated cubic reduction.

Definition 2.4 (Truncated cubic graph)**.** A *truncated cubic 3-edge-connected graph* is a graph $Q = Q^+ - v$, where Q^+ is a cubic 3-edge-connected graph on at least four vertices and v is a vertex of Q^+.

Such a graph is simple and connected; three of its vertices have degree 2 and all others have degree 3 (Lemma A.2 (a)).

A *cut vertex* of a connected graph is a vertex whose deletion leaves a graph that is not connected. A *block* of a graph is a maximal connected subgraph without a cut vertex. A graph with two vertices and one or more edges between them has no cut vertex. We use the standard facts: every edge of a graph lies in exactly one block, and every cycle lies in one block.

Lemma 2.5 (Truncated cubic reduction)**.** (a) *Let H_0 be a loopless 3-edge-connected multigraph on at least two vertices. Then every block K of H_0 is loopless and 3-edge-connected, has at least two vertices, and has no cut vertex.*

(b) *Let K be a loopless 3-edge-connected multigraph on at least two vertices without a cut vertex. Then there is a truncated cubic 3-edge-connected graph Q_K with these properties: $E(K)$ is a subset of $E(Q_K)$; the edges of Q_K that are not in $E(K)$ form pairwise disjoint cycles; and K is obtained from Q_K by the contraction of each of these cycles to one vertex, that is, by the contraction of the edges of Q_K that are not in $E(K)$.*

Proof. (a) H_0 is connected and loopless and has an edge at every vertex; so a block has an edge, it has at least two vertices, and it is loopless. It has no cut vertex by definition. Let

$V(K)$ be the disjoint union of two non-empty sets V_1, V_2. For a vertex x of K let $V[x]$ be the set of the vertices y of H_0 that are joined to x by a path without an edge of K (so x is in $V[x]$). Every vertex of H_0 is in some $V[x]$: take a path to a vertex of K and stop at the first vertex of K. No vertex is in $V[x]$ and in $V[x']$ with $x \neq x'$: otherwise there is a path without an edge of K between two different vertices of K, and with no other vertex in K; the graph K together with this path is connected, has no cut vertex, and is larger than K. An edge yy' of H_0 that is not in K has both ends in one set $V[x]$: a path from x to y without an edge of K, together with this edge, shows that y' is in $V[x]$. So the edges of H_0 between the union of the $V[x]$ with x in V_1 and the union of the $V[x]$ with x in V_2 are exactly the edges of K between V_1 and V_2. There are at least three of them, because H_0 is 3-edge-connected.

(b) *Step 1.* Add to K a new vertex v and three new edges from v to vertices of K, not all three to the same vertex. Call the result K^+. It is loopless. It has no cut vertex: for a vertex u of K the graph $K - u$ is connected, and v has a neighbour other than u; so $K^+ - u$ is connected; and $K^+ - v = K$ is connected. It is 3-edge-connected: let $V(K^+)$ be the disjoint union of two non-empty sets; if both contain vertices of K, at least three edges of K join them; if not, one of them is $\{v\}$, and three edges join them. So every vertex of K^+ has degree at least 3. A neighbour of v has degree at least 3 in K, so it has degree at least 4 in K^+.

Step 2. For every vertex $u \neq v$ of K^+ of degree $j \geq 4$ take a cycle O_u of length j with new vertices and new edges. Replace u by O_u: the j edges at u end now at the j vertices of O_u, one edge at each vertex. Call the result K^c. It is cubic and loopless, the cycles O_u are pairwise disjoint, and the contraction of every O_u to one vertex gives K^+ back. Every vertex of O_u has exactly one edge that is not on O_u, and the other end of this edge is not on O_u, because K^+ is loopless.

Step 3: K^c is 3-edge-connected. Let $V(K^c)$ be the disjoint union of two non-empty sets V_1, V_2 such that at most two edges join them. First case: every cycle O_u is in one of the two sets. Then the contraction of the cycles gives two non-empty sets of vertices of K^+ with the same edges between them; this contradicts Step 1. Second case: some cycle O_u has vertices in both sets. Then at least two edges of O_u join the two sets. So exactly two edges join them, both are on O_u, and every other cycle $O_{u'}$ is in one of the two sets. A vertex of O_u in V_1 has a neighbour outside O_u, and this neighbour is in V_1; the same for V_2. Now contract all cycles other than O_u and delete O_u. The result is $K^+ - u$. Its vertex set is split into two non-empty sets, which come from V_1 and from V_2, with no edge between them. So $K^+ - u$ is not connected; this contradicts Step 1.

Step 4. By Step 1 the ends in K of the three edges at v have degree at least 4 in K^+. So in K^c the three edges at v end at three different vertices, and K^c has at least four vertices. Put $Q_K = K^c - v$. This is a truncated cubic 3-edge-connected graph. Its edges are the edges of K and the edges of the cycles O_u. The contraction of every O_u to one vertex gives $K^+ - v = K$. □

Threads and the auxiliary graph. In the rest of this section H is a subdivision as in Definition 2.1 (SUB), and A is an abelian group. (In the later sections A is the quotient Γ/B of a group Γ by a subgroup B.)

Definition 2.6 (Threads and the auxiliary graph)**.** For an edge t of H_0, the path P_t of H that replaces t is the *thread* of t, and $\ell(t) \in \{1, 2, 3\}$ is its number of edges. Every thread is directed as a path; its edges and the edge t have this direction as reference orientation. Let $K_1, \ldots, K_r$ be the blocks of H_0. For every block K_b let Q_b be a truncated cubic 3-edge-connected graph (a cubic 3-edge-connected graph on at least four vertices minus one vertex)

such that $E(K_b)$ is a subset of $E(Q_b)$ and K_b is obtained from Q_b by the contraction of the other edges of Q_b. Q is the disjoint union of the Q_b. The edges of H_0 are the *original* edges of Q; the other edges of Q are *expansion* edges, and $\ell(t) = 0$ for an expansion edge t. For an A-flow Y of Q, $\pi(Y)$ is the A-flow of H that has the value $Y(t)$ on every edge of P_t, for every original edge t.

The graphs Q_b exist by Lemma 2.5. We fix them, one for every block, with disjoint vertex sets. A vertex x of K_b is a set V_x of vertices of Q_b: a component of the graph formed by the expansion edges of Q_b. The map $\pi(Y)$ is defined in the same way for every abelian group in the place of A, also for $\mathbb{Z}$; Lemma 2.7 (c) says that it is a flow.

Lemma 2.7 (The flows of H, of H_0 and of Q)**.** *Let A be an abelian group, finite or not.*

(a) *Every cycle of H is a union of threads. If O is a cycle of H with a direction, then χ_O is constant, 1 or -1, on the edges of each thread of O. Every A-flow of H is constant on every thread. So the A-flows of H are the maps that have the value $h_0(t)$ on every edge of P_t, for an A-flow h_0 of H_0.*

(b) *The restriction of an A-flow of H_0 to a block K_b is an A-flow of K_b.*

(c) *For every A-flow Y of Q, the restriction of Y to the original edges is an A-flow of H_0, and $\pi(Y)$ is an A-flow of H. Every A-flow of H is $\pi(Y)$ for some A-flow Y of Q.*

(d) *Let C be a cycle of Q with a direction. Then there are pairwise edge-disjoint directed cycles $O_1, \dots, O_j$ of H ($j \geq 0$) such that $\pi(x\chi_C) = x\,(\chi_{O_1} + \cdots + \chi_{O_j})$ for every x in A.*

Proof. (a) An interior vertex of a thread has degree 2 in H, and the thread is directed as a path. So a cycle that contains one edge of a thread contains the whole thread and passes it in one direction; this gives the first two statements. And conservation at an interior vertex says that its two edges have the same value. For a map that is constant on every thread, conservation at a vertex of H_0 is the same condition in H and in H_0.

(b) By Lemma 1.1 (a) an A-flow of H_0 is a sum of flows $x\chi_C$ with cycles C of H_0. Every cycle lies in one block. So the restriction to K_b is the sum of the terms with C in K_b.

(c) Let x be a vertex of K_b. Add the conservation conditions of Y at the vertices of V_x. Every edge with both ends in V_x is in this sum with both signs. What is left is conservation at x for the restriction of Y to $E(K_b)$. So this restriction is an A-flow of K_b. At a vertex of H_0, conservation is the sum of the conservation conditions in the blocks that contain this vertex. So the restriction of Y to the original edges is an A-flow of H_0, and by (a) $\pi(Y)$ is an A-flow of H.

Now let an A-flow of H be given. By (a) and (b) it comes from an A-flow h_0 of H_0 whose restriction to every block is a flow of the block. Let x be a vertex of K_b. For a vertex y in V_x let $\nu(y)$ be the boundary at y of the map that is h_0 on the original edges and 0 on the expansion edges. The sum of $\nu(y)$ over y in V_x is the boundary of h_0 at x in K_b, which is 0. The expansion edges connect V_x; choose a spanning tree of V_x that consists of expansion edges. There are values on the edges of this tree whose boundary at every y in V_x is $-\nu(y)$. (Induction on $|V_x|$: for one vertex, $\nu = 0$. Otherwise let y be a vertex that meets only one edge yy' of the tree; the value on this edge is fixed by the condition at y; delete y and add $\nu(y)$ to $\nu(y')$.) Give all other expansion edges the value 0. Do this for every x and every b. The result is an A-flow Y of Q that is equal to h_0 on the original edges; so $\pi(Y)$ is the given flow.

(d) By (c), used for the group $\mathbb{Z}$, the map $\pi(\chi_C)$ is an integer flow of H. Its values are in $\{-1, 0, 1\}$. By Lemma 1.1 (b) it is $\chi_{O_1} + \cdots + \chi_{O_j}$ with pairwise edge-disjoint directed cycles of H. And $\pi(x\chi_C) = x\,\pi(\chi_C)$ edge by edge. □

Frames.

Definition 2.8 (2-joins and 2-constructions)**.** A 2-*join* of two graphs U' and U'' with disjoint vertex sets is a graph that consists of U', of U'' and of exactly two new edges, each with one end in U' and one end in U'' (the two new edges can be parallel). A graph is 2-*constructible* if it has one vertex and no edge, or if it is a 2-join of two 2-constructible graphs.

A 2-*construction* of a 2-constructible graph U_0 fixes one way to obtain U_0. If U_0 has more than one vertex, it names two graphs U_0', U_0'' of which U_0 is a 2-join, and it contains a 2-construction of U_0' and a 2-construction of U_0''. The *stages* of the 2-construction are U_0 and, if U_0 has more than one vertex, the stages of the 2-constructions of U_0' and of U_0''. So the stages are subgraphs of U_0, and the stages with one vertex are the vertices of U_0. A stage U with more than one vertex is a *node*; it stands for the 2-join that makes it. The two stages U', U'' that it joins are its *operands*, and its two new edges are the *edges of the node*.

Definition 2.9 (Frame)**.** A *frame* of H consists of the following, for every component Q_b of Q:

(F1) a *well colouring* of Q_b: every edge is blue or red, the blue edges form a spanning tree T_b of Q_b, and the graph Q_b/red (its vertices are the components of the red edges, its edges are the blue edges) is 2-constructible;

(F2) a 2-construction of Q_b/red from single vertices; its 2-joins are the *nodes*, and the two graphs that a node joins are its *operands*;

(F3) for every node i, names e_i and f_i for its two new edges. The set $N_i = \{e_i, f_i\}$ is the *pair* of the node i.

The nodes of all components are numbered $1, \dots, s$: component after component, and in every component every node after all nodes that are subgraphs of its two operands. T is the union of the trees T_b; its edge set is the disjoint union of the pairs $N_1, \dots, N_s$. Every truncated cubic 3-edge-connected graph has a well colouring (Lemma A.5), so every H has a frame. The names in (F3) are free: at every node the two names can be exchanged.

In (F1), "the components of the red edges" are the *red components*: the components of the graph with the vertex set $V(Q_b)$ and the red edges. The ends of a blue edge in Q_b/red are the red components of its two ends.

Notation for a frame. Let a frame of H be given. For a stage U of the 2-construction of Q_b/red, V_U is the union of the red components that are the vertices of U, and Q_U is the subgraph of Q_b induced on V_U, the *graph of* U. For a node i, U_i is its stage, and U_i', U_i'' are its operands. A node h is *below* the node i if $h \neq i$ and U_h is a subgraph of U_i; then U_h lies in U_i' or in U_i'' and is a stage of its 2-construction (two stages are disjoint or nested, Lemma A.2 (b), and $U_h \neq U_i$), and $h < i$ by the numbering of Definition 2.9. The following facts are proved in the appendix (Lemma A.2 (b), (c) and Lemma A.3).

(G1) The red edges of Q_b form a forest. No blue edge has both ends in one red component.

(G2) Every blue edge is an edge of exactly one node. So $E(T)$ is the disjoint union of the pairs.

(G3) The edges of Q_U are the red edges with both ends in V_U and the edges of the pairs N_h of the nodes h with U_h a subgraph of U. Q_U is connected.

(G4) For a node i: V_{U_i} is the disjoint union of $V_{U_i'}$ and $V_{U_i''}$; each of e_i, f_i has one end in $V_{U_i'}$ and one end in $V_{U_i''}$; and $E(Q_{U_i}) = E(Q_{U_i'}) \cup E(Q_{U_i''}) \cup \{e_i, f_i\}$.

Protected and free threads; the official cycles.

Definition 2.10 (Protected and free threads; official cycles)**.** Let a frame of H be given. The thread P_t is *protected* if t is an original blue edge, and *free* if t is an original red edge; then t is a *free edge*. H_P is the union of the protected threads. For a free edge ε, D_ε is a fixed cycle of H that contains P_ε and no other free thread, directed along P_ε. T^e is the subgraph of Q formed by all red edges and the edges $e_1, \dots, e_s$. The *official cycle* C_i is the fundamental cycle of f_i in T^e, directed along f_i.

Lemma 2.11 (Official cycles)**.** *For every frame of H:*

(O1) *The red edges of Q form a forest. For every choice of one edge g_i in every pair N_i, the red edges and the edges $g_1, \dots, g_s$ form a spanning tree T^g of every component Q_b, and of the graph of every stage. In particular T^e is such a tree.*

(O2) *C_i contains e_i and f_i.*

(O3) *If $l < i$, then C_l contains no edge of N_i.*

(O4) *If $l > i$, then C_l does not contain f_i. (It can contain e_i.)*

(O5) *For every abelian group A, every A-flow of Q is a sum $x_1\chi_{C_1} + \dots + x_s\chi_{C_s}$ with x_i in A, in exactly one way.*

(O6) *H_P is connected and contains every vertex of H that is not an interior vertex of a free thread; and the cycle D_ε exists for every free edge ε.*

In (O1), "T^g is a spanning tree of the graph Q_U of a stage U" means: the edges of T^g in Q_U form a spanning tree of Q_U. This holds for every stage, also for the stages with one vertex.

Proof. (O1) The red edges form a forest by (G1). Let one edge g_i in every pair N_i be chosen. For a stage U let T^g_U be the graph with the vertex set V_U whose edges are the red edges with both ends in V_U and the edges g_h of the nodes h with U_h a subgraph of U. We show by induction on the 2-construction that T^g_U is a spanning tree of Q_U. If U has one vertex, then V_U is one red component; it is connected by its red edges, and they form a forest; so it is a tree. Let $U = U_i$ be a node. By induction $T^g_{U_i'}$ and $T^g_{U_i''}$ are trees; by (G4) their vertex sets are disjoint with the union V_{U_i}, and g_i has one end in each of them. A red edge with both ends in V_{U_i} has both ends in one red component, so in $V_{U_i'}$ or in $V_{U_i''}$. A node h with U_h a subgraph of U_i is i, or U_h is a subgraph of U_i' or of U_i''. So $T^g_{U_i}$ is the union of the two trees and the edge g_i: a spanning tree of Q_{U_i}. For $U = Q_b$/red this is the statement on Q_b.

Consequence. For every stage U and two vertices of V_U, the path of T^g between them lies in Q_U: it is the path of the tree T^g_U.

(O2) By (G2) the edges of Q outside T^e are $f_1, \dots, f_s$. So C_i consists of f_i and of the path of T^e between the ends of f_i. Both ends are in V_{U_i}, so by the consequence of (O1) this path lies in Q_{U_i}. By (G4) one end of f_i is in $V_{U_i'}$ and the other one is in $V_{U_i''}$, and the only edges of Q_{U_i} between these two sets are e_i and f_i. The path does not contain f_i. So it

contains e_i. We note the result as $(*)$: *C_i is a cycle of Q_{U_i}; it consists of f_i, of e_i, of a path of T^e in $Q_{U_i'}$ and of a path of T^e in $Q_{U_i''}$.*

(O3) Let $l < i$. By $(*)$, C_l is a cycle of Q_{U_l}. By (G3) the blue edges of Q_{U_l} are the edges of the pairs N_h with $h = l$ or h below l; so $h \leq l < i$. The pairs are disjoint. So no edge of N_i is an edge of Q_{U_l}.

(O4) All edges of C_l other than f_l are in T^e. The edge f_i is not in T^e, and $f_i \neq f_l$.

(O5) By (O1), T^e is a spanning tree in every component of Q. The edges outside T^e are $f_1, \dots, f_s$, and C_i is the fundamental cycle of f_i, directed along f_i. This is Lemma 1.1 (a); the coefficients are $x_i = Y(f_i)$ for the flow Y.

(O6) T_b is a spanning tree of Q_b. The contraction of the expansion edges maps it onto a connected spanning subgraph of K_b; its edges are the original edges of T_b, that is the original blue edges of Q_b. The graph H_0 is connected, and every edge of H_0 is in a block. So the original blue edges of Q form a connected spanning subgraph of H_0: two vertices of H_0 are joined by a path of H_0, and the two ends of every edge of this path are joined by original blue edges in the block of this edge. Replace the edges by their threads: H_P is connected, and it contains all vertices of H_0 and the interior vertices of the protected threads. The other vertices of H are interior vertices of free threads.

Let ε be a free edge, in the block K_b. It is an original edge of Q_b that is not in T_b. Its fundamental cycle in T_b is mapped by the contraction of the expansion edges to a closed trail of K_b that contains ε once and, apart from ε, original blue edges only. The edges of a closed trail form a graph in which every vertex has an even degree; so every edge of it lies on a cycle of this graph. Take such a cycle through ε. K_b is loopless, so this cycle has more edges than ε, and they are original blue edges. Replace the edges of this cycle by their threads. This gives a cycle of H that contains P_ε and, apart from it, protected threads only. □

By (O2), (O3) and by $(*)$ in the proof above, the official cycles are a closure sequence in the sense of Lemma A.6: every edge of C_i other than e_i and f_i is a red edge or an edge of a pair N_h with $h < i$, and no edge of N_i is on a cycle C_l with $l < i$.

Definition 2.12 (The two cycles of a node)**.** Let a frame of H be given. T^f is the tree T^g of (O1) for the choice $g_i = f_i$ for every i: all red edges and the edges $f_1, \dots, f_s$. For a node i put $C_i^e = C_i$, and let C_i^f be the fundamental cycle of e_i in T^f, with the direction in which it passes f_i forward (by Lemma 2.13 (a) it contains f_i).

Lemma 2.13 (The two cycles of a node)**.** *Let a frame of H be given and let i be a node.*

(a) *C_i^f is a cycle of Q_{U_i}. It consists of e_i, of f_i, of a path of T^f in $Q_{U_i'}$ and of a path of T^f in $Q_{U_i''}$. The cycles C_i^e and C_i^f pass f_i forward, and they pass e_i in the same direction. So $\chi_{C_i^e}$ and $\chi_{C_i^f}$ are equal on N_i.*

(b) *Let $h \neq i$ be a node. If h is below i, then C_i^e does not contain f_h, and C_i^f does not contain e_h. If h is not below i, then no edge of N_h is on C_i^e or on C_i^f; this is so for every $h > i$.*

(c) *$\Delta_i = \chi_{C_i^f} - \chi_{C_i^e}$ is an integer flow of Q. It is 0 on N_i and outside Q_{U_i}. So it is an integer flow of the graph $Q_{U_i'} \cup Q_{U_i''}$, whose edges are red edges and edges of pairs N_h with h below i. In particular Δ_i is 0 on $N_i \cup N_{i+1} \cup \dots \cup N_s$. For every abelian group A and every x in A, the A-flow $x\Delta_i$ is a sum of flows $x'\chi_C$ with x' in A and C a cycle of $Q_{U_i'} \cup Q_{U_i''}$.*

Proof. (a) The proof of (O2), with the tree T^f in the place of T^e and with the names e_i and f_i exchanged, gives the first two sentences. By $(*)$ in the proof of Lemma 2.11, the cycle C_i^e has the same form, with paths of T^e. Let both cycles pass f_i forward, say from its end in $V_{U_i'}$ to its end in $V_{U_i''}$ (the other case is the same). After f_i each of the two cycles stays in $Q_{U_i''}$ until it passes e_i, from the end of e_i in $V_{U_i''}$ to its end in $V_{U_i'}$; then it goes back in $Q_{U_i'}$. So both pass e_i in the same direction.

(b) Let h be below i. All edges of C_i^e other than f_i are in T^e, and f_h is not in T^e. All edges of C_i^f other than e_i are in T^f, and e_h is not in T^f. Let $h \neq i$ be not below i. Both cycles are cycles of Q_{U_i}, and by (G3) the pair N_h has no edge in Q_{U_i}. A node $h > i$ is not below i.

(c) Δ_i is a difference of two integer flows. It is 0 on N_i by (a), and outside Q_{U_i} because both cycles are in Q_{U_i}. By (G4) it is therefore 0 outside $Q_{U_i'} \cup Q_{U_i''}$; so it is an integer flow of this graph. By (G3) the edges of this graph are red edges and edges of pairs N_h with U_h a subgraph of U_i' or of U_i'', that is with h below i. The last sentence is Lemma 1.1 (a) for the A-flow $x\Delta_i$ of the graph $Q_{U_i'} \cup Q_{U_i''}$. □

Lists and counts. In this subsection Γ is a group, B is a subgroup, $A = \Gamma/B$, and $x \mapsto \overline{x}$ is the quotient map. The *A-coordinate* of a Γ-flow f is the A-flow u with the values $u(e) = \overline{f(e)}$. The *A-part* of an element x of Γ is $\overline{x}$.

Definition 2.14 (Lists; H-good flows)**.** Let a frame of H be given. A *list assignment* Λ gives every blue edge t of Q a set $\Lambda(t)$ of elements of A, with $\Lambda(t)$ empty for every expansion edge. $P(\Lambda)$ is the set of the A-flows Y of Q with $Y(t) \notin \Lambda(t)$ for every blue edge t.

Let G be a multigraph that contains H, and f a nowhere-zero Γ-flow of G with A-coordinate u. In the case (SUB), f is *H-good* if $u(e) \neq 0$ for every edge e of every protected thread; then

$$\Lambda_f(t) = \{\, -u(e) : e \in E(P_t) \,\} \qquad \text{for every original blue edge } t.$$

So $|\Lambda_f(t)| \leq \ell(t)$, 0 is in $P(\Lambda_f)$, and for an A-flow Y of Q: Y is in $P(\Lambda_f)$ if and only if $u + \pi(Y)$ is non-zero on all protected threads. ($u + \pi(Y)$ means: add $\pi(Y)$ on $E(H)$ and nothing outside $E(H)$.) In the case (CYC), f is *H-good* if $u(e) \neq 0$ for every edge e of H.

The three claims after the display are clear: the thread P_t has $\ell(t)$ edges; no element $-u(e)$ of a list is 0; and $Y(t) \neq -u(e)$ for all edges e of P_t says that $u + \pi(Y)$ is not 0 on P_t.

Definition 2.15 (The counts; counted frame)**.** Let a frame of H be given. The *counts* for A and B are the following inequalities.

(C1) $2\,\ell(e_i) + \ell(f_i) < |A|$ for every node i (the walk lemma).

(C2) $\ell(e_i) + \ell(f_i) + \max\{\ell(e_i), \ell(f_i)\} < |A|$ for every node i (the reverse choice for a step with non-zero A-part).

(C3) $\ell(e_i) + \ell(f_i) + 2 < |A|$ for every node i (the reverse choice for a step in B).

(C4) $\ell(\varepsilon) < |B|$ for every free edge ε (the preparation in B).

The frame is *counted* (for A and B) if (C1)–(C4) hold. Two conditions on a frame that do not use the groups:

(S) no pair N_i consists of two original edges whose threads both have 3 edges;

(NR) no free thread has 3 edges (every original edge whose thread has 3 edges is blue).

Remark 2.16. The left sides of (C1), (C2), (C3), (C4) are at most 9, 9, 8, 3. So every frame is counted if $|A| \geq 10$ and $|B| \geq 4$. If the names are chosen with $\ell(e_i) \leq \ell(f_i)$ for every

i, then (C2) implies (C1). With these names: a frame with (S) is counted if $|A| \geq 9$ and $|B| \geq 4$ (left sides at most 7, 8, 7, 3); a frame with (NR) is counted if $|A| \geq 10$ and $|B| \geq 3$ (the left side of (C4) is at most 2).

3. Step systems and the master theorem

This section proves the master theorem in its two forms (Theorems 3.11 and 3.12), three corollaries, and the integer form (Theorem 3.19). All proofs are complete here. What is taken from other sections is cited at the place of use, with the property that is used: liftable steps (Definition 1.7, Lemma 1.12); good subgraphs (Definition 2.1) and the extraction lemma (Lemma 2.2); threads, the auxiliary graph, frames, official cycles, lists and counts (Definitions 2.6, 2.9, 2.10, 2.14, 2.15 and Lemma 2.11). The averaging conditions (AV) and (AV″) are hypotheses in this section; Lemma 4.1 proves them for products.

Two rules hold in every proof of this section.

(a) Every use of $|A|$ or of $|B|$, other than in the hypotheses of Lemmas 3.4 and 3.5, names the count that it uses: (C0) of Theorem 3.11, or one of (C1)–(C4) of Definition 2.15; every use of the bound $\ell(t) \leq 3$ names (i). Remark 3.14 lists the places.

(b) (R1) with six deleted elements, the count (C1) and the walk lemma (Lemma 3.5) are used only in the proof of Lemma 3.8. Every later proof uses Lemma 3.8 only through its conclusion. This is why the frame form of the theorem holds.

Step systems. Throughout, Γ is a finite abelian group, B a subgroup, $A = \Gamma/B$, and $x \mapsto \overline{x}$ is the quotient map. For a Γ-flow f of a multigraph the A-flow $u = \overline{f}$ (edge by edge) is the *A-coordinate* of f.

Definition 3.1 (Step system)**.** A *step system* is a set Σ of non-zero elements of Γ with $\Sigma = -\Sigma$. Put

$$\Sigma_B = \Sigma \cap B, \qquad \Sigma_A = \{\, \overline{\lambda} : \lambda \in \Sigma \,\} \setminus \{0\},$$

and for every s in Σ_A fix one $\gamma(s)$ in Σ with $\overline{\gamma(s)} = s$. A *Σ-move* of a multigraph G is a move $f \to f + \lambda\chi_C$ of $F(G,\Gamma)$ with λ in Σ. Two flows are *Σ-joined* if a sequence of Σ-moves leads from one to the other.

So a Σ-move leads from a nowhere-zero flow to a nowhere-zero flow. The reverse of a Σ-move is a Σ-move, because $\Sigma = -\Sigma$. A flow is Σ-joined to itself (the empty sequence).

Definition 3.2 (The conditions on a step system)**.** (R1) For every set X of at most 6 elements of A, any two elements of $A \setminus X$ are joined by a walk in $A \setminus X$ all of whose steps are in Σ_A.

(R1$^-$) The same for every set X of at most 5 elements of A.

(R2) For every set X of at most 2 elements of B, any two elements of $B \setminus X$ are joined by a walk in $B \setminus X$ all of whose steps are in Σ_B.

(R3*) Any two non-zero elements of Γ are joined by a walk in $\Gamma \setminus \{0\}$ all of whose steps are in Σ.

(AV) For every loopless 3-edge-connected multigraph G with $n \geq 2$ vertices and m edges, every nowhere-zero Γ-flow of G is Σ-joined to a nowhere-zero Γ-flow whose A-coordinate has at most $m/9$ zeros.

(AV″) For every loopless 3-edge-connected multigraph G with $n \geq 2$ vertices, every nowhere-zero Γ-flow of G is Σ-joined to a nowhere-zero Γ-flow f' such that the multigraph $(V(G), \mathrm{supp}(\overline{f'}))$ contains a good subgraph.

A *step* of a walk is the difference of two consecutive terms. (R1) implies (R1$^-$). (R1) or (R1$^-$) with X empty says: every element of A is a sum of elements of Σ_A. (R2) with X empty says: every element of B is a sum of elements of Σ_B.

Five elementary facts. Here G is a multigraph, f a Γ-flow of G and $u = \overline{f}$. An *integer flow* of G is a map $E(G) \to \mathbb{Z}$ with conservation at every vertex.

(E1) $f(e) \neq 0$ if and only if $u(e) \neq 0$, or $u(e) = 0$ and $f(e)$ is an element of B other than 0. (If $u(e) = 0$, then $f(e)$ is in B.)

(E2) If y is in B and C is a cycle of G, then $f + y\chi_C$ has the A-coordinate u.

(E3) *Cycle decomposition* (Lemma 1.1). Every flow of G with values in an abelian group is a sum of flows $x\chi_C$, where C is a cycle of G and x is in the group. Every integer flow of G with values in $\{0, 1, -1\}$ is a sum $\chi_{O_1} + \cdots + \chi_{O_\nu}$ with pairwise edge-disjoint cycles $O_1, \ldots, O_\nu$ of G ($\nu = 0$ for the zero flow).

(E4) Every A-flow of G is the A-coordinate of some Γ-flow of G.

(E5) *Contraction.* Let G' be obtained from G in this way: the vertices of each class of a partition of $V(G)$ are identified to one vertex, and some edges that have both ends in one class are deleted. Then the restriction of a flow of G to $E(G')$ is a flow of G' (values in any abelian group).

Proofs. (E1) and (E2) hold because B is the kernel of the quotient map.

(E3) The first statement is Lemma 1.1 (a). The second one is Lemma 1.1 (b): for an integer flow with values in $\{0, 1, -1\}$ the cycles of the signed form are pairwise edge-disjoint.

(E4) By (E3) the A-flow is a sum $\sum_r a_r \chi_{C_r}$. Choose x_r in Γ with $\overline{x_r} = a_r$. The Γ-flow $\sum_r x_r \chi_{C_r}$ has this A-coordinate.

(E5) Add the conservation equations of G over the vertices of one class. An edge with both ends in the class enters the class once and leaves it once, so it drops out. What is left is the conservation equation of G' at the vertex made from the class. □

Lemma 3.3. (AV) *implies* (AV″). (*A loopless multigraph with $n \geq 3$ vertices and more than $(4/3)\,n - 2$ edges contains a good subgraph; for $n = 2$ the support has two parallel edges.*)

Proof. Let G be loopless and 3-edge-connected with $n \geq 2$ vertices and m edges. Every vertex has degree at least 3, so $m \geq 3n/2$. Let f' be the flow that (AV) gives, with A-coordinate u'. Then

$$|\,\mathrm{supp}(u')| \;\geq\; m - m/9 \;\geq\; (8/9)(3/2)\,n \;=\; (4/3)\,n \;>\; (4/3)\,n - 2.$$

Let $n \geq 3$. The multigraph $(V(G), \mathrm{supp}(u'))$ is loopless and has n vertices and more than $(4/3)\,n - 2$ edges. By Lemma 2.2 (a) it contains a cycle with at most 5 edges or a subdivision of a loopless 3-edge-connected multigraph on at least two vertices in which every edge is replaced by a path with at most 3 edges. The cycle has at least 2 vertices, because the multigraph is loopless. This is a good subgraph (Definition 2.1). Let $n = 2$. Then $m \geq 3$, so $\mathrm{supp}(u')$ has at least $8m/9 > 2$ edges. G is loopless, so two of them are parallel. They form a cycle with 2 edges and 2 vertices, a good subgraph of the kind (CYC). □

The subdivision: what is used. From here to the end of the proof of Lemma 3.9, H is a subdivision as in Definition 2.1 (SUB), with the graphs H_0, K_b, Q_b, Q of Definition 2.6,

and a frame of H is given (Definition 2.9). The proofs use the following properties of these objects, and no other property.

(i) *Threads* (Definition 2.6). Every edge of H lies on exactly one thread P_t, where t is an edge of H_0, and $1 \leq \ell(t) \leq 3$. The interior vertices of a thread have degree 2 in H; the other vertices of H are the vertices of H_0. H_0 is connected and loopless, every edge of H_0 lies in exactly one block K_b, and every cycle of H_0 lies in one block.

(ii) *The auxiliary graph* (Definition 2.6). $E(K_b)$ is a subset of $E(Q_b)$, and K_b is obtained from Q_b by the contraction of the expansion edges: the vertices of K_b are the components of the subgraph of Q_b formed by all vertices and the expansion edges, and an original edge joins the components of its two ends. $\ell(t) = 0$ for an expansion edge t.

(iii) *The frame* (Definition 2.9). The blue edges of Q_b form a spanning tree T_b of Q_b. Every blue edge of Q lies in exactly one pair N_i. A *stage* is a graph U of the 2-construction (Definition 2.8); its vertices are components of the red edges. V_U is the union of these components, and Q_U is the subgraph of Q induced on V_U. If U' and U'' are the two operands of the node i, then $V_{U'}$ and $V_{U''}$ are disjoint, and each of the edges e_i, f_i has one end in $V_{U'}$ and the other end in $V_{U''}$ (a 2-join adds two edges, each with one end in each of the two graphs; Lemma A.3 (c)). The names e_i, f_i can be exchanged at every node; the result is a frame again.

(iv) *Official cycles* (Definition 2.10, Lemma 2.11). The statements (O1)–(O6). Of (O1) we use, with Q_U as in the notation for a frame after Definition 2.9 and the paragraph after Lemma 2.11: for every choice of one edge g_i in every pair, T^g is a spanning tree of every component Q_b; and for every stage U, the edges of T^e in Q_U form a spanning tree of Q_U (the proof of (O1) shows this for every stage, by induction on the 2-construction, from Lemmas A.2 and A.3). The blue edges of T^g are the edges g_i. The cycle D_ε contains P_ε and no other free thread, and it is directed along P_ε.

(v) *Lists and counts* (Definitions 2.14, 2.15). A list assignment is empty on every expansion edge. An H-good flow is nowhere zero. For an H-good flow f with A-coordinate u: $|\Lambda_f(t)| \leq \ell(t)$; 0 is in $P(\Lambda_f)$; and an A-flow Y of Q is in $P(\Lambda_f)$ if and only if $u + \pi(Y)$ is non-zero on every edge of every protected thread. Here and below, $u + \pi(Y)$ means: add $\pi(Y)$ on $E(H)$ and nothing outside $E(H)$.

Whenever H is a subgraph of a multigraph G, the reference orientation of G on $E(H)$ is the direction of the threads. This is no restriction: a change of the reference orientation of an edge changes the sign of all flow values on this edge and nothing else. For a flow Y of Q, with values in A or in $\mathbb{Z}$, $\pi(Y)$ has the value $Y(t)$ on every edge of P_t, for every original edge t.

Lemma 2.7 describes the flows of H and of Q. Below we use its parts (a), (c) and (d): every cycle of H is a union of threads, and for a cycle O of H with a direction, χ_O is constant, 1 or -1, on the edges of each thread of O; every flow of H is constant on the edges of each thread; $\pi(Y)$ is an A-flow of H for every A-flow Y of Q, and every A-flow of H is $\pi(Y)$ for some A-flow Y of Q; and for every cycle C of Q with a direction there are pairwise edge-disjoint directed cycles $O_1, \dots, O_\nu$ of H ($\nu = 0$ is possible) with $\pi(x\chi_C) = x\,(\chi_{O_1} + \cdots + \chi_{O_\nu})$ for every x in A.

The reverse choice and the walk lemma.

Lemma 3.4 (Reverse choice)**.** *Let Λ be a list assignment.*

(a) *If* $|\Lambda(e_i)| + |\Lambda(f_i)| < |A|$ *for every node* i*, then for every* A*-flow* Y_0 *of* Q *there are* $x_1, \dots, x_s$ *in* A *with* $Y_0 + x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$ *in* $P(\Lambda)$.

(b) *Let* E' *be a set of red edges of* Q*, and for every* ε *in* E' *let* $\Phi(\varepsilon)$ *be a set of elements of* A*, not all of* A*. Let* $i(\varepsilon)$ *be the least* i *with* ε *on* C_i*, if there is one. If for every node* i

$$|\Lambda(e_i)| + |\Lambda(f_i)| + \sum_{\varepsilon\in E',\ i(\varepsilon)=i} |\Phi(\varepsilon)| \ < \ |A|,$$

then there is Y *in* $P(\Lambda)$ *with* $Y(\varepsilon) \notin \Phi(\varepsilon)$ *for every* ε *in* E'.

Proof. Every blue edge of Q lies in exactly one pair (iii). By (O2) the cycle C_i contains e_i and f_i; so χ_{C_i} has the value 1 or -1 on each of them. By (O3) no cycle C_l with $l < i$ contains an edge of N_i.

(a) Choose $x_s, x_{s-1}, \dots, x_1$ in this order. Let $x_s, \dots, x_{i+1}$ be chosen, and put $Z_i = Y_0 + \sum_{l>i} x_l\chi_{C_l}$. For an edge t of N_i the flow $Y = Y_0 + x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$ has the value

$$Y(t) = Z_i(t) + x_i\,\chi_{C_i}(t),$$

whatever $x_{i-1}, \dots, x_1$ are (O3). Since $\chi_{C_i}(t)$ is 1 or -1, the condition $Y(t) \notin \Lambda(t)$ forbids exactly $|\Lambda(t)|$ values of x_i. The two edges of N_i together forbid at most $|\Lambda(e_i)| + |\Lambda(f_i)| < |A|$ values. So x_i can be chosen. In the end $Y(t) \notin \Lambda(t)$ for every blue edge t.

(b) Let E'_0 be the set of the edges of E' that lie on no official cycle. For ε in E'_0 choose y_ε in A outside $\Phi(\varepsilon)$. The blue edges form a spanning tree of the component of Q that contains ε (iii). Let F_ε be the cycle formed by ε and the path of this tree between the ends of ε, directed along ε. Its only red edge is ε. Put $Y_0 = \sum_{\varepsilon\in E'_0} y_\varepsilon\chi_{F_\varepsilon}$; so $Y_0(\varepsilon) = y_\varepsilon$ for every ε in E'_0. Choose $x_s, \dots, x_1$ as in (a), for the flow $Y = Y_0 + x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$, with one more condition on x_i: $Y(\varepsilon) \notin \Phi(\varepsilon)$ for every ε in E' with $i(\varepsilon) = i$. For such an ε the value $Y(\varepsilon) = Z_i(\varepsilon) + x_i\,\chi_{C_i}(\varepsilon)$ does not depend on $x_{i-1}, \dots, x_1$, because no cycle C_l with $l < i(\varepsilon)$ contains ε. And $\chi_{C_i}(\varepsilon)$ is 1 or -1, so the condition forbids $|\Phi(\varepsilon)|$ values of x_i. By the hypothesis fewer than $|A|$ values of x_i are forbidden in all. So x_i can be chosen. The flow Y is in $P(\Lambda)$; $Y(\varepsilon) \notin \Phi(\varepsilon)$ for every ε in E' that lies on an official cycle; and $Y(\varepsilon) = Y_0(\varepsilon) = y_\varepsilon \notin \Phi(\varepsilon)$ for ε in E'_0, because no official cycle contains ε. □

Lemma 3.5 (Walk lemma)**.** *Assume* (R1)*. Let* Λ *be a list assignment such that for every node* i

$$|\Lambda(e_i)| + |\Lambda(f_i)| \le 6 \qquad \textit{and} \qquad 2\,|\Lambda(e_i)| + |\Lambda(f_i)| < |A|.$$

Then any two Y*,* Y' *in* $P(\Lambda)$ *are joined by a sequence* $Y = Y_0, Y_1, \dots, Y_n = Y'$ *in* $P(\Lambda)$ *in which every difference* $Y_r - Y_{r-1}$ *is* $\sigma\chi_{C_l}$ *with* σ *in* Σ_A *and an official cycle* C_l*. In particular this holds for* $\Lambda = \Lambda_f$*, for every* H*-good flow* f *(Definition 2.14), if the frame has* (C1).

Proof. A *good sequence* is a sequence of A-flows of Q in which every difference of two consecutive terms is $\sigma\chi_{C_l}$ with σ in Σ_A and an official cycle C_l. For $i = 1, \dots, s+1$ let Λ_i be the list assignment that is equal to Λ on the edges of $N_i \cup \cdots \cup N_s$ and empty on all other blue edges. So $\Lambda_1 = \Lambda$ (iii), and $P(\Lambda_{s+1})$ is the set of all A-flows of Q. Claim(i): any two members of $P(\Lambda_i)$ are joined by a good sequence in $P(\Lambda_i)$. Claim(1) is the lemma.

(1) *Claim*$(s+1)$. Let Y, Y' be A-flows of Q. By (O5), $Y' - Y = x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$ with x_i in A. By (R1) with X empty every x_i is a sum of elements of Σ_A. Add these terms one by one, cycle after cycle. This is a good sequence from Y to Y'.

(2) *Notation for the step from Claim$(i+1)$ to Claim(i)*, $1 \le i \le s$. Write $\chi = \chi_{C_i}$. By (O2), $\chi(e_i)$ and $\chi(f_i)$ are 1 or -1. For an A-flow Z of Q put

$$X_i^e(Z) = \{\, c \in A : Z(e_i) + c\,\chi(e_i) \in \Lambda(e_i) \,\}, \qquad X_i^f(Z) = \{\, c \in A : Z(f_i) + c\,\chi(f_i) \in \Lambda(f_i) \,\},$$

and $X_i(Z) = X_i^e(Z) \cup X_i^f(Z)$. Then $|X_i^e(Z)| = |\Lambda(e_i)|$ and $|X_i^f(Z)| = |\Lambda(f_i)|$; $X_i^e(Z)$ depends only on $Z(e_i)$, and $X_i^f(Z)$ only on $Z(f_i)$; and

$$|X_i(Z)| \;\le\; |\Lambda(e_i)| + |\Lambda(f_i)| \;\le\; 6.$$

By (O3) the cycle C_i contains no edge of $N_{i+1} \cup \cdots \cup N_s$. So for Z in $P(\Lambda_{i+1})$ and c in A:

$$(*) \qquad Z + c\chi \in P(\Lambda_{i+1}); \qquad \text{and} \qquad Z + c\chi \in P(\Lambda_i) \;\text{ if and only if }\; c \notin X_i(Z).$$

(3) *The step.* Assume Claim$(i+1)$, and let Y, Y' be in $P(\Lambda_i)$. They are in $P(\Lambda_{i+1})$. Take a good sequence $Y = Y_0, \dots, Y_n = Y'$ in $P(\Lambda_{i+1})$, with $Y_r - Y_{r-1} = \sigma_r \chi_{C_{l_r}}$. We build a good sequence in $P(\Lambda_i)$ that starts with Y and passes through flows $Y_r + c_r\chi$, $r = 0, 1, \dots, n$, with $c_0 = 0$ and $c_r \notin X_i(Y_r)$. For $r = 0$: $0 \notin X_i(Y_0)$ by $(*)$, because Y is in $P(\Lambda_i)$. Let $r \ge 1$, and let the sequence be built up to $Y_{r-1} + c_{r-1}\chi$.

(3a) $l_r = i$. Then $Y_{r-1} + c_{r-1}\chi = Y_r + (c_{r-1} - \sigma_r)\chi$. Put $c_r = c_{r-1} - \sigma_r$ and add no term. This flow is in $P(\Lambda_i)$, so $c_r \notin X_i(Y_r)$ by $(*)$. No count is used.

(3b) $l_r < i$. By (O3), C_{l_r} contains no edge of N_i. So $Y_r = Y_{r-1}$ on N_i and $X_i(Y_r) = X_i(Y_{r-1})$. Choose c_r outside this set. This is possible: $|\Lambda(e_i)| + |\Lambda(f_i)| \le 2\,|\Lambda(e_i)| + |\Lambda(f_i)| < |A|$, by the second hypothesis of the lemma.

(3c) $l_r > i$. By (O4), C_{l_r} does not contain f_i. So $X_i^f(Y_r) = X_i^f(Y_{r-1})$, and

$$|X_i(Y_{r-1}) \cup X_i(Y_r)| \;\le\; |X_i^e(Y_{r-1})| + |X_i^e(Y_r)| + |X_i^f(Y_{r-1})| \;=\; 2\,|\Lambda(e_i)| + |\Lambda(f_i)| \;<\; |A|.$$

The last inequality is the second hypothesis of the lemma; for $\Lambda = \Lambda_f$ it is the count (C1), see (5). Choose c_r outside $X_i(Y_{r-1}) \cup X_i(Y_r)$.

In (3b) and (3c): c_{r-1} and c_r are outside $X_i(Y_{r-1})$, a set of at most 6 elements by the first hypothesis of the lemma. By (R1) there is a walk from c_{r-1} to c_r in $A \setminus X_i(Y_{r-1})$ with steps in Σ_A. Add the flows $Y_{r-1} + c\chi$ for the terms c of this walk after the first term. They are in $P(\Lambda_i)$ by $(*)$, and two consecutive ones differ by $\sigma\chi_{C_i}$ with σ in Σ_A. Then add $Y_r + c_r\chi$. It is in $P(\Lambda_i)$ by $(*)$, because $c_r \notin X_i(Y_r)$; it differs from the term before it, $Y_{r-1} + c_r\chi$, by $\sigma_r\chi_{C_{l_r}}$.

(4) *The end of the step.* The sequence has reached $Y_n + c_n\chi$ with $c_n \notin X_i(Y_n)$. Also $0 \notin X_i(Y_n)$ by $(*)$, because $Y_n = Y'$ is in $P(\Lambda_i)$. By (R1) there is a walk from c_n to 0 in $A \setminus X_i(Y_n)$ with steps in Σ_A. Add the flows $Y_n + c\chi$ for its terms after the first term. The last one is Y'. This proves Claim(i).

(5) *The case* $\Lambda = \Lambda_f$. $|\Lambda_f(t)| \le \ell(t) \le 3$ for every blue edge t (v), (i). So $|\Lambda_f(e_i)| + |\Lambda_f(f_i)| \le 6$, and $2\,|\Lambda_f(e_i)| + |\Lambda_f(f_i)| \le 2\,\ell(e_i) + \ell(f_i) < |A|$ by (C1). □

A change of the A-coordinate inside H. Lemma 3.8 is put together from three parts: the walk lemma, which gives a sequence of steps $\sigma\chi_{C_l}$; Lemma 3.7, which makes one such step in G; and Lemma 3.6, which prepares the free threads for the step. Lemmas 3.6 and 3.7 do not use (R1), (C1) or the walk lemma, and Lemma 3.7 holds for every cycle C of Q.

Lemma 3.6 (Preparation in B)**.** *Assume* (R2)*, and let the frame have* (C4)*. Let G be a multigraph that contains H, and f an H-good Γ-flow of G with A-coordinate u. Let E° be a set of free edges. For every ε in E° put*

$$W_\varepsilon = \{\, -f(e) : e \in E(P_\varepsilon),\ u(e) = 0 \,\},$$

and let T_ε be a set of at most $\ell(\varepsilon)$ elements of B that contains W_ε. Then f is Σ-joined, by Σ-moves $y\chi_{D_\varepsilon}$ with y in Σ_B and ε in E°, to an H-good flow f° with A-coordinate u such that for every ε in E°

$$f^\circ = f + y_\varepsilon \quad \text{on } E(P_\varepsilon), \qquad \text{with an element } y_\varepsilon \text{ of } B \text{ outside } T_\varepsilon.$$

f° is equal to f outside $E(H)$, and on every free thread P_ε with ε not in E°.

Proof. W_ε is a subset of B by (E1), and 0 is not in W_ε, because f is nowhere zero.

(1) *One free edge ε.* If 0 is not in T_ε, put $y_\varepsilon = 0$ and make no move. Let 0 be in T_ε. By (C4), $|T_\varepsilon| \le \ell(\varepsilon) < |B|$. So there is y_ε in B outside T_ε. 0 is in T_ε and not in W_ε, so $|W_\varepsilon| \le |T_\varepsilon| - 1 \le \ell(\varepsilon) - 1 \le 2$; the last step holds because a thread has at most 3 edges (i). By (R2) there is a walk $0 = z_0, z_1, \dots, z_\nu = y_\varepsilon$ in $B \setminus W_\varepsilon$ with steps in Σ_B. Make the moves $(z_r - z_{r-1})\chi_{D_\varepsilon}$, $r = 1, \dots, \nu$. After the r-th move the flow is $f + z_r\chi_{D_\varepsilon}$. Its A-coordinate is u (E2). It is nowhere zero. Indeed, D_ε is a union of threads (Lemma 2.7 (a)); it contains P_ε and no other free thread (iv); so its edges outside P_ε lie on protected threads. An edge e of a protected thread has $u(e) \neq 0$, because f is H-good. An edge e of P_ε with $u(e) = 0$ has the value $f(e) + z_r$, because D_ε is directed along P_ε; this is not 0, because z_r is not in W_ε. The other edges of P_ε have $u(e) \neq 0$. The edges outside D_ε do not change. With (E1): every move is a Σ-move. The result $f + y_\varepsilon\chi_{D_\varepsilon}$ is H-good, it is equal to $f + y_\varepsilon$ on $E(P_\varepsilon)$, and it is equal to f outside $E(D_\varepsilon)$: on every other free thread and outside $E(H)$.

(2) *All edges of E°, one after the other.* Step (1) for ε does not change the A-coordinate, and it changes no value on a free thread other than P_ε. So when the turn of ε comes, the flow at hand is H-good with A-coordinate u, it is still equal to f on P_ε, its set W_ε is the set W_ε of the statement, and Step (1) can be made with the same T_ε. Later steps do not change the values on P_ε. □

Lemma 3.7 (One step inside H)**.** *Assume* (R2)*, and let the frame have* (C4)*. Let G be a multigraph that contains H, and f an H-good Γ-flow of G with A-coordinate u. Let C be a cycle of Q with a direction and σ an element of Σ_A such that $u' = u + \pi(\sigma\chi_C)$ is non-zero on every edge of every protected thread. Then f is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate u' that is equal to f outside $E(H)$.*

Proof. Put $\gamma = \gamma(\sigma)$; so γ is in Σ and $\overline{\gamma} = \sigma$. By Lemma 2.7 (d) and (a),

$$u' = u + \sigma\,(\chi_{O_1} + \cdots + \chi_{O_\nu}) \quad \text{on } E(H),$$

with pairwise edge-disjoint cycles $O_1, \dots, O_\nu$ of H; every O_j is a union of threads, and χ_{O_j} is constant, 1 or -1, on each of its threads. A thread lies on at most one O_j. For a free edge ε put

$$\delta_\varepsilon = \gamma \cdot (\text{the value of } \chi_{O_j} \text{ on } P_\varepsilon) \ \text{ if } P_\varepsilon \text{ lies on } O_j, \qquad \delta_\varepsilon = 0 \ \text{ if } P_\varepsilon \text{ lies on no } O_j.$$

So δ_ε is an element of Γ with

$$(**) \qquad \overline{\delta_\varepsilon} = u'(e) - u(e) \quad \text{for every edge } e \text{ of } P_\varepsilon.$$

(i) *Preparation in B.* For every free edge ε let W_ε be as in Lemma 3.6, and

$$T_\varepsilon = W_\varepsilon \cup \{\, -f(e) - \delta_\varepsilon : e \in E(P_\varepsilon),\ u'(e) = 0 \,\}.$$

T_ε is a subset of B: if $u'(e) = 0$, then $\overline{f(e) + \delta_\varepsilon} = u(e) + u'(e) - u(e) = 0$ by $(**)$. Every edge e of P_ε gives at most one element of T_ε. Indeed, let $u(e) = 0$ and $u'(e) = 0$. Then $\overline{\delta_\varepsilon} = 0$ by $(**)$. If P_ε were on some O_j, then $\overline{\delta_\varepsilon}$ would be σ or $-\sigma$, which is not 0. So P_ε lies on no O_j, $\delta_\varepsilon = 0$, and the two elements $-f(e)$ and $-f(e) - \delta_\varepsilon$ are equal. Hence $|T_\varepsilon| \le \ell(\varepsilon)$. Lemma 3.6, with E° the set of all free edges, gives an H-good flow f° with A-coordinate u, equal to f outside $E(H)$, with $f^\circ = f + y_\varepsilon$ on $E(P_\varepsilon)$ and $y_\varepsilon \notin T_\varepsilon$. So for every edge e of a free thread P_ε:

$$\text{(T)} \qquad \text{if } u'(e) = 0, \quad \text{then } f^\circ(e) + \delta_\varepsilon \neq 0.$$

($-f(e) - \delta_\varepsilon$ is in T_ε, so it is not y_ε.)

(ii) *The step.* Make the moves $\gamma\chi_{O_1}, \dots, \gamma\chi_{O_\nu}$, in this order. After the first j of them an edge e of H carries the value $f^\circ(e) + \gamma\chi_O(e)$, with the A-coordinate $u(e) + \sigma\chi_O(e) = u'(e)$, if e lies on a cycle O among $O_1, \dots, O_j$; and it carries the value $f^\circ(e)$, with the A-coordinate $u(e)$, if not. These values are not 0:

- on a protected thread, $u(e) \neq 0$ because f is H-good, and $u'(e) \neq 0$ by hypothesis; use (E1);
- on a free thread P_ε that lies on one of $O_1, \dots, O_j$, the value is $f^\circ(e) + \delta_\varepsilon$; it is not 0 by (E1) if $u'(e) \neq 0$, and by (T) if $u'(e) = 0$;
- on every other free thread the value is $f^\circ(e)$, and f° is nowhere zero.

The edges outside $E(H)$ do not change. So all these moves are Σ-moves on cycles of H. After the last one every edge e of H has the A-coordinate $u'(e)$; for an edge on no O_j note that $u'(e) = u(e)$. The result is nowhere zero and its A-coordinate u' is non-zero on the protected threads: it is H-good. It is equal to f outside $E(H)$. □

Lemma 3.8 (A change of the A-coordinate inside H)**.** *Assume* (R1) *and* (R2)*, and let the frame have* (C1) *and* (C4)*. Let G be a multigraph that contains H, f an H-good Γ-flow of G with A-coordinate u, and Y a flow in $P(\Lambda_f)$. Then f is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate $u + \pi(Y)$ that is equal to f outside $E(H)$.*

Proof. 0 and Y are in $P(\Lambda_f)$ (v). The frame has (C1), so the walk lemma (Lemma 3.5, its last sentence) gives a sequence $0 = Y_0, Y_1, \dots, Y_n = Y$ in $P(\Lambda_f)$ with $Y_r - Y_{r-1} = \sigma_r\chi_{C_{l_r}}$, σ_r in Σ_A. We build H-good flows $f = f_0, f_1, \dots, f_n$ of G, each Σ-joined to the one before it by Σ-moves on cycles of H and equal to f outside $E(H)$, such that f_r has the A-coordinate $u_r = u + \pi(Y_r)$. Let f_{r-1} be built. Then

$$u_{r-1} + \pi(\sigma_r\chi_{C_{l_r}}) = u + \pi(Y_{r-1}) + \pi(Y_r - Y_{r-1}) = u_r,$$

and u_r is non-zero on every edge of every protected thread, because Y_r is in $P(\Lambda_f)$ (v). So Lemma 3.7, for the flow f_{r-1}, the cycle C_{l_r} and σ_r, gives f_r; its hypotheses (R2) and (C4) are assumed. The flow f_n has the A-coordinate $u + \pi(Y)$. □

In the rest of this section we write (L1) for the conclusion of Lemma 3.8, for a given subdivision H with a given frame: *for every multigraph G that contains H, every H-good Γ-flow f of G with A-coordinate u, and every Y in $P(\Lambda_f)$, the flow f is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate $u + \pi(Y)$ that is equal to f outside $E(H)$.* By Lemma 3.8, (L1) holds if (R1) and (R2) hold and the frame has (C1) and (C4).

The path.

Lemma 3.9 (The path)**.** *Let $p \neq p'$ be vertices of H. There is a path J of H from p to p' such that, for every node i, at most $\max\{\ell(e_i), \ell(f_i)\}$ edges of J lie on the threads of the edges of N_i (both threads together; so at most 3). J can contain any number of free threads.*

Proof. An expansion edge has no thread. So "the threads of the edges of N_i" are the threads of the original edges of N_i; there are two, one or none of them.

(1) *The subgraph H_g.* Let $g = (g_1, \dots, g_s)$ be a choice of one edge g_i in every pair N_i. Let H_g be the subgraph of H formed by the vertices of H_0 and the threads of all original edges of the trees T^g of all components Q_b. Then:

   (1a) H_g is connected. Reason: T^g is a spanning tree of Q_b (iv). Under the contraction of the expansion edges (ii) it becomes a connected subgraph of K_b that contains all vertices of K_b; its edges are the original edges of T^g. Any two vertices of H_0 are joined by a path of H_0; the two ends of an edge of this path lie in one block K_b (i), and they are joined by original edges of T^g. So the original edges of the trees T^g form a connected subgraph of H_0 on all vertices of H_0. Replace these edges by their threads.

   (1b) A vertex of H is outside H_g only if it is an interior vertex of the thread of an original edge that is in some pair N_i and is not g_i. Reason: T^g contains all red edges and all edges g_i, and every blue edge is in a pair (iii).

   (1c) A path of H_g has, on the threads of the edges of N_i, only edges of the thread of g_i: at most $\ell(g_i) \leq \max\{\ell(e_i), \ell(f_i)\}$ edges. Reason: g_i is the only edge of N_i in T^g (iv).

(2) *Case 1: p and p' are interior vertices of the same thread P_t.* Let J be the part of P_t between them. A thread has at most 3 edges (i), so at most 2 interior vertices; hence J has one edge. If t is in a pair N_i, then J has $1 \leq \ell(t) \leq \max\{\ell(e_i), \ell(f_i)\}$ edge on the threads of N_i. J has no edge on the threads of the other pairs.

(3) *Case 2: not Case 1, and there is no pair N_i such that p is an interior vertex of the thread of one edge of N_i and p' an interior vertex of the thread of the other edge.* A vertex is an interior vertex of at most one thread. So, by the case, every pair N_i has at most one edge whose thread has p or p' as an interior vertex. If there is such an edge, let it be g_i; if not, put $g_i = e_i$. By (1b), p and p' are in H_g. By (1a) there is a path J of H_g from p to p'. By (1c) it is as stated.

(4) *Case 3: p is an interior vertex of P_t and p' an interior vertex of $P_{t'}$, where $t \neq t'$ are the two edges of one pair N_i.* Let U', U'' be the two operands of the node i. By (iii), t has an end x_1 in $V_{U'}$ and an end in $V_{U''}$, and t' has an end x_2 in $V_{U'}$ and an end in $V_{U''}$. So each of the threads P_t, $P_{t'}$ has one end *on the side of U'* (the vertex of H_0 that comes from x_1, from x_2) and one end on the side of U''. Let c be the number of edges of P_t between p and the end on the side of U', and c' the same for $P_{t'}$ and p'. The numbers of edges to the ends on the side of U'' are $\ell(t) - c$ and $\ell(t') - c'$. The two sums $c + c'$ and $(\ell(t) - c) + (\ell(t') - c')$ add up to $\ell(t) + \ell(t')$. So one of them is at most $(\ell(t) + \ell(t'))/2 \leq \max\{\ell(t), \ell(t')\}$. Let it be $c + c'$; the other case is the same with U' and U'' exchanged.

   By (iv) the edges of T^e in $Q_{U'}$ form a spanning tree of $Q_{U'}$. It contains a path R from x_1 to x_2. The edges t, t' are not edges of $Q_{U'}$: each has an end outside $V_{U'}$. So R contains

neither t nor t', and every blue edge of R is an edge e_l with $l \neq i$. Contract the expansion edges and replace the original edges of R by their threads. The result L is a connected subgraph of H (one vertex, if R has no original edge). It contains the ends of P_t and of $P_{t'}$ on the side of U'. It has no edge on P_t or $P_{t'}$. For a pair N_l with $l \neq i$ it has edges on the thread of e_l only.

Let J be a path from p to p' in the union of L, the part of P_t from p to its end on the side of U', and the part of $P_{t'}$ from p' to its end on the side of U'. This union is connected, so J exists. On the threads of N_i the path J has at most $c + c' \leq \max\{\ell(e_i), \ell(f_i)\}$ edges. On the threads of a pair N_l, $l \neq i$, it has at most $\ell(e_l) \leq \max\{\ell(e_l), \ell(f_l)\}$ edges.

The three cases cover all pairs $p \neq p'$. The bound "at most 3" holds because $\ell(t) \leq 3$ (i). □

The lift of one move of the contracted graph. Now H is a cycle or a subdivision. If H is a cycle of a multigraph G, a nowhere-zero Γ-flow f of G is *H-good* if $\overline{f}(e) \neq 0$ for every edge e of H (Definition 2.14).

Lemma 3.10 (The lift of one move of the contracted graph)**.** *Assume* $|A| \geq 8$, (R1) *and* (R2)*. Let G be a bridgeless multigraph and H a subgraph of G that is a cycle with at most* 5 *edges* (*case* CYC)*, or a subdivision with a counted frame* (*case* SUB)*. Let $G^* = G/H$ be the multigraph obtained from G by the identification of $V(H)$ to one vertex and the deletion of $E(H)$. Let f be an H-good Γ-flow of G with restriction f^* to G^*, and let $g^* = f^* + \lambda\chi_{C^*}$ be the result of a Σ-move of G^*. Then f is Σ-joined to an H-good Γ-flow g of G whose restriction to G^* is g^*.*

(*The frame form: the same holds with* (R1$^-$) *in the place of* (R1) *and, in the place of* (C1)*, the conclusion of Lemma 3.8 for this H and all f, Y.*)

Proof. We prove the frame form. It contains the first form: (R1) implies (R1$^-$), and for a subdivision with a counted frame (L1) holds by Lemma 3.8. So we assume: $|A| \geq 8$; (R1$^-$); (R2); and in the case SUB the frame has (C2), (C3), (C4), and (L1) holds. Let u be the A-coordinate of f.

(1) *The contracted graph.* $E(G^*) = E(G) \setminus E(H)$. Let v^* be the vertex of G^* made from $V(H)$. By (E5) the restriction of a flow of G to $E(G^*)$ is a flow of G^*. C^* is a cycle of G^* with a direction, λ is in Σ, and f^* and g^* are nowhere zero.

(2) *The cycle C.* If C^* does not contain v^*, its edges form a cycle of G. If C^* contains v^*, its edges form in G a path between two vertices p, p' of H whose other vertices are outside $V(H)$, or, if $p = p'$, a cycle of G. (C^* is a loop at v^*, or it has exactly two edges at v^*.)

If the edges of C^* form a cycle of G, direct it as C^* and put $g = f + \lambda\chi_{C^*}$. This flow is equal to g^* on $E(G^*)$ and to f on $E(H)$. So it is nowhere zero, the move is a Σ-move, g is H-good, and its restriction is g^*.

From now on the edges of C^* form a path between $p \neq p'$. For every path J of H from p to p' the union $C = C^* \cup J$ is a cycle of G, because all vertices of J are in $V(H)$. Direct C so that $\chi_C = \chi_{C^*}$ on $E(G^*)$. The path J is chosen below. In each case we reach an H-good flow f' that is equal to f outside $E(H)$, and then make the move $g = f' + \lambda\chi_C$. On $E(G^*)$ the flow g is equal to $f^* + \lambda\chi_{C^*} = g^*$; so it is not 0 there, and the restriction of g is g^*. It remains to look at the edges of H.

(3) *Case* CYC. Direct the cycle H. Let J be an arc of H between p and p' with at most 2 edges; it exists because H has at most 5 edges.

$\overline{\lambda} = 0$. Put $f' = f$. By (E2) the move $\lambda\chi_C$ does not change the A-coordinate, which is non-zero on $E(H)$. So g is nowhere zero (E1) and H-good.

$\overline{\lambda} \neq 0$. Put

$$X = \{\, c \in A : u + c\chi_H \text{ has a zero on } E(H) \,\},$$
$$X' = \{\, c \in A : u + c\chi_H + \overline{\lambda}\chi_C \text{ has a zero on } E(J) \,\}.$$

Every edge of H excludes one value of c, because χ_H is 1 or -1 on it. So $|X| \leq 5$, $|X'| \leq 2$, and 0 is not in X, because f is H-good. By the hypothesis $|A| \geq 8$ (it is (C0) of Theorem 3.11), $|X \cup X'| \leq 5 + 2 < |A|$; choose c outside $X \cup X'$. By (R1$^-$) there is a walk $0 = c_0, c_1, \ldots, c_\nu = c$ in $A \setminus X$ with steps $d_r = c_r - c_{r-1}$ in Σ_A. Make the moves $\gamma(d_r)\chi_H$, $r = 1, \ldots, \nu$. After the r-th move the A-coordinate is $u + c_r\chi_H$; it is non-zero on $E(H)$, because c_r is not in X; outside $E(H)$ nothing changes. So these are Σ-moves (E1). The result f' is H-good and equal to f outside $E(H)$. Now $g = f' + \lambda\chi_C$ has, on $E(H)$, the A-coordinate $u + c\chi_H$ outside J and $u + c\chi_H + \overline{\lambda}\chi_C$ on J. Both are non-zero, because c is not in X and not in X'. So g is nowhere zero and H-good.

(4) *Case* SUB, $\overline{\lambda} \neq 0$. Let J be the path of Lemma 3.9, and let j_i be the number of edges of J on the threads of the edges of N_i. So $j_i \leq \max\{\ell(e_i), \ell(f_i)\}$.

(4a) *The protected threads.* For an original blue edge t put

$$\Lambda_1(t) = \Lambda_f(t) \cup \{\, -u(e) - \overline{\lambda}\chi_C(e) : e \in E(P_t) \cap E(J) \,\},$$

and let Λ_1 be empty on the expansion edges. Then, for every node i,

$$|\Lambda_1(e_i)| + |\Lambda_1(f_i)| \;\leq\; \ell(e_i) + \ell(f_i) + j_i \;\leq\; \ell(e_i) + \ell(f_i) + \max\{\ell(e_i), \ell(f_i)\} \;<\; |A|$$

by (v) and (C2). By Lemma 3.4 (a), with $Y_0 = 0$, there is Y in $P(\Lambda_1)$. It is in $P(\Lambda_f)$, because Λ_1 contains Λ_f. By (L1), f is Σ-joined to an H-good flow f_1 with A-coordinate $u_1 = u + \pi(Y)$ that is equal to f outside $E(H)$. For an edge e of J on a protected thread P_t: $u_1(e) + \overline{\lambda}\chi_C(e) = u(e) + Y(t) + \overline{\lambda}\chi_C(e) \neq 0$, because $Y(t)$ is not in $\Lambda_1(t)$.

(4b) *The free threads.* Let E° be the set of the free edges ε such that P_ε has an edge on J. Their number is not bounded. For ε in E° put

$$W_\varepsilon = \{\, -f_1(e) : e \in E(P_\varepsilon),\ u_1(e) = 0 \,\},$$
$$T_\varepsilon = W_\varepsilon \cup \{\, -f_1(e) - \lambda\chi_C(e) : e \in E(P_\varepsilon) \cap E(J),\ u_1(e) + \overline{\lambda}\chi_C(e) = 0 \,\}.$$

T_ε is a subset of B: the A-coordinate of $f_1(e) + \lambda\chi_C(e)$ is $u_1(e) + \overline{\lambda}\chi_C(e)$. Every edge e of P_ε gives at most one element of T_ε: for an edge of J, $\chi_C(e)$ is 1 or -1 and $\overline{\lambda} \neq 0$, so $u_1(e) = 0$ and $u_1(e) + \overline{\lambda}\chi_C(e) = 0$ exclude each other. Hence $|T_\varepsilon| \leq \ell(\varepsilon)$. By Lemma 3.6 (it uses (R2) and (C4)), f_1 is Σ-joined to an H-good flow f_2 with A-coordinate u_1, equal to f_1 outside $E(H)$, with $f_2 = f_1 + y_\varepsilon$ on $E(P_\varepsilon)$ and $y_\varepsilon \notin T_\varepsilon$, for every ε in E°.

(4c) *The move.* Put $f' = f_2$ and $g = f_2 + \lambda\chi_C$. On an edge e of J the new A-coordinate is $u_1(e) + \overline{\lambda}\chi_C(e)$. If e is on a protected thread, this is not 0 by (4a). If e is on a free thread P_ε and this is 0, then $-f_1(e) - \lambda\chi_C(e)$ is in T_ε, so it is not y_ε, and $g(e) = f_1(e) + y_\varepsilon + \lambda\chi_C(e) \neq 0$. The edges of H outside J do not change. So g is nowhere zero (E1), the move is a Σ-move, and g is H-good: on the protected threads its A-coordinate is u_1 outside J and $u_1 + \overline{\lambda}\chi_C$ on J.

(5) *Case* SUB, $\overline{\lambda} = 0$. So λ is in Σ_B.

(5a) *The path J.* A thread has at most 3 edges (i). So every interior vertex of a thread is next to an end of the thread, and two interior vertices of one thread are next to each other. If p and p' are interior vertices of the same free thread, J is the edge of this thread between them. Otherwise: if p is an interior vertex of a free thread, let q be an end of this thread that is next to p, and let J start with the edge from p to q; if not, put $q = p$. In the same way p' gives q', and J ends with the edge from q' to p' if $q' \neq p'$. By (O6), q and q' are in H_P, and H_P is connected; between q and q' the path J runs inside H_P. (An interior vertex of a free thread is not in H_P; so J is a path.) A *free piece* of J is the part of J on one free thread. Hence: J has at most two free pieces; they lie on different free threads; *each has exactly one edge*; and all other edges of J are on protected threads.

(5b) *The A-coordinate on J.* Let E' be the set of the free edges ε such that P_ε has a free piece of J; so $|E'| \leq 2$. For ε in E' let e_ε be the edge of this free piece, and $\Phi(\varepsilon) = \{-u(e_\varepsilon)\}$: one element, not all of A by the hypothesis $|A| \geq 8$ ((C0)). For every node i,

$$|\Lambda_f(e_i)| + |\Lambda_f(f_i)| + \sum_{\varepsilon \in E',\ i(\varepsilon)=i} |\Phi(\varepsilon)| \ \leq\ \ell(e_i) + \ell(f_i) + 2 \ <\ |A|$$

by (v) and (C3). By Lemma 3.4 (b) there is Y in $P(\Lambda_f)$ with $Y(\varepsilon) \neq -u(e_\varepsilon)$ for every ε in E'. By (L1), f is Σ-joined to an H-good flow f_1 with A-coordinate $u_1 = u + \pi(Y)$ that is equal to f outside $E(H)$. u_1 is non-zero on the protected threads, and on a free piece $u_1(e_\varepsilon) = u(e_\varepsilon) + Y(\varepsilon) \neq 0$. So u_1 is non-zero on every edge of J.

(5c) *The move.* Put $f' = f_1$ and $g = f_1 + \lambda\chi_C$. By (E2) the A-coordinate does not change; it is non-zero on J. So g is nowhere zero (E1) and H-good. The group B and the condition (R2) are not used in this case, except inside (L1). □

The master theorem.

Theorem 3.11 (The master theorem)**.** *Let Γ be a finite abelian group, B a subgroup, $A = \Gamma/B$, and Σ a step system. Assume:*

(H1) *the step conditions* (R1), (R2), (R3*), *and the averaging condition* (AV″)*;*

(H2) *the counts:*

(C0) $|A| \geq 8$*;*

and in (AV″) *the good subgraph can always be chosen to be a cycle with at most* 5 *edges, or a subdivision that has a counted frame, that is a frame with* (C1), (C2), (C3), (C4).

Then on every connected bridgeless multigraph G (loops allowed) every nowhere-zero Γ-flow is Σ-joined to every Γ-flow whose A-coordinate is nowhere zero. Hence any two nowhere-zero Γ-flows of a bridgeless multigraph are Σ-joined, and $F(G,\Gamma)$ is connected for every finite bridgeless multigraph G.

Theorem 3.12 (The master theorem, frame form)**.** *The conclusion of Theorem 3.11 holds also if, in* (H1), (R1) *is replaced by* (R1⁻), *and, in* (H2), (C1) *is replaced by*

(L1) *the conclusion of Lemma 3.8 holds for the subdivisions H and the frames of* (H2)*, for every multigraph G that contains H.*

((R1) *for six deleted elements and* (C1) *are used only in the walk lemma, and the walk lemma is used only in Lemma 3.8.*)

Proof of Theorems 3.11 and 3.12. The hypotheses of Theorem 3.11 imply those of Theorem 3.12: (R1) implies (R1$^-$), and (L1) holds by Lemma 3.8, which uses (R1), (R2), (C1) and (C4). So let the hypotheses of the frame form hold: (R1$^-$), (R2), (R3*), (AV$''$), (C0), and the good subgraph in (AV$''$) can be chosen *admissible*: a cycle with at most 5 edges, or a subdivision with a frame that has (C2), (C3), (C4) and (L1). For an admissible H Lemma 3.10 holds in its frame form.

A *target* of a multigraph is a Γ-flow whose A-coordinate is nowhere zero. A target is nowhere zero (E1).

Claim. On every connected bridgeless multigraph G, every nowhere-zero Γ-flow is Σ-joined to every target.

(1) *The claim gives the theorem.* The first statement is the claim. Let G be a bridgeless multigraph. It has a nowhere-zero A-flow, because $|A| \geq 6$ by (C0) (Theorem 0.1, the theorems of Seymour and of Tutte: every bridgeless multigraph, loops allowed, has a nowhere-zero flow with values in any abelian group of order at least 6). By (E4) it is the A-coordinate of a Γ-flow h: a target. A cycle of G lies in one component of G, and a Σ-move of a component is a Σ-move of G. So, component after component, the claim joins every nowhere-zero Γ-flow of G to h. The reverse of a Σ-move is a Σ-move; so any two nowhere-zero Γ-flows are Σ-joined through h. Σ-moves are moves of $F(G, \Gamma)$; so $F(G, \Gamma)$ is connected.

(2) *The induction.* We prove the claim by induction on the number n of vertices of G; for each n first for the loopless graphs, then for the graphs with loops. A connected loopless multigraph with one vertex has no edge and only the empty flow. A connected bridgeless loopless multigraph with $n \geq 2$ has a cut with exactly 2 edges or is 3-edge-connected. So three cases remain: (a), (b), (c). In all of them f is a nowhere-zero Γ-flow and h a target of G.

(a) *G has a loop.* Let G^- be G without its loops. It is connected, bridgeless and loopless, and it has n vertices; so the claim holds for G^- (the loopless graphs with n vertices come first). The restrictions of f and h to G^- are a nowhere-zero flow and a target. A Σ-move of G^- is a Σ-move of G: the values on the loops do not change. So f is Σ-joined to the flow that is equal to h on $E(G^-)$ and to f on the loops. Now let e be a loop, with the values $x = f(e)$ and $x' = h(e)$. Both are non-zero; x' because $\overline{x'} \neq 0$. By (R3*) there is a walk from x to x' in $\Gamma \setminus \{0\}$ with steps in Σ. A loop is a cycle with one edge. The moves $\lambda\chi_e$, for the steps λ of the walk, are Σ-moves: the value on e is never 0, and no other edge changes. Do this for every loop. The result is h.

(b) *G is loopless and has a cut with exactly two edges e_1, e_2.* Let G/e_1 be obtained from G by the identification of the two ends of e_1 and the deletion of e_1. It is connected, it has $n-1$ vertices, and it is bridgeless (a cycle of G through an edge $b \neq e_1$ becomes a closed trail of G/e_1 through b, and a closed trail through b contains a cycle through b); it can have loops. By induction the claim holds for G/e_1. By (E5) the restriction of a flow of G to $E(G/e_1)$ is a flow of G/e_1. Every flow of G/e_1 is the restriction of exactly one flow of G: the value on e_1 is fixed by conservation at one end of e_1, and conservation at the

other end then follows from conservation at the vertex of G/e_1 made from the two ends. In a flow of G the value on e_1 is the value on e_2 or its negative (conservation over one side of the cut). So a flow of G is nowhere zero if and only if its restriction is, and it is a target if and only if its restriction is. Let $g' \to g' + \lambda\chi_{C'}$ be a Σ-move of G/e_1. The edges of C' form a cycle of G, or a path of G between the two ends of e_1; in the second case add e_1. This gives a cycle C of G with $\chi_C = \chi_{C'}$ on $E(G/e_1)$. If g is the flow of G with restriction g', then $g + \lambda\chi_C$ is the flow of G with restriction $g' + \lambda\chi_{C'}$. Both are nowhere zero. So every Σ-move of G/e_1 lifts to a Σ-move of G, and the claim for G/e_1 gives the claim for G.

(c) *G is loopless and 3-edge-connected, $n \geq 2$.* By (AV″) and (H2), f is Σ-joined to a nowhere-zero flow f' with A-coordinate u' such that $(V(G), \mathrm{supp}(u'))$ contains an admissible good subgraph H. If H is a subdivision, fix a frame of H as in (H2). u' is non-zero on $E(H)$, so f' is H-good. H has at least 2 vertices (Definition 2.1). So $G^* = G/H$ has fewer vertices than G. It is connected. It is bridgeless: the edges outside $E(H)$ of a cycle of G through an edge b of G^* form a closed trail of G^* through b, and it contains a cycle through b. It can have loops. By induction the claim holds for G^*. The restriction of f' to G^* is nowhere zero, and the restriction h^* of h is a target of G^*. So there are Σ-moves of G^* that lead from the restriction of f' through flows $g_0, g_1, \ldots, g_\nu = h^*$. Lemma 3.10, used ν times, gives H-good flows $f' = f_0, f_1, \ldots, f_\nu$ of G, each Σ-joined to the one before it, such that the restriction of f_r to G^* is g_r. So f_ν is H-good and equal to h outside $E(H)$. Let u_ν be its A-coordinate.

End, case CYC. Direct the cycle H. The A-flow $\overline{h} - u_\nu$ is 0 outside $E(H)$, so it is $c\chi_H$ with c in A. The set $X = \{\, c' \in A : u_\nu + c'\chi_H$ has a zero on $E(H)\,\}$ has at most 5 elements. 0 is not in X, because f_ν is H-good; c is not in X, because $\overline{h}$ is nowhere zero. By (R1$^-$) there is a walk from 0 to c in $A \setminus X$ with steps d_r in Σ_A. The moves $\gamma(d_r)\chi_H$ are Σ-moves: after each of them the A-coordinate is non-zero on $E(H)$, and nothing changes outside $E(H)$. The result f'' has the A-coordinate $\overline{h}$ and is equal to h outside $E(H)$. So $h - f'' = y\chi_H$ with $\overline{y} = 0$, that is with y in B. By (R2) with X empty, y is a sum of elements of Σ_B. The moves on H with these elements are Σ-moves: the A-coordinate stays $\overline{h}$ (E2), which is nowhere zero (E1). The result is h.

End, case SUB. The A-flow $\overline{h} - u_\nu$ is 0 outside $E(H)$, so it is an A-flow of H. By Lemma 2.7 (c) there is an A-flow Y of Q with $\pi(Y) = \overline{h} - u_\nu$. Y is in $P(\Lambda_{f_\nu})$, because $u_\nu + \pi(Y) = \overline{h}$ has no zero (v). By (L1), f_ν is Σ-joined to a flow f'' with A-coordinate $\overline{h}$ that is equal to f_ν, hence to h, outside $E(H)$. So $h - f''$ is a flow of H with values in B. By (E3) it is a sum of flows $y\chi_O$ with O a cycle of H and y in B, and by (R2) with X empty every y is a sum of elements of Σ_B. Make these moves one by one. They are Σ-moves: the A-coordinate stays $\overline{h}$ (E2), which is nowhere zero (E1). The result is h. □

Remark 3.13 (Where the deletions are used)**.** (R1): the walk lemma deletes at most $\ell(e_i) + \ell(f_i) \leq 6$ elements of A; the case CYC of Lemma 3.10 and the end of the proof of Theorem 3.11 in the case CYC delete at most 5. (R2): the preparation in B (Lemma 3.8, and Lemma 3.10 for a step with non-zero A-part) deletes at most $\ell(\varepsilon) - 1 \leq 2$ elements of B; the end of the proof uses only that Σ_B generates B. (R3*) is used for loops only. (C0) is the count $5 + 2 < |A|$ of the case CYC; the target flow needs $|A| \geq 6$.

Remark 3.14 (Where the counts are used)**.** The proofs of this section use the order of A, the order of B and the numbers $\ell(t)$ at the following places, and at no other place. j_i is the number of edges of the path J on the threads of the edges of N_i.

(C0) Lemma 3.10, case CYC: at most $5+2$ values of c are excluded. Case SUB, $\overline{\lambda}=0$: the set $\Phi(\varepsilon)$ of one element is not all of A. Theorem 3.11: the target needs $|A|\geq 6$.

(C1) The walk lemma: an offset c_r outside a set of at most $2\,|\Lambda_f(e_i)|+|\Lambda_f(f_i)| \leq 2\,\ell(e_i)+\ell(f_i)$ elements. It is used only through Lemma 3.8.

(C2) Lemma 3.10, case SUB, $\overline{\lambda}\neq 0$: the reverse choice for lists with $\ell(e_i)+\ell(f_i)+j_i$ elements at the node i, and $j_i \leq \max\{\ell(e_i),\ell(f_i)\}$ by Lemma 3.9.

(C3) Lemma 3.10, case SUB, $\overline{\lambda}=0$: the reverse choice for lists with $\ell(e_i)+\ell(f_i)$ elements and at most two free edges with one excluded value each.

(C4) Lemma 3.6: an element of B outside a set of at most $\ell(\varepsilon)$ elements. It is used in Lemma 3.7, hence in Lemma 3.8, and in Lemma 3.10, case SUB, $\overline{\lambda}\neq 0$.

The bound $\ell(t)\leq 3$ is used for: at most 6 deleted elements in the walk lemma; at most 2 deleted elements in Lemma 3.6; the free pieces with one edge in Lemma 3.10, case SUB, $\overline{\lambda}=0$; Case 1 of Lemma 3.9. The frame form uses (C0), (C2), (C3), (C4) and, in the place of (C1), the hypothesis (L1).

Three corollaries. Each corollary is Theorem 3.11 together with a check of (H2). By Remark 3.14 this check is a check of the five inequalities (C0)–(C4).

Corollary 3.15 (No condition on the colouring)**.** *Let $|A|\geq 10$ and $|B|\geq 4$, and let Σ be a step system with* (R1), (R2), (R3*) *and* (AV″)*. Then the conclusion of Theorem 3.11 holds.*

Proof. (H1) is assumed. (C0): $|A|\geq 10\geq 8$. Let H be a subdivision that (AV″) gives. It has a frame (Definition 2.9: every truncated cubic 3-edge-connected graph has a well colouring, Lemma A.5). In every frame $\ell(t)\leq 3$ for all t. So the left sides of (C1), (C2), (C3) are at most $2\cdot 3+3=9$, $3+3+3=9$ and $3+3+2=8$; all are below $10\leq|A|$. The left side of (C4) is at most $3<4\leq|B|$. So every frame is counted, and (H2) holds. □

Corollary 3.16 (Quotient of order 9, condition (S))**.** *Let $|A|\geq 9$ and $|B|\geq 4$, and let Σ be a step system with* (R1), (R2), (R3*) *and with* (AV″) *in this form: the good subgraph can always be chosen to be a cycle with at most* 5 *edges, or a subdivision that has a frame with* (S)*. Then the conclusion of Theorem 3.11 holds.*

Proof. (H1) is assumed. (C0): $|A|\geq 9\geq 8$. Let H be a subdivision with a frame that has (S). Exchange the names at the nodes so that $\ell(e_i)\leq\ell(f_i)$ for every node i; the result is a frame (iii), and (S) does not depend on the names. If $\ell(e_i)=3$, then $\ell(f_i)=3$, and N_i consists of two original edges whose threads have 3 edges; (S) excludes this. So $\ell(e_i)\leq 2$ and $\ell(f_i)\leq 3$ for every i. The left sides: (C1) at most $2\cdot 2+3=7$; (C2) at most $2+3+3=8$; (C3) at most $2+3+2=7$; all are below $9\leq|A|$. (C4): at most $3<4\leq|B|$. So this frame is counted, and (H2) holds. □

Corollary 3.17 (Subgroup of order 3, condition (NR))**.** *Let $|A|\geq 10$ and $|B|=3$, and let Σ be a step system with* (R1), (R2), (R3*) *and with* (AV″) *in this form: the good subgraph can always be chosen to be a cycle with at most* 5 *edges, or a subdivision that has a frame with* (NR)*. Then the conclusion of Theorem 3.11 holds.*

Proof. (H1) is assumed. (C0): $|A|\geq 10\geq 8$. Let H be a subdivision with a frame that has (NR). The left sides of (C1), (C2), (C3) are at most 9, 9, 8, as in the proof of Corollary 3.15;

all are below $10 \le |A|$. By (NR) every free thread has at most 2 edges; so the left side of (C4) is at most $2 < 3 = |B|$. So this frame is counted, and (H2) holds. □

Remark 3.18. Corollaries 3.16 and 3.17 are conditional: no statement of this paper says that a frame with (S), or a frame with (NR), exists for every subdivision that the averaging step gives, so as they stand they give no new modulus and no new group.

The integer form.

Theorem 3.19 (The integer form)**.** *Let $k = ab$ with integers $a, b \ge 2$, $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ (the multiples of a; b elements) and $A = \mathbb{Z}_a$ (reduction modulo a). Let Σ be a step system all of whose elements are liftable, and let Γ, B, Σ satisfy the hypotheses of Theorem 3.11 or of Theorem 3.12. Then*

(a) *$M_k(G)$ is connected for every finite bridgeless multigraph G (loops allowed);*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

Proof. The quotient $\mathbb{Z}_k/a\mathbb{Z}_k$ is $\mathbb{Z}_a$, and the quotient map is the reduction modulo a. (C0) gives $a \ge 8$, so $k \ge 16$.

(a) $M_k(G) = F(G, \mathbb{Z}_k)$. It is connected by Theorem 3.11 or Theorem 3.12.

(b) Let G be loopless, and let F, F' be two vertices of $I_k(G)$. Their reductions f, f' modulo k are nowhere-zero $\mathbb{Z}_k$-flows of G, because $1 \le |F(e)| \le k-1$. By Theorem 3.11 or Theorem 3.12 there are Σ-moves of G that lead from f through flows $f = f_0, f_1, \dots, f_\nu = f'$. For $r \ge 1$ the step from f_{r-1} to f_r is a move $f_{r-1} \to f_{r-1} + \lambda\chi_C$ of $M_k(G)$ with a liftable λ. By Definition 1.7, all integer lifts of f_{r-1} and all integer lifts of f_r lie in one component of $I_k(G)$. Every f_r has an integer lift (Lemma 1.12 (a)). So this is the same component for all r. It contains F and F'. For $\nu = 0$ use Lemma 1.12 (a): all integer lifts of one flow lie in one component.

Liftable steps are used only for the Σ-moves of the given loopless graph G. Graphs with loops occur only inside the induction in the proof of Theorem 3.11, where all flows have values in Γ. □

4. Averaging in a product

This section proves Lemma 4.1 and, as a first consequence, a theorem on groups (Theorem 4.5). Lemma 4.1 gives the averaging conditions (AV) and (AV″) of Definition 3.2 for every step system that contains generators of the two factors of a product, if $(|B|-1)/(|\Gamma|-1) < 1/9$. The subgroup B need not be a direct factor: the proof uses it only through this ratio.

The notation is that of Definition 3.1: Γ is a finite abelian group, B a subgroup, $A = \Gamma/B$, $x \mapsto \overline{x}$ the quotient map, and the A-coordinate of a Γ-flow f is the A-flow $\overline{f}$. A *zero* of the A-coordinate of f is an edge e with $\overline{f(e)} = 0$.

Lemma 4.1 (Averaging in a product)**.** *Let $\Gamma = \Gamma_1 \times \Gamma_2$ with both factors non-trivial, and let B be a subgroup of Γ with*

$$\frac{|B|-1}{|\Gamma|-1} < \frac{1}{9} \qquad \text{(true for every subgroup of index at least 9).}$$

Let Σ be a step system that contains a set of generators of $\Gamma_1 \times 0$ and a set of generators of $0 \times \Gamma_2$. Then on every multigraph G with m edges, every nowhere-zero Γ-flow is Σ-joined to a nowhere-zero Γ-flow whose A-coordinate has at most $m/9$ zeros. In particular (AV) *and* (AV″) *hold. (B need not be a direct factor, and it need not be one of the two factors.)*

Proof of Lemma 4.1. Let m be the number of edges of G and f a nowhere-zero Γ-flow of G. If $m = 0$, then f itself has no zero. Let $m \geq 1$, and put $g = |\Gamma|$.

We write an element x of Γ as $x = (x_{(1)}, x_{(2)})$ with $x_{(1)}$ in Γ_1 and $x_{(2)}$ in Γ_2, and a Γ-flow φ as $\varphi = (\varphi_{(1)}, \varphi_{(2)})$; here $\varphi_{(i)}$ is a Γ_i-flow. Put $\Sigma_1 = \Sigma \cap (\Gamma_1 \times 0)$ and $\Sigma_2 = \Sigma \cap (0 \times \Gamma_2)$. The given set of generators of $\Gamma_1 \times 0$ is a subset of Σ_1, and $\Sigma_1 = -\Sigma_1$ because $\Sigma = -\Sigma$. So every element of $\Gamma_1 \times 0$ is a finite sum of elements of Σ_1 (the empty sum for 0). In the same way every element of $0 \times \Gamma_2$ is a finite sum of elements of Σ_2.

For a nowhere-zero Γ-flow φ of G, let $G\langle\varphi_{(2)}\rangle$ be the spanning subgraph of G with the edge set $\mathrm{supp}(\varphi_{(2)})$, and $G\langle\varphi_{(1)}\rangle$ the one with the edge set $\mathrm{supp}(\varphi_{(1)})$. A Γ_1-flow of $G\langle\varphi_{(2)}\rangle$, with the value 0 on all other edges, is a Γ_1-flow of G; and in the same way for Γ_2.

(1) *Rounds.* A *round of type* 1 replaces a nowhere-zero Γ-flow φ by $(\varphi_{(1)} + \xi,\ \varphi_{(2)})$, where ξ is a Γ_1-flow of $G\langle\varphi_{(2)}\rangle$. A *round of type* 2 replaces φ by $(\varphi_{(1)},\ \varphi_{(2)} + \xi)$, where ξ is a Γ_2-flow of $G\langle\varphi_{(1)}\rangle$. In a *random* round, ξ is chosen uniformly at random in the finite group of these flows.

(2) *The result of a round is nowhere zero and Σ-joined to φ.* Take type 1; type 2 is the same with the coordinates exchanged. Let e be an edge. If e is in $\mathrm{supp}(\varphi_{(2)})$, the second coordinate of the new value is $\varphi_{(2)}(e) \neq 0$. If not, then $\xi(e) = 0$ and the value is $\varphi(e) \neq 0$. So the result is nowhere zero. By the cycle decomposition (Lemma 1.1 (a), for the graph $G\langle\varphi_{(2)}\rangle$ and the group Γ_1),

$$\xi = z_1\chi_{C_1} + \cdots + z_\nu\chi_{C_\nu}$$

with cycles $C_1, \ldots, C_\nu$ of $G\langle\varphi_{(2)}\rangle$, each with a direction, and z_r in Γ_1. Every $(z_r, 0)$ is a sum of elements of Σ_1. Add these elements one at a time, each one times χ_{C_r}, for $r = 1, \ldots, \nu$. Every flow in this sequence is $(\varphi_{(1)} + \xi',\ \varphi_{(2)})$, where ξ' is a Γ_1-flow that is 0 outside $\mathrm{supp}(\varphi_{(2)})$; as above, it is nowhere zero. So every addition is a Σ-move, and the last flow is the result of the round.

(3) *The law of one edge after one random round.* Define two matrices Π_1, Π_2 with rows and columns indexed by $\Gamma \setminus \{0\}$:

$$\Pi_1(x,y) = 1/|\Gamma_1| \ \text{ if } x_{(2)} = y_{(2)} \neq 0, \quad \Pi_1(x,x) = 1 \ \text{ if } x_{(2)} = 0, \quad \Pi_1(x,y) = 0 \ \text{ otherwise;}$$

$$\Pi_2(x,y) = 1/|\Gamma_2| \ \text{ if } x_{(1)} = y_{(1)} \neq 0, \quad \Pi_2(x,x) = 1 \ \text{ if } x_{(1)} = 0, \quad \Pi_2(x,y) = 0 \ \text{ otherwise.}$$

Claim: if φ' is the result of a random round of type i applied to φ, then for every edge e and every y in $\Gamma \setminus \{0\}$

$$\Pr[\,\varphi'(e) = y\,] = \Pi_i(\varphi(e), y).$$

Take type 1. If e is not in $\mathrm{supp}(\varphi_{(2)})$, then $\xi(e) = 0$, $\varphi'(e) = \varphi(e)$, and $\varphi(e)_{(2)} = 0$: this is the second case of the definition of Π_1. Let e be in $\mathrm{supp}(\varphi_{(2)})$. First, e lies on a cycle of $G\langle\varphi_{(2)}\rangle$. Indeed, suppose it does not. Then e is not a loop, because a loop is a cycle. Let X be the vertex set of the component of $G\langle\varphi_{(2)}\rangle - e$ that contains one end of e. The other end of e is not in X: otherwise a path of $G\langle\varphi_{(2)}\rangle - e$ between the two ends, together with e, is a cycle through e. So e is the only edge of $G\langle\varphi_{(2)}\rangle$ with exactly one end in X. Add the conservation equations of $\varphi_{(2)}$ at the vertices of X: an edge with both ends in X enters X once and leaves it once, so it drops out. The sum is $\pm\varphi_{(2)}(e)$ plus the values of $\varphi_{(2)}$ on the other edges of G with exactly one end in X; these edges are not in $\mathrm{supp}(\varphi_{(2)})$. So $\varphi_{(2)}(e) = 0$, a contradiction. Hence e lies on a cycle C of $G\langle\varphi_{(2)}\rangle$. For every x in Γ_1 the flow $x\chi_C$ has the value x or $-x$ on e. So the map $\xi \mapsto \xi(e)$ is a homomorphism of the group of the Γ_1-flows of $G\langle\varphi_{(2)}\rangle$ onto Γ_1. All

its fibres are cosets of its kernel, so they have the same size, and $\xi(e)$ is uniform on Γ_1. So $\varphi'(e) = (\varphi(e)_{(1)} + \xi(e),\ \varphi(e)_{(2)})$ is uniform on the $|\Gamma_1|$ elements y with $y_{(2)} = \varphi(e)_{(2)}$; these elements are not 0. This is the first case of the definition of Π_1.

(4) *The chain.* Put $f^{(0)} = f$. For $r \geq 0$ let $f^{(r+1)}$ be the result of a random round applied to $f^{(r)}$, of type 1 if r is even and of type 2 if r is odd, with a new independent choice. This is a finite probability space, and every sequence of choices has positive probability. By (2), every $f^{(r)}$ is a nowhere-zero Γ-flow that is Σ-joined to f, for every sequence of choices. Fix an edge e, and let μ_r be the law of $f^{(r)}(e)$: a row vector indexed by $\Gamma \setminus \{0\}$, with non-negative entries and sum 1. If $\Pr[f^{(r)} = \varphi] > 0$, then under the condition $f^{(r)} = \varphi$ the law of $f^{(r+1)}(e)$ is the row $\Pi_i(\varphi(e), \cdot)$, by (3). Summing over φ gives $\mu_{r+1} = \mu_r \Pi_i$, with $i = 1$ for even r and $i = 2$ for odd r. So, with $\Pi = \Pi_1 \Pi_2$,

$$\mu_{2h} = \mu_0 \Pi^h \qquad \text{for every integer } h \geq 0,$$

where μ_0 has the entry 1 at $f(e)$. Only the law of one edge is used, not the joint law of two edges.

(5) *Mixing.* The matrices Π_1, Π_2 have non-negative entries and row sums 1. They are symmetric: for $x \neq y$ the condition $x_{(2)} = y_{(2)} \neq 0$ is symmetric in x and y, and the same holds for Π_2. So their column sums are 1 too, and the same holds for Π. Let η be the row vector with all entries $1/(g-1)$. Then $\eta\Pi = \eta$.

Claim: every entry of Π^3 is positive. Four facts, for x, y in $\Gamma \setminus \{0\}$:

(i) $\Pi(x,x) > 0$. If $x_{(2)} \neq 0$: $\Pi_1(x,x) = 1/|\Gamma_1|$, and $\Pi_2(x,x)$ is $1/|\Gamma_2|$ or 1. If $x_{(2)} = 0$, then $x_{(1)} \neq 0$: $\Pi_1(x,x) = 1$ and $\Pi_2(x,x) = 1/|\Gamma_2|$.

(ii) If $x_{(2)} \neq 0$ and $y_{(1)} \neq 0$, then $\Pi(x,y) > 0$: Π_1 leads from x to $(y_{(1)}, x_{(2)})$, and Π_2 leads from there to y, because $y_{(1)} \neq 0$.

(iii) If $x_{(2)} = 0$, then $\Pi(x,x') > 0$ for $x' = (x_{(1)}, z)$ and every z in Γ_2: Π_1 keeps x, and Π_2 leads to x', because $x_{(1)} \neq 0$.

(iv) If $y_{(1)} = 0$, then $\Pi(x'',y) > 0$ for $x'' = (z, y_{(2)})$ and every z in Γ_1: Π_1 leads from x'' to y, because $y_{(2)} \neq 0$, and Π_2 keeps y, because $y_{(1)} = 0$.

Let x, y be given. There is x' with $x'_{(2)} \neq 0$ and $\Pi(x,x') > 0$: take $x' = x$ by (i) if $x_{(2)} \neq 0$, and otherwise $x' = (x_{(1)}, z)$ with $z \neq 0$ by (iii); such a z exists because Γ_2 is not trivial. If $y_{(1)} \neq 0$, then $\Pi(x',y) > 0$ by (ii), and $\Pi^3(x,y) \geq \Pi(x,x')\,\Pi(x',y)\,\Pi(y,y) > 0$ by (i). If $y_{(1)} = 0$, choose $z \neq 0$ in Γ_1 (it exists because Γ_1 is not trivial) and put $x'' = (z, y_{(2)})$. Then $\Pi(x',x'') > 0$ by (ii), and $\Pi(x'',y) > 0$ by (iv); so $\Pi^3(x,y) \geq \Pi(x,x')\,\Pi(x',x'')\,\Pi(x'',y) > 0$.

A bound. Every positive entry of Π_i is $1/|\Gamma_i|$ or 1, so it is at least $1/|\Gamma_i|$. A positive entry of Π is a sum of products $\Pi_1(x,z)\Pi_2(z,y)$, one of which is positive; so it is at least $1/(|\Gamma_1|\,|\Gamma_2|) = 1/g$. In the same way every entry of Π^3, which is positive, is at least $1/g^3$. Put

$$\delta = (g-1)/g^3, \qquad \text{so } 0 < \delta < 1,$$

and let Π_{u} be the matrix with all entries $1/(g-1)$. The matrix $\Pi^3 - \delta\Pi_{\mathrm{u}}$ has non-negative entries (each is at least $1/g^3 - \delta/(g-1) = 0$) and row sums $1-\delta$. So $\Pi^3 = \delta\Pi_{\mathrm{u}} + (1-\delta)\Pi_{\mathrm{r}}$ with a matrix Π_{r} that has non-negative entries and row sums 1. From $\eta\Pi^3 = \eta$ and $\eta\Pi_{\mathrm{u}} = \eta$ we get $\eta\Pi_{\mathrm{r}} = \eta$. For a row vector v put $\|v\| = \sum_y |v(y)|$. If R has non-negative

entries and row sums 1, then $\|vR\| \le \|v\|$. Let μ be a row vector with non-negative entries and sum 1. Then $\mu\Pi_{\mathrm{u}} = \eta$, and

$$\mu\Pi^3 - \eta = \delta\eta + (1-\delta)\,\mu\Pi_{\mathrm{r}} - \eta = (1-\delta)\,(\mu-\eta)\,\Pi_{\mathrm{r}},$$

and so $\|\mu\Pi^3 - \eta\| \le (1-\delta)\,\|\mu - \eta\|$. The vector $\mu\Pi^3$ again has non-negative entries and sum 1. By induction, for every integer $j \ge 0$,

$$\|\mu\Pi^{3j} - \eta\| \le (1-\delta)^j\,\|\mu - \eta\| \le 2\,(1-\delta)^j.$$

(6) *Zeros.* Every $f^{(r)}$ is nowhere zero. So an edge e is a zero of the A-coordinate of $f^{(r)}$ if and only if $f^{(r)}(e)$ is in $B \setminus \{0\}$. Let $j \ge 0$. After $6j$ rounds, that is $3j$ pairs of rounds, the law of $f^{(6j)}(e)$ is $\mu_{6j} = \mu_0\Pi^{3j}$ by (4). Let $\mathcal{E}_e$ be the event that e is a zero of the A-coordinate of $f^{(6j)}$. By (5),

$$\begin{aligned}\Pr[\mathcal{E}_e] &= \sum_{y\in B\setminus\{0\}} \mu_{6j}(y) \le \sum_{y\in B\setminus\{0\}} \Big(\frac{1}{g-1} + \Big|\mu_{6j}(y) - \frac{1}{g-1}\Big|\Big)\\ &\le \frac{|B|-1}{|\Gamma|-1} + \|\mu_{6j} - \eta\| \le \frac{|B|-1}{|\Gamma|-1} + 2\,(1-\delta)^j.\end{aligned}$$

By hypothesis $(|B|-1)/(|\Gamma|-1) < 1/9$. Fix j so large that $2\,(1-\delta)^j < 1/9 - (|B|-1)/(|\Gamma|-1)$. Then every edge is a zero with probability less than $1/9$. By linearity of expectation, the expected number of zeros of the A-coordinate of $f^{(6j)}$ is less than $m/9$. So some sequence of choices gives a flow $f^{(6j)}$ with fewer than $m/9$ zeros. By (4) it is a nowhere-zero Γ-flow that is Σ-joined to f. This proves the first statement.

(AV) is the first statement for the loopless 3-edge-connected multigraphs with at least two vertices. (AV″) follows from (AV) by Lemma 3.3. The group B is used only in step (6), through its order; it need not be a direct factor. □

Remark 4.2 (The estimate behind Lemma 4.1)**.** Steps (1)–(5) and the first display of step (6) of the proof of Lemma 4.1 do not use the hypothesis $(|B|-1)/(|\Gamma|-1) < 1/9$. They show: let $\Gamma = \Gamma_1 \times \Gamma_2$ with both factors non-trivial, let B be any subgroup of Γ, and let Σ be as in Lemma 4.1. Then for every multigraph G, every nowhere-zero Γ-flow f of G and every integer $j \ge 0$, every outcome of the random flow $f^{(6j)}$ of the proof is a nowhere-zero Γ-flow that is Σ-joined to f, and every edge of G is a zero of the A-coordinate of $f^{(6j)}$ with probability at most

$$\frac{|B|-1}{|\Gamma|-1} + 2\,(1-\delta)^j, \qquad \delta = \frac{|\Gamma|-1}{|\Gamma|^3}.$$

Part II uses this estimate also at the ratio $1/9$ (Lemma 14.3).

Remark 4.3 (The ratio)**.** Put $a = |A|$ and $b = |B|$, so $|\Gamma| = ab$. The condition $(b-1)/(ab-1) < 1/9$ is the inequality $9b - 9 < ab - 1$, that is $(9-a)\,b < 8$. It holds for every $a \ge 9$. More precisely, $(b-1)/(ab-1) < 1/a$ for every $a \ge 2$, because this inequality is $a(b-1) < ab - 1$, that is $a > 1$.

Remark 4.4 (Internal direct sums)**.** Lemma 4.1 speaks only of the group Γ, its subgroup B, the set Σ and flows; an isomorphism of groups carries all of them along. So the lemma applies to a group Γ with two non-trivial subgroups Γ_1, Γ_2 such that $\Gamma_1 \cap \Gamma_2 = \{0\}$ and $\Gamma_1 + \Gamma_2 = \Gamma$: the map $(x_1, x_2) \mapsto x_1 + x_2$ is an isomorphism of $\Gamma_1 \times \Gamma_2$ onto Γ, and Σ must contain a set of generators of Γ_1 and a set of generators of Γ_2. For $\Gamma = \mathbb{Z}_k$ with $k = cc'$, where c, $c' \ge 2$ are coprime, take $\Gamma_1 = c'\mathbb{Z}_k$ and $\Gamma_2 = c\mathbb{Z}_k$. Then Γ_1 has order c and is

generated by c', and Γ_2 has order c' and is generated by c. Their intersection is $\{0\}$, because its order divides c and c'; so $|\Gamma_1 + \Gamma_2| = cc' = k$.

A consequence for groups.

Theorem 4.5 (The group theorem for a subgroup of index at least 10)**.** *Let Γ be a finite abelian group that is not cyclic of prime-power order, and let $|\Gamma|$ have a divisor d with $4 \le d \le |\Gamma|/10$. Then $F(G,\Gamma)$ is connected for every finite bridgeless multigraph G (loops allowed). The smallest orders are 40, 44, 48, 50. (This is Corollary 3.15 with $\Sigma = \Gamma \setminus \{0\}$ and a subgroup B of order d; (AV″) by Lemma 4.1.)*

Proof. $|\Gamma| \ge 10\,d \ge 40$. The group Γ is a direct product of cyclic groups of prime-power order. It is not cyclic of prime-power order and not trivial, so there are at least two factors, and $\Gamma = \Gamma_1 \times \Gamma_2$ with both factors non-trivial. Let B be a subgroup of Γ of order d; it exists, because d divides $|\Gamma|$. Put $A = \Gamma/B$. Then $|B| = d \ge 4$ and $|A| = |\Gamma|/d \ge 10$.

Put $\Sigma = \Gamma \setminus \{0\}$. This is a step system. Every move of $F(G,\Gamma)$ is a Σ-move. Here $\Sigma_B = B \setminus \{0\}$, and $\Sigma_A = A \setminus \{0\}$, because every non-zero element of A is the image of a non-zero element of Γ. So (R1), (R2) and (R3*) hold with walks of one step: two different elements of $A \setminus X$, of $B \setminus X$, or of $\Gamma \setminus \{0\}$, differ by an element of Σ_A, of Σ_B, or of Σ. The set Σ contains all non-zero elements of $\Gamma_1 \times 0$ and of $0 \times \Gamma_2$, and the index of B is $|A| \ge 10$; so (AV″) holds by Lemma 4.1. Corollary 3.15 gives the connectivity of $F(G,\Gamma)$.

The orders: an order n of such a group has a divisor d with $4 \le d \le n/10$, so $n \ge 40$. For $40 \le n \le 50$ such a divisor exists exactly for $n = 40$, 44, 48 ($d = 4$) and $n = 50$ ($d = 5$); for these n the cyclic group $\mathbb{Z}_n$ is not of prime-power order. □

For the cyclic group $\mathbb{Z}_k$ the theorem is a statement on $M_k(G) = F(G,\mathbb{Z}_k)$ only. It says nothing about the integer graph $I_k(G)$: Theorem 3.19 needs a step system of liftable elements, and the lemmas of Section 1 do not show that every element of $\mathbb{Z}_k \setminus \{0\}$ is liftable (Remark 8.11).

5. The moduli 40, 48 and 66; the divisor criterion

This section proves that $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G when $k = 40$, $k = 48$ or $k = 66$ (Theorems 5.3 and 5.4, Corollary 5.5), and Criterion 5.11 for $k = ab$ with $a \ge 10$ and a composite $b \ge 4$. Every result comes from Theorem 3.11, in the form of Corollary 3.15, and from Theorem 3.19. Lemma 5.1 says which hypotheses must be checked; for each modulus they are checked one by one. Criterion 5.11 uses the theorems of Cauchy and Davenport and of Mader and Watkins, but only in its last sentence; the proofs for 40, 48 and 66 use neither of them.

Setting. Throughout this section $k = ab$ with integers $a, b \ge 2$, $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ (the multiples of a; b elements) and $A = \mathbb{Z}_a$; the quotient map is the reduction modulo a. This is the setting of Theorem 3.19. For an integer z we write z also for its residue in $\mathbb{Z}_k$. Because a divides k, the image in A of the residue of z is the residue of z modulo a.

Let D be a set of non-zero elements of $\mathbb{Z}_a$ with $D = -D$. The *graph on $\mathbb{Z}_a$ with the step set D* has the vertex set $\mathbb{Z}_a$; two vertices x, y are adjacent if $y - x$ is in D. It *stays connected after the deletion of at most 6 elements* if for every set X of at most 6 elements of $\mathbb{Z}_a$, any two elements of $\mathbb{Z}_a \setminus X$ are joined by a walk in $\mathbb{Z}_a \setminus X$ all of whose steps are in D. For $D = \Sigma_A$ this is condition (R1) of Definition 3.2.

The master theorem for $\mathbb{Z}_k$ with a quotient of order at least 10.

Lemma 5.1 (The master theorem for $\mathbb{Z}_k$ with a quotient of order at least 10)**.** *Let $k = ab$ with integers $a \geq 10$ and $b \geq 4$, $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ and $A = \mathbb{Z}_a$. Let Σ be a step system with the following three properties.*

(a) *Every element of Σ is liftable.*

(b) *Σ has the properties* (R1), (R2) *and* (R3*).

(c) *$\mathbb{Z}_k$ has two non-trivial subgroups Γ_1 and Γ_2 whose orders are coprime and have the product k, and Σ contains a generator of Γ_1 and a generator of Γ_2.*

Then Γ, B and Σ satisfy the hypotheses of Theorem 3.11. Hence $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G, and $M_k(G)$ is connected also for every finite bridgeless multigraph G with loops.

Proof. The averaging condition. The subgroups of $\mathbb{Z}_k$ are cyclic; so by (c) the set Σ contains a set of generators of Γ_1 and a set of generators of Γ_2. The order of $\Gamma_1 \cap \Gamma_2$ divides the two coprime orders, so $\Gamma_1 \cap \Gamma_2 = \{0\}$, and $|\Gamma_1 + \Gamma_2| = |\Gamma_1|\,|\Gamma_2| = k$, so $\Gamma_1 + \Gamma_2 = \mathbb{Z}_k$. By Remark 4.4, Lemma 4.1 applies to these two factors. Its ratio condition holds:

$$\frac{|B|-1}{|\Gamma|-1} = \frac{b-1}{ab-1} < \frac{1}{a} \leq \frac{1}{10} < \frac{1}{9};$$

the first inequality is $a(b-1) < ab-1$, that is $a > 1$. So (AV$''$) holds by Lemma 4.1.

(H1). (R1), (R2) and (R3*) hold by (b), and (AV$''$) holds.

(H2). (C0): $|A| = a \geq 10 \geq 8$. Let H be a good subgraph that (AV$''$) gives. If H is a cycle with at most 5 edges, nothing is to be checked. If H is a subdivision, it has a frame (Lemma A.5; see Definition 2.9). Every frame of H is counted, because $|A| = a \geq 10$ and $|B| = b \geq 4$: the left sides of (C1), (C2), (C3), (C4) are at most 9, 9, 8, 3 (Remark 2.16). So (H2) holds. (This is the proof of Corollary 3.15.)

Conclusion. By (a), every element of Σ is liftable. So Theorem 3.19 applies: part (a) gives $M_k(G)$ for every finite bridgeless multigraph G, loops allowed, and part (b) gives $I_k(G)$ for every finite bridgeless loopless multigraph G. □

The moduli 40, 48 and 66. For these three moduli, every non-zero element of A except two is the image of a step. The following lemma gives (R1) in this case.

Lemma 5.2 (At most two non-steps)**.** *Let $a \geq 10$, and let D be a set of non-zero elements of $\mathbb{Z}_a$ with $D = -D$ and $|D| \geq a-3$. Then the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements.*

Proof. Let D^c be the set of the non-zero elements of $\mathbb{Z}_a$ that are not in D. Then $|D^c| \leq 2$ and $D^c = -D^c$. Let X be a set of at most 6 elements of $\mathbb{Z}_a$, and suppose that two elements of $\mathbb{Z}_a \setminus X$ are not joined by a walk in $\mathbb{Z}_a \setminus X$ with steps in D. Let U be the set of the elements of $\mathbb{Z}_a \setminus X$ that are joined to the first of them, and $V = \mathbb{Z}_a \setminus (X \cup U)$. Both are not empty, and no step of D leads from U to V. So $v - u$ is in D^c for all u in U and v in V. For a fixed u this gives $|V| \leq |D^c| \leq 2$, and for a fixed v it gives $|U| \leq 2$. But $|U| + |V| = |\mathbb{Z}_a \setminus X| \geq a-6 \geq 4$. Hence $a = 10$ and $|U| = |V| = |D^c| = 2$. The group $\mathbb{Z}_a$ has at most one non-zero element x with $x = -x$ (namely $a/2$, if a is even). So $D^c = \{d_0, -d_0\}$ with $d_0 \neq -d_0$. Write $U = \{u_1, u_2\}$. Then $V = u_1 + D^c = u_2 + D^c$. The element $u_2 + d_0$ is in $u_1 + D^c$ and differs from $u_1 + d_0$; so $u_2 + d_0 = u_1 - d_0$. In the same way $u_2 - d_0 = u_1 + d_0$.

Subtracting the two equations gives $4d_0 = 0$ in $\mathbb{Z}_{10}$. So 5 divides $2d_0$, hence 5 divides d_0, and $2d_0 = 0$ in $\mathbb{Z}_{10}$, that is $d_0 = -d_0$. This is a contradiction. □

Theorem 5.3 (The modulus 40)**.** *$M_{40}(G)$ and $I_{40}(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Here $k = 40$, $a = 10$, $b = 4$, $\Gamma = \mathbb{Z}_{40}$, $B = 10\mathbb{Z}_{40} = \{0, 10, 20, 30\}$ and $A = \mathbb{Z}_{10}$. We check the hypotheses of Lemma 5.1.

(Set) $\Sigma = \{\pm1, \pm2, \pm4, \pm5, \pm8, \pm10, 20\}$, a set of 13 elements of $\mathbb{Z}_{40}$. They are not 0, and $\Sigma = -\Sigma$ because $-20 = 20$. So Σ is a step system.

(Lifts) ±1 are liftable by Lemma 1.12 (b). The numbers 2, 4, 5, 8, 10, 20 are divisors of 40 smaller than 40; so ±2, ±4, ±5, ±8, ±10 and 20 are liftable by Lemma 1.12 (d). This is hypothesis (a) of Lemma 5.1.

(Σ_A) The residues modulo 10 of $1, -1, 2, -2, 4, -4, 5, -5, 8, -8$ are $1, 9, 2, 8, 4, 6, 5, 5, 8, 2$; those of ±10 and 20 are 0. So $\Sigma_A = \{1, 2, 4, 5, 6, 8, 9\}$: all non-zero elements of $\mathbb{Z}_{10}$ except 3 and 7.

(Σ_B) $\Sigma_B = \Sigma \cap B = \{10, 20, 30\} = B \setminus \{0\}$.

(R1) $\Sigma_A = -\Sigma_A$ and $|\Sigma_A| = 7 = a - 3$. Lemma 5.2. With (R2) and (R3*) below, this is hypothesis (b).

(R2) Two different elements of B differ by an element of $B \setminus \{0\} = \Sigma_B$. So for every set X, any two elements of $B \setminus X$ are joined by a walk of one step.

(R3*) 1 is in Σ. For $1 \le x < x' \le 39$ the walk $x, x+1, \ldots, x'$ joins x to x' in $\mathbb{Z}_{40} \setminus \{0\}$.

(Product) $\Gamma_1 = 5\mathbb{Z}_{40}$ has order 8 and is generated by 5; $\Gamma_2 = 8\mathbb{Z}_{40}$ has order 5 and is generated by 8. The orders 8 and 5 are coprime, $8 \cdot 5 = 40$, and 5, 8 are in Σ. The ratio of Lemma 4.1 is $(|B| - 1)/(|\Gamma| - 1) = 3/39 = 1/13 < 1/9$. This is hypothesis (c) of Lemma 5.1.

(Counts) $a = 10 \ge 10$ and $b = 4 \ge 4$.

Lemma 5.1 gives the theorem. □

Theorem 5.4 (The modulus 48)**.** *$M_{48}(G)$ and $I_{48}(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Here $k = 48$, $a = 12$, $b = 4$, $\Gamma = \mathbb{Z}_{48}$, $B = 12\mathbb{Z}_{48} = \{0, 12, 24, 36\}$ and $A = \mathbb{Z}_{12}$.

(Set) $\Sigma = \{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, 24\}$, a set of 17 elements of $\mathbb{Z}_{48}$. They are not 0, and $\Sigma = -\Sigma$ because $-24 = 24$. So Σ is a step system.

(Lifts) ±1: Lemma 1.12 (b). The numbers 2, 3, 4, 6, 8, 12, 16, 24 are divisors of 48 smaller than 48; so the other elements are liftable by Lemma 1.12 (d). This is hypothesis (a).

(Σ_A) The residues modulo 12 of ±1, ±2, ±3, ±4 are $1, 11, 2, 10, 3, 9, 4, 8$; those of $6, -6, 8, -8, 16, -16$ are $6, 6, 8, 4, 4, 8$; those of ±12 and 24 are 0. So $\Sigma_A = \{1, 2, 3, 4, 6, 8, 9, 10, 11\}$: all non-zero elements of $\mathbb{Z}_{12}$ except 5 and 7.

(Σ_B) $\Sigma_B = \Sigma \cap B = \{12, 24, 36\} = B \setminus \{0\}$.

(R1) $\Sigma_A = -\Sigma_A$ and $|\Sigma_A| = 9 = a - 3$. Lemma 5.2.

(R2) As for $k = 40$: $\Sigma_B = B \setminus \{0\}$.

(R3*) 1 is in Σ; the walk with steps $+1$ inside $\{1, \dots, 47\}$.

(Product) $\Gamma_1 = 3\mathbb{Z}_{48}$ has order 16 and is generated by 3; $\Gamma_2 = 16\mathbb{Z}_{48}$ has order 3 and is generated by 16. The orders 16 and 3 are coprime, $16 \cdot 3 = 48$, and 3, 16 are in Σ. The ratio of Lemma 4.1 is $3/47 < 1/9$, because $27 < 47$. The subgroup B is not one of the two factors; Lemma 4.1 does not ask this. This is hypothesis (c).

(Counts) $a = 12 \geq 10$ and $b = 4 \geq 4$.

Lemma 5.1 gives the theorem. □

Corollary 5.5 (The modulus 66)**.** *$M_{66}(G)$ and $I_{66}(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Here $k = 66$, $a = 11$, $b = 6$, $\Gamma = \mathbb{Z}_{66}$, $B = 11\mathbb{Z}_{66} = \{0, 11, 22, 33, 44, 55\}$ and $A = \mathbb{Z}_{11}$.

(Set) $\Sigma = \{\pm 1, \pm 2, \pm 3, \pm 6, \pm 11, \pm 22, 33\}$, a set of 13 elements of $\mathbb{Z}_{66}$. They are not 0, and $\Sigma = -\Sigma$ because $-33 = 33$. So Σ is a step system.

(Lifts) ± 1: Lemma 1.12 (b). The numbers 2, 3, 6, 11, 22, 33 are divisors of 66 smaller than 66; Lemma 1.12 (d). This is hypothesis (a).

(Σ_A) The residues modulo 11 of $1, -1, 2, -2, 3, -3, 6, -6$ are $1, 10, 2, 9, 3, 8, 6, 5$; those of ± 11, ± 22 and 33 are 0. So $\Sigma_A = \{1, 2, 3, 5, 6, 8, 9, 10\}$: all non-zero elements of $\mathbb{Z}_{11}$ except 4 and 7.

(Σ_B) $\Sigma_B = \Sigma \cap B = \{11, 22, 33, 44, 55\} = B \setminus \{0\}$.

(R1) $\Sigma_A = -\Sigma_A$ and $|\Sigma_A| = 8 = a - 3$. Lemma 5.2.

(R2) As for $k = 40$: $\Sigma_B = B \setminus \{0\}$.

(R3*) 1 is in Σ; the walk with steps $+1$ inside $\{1, \dots, 65\}$.

(Product) $\Gamma_1 = 6\mathbb{Z}_{66}$ has order 11 and is generated by 6; $\Gamma_2 = 11\mathbb{Z}_{66}$ has order 6 and is generated by 11. The orders 11 and 6 are coprime, $11 \cdot 6 = 66$, and 6, 11 are in Σ. The ratio of Lemma 4.1 is $5/65 = 1/13 < 1/9$. This is hypothesis (c).

(Counts) $a = 11 \geq 10$ and $b = 6 \geq 4$.

Lemma 5.1 gives the corollary. □

The three proofs in one table (D^c is the set of the non-zero elements of A that are not in Σ_A; the last column is the ratio $(|B| - 1)/(|\Gamma| - 1)$ of Lemma 4.1):

k	a	b	Σ	D^c	Σ_B	generators	ratio
40	10	4	$\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, 20$	± 3	$B \setminus \{0\}$	5, 8	3/39
48	12	4	$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, 24$	± 5	$B \setminus \{0\}$	3, 16	3/47
66	11	6	$\pm 1, \pm 2, \pm 3, \pm 6, \pm 11, \pm 22, 33$	± 4	$B \setminus \{0\}$	6, 11	5/65

Remark 5.6 (What this section does not give)**.** (a) *The modulus* 44. The only factorization $44 = ab$ with $a \geq 10$ and $b \geq 4$ is $a = 11$, $b = 4$. The divisors of 44 smaller than 44 are 1, 2, 4, 11, 22; so the step set of the kind used here is $\{\pm 1, \pm 2, \pm 4, \pm 11, 22\}$, and its set Σ_A in $\mathbb{Z}_{11}$ is $\{\pm 1, \pm 2, \pm 4\}$, six elements. The deletion of the six neighbours 1, 2, 4, 7, 9, 10 of 0 leaves 0 without a neighbour. So (R1) fails for this step set, and Criterion 5.11 does not apply. No statement of this section concerns $I_{44}(G)$. (The group $\mathbb{Z}_{44}$ is covered by Theorem 4.5, with a subgroup of order 4; that is a statement on $M_{44}(G) = F(G, \mathbb{Z}_{44})$ only.)

(b) *Primes and prime powers.* Lemma 5.1 needs $k = ab$ with $a \geq 10$ and $b \geq 4$, so k is not a prime. It also needs two non-trivial subgroups of $\mathbb{Z}_k$ of coprime orders with the product k; a prime power k has no such subgroups. So no prime and no prime power is reached in this section.

The divisor criterion. The criterion uses one more elementary lemma, for (R2), and two quoted theorems, for its last sentence.

Lemma 5.7 ($\mathbb{Z}_b$ minus two elements)**.** *Let $b \geq 4$, let d be a divisor of b with $1 < d < b$, and let X be a set of at most 2 elements of $\mathbb{Z}_b$. Then any two elements of $\mathbb{Z}_b \setminus X$ are joined by a walk in $\mathbb{Z}_b \setminus X$ all of whose steps are in $\{1, -1, d, -d\}$.*

Proof. The steps ± 1 make $\mathbb{Z}_b$ a cycle of length b. If $|X| \leq 1$, then $\mathbb{Z}_b \setminus X$ is this cycle or a path, and the steps ± 1 are enough. Let $X = \{x, x'\}$ with $x \neq x'$. The cycle without x and x' consists of two arcs, and the steps ± 1 join the elements of each arc. If one arc is empty, we are done. Otherwise, exchange the names x, x' if necessary, and translate (a translation keeps the steps), so that $x = 0$, $x' = n + 1$, and the two arcs are $I = \{1, \ldots, n\}$ and $I' = \{n + 2, \ldots, b - 1\}$ with $1 \leq |I| \leq |I'|$. Then $n \leq b - n - 2$, that is $n \leq (b - 2)/2$. Because d is a divisor of b with $d < b$, $d \leq b/2$. So

$$n + 2 \;\leq\; n + d \;\leq\; \frac{b-2}{2} + \frac{b}{2} \;=\; b - 1,$$

and $n + d$ is in I'. The step d joins n in I to $n + d$ in I'. □

The two quoted theorems are those of Cauchy and Davenport (Theorem 0.2) and of Mader and Watkins (Theorem 0.4); both are stated in the list of quoted theorems at the end of Section 0.

Lemma 5.8 (Seven steps, prime quotient)**.** *Let a be a prime, and let D be a set of non-zero elements of $\mathbb{Z}_a$ with $D = -D$ and $|D| \geq 7$. Then the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements.*

Proof. Let X be a set of at most 6 elements of $\mathbb{Z}_a$, and suppose that two elements of $\mathbb{Z}_a \setminus X$ are not joined by a walk in $\mathbb{Z}_a \setminus X$ with steps in D. Let U be the set of the elements of $\mathbb{Z}_a \setminus X$ that are joined to the first of them, and $V = \mathbb{Z}_a \setminus (X \cup U)$; both are not empty. Put $D_0 = D \cup \{0\}$, so $|D_0| = |D| + 1 \geq 8$. Let u be in U and z in D_0. If $z = 0$, then $u + z = u$ is in U. If z is in D and $u + z$ is not in X, then $u + z$ is joined to u by one step, so it is in U. Hence $U + D_0$ is a subset of $U \cup X$, and

$$|U + D_0| \;\leq\; |U| + 6.$$

The set $U + D_0$ misses V, so $|U + D_0| < a$. By Theorem 0.2 (with $p = a$), $|U + D_0| \geq \min\{a,\ |U| + |D_0| - 1\}$; the minimum is not a, because $|U + D_0| < a$. So $|U + D_0| \geq |U| + |D_0| - 1 = |U| + |D| \geq |U| + 7$. This contradicts $|U + D_0| \leq |U| + 6$. □

Lemma 5.9 (Nine steps)**.** *Let $a \geq 10$, and let D be a set of non-zero elements of $\mathbb{Z}_a$ with $D = -D$, $1 \in D$ and $|D| \geq 9$. Then the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements.*

Proof. Let Δ be the graph on $\mathbb{Z}_a$ with the step set D. It has no loop, because 0 is not in D. Two vertices x, y are joined by at most one edge, and x is adjacent to y if and only if y is adjacent to x, because $D = -D$. The neighbours of x are the $|D|$ different elements $x + z$ with z in D; so every vertex has $h = |D| \geq 9$ neighbours. Δ is connected: 1 is in D, and the

walk $x, x+1, x+2, \dots$ reaches every vertex. Δ is vertex-transitive: for every c in $\mathbb{Z}_a$ the translation $y \mapsto y+c$ maps edges to edges, and it maps 0 to c. By Theorem 0.4, the deletion of fewer than $2(h+1)/3$ vertices leaves Δ connected; and $2(h+1)/3 \geq 20/3 > 6$. So the deletion of a set X of at most 6 elements leaves a connected graph on $\mathbb{Z}_a \setminus X$; a path in it is a walk in $\mathbb{Z}_a \setminus X$ with steps in D.

The proof uses a only through the fact that $\mathbb{Z}_a$ has a set D as in the lemma; such a set exists only if $a - 1 \geq |D| \geq 9$, that is $a \geq 10$. No larger lower bound on a is used. □

Remark 5.10 (Nine steps for $a \leq 12$ without a quoted theorem)**.** For $a = 10$, 11, 12 Theorem 0.4 is not needed in Lemma 5.9: there $|D| \geq 9 \geq a - 3$, and Lemma 5.2 gives the same conclusion.

Criterion 5.11 (The divisor criterion)**.** *Let k be not a prime power, and $k = ab$ with $a \geq 10$ and a composite number $b \geq 4$. Let D be the set of the non-zero residues modulo a of ± 1, ± 2 and $\pm q$, for all divisors q of k with $1 \leq q < k$. If the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements, then $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G. This holds if D has at least 9 elements; and if a is a prime, it holds if D has at least 7 elements.*

Proof. Let $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$, $A = \mathbb{Z}_a$, and let Σ be the set of the residues in $\mathbb{Z}_k$ of the integers ± 1, ± 2 and $\pm q$, for all divisors q of k with $1 \leq q < k$. We check the hypotheses of Lemma 5.1.

(Set) $k = ab \geq 40$. So the residues of ± 1 and ± 2 are not 0, and the residue of $\pm q$ is not 0 because $1 \leq q < k$. $\Sigma = -\Sigma$ by its definition. So Σ is a step system.

(Lifts) ± 1: Lemma 1.12 (b). ± 2: Lemma 1.12 (c). $\pm q$: Lemma 1.12 (d). This is hypothesis (a).

(Σ_A) The image in A of the residue of an integer z is the residue of z modulo a. So Σ_A, the set of the non-zero images, is D.

(Σ_B) Because b is composite, it has a divisor d with $1 < d < b$. The numbers a and ad are divisors of k, and $1 \leq a < ad < ab = k$. So $\pm a$ and $\pm ad$ are in Σ. They are multiples of a, so they are in Σ_B.

(R1) It is the hypothesis, because $\Sigma_A = D$.

(R2) The map $j \mapsto aj$ is an isomorphism of $\mathbb{Z}_b$ onto B: it is well defined and additive, because $k = ab$; it is injective, because $aj = 0$ in $\mathbb{Z}_k$ means that b divides j; and its image is B. It maps 1, -1, d, $-d$ to a, $-a$, ad, $-ad$. Let X be a set of at most 2 elements of B, and X' its preimage in $\mathbb{Z}_b$. By Lemma 5.7 ($b \geq 4$), any two elements of $\mathbb{Z}_b \setminus X'$ are joined by a walk in $\mathbb{Z}_b \setminus X'$ with steps ± 1, $\pm d$. Its image is a walk in $B \setminus X$ with steps $\pm a$, $\pm ad$, which are in Σ_B.

(R3*) 1 is in Σ. For $1 \leq x < x' \leq k-1$ the walk $x, x+1, \dots, x'$ joins x to x' in $\mathbb{Z}_k \setminus \{0\}$.

(Product) Let p be a prime divisor of k, c the largest power of p that divides k, and $c' = k/c$. Then $c \geq 2$; $c' \geq 2$, because k is not a power of p; and c, c' are coprime. Both are divisors of k smaller than k, so c and c' are in Σ. The subgroup $c'\mathbb{Z}_k$ has order c and is generated by c'; the subgroup $c\mathbb{Z}_k$ has order c' and is generated by c (Remark 4.4). The ratio of Lemma 4.1 is $(b-1)/(ab-1) < 1/a \leq 1/10$. This is hypothesis (c).

(Counts) $a \geq 10$ and $b \geq 4$.

Lemma 5.1 gives the first statement. For the last sentence: 1 is in D, because $a \geq 10$; and $D = -D$. If D has at least 9 elements, Lemma 5.9 gives the deletion condition. If a is a prime and D has at least 7 elements, Lemma 5.8 gives it. □

Remark 5.12 (The three moduli as cases of the criterion)**.** The moduli 40, 48 and 66 are cases of Criterion 5.11. In each of the three proofs above, Σ is the set of the residues of ± 1, ± 2 and $\pm q$ for the divisors q of k with $1 \leq q < k$; the number b is composite (4, 4, 6); and $D = \Sigma_A$. For $k = 48$ the set D has 9 elements. For $k = 66$ the number $a = 11$ is a prime and D has 8 elements. For $k = 40$ the set D has 7 elements and $a = 10$ is not a prime, so the last sentence of the criterion does not apply; the deletion condition is Lemma 5.2. The proofs above use neither of the two quoted theorems.

6. The modulus $4p$ for a prime $p \geq 13$: the double step

This section proves Theorem 6.9: for every prime $p \geq 13$ and $k = 4p$, the graphs $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G ($M_k(G)$ also with loops). The theorem is Theorem 3.19 together with the frame form of the master theorem, Theorem 3.12.

The step system Σ of this section has only six A-parts: ± 1, ± 2, ± 4 modulo p. If six elements of A are deleted, one element can be isolated (Remark 6.2). So (R1) fails, and the walk lemma and Lemma 3.8 of Section 3 cannot be used. The frame form asks instead for (R1$^-$), which holds (Lemma 6.1), and for the conclusion (L1) of Lemma 3.8, which Lemma 6.7 proves. Its proof walks in the flows of the auxiliary graph that avoid the lists, with ordinary steps and with *double steps* (Lemma 6.5). Lemma 6.6 makes a double step in the graph. A double step leaves an isolated value through a middle flow that has exactly one zero on the protected threads. It has two routes, and its step ± 1 has two lifts, ± 1 and $\pm(1 + 2p)$. A parity argument in B shows that one of the four choices works.

Two conventions of Section 3 hold in this section. If H is a subgraph of a multigraph G, the reference orientation of G on $E(H)$ is the direction of the threads. And $u + \pi(Y)$ means: add $\pi(Y)$ on $E(H)$ and nothing outside $E(H)$.

The setting. Throughout this section p is a prime with $p \geq 13$, and

$$k = 4p, \qquad \Gamma = \mathbb{Z}_k, \qquad B = p\mathbb{Z}_k = \{0, p, 2p, 3p\}, \qquad A = \Gamma/B = \mathbb{Z}_p,$$

where the quotient map $x \mapsto \overline{x}$ is the reduction modulo p. So $k = ab$ with $a = p$ and $b = 4$, as in Theorem 3.19. As in Section 2, the A-part of x in Γ is $\overline{x}$. A *protected edge* is an edge of a protected thread. The step system of this section is

$$\Sigma = \{\pm 1,\ \pm(1 + 2p),\ \pm 2,\ \pm 4,\ \pm p,\ 2p\} \subseteq \Gamma.$$

(S1) *The group B.* B is cyclic of order 4, generated by p. Its only element of order 2 is $2p$. So $y + 2p \neq y$ for y in B, $2p + 2p = 0$, and $2p\,\xi = 2p$ for every odd integer ξ. For x in Γ: if $\overline{x} \neq 0$, then $x \neq 0$; if $\overline{x} = 0$, then x is in B. Adding an element of B does not change the A-part.

(S2) *Small integers.* An integer ξ with $1 \leq |\xi| \leq 12$ is not 0 modulo p, because $p \geq 13$.

(S3) *Σ is a step system of liftable elements.* 0 is not in Σ: $1 + 2p$ is odd, and 1, 2, 4, p, $2p$ and their negatives are not multiples of $4p$. $\Sigma = -\Sigma$, because $-2p = 2p$. Every element of Σ is liftable (Definition 1.7), by Corollary 1.13; in detail, by Lemma 1.12: ± 1 by (b); $\pm(1 + 2p)$ by (e); ± 2 by (c), and also by (d); ± 4, $\pm p$, $2p$ by (d), because 4, p, $2p$ are divisors of k smaller than k, $-p = 3p$ and $-2p = 2p$.

(S4) Σ_B, Σ_A *and the lifts.* The elements ±1, $\pm(1+2p)$, ±2, ±4 are ±1, ±2, ±4 modulo p, so they are not in B (S2). Hence $\Sigma_B = \Sigma \cap B = \{p, 2p, 3p\} = B \setminus \{0\}$. The A-parts are

$$\Sigma_A = \{1, -1, 2, -2, 4, -4\} \subseteq \mathbb{Z}_p$$

(the A-part of $1+2p$ is 1). These six elements are different and non-zero: they and their differences are integers ξ with $1 \le |\xi| \le 8$ (S2). The fixed lifts (Definition 3.1) are $\gamma(\sigma) = \sigma$, the integer σ in $\{\pm1, \pm2, \pm4\}$ read modulo k. The elements $\pm(1+2p)$ are used only in the double step (Lemma 6.6).

(S5) *The two lifts of* ±1. Let μ_1 be 1 or -1. The elements μ_1 and $\mu_1(1+2p)$ of Σ both have the A-part μ_1, and $\mu_1(1+2p) - \mu_1 = 2p\,\mu_1 = 2p$ by (S1).

(S6) *Condition* (R2). Let X be a set of at most 2 elements of B. Two elements of $B \setminus X$ differ by an element of $B \setminus \{0\} = \Sigma_B$: they are joined by a walk with one step. So (R2) holds. Also $|B| = 4$.

Six steps modulo p. The proof of the next lemma uses two quoted theorems on sums of sets in $\mathbb{Z}_p$: the theorem of Cauchy and Davenport (Theorem 0.2) and the theorem of Vosper (Theorem 0.3). For subsets $\mathcal{U}$, $\mathcal{V}$ of $\mathbb{Z}_p$ we write $\mathcal{U} + \mathcal{V} = \{\, x + x' : x \in \mathcal{U},\ x' \in \mathcal{V} \,\}$.

Lemma 6.1 (Six steps modulo p)**.** *Let X be a set of at most 6 elements of $A = \mathbb{Z}_p$, and consider the graph on $A \setminus X$ in which x and x' are adjacent if $x' - x$ is in $\Sigma_A = \{\pm1, \pm2, \pm4\}$.*

(a) *If $|X| \le 5$, this graph is connected. So* (R1$^-$) *holds for Σ.*

(b) *If $|X| = 6$ and $X \ne c + \Sigma_A$ for every c in A, this graph is connected.*

(c) *If $X = c + \Sigma_A$, then c is not in X, no other element c' has $X = c' + \Sigma_A$, and the graph has exactly two components: $\{c\}$ and $A \setminus (X \cup \{c\})$.*

Proof. Put $\Sigma_A^0 = \Sigma_A \cup \{0\}$; it has 7 elements (S4). Let Ω be the vertex set of a component of the graph, and assume $\Omega \ne A \setminus X$.

(1) $\Omega + \Sigma_A^0 \subseteq \Omega \cup X$. Reason: for x in Ω and σ in Σ_A^0, the element $x + \sigma$ is in X, or it is a vertex that is equal or adjacent to x, hence in Ω. So $|\Omega + \Sigma_A^0| \le |\Omega| + |X|$. Some vertex is not in Ω, so $\Omega \cup X \ne A$ and $|\Omega + \Sigma_A^0| \le p - 1$.

(2) By the theorem of Cauchy and Davenport, $|\Omega + \Sigma_A^0| \ge \min\{p,\ |\Omega| + 6\}$. By (1) this minimum is not p; so $|\Omega + \Sigma_A^0| \ge |\Omega| + 6$. With (1): $|X| \ge 6$. This proves (a): if $|X| \le 5$, no such Ω exists. If $|X| = 6$, then $|\Omega + \Sigma_A^0| = |\Omega| + 6$, and $\Omega + \Sigma_A^0 = \Omega \cup X$, because the two sets have the same number of elements (Ω and X are disjoint). From now on $|X| = 6$.

(3) Σ_A^0 is not an arithmetic progression. Suppose that $\Sigma_A^0 = \{\, b' + rd : r = 0, 1, \dots, 6 \,\}$ with $d \ne 0$. The sum of the seven elements is 0, because $\Sigma_A = -\Sigma_A$; it is also $7b' + 21d = 7(b' + 3d)$. Since $p \ne 7$, $b' = -3d$, and $\Sigma_A^0 = \{\, rd : -3 \le r \le 3 \,\}$. Let x be the inverse of d in $\mathbb{Z}_p$. Then $x\,\Sigma_A^0 = \{-3, \dots, 3\}$, so x, $2x$ and $4x$ are in $\{\pm1, \pm2, \pm3\}$. If $x = \pm1$, then $4x = \pm4$; if $x = \pm2$, then $2x = \pm4$; if $x = \pm3$, then $2x = \pm6$. But ±4 and ±6 are not in $\{\pm1, \pm2, \pm3\}$: the differences are integers ξ with $1 \le |\xi| \le 9$ (S2). This is a contradiction.

(4) By (2), $|\Omega + \Sigma_A^0| = |\Omega| + |\Sigma_A^0| - 1$. If $|\Omega| \ge 2$ and $|\Omega| + 6 \le p - 2$, the theorem of Vosper says that Σ_A^0 is an arithmetic progression, against (3). So $|\Omega| = 1$, or $|\Omega| + 6 \ge p - 1$; in the second case $|\Omega| + 6 = p - 1$ by (1).

(5) If $|\Omega| = 1$, say $\Omega = \{c\}$, then (2) gives $\{c\} \cup (c + \Sigma_A) = \{c\} \cup X$. Here c is not in X (it is a vertex) and not in $c + \Sigma_A$ (0 is not in Σ_A). So $X = c + \Sigma_A$. If $|\Omega| = p - 7$, then $|\Omega \cup X| = p - 1$, so exactly one element c of A is outside $\Omega \cup X$. It is a vertex. Its component is not Ω, and it lies in $A \setminus (\Omega \cup X) = \{c\}$. So $\{c\}$ is a component, and the first case, for this component, gives $X = c + \Sigma_A$.

(b) By (5), if the graph is not connected, then $X = c + \Sigma_A$ for some c.

(c) Let $X = c + \Sigma_A$. Then c is not in X, because 0 is not in Σ_A. If also $X = c' + \Sigma_A$, then the sum of the elements of X is $6c = 6c'$, so $c = c'$ by (S2). All neighbours $c + \sigma$ of c are in X, so $\{c\}$ is a component. The set $A \setminus (X \cup \{c\})$ has $p - 7 \geq 6$ elements; let Ω be a component that meets it. $\Omega \neq A \setminus X$, because $\{c\}$ is another component. By (4) and (5), $|\Omega| = 1$ or $|\Omega| = p - 7$. If $|\Omega| = 1$, then (5) gives $\Omega = \{c'\}$ with $X = c' + \Sigma_A$, so $c' = c$; but Ω meets $A \setminus (X \cup \{c\})$. So $|\Omega| = p - 7$, and $\Omega \subseteq A \setminus (X \cup \{c\})$ is all of this set. Hence the components are $\{c\}$ and $A \setminus (X \cup \{c\})$. □

Remark 6.2 (Condition (R1) fails)**.** Let $X = \Sigma_A$. Then 0 is not in X, every element $0 + \sigma$ with σ in Σ_A is in X, and $p - 7 \geq 6$ elements other than 0 are outside X. So 0 is isolated, and (R1) is false for Σ. Theorem 3.11 and Corollary 3.15 do not apply. By Lemma 6.1, this is the only way in which six deleted elements disconnect A: $X = c + \Sigma_A$, with the one isolated value c.

Double steps on the auxiliary graph. From here to the end of the proof of Lemma 6.7, H is a subdivision as in Definition 2.1 (SUB), with the graphs H_0, Q of Definition 2.6, and a frame of H is given (Definition 2.9). $C_1, \dots, C_s$ are the official cycles (Definition 2.10). For a node j, $C_j^e = C_j$ and C_j^f are the two cycles of the node j (Definition 2.12), and $\Delta_j = \chi_{C_j^f} - \chi_{C_j^e}$. We use three facts of Lemma 2.13:

(TC1) $\chi_{C_j^e}$ and $\chi_{C_j^f}$ are equal on N_j (part (a)).

(TC2) If $h > j$, no edge of N_h lies on C_j^e or on C_j^f. If $i < j$, then C_j^e does not contain f_i, and C_j^f does not contain e_i. (Part (b): if i is below j, this is its first sentence; if not, no edge of N_i lies on the two cycles. A node $h > j$ is not below j.)

(TC3) Δ_j is 0 on $N_j \cup N_{j+1} \cup \cdots \cup N_s$. For every x in A, the A-flow $x\Delta_j$ is a sum of flows $x'\chi_C$ with x' in A and C a cycle of Q that has no edge in $N_j \cup \cdots \cup N_s$. (Part (c): the cycles C lie in $Q_{U_j'} \cup Q_{U_j''}$, whose edges are red edges and edges of pairs N_h with h below j, so $h < j$.)

The sets X_i. Let Λ be a list assignment (Definition 2.14). For a node i, an edge t of N_i and an A-flow Z of Q put

$$X_i^t(Z) = \{\, c \in A : Z(t) + c\,\chi_{C_i}(t) \in \Lambda(t) \,\}, \qquad X_i(Z) = X_i^{e_i}(Z) \cup X_i^{f_i}(Z).$$

These are the sets of the proof of the walk lemma of Section 3. By (O2) of Lemma 2.11, $\chi_{C_i}(t)$ is 1 or -1. Hence:

(Y1) $|X_i^t(Z)| = |\Lambda(t)|$, and $X_i^t(Z)$ depends only on the value $Z(t)$.

(Y2) For c in A, the value of $Z + c\chi_{C_i}$ on t is in $\Lambda(t)$ if and only if c is in $X_i^t(Z)$. In particular $Z(e_i) \notin \Lambda(e_i)$ and $Z(f_i) \notin \Lambda(f_i)$ if and only if 0 is not in $X_i(Z)$.

(Y3) $X_i(Z + c\chi_{C_i}) = X_i(Z) - c$ for c in A.

(Y4) If an A-flow Z' is equal to Z on N_i, then $X_i^t(Z') = X_i^t(Z)$ for both edges t of N_i.

Definition 6.3 (Charged node; double step)**.** Let Λ be a list assignment with $|\Lambda(t)| \le 3$ for every blue edge t, let Z be an A-flow of Q, and let j be a node.

(a) The node j is *charged* at Z if $X_j(Z) = \Sigma_A$. Then exactly one edge t of N_j has 1 in $X_j^t(Z)$ (Lemma 6.4 (a)). It is the *charged edge*, written t^*; it is an original blue edge. If $\Lambda = \Lambda_f$ for an H-good flow f, the protected thread P_{t^*} is the *charged thread*; it has 3 edges, because $3 = |\Lambda_f(t^*)| \le \ell(t^*) \le 3$.

(b) Let j be charged at Z. An *admissible pair* is a pair (μ_1, μ_2) of integers with $\mu_1 \in \{1, -1\}$ and $\mu_2 \in \{2, -2, 4, -4\}$ such that μ_1 and μ_2, read modulo p, are in $X_j^{t^*}(Z)$, and $\mu_1 + \mu_2$, read modulo p, is not in Σ_A.

(c) A *double step* of the node j at Z is given by Z, a node j that is charged at Z, and an admissible pair (μ_1, μ_2). Its four *flows* are

$$Z \text{ (start)}, \qquad Z^{(1)} = Z + \mu_1 \chi_{C_j^e} \quad \text{and} \quad Z^{(2)} = Z + \mu_2 \chi_{C_j^f} \text{ (middle)},$$
$$Z^{(3)} = Z + \mu_1 \chi_{C_j^e} + \mu_2 \chi_{C_j^f} \text{ (end)}.$$

The number $z = \mu_1 + \mu_2$ is its *end value*.

(d) The double step is *good* for Λ if Z and $Z^{(3)}$ are in $P(\Lambda)$, and $Z^{(1)}(t) \notin \Lambda(t)$ and $Z^{(2)}(t) \notin \Lambda(t)$ for every blue edge t other than t^*.

Lemma 6.4 (The four flows of a double step)**.** *Let Λ be a list assignment with $|\Lambda(t)| \le 3$ for every blue edge t, let Z be an A-flow of Q, and let the node j be charged at Z.*

(a) *Both edges of N_j are original, $|\Lambda(e_j)| = |\Lambda(f_j)| = 3$, and the sets $X_j^{e_j}(Z)$, $X_j^{f_j}(Z)$ are disjoint. So the charged edge t^* is well defined. An admissible pair exists. Every admissible pair has the end value $z \in \{3, -3, 5, -5\}$, and z is not in $\Sigma_A \cup \{0\}$.*

(b) *Let (μ_1, μ_2) be an admissible pair. On the edges of N_j the four flows Z, $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$ are equal to $Z + x\chi_{C_j}$ with $x = 0$, μ_1, μ_2, z. Hence Z and $Z^{(3)}$ have no value in Λ on N_j; $Z^{(1)}$ and $Z^{(2)}$ have a value in $\Lambda(t^*)$ on t^* and no value in Λ on the other edge of N_j. On N_h with $h > j$, the four flows are equal to Z.*

(c) *Let $i < j$ be a node. Then $Z^{(2)}(e_i) = Z(e_i)$, $Z^{(3)}(e_i) = Z^{(1)}(e_i)$, $Z^{(1)}(f_i) = Z(f_i)$ and $Z^{(3)}(f_i) = Z^{(2)}(f_i)$. So each edge of N_i has at most two values in the four flows.*

(d) $Z^{(3)} = Z + z\chi_{C_j} + \mu_2 \Delta_j$.

Proof. (a) By (Y1), $|X_j(Z)| \le |\Lambda(e_j)| + |\Lambda(f_j)| \le 6 = |\Sigma_A|$ (S4). Since $X_j(Z) = \Sigma_A$, equality holds everywhere: both lists have 3 elements, and the two sets are disjoint. A list assignment is empty on the expansion edges (Definition 2.14); so e_j and f_j are original. The element 1 of Σ_A lies in exactly one of the two disjoint sets; so t^* is well defined. Now $X_j^{t^*}(Z) = \{1, x, x'\}$ with two different elements x, x' of $\Sigma_A \setminus \{1\} = \{-1, 2, -2, 4, -4\}$. If one of x, x' is in $\{2, 4, -4\}$, the pair $(1, \text{this element})$ is admissible: its end value is 3, 5 or -3, not in Σ_A. If not, $\{x, x'\} \subseteq \{-1, -2\}$, so x and x' are -1 and -2, and $(-1, -2)$ is admissible: its end value is -3. For every pair (μ_1, μ_2) with $\mu_1 = \pm 1$ and $\mu_2 \in \{\pm 2, \pm 4\}$, the sum $\mu_1 + \mu_2$ is one of ± 1, ± 3, ± 5; for an admissible pair it is not ± 1, because ± 1 is in Σ_A. So z is one of ± 3, ± 5. Each of ± 3, ± 5 differs from each of 0, ± 1, ± 2, ± 4 by an integer ξ with $1 \le |\xi| \le 9$; so z is not in $\Sigma_A \cup \{0\}$ (S2).

(b) $C_j^e = C_j$, and by (TC1) $\chi_{C_j^f} = \chi_{C_j}$ on N_j. This gives the first sentence. By (Y2), $Z + x\chi_{C_j}$ has a value in $\Lambda(t)$ on the edge t of N_j if and only if x is in $X_j^t(Z)$. The number 0

is not in $\Sigma_A = X_j(Z)$, and z is not in Σ_A by (a). μ_1 and μ_2 are in $X_j^{t^*}(Z)$, and not in the set of the other edge, because the two sets are disjoint. The last sentence is (TC2).

(c) By (TC2), C_j^f does not contain e_i and C_j^e does not contain f_i. So the value on e_i changes only by the term $\mu_1\chi_{C_j^e}$, which $Z^{(1)}$ and $Z^{(3)}$ have and Z, $Z^{(2)}$ do not have; the value on f_i changes only by the term $\mu_2\chi_{C_j^f}$, which $Z^{(2)}$ and $Z^{(3)}$ have.

(d) $\mu_1\chi_{C_j^e} + \mu_2\chi_{C_j^f} = (\mu_1+\mu_2)\chi_{C_j} + \mu_2(\chi_{C_j^f} - \chi_{C_j^e})$, because $C_j^e = C_j$. □

The walk lemma with double steps.

Lemma 6.5 (Walk lemma with double steps)**.** *Let Λ be a list assignment with $|\Lambda(t)| \le 3$ for every blue edge t. Then any two Y, Y' in $P(\Lambda)$ are joined by a sequence $Y = Y_0, Y_1, \dots, Y_\nu = Y'$ in $P(\Lambda)$ in which every pair Y_{r-1}, Y_r is*

- *an* ordinary step*: $Y_r - Y_{r-1} = \sigma\chi_C$ with σ in Σ_A and a cycle C of Q with a direction; or*
- *a* double step*: Y_{r-1} and Y_r are the start and the end, in one of the two orders, of a double step that is good for Λ.*

Proof. For $i = 1, \dots, s+1$ let Λ_i be the list assignment that is equal to Λ on the edges of $N_i \cup \dots \cup N_s$ and empty on all other blue edges. So $\Lambda_1 = \Lambda$, because every blue edge lies in a pair, and $P(\Lambda_{s+1})$ is the set of all A-flows of Q. For $j \ge i$ the assignments Λ_i and Λ agree on N_j; so the sets X_j, the charged nodes j, their charged edges and the double steps of the node j are the same for Λ_i and for Λ. Claim(i): any two members of $P(\Lambda_i)$ are joined by a sequence in $P(\Lambda_i)$ whose steps are ordinary steps and double steps of nodes $j \ge i$ that are good for Λ_i. Claim(1) is the lemma.

(1) *Claim*$(s+1)$. Let Y, Y' be A-flows of Q. By (O5) of Lemma 2.11, $Y' - Y = x_1\chi_{C_1} + \dots + x_s\chi_{C_s}$ with x_l in A. Every x_l is a sum of copies of 1, and 1 is in Σ_A. Add these terms one by one: this is a sequence of ordinary steps from Y to Y'.

(2) *Notation.* Let $1 \le i \le s$ and write $\chi = \chi_{C_i}$. By (O3) of Lemma 2.11, C_i has no edge in $N_{i+1} \cup \dots \cup N_s$. With (Y2): for Z in $P(\Lambda_{i+1})$ and c in A,

$$(*) \qquad Z + c\chi \in P(\Lambda_{i+1}); \qquad \text{and} \qquad Z + c\chi \in P(\Lambda_i) \ \text{ if and only if } \ c \notin X_i(Z).$$

Also $|X_i(Z)| \le 6$ by (Y1).

(3) *From Claim*$(i+1)$ *to Claim*(i). Let Y, Y' be in $P(\Lambda_i)$. They are in $P(\Lambda_{i+1})$. By Claim$(i+1)$ there is a sequence $Y = Y_0, \dots, Y_\nu = Y'$ in $P(\Lambda_{i+1})$ whose steps are ordinary steps and double steps of nodes $j \ge i+1$ that are good for Λ_{i+1}.

(3a) *The corrections.* Choose $d_1, \dots, d_\nu$ in A as follows. If the step r is ordinary: d_r outside $X_i(Y_{r-1}) \cup X_i(Y_r)$, a set of at most 12 elements. If the step r is a double step of a node $j \ge i+1$, with the four flows Z, $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$: d_r outside $X_i(Z) \cup X_i(Z^{(1)}) \cup X_i(Z^{(2)}) \cup X_i(Z^{(3)})$. This set has at most 12 elements. Reason: by (Y1) the part $X_i^{e_i}$ of each of the four sets depends only on the value on e_i; by Lemma 6.4 (c), with $i < j$, the four flows have at most two values on e_i; so the four parts $X_i^{e_i}$ have at most $2\,|\Lambda(e_i)| \le 6$ elements together. The same holds for f_i. In both cases d_r exists, because $p \ge 13 > 12$. (This is the count $12 < p$. The proof uses $p \ge 13$ also through Lemmas 6.1 and 6.4.)

(3b) *The shifted steps.* Let $1 \le r \le \nu$. The flows $Y_{r-1} + d_r\chi$ and $Y_r + d_r\chi$ are in $P(\Lambda_i)$, by $(*)$: d_r is not in $X_i(Y_{r-1}) \cup X_i(Y_r)$ (for a double step, $\{Y_{r-1}, Y_r\} = \{Z, Z^{(3)}\}$).

If the step r is ordinary, their difference is the same step $\sigma\chi_C$. Let the step r be a double step of the node $j \geq i+1$ with start Z and pair (μ_1, μ_2). Then $Z + d_r\chi$ is the start of a double step of the node j with the same pair, and this double step is good for Λ_i. Reasons: C_i has no edge in N_j (O3), so $Z + d_r\chi = Z$ on N_j; by (Y4), $X_j^t(Z + d_r\chi) = X_j^t(Z)$ for both edges t of N_j; so j is charged at $Z + d_r\chi$, with the same charged edge t^*, and (μ_1, μ_2) is admissible. The four flows of the new double step are $Z^{(x)} + d_r\chi$, $x = 0, 1, 2, 3$ (with $Z^{(0)} = Z$). Start and end are in $P(\Lambda_i)$, as shown. The middle flow $Z^{(1)} + d_r\chi$ has no value in Λ on N_i, because $d_r \notin X_i(Z^{(1)})$ (Y2); on $N_{i+1} \cup \cdots \cup N_s$ it is equal to $Z^{(1)}$ (O3), which has no value in $\Lambda_{i+1} = \Lambda$ there except on t^*, because the old double step is good for Λ_{i+1}; and Λ_i is empty on the other blue edges. The same holds for $Z^{(2)} + d_r\chi$. So the shifted pair $Y_{r-1} + d_r\chi$, $Y_r + d_r\chi$ is a double step of a node $j \geq i$ that is good for Λ_i.

(3c) *The connections.* Put $d_0 = d_{\nu+1} = 0$. For $r = 0, \ldots, \nu$ a *connection at* Y_r is a sequence in $P(\Lambda_i)$ from $Y_r + d_r\chi$ to $Y_r + d_{r+1}\chi$ whose steps are ordinary steps and double steps of the node i that are good for Λ_i. We construct one. Write $d = d_r$, $d' = d_{r+1}$ and $X = X_i(Y_r)$. Then d and d' are not in X: for $r \geq 1$ and for $r + 1 \leq \nu$ by (3a); and $d_0 = 0$ is not in $X_i(Y_0)$, $d_{\nu+1} = 0$ is not in $X_i(Y_\nu)$, by $(*)$, because Y and Y' are in $P(\Lambda_i)$. For every c outside X the flow $Y_r + c\chi$ is in $P(\Lambda_i)$ $(*)$, and $Y_r + c'\chi - (Y_r + c\chi) = (c' - c)\chi$ is an ordinary step if $c' - c$ is in Σ_A.

- If $d = d'$, the connection has no step.
- If $A \setminus X$ has a walk from d to d' with steps in Σ_A, take the flows $Y_r + c\chi$ for the terms c of the walk. By Lemma 6.1 ($|X| \leq 6$), such a walk exists unless $X = c + \Sigma_A$ for some c and exactly one of d, d' is c.
- Let $X = c + \Sigma_A$ and $d = c \neq d'$. Put $Z = Y_r + c\chi$; it is in $P(\Lambda_i)$. By (Y3), $X_i(Z) = X - c = \Sigma_A$: the node i is charged at Z. Take an admissible pair (μ_1, μ_2) (Lemma 6.4 (a)) and the double step of the node i at Z with this pair. It is good for Λ_i: Z is in $P(\Lambda_i)$; by Lemma 6.4 (b), $Z^{(3)}$ has no value in Λ on N_i, $Z^{(1)}$ and $Z^{(2)}$ have none on N_i except on t^*, and on $N_{i+1} \cup \cdots \cup N_s$ all four flows are equal to Z; and Λ_i is empty on the other blue edges. By Lemma 6.4 (d), $Z^{(3)} = Y_r + (c + z)\chi + \mu_2\Delta_i$. The connection is: the double step from Z to $Z^{(3)}$, then Part 1, then Part 2.

  Part 1. $c + z$ is not in $c + (\Sigma_A \cup \{0\}) = X \cup \{c\}$ (Lemma 6.4 (a)), and d' is not in $X \cup \{c\}$. By Lemma 6.1 (c), $A \setminus (X \cup \{c\})$ is a component of the graph of that lemma; so it has a walk from $c + z$ to d' with steps in Σ_A. Take the flows $Y_r + w\chi + \mu_2\Delta_i$ for the terms w of this walk. By (TC3), Δ_i is 0 on $N_i \cup \cdots \cup N_s$; so on these edges such a flow is equal to $Y_r + w\chi$, which has no value in Λ there, because $w \notin X$ $(*)$. Λ_i is empty on the other blue edges. So these flows are in $P(\Lambda_i)$, and two consecutive ones differ by an ordinary step on C_i.

  Part 2. By (TC3), $\mu_2\Delta_i$ is a sum of flows $x'\chi_{C'}$ with x' in A and cycles C' of Q that have no edge in $N_i \cup \cdots \cup N_s$. Write every x' as a sum of copies of 1 and subtract the flows $\chi_{C'}$ one by one: these are ordinary steps with $\sigma = -1$. Every flow on the way is equal to $Y_r + d'\chi$ on $N_i \cup \cdots \cup N_s$, so it is in $P(\Lambda_i)$. The last flow is $Y_r + d'\chi$.
- Let $X = c + \Sigma_A$ and $d' = c \neq d$. Do the same in the reverse order: from $Y_r + d\chi$ add the flows $\chi_{C'}$ of Part 2 one by one (ordinary steps with $\sigma = 1$), then walk

from d to $c+z$ as in Part 1 (d is not in $X \cup \{c\}$), and end with the double step from $Z^{(3)}$ to $Z = Y_r + c\chi$.

(3d) *The new sequence.* Put together: the connection at Y_0, the shifted step 1, the connection at $Y_1, \dots$, the shifted step ν, the connection at Y_ν. Consecutive pieces fit: the shifted step r goes from $Y_{r-1} + d_r\chi$ to $Y_r + d_r\chi$. The sequence goes from $Y_0 + d_0\chi = Y$ to $Y_\nu + d_{\nu+1}\chi = Y'$, all its terms are in $P(\Lambda_i)$, and every step is an ordinary step or a double step of a node $j \geq i$ that is good for Λ_i. This is Claim(i). □

The double step in the graph. The next lemma makes a double step in a multigraph G that contains H. Its setting:

– G is a multigraph that contains H, and u_0 is an A-flow of G; in part (a), f_0 is an H-good Γ-flow of G with A-coordinate u_0;

– $\mathcal{O}^e$ and $\mathcal{O}^f$ are two sets of directed cycles of H; the cycles in $\mathcal{O}^e$ are pairwise edge-disjoint, and so are the cycles in $\mathcal{O}^f$ (a cycle of $\mathcal{O}^e$ and a cycle of $\mathcal{O}^f$ can have common edges); $\chi^e = \sum_{O\in\mathcal{O}^e} \chi_O$ and $\chi^f = \sum_{O\in\mathcal{O}^f} \chi_O$, integer flows of H with values in $\{0, 1, -1\}$;

– $\mu_1 \in \{1, -1\}$ and $\mu_2 \in \{2, -2, 4, -4\}$ are integers, read modulo p in A and modulo k in Γ; and, on $E(G)$ (all three are equal to u_0 outside $E(H)$),

$$u_1 = u_0 + \mu_1\chi^e, \qquad u_2 = u_0 + \mu_2\chi^f, \qquad u_3 = u_0 + \mu_1\chi^e + \mu_2\chi^f.$$

Hypothesis (J):

(J1) u_3 is non-zero on every edge of every protected thread;

(J2) there are edges e_1^* and e_2^* of one protected thread P_{t^*} such that $u_1(e_1^*) = 0$ and $u_2(e_2^*) = 0$; u_1 is non-zero on every other edge of every protected thread; and u_2 is non-zero on every other edge of every protected thread.

Lemma 6.6 (The double step in G)**.** *In this setting:*

(a) *Assume* (J). *Then f_0 is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate u_3 that is equal to f_0 outside $E(H)$.*

(b) *Let g be an H-good Γ-flow of G with A-coordinate u_3. Assume* (J2)*, and assume that u_0 is non-zero on every edge of every protected thread. Then g is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate u_0 that is equal to g outside $E(H)$.*

Proof. Part (b) *from part* (a). Use (a) for the flow g, with $-\mu_1$, $-\mu_2$ and the same $\mathcal{O}^e$, $\mathcal{O}^f$. The three flows of (a) are now $u_3 - \mu_1\chi^e = u_2$, $u_3 - \mu_2\chi^f = u_1$ and $u_3 - \mu_1\chi^e - \mu_2\chi^f = u_0$. (J1) for this use says that u_0 is non-zero on the protected threads: this is assumed. (J2) for this use holds with e_2^* in the place of e_1^* and e_1^* in the place of e_2^*. And $-\mu_1 \in \{1, -1\}$, $-\mu_2 \in \{2, -2, 4, -4\}$. So (a) gives (b).

Part (a). Assume (J). The proof has fourteen steps.

(1) *Constant values on threads.* Every cycle O of H is a union of threads, and χ_O is constant, 1 or -1, on each thread of O (Lemma 2.7 (a)). The cycles of $\mathcal{O}^e$ are pairwise edge-disjoint; so a thread lies on at most one of them. Hence χ^e is constant on every thread, with a value in $\{0, 1, -1\}$; the same holds for χ^f. For a free edge ε let c_ε^e and c_ε^f be the values of χ^e and χ^f on P_ε. Moreover $\chi^e(e_1^*) = \pm 1$: $u_0(e_1^*) \neq 0$ because f_0 is H-good, and $u_1(e_1^*) = 0$; so $\mu_1\chi^e(e_1^*) = -u_0(e_1^*) \neq 0$.

(2) *Moves, routes, choices.* Let γ be one of the two lifts μ_1, $\mu_1(1+2p)$ of μ_1 (not necessarily the fixed lift of μ_1, which is μ_1); both are in Σ and have the A-part μ_1, and they differ by $2p$ (S5). The integer μ_2 is in Σ, with the A-part μ_2. The moves are, in this order:

(i) the *preparation*: for every free edge ε the move $y_\varepsilon\chi_{D_\varepsilon}$, with y_ε in B (no move if $y_\varepsilon = 0$); in Case 1 below, also one move $y\chi_D$ with y in B and a cycle D of H_P;

(ii) *route 1*: the moves $\gamma\chi_O$ for O in $\mathcal{O}^e$, one after the other, and then the moves $\mu_2\chi_O$ for O in $\mathcal{O}^f$; *route 2*: first the moves $\mu_2\chi_O$ for O in $\mathcal{O}^f$, and then the moves $\gamma\chi_O$ for O in $\mathcal{O}^e$.

A non-zero element of B is in Σ_B (S4). So every move is $\lambda\chi_O$ with λ in Σ and a cycle O of H. A *choice* is a route together with a lift γ: there are four choices. If all flows reached are nowhere zero, all moves are Σ-moves, and the last flow has the A-coordinate $u_0 + \mu_1\chi^e + \mu_2\chi^f = u_3$ (the moves (i) do not change the A-coordinate, (S1)); it is equal to f_0 outside $E(H)$; and it is H-good by (J1). We show: for some choice there are y_ε (and y) such that all flows reached are nowhere zero.

(3) *The preparation.* Let f° be the flow after (i). Its A-coordinate is u_0 (S1). The cycle D_ε contains P_ε and no other free thread, and it is directed along P_ε (Definition 2.10); D lies in H_P. So the move of ε is the only move of (i) that changes an edge of P_ε, and $f^\circ = f_0 + y_\varepsilon$ on $E(P_\varepsilon)$. All other edges that (i) changes are on protected threads, where the A-coordinate u_0 is non-zero. So all flows of (i) are nowhere zero if

(W0) $f_0(e) + y_\varepsilon \neq 0$ for every edge e of every free thread P_ε with $u_0(e) = 0$.

(An edge with $u_0(e) \neq 0$ is never 0 in (i), by (S1).)

(4) *The routes.* Assume (W0). In route 1, after some of the moves $\gamma\chi_O$, an edge e has the value $f^\circ(e) + \gamma\chi^e(e)$, with the A-coordinate $u_1(e)$, if it lies on a cycle O already used, and the value $f^\circ(e)$ otherwise; this holds because the cycles of $\mathcal{O}^e$ are pairwise edge-disjoint. After all moves $\gamma\chi_O$ and some of the moves $\mu_2\chi_O$, an edge e has the value $f^\circ(e) + \gamma\chi^e(e) + \mu_2\chi^f(e)$, with the A-coordinate $u_3(e)$, if it lies on a cycle of $\mathcal{O}^f$ already used, and $f^\circ(e) + \gamma\chi^e(e)$ otherwise. Edges outside $E(H)$ do not change. A value with a non-zero A-coordinate is not 0 (S1). So all flows of route 1 are nowhere zero if

(W1) $f^\circ(e) + \gamma\chi^e(e) \neq 0$ for every edge e of H with $u_1(e) = 0$, and

(W3) $f^\circ(e) + \gamma\chi^e(e) + \mu_2\chi^f(e) \neq 0$ for every edge e of H with $u_3(e) = 0$.

In the same way all flows of route 2 are nowhere zero if (W3) holds and

(W2) $f^\circ(e) + \mu_2\chi^f(e) \neq 0$ for every edge e of H with $u_2(e) = 0$.

We say that a choice *works* for the given y_ε (and y) if (W0), (W3) and (W1) (route 1) or (W2) (route 2) hold. By (2)–(4) it suffices to find a choice and values that work.

(5) *The protected edges.* A protected edge e has $u_0(e) \neq 0$ (f_0 is H-good) and $u_3(e) \neq 0$ (J1). By (J2), $u_1(e) = 0$ only for $e = e_1^*$, and $u_2(e) = 0$ only for $e = e_2^*$. So on the protected threads the only conditions are (W1) at e_1^* in route 1 and (W2) at e_2^* in route 2. All other conditions are at edges of free threads.

(6) *One excluded value for each edge of a free thread.* Let e be an edge of P_ε, and write $c^e = c^e_\varepsilon$, $c^f = c^f_\varepsilon$. Then $f^\circ(e) = f_0(e) + y_\varepsilon$ by (3), and

$$u_1(e) = u_0(e) + \mu_1 c^e, \qquad u_2(e) = u_0(e) + \mu_2 c^f, \qquad u_3(e) = u_0(e) + \mu_1 c^e + \mu_2 c^f.$$

In route 1 the conditions at e say that y_ε is not

$$-f_0(e) \text{ if } u_0(e) = 0 \text{ (W0)}; \qquad -f_0(e) - \gamma c^e \text{ if } u_1(e) = 0 \text{ (W1)};$$
$$-f_0(e) - \gamma c^e - \mu_2 c^f \text{ if } u_3(e) = 0 \text{ (W3)}.$$

Each of these is an element of B, because its A-part is $-u_0(e)$, $-u_1(e)$ or $-u_3(e)$, which is 0 (S1). If two of the numbers $u_0(e)$, $u_1(e)$, $u_3(e)$ are 0, the two excluded elements are equal:

- $u_0(e) = u_1(e) = 0$: then $\mu_1 c^e = 0$, so $c^e = 0$;
- $u_1(e) = u_3(e) = 0$: then $\mu_2 c^f = 0$, so $c^f = 0$ ($\mu_2 \neq 0$ modulo p, (S2));
- $u_0(e) = u_3(e) = 0$: then $\mu_1 c^e + \mu_2 c^f = 0$. If $c^e \neq 0$ and $c^f \neq 0$, this number is one of $\pm 1 \pm 2$, $\pm 1 \pm 4$; if exactly one of them is 0, it is one of ± 1, ± 2, ± 4. None of these is 0 modulo p (S2). So $c^e = c^f = 0$.

So in route 1 every edge of a free thread excludes at most one value of y_ε. In route 2 the conditions exclude $-f_0(e)$ if $u_0(e) = 0$, $-f_0(e) - \mu_2 c^f$ if $u_2(e) = 0$, and $-f_0(e) - \gamma c^e - \mu_2 c^f$ if $u_3(e) = 0$; in the same way ($u_0(e) = u_2(e) = 0$ gives $c^f = 0$, $u_2(e) = u_3(e) = 0$ gives $c^e = 0$, $u_0(e) = u_3(e) = 0$ gives $c^e = c^f = 0$) every edge excludes at most one value. For a choice let AL_ε be the set of the elements of B that no edge of P_ε excludes. A thread has at most 3 edges and $|B| = 4$, so AL_ε is not empty. The free thread P_ε is *rigid* for the choice if AL_ε has exactly one element; then P_ε has 3 edges, and they exclude three different elements. The conditions at the edges of P_ε concern y_ε only.

(7) *The condition at e_1^* or e_2^*.* By (3), $f^\circ(e_1^*) = f_0(e_1^*) + \sum y_\varepsilon \chi_{D_\varepsilon}(e_1^*)$ ($+ y\chi_D(e_1^*)$ in Case 1), the sum over the free edges ε with e_1^* on D_ε; the same for e_2^*. In route 1 the condition (W1) at e_1^* is $\Psi_1 \neq 0$, and in route 2 the condition (W2) at e_2^* is $\Psi_2 \neq 0$, where

$$\Psi_1 = f^\circ(e_1^*) + \gamma\chi^e(e_1^*), \qquad \Psi_2 = f^\circ(e_2^*) + \mu_2\chi^f(e_2^*).$$

Both are in B, because their A-parts are $u_1(e_1^*) = 0$ and $u_2(e_2^*) = 0$. If D_ε contains P_{t^*}, then χ_{D_ε} is 1 or -1, the same on all edges of P_{t^*} (1). By (4)–(6): a choice works if one can take y_ε in AL_ε for every free edge ε (and y) such that $\Psi_1 \neq 0$ (route 1) or $\Psi_2 \neq 0$ (route 2).

(8) *Case 1: P_{t^*} lies on a cycle D of H_P.* Take route 1 and $\gamma = \mu_1$. Take any y_ε in AL_ε, for every free edge ε. Then Ψ_1 is a fixed element plus $y\chi_D(e_1^*)$, and $\chi_D(e_1^*) = \pm 1$. As y runs through B, Ψ_1 runs through B; choose y with $\Psi_1 \neq 0$. The move $y\chi_D$ changes only protected edges, so the conditions at the edges of the free threads still hold. This choice works.

(9) *Case 2: P_{t^*} lies on no cycle of H_P. Two sides.* Let x_1, x_2 be the ends of P_{t^*}; they are different vertices of H_0, because H_0 is loopless. H_P is connected ((O6) of Lemma 2.11). Let H_P^- be H_P without the edges and the interior vertices of P_{t^*}. An interior vertex of P_{t^*} has degree 2 in H, with both edges on P_{t^*}. So a path of H_P from a vertex of H_P^- to x_1 stays in H_P^- until it first meets x_1 or x_2; hence every vertex of H_P^- is joined in H_P^- to x_1 or to x_2. And x_1, x_2 are not joined in H_P^-: such a path and P_{t^*} would

form a cycle of H_P through P_{t^*}. So H_P^- has exactly two components; let $V_1 \ni x_1$ and $V_2 \ni x_2$ be their vertex sets. A path of H_P with both ends in $V_1 \cup V_2$ contains P_{t^*} if and only if it has one end in V_1 and the other in V_2. Reason: a path in H_P^- stays in one component, so a path from V_1 to V_2 has an edge of P_{t^*}; its ends are not interior vertices of P_{t^*}, so it contains all of P_{t^*}. Conversely, if the path contains P_{t^*}, the rest of it consists of a path of H_P^- that ends at x_1 and a path of H_P^- that ends at x_2; so one end is in V_1 and the other in V_2.

The ends of a free thread P_ε are vertices of H_0; so they are in H_P (O6) and are not interior vertices of P_{t^*}: they are in $V_1 \cup V_2$. Call ε *crossing* if P_ε has one end in V_1 and the other in V_2. The cycle D_ε without the edges and the interior vertices of P_ε is a path of H_P between the ends of P_ε, because its other threads are protected (Definition 2.10). So D_ε contains P_{t^*} if and only if ε is crossing; and e_1^*, e_2^*, which are edges of P_{t^*}, lie on D_ε if and only if ε is crossing. In Case 2 no move $y\chi_D$ is made. So, by (7),

$$\Psi_1 = f_0(e_1^*)+\gamma\chi^e(e_1^*)+\sum_{\varepsilon \text{ crossing}} y_\varepsilon \chi_{D_\varepsilon}(e_1^*), \qquad \Psi_2 = f_0(e_2^*)+\mu_2\chi^f(e_2^*)+\sum_{\varepsilon \text{ crossing}} y_\varepsilon \chi_{D_\varepsilon}(e_2^*).$$

(10) *Claim: if, for some choice, some crossing ε_0 is not rigid, then this choice works.* Take y_ε in AL_ε for every free edge $\varepsilon \neq \varepsilon_0$. Then Ψ_r (r the route) is a fixed element plus $y_{\varepsilon_0}\chi_{D_{\varepsilon_0}}(e_r^*)$, with $\chi_{D_{\varepsilon_0}}(e_r^*) = \pm 1$ by (9) and (7). Two different elements y_{ε_0} of $\mathrm{AL}_{\varepsilon_0}$ give two different values of Ψ_r; at most one of them is 0. Take the other one; by (7) the choice works.

(11) *From now on assume that no choice works* (in Case 2). By (10), every crossing thread is rigid for each of the four choices. Let ε be crossing and e an edge of P_ε. By (6), e excludes a value in each choice. We show: $u_0(e) = 0$ or $u_3(e) = 0$. In route 1, e excludes a value, so one of $u_0(e)$, $u_1(e)$, $u_3(e)$ is 0; in route 2, one of $u_0(e)$, $u_2(e)$, $u_3(e)$ is 0.

- $c^e \neq 0$ and $c^f \neq 0$. The differences of the four numbers $u_0(e)$, $u_1(e)$, $u_2(e)$, $u_3(e)$ are $\pm\mu_1 c^e$, $\pm\mu_2 c^f$, $\pm(\mu_1 c^e + \mu_2 c^f)$ and $\pm(\mu_2 c^f - \mu_1 c^e)$: integers ξ with $1 \leq |\xi| \leq 5$, not 0 modulo p (S2). So at most one of the four numbers is 0. If it is $u_1(e)$, route 2 excludes nothing at e; if it is $u_2(e)$, route 1 excludes nothing at e. So it is $u_0(e)$ or $u_3(e)$.
- $c^e \neq 0$ and $c^f = 0$. Then $u_2(e) = u_0(e)$ and $u_3(e) = u_1(e)$; route 2 gives $u_0(e) = 0$ or $u_3(e) = 0$.
- $c^e = 0$ and $c^f \neq 0$. Then $u_1(e) = u_0(e)$ and $u_3(e) = u_2(e)$; route 1 gives $u_0(e) = 0$ or $u_3(e) = 0$.
- $c^e = c^f = 0$. Then the four numbers are equal, and route 1 gives $u_0(e) = 0$.

(12) *The same allowed set in both routes.* Let ε be crossing and e an edge of P_ε. By (6) and (11), the value that e excludes is

- $-f_0(e)$ if $u_0(e) = 0$, in both routes (if also $u_1(e)$, $u_2(e)$ or $u_3(e)$ is 0, (6) gives the same value);
- $-f_0(e) - \gamma c^e - \mu_2 c^f$ if $u_0(e) \neq 0$ and $u_3(e) = 0$, in both routes (if also $u_1(e) = 0$ in route 1, or $u_2(e) = 0$ in route 2, (6) gives the same value).

So, for each lift γ, the set AL_ε is the same in the two routes. Call e *shifting* if $u_0(e) \neq 0$, $u_3(e) = 0$ and $c^e \neq 0$, and let κ_ε be the number of the shifting edges of P_ε. It depends

neither on the route nor on the lift. (All edges of P_ε have the same c^e; if $c^e = 0$, then $\kappa_\varepsilon = 0$.)

(13) *The other lift.* Replace γ by the other lift $\gamma + 2p$ (S5). The value that a shifting edge excludes changes by $-2p\,c^e = 2p$ (S1). The value that any other edge of P_ε excludes does not change: it is $-f_0(e)$, or $-f_0(e) - \mu_2 c^f$ because $c^e = 0$. The thread is rigid for both lifts: for each lift its three excluded values are different, and the allowed value is the fourth element of B.

- $\kappa_\varepsilon = 0$: the set of the excluded values does not change; so the allowed value does not change.
- $\kappa_\varepsilon = 3$: the three excluded values change by $2p$; so the allowed value changes by $2p$.
- $\kappa_\varepsilon = 1$: the shifting edge excludes w, then $w + 2p$; the other two edges exclude w' and w'' for both lifts. By rigidity, $w' \neq w''$, and w', w'' are not in $\{w, w + 2p\}$. So $\{w', w''\} = B \setminus \{w, w + 2p\}$, and the allowed value is $w + 2p$, then w: it changes by $2p$.
- $\kappa_\varepsilon = 2$: the shifting edges exclude w, w', then $w + 2p$, $w' + 2p$; the third edge excludes w'' for both lifts. Suppose $w' \neq w + 2p$. Then w, w', $w + 2p$, $w' + 2p$ are four different elements: $w \neq w'$ by rigidity, and $w' \neq w + 2p$, which is the same as $w \neq w' + 2p$ (S1). So they are all of B, and w'' is one of them; this contradicts rigidity for one of the two lifts. So $w' = w + 2p$, the set of the excluded values $\{w, w + 2p, w''\}$ does not change, and the allowed value does not change.

So the allowed value of P_ε changes by $2p$ if κ_ε is odd, and it does not change if κ_ε is even; by (12) this holds in both routes.

(14) *Parity.* Let ω be the number of the crossing threads with κ_ε odd. Every crossing thread is rigid, so in each choice its y_ε must be its allowed value. For the other free edges take any y_ε in AL_ε; they do not enter Ψ_1, Ψ_2 (9). Since no choice works, (7) gives $\Psi_1 = 0$ for both lifts in route 1 and $\Psi_2 = 0$ for both lifts in route 2, with these values of y_ε. When γ is replaced by $\gamma + 2p$, the sum over the crossing threads in (9) changes by $2p$ for every thread with κ_ε odd ($2p \cdot (\pm 1) = 2p$, (S1)) and by 0 for the others: by $2p\,\omega$ in total, in both routes.

Route 1: the term $\gamma\chi^e(e_1^*)$ changes by $2p\,\chi^e(e_1^*) = 2p$, by (1) and (S1). So Ψ_1 changes by $2p\,(1 + \omega)$. It is 0 for both lifts; so $2p\,(1 + \omega) = 0$, and ω is odd (S1).

Route 2: the term $\mu_2\chi^f(e_2^*)$ does not depend on γ. So Ψ_2 changes by $2p\,\omega$. It is 0 for both lifts; so $2p\,\omega = 0$, and ω is even.

This is a contradiction. So in Case 2 some choice works; in Case 1 a choice works by (8). By (2) this proves (a). □

In the proof, (W1) or (W2) at e_1^* or e_2^* is the only condition that involves more than one free thread. The preparation is the preparation in B of Lemma 3.6 with one move for each free thread; it is written out here because the values y_ε must be chosen, not only avoided.

Lemmas 3.8 and 3.10 for $k = 4p$.

Lemma 6.7 (Lemma 3.8 for $k = 4p$)**.** *Let H be a subdivision as in Definition 2.1* (SUB) *with a frame. Let G be a multigraph that contains H, f an H-good Γ-flow of G with A-coordinate u, and Y a flow in $P(\Lambda_f)$. Then f is Σ-joined, by Σ-moves on cycles of H, to an H-good*

flow with A-coordinate $u+\pi(Y)$ that is equal to f outside $E(H)$. So the conclusion (L1) *of Lemma 3.8 holds for every subdivision H and every frame of H.*

Proof. (1) *The sequence.* $|\Lambda_f(t)| \le \ell(t) \le 3$ for every original blue edge t, and Λ_f is empty on the expansion edges (Definition 2.14). 0 and Y are in $P(\Lambda_f)$. Lemma 6.5, for $\Lambda = \Lambda_f$, gives a sequence $0 = Y_0, Y_1, \dots, Y_\nu = Y$ in $P(\Lambda_f)$ of ordinary steps and double steps that are good for Λ_f. We build H-good flows $f = g_0, g_1, \dots, g_\nu$ of G, each Σ-joined to the one before it by Σ-moves on cycles of H and equal to f outside $E(H)$, such that g_r has the A-coordinate $u+\pi(Y_r)$. Let g_{r-1} be built.

(2) *An ordinary step,* $Y_r - Y_{r-1} = \sigma\chi_C$. Then $u+\pi(Y_r) = u+\pi(Y_{r-1}) + \pi(\sigma\chi_C)$, and it is non-zero on every edge of every protected thread, because Y_r is in $P(\Lambda_f)$ (Definition 2.14). Lemma 3.7, for the flow g_{r-1}, the cycle C and σ, gives g_r. Its hypotheses hold: (R2) by (S6), and (C4) because $\ell(\varepsilon) \le 3 < 4 = |B|$.

(3) *A double step.* Y_{r-1} and Y_r are Z and $Z^{(3)}$, in one of the two orders, for a double step of a node j at Z with a pair (μ_1, μ_2) that is good for Λ_f. Let t^* be its charged edge. By Lemma 2.7 (d) there are sets $\mathcal{O}^e$, $\mathcal{O}^f$ of pairwise edge-disjoint directed cycles of H with $\pi(x\chi_{C_j^e}) = x\chi^e$ and $\pi(x\chi_{C_j^f}) = x\chi^f$ for every x in A, where $\chi^e = \sum_{O\in\mathcal{O}^e}\chi_O$ and $\chi^f = \sum_{O\in\mathcal{O}^f}\chi_O$. Put $u_0 = u+\pi(Z)$. Then

$$u+\pi(Z^{(1)}) = u_0+\mu_1\chi^e = u_1, \qquad u+\pi(Z^{(2)}) = u_0+\mu_2\chi^f = u_2, \qquad u+\pi(Z^{(3)}) = u_3,$$

in the notation of Lemma 6.6. We check its hypotheses.

(3a) u_0 and u_3 are non-zero on every edge of every protected thread, because Z and $Z^{(3)}$ are in $P(\Lambda_f)$ (Definition 2.14). This is (J1), and the extra hypothesis of Lemma 6.6 (b).

(3b) For an original blue edge t and an edge e of P_t: $u_1(e) = u(e)+Z^{(1)}(t)$, so $u_1(e) = 0$ if and only if $Z^{(1)}(t) = -u(e)$, an element of $\Lambda_f(t)$. For $t \ne t^*$, $Z^{(1)}(t)$ is not in $\Lambda_f(t)$, because the double step is good; so u_1 is non-zero on P_t. On t^*, $Z^{(1)}(t^*)$ is in $\Lambda_f(t^*)$ (Lemma 6.4 (b)). The set $\Lambda_f(t^*) = \{-u(e) : e \in E(P_{t^*})\}$ has 3 elements (Lemma 6.4 (a)), and P_{t^*} has at most 3 edges; so the edges of P_{t^*} give three different elements, and exactly one edge e_1^* of P_{t^*} has $u_1(e_1^*) = 0$.

(3c) In the same way, u_2 is 0 on exactly one edge e_2^* of the protected threads, and e_2^* lies on P_{t^*}. So (J2) holds, with the charged thread P_{t^*}.

If $Y_{r-1} = Z$: g_{r-1} is H-good with A-coordinate u_0, and Lemma 6.6 (a) gives an H-good flow g_r with A-coordinate $u_3 = u+\pi(Y_r)$, equal to g_{r-1} outside $E(H)$. If $Y_{r-1} = Z^{(3)}$: g_{r-1} has A-coordinate u_3, and Lemma 6.6 (b) gives g_r with A-coordinate $u_0 = u+\pi(Y_r)$, equal to g_{r-1} outside $E(H)$.

(4) *The end.* g_ν is H-good, equal to f outside $E(H)$, and has the A-coordinate $u+\pi(Y)$. □

Corollary 6.8 (Lemma 3.10 for $k = 4p$)**.** *Let G be a bridgeless multigraph and H a subgraph of G that is a cycle with at most 5 edges (case* CYC*), or a subdivision as in Definition 2.1* (SUB) *with any frame (case* SUB*). Let $G^* = G/H$ be the multigraph obtained from G by the identification of $V(H)$ to one vertex and the deletion of $E(H)$. Let f be an H-good Γ-flow of G with restriction f^* to G^*, and let $g^* = f^* + \lambda\chi_{C^*}$ be the result of a Σ-move of G^*. Then f is Σ-joined to an H-good Γ-flow g of G whose restriction to G^* is g^*. This holds for every λ in Σ, also for $\lambda = \pm(1+2p)$.*

Proof. This is the frame form of Lemma 3.10. Its hypotheses: $|A| = p \geq 13 \geq 8$; (R1$^-$) by Lemma 6.1 (a); (R2) by (S6). In the case SUB the frame must have (C2), (C3), (C4) and (L1). Every thread has at most 3 edges, so the left sides of (C2), (C3), (C4) are at most $3+3+3 = 9 < 13 \leq |A|$, $3+3+2 = 8 < 13 \leq |A|$ and $3 < 4 = |B|$; and (L1) holds by Lemma 6.7. Lemma 3.10 is stated for every λ in Σ. □

The theorem.

Theorem 6.9 (The modulus $4p$)**.** *Let $p \geq 13$ be a prime and $k = 4p$. Then*

(a) *$M_k(G)$ is connected for every finite bridgeless multigraph G (loops allowed);*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

This is Theorem 3.19 with the frame form of the master theorem (Theorem 3.12), for $a = p$, $b = 4$ and $\Sigma = \{\pm 1, \pm(1+2p), \pm 2, \pm 4, \pm p, 2p\}$.

Proof. We check the hypotheses of Theorem 3.19 with Theorem 3.12, one by one.

(1) *The groups.* $k = ab$ with $a = p \geq 2$ and $b = 4 \geq 2$; $\Gamma = \mathbb{Z}_k$, $B = p\mathbb{Z}_k$ (the multiples of p; 4 elements), $A = \mathbb{Z}_p$ (reduction modulo p). This is the setting of this section.

(2) *The step system.* Σ is a step system, and all its elements are liftable: (S3). Which statement lifts each element: ± 1 by Lemma 1.12 (b), $\pm(1+2p)$ by (e), ± 2 by (c), ± 4, $\pm p$, $2p$ by (d).

(3) *Σ_A and Σ_B.* $\Sigma_A = \{\pm 1, \pm 2, \pm 4\}$ modulo p, six different elements, with the lifts $\gamma(\sigma) = \sigma$; $\Sigma_B = \{p, 2p, 3p\} = B \setminus \{0\}$: (S4).

(4) (R1$^-$) by Lemma 6.1 (a). ((R1) is false, Remark 6.2; the frame form does not need it.)

(5) (R2) by (S6).

(6) (R3*). 1 is in Σ. In $\Gamma \setminus \{0\} = \{1, 2, \ldots, k-1\}$ any two elements are joined by a walk with steps 1 or -1 that stays in $\{1, \ldots, k-1\}$.

(7) (AV$''$). By the Chinese remainder theorem, $x \mapsto (x \bmod p,\ x \bmod 4)$ is an isomorphism from Γ onto $\mathbb{Z}_p \times \mathbb{Z}_4$. It maps B onto $0 \times \mathbb{Z}_4$, the element 4 of Σ to $(4, 0)$, which generates $\mathbb{Z}_p \times 0$ (4 is invertible modulo p), and the element p of Σ to $(0, p \bmod 4)$, which generates $0 \times \mathbb{Z}_4$ (p is odd). And

$$\frac{|B|-1}{|\Gamma|-1} = \frac{3}{4p-1} \leq \frac{3}{51} < \frac{1}{9}.$$

So Lemma 4.1, with $\Gamma_1 = \mathbb{Z}_p$ and $\Gamma_2 = \mathbb{Z}_4$, gives (AV) and (AV$''$). (A group isomorphism maps flows to flows and moves to moves, so the lemma applies to Γ.)

(8) (C0): $|A| = p \geq 13 \geq 8$.

(9) *The good subgraphs and their frames* ((H2) in the frame form). A good subgraph is a cycle with at most 5 edges or a subdivision as in Definition 2.1 (SUB). Every subdivision has a frame (Lemma A.5; see Definition 2.9). We show that every frame has (C2), (C3), (C4) and (L1). Every thread has at most 3 edges. So: (C2): $\ell(e_i)+\ell(f_i)+\max\{\ell(e_i), \ell(f_i)\} \leq 9 < 13 \leq |A|$; (C3): $\ell(e_i) + \ell(f_i) + 2 \leq 8 < 13 \leq |A|$; (C4): $\ell(\varepsilon) \leq 3 < 4 = |B|$; (L1) by Lemma 6.7. ((C1) is not needed in the frame form; it would hold, $2 \cdot 3 + 3 = 9 < 13$, but (R1) fails.)

(10) *Conclusion.* By (1)–(9) the hypotheses of Theorem 3.12 hold for Γ, B and Σ, and all elements of Σ are liftable. Theorem 3.19 gives (a) and (b). □

Remark 6.10 (Where $p \geq 13$ is used)**.** The size of p is used only at these places: (S4) (six different elements of Σ_A, $|\xi| \leq 8$); Lemma 6.1 (p prime for the two quoted theorems, $p \neq 7$, (S2) for $|\xi| \leq 9$, and $p - 7 \geq 6$ in part (c)); Remark 6.2 ($p - 7 \geq 6$); Lemma 6.4 (a) ($z \notin \Sigma_A \cup \{0\}$, $|\xi| \leq 9$); Lemma 6.5, step (3a) (the count $12 < p$, the largest count of this section); Lemma 6.6 ((S2) for $|\xi| \leq 5$); Corollary 6.8 and the proof of Theorem 6.9: (C0), (C2), (C3) and $3/(4p-1) < 1/9$. For $p \leq 11$ the count $12 < p$ fails, and this section says nothing about $k = 4p$ with $p \leq 11$. The smallest modulus of the theorem is $k = 52$, with $\Sigma = \{\pm 1, \pm 27, \pm 2, \pm 4, \pm 13, 26\}$ in $\mathbb{Z}_{52}$. Theorem 6.9 is the case (B3) of Theorem 9.2.

7. Prime powers

This section proves that $M_k(G)$ and $I_k(G)$ are connected for every prime power $k = p^N$ with $N \geq 6$ (Theorem 7.4). The proof is Theorem 3.19 with Theorem 3.11, for $\Gamma = \mathbb{Z}_k$ and the subgroup $B = p^i\mathbb{Z}_k$ of the multiples of p^i, for a suitable i. The step conditions are checked by hand. The averaging condition needs a new argument. Lemma 4.1 asks for a product of two non-trivial groups, and a cyclic group of prime-power order is not such a product. Here (AV″) is proved by a deterministic argument: moves by the steps $1, p, \ldots, p^{i-1}$ lower the p-adic valuations of the values of the flow (the *flattening*), and a count over four levels of the valuation gives a good subgraph (Lemma 7.2).

Throughout this section p is a prime, $N \geq 2$ is an integer, and $k = p^N$.

Valuations and the flattening process. For an element $x \neq 0$ of $\mathbb{Z}_k$, its *valuation* $\mathrm{val}(x)$ is the largest j in $\{0, \ldots, N-1\}$ such that x is in $p^j\mathbb{Z}_k$. So $\mathrm{val}(-x) = \mathrm{val}(x)$, and for $x \neq 0$ and $0 \leq j \leq N-1$: x is in $p^j\mathbb{Z}_k$ if and only if $\mathrm{val}(x) \geq j$.

Fact (V). *Let x, y be non-zero elements of $\mathbb{Z}_k$ with $\mathrm{val}(x) \neq \mathrm{val}(y)$. Then $x + y \neq 0$ and $\mathrm{val}(x+y) = \min\{\mathrm{val}(x), \mathrm{val}(y)\}$.*

Proof. First, let z be an integer that p does not divide, and let $0 \leq j \leq N-1$. Then $p^j z$ is not 0 in $\mathbb{Z}_k$ and has the valuation j. Indeed, it is in $p^j\mathbb{Z}_k$. If $p^j z = p^{j+1}w$ in $\mathbb{Z}_k$ for an integer w (this includes the case $p^j z = 0$), then p^N divides $p^j(z - pw)$, so p divides $z - pw$ and z, which is false.

Next, an element $x \neq 0$ with $\mathrm{val}(x) = j$ is $x = p^j z$ for an integer z that p does not divide: x is $p^j z$ for some integer z; if p divided z, then x would be in $p^{j+1}\mathbb{Z}_k$, which contradicts the choice of j if $j \leq N-2$, and gives $x = 0$ if $j = N-1$.

Now let $\mathrm{val}(x) = j < \mathrm{val}(y) = l$, and write $x = p^j z$, $y = p^l w$ with integers z, w that p does not divide. Then $x + y = p^j(z + p^{l-j}w)$, and p does not divide $z + p^{l-j}w$. By the first paragraph, $x + y \neq 0$ and $\mathrm{val}(x+y) = j$. □

For a nowhere-zero $\mathbb{Z}_k$-flow g of a multigraph G and $0 \leq j \leq N-1$ put

$$E_j(g) = \{\, e \in E(G) : \mathrm{val}(g(e)) = j \,\},$$

and define $E_{<j}(g)$, $E_{>j}(g)$, $E_{\geq j}(g)$ in the same way. The valuation of $g(e)$ does not depend on the reference orientation of e, because $\mathrm{val}(-x) = \mathrm{val}(x)$. Every edge lies in exactly one set $E_j(g)$, and $E_{\geq j}(g) = E_{>j-1}(g)$ for $j \geq 1$.

Lemma 7.1 (The flattening process)**.** *Let $1 \leq i \leq N-1$, and let Σ be a step system of $\mathbb{Z}_k$ that contains 1 and p^j for $1 \leq j \leq i-1$. Let G be a finite loopless multigraph and f a nowhere-zero $\mathbb{Z}_k$-flow of G. Then f is Σ-joined to a nowhere-zero $\mathbb{Z}_k$-flow f' such that for every j in $\{0, \ldots, i-1\}$:*

(a) *every cycle of G with no edge in $E_j(f')$ has all its edges in $E_{<j}(f')$;*

(b) $|E_{>j}(f')| \le n_j - 1$, *where* n_j *is the number of components of the multigraph* $(V(G), E_{<j}(f'))$.

Proof. The move. Let g be a nowhere-zero $\mathbb{Z}_k$-flow of G, let $0 \le j \le i-1$, and let C be a cycle of G with a direction and with no edge in $E_j(g)$. Put $g' = g + p^j\chi_C$. The element p^j is in Σ (for $j = 0$ it is the step 1), and $p^j \ne 0$ with $\mathrm{val}(p^j) = j$, because $j \le i-1 \le N-2$ (Fact (V), first paragraph, with $z = 1$). For an edge e of C we have $g'(e) = g(e) \pm p^j$ and $\mathrm{val}(g(e)) \ne j = \mathrm{val}(\pm p^j)$. By Fact (V), $g'(e) \ne 0$ and $\mathrm{val}(g'(e)) = \min\{\mathrm{val}(g(e)), j\}$. The edges outside C keep their values. So g' is nowhere zero, and $g \to g'$ is a Σ-move. On C, an edge with valuation less than j keeps its valuation, and an edge with valuation greater than j gets the valuation j.

The process. Start with $g = f$. As long as there are j in $\{0, \dots, i-1\}$ and a cycle C of G with no edge in $E_j(g)$ and with at least one edge in $E_{>j}(g)$, make the move above with these j and C (in either direction of C). The sum of the valuations of all values of g is a non-negative integer, and every move lowers it by at least 1. So the process stops. Let f' be the last flow. It is Σ-joined to f. From now on E_j, $E_{<j}$, $E_{>j}$ are the sets of f'.

(a). Let C be a cycle with no edge in E_j. The process has stopped, so C has no edge in $E_{>j}$. Every edge has a valuation, so all edges of C are in $E_{<j}$.

(b). Let G_j be the multigraph whose vertices are the n_j components of $(V(G), E_{<j})$ and whose edges are the edges of $E_{>j}$; an edge of $E_{>j}$ joins the components of its two ends (it is a loop of G_j if they are the same component). We show that G_j has no cycle; then G_j is a forest with n_j vertices, and $|E_{>j}| = |E(G_j)| \le n_j - 1$.

Suppose that G_j has a cycle O, and let e_1 be an edge of O. Go once round O. At every vertex of O (a component of $(V(G), E_{<j})$) the edge of O that arrives and the edge of O that leaves have ends in this component; join these two ends by a path inside the component. (If O is a loop, join the two ends of e_1 by a path inside its component.) This gives a closed walk W of G that uses every edge of O exactly once and otherwise only edges of $E_{<j}$. So W uses e_1 exactly once, and it uses no edge of E_j. The two ends x, y of e_1 are different, because G is loopless. The walk W without its passage through e_1 is a walk from y to x that does not use e_1. It contains a path P from y to x, and P does not contain e_1. Then P and e_1 form a cycle of G that has no edge in E_j and that has the edge e_1 of $E_{>j}$. This contradicts (a). □

Four levels give a good subgraph.

Lemma 7.2 (Four levels)**.** *Let* $4 \le i \le N-1$ *(so* $N \ge 5$*), and let* Σ *be a step system of* $\mathbb{Z}_k$ *that contains* 1 *and* p^j *for* $1 \le j \le i-1$*. Let* G *be a loopless* 3*-edge-connected multigraph with* $n \ge 2$ *vertices, and* f *a nowhere-zero* $\mathbb{Z}_k$*-flow of* G*. Then* f *is* Σ*-joined to a nowhere-zero* $\mathbb{Z}_k$*-flow* f' *such that the multigraph*

$$\big(V(G),\ \{\, e : f'(e) \notin p^i\mathbb{Z}_k \,\}\big)$$

contains a good subgraph (Definition 2.1). So, for $\Gamma = \mathbb{Z}_k$*,* $B = p^i\mathbb{Z}_k$*,* $A = \Gamma/B$ *and this* Σ*, the condition* (AV″) *of Definition 3.2 holds.*

Proof. Let f' be the flow of Lemma 7.1 for this i, and let E_j, $E_{<j}$, $E_{>j}$ be the sets of f'. For $0 \le j \le i$ let n_j be the number of components of $(V(G), E_{<j})$; for $j \le i-1$ this is the number n_j of Lemma 7.1 (b). Then $n_0 = n$, because $E_{<0}$ is empty. Let m be the number of edges of G. Every vertex of G has degree at least 3, because G is loopless and 3-edge-connected with $n \ge 2$; so $m \ge 3n/2$.

An edge e has $f'(e) \notin p^i\mathbb{Z}_k$ if and only if $\mathrm{val}(f'(e)) < i$. So the multigraph of the lemma is $(V(G), E_{<i})$. Suppose that it contains no good subgraph. Let $1 \le j \le i$. The multigraph

$(V(G), E_{<j})$ is a subgraph of $(V(G), E_{<i})$, so it contains no good subgraph either; it is loopless and has n_j components.

Step 1: many edges. We have $E_{\geq j} = E_{>j-1}$, and $j-1$ is in $\{0, \dots, i-1\}$. By Lemma 7.1 (b),

$$|E_{<j}| \;=\; m - |E_{>j-1}| \;\geq\; \tfrac{3}{2}\, n - n_{j-1} + 1.$$

Step 2: few edges. Let a component of $(V(G), E_{<j})$ have n' vertices and m' edges. If $n' \geq 3$, then $m' \leq \frac{4}{3}n' - 2$: otherwise Lemma 2.2 (a), applied to this component (a loopless multigraph with $n' \geq 3$ vertices and more than $\frac{4}{3}n' - 2$ edges), gives a good subgraph of the component, and so of $(V(G), E_{<i})$. If $n' = 2$, then $m' \leq 1$: two parallel edges form a cycle with 2 edges and 2 vertices, which is a good subgraph of the kind (CYC). If $n' = 1$, then $m' = 0$, because there is no loop. In all three cases $m' \leq \frac{4}{3}\,(n' - 1)$. The sum over the n_j components gives

$$|E_{<j}| \;\leq\; \tfrac{4}{3}\,(n - n_j).$$

Step 3: the recursion. Steps 1 and 2 give $\frac{3}{2}n - n_{j-1} + 1 \leq \frac{4}{3}n - \frac{4}{3}n_j$, that is

$$n_j \;\leq\; \tfrac{3}{4}\big(n_{j-1} - \tfrac{1}{6}\, n - 1\big) \qquad (1 \leq j \leq i).$$

The right side grows with n_{j-1}. Define numbers α_j, β_j by $\alpha_0 = 1$, $\beta_0 = 0$ and $\alpha_j = \frac{3}{4}(\alpha_{j-1} - \frac{1}{6})$, $\beta_j = \frac{3}{4}(\beta_{j-1} + 1)$. Then $n_0 = \alpha_0 n - \beta_0$, and by induction on j, $n_j \leq \alpha_j n - \beta_j$ for $0 \leq j \leq 4$ (here $4 \leq i$ is used). The values are:

j	0	1	2	3	4
α_j	1	$\frac{5}{8}$	$\frac{11}{32}$	$\frac{17}{128}$	$-\frac{13}{512}$
β_j	0	$\frac{3}{4}$	$\frac{21}{16}$	$\frac{111}{64}$	$\frac{525}{256}$

(For example $\alpha_4 = \frac{3}{4}\big(\frac{17}{128} - \frac{1}{6}\big) = \frac{3}{4} \cdot \big(-\frac{13}{384}\big)$ and $\beta_4 = \frac{3}{4} \cdot \frac{175}{64}$.) So $n_4 \leq -\frac{13}{512}\, n - \frac{525}{256} < 0$. But $n_4 \geq 1$. This contradiction shows that $(V(G), E_{<i})$ contains a good subgraph.

The last sentence of the lemma: the A-coordinate of f' is 0 exactly on the edges e with $f'(e) \in p^i\mathbb{Z}_k = B$, so $\operatorname{supp}(\overline{f'}) = \{\, e : f'(e) \notin p^i\mathbb{Z}_k \,\}$; and f' is Σ-joined to f. □

The count uses part (b) of Lemma 7.1 only for $j \leq 3$. Three levels would not be enough for this count, because $\alpha_3 > 0$.

The steps $\pm 2^j$ in $\mathbb{Z}_{2^r}$.

Lemma 7.3 (The powers of 2)**.** *Let $r \geq 2$, and let Δ_r be the graph on $\mathbb{Z}_{2^r}$ in which x and y are adjacent if $y - x$ is 2^j or -2^j for some j with $0 \leq j \leq r-1$. Then for every set X of at most $2r-2$ elements of $\mathbb{Z}_{2^r}$, the graph $\Delta_r - X$ is connected.*

Proof. The degree. Let $r' \geq 2$. In $\mathbb{Z}_{2^{r'}}$ the elements $\pm 2^j$, $0 \leq j \leq r'-1$, are $2r'-1$ different non-zero elements: they are not 0; $2^j \neq 2^l$ for $j < l$, since $1 \leq 2^j < 2^l \leq 2^{r'-1}$; and $2^j = -2^l$ with $j \leq l$ means $2^j + 2^l = 2^{r'}$ (the sum is between 2 and $2^{r'}$), so $2^j(1 + 2^{l-j}) = 2^{r'}$; for $j < l$ the factor $1 + 2^{l-j}$ is odd and at least 3, which is impossible; so $j = l = r' - 1$. Hence every vertex of $\Delta_{r'}$ has degree $2r' - 1$.

Induction on r. Δ_2 is the complete graph on 4 vertices (the steps are ± 1 and 2); after the deletion of at most 2 vertices it is connected. Let $r \geq 3$, and let X be a set of at most $2r-2$ elements. Each of the two classes of parity of $\mathbb{Z}_{2^r}$ has $2^{r-1} \geq 4$ elements. One of the two classes of parity contains at most $r-1$ elements of X. The translation $x \mapsto x+1$ is an automorphism of Δ_r that exchanges the two classes. So we may assume that the even class contains $w \leq r-1$ elements of X.

The two classes. The map $2x \mapsto x$ is a bijection from the even elements of $\mathbb{Z}_{2^r}$ onto $\mathbb{Z}_{2^{r-1}}$. Two even elements $2x$, $2y$ are adjacent in Δ_r if and only if $2(y-x) = \pm 2^j$ in $\mathbb{Z}_{2^r}$ for some $j \geq 1$ (± 1 is odd), that is $y - x = \pm 2^{j-1}$ in $\mathbb{Z}_{2^{r-1}}$ with $0 \leq j-1 \leq r-2$. So the even class induces a copy of Δ_{r-1}. In the same way the odd class induces a copy of Δ_{r-1} (by $2x+1 \mapsto x$).

The even survivors. We have $w \leq r-1 \leq 2(r-1)-2$, because $r \geq 3$. By induction the even elements outside X induce a connected graph. There are $2^{r-1} - w > 0$ of them.

Attached odd survivors. An odd element x has exactly two even neighbours, $x-1$ and $x+1$ (the only odd step is ± 1, and $x-1 \neq x+1$). Call an odd element x outside X *attached* if $x-1$ or $x+1$ is outside X. An attached element is joined to the connected set of the even survivors. If x is not attached, then $x-1$ and $x+1$ are both in X. The even elements form a cycle of length $2^{r-1} \geq 4$ in which y and $y+2$ are consecutive; so $\{x-1, x+1\}$ is an edge of this cycle with both ends in X, and different x give different edges. The $w < 2^{r-1}$ even elements of X induce a union of paths in this cycle, with at most $\max\{0, w-1\}$ edges. So at most $\max\{0, w-1\}$ odd elements outside X are not attached.

The end. If $w \leq 1$, every odd element outside X is attached, and $\Delta_r - X$ is connected. Let $w \geq 2$. There are at least $2^{r-1} - (|X| - w) \geq 2^{r-1} - 2r + 2 + w$ odd elements outside X, and at most $w-1$ of them are not attached; so at least $2^{r-1} - 2r + 3$ of them are attached. In the odd class, which induces a copy of Δ_{r-1}, every element has degree $2(r-1)-1 = 2r-3$, so it has exactly $(2^{r-1}-1) - (2r-3) = 2^{r-1} - 2r + 2$ non-neighbours in the odd class other than itself. Let x be an odd element outside X that is not attached. The attached elements are at least $2^{r-1} - 2r + 3$ odd elements other than x, so one of them is a neighbour of x. So every element outside X is joined to the even survivors, and $\Delta_r - X$ is connected. □

The theorem.

Theorem 7.4 (Prime powers)**.** *Let p be a prime, $N \geq 6$ and $k = p^N$. Then*

(a) *$M_k(G) = F(G, \mathbb{Z}_k)$ is connected for every finite bridgeless multigraph G (loops allowed);*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

For example $k = 64$, 128, 256, 729, 15625.

Proof. We apply Theorem 3.19 with $\Gamma = \mathbb{Z}_k$, $k = ab$, $B = a\mathbb{Z}_k$, $A = \mathbb{Z}_a$, and with the hypotheses of Theorem 3.11. The data of the two cases:

	$p = 2$	p odd
$a = \lvert A \rvert$	2^{N-2} (≥ 16)	p^4 (≥ 81)
$b = \lvert B \rvert$	4	p^{N-4} $(\geq p^2 \geq 9)$
i with $a = p^i$	$N-2$	4
Σ	$\{\pm 2^j : 0 \leq j \leq N-1\}$	$\{\pm 1, \pm 2\} \cup \{\pm p^j : 1 \leq j \leq N-1\}$

In both cases $B = a\mathbb{Z}_k = p^i\mathbb{Z}_k$, and $4 \leq i \leq N-1$ (for $p = 2$ because $N \geq 6$).

(Σ) *A step system.* No element of Σ is 0 in $\mathbb{Z}_k$: the integers 1, 2 and p^j with $j \leq N-1$ lie between 1 and $k-1$. And $\Sigma = -\Sigma$. So Σ is a step system (Definition 3.1).

(Lift) *Liftable.* The elements ± 1 are liftable by Lemma 1.12 (b), ± 2 by Lemma 1.12 (c), and $\pm p^j$ for $0 \leq j \leq N-1$ by Lemma 1.12 (d), because p^j is a divisor of k with $1 \leq p^j < k$. So every element of Σ is liftable.

(Σ_A) *The A-parts.* The quotient map is the reduction modulo $a = p^i$, and p^j is 0 modulo p^i for $j \geq i$. For $p = 2$: Σ_A is the set of the residues of $\pm 2^j$, $0 \leq j \leq N-3$, modulo

2^{N-2}. For p odd: Σ_A is the set of the residues of ±1, ±2, $\pm p$, $\pm p^2$, $\pm p^3$ modulo p^4.

(Σ_B) *The steps in B.* For $p = 2$: $B = \{0, a, 2a, 3a\}$, and $a = 2^{N-2}$, $3a = -2^{N-2}$, $2a = 2^{N-1}$ are in Σ; so $\Sigma_B = B \setminus \{0\}$. For p odd: $\Sigma_B = \{\pm p^j : 4 \le j \le N-1\}$, because ±1, ±2, $\pm p$, $\pm p^2$, $\pm p^3$ are not 0 modulo p^4; it contains $\pm a = \pm p^4$ and $\pm ap = \pm p^5$ ($5 \le N-1$).

(R3*) Let $x \neq x'$ be non-zero elements of $\mathbb{Z}_k$; take their representatives in $\{1, \dots, k-1\}$, with $x < x'$. The walk $x, x+1, \dots, x'$ stays in $\{1, \dots, k-1\}$, so it avoids 0, and its steps are $1 \in \Sigma$.

(R2) For $p = 2$: every two different elements of B differ by a non-zero element of B, which is in Σ_B; so any two elements of $B \setminus X$ are joined by one step, for every X. For p odd: the map $x \mapsto ax$ is an isomorphism of $\mathbb{Z}_b$ onto B, and it maps ±1, $\pm p$ into Σ_B. The prime p divides $b = p^{N-4}$, and $1 < p < b$ because $N - 4 \ge 2$. By Lemma 5.7 (for $b \ge 4$ and a divisor d of b with $1 < d < b$, the set $\mathbb{Z}_b$ minus at most 2 elements stays connected under the steps ±1, $\pm d$), used with $d = p$, the set B minus at most 2 elements stays connected under the steps of Σ_B.

(R1) For $p = 2$: the graph on $A = \mathbb{Z}_{2^{N-2}}$ with the steps Σ_A is the graph Δ_{N-2} of Lemma 7.3. Put $r = N - 2 \ge 4$; then $2r - 2 = 2N - 6 \ge 6$. By Lemma 7.3 it stays connected after the deletion of at most 6 elements. This is (R1).

For p odd: the ten residues in Σ_A are different and not 0. Indeed: $0 < p^j < p^4$ for $0 \le j \le 3$, so $p^j \neq 0$, and $p^j \neq p^l$ for $j \neq l$. If $p^j = -p^l$ modulo p^4 with $j \le l \le 3$, then $p^j + p^l = p^4$, because the sum is between 2 and $2p^3 < 2p^4$; so $p^j(1 + p^{l-j}) = p^4$; for $j < l$ the factor $1 + p^{l-j}$ is not divisible by p and is greater than 1, which is impossible; and $j = l$ gives $2p^j = p^4$, which is impossible for odd p. Next, $2 = \pm p^j$ modulo p^4 would mean that p^4 divides $p^j \mp 2$; this number is not 0, because p is odd, and its absolute value is at most $p^3 + 2 < p^4$. The same holds for -2. Finally $2 \neq 0$ and $2 \neq -2$ modulo p^4, because p^4 divides neither 2 nor 4. So $\Sigma_A = -\Sigma_A$, 1 is in Σ_A, and $|\Sigma_A| = 10 \ge 9$. By Lemma 5.9 (for $a \ge 10$, a set of at least 9 non-zero elements of $\mathbb{Z}_a$ that is closed under $x \mapsto -x$ and contains 1, used as a step set: $\mathbb{Z}_a$ minus at most 6 elements stays connected), used with $a = p^4 \ge 81$ and the step set Σ_A, (R1) holds.

(AV″) Σ contains 1 and p^j for $1 \le j \le i-1$: for $p = 2$ the elements 2^j with $j \le N-3$; for p odd the elements p, p^2, p^3. So Lemma 7.2, with this i, gives (AV″) for $\Gamma = \mathbb{Z}_k$, $B = p^i\mathbb{Z}_k = a\mathbb{Z}_k$.

(Counts) *The counts.* $|A| = a \ge 16 \ge 10$ and $|B| = b \ge 4$. So (C0) holds, and by Remark 2.16 every frame is counted. Every subdivision has a frame (Lemma A.5; see Definition 2.9). So in (AV″) the good subgraph is a cycle with at most 5 edges or a subdivision with a counted frame: (H2) holds. This is the check in the proof of Corollary 3.15.

So Γ, B, Σ satisfy the hypotheses (H1) and (H2) of Theorem 3.11, and every element of Σ is liftable. Theorem 3.19 gives (a) and (b). □

Remark 7.5. (1) Where $N \ge 6$ is used. For $p = 2$: Lemma 7.2 needs $i = N - 2 \ge 4$, Remark 2.16 needs $|A| = 2^{N-2} \ge 10$, and (R1) needs $2(N-2) - 2 \ge 6$ in Lemma 7.3. For p

odd: Lemma 5.7 needs a divisor of $b = p^{N-4}$ strictly between 1 and b, so $N - 4 \geq 2$. The case $N \leq 5$ is not treated here.

(2) What is quoted. For $p = 2$ the proof uses no quoted theorem beyond those of Sections 1 to 3. For p odd, Lemma 5.9 rests on the theorem of Mader and Watkins on vertex-transitive graphs. For $p = 2$ the steps ± 2 are of the form $\pm p^j$; for p odd they are needed in this argument, because without them $|\Sigma_A| = 8$.

8. Two coprime factors

This section proves connectivity for a modulus $k = ab$ whose two factors a and b are coprime and each large enough (Theorem 8.8 and Corollary 8.9). It gives every multiple of $1120 = 32 \cdot 35$, every product pq of two different primes $p, q \geq 43$, and every p^2q with different primes $p \geq 7$ and $q \geq 43$. For a product of two different primes it is the only result of this paper that gives the integer graphs $I_k(G)$ for all graphs G.

The proof is not a case of Theorem 3.11. Every move changes only one of the two residues of the flow, modulo a or modulo b, on a cycle on which the other residue has no zero. We call such a move a *guarded move*. The divisor extension (Theorem 1.5) lifts every guarded move to the integer flows (Lemma 8.2). Remark 8.11 says what Theorem 3.11 gives for these moduli, and why Theorem 3.19 does not give the integer flows here.

Guarded moves. In this section $a, b \geq 2$ are coprime integers and $k = ab$, except in the subsection 'List avoidance on a 3-edge-connected graph'. For x in $\mathbb{Z}_k$, $\rho_a(x)$ and $\rho_b(x)$ are its residues modulo a and modulo b. By the Chinese remainder theorem the map $x \mapsto (\rho_a(x), \rho_b(x))$ is a bijection from $\mathbb{Z}_k$ onto $\mathbb{Z}_a \times \mathbb{Z}_b$, and it is additive. The map ρ_a is a bijection from the subgroup $b\mathbb{Z}_k$ (it has a elements) onto $\mathbb{Z}_a$: if $\rho_a(bx) = 0$, then a divides x, because a and b are coprime, and so $bx = 0$ in $\mathbb{Z}_k$. Let $\iota_a \colon \mathbb{Z}_a \to b\mathbb{Z}_k$ be its inverse. So $\rho_a(\iota_a(z)) = z$ and $\rho_b(\iota_a(z)) = 0$ for every z in $\mathbb{Z}_a$. In the same way $\iota_b \colon \mathbb{Z}_b \to a\mathbb{Z}_k$ is the inverse of the restriction of ρ_b to $a\mathbb{Z}_k$. Every x in $\mathbb{Z}_k$ is $\iota_a(\rho_a(x)) + \iota_b(\rho_b(x))$.

For a map f from $E(G)$ to $\mathbb{Z}_k$, the maps $\rho_a(f)$ and $\rho_b(f)$ are taken edge by edge. The map f is a flow if and only if $\rho_a(f)$ and $\rho_b(f)$ are flows, and $f(e) = 0$ if and only if $\rho_a(f)(e) = 0$ and $\rho_b(f)(e) = 0$. For a $\mathbb{Z}_a$-flow w of G, the map $\iota_a(w)$ is a $\mathbb{Z}_k$-flow with values in $b\mathbb{Z}_k$, because ι_a is additive; in the same way for ι_b.

Definition 8.1 (Guarded moves)**.** Let f be a nowhere-zero $\mathbb{Z}_k$-flow of a multigraph G. A move $f \to f + \lambda\chi_C$ of $M_k(G)$ is a *guarded a-move* if λ is in $b\mathbb{Z}_k$ and $\rho_b(f)(e) \neq 0$ for every edge e of C. It is a *guarded b-move* if λ is in $a\mathbb{Z}_k$ and $\rho_a(f)(e) \neq 0$ for every edge e of C. A *guarded move* is a guarded a-move or a guarded b-move.

A guarded a-move does not change the residues modulo b, so its reverse is a guarded a-move. If $\lambda \neq 0$ is in $b\mathbb{Z}_k$ and $\rho_b(f)$ has no zero on C, then $f + \lambda\chi_C$ is nowhere zero: on C its residue modulo b is that of f, and outside C nothing changes. So $f \to f + \lambda\chi_C$ is then a guarded a-move. The same holds with a and b exchanged.

Lemma 8.2 (Guarded moves)**.** (a) *Let f be a nowhere-zero $\mathbb{Z}_k$-flow of a multigraph G, let E' be a set of edges with $\rho_b(f)(e) \neq 0$ for every e in E', and let w be a $\mathbb{Z}_a$-flow of G that is 0 outside E'. Then f is joined to $f + \iota_a(w)$ by a sequence of guarded a-moves. All flows of the sequence are equal to f outside E', and their residues modulo b are those of f. The same holds with a and b exchanged.*

(b) *Let $f \to g$ be a guarded move of $M_k(G)$. Then all integer lifts of f and all integer lifts of g lie in one component of $I_k(G)$.*

Proof. (a) The restriction of w to the graph $(V(G), E')$ is a $\mathbb{Z}_a$-flow of this graph. By Lemma 1.1 (a), $w = x_1\chi_{C_1} + \cdots + x_\nu\chi_{C_\nu}$ with x_r in $\mathbb{Z}_a$ and cycles C_r all of whose edges are in E'. Put $f_r = f + \sum_{r' \le r} \iota_a(x_{r'})\chi_{C_{r'}}$. Then $f_0 = f$ and $f_\nu = f + \iota_a(w)$, because ι_a is additive. Every f_r has the residues of f modulo b, because the values of ι_a are in $b\mathbb{Z}_k$; so $f_r(e) \neq 0$ for e in E'. Outside E' we have $f_r = f$, which is not 0. If $x_r \neq 0$, the step from f_{r-1} to f_r is the move with $\lambda = \iota_a(x_r) \neq 0$ on C_r, and $\rho_b(f_{r-1}) = \rho_b(f)$ has no zero on C_r: a guarded a-move. If $x_r = 0$, then $f_r = f_{r-1}$, and we skip the step.

(b) Let $g = f + \lambda\chi_C$ be a guarded a-move. Then f and g have the same reduction $w = \rho_b(f)$ modulo b. If $w(e) = 0$, then e is not on C, so $f(e) = g(e)$. Theorem 1.5 with $m = a$ and $q = b$ gives the claim. For a guarded b-move use Theorem 1.5 with $m = b$ and $q = a$. □

Lemma 8.3 (Averaging in one coordinate)**.** *Let G be a multigraph with m edges and f a nowhere-zero $\mathbb{Z}_k$-flow of G. Then f is joined by guarded a-moves to a flow f' with $\rho_b(f') = \rho_b(f)$ such that $\rho_a(f')$ has at most m/a zeros.*

Proof. Let E' be the set of the edges e with $\rho_b(f)(e) \neq 0$, and $G' = (V(G), E')$. Every edge e of E' lies on a cycle of G'. If not, e is not a loop, and its two ends are in different components of $G' - e$. Let X be the vertex set of the component of $G' - e$ that contains the head of e. Add the conservation equations of the $\mathbb{Z}_b$-flow $\rho_b(f)$ over the vertices of X. An edge with both ends in X drops out, the edges outside E' carry 0, and e is the only edge of E' with exactly one end in X. So $\rho_b(f)(e) = 0$, a contradiction.

Let $\mathcal{F}$ be the finite group of the $\mathbb{Z}_a$-flows of G'. For an edge e of E' the map $y \mapsto y(e)$ is a homomorphism from $\mathcal{F}$ onto $\mathbb{Z}_a$: e lies on a cycle C of G', and $x\chi_C$ is in $\mathcal{F}$ with the value x or $-x$ on e. So for every e in E' exactly $|\mathcal{F}|/a$ flows y in $\mathcal{F}$ have $\rho_a(f)(e) + y(e) = 0$. Summed over e in E': the number of pairs (y, e) with y in $\mathcal{F}$, e in E' and $\rho_a(f)(e) + y(e) = 0$ is $|E'|\,|\mathcal{F}|/a$. So some y in $\mathcal{F}$ has $\rho_a(f)(e) + y(e) = 0$ for at most $|E'|/a \le m/a$ edges e of E'. Extend y by 0 to a $\mathbb{Z}_a$-flow of G. For an edge e outside E' we have $\rho_b(f)(e) = 0$, so $\rho_a(f)(e) \neq 0$, because $f(e) \neq 0$; and $y(e) = 0$. So $\rho_a(f) + y$ has at most m/a zeros. By Lemma 8.2 (a) with this E', f is joined by guarded a-moves to $f' = f + \iota_a(y)$, with $\rho_a(f') = \rho_a(f) + y$ and $\rho_b(f') = \rho_b(f)$. □

List avoidance on a 3-edge-connected graph. In this subsection Γ is a finite abelian group and B is a subgroup of Γ; a, b and k are not used.

Definition 8.4 (One-value avoidance)**.** A finite abelian group Γ has *one-value avoidance* if for every good subgraph H of a multigraph (Definition 2.1) and every map $z\colon E(H) \to \Gamma$ there is a Γ-flow w of H with $w(e) \neq z(e)$ for every edge e of H.

If Γ has one-value avoidance, then $|\Gamma| \ge 6$: on a cycle C with 5 edges the flows are the maps $x\chi_C$, and every edge excludes one value of x.

The next two lemmas use the objects of Section 2 for a loopless 3-edge-connected multigraph G on at least two vertices, seen as the subdivision $H = G$ of $H_0 = G$ in which every thread has one edge; this is a good subgraph of the kind (SUB) of Definition 2.1. So there are the auxiliary graph Q (Definition 2.6), a frame (Definition 2.9) and its official cycles $C_1, \ldots, C_s$ (Definition 2.10). The original edges of Q are the edges of G; the free edges are the original red edges. The reference orientation of G is the direction of the threads; a change of the reference orientation of an edge e replaces the values of all flows on e, and a set of values excluded on e, by their negatives, and changes nothing else. By Lemma 2.7 (c), the restriction of a Γ-flow of Q to the original edges is a Γ-flow of G. The reverse choice, Lemma 3.4,

applies with the trivial subgroup $\{0\}$ in the place of B, so that Γ is in the place of A; we use its part (a) in this form. For a free edge ε, let F_ε be the cycle formed by ε and the path between the ends of ε in the blue spanning tree of the component of Q that contains ε (Definition 2.9 (F1)), directed along ε. Its only red edge is ε.

Lemma 8.5 (Subgroup list avoidance)**.** *Let $j \geq 1$ be an integer, and let $|B| \geq j+1$ and $|\Gamma| > 2j\,|B|$. Let G be a loopless 3-edge-connected multigraph on at least two vertices, and for every edge e of G let $\Phi(e)$ be a set of at most j elements of Γ. Then G has a Γ-flow w with $w(e) \notin \Phi(e)$ for every edge e.*

Proof. Use the frame and the notation above. Define a list assignment: $\Lambda(t) = \Phi(t) + B = \{\, z + y : z \in \Phi(t),\ y \in B \,\}$ for every original blue edge t, and $\Lambda(t)$ empty for every blue expansion edge. Then $|\Lambda(t)| \leq j\,|B|$, so $|\Lambda(e_i)| + |\Lambda(f_i)| \leq 2j\,|B| < |\Gamma|$ for every node i. By Lemma 3.4 (a) with $Y_0 = 0$ there is a Γ-flow Y of Q with $Y(t) \notin \Lambda(t)$ for every blue edge t. So $Y(t) + y \notin \Phi(t)$ for every original blue edge t and every y in B.

Let ε be a free edge. The $|B| \geq j+1$ elements $Y(\varepsilon)+y$, y in B, are different, and at most j of them are in $\Phi(\varepsilon)$. Choose y_ε in B with $Y(\varepsilon)+y_\varepsilon \notin \Phi(\varepsilon)$. Put $Y' = Y + \sum_\varepsilon y_\varepsilon \chi_{F_\varepsilon}$, the sum over all free edges. A free edge ε lies on F_ε and on no other cycle $F_{\varepsilon'}$, so $Y'(\varepsilon) = Y(\varepsilon) + y_\varepsilon \notin \Phi(\varepsilon)$. On a blue edge t the difference $Y'(t) - Y(t)$ is a sum of elements $\pm y_\varepsilon$, so it is in B; hence $Y'(t) \notin \Phi(t)$ if t is original. Every original edge is blue or free. By Lemma 2.7 (c) the restriction w of Y' to the original edges is a Γ-flow of G, and $w(e) \notin \Phi(e)$ for every edge e of G. □

Lemma 8.6 (Cyclic list avoidance)**.** *Let $j \geq 1$ and $N > 2j(2j+1)$ be integers. Let G be a loopless 3-edge-connected multigraph on at least two vertices, and for every edge e of G let $\Phi(e)$ be a set of at most j elements of $\mathbb{Z}_N$. Then G has a $\mathbb{Z}_N$-flow w with $w(e) \notin \Phi(e)$ for every edge e.*

Proof. Use the frame and the notation above, with $\Gamma = \mathbb{Z}_N$, and put $d = j + 1$. For every original blue edge t let $\Lambda(t)$ be the set of the elements $z - y$ of $\mathbb{Z}_N$ with z in $\Phi(t)$ and an integer y with $|y| \leq j$; for a blue expansion edge $\Lambda(t)$ is empty. Then $|\Lambda(t)| \leq j(2j+1)$, so $|\Lambda(e_i)| + |\Lambda(f_i)| \leq 2j(2j+1) < N$ for every node i. By Lemma 3.4 (a) with $Y_0 = 0$ there is a $\mathbb{Z}_N$-flow Y of Q with $Y(t) \notin \Lambda(t)$ for every blue edge t. So $Y(t) + y \notin \Phi(t)$ for every original blue edge t and every integer y with $|y| \leq j$.

The $2j+1$ integers y with $|y| \leq j$ have different residues modulo N, because $2j < N$. Let ε be a free edge, and let $\Psi(\varepsilon)$ be the set of the residues modulo d of the integers y with $|y| \leq j$ and $Y(\varepsilon) + y$ in $\Phi(\varepsilon)$. For every z in $\Phi(\varepsilon)$ at most one such y has $Y(\varepsilon) + y = z$; so $|\Psi(\varepsilon)| \leq j < d$. Choose x_ε in $\mathbb{Z}_d$ outside $\Psi(\varepsilon)$. Let η be the $\mathbb{Z}_d$-flow $\sum_\varepsilon x_\varepsilon \chi_{F_\varepsilon}$ of Q, the sum over all free edges; so $\eta(\varepsilon) = x_\varepsilon$ for every free edge ε. By Corollary 1.3 (for the modulus $d \geq 2$, zeros allowed) there is an integer flow L of Q with $L = \eta$ modulo d and $|L(e)| \leq d - 1 = j$ for every edge e. Put $Y' = Y + L$, where L is read modulo N; this is a $\mathbb{Z}_N$-flow of Q.

On an original blue edge t: $|L(t)| \leq j$, so $Y'(t) = Y(t) + L(t) \notin \Phi(t)$. On a free edge ε: if $Y(\varepsilon) + L(\varepsilon)$ were in $\Phi(\varepsilon)$, the residue of $L(\varepsilon)$ modulo d would be in $\Psi(\varepsilon)$, because $|L(\varepsilon)| \leq j$; but this residue is x_ε. So $Y'(\varepsilon) \notin \Phi(\varepsilon)$. By Lemma 2.7 (c) the restriction w of Y' to the original edges is a $\mathbb{Z}_N$-flow of G with $w(e) \notin \Phi(e)$ for every edge e of G. □

The integer flow L is what makes the corrections on the free edges work in the two moduli d and N at once: it is one integer flow, and it is small on every edge.

Corollary 8.7 (One-value avoidance)**.** *Let $N = de$ with integers $d \geq 4$ and $e \geq 7$, or let $N \geq 43$. Then $\mathbb{Z}_N$ has one-value avoidance.*

Proof. Let H be a good subgraph of a multigraph, and $z\colon E(H) \to \mathbb{Z}_N$. In both cases $N \geq 28$.

H is a cycle with at most 5 edges. Direct it. Its $\mathbb{Z}_N$-flows are the maps $x\chi_H$, x in $\mathbb{Z}_N$ (conservation at its vertices of degree 2). An edge e excludes the one value $x = \chi_H(e)\, z(e)$. At most $5 < N$ values are excluded; take another one.

H is a subdivision as in Definition 2.1 (SUB). Give the edges of every thread the direction of the thread; this changes z by signs only. By Lemma 2.7 (a) the $\mathbb{Z}_N$-flows of H are the maps with the value $w_0(t)$ on every edge of P_t, for a $\mathbb{Z}_N$-flow w_0 of H_0. Put $\Phi(t) = \{\, z(e) : e \in E(P_t) \,\}$; it has at most 3 elements. H_0 is loopless and 3-edge-connected, with at least two vertices. If $N = de$ with $d \geq 4$ and $e \geq 7$, let B be the subgroup of $\mathbb{Z}_N$ of order d (the multiples of e): $|B| = d \geq 4 = 3+1$ and $N = de \geq 7d > 6d = 2\cdot 3\cdot |B|$. Lemma 8.5 with $j = 3$ gives w_0 with $w_0(t) \notin \Phi(t)$ for every edge t of H_0. If $N \geq 43$, then $N > 42 = 2\cdot 3\cdot (2\cdot 3+1)$, and Lemma 8.6 with $j = 3$ gives such a w_0. The flow of H that has the value $w_0(t)$ on P_t avoids z. □

The theorem of two coprime factors. From here on a, b and $k = ab$ are as in the subsection 'Guarded moves'.

Theorem 8.8 (Two coprime factors)**.** *Let $a \geq 9$ and $b \geq 6$ be coprime integers such that $\mathbb{Z}_a$ and $\mathbb{Z}_b$ have one-value avoidance, and let $k = ab$.*

(a) *On every bridgeless multigraph G (loops allowed) any two nowhere-zero $\mathbb{Z}_k$-flows are joined by a sequence of guarded moves. In particular $M_k(G)$ is connected.*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

Proof. A *target* of a multigraph is a $\mathbb{Z}_k$-flow h such that $\rho_a(h)$ and $\rho_b(h)$ are both nowhere zero. A target is nowhere zero.

Claim. On every connected bridgeless multigraph G (loops allowed), every nowhere-zero $\mathbb{Z}_k$-flow is joined to every target by a sequence of guarded moves.

(1) *The claim gives the theorem.* Let G be bridgeless. By Theorem 0.1 (Seymour, Tutte), every bridgeless multigraph, loops allowed, has a nowhere-zero flow with values in any abelian group of order at least 6. So G has a nowhere-zero $\mathbb{Z}_a$-flow h_a and a nowhere-zero $\mathbb{Z}_b$-flow h_b, because $a \geq 9$ and $b \geq 6$. Then $h = \iota_a(h_a) + \iota_b(h_b)$ is a target. A cycle of G lies in one component of G, and a guarded move of a component is a guarded move of G. So, component after component, the claim joins every nowhere-zero $\mathbb{Z}_k$-flow of G to h by guarded moves. The reverse of a guarded move is a guarded move. This gives (a), because guarded moves are moves of $M_k(G)$.

For (b) let G be loopless, and let F, F' be vertices of $I_k(G)$. Their reductions f, f' modulo k are nowhere-zero $\mathbb{Z}_k$-flows, because $1 \leq |F(e)| \leq k-1$. By (a) a sequence of guarded moves leads from f to f'. By Lemma 8.2 (b), all integer lifts of two consecutive flows of the sequence lie in one component of $I_k(G)$. Every flow of the sequence has an integer lift, and all integer lifts of one flow lie in one component (Lemma 1.12 (a); $k \geq 54$). So all of them lie in one component, and it contains F and F'.

(2) *The induction.* We prove the claim by induction on the number n of vertices of G; for each n first for the loopless graphs, then for the graphs with loops. A connected loopless multigraph with one vertex has no edge and only the empty flow. A connected

bridgeless loopless multigraph with $n \geq 2$ vertices has a cut with exactly 2 edges or is 3-edge-connected: a cut with one edge consists of a bridge. So three cases remain: (I), (II), (III). In all of them f is a nowhere-zero $\mathbb{Z}_k$-flow and h is a target of G.

(I) *G has a loop.* Let G^- be G without its loops. It is connected, bridgeless and loopless, and it has n vertices; so the claim holds for G^-. The restrictions of f and h to G^- are a nowhere-zero flow and a target. A guarded move of G^- is a guarded move of G: its cycle has no loop, and the values on the loops do not change. So f is joined by guarded moves to the flow that is equal to h on $E(G^-)$ and to f on the loops. Now let e be a loop, directed, with the values $x = f(e) \neq 0$ and $x' = h(e)$; a loop is a cycle with one edge. If $\rho_b(x) \neq 0$, first add $\iota_a(\rho_a(x') - \rho_a(x))\,\chi_e$: a guarded a-move (if it is not 0), because $\rho_b(x) \neq 0$; the new value x_1 has $\rho_a(x_1) = \rho_a(x') \neq 0$. Then add $\iota_b(\rho_b(x') - \rho_b(x_1))\,\chi_e$: a guarded b-move, because $\rho_a(x_1) \neq 0$. The new value is x'. If $\rho_b(x) = 0$, then $\rho_a(x) \neq 0$, and the two moves are made in the other order: first the b-move, guarded by $\rho_a(x) \neq 0$, then the a-move, guarded by $\rho_b(x') \neq 0$. No other edge changes. Do this for every loop. The result is h.

(II) *G is loopless and has a cut with exactly two edges e_1, e_2.* Let G/e_1 be obtained from G by the identification of the two ends of e_1 and the deletion of e_1. It is connected and has $n-1$ vertices. It is bridgeless: if an edge $e \neq e_1$ lies on a cycle O of G, then the edges of O other than e_1 form a closed trail of G/e_1 through e, and every edge of a closed trail lies on a cycle of the trail. It can have loops. By induction the claim holds for G/e_1. The restriction of a flow of G to $E(G/e_1)$ is a flow of G/e_1 (add the two conservation equations at the ends of e_1), and every flow of G/e_1 is the restriction of exactly one flow of G: conservation at one end of e_1 fixes the value on e_1. In a flow of G, with values in any abelian group, the value on e_1 is the value on e_2 or its negative (conservation over one side of the cut). So a flow of G is nowhere zero, or a target, if and only if its restriction is. Let $g' \to g' + \lambda\chi_{C'}$ be a guarded move of G/e_1. The edges of C' form a cycle of G, or a path of G between the two ends of e_1; in the second case add e_1. This gives a cycle C of G with $\chi_C = \chi_{C'}$ on $E(G/e_1)$. In the second case the path crosses the cut and does not use e_1, so C' contains e_2. Let g be the flow of G with restriction g'. Then $g + \lambda\chi_C$ is the flow of G with restriction $g' + \lambda\chi_{C'}$, and $g \to g + \lambda\chi_C$ is a guarded move of the same kind: the residue that guards is not 0 on the edges of C', and, if e_1 is on C, also on e_1, because its value there is $\pm$ its value on e_2. So the claim for G/e_1 gives the claim for G.

(III) *G is loopless and 3-edge-connected, $n \geq 2$.* Let m be the number of edges. Every vertex has degree at least 3, so $m \geq 3n/2$. By Lemma 8.3, f is joined by guarded a-moves to a flow f' such that $\rho_a(f')$ has at most $m/a \leq m/9$ zeros. So $\operatorname{supp}(\rho_a(f'))$ has at least $(8/9)\,m \geq (4/3)\,n > (4/3)\,n - 2$ edges. If $n \geq 3$, Lemma 2.2 (a) gives a good subgraph H of the loopless multigraph $(V(G), \operatorname{supp}(\rho_a(f')))$. If $n = 2$, then $m \geq 3$, and the support has at least $8m/9 > 2$ edges; two of them are parallel and form a good subgraph H of the kind (CYC). In both cases H is connected, it has at least two vertices, and $\rho_a(f')$ has no zero on $E(H)$.

Let $G^* = G/H$ be obtained from G by the identification of $V(H)$ to one vertex v_H and the deletion of $E(H)$. It has fewer vertices than G, and it is connected. It is bridgeless: if an edge e of G^* lies on a cycle O of G, the edges of O outside $E(H)$ form a closed trail of G^* through e, and every edge of a closed trail lies on a cycle of the trail. It can have loops. By induction the claim holds for G^*. The restriction of a flow of G to $E(G^*)$ is a

flow of G^* (add the conservation equations at the vertices of H; the edges of H drop out). So the restriction of f' is a nowhere-zero flow of G^*, and the restriction h^* of h is a target of G^*. So there are guarded moves of G^* that lead from the restriction of f' through flows $g_0, g_1, \dots, g_\nu = h^*$.

The cycles. For a cycle C^* of G^* we fix a cycle C of G with $\chi_C = \chi_{C^*}$ on $E(G^*)$ and $E(C) \subseteq E(C^*) \cup E(H)$. If C^* does not pass v_H, its edges form a cycle of G; take it. Otherwise C^* passes v_H once, and its edges form, in G, a cycle or a path P between two different vertices v_1, v_2 of H whose inner vertices are not in $V(H)$; a loop of G^* at v_H is such a path with one edge, because G is loopless. Add a path of H between v_2 and v_1; H is connected. Direct C along C^*.

The invariant. We build flows $x_0 = f', x_1, \dots, x_\nu$ of G, each joined to the one before it by guarded moves, such that the restriction of x_r to $E(G^*)$ is g_r and $\rho_a(x_r)$ has no zero on $E(H)$. Such a flow is nowhere zero. $x_0 = f'$ has these properties. Let $x = x_{r-1}$ be built, and let $g_r = g_{r-1} + \lambda\chi_{C^*}$, with the cycle C of C^*.

A guarded b-move of G^* (λ in $a\mathbb{Z}_k$, $\rho_a(g_{r-1})$ has no zero on C^*). Then $\rho_a(x)$ has no zero on $E(C)$: not on $E(H)$ by the invariant, and not on $E(C^*)$, where x is equal to g_{r-1}. So $x \to x + \lambda\chi_C$ is a guarded b-move. Put $x_r = x + \lambda\chi_C$. It has the residues of x modulo a, and its restriction to $E(G^*)$ is g_r.

A guarded a-move of G^* (λ in $b\mathbb{Z}_k$, $\rho_b(g_{r-1})$ has no zero on C^*).

(i) $\mathbb{Z}_b$ has one-value avoidance. So there is a $\mathbb{Z}_b$-flow w_b of H with $w_b(e) \neq -\rho_b(x)(e)$ for every edge e of H. Extend it by 0 to a flow of G. By Lemma 8.2 (a), with a and b exchanged and $E' = E(H)$ ($\rho_a(x)$ has no zero there), x is joined by guarded b-moves to $x' = x + \iota_b(w_b)$. Now $\rho_b(x')$ has no zero on $E(H)$; $\rho_a(x') = \rho_a(x)$; and $x' = x$ outside $E(H)$.

(ii) $\mathbb{Z}_a$ has one-value avoidance. So there is a $\mathbb{Z}_a$-flow w_a of H with $w_a(e) \neq -\rho_a(x')(e) - \rho_a(\lambda)\chi_C(e)$ for every edge e of H. Extend it by 0. By Lemma 8.2 (a) with $E' = E(H)$ ($\rho_b(x')$ has no zero there), x' is joined by guarded a-moves to $x'' = x' + \iota_a(w_a)$. Now $\rho_b(x'') = \rho_b(x')$, and $x'' = x$ outside $E(H)$. ($\rho_a(x'')$ can have zeros on $E(H)$.)

(iii) $\rho_b(x'')$ has no zero on $E(C)$: not on $E(H)$ by (i), and not on $E(C^*)$, where x'' is equal to g_{r-1}. So $x'' \to x'' + \lambda\chi_C$ is a guarded a-move. Put $x_r = x'' + \lambda\chi_C$. Its restriction to $E(G^*)$ is $g_{r-1} + \lambda\chi_{C^*} = g_r$. On $E(H)$, $\rho_a(x_r) = \rho_a(x') + w_a + \rho_a(\lambda)\chi_C$, which has no zero by the choice of w_a.

The end. The flow x_ν is equal to h outside $E(H)$, and $\rho_a(x_\nu)$ has no zero on $E(H)$. So $\rho_a(x_\nu)$ has no zero at all, because $\rho_a(h)$ has none. By Lemma 8.2 (a), with a and b exchanged and $E' = E(G)$, x_ν is joined by guarded b-moves to the flow x' with $\rho_b(x') = \rho_b(h)$ and $\rho_a(x') = \rho_a(x_\nu)$ (add ι_b of the $\mathbb{Z}_b$-flow $\rho_b(h) - \rho_b(x_\nu)$). Now $\rho_b(x') = \rho_b(h)$ has no zero. By Lemma 8.2 (a) with $E' = E(G)$, x' is joined by guarded a-moves to the flow with the residues $\rho_a(h)$ and $\rho_b(h)$, that is to h. Together: f is joined to f', to x_ν, and to h by guarded moves. □

Corollary 8.9 (Two admitted factors)**.** *Call an integer $N \geq 2$ an* admitted factor *if $N = de$ with integers $d \geq 4$ and $e \geq 7$, or if $N \geq 43$.*

(a) *Let $k = ab$ with coprime admitted factors a and b. Then $M_k(G)$ is connected for every finite bridgeless multigraph G (loops allowed), and $I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

(b) *Every multiple of a product of two coprime admitted factors is again such a product.*

(c) *In particular* (a) *holds for every multiple of* $1120 = 32 \cdot 35$*, for* $k = pq$ *with different primes* $p, q \geq 43$*, and for* $k = p^2q$ *with different primes* $p \geq 7$ *and* $q \geq 43$*.*

Proof. (a) An admitted factor is at least $4 \cdot 7 = 28$. So $a \geq 9$ and $b \geq 6$. By Corollary 8.7, $\mathbb{Z}_a$ and $\mathbb{Z}_b$ have one-value avoidance. Theorem 8.8 gives the claim.

(b) Let $k = ab$ with coprime admitted factors a, b, and let k' be a multiple of k. For every prime p that divides k'/k, let p^v be the largest power of p that divides k'/k. Put it to b if p divides b, and to a otherwise. This gives $k' = a'b'$, where a' is a multiple of a and b' is a multiple of b. A prime that divides a' divides a or was put to a; in both cases it does not divide b, and it was not put to b. So a' and b' are coprime. A multiple of an admitted factor is an admitted factor: a multiple of de is de' with a multiple e' of e, so $e' \geq 7$; and a multiple of a number $N \geq 43$ is at least 43.

(c) $32 = 4 \cdot 8$ and $35 = 5 \cdot 7$ are coprime admitted factors; use (b). A prime $p \geq 43$ is an admitted factor. For a prime $p \geq 7$, $p^2 = p \cdot p$ with $p \geq 4$ and $p \geq 7$ is an admitted factor. □

Remark 8.10 (The admitted factors)**.** The admitted factors below 43 are 28, 32, 35, 36, 40, 42. So the admitted factors are the integers $N \geq 28$ other than 29, 30, 31, 33, 34, 37, 38, 39, 41. The smallest product of two coprime admitted factors is 1120. Indeed, let $ab < 1120$ with admitted factors $a \leq b$. Then $a^2 < 1120$ gives $a \leq 33$, so a is 28 or 32, and $b < 1120/28 = 40$ gives b in $\{28, 32, 35, 36\}$; of these pairs only 32, 35 is coprime, and $32 \cdot 35$ is not smaller than 1120. Every multiple of 1120 is also a multiple of 4 from 52 on, so Theorem 9.2 gives it as well. The two lemmas give one-value avoidance only for the admitted factors; other orders are not treated.

Remark 8.11 (The master theorem for these moduli)**.** Let $a \geq 10$ and $b \geq 4$ be coprime and $k = ab$. The connectivity of $M_k(G)$ in Theorem 8.8 (a) then also follows from Theorem 3.11. Put $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ (order b), $A = \mathbb{Z}_a$, and $\Sigma = \Gamma \setminus \{0\}$. Then $\Sigma_A = A \setminus \{0\}$ and $\Sigma_B = B \setminus \{0\}$, so (R1), (R2) and (R3*) hold with walks of one step. By the Chinese remainder theorem $\mathbb{Z}_k = \mathbb{Z}_a \times \mathbb{Z}_b$; the element b of Σ generates the factor $\mathbb{Z}_a \times 0$ and the element a generates $0 \times \mathbb{Z}_b$, and $(|B| - 1)/(|\Gamma| - 1) = (b - 1)/(ab - 1) < 1/9$ because $a \geq 10$. So Lemma 4.1 gives (AV) and (AV″). And $|A| = a \geq 10$, $|B| = b \geq 4$: Corollary 3.15 applies. This is a special case of Theorem 4.5, for $\Gamma = \mathbb{Z}_k$.

Theorem 3.19 does not give the integer graphs $I_k(G)$ in this way, because it asks that every element of $\Sigma = \Gamma \setminus \{0\}$ is liftable (Definition 1.7), and this is not shown. For $k = pq$ a smaller Σ does not help either. Let $k = pq$ with different primes $p, q \geq 5$. The elements that Lemma 1.12 shows to be liftable are ± 1, ± 2, $\pm p$, $\pm q$: in part (e), $c = 1$ modulo p and modulo q means $c = 1$ modulo k. With $B = p\mathbb{Z}_k$, the only ones of them in B are $\pm p$. The map $jp \mapsto j$ identifies B with $\mathbb{Z}_q$ and $\pm p$ with the steps ± 1, and $\mathbb{Z}_q$ without the two elements 1, 3 is not connected by these steps. So (R2) fails; the same holds for $B = q\mathbb{Z}_k$. The guarded moves do not need liftable steps: Lemma 8.2 (b) lifts every guarded move, whatever its step.

9. Every multiple of four from 52 on; the moduli by their number of prime factors

This section collects the integer moduli. Theorem 9.2 gives $k = 40$, $k = 48$ and every multiple of 4 from 52 on. Theorem 9.5 gives every k with at least four prime factors that is not a prime power, except 24, 36 and 54. Proposition 9.7 and Corollary 9.8 say exactly

which k these results reach (Definition 9.6), by the number of prime factors. Corollary 9.9: the moduli k for which $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G have natural density one.

$\Omega(k)$ is the number of prime factors of k counted with multiplicity, $\omega(k)$ the number of different prime factors, and $\tau(a)$ the number of positive divisors of a. For a set D of non-zero elements of $\mathbb{Z}_a$ with $D = -D$, the *graph on $\mathbb{Z}_a$ with the step set D* has the vertex set $\mathbb{Z}_a$; two vertices x, y are adjacent if $y - x$ is in D. The *sign class* of an element x is $\{x, -x\}$. For a split $k = ab$, D always denotes the step set of Criterion 5.11: the non-zero residues modulo a of ± 1, ± 2 and $\pm q$, for all divisors q of k with $1 \le q < k$.

How Criterion 5.11 is used. Most cases below are uses of Criterion 5.11, which is Theorem 3.19 together with Corollary 3.15. For every use we name the split $k = ab$ (with k not a prime power, $a \ge 10$ and a composite $b \ge 4$) and the reason why the deletion condition holds. The other data of the proof do not depend on the case. We list them once, so that each hypothesis of Theorem 3.11 can be checked.

(D1) $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ (order b), $A = \mathbb{Z}_a$. Σ is the set of the elements ± 1, ± 2, $\pm q$ of $\mathbb{Z}_k$, for the divisors q of k with $1 \le q < k$. They are liftable by Lemma 1.12 (b), (c), (d).

(D2) $\Sigma_A = D$, and (R1) is the deletion condition: the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements.

(D3) Let d be the least prime factor of b; so $1 < d < b$. The divisors a and ad of k give $\pm a$, $\pm ad$ in Σ_B; under $j \mapsto aj$ these are the steps ± 1, $\pm d$ of $\mathbb{Z}_b$, and (R2) is Lemma 5.7.

(D4) (R3*): two non-zero elements of $\mathbb{Z}_k$ are joined by steps 1 or -1 without passing 0.

(D5) (AV): let p be a prime factor of k, k_1 the largest power of p that divides k, and $k_2 = k/k_1$. Both are divisors of k with $1 < k_1, k_2 < k$, because k is not a prime power. By the Chinese remainder theorem $\mathbb{Z}_k = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$; the element k_2 of Σ generates the factor $\mathbb{Z}_{k_1} \times 0$, and k_1 generates $0 \times \mathbb{Z}_{k_2}$. And $(|B|-1)/(|\Gamma|-1) = (b-1)/(ab-1) < 1/9$, because $a \ge 10$. So Lemma 4.1 gives (AV) and (AV″).

(D6) The counts: $|A| = a \ge 10$ and $|B| = b \ge 4$. So (C0) holds, and every frame is counted: the left sides of (C1), (C2), (C3), (C4) are at most 9, 9, 8, 3 (Remark 2.16). This is Corollary 3.15, and Theorem 3.19 gives the connectivity of $M_k(G)$ and $I_k(G)$.

Criterion 5.11 says that the deletion condition holds if D has at least 9 elements, and, if a is a prime, if D has at least 7 elements. If D has at most 6 elements, the deletion condition fails: delete the elements of D, the neighbours of 0; then 0 has no neighbour, and other vertices remain, because $a \ge 10$.

Lemma 9.1 (Many divisors)**.** *Let k be not a prime power, and $k = ab$ with $a \ge 10$ and a composite number $b \ge 4$. If $\tau(a) \ge 6$, or if a is odd and $\tau(a) \ge 5$, then D has at least 9 elements. So, by Criterion 5.11, $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Every divisor q of a with $1 \le q < a$ is a divisor of k with $q < k$, so the residues $\pm q$ modulo a are in D. The residue of q is q itself, not 0, and different q give different residues. Let $q \equiv -q'$ modulo a for two such divisors. Then $q + q' = a$, because $2 \le q + q' \le 2a - 2$. So q divides $a - q = q'$ and q' divides q, hence $q = q' = a/2$. Therefore the residues $\pm q$ form a set R_0 with $2\,(\tau(a) - 1)$ elements if a is odd, and with $2\,(\tau(a) - 1) - 1$ elements if a is even (only $a/2$ is its own negative). If $\tau(a) \ge 6$, then $|R_0| \ge 9$.

Let a be odd and $\tau(a) \geq 5$; so $|R_0| \geq 8$. The residues 2 and -2 are in D. They are different and not 0, because a is odd and $a \geq 10$. They are not in R_0: 2 is not a divisor of the odd number a; and $2 \equiv -q$ with a divisor $q < a$ gives $q = a - 2$, so q divides 2, and $a = q + 2$ is 3 or 4, which is not the case. As $R_0 = -R_0$, also -2 is not in R_0. So $|D| \geq 10$. □

Every multiple of four from 52 on.

Theorem 9.2 (The bases)**.** *Let $k = 40$, or $k = 48$, or let k be a multiple of 4 with $k \geq 52$. Then $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G. The cases and the theorems that give them:*

(B1) *$k = 40$: Theorem 5.3, that is Theorem 3.19 with Corollary 3.15, $a = 10$, $b = 4$, $\Sigma = \{\pm1, \pm2, \pm4, \pm5, \pm8, \pm10, 20\}$.*

(B2) *$k = 48$: Theorem 5.4, that is Theorem 3.19 with Corollary 3.15, $a = 12$, $b = 4$, $\Sigma = \{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, 24\}$.*

(B3) *$k = 4p$ with a prime $p \geq 13$: the theorem of the double step (Theorem 6.9), that is Theorem 3.19 with Theorem 3.12, $a = p$, $b = 4$, $\Sigma = \{\pm1, \pm(1 + 2p), \pm2, \pm4, \pm p, 2p\}$.*

(B4) *$k = 2^N$ with $N \geq 6$: the theorem of the prime powers (Theorem 7.4), that is Theorem 3.19 with Corollary 3.15, $a = 2^{N-2}$, $b = 4$, $\Sigma = \{\pm 2^j : 0 \leq j \leq N - 1\}$, and (AV″) by the flattening of valuations.*

(B5) *k a multiple of 8, $k \geq 56$, k not a power of 2: Criterion 5.11 with $a = k/4$, $b = 4$ (the nine residues ±1, ±2, ±4, ±8, $a/2$).*

(B6) *$k = 4m$ with m odd and composite (so $m \geq 15$): Criterion 5.11 with $a = m$, $b = 4$ (the ten residues ±1, ±2, ±4, $\pm d_0$, $\pm 2d_0$, d_0 the least prime factor of m).*

Every multiple of 4 from 52 on is in one of the cases (B3)–(B6).

Proof. (B1) to (B4) are the theorems named there, with their proofs in their sections.

(B5) Let k be a multiple of 8 with $k \geq 56$ that is not a power of 2. It is not a prime power: it is even and has an odd prime factor. Use Criterion 5.11 with $a = k/4$ and $b = 4$ (composite; in (D3) $d = 2$; in (D5) the prime 2). Here a is even, $a \geq 14$, and $a \neq 16$, because $k \neq 64$. The numbers 4, 8 and $a/2 = k/8$ are divisors of k smaller than k. So the nine residues ±1, ±2, ±4, ±8, $a/2$ modulo a are in D. They are different and not 0. Indeed, $0 < a/2 < a$ and $8 < a$. The difference of two of the eight elements ±1, ±2, ±4, ±8 is, up to its sign, one of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16; the number a divides none of them, because $a \geq 14$ and $a \neq 16$. And $a/2 \equiv x$ modulo a, with x one of the eight, gives $a - 2x = 2ja$ for an integer j, so a divides $2x$; with $2 \leq |2x| \leq 16$ and $a \geq 14$ this means $a = 16$, which is excluded. So $|D| \geq 9$, and Criterion 5.11 applies.

(B6) Let $k = 4m$ with m odd and composite, and $k \geq 52$. Then $m \geq 13$, and so $m \geq 15$, because 13 is a prime. k is not a prime power. Use Criterion 5.11 with $a = m$ and $b = 4$ (in (D3) $d = 2$; in (D5) the prime 2, $\mathbb{Z}_k = \mathbb{Z}_4 \times \mathbb{Z}_m$). Let d_0 be the least prime factor of m; it is odd, $d_0 \geq 3$, and $d_0^2 \leq m$. The numbers 4, d_0 and $2d_0$ are divisors of $k = 4m$ smaller than k. The five integers 1, 2, 4, d_0, $2d_0$ are different, because d_0 is odd and at least 3, and $2d_0 \geq 6$. They are smaller than $m/2$: $4 < 15/2 \leq m/2$; if $m = 15$, then $2d_0 = 6 < m/2$; if $m > 15$, then $m \geq 21$ and $2d_0 \leq 2\sqrt{m} < m/2$. For two integers x, y with $0 < x, y < m/2$, the residues of x and $-y$ modulo m are different, because $0 < x + y < m$; and those of x and y are different if $x \neq y$. So the ten residues ±1, ±2, ±4, $\pm d_0$, $\pm 2d_0$ are different and not 0: $|D| \geq 10$, and Criterion 5.11 applies.

The last sentence. Let $k = 4m \geq 52$, so $m \geq 13$. If m is odd and a prime, then $k = 4p$ with $p = m \geq 13$: case (B3). If m is odd and composite: case (B6). If m is even, then 8 divides k and $k \geq 56$; if k is a power of 2, then $k = 2^N$ with $2^N \geq 64$, so $N \geq 6$: case (B4); otherwise case (B5). □

Criterion 5.11 also gives, in the same way as (B5) and (B6), every $k = 2^e m$ with $e \geq 2$ and an odd composite $m \geq 15$ ($a = m$, $b = 2^e$); all these k are already in (B5) or (B6).

Twice a prime cube.

Lemma 9.3 (The steps ± 1, ± 2, $\pm p$, $\pm 2p$ in $\mathbb{Z}_{p^2}$)**.** *Let $p \geq 5$ be an integer. The graph on $\mathbb{Z}_{p^2}$ with the step set $\{\pm 1, \pm 2, \pm p, \pm 2p\}$ stays connected after the deletion of at most 6 elements.*

Proof. Let X be the set of the deleted elements, $|X| \leq 6$. The *rows* are the p residue classes modulo p; each row has p elements. For a row i let d_i be the number of its elements in X.

A row with at most 3 deleted elements stays connected. Order the elements of a row as $x, x + p, x + 2p, \ldots, x + (p - 1)p$ in a cycle. The steps $\pm p$, $\pm 2p$ join two elements at distance 1 or 2 along this cycle. Between two consecutive remaining elements (along the cycle) lie the deleted elements of a gap. If at most one gap has 2 or more elements, the remaining elements are joined along the cycle, around the one long gap. Two gaps with 2 or more elements need at least 4 deleted elements. Every such row keeps at least $p - 3 \geq 2$ elements. (For $p = 5$ the row is a complete graph.)

Call a row *good* if $d_i \leq 3$. At most one row is not good, because two such rows need at least 8 deleted elements. The step $+1$ maps the row i onto the row $i + 1$ (modulo p), one to one; this is a set of p disjoint edges between the two rows, also from the row $p - 1$ to the row 0. A deleted element meets at most one of these edges.

Case 1: one row i_0 is not good. Then $d_{i_0} \geq 4$, and the other rows together contain at most 2 deleted elements. The good rows form a path in the cycle of the rows $0, 1, \ldots, p - 1$ (the cycle without the row i_0). Two consecutive good rows are joined by an edge between two remaining elements, because $2 < p$. Each good row is connected. So all remaining elements of the good rows lie in one component. A remaining element x of the row i_0 has the four different neighbours $x + 1$, $x - 1$, $x + 2$, $x - 2$; they lie in other rows, because $p \geq 5$ does not divide 1 or 2. At most 2 of them are deleted. So x lies in the same component.

Case 2: all rows are good. Then every row is connected and has remaining elements. Say that the link between the rows i and $i + 1$ *fails* if none of the p edges between them has two remaining ends. Then $d_i + d_{i+1} \geq p \geq 5$. Two failed links without a common row need at least 10 deleted elements. Two failed links with the common row i need $d_{i-1} + d_i + d_{i+1} \geq 2p - d_i \geq 10 - 3 = 7$ deleted elements. So at most one link fails. A cycle without one edge is connected. So all remaining elements lie in one component. □

Proposition 9.4 ($k = 2p^3$)**.** *Let $p \geq 5$ be a prime and $k = 2p^3$. Then $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Use Criterion 5.11 with $a = p^2 \geq 25$ and $b = 2p$ (composite, $b \geq 10$; in (D3) $d = 2$; in (D5) the prime 2, $\mathbb{Z}_k = \mathbb{Z}_2 \times \mathbb{Z}_{p^3}$, with the generators p^3 and 2). k is not a prime power. The divisors q of k with $q < k$ are 1, 2, p, $2p$, p^2, $2p^2$, p^3; the last three are 0 modulo p^2. So $D = \{\pm 1, \pm 2, \pm p, \pm 2p\}$ modulo p^2, and Lemma 9.3 is the deletion condition. □

At least four prime factors.

Theorem 9.5 (Four prime factors)**.** *Let k be not a prime power, $\Omega(k) \geq 4$, and $k \notin \{24, 36, 54\}$. Then $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. In every case below we give a split $k = ab$ for Criterion 5.11 with $a \geq 10$ and a composite $b \geq 4$, and we show $|D| \geq 9$; or we name another theorem. The data (D1)–(D6) are those of the subsection 'How Criterion 5.11 is used'.

Step 1: $\Omega(k) \geq 5$. Write k as a product of $\Omega(k)$ primes, the *entries*; not all entries are equal. If $k = 48$, use Theorem 9.2 (B2). Otherwise we put two entries into b and the others into a; then b is a product of two primes, so $b \geq 4$ and b is composite. We use one fact: if a has two different prime factors and $\Omega(a) \geq 3$, then $\tau(a) \geq 6$. (If $a = y^e y'^f \cdots$ with different primes y, y', then $\tau(a) \geq (e+1)(f+1)$; with exactly two different primes $e + f \geq 3$ gives $(e+1)(f+1) \geq 6$; with three or more different primes $\tau(a) \geq 8$.)

(1a) $\Omega(k) \geq 6$. Choose two different primes y, y' among the entries. Put y, y' and $\Omega(k)-4 \geq 2$ further entries into a, and the two remaining entries into b. Then a has two different prime factors, $\Omega(a) \geq 4$, and $a \geq 2 \cdot 3 \cdot 2 \cdot 2 = 24$.

(1b) $\Omega(k) = 5$, and some entry y is at least 5. Choose an entry $y' \neq y$ and a third entry y''. Put $a = yy'y''$, and the two remaining entries into b. Then a has two different prime factors, $\Omega(a) = 3$, and $a \geq 5 \cdot 2 \cdot 2 = 20$.

(1c) $\Omega(k) = 5$, and all entries are 2 or 3. Then k is $2^4 \cdot 3 = 48$, $2^3 \cdot 3^2 = 72$, $2^2 \cdot 3^3 = 108$ or $2 \cdot 3^4 = 162$. For 72, 108, 162 put $a = 18$ and $b = 4$, 6, 9; $\tau(18) = 6$.

In all three cases $\tau(a) \geq 6$ and $a \geq 18$, and Lemma 9.1 applies.

Step 2: $\Omega(k) = 4$. There are four types; p, q, r, u, v, w, z are primes.

Type p^3q, $p \neq q$. If $p = 2$, then $k = 8q$: $q = 3$ gives 24, excluded; $q = 5$ gives 40, which is Theorem 9.2 (B1); $q \geq 7$ gives a multiple of 4 with $k \geq 56$, which is Theorem 9.2. If $q = 2$, then $k = 2p^3$: $p = 3$ gives 54, excluded; $p \geq 5$ is Proposition 9.4. Now let p and q be odd.

If $p \not\equiv \pm 1$ modulo q, put $a = pq \geq 15$ and $b = p^2 \geq 9$. The six residues ± 1, ± 2, $\pm q$ modulo pq are in D (q divides k). They are different: 1, 2, q are different and smaller than $pq/2$. None of them is a multiple of p. The four residues $\pm p$, $\pm p^2$ are in D; they are non-zero multiples of p, because q does not divide p or p^2. They are different: $p \equiv -p$ or $p^2 \equiv -p^2$ modulo pq would mean that q divides 2 or $2p$, and $p \equiv \pm p^2$ modulo pq means that q divides $1 \mp p$, that is $p \equiv \pm 1$ modulo q. So $|D| \geq 10$.

If $p \equiv \pm 1$ modulo q, then $p > q$: for $p < q$ it would mean $p = 1$ or $p = q - 1$, and $q - 1$ is even. Also $q \neq p - 1$, because $p - 1$ is even; so $q \leq p - 2$. Put $a = p^2 \geq 25$ and $b = pq \geq 15$. The six residues ± 1, ± 2, $\pm q$ modulo p^2 are different (1, 2, q are different and smaller than $p^2/2$) and not multiples of p. The four residues $\pm p$, $\pm pq$ are in D. They are the residues $p \cdot x$ with x in $\{1, -1, q, -q\}$, and these four values of x are different modulo p, because $3 \leq q \leq p - 2$ and p is odd. So the four residues are different non-zero multiples of p, and $|D| \geq 10$.

Type p^2q^2, $p < q$. If $p = 2$ and $q = 3$, then $k = 36$, excluded. If $p = 2$ and $q \geq 5$, then k is a multiple of 4 with $k \geq 100$: Theorem 9.2. Now let p and q be odd. Put $a = b = pq \geq 15$. As above, ± 1, ± 2, $\pm q$ are six different residues modulo pq, none a multiple of p. The four residues $\pm p$, $\pm p^2$ are different non-zero multiples of p, because $p \not\equiv \pm 1$ modulo q: $3 \leq p \leq q - 2$. So $|D| \geq 10$.

Type p^2qr, p, q, r different. If $p = 2$, then $k = 4qr \geq 60$: Theorem 9.2. If one of q, r is 2, say $r = 2$, then $k = 2p^2q$ with different odd primes p, q; put $a = pq \geq 15$ and $b = 2p \geq 6$.

The residues ±1, ±2 modulo pq are four different units. The residues $\pm p$, $\pm 2p$ are non-zero multiples of p, and they are four different residues unless $q = 3$, when they are two; in the same way $\pm q$, $\pm 2q$ give four residues, or two if $p = 3$. Multiples of p and multiples of q have only 0 in common, and units are neither. At most one of p, q is 3. So $|D| \geq 4 + 2 + 4 = 10$.

Now let p, q, r be odd. If $p \not\equiv \pm1$ modulo qr, put $a = qr \geq 15$ and $b = p^2 \geq 9$. The eight residues ±1, ±2, $\pm q$, $\pm r$ are different: the first four are units, the others are non-zero multiples of q or of r, and qr is odd. The residue of p is a unit, so it is not $\pm q$ or $\pm r$. If $p \not\equiv \pm2$ modulo qr, then $\pm p$ are two new residues. If $p \equiv \pm2$, then $p^2 \equiv 4$, and ±4 are two new residues: 4 is a unit, and $4 \not\equiv \pm1, \pm2, -4$ modulo $qr \geq 15$. So $|D| \geq 10$. If $p \equiv \pm1$ modulo qr, then $p \geq qr - 1 > q, r$. Put $a = pq \geq 15$ and $b = pr \geq 15$. The eight residues ±1, ±2, $\pm p$, $\pm q$ modulo pq are different. The residues $\pm r$ are two more: r is a unit modulo pq, so not $\pm p$, $\pm q$; and $3 \leq r < p$ gives $r \not\equiv \pm1, \pm2$ ($pq - 2 > p$). So $|D| \geq 10$.

Type uvwz, $u < v < w < z$. Put $a = wz \geq 35$ and $b = uv \geq 6$. The eight residues ±1, ±2, $\pm w$, $\pm z$ modulo wz are different: wz is odd, the first four are units, and $\pm w$, $\pm z$ are non-zero multiples of w or of z. The divisor b of k is a unit modulo a, so it is not $\pm w$, $\pm z$. We have $w \geq u + 2$ and $z \geq v + 2$, so $a - b \geq (u + 2)(v + 2) - uv = 2u + 2v + 4 > 2$; with $6 \leq b < a$ this gives $b \not\equiv \pm1, \pm2$ modulo a. And $b \not\equiv -b$, because a is odd and $0 < b < a$. So $\pm b$ are two new residues, and $|D| \geq 10$.

In every case of Step 2 with a split, $a \geq 15$ and $b \geq 6$ is composite; $|D| \geq 10 \geq 9$, and Criterion 5.11 applies. The four types are all the non-prime-powers with $\Omega(k) = 4$. □

The moduli by their number of prime factors.

Definition 9.6 (Covered)**.** An integer $k \geq 2$ is *covered* if one of the following four results applies to it:

(1) Theorem 9.2: $k = 40$, $k = 48$, or k is a multiple of 4 with $k \geq 52$;

(2) Theorem 7.4: $k = p^N$ with a prime p and $N \geq 6$;

(3) Corollary 8.9 (a): k is the product of two coprime admitted factors;

(4) Criterion 5.11: k is not a prime power, and some split $k = ab$ with $a \geq 10$ and a composite $b \geq 4$ has the deletion condition.

For every covered k, $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G. The proofs of Theorem 9.5, Lemma 9.1 and Proposition 9.4 use only (1) and (4); so every k that they give is covered.

Proposition 9.7 (Three prime factors)**.** *Let p, q, r be primes.*

(i) *Let $p \neq q$. Then $k = p^2q$ is covered if and only if*

  (α) *$p = 2$ and $q \geq 13$; or*

  (β) *$p \geq 7$ and $q \geq 43$; or*

  (γ) *$q \geq 11$, and none of p, p^2 is congruent to 1, -1, 2 or -2 modulo q.*

(ii) *Let $p < q < r$ be odd. Then $k = pqr$ is covered, except $k = 105 = 3 \cdot 5 \cdot 7$, which is not covered.*

(iii) *Let $3 \leq p < q$ be odd. Then $k = 2pq$ is covered, except the ten values 30, 42, 70, 110, 182, 286, 506, 646, 1406, 1798, which are not covered.*

In particular: $4q$ is covered exactly for the primes $q \geq 13$; $9q$ exactly for the primes $q \geq 13$; $25q$ exactly for the primes $q \geq 11$ other than 13 and 23; no p^2q with q in $\{2, 3, 5, 7\}$ is covered;

and among the products of three different primes exactly 30, 42, 70, 105, 110, 182, 286, 506, 646, 1406, 1798 *are not covered.*

Proof. Let $\Omega(k) = 3$ and let k not be a prime power. We first see which of the four results can apply. (1) needs $4 \mid k$, so $k = 4q$ with an odd prime q; and $4q$ is not 40 or 48; so (1) applies exactly to $k = 4q$ with $q \geq 13$. (2) does not apply. For (3), the coprime splits of k come from its prime powers. A prime is an admitted factor if and only if it is at least 43, because a number de with $d, e \geq 2$ is not a prime. For (4), b is composite, so $\Omega(b) \geq 2$ and $\Omega(a) \leq 1$; with $a \geq 10$, a is a prime, $a \geq 11$, and $\Omega(b) = 2$. A divisor q' of k that is a multiple of a gives the residue 0. So D is the set of the residues of ± 1, ± 2 and $\pm q'$ for the divisors q' of k that are not multiples of a. As a is an odd prime, D is a union of sign classes with two elements each. If D has at least four sign classes, then $|D| \geq 8 \geq 7$ and the deletion condition holds by Criterion 5.11; if it has at most three, then $|D| \leq 6$, and the deletion condition fails (the paragraph before Lemma 9.1).

(i) $k = p^2 q$. (1) is exactly (α). (3): the only coprime split is p^2 and q. The prime q is admitted exactly if $q \geq 43$. The number p^2 is admitted exactly if $p \geq 7$: 4, 9, 25 are smaller than 28, and for $p \geq 7$, $p^2 = p \cdot p$ with $p \geq 4$ and $p \geq 7$. So (3) is exactly (β). (4): $a = p$ or $a = q$. For $a = p$ the divisors that are not multiples of p are 1 and q, so D has at most the three sign classes of 1, 2, q: no. For $a = q \geq 11$ they are 1, p, p^2, so $D = \{\pm 1, \pm 2, \pm p, \pm p^2\}$ modulo q; all are non-zero, because $q \geq 11$ and $q \neq p$. The sign classes of 1 and 2 are different, because $q \geq 5$. The class of p^2 is that of p exactly if $p \equiv \pm 1$, since p is invertible modulo q. So there are four classes exactly if neither p nor p^2 is ± 1 or ± 2 modulo q, that is (γ).

The readings. For $p = 2$: (β) fails, and (γ) fails because $p = 2$. For $p = 3$, (α) and (β) fail; for $q \geq 11$, 3 is not ± 1, ± 2, and 9 is ± 1 or ± 2 exactly if q divides 8, 10, 7 or 11, that is $q = 11$. For $p = 5$: 5 is not ± 1, ± 2 modulo $q \geq 11$, and 25 is exactly if q divides 24, 26, 23 or 27, that is $q = 13$ or $q = 23$. For $q \leq 7$ none of (α), (β), (γ) holds.

(ii) $k = pqr$ with odd $p < q < r$. (1) does not apply. If $r < 11$, then $k = 3 \cdot 5 \cdot 7 = 105$: (4) needs a prime $a \geq 11$, and (3) needs an admitted factor among 3, 5, 7, 15, 21, 35, all smaller than 28; so 105 is not covered. Let $r \geq 11$, and take $a = r$, $b = pq$. The divisors that are not multiples of r are 1, p, q, pq. The classes of 1, 2, p, q modulo r are different unless $q = r - 2$: p and q are odd and smaller than r, so neither is $\equiv 1$ or $\equiv 2$, neither is $r - 1$, $p = r - 2$ is impossible because $p < q < r$ and q is odd, and $p \equiv -q$ would mean $p + q = r$, but $p + q$ is even. So if $q \neq r - 2$, there are four classes. Let $q = r - 2$. Then $pq \equiv -2p$, and its class is new unless $2p \equiv \pm 1$: $2p \equiv \pm 2$ would give $p \equiv \pm 1$, and $2p \equiv \pm p$ would give $r \mid p$ or $r \mid 3$. As $2p < 2r$, $2p \equiv \pm 1$ means $r = 2p - 1$ or $r = 2p + 1$. If $r = 2p + 1$, then $q = 2p - 1$. One of the three primes p, $2p - 1$, $2p + 1$ is divisible by 3, so it is 3; as $2p - 1$ and $2p + 1$ are larger than 3, $p = 3$, and $r = 7 < 11$, which is not the case. If $r = 2p - 1$, then $q = 2p - 3$, and $p \geq 7$ ($p = 3$ gives $q = p$, $p = 5$ gives $r = 9$), so $q \geq 11$. Take instead $a = q$ and $b = pr$. Modulo q: $r \equiv 2$ and $pr \equiv 2p \equiv 3$. The classes of 1, 2, 3, p are different, because $q \geq 11$ and $p \equiv \pm 1, \pm 2, \pm 3$ modulo $q = 2p - 3$ would give $p \leq 6$. So in all cases with $r \geq 11$ there are four classes, and (4) applies.

(iii) $k = 2pq$ with odd $3 \leq p < q$. (1) does not apply, because $4 \nmid k$. (3): the coprime splits are 2 and pq, p and $2q$, q and $2p$. The factor 2 is not admitted. A number $2s$ with an odd prime s is admitted exactly if $2s \geq 43$, because $2s = de$ with $d \geq 4$ and $e \geq 7$ is impossible. So (3) applies exactly if $p \geq 43$, or $q \geq 43$ and $p \geq 23$; that is, exactly if $p \geq 23$ and $q \geq 43$. (4): a is p or q ($a = 2$ is too small). For $a = q \geq 11$, $D = \{\pm 1, \pm 2, \pm p, \pm 2p\}$ modulo q. Its four classes are different unless $q = p + 2$, $q = 2p - 1$ or $q = 2p + 1$: $p \equiv \pm 1$

is impossible; $p \equiv \pm 2$ only as $p = q - 2$; $2p \equiv \pm 1$ only as $q = 2p \mp 1$; $2p \equiv \pm 2$ would give $p \equiv \pm 1$; $2p \equiv \pm p$ would give $q \mid p$ or $q \mid 3$. In these three exceptional forms, also $a = p$ fails if $p \geq 11$: $D = \{\pm 1, \pm 2, \pm q, \pm 2q\}$ modulo p, and $q \equiv 2$ or $q \equiv \pm 1$ modulo p, so there are at most three classes. If $q < 11$, then $p < 11$ too, and (4) needs $a \geq 11$. So (4) applies exactly if $q \geq 11$ and q is none of $p + 2$, $2p - 1$, $2p + 1$.

So k is not covered exactly if $[q < 11$, or $q \in \{p + 2, 2p - 1, 2p + 1\}]$ and $[p < 23$ or $q < 43]$. With $q < 11$: $(3, 5)$, $(3, 7)$, $(5, 7)$, that is 30, 42, 70. With $q = p + 2 \geq 11$ and $q < 43$ (if $p < 23$, then $q < 43$ as well): $(11, 13)$, $(17, 19)$, $(29, 31)$, that is 286, 646, 1798; the pair $(41, 43)$ has $q = 43$ and $p \geq 23$. With $q = 2p - 1 \geq 11$ and $p \leq 19$: $p = 7$ and $p = 19$ ($q = 13$, 37), that is 182, 1406; for $p = 11$, 13, 17 the number $2p - 1$ is not a prime. With $q = 2p + 1 \geq 11$ and $p \leq 19$: $p = 5$ and $p = 11$ ($q = 11$, 23), that is 110, 506; for $p = 7$, 13, 17, 19 the number $2p + 1$ is not a prime. For $p \geq 23$ the forms $q = 2p \pm 1$ have $q \geq 45$. These are the ten values. □

Corollary 9.8 (The covered moduli)**.** *Let $k \geq 2$.*

(a) *A prime power $k = p^e$ is covered if and only if $e \geq 6$. In particular no prime is covered.*

(b) *A product $k = pq$ of two different primes is covered if and only if $p \geq 43$ and $q \geq 43$.*

(c) *For $\Omega(k) = 3$, k not a prime power: Proposition 9.7.*

(d) *If $\Omega(k) \geq 4$ and k is not a prime power, then k is covered if and only if $k \notin \{24, 36, 54\}$.*

Proof. (a) (1) needs $4 \mid k$, so $k = 2^e$; 2^e is not 40 or 48, and $2^e \geq 52$ means $e \geq 6$. (2) is $e \geq 6$. (3) and (4) do not apply to a prime power.

(b) (1) does not apply: $4 \nmid pq$. (2) does not apply. (4) needs a composite b, so $\Omega(b) \geq 2$ and $a = 1$: no. (3): the only coprime split is p and q, and a prime is an admitted factor exactly if it is at least 43.

(d) Theorem 9.5 and the sentence after Definition 9.6 give every other k. The three values are not covered: (1) does not apply, because 24 and 36 are smaller than 40, and $4 \nmid 54$; (2) does not apply; (3) needs two admitted factors, so $k \geq 28 \cdot 28$; (4) needs $k = ab \geq 40$ with $4 \leq b \leq k/10$, which fails for 24 and 36; for 54 it gives $b = 4$ or $b = 5$, and neither divides 54. □

Density one.

Corollary 9.9 (Density one)**.** *Let S^* be the set of the integers $k \geq 2$ such that $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G. Then S^* has natural density 1: the number of elements of S^* that are at most x, divided by x, tends to 1 as x tends to infinity.*

Proof. If $\omega(k) \geq 5$, then $\Omega(k) \geq 5$, k is not a prime power, and $k \geq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$; so k is in S^* by Theorem 9.5. So it is enough to show that the set T of the positive integers n with $\omega(n) \leq 4$ has density 0.

Fix $y \geq 2$, and let P_y be the product of the primes $p \leq y$. Let T_y be the set of the positive integers that are divisible by at most 4 of the primes $p \leq y$. Then T is a subset of T_y. Whether n is in T_y depends only on the residue of n modulo P_y; so T_y has the density $\delta_y = |T_y \cap [1, P_y]|/P_y$. Let n be a uniformly random element of $\{1, \ldots, P_y\}$. By the Chinese remainder theorem the events "p divides n", for the primes $p \leq y$, are independent, and the event for p has the probability $1/p$. Let X be the number of these events that occur. Then its mean is $\mu_y = \sum_{p \leq y} 1/p$, and its variance is $\sum_{p \leq y} (1/p)(1 - 1/p) \leq \mu_y$. If $\mu_y > 4$,

Chebyshev's inequality (Theorem 0.7) gives

$$\delta_y = \Pr[X \le 4] \;\le\; \Pr\big[\,|X-\mu_y| \ge \mu_y - 4\,\big] \;\le\; \frac{\mu_y}{(\mu_y-4)^2}.$$

The sum of $1/p$ over all primes diverges (Euler; Theorem 0.6), so μ_y tends to infinity, and δ_y tends to 0. For every y the upper density of T is at most δ_y. So T has density 0, and S^* has density 1. □

Remark 9.10 (Moduli that are not covered)**.** The moduli that are not covered include the primes, the prime powers p^e with $e \le 5$, the values 24, 36, 54 and the eleven products of three different primes of Proposition 9.7. For $k \ge 6$ in these cases, the connectivity of $I_k(G)$ for all graphs is open (Section 12). For example $k = 44 = 4 \cdot 11$ is not covered (Proposition 9.7 (i): $4q$ is covered only for $q \ge 13$): it is smaller than 52 and is not 40 or 48; its only split for Criterion 5.11 is $a = 11$, $b = 4$, with $D = \{\pm1, \pm2, \pm4\}$ modulo 11; and 44 is not the product of two coprime admitted factors. (The modular graph $M_k(G)$ with $k = 44$ is connected by Theorem 4.5.)

10. One graph: larger moduli, few edges, few vertices, short cycles

Theorem 9.2 gives moduli k for which $I_k(G)$ and $M_k(G)$ are connected for *every* finite bridgeless loopless multigraph G: $k = 40$, $k = 48$ and every multiple of 4 from 52 on. We call them the *bases*. This section fixes one graph G and gives *every* k from a bound on, prime numbers k included. The bound depends on G:

(a) on the length of a longest cycle: every $k \ge \max(48, c(G)+4)$, and every $k \ge 40$ if every cycle has at most 39 edges (Corollary 10.6);

(b) on the number of edges: every $k \ge 40$ if G has at most 189 edges (Corollary 10.12);

(c) on the number of vertices: every $k \ge 40$ if G has at most 149 vertices (Corollary 10.16);

(d) on short cycles: every $k \ge 40$ if every edge lies on a cycle with at most 8 edges, or if G is connected with diameter at most 3 (Corollaries 10.22, 10.23, 10.24).

All four go up one level at a time, from $k-1$ to k, starting at a base. This section does *not* prove that $I_k(G)$ is connected for every bridgeless G and every large k.

Setting. In this section G is a finite bridgeless loopless multigraph, and $k \ge 3$ is an integer, unless something else is said. The conventions are those of Section 1. In particular $I_k(G)$ is the graph on the integer flows F of G with $1 \le |F(e)| \le k-1$ for every edge e, and F, F' are adjacent if $F' - F = \lambda\chi_C$ for a cycle C with a direction and an integer $\lambda \ne 0$ (a move).

For a flow F of $I_k(G)$, G_F is the orientation of G in which F is positive: every edge is directed so that its value is in $[1, k-1]$. We change the reference orientation freely (Section 1). In a statement about one flow F the values are read in G_F; an edge of G with this direction is an *arc*, and its *value* $F(e)$ is its size $|F(e)|$. An arc is a *top* if its value is $k-1$, a 1-*arc* if its value is 1, and a *middle arc* if its value is in $[2, k-2]$. For a set X of vertices, $\mathrm{in}(X)$ and $\mathrm{out}(X)$ are the numbers of arcs of G_F that enter X and that leave X; $\mathrm{in}_v(X)$ and $\mathrm{out}_v(X)$ count only the arcs of value v. The sum of the conservation equations over the vertices of X gives

$$\sum_{e \text{ leaves } X} F(e) \;=\; \sum_{e \text{ enters } X} F(e) \qquad \text{(the flow out of } X \text{ is the flow into } X\text{)}.$$

By Lemma 1.1 (b), applied to F with the reference orientation G_F, every arc of G_F lies on a *directed cycle* of G_F, that is a cycle that passes all its arcs in their direction.

Joined means: in the same component of $I_k(G)$, for the k of the context. A flow of $I_k(G)$ *descends* if it is joined to a flow of $I_{k-1}(G)$. $c(G)$ is the number of edges of a longest cycle of G, and $c(G) = 0$ if G has no cycle. For an integer $k_0 \geq 2$ put

$$\sigma(k_0) = \sum_{\lambda=1}^{k_0-1} \left\lceil \frac{k_0}{\lambda} \right\rceil.$$

One quoted theorem is used: Hoffman's circulation theorem, Theorem 0.5.

Lemma 10.1 (One level)**.** *Let G be any finite multigraph and $k \geq 3$.*

(a) *The vertices of $I_{k-1}(G)$ are the flows of $I_k(G)$ with all sizes at most $k-2$, and two of them are adjacent in $I_{k-1}(G)$ if and only if they are adjacent in $I_k(G)$.*

(b) *If $I_{k-1}(G)$ is connected and every flow of $I_k(G)$ descends, then $I_k(G)$ is connected.*

Proof. (a) holds by the definitions: the adjacency is the same rule in both graphs. (b) If $I_{k-1}(G)$ has no vertex, then no flow of $I_k(G)$ descends, so $I_k(G)$ has no vertex and is connected. Otherwise, by (a), the connected graph $I_{k-1}(G)$ lies in one component of $I_k(G)$, and every flow of $I_k(G)$ is joined to it. □

Lemma 10.2 (From integers to residues)**.** *Let G be any finite multigraph and $k \geq 2$. If $I_k(G)$ is connected, then $M_k(G)$ is connected. So each statement of this section that $I_k(G)$ is connected gives that $M_k(G)$ is connected.*

Proof. Let f, f' be vertices of $M_k(G)$. By Corollary 1.3 they have integer lifts F, F'. Take a path $F = F_0, F_1, \ldots, F_\nu = F'$ in $I_k(G)$, and let f_r be the reduction of F_r modulo k. It is a nowhere-zero $\mathbb{Z}_k$-flow, because $1 \leq |F_r(e)| \leq k-1$. If $F_r - F_{r-1} = \lambda\chi_C$, then $f_r - f_{r-1}$ is $\lambda\chi_C$ modulo k: either $f_r = f_{r-1}$, or f_{r-1} and f_r are adjacent in $M_k(G)$. So $f = f_0$ and $f' = f_\nu$ are joined in $M_k(G)$. □

Lemma 10.3 (Same orientation)**.** *Let F and T be flows of $I_k(G)$ with $G_T = G_F$. Then F and T are joined, by moves of the form χ_O.*

Proof. Use the reference orientation G_F. By Lemma 1.1 (b), $T - F = \chi_{O_1} + \cdots + \chi_{O_j}$, where every directed cycle O_r passes its edges in the direction of the sign of $T - F$. So on every edge e, every partial sum $F + \chi_{O_1} + \cdots + \chi_{O_r}$ has a value between $F(e)$ and $T(e)$, that is in $[1, k-1]$. All partial sums are in $I_k(G)$, and two consecutive ones differ by a move. □

Larger moduli on one graph.

Lemma 10.4 (Top-value reduction)**.** *Let F be a flow of $I_k(G)$ with a top e, and let C be a directed cycle of G_F through e. Let λ be an integer with $1 \leq \lambda \leq k-2$ that is not the value of any arc of C. Then $F - \lambda\chi_C$ is a flow of $I_k(G)$ that is adjacent to F (χ_C is read in G_F, so it is 1 on the arcs of C), and its tops are the tops of F that are not on C.*

Proof. An arc of C has a value v with $1 \leq v \leq k-1$ and $v \neq \lambda$. Its new value $v - \lambda$ is not 0, and $1 - \lambda \leq v - \lambda \leq k - 1 - \lambda$; so $|v - \lambda| \leq \max\{\lambda - 1, k - 1 - \lambda\} \leq k - 2$. The arcs outside C keep their values. So $F - \lambda\chi_C$ is in $I_k(G)$, no arc of C is a top of it, and the tops outside C do not change. It is adjacent to F because $\lambda \neq 0$. □

Theorem 10.5 (Larger moduli on one graph)**.** *Let $k_0 \geq 2$ be an integer with $k_0 \geq c(G) + 1$ such that $I_{k_0}(G)$ is connected. Then $I_k(G)$ and $M_k(G)$ are connected for every $k \geq k_0$.*

Proof. Induction on k; for $k = k_0$ this is the hypothesis and Lemma 10.2. Let $k > k_0$, so $k \geq 3$, and let $I_{k-1}(G)$ be connected. A flow of $I_k(G)$ without a top is in $I_{k-1}(G)$. Let F be a flow of $I_k(G)$ with a top e. Take a directed cycle C of G_F through e. It has at most $c(G) \leq k_0 - 1 \leq k - 2$ arcs, so at most $k - 3$ arcs other than e. The value of e is $k - 1$. So at most $k - 3$ of the $k - 2$ integers $1, \dots, k - 2$ are values of arcs of C, and some λ in $[1, k - 2]$ is not. By Lemma 10.4, F is adjacent to a flow with fewer tops. Repeat this with the new flow and its own orientation. The number of tops goes down at every step; so F is joined to a flow without tops, that is a flow of $I_{k-1}(G)$. So every flow of $I_k(G)$ descends, and $I_k(G)$ is connected by Lemma 10.1 (b). $M_k(G)$ is connected by Lemma 10.2. □

Corollary 10.6 (One graph: the bound by the longest cycle)**.** *Let G be a finite bridgeless loopless multigraph, and $c = c(G)$. Then $I_k(G)$ and $M_k(G)$ are connected for every $k \geq \max(48, c + 4)$. If $c \leq 39$, they are connected for every $k \geq 40$. Since $c \leq |V(G)|$, this holds in particular for every $k \geq \max(48, |V(G)| + 4)$.*

Proof. Let k_0 be the least multiple of 4 with $k_0 \geq \max(48, c + 1)$. It is a base, so $I_{k_0}(G)$ is connected by Theorem 9.2, and $k_0 \geq c + 1$. If $c + 1 \leq 48$, then $k_0 = 48$; otherwise $k_0 \leq (c + 1) + 3 = c + 4$. So $k_0 \leq \max(48, c + 4)$, and Theorem 10.5 gives every $k \geq k_0$. If $c \leq 39$, take the base $k_0 = 40 \geq c + 1$ in Theorem 10.5. □

One missing value; few edges.

Theorem 10.7 (One missing value)**.** *Let λ be an integer with $1 \leq \lambda \leq k - 2$, and let F be a flow of $I_k(G)$ with no arc of size λ. Then F is joined, by moves of the form $-\lambda\chi_O$ with directed cycles O of G_F, to a flow with all sizes at most $k - 2$. In particular F descends.*

Proof. Claim $(*)$. Let F be any flow of $I_k(G)$ and $1 \leq \lambda \leq k - 2$. Let Y be an integer flow of G with values in $\{0, 1\}$ in the orientation G_F, with $Y(e) = 1$ on every top and $Y(e) = 0$ on every arc of value λ. Then F is joined, by moves $-\lambda\chi_O$ with directed cycles O of G_F, to a flow with all sizes at most $k - 2$.

Proof of the claim. Use the reference orientation G_F, and write $v(e) = F(e)$, so $1 \leq v(e) \leq k - 1$. By Lemma 1.1 (b), $Y = \chi_{O_1} + \cdots + \chi_{O_\nu}$ with pairwise edge-disjoint directed cycles $O_1, \dots, O_\nu$ of G_F (Y has values in $\{0, 1\}$), and every arc with $Y(e) = 1$ is on exactly one of them. Put $F_r = F - \lambda(\chi_{O_1} + \cdots + \chi_{O_r})$ for $r = 0, \dots, \nu$. An arc on one of $O_1, \dots, O_r$ has in F_r the value $v(e) - \lambda$. It is not 0, because $Y(e) = 1$ and so $v(e) \neq \lambda$. And $v(e) - \lambda \leq k - 1 - \lambda \leq k - 2$, $\lambda - v(e) \leq \lambda - 1 \leq k - 3$. Every other arc keeps its value. So every F_r is in $I_k(G)$, and $F_r - F_{r-1} = -\lambda\chi_{O_r}$ is a move. In F_ν an arc on some O_r has a size at most $k - 2$, and an arc on no O_r has $Y(e) = 0$; so it is not a top of F, and its size $v(e)$ is at most $k - 2$. This proves the claim.

Now let F have no arc of size λ. In the orientation G_F put $x(e) = F(e)/(k - 1)$. Then $0 < x(e) \leq 1$, $x(e) = 1$ exactly on the tops, and $\partial x = 0$. By Lemma 1.2 there is $Y\colon E(G) \to \{0, 1\}$ with $\partial Y = 0$ and $Y(e) = 1$ on every top. This Y is an integer flow, and the condition on the arcs of value λ is empty. The claim gives the theorem. □

Corollary 10.8 (Every size occurs)**.** *Let F be a flow of $I_k(G)$ that does not descend. Then F, and every flow that is joined to F, has arcs of every size $1, 2, \dots, k - 1$.*

Proof. A flow joined to F does not descend either. If it has no arc of a size $\lambda \leq k - 2$, it descends by Theorem 10.7. If it has no arc of size $k - 1$, it is in $I_{k-1}(G)$, so it descends. □

Corollary 10.9 (The cut for one size)**.** *Let $1 \le \lambda \le k-2$, and let F be a flow of $I_k(G)$ that is* not *joined by moves of the form $-\lambda\chi_O$, with directed cycles O of G_F, to a flow with all sizes at most $k-2$. Then there is a set X of vertices with*

$$\mathrm{out}_{k-1}(X) \;\ge\; \mathrm{in}(X) - \mathrm{in}_\lambda(X) + 1 \qquad \textit{and} \qquad \lambda \cdot \mathrm{in}_\lambda(X) \;\ge\; k-1.$$

So at least $\lceil (k-1)/\lambda \rceil$ arcs of value λ enter X.

Proof. On the arcs of G_F put $l(e) = u(e) = 1$ on the tops, $l(e) = u(e) = 0$ on the arcs of value λ, and $l(e) = 0$, $u(e) = 1$ on all other arcs (a top does not have the value $\lambda \le k-2$). An integer flow Y with $l \le Y \le u$ is a flow as in Claim $(*)$ of the proof of Theorem 10.7; the claim would join F by moves $-\lambda\chi_O$ to a flow with all sizes at most $k-2$. This is excluded. So by Theorem 0.5 some set X has

$$(\text{sum of } l \text{ over the arcs that leave } X) > (\text{sum of } u \text{ over the arcs that enter } X),$$

that is $\mathrm{out}_{k-1}(X) > \mathrm{in}(X) - \mathrm{in}_\lambda(X)$. All values are in $[1, k-1]$, and the flow out of X is the flow into X:

$$(k-1)\,\mathrm{out}_{k-1}(X) \;\le\; \lambda\,\mathrm{in}_\lambda(X) + (k-1)\big(\mathrm{in}(X) - \mathrm{in}_\lambda(X)\big) \;\le\; \lambda\,\mathrm{in}_\lambda(X) + (k-1)\big(\mathrm{out}_{k-1}(X) - 1\big).$$

So $\lambda\,\mathrm{in}_\lambda(X) \ge k-1$. □

Corollary 10.10 (The count of the edges)**.** *Let F be a flow of $I_k(G)$ that does not descend. Then for every λ with $1 \le \lambda \le k-2$ the flow F has at least $\lceil (k-1)/\lambda \rceil$ arcs of size λ, and it has a top. So $|E(G)| \ge 1 + \sigma(k-1)$.*

Proof. F is not joined to any flow with all sizes at most $k-2$, because such a flow is in $I_{k-1}(G)$. Corollary 10.9 gives the arcs of size λ, for every λ. Arcs of different sizes are different arcs. F has a top, because F is not in $I_{k-1}(G)$. So $|E(G)| \ge 1 + \sum_{\lambda=1}^{k-2} \lceil (k-1)/\lambda \rceil = 1 + \sigma(k-1)$. □

Corollary 10.11 (One level at a time)**.** *Let $k_0 \ge 3$, let $I_{k_0}(G)$ be connected, and let $|E(G)| \le \sigma(k_0)$. Then $I_k(G)$ and $M_k(G)$ are connected for every $k \ge k_0$.*

Proof. First, $\sigma(k+1) \ge \sigma(k)$ for $k \ge 2$: for $\lambda \le k-1$ the term $\lceil (k+1)/\lambda \rceil$ is at least $\lceil k/\lambda \rceil$, and there is one more term. Induction on k; $k = k_0$ is the hypothesis. Let $k > k_0$, and let $I_{k-1}(G)$ be connected. If a flow of $I_k(G)$ does not descend, Corollary 10.10 gives $|E(G)| \ge 1 + \sigma(k-1) \ge 1 + \sigma(k_0)$, a contradiction. So every flow descends, and Lemma 10.1 (b) gives the step. $M_k(G)$: Lemma 10.2. □

Corollary 10.12 (At most 189 edges)**.** *Let G be a finite bridgeless loopless multigraph.*

(a) *If G has at most 189 edges, then $I_k(G)$ and $M_k(G)$ are connected for every $k \ge 40$.*

(b) *If G has at most 235 edges, then they are connected for every $k \ge 48$.*

(c) *In general, they are connected for every $k \ge k_0$, for every base k_0 with $\sigma(k_0) \ge |E(G)|$.*

Proof. (c) is Corollary 10.11 with Theorem 9.2. (a) and (b) are (c) with $\sigma(40) = 189$ and $\sigma(48) = 235$. The terms $\lceil 40/\lambda \rceil$ and $\lceil 48/\lambda \rceil$ are:

λ	1	2	3	4	5	6	7	8, 9	10, …, 13	14, …, 19	20, …, 39
$\lceil 40/\lambda \rceil$	40	20	14	10	8	7	6	5	4	3	2

λ	1	2	3	4	5	6	7	8, 9	10, 11	12, …, 15	16, …, 23	24, …, 47
$\lceil 48/\lambda \rceil$	48	24	16	12	10	8	7	6	5	4	3	2

So $\sigma(40) = 40+20+14+10+8+7+6+2\cdot5+4\cdot4+6\cdot3+20\cdot2 = 115+16+18+40 = 189$, and $\sigma(48) = 48+24+16+12+10+8+7+2\cdot6+2\cdot5+4\cdot4+8\cdot3+24\cdot2 = 147+16+24+48 = 235$. □

Few vertices. In this subsection $k \geq 4$. For a flow F of $I_k(G)$ let $\Phi(F) = \sum_e |F(e)|$, the sum of the sizes. A *failure* (for G and k) is a component of $I_k(G)$ that contains no flow of $I_{k-1}(G)$. Put

$$S(k) = \sum_{\lambda=2}^{k-2} \left\lceil \frac{k-1}{\lambda} \right\rceil = \sigma(k-1) - (k-1).$$

Lemma 10.13 (A potential)**.** *Let $\mathcal{U}$ be a component of $I_k(G)$, and let F be a flow of $\mathcal{U}$ with the least Φ in $\mathcal{U}$. Then there is a map $\theta\colon V(G) \to \mathbb{Z}$ such that for every arc e of G_F from x to y:*

$$\begin{aligned} &\theta(y) - \theta(x) = 1 && \text{if } e \text{ is a middle arc;} \\ &\theta(y) - \theta(x) \leq 1 && \text{if } e \text{ is a 1-arc;} \\ &\theta(y) - \theta(x) \geq 1 && \text{if } e \text{ is a top.} \end{aligned}$$

Proof. Let R be the directed multigraph on $V(G)$ with the following arcs and costs. Every arc e of G_F from x to y gives an arc $x \to y$ of cost $+1$ if $F(e) \leq k-2$, and an arc $y \to x$ of cost -1 if $F(e) \geq 2$. A middle arc gives both.

Every directed cycle of R has a cost of at least 0. Let O be a directed cycle of R (distinct vertices; R has no loop, because G has none). If O uses both arcs that come from one arc e of G_F, then after the first of them O is at an end of e, and the only arc of O that leaves this vertex is the second one; so O consists of these two arcs, and its cost is 0. Otherwise O uses pairwise different edges of G and passes pairwise different vertices, so it is a cycle C of G with a direction (two parallel edges give a cycle with 2 edges). Put $F' = F + \chi_C$, with χ_C read in G_F. If O uses the arc $x \to y$ of e, then C passes e in its direction and $F'(e) = F(e) + 1 \leq k-1$. If O uses the arc $y \to x$ of e, then C passes e against its direction and $F'(e) = F(e) - 1 \geq 1$. All other values are equal. So F' is in $I_k(G)$, positive on G_F, and adjacent to F; so F' is in $\mathcal{U}$, and $\Phi(F') - \Phi(F)$ is the cost of O. Since F has the least Φ in $\mathcal{U}$, the cost is at least 0.

The potential. For a vertex y let $\theta(y)$ be the least cost of a directed path of R that ends at y (the path with the one vertex y and no arc has cost 0; there are finitely many paths). Let $x \to y$ be an arc of R with cost κ, and let P be a path that ends at x with cost $\theta(x)$. If y is not on P, then P followed by the arc is a path that ends at y, so $\theta(y) \leq \theta(x) + \kappa$. If y is on P, let P_1 be the part of P up to y and P_2 the part from y to x. Then P_2 followed by the arc $x \to y$ is a directed cycle of R, so $\text{cost}(P_2) + \kappa \geq 0$, and $\theta(y) \leq \text{cost}(P_1) = \theta(x) - \text{cost}(P_2) \leq \theta(x) + \kappa$. In both cases $\theta(y) \leq \theta(x) + \kappa$.

An arc e of G_F from x to y with $F(e) \leq k-2$ gives $\theta(y) - \theta(x) \leq 1$. If $F(e) \geq 2$, the arc $y \to x$ of cost -1 gives $\theta(x) \leq \theta(y) - 1$. A middle arc has both; a 1-arc has the first $(1 \leq k-2)$; a top has the second $(k-1 \geq 2)$. □

Lemma 10.14 (A forest of middle arcs)**.** *Every component $\mathcal{U}$ of $I_k(G)$ contains a flow F^* that has the least Φ in $\mathcal{U}$ and whose middle arcs contain no cycle of G.*

Proof. Among the flows of $\mathcal{U}$ with the least Φ, take a flow F with the least number of middle arcs. Suppose that C is a cycle of G all of whose edges are middle arcs of F. Give C a direction, and take θ from Lemma 10.13. Every arc of C has $\theta(y) - \theta(x) = 1$. The sum of $\theta(y) - \theta(x)$ over the arcs that C passes in their direction, minus the same sum over the arcs that C passes against their direction, is 0, because it is a sum of differences of θ once round C. So C passes as many arcs in their direction as against it. C has at least 2 edges, so it passes at least one arc in its direction.

Put $F_j = F + j\chi_C$ (χ_C read in G_F) for $j = 0, 1, 2, \dots$, and let $j_0 \geq 1$ be the least j such that F_j has the value 1 or $k-1$ on some arc of C. It exists, because the value of an arc that C passes in its direction grows by 1 at every step. For $j < j_0$ all values of F_j on C are in $[2, k-2]$, so all values of F_{j_0} on C are in $[1, k-1]$. The values outside C do not change. So $F_0, \dots, F_{j_0}$ are in $I_k(G)$, all positive on G_F, and two consecutive ones differ by a move. Each step adds 1 to the values of the arcs that C passes in their direction and subtracts 1 from the values of the other arcs of C; there are as many arcs of the first kind as of the second, so $\Phi(F_j) = \Phi(F)$ for $0 \leq j \leq j_0$. So F_{j_0} is in $\mathcal{U}$ and has the least Φ. Its middle arcs are among those of F, and an arc of C is not a middle arc of F_{j_0}. So F_{j_0} has fewer middle arcs than F, a contradiction. □

Theorem 10.15 (A failure needs many vertices)**.** *Let $k \geq 4$, and let $\mathcal{U}$ be a failure for G and k. Then*

$$|V(G)| \geq 1 + S(k) \qquad \text{and} \qquad |E(G)| \geq k + S(k).$$

Proof. Take F^* in $\mathcal{U}$ from Lemma 10.14. Let $1 \leq \lambda \leq k-2$. The flow F^* is not joined to any flow with all sizes at most $k-2$ (such a flow is in $I_{k-1}(G)$), in particular not by moves $-\lambda\chi_O$ with directed cycles O of G_{F^*}. By Corollary 10.9, F^* has at least $\lceil (k-1)/\lambda \rceil$ arcs of size λ. For $2 \leq \lambda \leq k-2$ these are middle arcs, and arcs of different sizes are different. So F^* has at least $S(k)$ middle arcs. They form a forest of G, which has at most $|V(G)| - 1$ edges. So $|V(G)| - 1 \geq S(k)$. For the edges: F^* has in addition at least $k-1$ arcs of size 1 ($\lambda = 1$) and at least one top (F^* is not in $I_{k-1}(G)$). □

Corollary 10.16 (At most 149 vertices)**.** *Let G be a finite bridgeless loopless multigraph with n vertices.*

(a) *If $n \leq 149$, then $I_k(G)$ and $M_k(G)$ are connected for every $k \geq 40$ (with any number of edges). If $n \leq 187$, they are connected for every $k \geq 48$.*

(b) *They are connected for every $k \geq k_0$, for every base k_0 with $S(k_0 + 1) = \sigma(k_0) - k_0 \geq n$.*

(c) *$S(k_0 + 1) \geq k_0(\ln k_0 - 1)$ for every $k_0 \geq 4$. So* (b) *holds for every base k_0 with $k_0(\ln k_0 - 1) \geq n$.*

(d) *If $n \geq 150$, they are connected for every $k \geq 2n/\ln n + 4$.*

Proof. First, $S(k+1) \geq S(k)$ for $k \geq 4$: $S(k+1) = \sum_{\lambda=2}^{k-1} \lceil k/\lambda \rceil$, every term $\lceil k/\lambda \rceil$ with $\lambda \leq k-2$ is at least $\lceil (k-1)/\lambda \rceil$, and there is one more term.

(b) Induction on $k \geq k_0$; for $k = k_0$, $I_{k_0}(G)$ is connected by Theorem 9.2. Let $k > k_0$, and let $I_{k-1}(G)$ be connected. Then $S(k) \geq S(k_0 + 1) \geq n$, so by Theorem 10.15 there is no failure for G and k: every flow of $I_k(G)$ descends. Lemma 10.1 (b) gives the step, and Lemma 10.2 gives $M_k(G)$.

(a) $S(41) = \sigma(40) - 40 = 149$ and $S(49) = \sigma(48) - 48 = 187$ (Corollary 10.12). Use (b) with the bases 40 and 48.

(c) $S(k_0+1) = \sum_{\lambda=2}^{k_0-1} \lceil k_0/\lambda \rceil \geq k_0\left(\frac{1}{2} + \dots + \frac{1}{k_0-1}\right) \geq k_0(\ln k_0 - 1)$, because $1 + \frac{1}{2} + \dots + \frac{1}{k_0-1} \geq \int_1^{k_0} \frac{dx}{x} = \ln k_0$.

(d) Put $\eta = 2n/\ln n$, and let k_0 be the least multiple of 4 with $k_0 \geq \eta$; so $k_0 < \eta + 4$. The function $\xi/\ln\xi$ grows for $\xi \geq 3$, and $\ln 150 < 5.02$; so $\eta \geq 300/5.02 > 59$, and $k_0 \geq 60$ is a base. The function $\xi(\ln\xi - 1)$ grows for $\xi \geq 1$, so $k_0(\ln k_0 - 1) \geq \eta(\ln\eta - 1)$. And $\eta(\ln\eta - 1) \geq n$ means $2(\ln 2 + \ln n - \ln\ln n - 1) \geq \ln n$, that is

$$\ln n - 2\ln\ln n \;\geq\; 2 - 2\ln 2.$$

Here $\ln n \geq \ln 150 > 5$. The function $\xi - 2\ln\xi$ grows for $\xi > 2$, and at $\xi = 5$ it is $5 - 2\ln 5 > 1.7$, while $2 - 2\ln 2 < 0.62$. So $k_0(\ln k_0 - 1) \geq n$, and (c) gives every $k \geq k_0$, in particular every $k \geq \eta + 4$. □

Short cycles. In this subsection F is a flow of $I_k(G)$. A *lift* of F is a flow $\widetilde{F}$ of $I_k(G)$ with $\widetilde{F} = F$ modulo k (edge by edge). Let $\Psi(F)$ be the number of arcs of G_F that are 1-arcs or tops. An edge e is counted in $\Psi(F)$ if and only if $F(e)$ is 1 or -1 modulo k; so $\Psi(F)$ depends only on F modulo k.

Residues. Let C be a cycle of G with a direction. For an arc e of G_F on C put $r(e) = F(e)$ if C passes e in its direction, and $r(e) = k - F(e)$ if C passes e against its direction. So $1 \leq r(e) \leq k-1$, and $r(e)$ is the value of F on e measured in the direction of C, taken modulo k. It is the same for all lifts of F. If the direction of C is reversed, $r(e)$ becomes $k - r(e)$. Write $C_{<\lambda}$ and $C_{>\lambda}$ for the sets of the arcs e of C with $r(e) < \lambda$ and with $r(e) > \lambda$. For a set X of vertices and an arc e of the cut of X (the edges with one end in X) put $r_X(e) = F(e)$ if e leaves X and $r_X(e) = k - F(e)$ if e enters X. If C passes the arc e of the cut *into* X, then $r_X(e) = k - r(e)$; if C passes e *out of* X, then $r_X(e) = r(e)$. (Check the four cases: e in the direction of C or against it, and e entering or leaving X in G_F.)

Lemma 10.17 (Lifts)**.** (a) *The lifts of F are exactly the flows $F - kY$, where Y is an integer flow of G with values in $\{0,1\}$ in the orientation G_F. All lifts of F are joined to F.*

(b) *For every set X of vertices, $k\,\mathrm{in}(X)$ is the sum of $r_X(e)$ over the arcs e of the cut of X.*

Proof. (a) Let $\widetilde{F}$ be a lift, read in G_F. On an arc e the number $\widetilde{F}(e)$ is $F(e)$ modulo k and lies in $[-(k-1), k-1] \setminus \{0\}$; so it is $F(e)$ or $F(e) - k$. So $Y = (F - \widetilde{F})/k$ is an integer flow with values in $\{0,1\}$. Conversely, for such a Y the flow $F - kY$ has on every arc the value $F(e)$ in $[1, k-1]$ or $F(e) - k$ in $[-(k-1), -1]$; so it is a lift of F. The lifts of F are the integer lifts of the reduction of F modulo k, so they are joined by Lemma 1.4.

(b) The flow out of X is the flow into X. So the sum of r_X is $\sum_{e \text{ leaves } X} F(e) + \sum_{e \text{ enters } X} \big(k - F(e)\big) = k\,\mathrm{in}(X)$. □

Theorem 10.18 (A step after a change of lift)**.** *Let C be a cycle of G with a direction, and λ an integer with $1 \leq \lambda \leq k-1$ such that $r(e) \neq \lambda$ for every arc e of C. The following are equivalent:*

(i) *some lift $\widetilde{F}$ of F has $\widetilde{F} - \lambda\chi_C$ in $I_k(G)$;*

(ii) *for every set X of vertices, the number of arcs of $C_{<\lambda}$ that C passes into X is at most $\mathrm{in}(X)$.*

If (i) *holds, then F is joined to $\widetilde{F} - \lambda\chi_C$, and this flow is $F - \lambda\chi_C$ modulo k.*

Proof. Measure every value on an arc e of C in the direction of C. Measured so, F has a value that is $r(e)$ modulo k. A lift $\widetilde{F}$ has a value that is $r(e)$ modulo k and whose absolute value is in $[1, k-1]$; so it has the value $r(e)$ or $r(e) - k$. After the step it has $r(e) - \lambda$ or $r(e) - k - \lambda$. The first number is not 0 and lies in $[-(k-2), k-2]$. The second is negative, and it is at least $-(k-1)$ exactly if $r(e) \geq \lambda + 1$. The values outside C do not change. So $\widetilde{F} - \lambda\chi_C$ is in $I_k(G)$ if and only if $\widetilde{F}$ has the value $r(e)$, measured in the direction of C, on every arc e of $C_{<\lambda}$.

Write $\widetilde{F} = F - kY$ as in Lemma 10.17 (a). Let C^+ be the set of the arcs of $C_{<\lambda}$ that C passes in their direction, and C^- the set of those that C passes against their direction. On an arc of C^+, F has the value $r(e)$ along C, and $\widetilde{F}$ has it if and only if $Y(e) = 0$. On an arc

of C^-, F has the value $-F(e) = r(e) - k$ along C, and $\widetilde{F}$ has the value $r(e)$ if and only if $Y(e) = 1$. So (i) holds if and only if there is an integer flow Y in G_F with $l \le Y \le u$, where $l = u = 0$ on C^+, $l = u = 1$ on C^-, and $l = 0$, $u = 1$ on all other arcs. By Theorem 0.5 this holds if and only if for every X:

$$(\text{number of arcs of } C^- \text{ that leave } X) \;\le\; \mathrm{in}(X) - (\text{number of arcs of } C^+ \text{ that enter } X).$$

An arc of C^- that leaves X is passed by C into X, and so is an arc of C^+ that enters X; and every arc of $C_{<\lambda}$ that C passes into X is of one of these two kinds. So this is (ii).

The last sentence: F is joined to $\widetilde{F}$ by Lemma 10.17 (a), and $\widetilde{F} \to \widetilde{F} - \lambda\chi_C$ is a move. □

Corollary 10.19 (A sufficient condition)**.** *Let C and λ be as in Theorem 10.18. If the sum of $r(e) - 1$ over the arcs e of $C_{<\lambda}$ is less than k, then* (ii) *of Theorem 10.18 holds. In particular* (ii) *holds for $\lambda = 1$ ($C_{<1}$ is empty) and for $\lambda = 2$ (every arc of $C_{<2}$ has $r(e) = 1$).*

Proof. Fix X. The cycle C passes arcs of the cut of X into X as often as out of X; say j times each. Let j' of the j arcs that C passes into X be in $C_{<\lambda}$. By Lemma 10.17 (b), and because $r_X \ge 1$ on every arc of the cut,

$$\begin{aligned} k\,\mathrm{in}(X) \;&\ge \sum_{\substack{e\in C_{<\lambda}\\ \text{passed into } X}} \big(k - r(e)\big) + (j - j') + j \;=\; j'k - \sum_{\substack{e\in C_{<\lambda}\\ \text{passed into } X}} \big(r(e) - 1\big) + 2(j - j') \\ &\ge\; j'k - \sum_{e\in C_{<\lambda}} \big(r(e) - 1\big) \;>\; j'k - k. \end{aligned}$$

(The j' arcs of $C_{<\lambda}$ passed into X have $r_X = k - r(e)$; the other $j - j'$ arcs passed into X, the j arcs passed out of X, and the arcs of the cut that are not on C have $r_X \ge 1$, so they add at least $(j - j') + j$; and every term $r(e) - 1$ is at least 0.) So $\mathrm{in}(X) > j' - 1$, that is $\mathrm{in}(X) \ge j'$. □

Theorem 10.20 (The cleaning step)**.** *Let e_0 be a 1-arc or a top of F, and let C be a cycle through e_0, directed so that $r(e_0) = 1$ (in the direction of a 1-arc, against the direction of a top).*

(a) *Let λ be an integer with $3 \le \lambda \le k - 3$ such that no arc of C has $r(e)$ in $\{\lambda - 1, \lambda, \lambda + 1\}$, and such that the sum of $r(e) - 1$ over $C_{<\lambda}$ is less than k, or the sum of $k - 1 - r(e)$ over $C_{>\lambda}$ is less than k. Then F is joined to a flow F' with $F' = F - \lambda\chi_C$ modulo k and $\Psi(F') \le \Psi(F) - 1$.*

(b) *Let $h \ge 0$ be an integer with $3h^2 < k$ and $6h + 8 < k$, and put $I_{\mathrm{low}} = [2, 3h + 4]$ and $I_{\mathrm{high}} = [k - 3h - 4, k - 2]$ (sets of integers; they are disjoint, because $3h + 4 < k - 3h - 4$). If at most h arcs of C have $r(e)$ in I_{low}, or at most h arcs of C have $r(e)$ in I_{high}, then F is joined to a flow F' with $\Psi(F') \le \Psi(F) - 1$.*

Proof. (a) First case: the sum over $C_{<\lambda}$ is less than k. No arc of C has $r(e) = \lambda$. By Theorem 10.18 and Corollary 10.19, F is joined to $F' = \widetilde{F} - \lambda\chi_C$, which is $F - \lambda\chi_C$ modulo k. Second case: the sum over $C_{>\lambda}$ is less than k. Let C' be C with the other direction; its residues are $k - r(e)$, none of them is $k - \lambda$, and $3 \le k - \lambda \le k - 3$. The arcs with $k - r(e) < k - \lambda$ are the arcs of $C_{>\lambda}$, and the sum of $(k - r(e)) - 1$ over them is less than k. Theorem 10.18 and Corollary 10.19 for C' and $k - \lambda$ give F' joined to F with $F' = F - (k - \lambda)\chi_{C'} = F + (k - \lambda)\chi_C = F - \lambda\chi_C$ modulo k.

The count. An edge outside C keeps its value modulo k. An arc e of C has in F' the value $r(e) - \lambda$ modulo k, measured in the direction of C. It is counted in $\Psi(F')$ if and only

if $r(e) - \lambda$ is 1 or -1 modulo k. Since $r(e) - \lambda$ lies in $[4-k, k-4]$, this means $r(e) = \lambda + 1$ or $r(e) = \lambda - 1$, which is excluded. So no edge of C is counted in $\Psi(F')$, while e_0 is counted in $\Psi(F)$.

(b) First case: at most h arcs of C have $r(e)$ in I_{low}. Look at the $3h+1$ integers λ in $[3, 3h+3]$. An arc with $r(e)$ in I_{low} has $r(e)$ in $\{\lambda-1, \lambda, \lambda+1\}$ for at most 3 of them. An arc with $r(e) = 1$ or $r(e) \geq 3h+5$ has this for none of them, because $\lambda - 1 \geq 2$ and $\lambda + 1 \leq 3h+4$. So at most $3h$ values of λ are excluded, and some λ in $[3, 3h+3]$ has no arc of C with $r(e)$ in $\{\lambda-1, \lambda, \lambda+1\}$. For this λ, an arc of $C_{<\lambda}$ has $r(e) = 1$, or $2 \leq r(e) \leq \lambda - 2$; the arcs of the second kind have $r(e)$ in I_{low}, so there are at most h of them, each with $r(e) - 1 \leq \lambda - 3 \leq 3h$. So the sum of $r(e) - 1$ over $C_{<\lambda}$ is at most $3h^2 < k$. And $3 \leq \lambda \leq 3h + 3 \leq k - 3$, because $6h + 8 < k$. Part (a) applies.

Second case: at most h arcs of C have $r(e)$ in I_{high}. Look at the $3h+1$ integers λ in $[k-3h-3, k-3]$. An arc with $r(e) = k-1$ or $r(e) \leq k-3h-5$ has $r(e)$ in $\{\lambda-1, \lambda, \lambda+1\}$ for none of them, because $\lambda + 1 \leq k-2$ and $\lambda - 1 \geq k - 3h - 4$; an arc with $r(e)$ in I_{high} for at most 3. So some λ has no arc of C with $r(e)$ in $\{\lambda-1, \lambda, \lambda+1\}$. An arc of $C_{>\lambda}$ has $r(e) = k-1$, or $\lambda + 2 \leq r(e) \leq k-2$; the arcs of the second kind have $r(e)$ in I_{high}, so there are at most h of them, each with $k - 1 - r(e) \leq k - 3 - \lambda \leq 3h$. So the sum of $k - 1 - r(e)$ over $C_{>\lambda}$ is at most $3h^2 < k$. And $k - 3 \geq \lambda \geq k - 3h - 3 \geq 3$. Part (a) applies. □

Theorem 10.21 (Short cycles)**.** *Let $h \geq 0$ be an integer with $3h^2 < k$ and $6h + 8 < k$. If every edge of G lies on a cycle of G with at most $2h+2$ edges, then every flow of $I_k(G)$ is joined to a flow with all sizes in $[2, k-2]$. In particular every flow of $I_k(G)$ descends.*

Proof. Induction on $\Psi(F)$. If $\Psi(F) = 0$, then F has no 1-arc and no top. Otherwise take a 1-arc or a top e_0, and a cycle C through e_0 with at most $2h+2$ edges, directed so that $r(e_0) = 1$. The residue 1 is in neither I_{low} nor I_{high}. The other arcs of C are at most $2h+1$, and the two sets are disjoint; so one of them contains the residues of at most h arcs of C. Theorem 10.20 (b) gives F' joined to F with $\Psi(F') < \Psi(F)$. A flow with all sizes in $[2, k-2]$ is in $I_{k-1}(G)$. □

Corollary 10.22 (Every edge on a short cycle)**.** (a) *If every edge of G lies on a cycle with at most 8 edges, then every flow of $I_k(G)$ descends for every $k \geq 28$, and $I_k(G)$ and $M_k(G)$ are connected for every $k \geq 40$.*

(b) *Let $h \geq 4$. If every edge of G lies on a cycle with at most $2h+2$ edges, then $I_k(G)$ and $M_k(G)$ are connected for every $k \geq k_h$, where k_h is the least multiple of 4 with $k_h \geq 3h^2$ (so $48 \leq k_h \leq 3h^2 + 3$).*

Proof. (a) Take $h = 3$: $3h^2 = 27 < k$ and $6h + 8 = 26 < k$ for $k \geq 28$, so Theorem 10.21 applies. $I_{40}(G)$ is connected by Theorem 9.2. For $k \geq 41$, every flow of $I_k(G)$ descends, and Lemma 10.1 (b) gives every $k \geq 40$ by induction. $M_k(G)$: Lemma 10.2.

(b) $k_h \geq 3 \cdot 16 = 48$ is a multiple of 4, so $I_{k_h}(G)$ is connected by Theorem 9.2. For $k \geq k_h + 1$: $3h^2 \leq k_h < k$, and $6h + 8 \leq 3h^2 < k$, because $3h^2 - 6h - 8 \geq 16$ for $h \geq 4$. So Theorem 10.21 applies, and the induction is as in (a). □

Examples for (a): complete graphs on at least 3 vertices (every edge is on a triangle), complete bipartite graphs with at least 2 vertices on each side and hypercubes of dimension at least 2 (every edge is on a cycle with 4 edges), and every bridgeless loopless multigraph whose cycles all have at most 8 edges.

Corollary 10.23 (Diameter)**.** *Let G be connected, with at least one edge, and let* $\operatorname{diam}(G)$ *be its diameter. Then every edge of G lies on a cycle with at most* $2\operatorname{diam}(G)+1$ *edges. So, if* $\operatorname{diam}(G)\le 3$*, then $I_k(G)$ and $M_k(G)$ are connected for every $k\ge 40$; if* $\operatorname{diam}(G)\ge 4$*, they are connected for every $k\ge k_{\operatorname{diam}(G)}$ (Corollary 10.22 (b)).*

Proof. Write dist for the distance in G, and $d_G=\operatorname{diam}(G)$. Let e be an edge with ends $x\ne y$.

Case 1: some vertex w has $\operatorname{dist}(w,x)=\operatorname{dist}(w,y)=j$. Then $j\le d_G$. Take shortest paths P from w to x and P' from w to y. Every vertex of a shortest path from w has, as its distance from w, its position on the path. So P does not contain y (y has the distance j, and the only vertex of P with the distance j is x), P' does not contain x, and neither path uses e. Let z be the last vertex of P that lies on P' (w lies on both); $z\ne x$, $z\ne y$. The part of P from z to x, the edge e, and the part of P' from y back to z form a cycle through e: the first part meets P' only in z. It has at most $2j+1\le 2d_G+1$ edges.

Case 2: there is no such w. Then $V(G)$ is the disjoint union of $V_x=\{w:\operatorname{dist}(w,x)<\operatorname{dist}(w,y)\}$ and $V_y=\{w:\operatorname{dist}(w,y)<\operatorname{dist}(w,x)\}$, and the two distances of a vertex differ by exactly 1, because x and y are adjacent. The edge e is not a bridge, so $G-e$ has a path from x in V_x to y in V_y, and some edge $e'\ne e$ of this path joins a vertex x_1 in V_x to a vertex y_1 in V_y. Now $\operatorname{dist}(y_1,y)+1=\operatorname{dist}(y_1,x)\le\operatorname{dist}(x_1,x)+1$ and $\operatorname{dist}(x_1,x)+1=\operatorname{dist}(x_1,y)\le\operatorname{dist}(y_1,y)+1$; so $\operatorname{dist}(x_1,x)=\operatorname{dist}(y_1,y)=i'$, and $i'+1=\operatorname{dist}(x_1,y)\le d_G$. A shortest path from x to x_1 stays in V_x: a vertex z on it has $\operatorname{dist}(z,y)\ge\operatorname{dist}(x_1,y)-\operatorname{dist}(z,x_1)=i'+1-(i'-\operatorname{dist}(z,x))=\operatorname{dist}(z,x)+1$. In the same way a shortest path from y_1 to y stays in V_y. These two paths, with e' and e, form a cycle through e with $2i'+2\le 2d_G$ edges.

The rest: if $d_G\le 3$, every edge lies on a cycle with at most $7\le 8$ edges, and Corollary 10.22 (a) applies. If $d_G\ge 4$, use Corollary 10.22 (b) with $h=d_G$, because $2d_G+1\le 2d_G+2$. □

Corollary 10.24 (All but a few edges on short cycles)**.** *Let $h\ge 0$ be an integer with $3h^2<k$ and $6h+8<k$. Call an edge* short *if it lies on a cycle with at most $2h+2$ edges. An* edge cut *is the set of the edges with one end in X and one end outside X, for a set X of vertices.*

(a) *Every flow of $I_k(G)$ is joined to a flow F' in which no short edge is a 1-arc or a top.*

(b) *If no edge cut of G contains k or more edges that are not short, then every flow of $I_k(G)$ descends. This holds in particular if G has at most $k-1$ edges that are not short.*

(c) *If all edges of G except at most 40 lie on cycles with at most 8 edges, then $I_k(G)$ and $M_k(G)$ are connected for every $k\ge 40$.*

Proof. (a) As long as a short edge e_0 is a 1-arc or a top, apply Theorem 10.20 (b) with a cycle through e_0 that has at most $2h+2$ edges, as in the proof of Theorem 10.21. Every time Ψ goes down.

(b) Take F' from (a). In this proof in, out are counted in $G_{F'}$. If $G_{F'}$ has an integer flow T with $1\le T(e)\le k-2$ on every arc, then T is in $I_{k-1}(G)$ and $G_T=G_{F'}$, so T is joined to F' by Lemma 10.3, and F' descends. Otherwise, by Theorem 0.5 with $l=1$ and $u=k-2$ on all arcs, some set X has $\operatorname{out}(X)>(k-2)\operatorname{in}(X)$. Then $\operatorname{out}(X)\ge 1$, the flow out of X is positive, so is the flow into X, and $\operatorname{in}(X)\ge 1$; hence $\operatorname{out}(X)\ge k-1$. Every arc that leaves X and is not a 1-arc has a value of at least 2:

$$(k-1)\operatorname{in}(X)\ \ge\ \text{flow into } X\ =\ \text{flow out of } X\ \ge\ \operatorname{out}_1(X)+2\big(\operatorname{out}(X)-\operatorname{out}_1(X)\big).$$

With $\mathrm{in}(X) \le (\mathrm{out}(X)-1)/(k-2)$ this gives

$$\begin{aligned}\mathrm{out}_1(X) &\ge 2\,\mathrm{out}(X) - \frac{(k-1)(\mathrm{out}(X)-1)}{k-2} = \frac{(k-3)\,\mathrm{out}(X)+(k-1)}{k-2}\\ &\ge \frac{(k-3)(k-1)+(k-1)}{k-2} = k-1.\end{aligned}$$

Also the flow into X is at least $\mathrm{out}(X) > (k-2)\,\mathrm{in}(X)$, so some arc that enters X has a value greater than $k-2$: it is a top. So the cut of X contains at least k arcs that are 1-arcs or tops of F'. By (a) none of them is short. This contradicts the hypothesis. So F', and hence every flow, descends.

(c) Take $h = 3$. For every $k \ge 41$: $27 < k$, $26 < k$, and G has at most $40 \le k-1$ edges that are not short; so by (b) every flow of $I_k(G)$ descends. $I_{40}(G)$ is connected by Theorem 9.2. Lemma 10.1 (b) and induction give every $k \ge 40$, and Lemma 10.2 gives $M_k(G)$. □

Remarks.

Remark 10.25 (How the bounds compare)**.** The four kinds of bound do not compare in general. For a graph with a Hamiltonian cycle on n vertices, Theorem 10.5, with the base k_0 of the proof of Corollary 10.6, gives every $k \ge \max(48, 4\lceil (n+1)/4\rceil)$, and Corollary 10.6 gives every $k \ge 40$ if $n \le 39$; Corollary 10.16 (d) gives every $k \ge 2n/\ln n + 4$ for $n \ge 150$. A graph with many vertices and only short cycles is better for Corollary 10.6 or 10.22. A graph with at most 149 vertices can have any number of edges, and a graph with at most 189 edges can have a cycle with more than 39 edges. Corollary 10.22 is a local condition: the graph can be as large as we like and can have long cycles.

Remark 10.26 (Computations; not part of a proof)**.** No proof of this section uses a computer. The following values were computed by a short program, as a check of the arithmetic only: $\sigma(52) = 262$, $\sigma(56) = 286$, $\sigma(60) = 308$, $\sigma(100) = 572$; and $\sigma(k_0) - k_0 = 149$, 187, 210, 230, 248, 472, 1085 for $k_0 = 40$, 48, 52, 56, 60, 100, 200. The least base $k_0 \ge 48$ with $\sigma(k_0) - k_0 \ge n$ (Corollary 10.16 (b)) is 188 for $n = 1000$, 1360 for $n = 10{,}000$, and 10,612 for $n = 100{,}000$.

11. The limits of the method

This section says where the method of Part I stops. It contains no new result. As in Section 0, "k is reached" means that one of the results of this paper applies to k. We do not know a finite bridgeless loopless multigraph G and an integer $k \ge 6$ such that $I_k(G)$ is not connected (the modulus 5 fails [EHLMNS]).

The bound 40. Corollary 3.15 needs $|A| \ge 10$ and $|B| \ge 4$. The reason is in the counts (Definition 2.15, Remark 2.16): a thread has up to 3 edges, so the left sides of (C1), (C2), (C3) can be 9, 9, 8, and the left side of (C4) can be 3. For $\Gamma = \mathbb{Z}_k$ with $B = a\mathbb{Z}_k$ this means $k = ab \ge 40$. Theorem 3.11 itself asks only $|A| \ge 8$ and counted frames; but no statement of this paper controls the frames of the good subgraphs that the averaging step gives. So all results for all graphs have $k \ge 40$; for $6 \le k < 40$ the question for all graphs is open (Problem (P1) of Section 12). For a group with $2A = 0$ and $2B = 0$, Part II avoids these counts (Theorem 13.1: $|A| \ge 8$ and any frame) and reaches the order 32 (Corollary 13.2). This does not help $\mathbb{Z}_k$: its elements x with $2x = 0$ form a subgroup with at most 2 elements, so $\mathbb{Z}_k$ has no subgroup B with $2B = 0$ and $|B| \ge 4$.

What a condition on the colouring would give. Corollaries 3.16 and 3.17 are conditional (Remark 3.18). Suppose that every subdivision that the averaging step gives had a frame with (S) (no pair consists of two original edges whose threads both have 3 edges). Then Corollary 3.16 would apply with $|A| = 9$. For $k = 36 = 9 \cdot 4$, the residues of ± 1, ± 2 and $\pm q$ for the divisors $q < 36$ of 36 form the step system $\Sigma = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, 18\}$ of liftable elements (Lemma 1.12); modulo 9 it gives every non-zero residue, so (R1) holds with walks of one step; $\Sigma_B = \{9, 18, 27\} = B \setminus \{0\}$; and $\mathbb{Z}_{36} = \mathbb{Z}_4 \times \mathbb{Z}_9$ with the generators 9 and 4 and the ratio $3/35 < 1/9$ gives (AV″) by Lemma 4.1. So the frame condition (S) would give $k = 36$. In the same way, a frame with (NR) (no free thread with 3 edges) would, by Corollary 3.17, give $k = 30 = 10 \cdot 3$ (Σ_A is all of $\mathbb{Z}_{10} \setminus \{0\}$, $\Sigma_B = \{10, 20\}$, $\mathbb{Z}_{30} = \mathbb{Z}_2 \times \mathbb{Z}_{15}$, ratio $2/29$) and $k = 36 = 12 \cdot 3$ (Σ_A misses only ± 5, Lemma 5.2; $\Sigma_B = \{12, 24\}$; ratio $2/35$). In a graph without cycles with at most 5 edges, Lemma 2.2 (b) gives a good subgraph (a vertex-minimal dense subgraph) with the properties (X2)–(X4), which limit the threads with 3 edges; they do not give (S) or (NR). So $k = 30$ and $k = 36$ remain open.

The modulus 44 and $4p$ with $p \leq 11$. The only split $44 = ab$ with $a \geq 10$ and $b \geq 4$ is $a = 11$, $b = 4$. The liftable steps of Section 1 give $\Sigma_A = \{\pm 1, \pm 2, \pm 4\}$ in $\mathbb{Z}_{11}$, and the deletion of the six neighbours of 0 isolates 0; so (R1) fails (Remark 5.6). The double step of Section 6 replaces (R1) for $k = 4p$, but its walk lemma needs the count $12 < p$ (Remark 6.10); so it does not reach $4p$ with $p \leq 11$, that is 8, 12, 20, 28 and 44. And 44 is not a product of two coprime admitted factors. So $I_{44}(G)$ is not reached. The modular graph $M_{44}(G)$ is connected for every bridgeless G (Theorem 4.5); whether this implies the connectivity of $I_{44}(G)$ is a case of Problem 2.6 of [EHLMNS].

Primes and prime powers. Every result of Sections 5–9 for all graphs uses a factorization of k. Theorem 3.19 needs $k = ab$ with $a, b \geq 2$; Lemma 4.1 needs a product of two non-trivial groups; the guarded moves of Section 8 need two coprime factors. So no prime k is reached for all graphs (Corollary 9.8 (a)). For a prime k, Section 10 gives the connectivity only for single graphs, from a base on.

A cyclic group of prime-power order is not a product of two non-trivial groups, so Lemma 4.1 does not apply to $\mathbb{Z}_{p^e}$. Section 7 replaces it by the flattening of valuations; the proof of Theorem 7.4 needs $e \geq 6$ (Remark 7.5: for $p = 2$, Lemma 7.2 needs $e - 2 \geq 4$, the count needs $|A| = 2^{e-2} \geq 10$, and (R1) needs $2(e-2) - 2 \geq 6$; for an odd p, Lemma 5.7 needs a divisor of p^{e-4} strictly between 1 and p^{e-4}). So p^e with $e \leq 5$ is not reached; for example 81, 125 and 243 are not reached.

Two prime factors. For $k = pq$ with different primes, Criterion 5.11 does not apply, because it needs a composite factor b. Theorem 8.8 needs factors with one-value avoidance, and the lemmas of Section 8 give it for a prime only from 43 on. For the modular graph alone, more moduli are known: for coprime $a \geq 10$ and $b \geq 4$, Theorem 3.11 gives the connectivity of $M_k(G)$ (Remark 8.11); but for primes $p, q \geq 5$ the liftable steps that Section 1 provides in $B = p\mathbb{Z}_k$ are only $\pm p$, so (R2) fails, and Theorem 3.19 does not give $I_k(G)$.

One graph. The bounds of Section 10 depend on G: on its longest cycle, its number of edges or of vertices, or its short cycles. They do not give one bound for all graphs beyond the bases; in particular they do not give every $k \geq 41$ for all graphs.

12. Open problems

We end with the questions that this paper leaves open. We know no finite bridgeless loopless multigraph G and no $k \geq 6$ for which $I_k(G)$ or $M_k(G)$ is not connected.

(P1) *The least modulus.* Is $I_6(G)$ connected for every finite bridgeless loopless multigraph G? Is there any k with $6 \leq k < 40$ with this property? The modulus 5 fails [EHLMNS, Theorem 3.4].

(P2) *All moduli from 40 on.* Is $I_k(G)$ connected for every $k \geq 40$ and every such G? The moduli from 41 to 47 are not reached, and no prime is reached (Proposition 9.7, Corollary 9.8 and Section 11). For $k = 44$ the graph $M_{44}(G)$ is connected (Theorem 4.5), and the question for $I_{44}(G)$ is a case of Problem 2.6 of [EHLMNS].

(P3) *Small groups.* What is the least order $n \geq 6$ of a finite abelian group Γ such that $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G? By Corollary 13.2 of Part II it is at most 32. Is $F(G, \mathbb{Z}_6)$ connected for every such G? We ask for $n \geq 6$ for these reasons. A group with at most two elements has at most one nowhere-zero flow on G. For order 3, orient $K_{3,3}$ from one side to the other: its only nowhere-zero $\mathbb{Z}_3$-flows are the two constant flows, which differ on all nine edges and hence are not adjacent. Neither group of order 4, nor the group of order 5, has the property [EHLMNS, Sections 2.4, 3 and 4]. By [CLSWX, Theorem 5.1], for each fixed abelian group Γ of order at least five it suffices to test this connectivity on simple, 2-connected, 3-edge-connected cubic graphs. For $\mathbb{F}_2^3$ and $\mathbb{F}_2^4$, Part II proves the connectivity for every 4-edge-connected G (Theorem 13.3), and for every G whose components have at most 44, respectively 19022, vertices of odd degree (Theorems 22.1 and 28.6; these two use the unpublished manuscript [Go] only through its Theorem 1.2(c)).

Part II. Binary extensions and edge connectivity

13. Results and scope

Graphs, flows, the signed indicators χ_C of cycles with a direction, moves, bridgeless graphs and the reconfiguration graph $F(G, \Gamma)$ are as in Section 1. In particular, loops and parallel edges are allowed, and a move from f to $f + \lambda\chi_C$ may use every nonzero coefficient $\lambda \in \Gamma$. Two nowhere-zero flows are *joined* if a sequence of moves leads from one to the other, that is, if they lie in one component of $F(G, \Gamma)$; in the language of Section 3, they are Σ-joined for the step system $\Sigma = \Gamma \setminus \{0\}$. The terms "general step", "quotient step" and "kernel step" below are defined where they are used; they are not elements of a step system. We write $\mathbb{F}_2^m$ for the elementary abelian group of order 2^m.

Theorem 13.1 (Binary kernel and quotient)**.** *Let Γ be a finite abelian group and $B \leq \Gamma$. Put $A = \Gamma/B$. Suppose*

$$2B = 0, \qquad 2A = 0, \qquad |B| \geq 4, \qquad |A| \geq 8, \qquad \frac{|B| - 1}{|\Gamma| - 1} \leq \frac{1}{9}.$$

Then $F(G, \Gamma)$ is nonempty and connected for every finite bridgeless multigraph G.

The two exponent conditions are equivalent to $2\Gamma \subseteq B \subseteq \Gamma[2]$, where $\Gamma[2] = \{x : 2x = 0\}$. They do not require B to be a direct factor. A lift in Γ of an element of A can have order four, so ambient cycle additions must retain their signs.

Corollary 13.2. *If the exponent of a finite abelian group Γ divides 4 and $|\Gamma| \geq 32$, then $F(G,\Gamma)$ is nonempty and connected for every finite bridgeless multigraph G. In particular this holds for $\Gamma = \mathbb{F}_2^n$ whenever $n \geq 5$.*

At order 32 these groups are exactly

$$\mathbb{F}_2^5, \qquad \mathbb{Z}_4 \times \mathbb{F}_2^3, \qquad \mathbb{Z}_4^2 \times \mathbb{Z}_2.$$

This is a statement about these group flow graphs. For the cyclic group $\mathbb{Z}_{32}$ and for integer 32-flows, the connectivity remains open.

Esperet, Hendrey, Lagoutte, Marseloo, Norin and Steiner introduced the problem and proved, among other results, universal connectivity for $\mathbb{F}_2^8$ [EHLMNS]. Cranston, Li, Su, Wang and Xu developed further structural and contraction results [CLSWX]. Their Theorem 5.6 covers every abelian group: a bridgeless graph on n vertices, exactly n_2 of which have degree two, has a connected flow graph over every abelian group A with $|A| \geq \lfloor 3\log_2(n - n_2 + 2)\rfloor$. For groups of order 8, 16 and 32 this is the range $n - n_2 \leq 5$, $n - n_2 \leq 48$ and $n - n_2 \leq 2045$, respectively. Part I gives universal connectivity for finite abelian groups that are not cyclic of prime-power order and whose order has a divisor d with $4 \leq d \leq |\Gamma|/10$ (Theorem 4.5). Consequently the exponent-dividing-4 groups of order at least 64, including all $\mathbb{F}_2^n$ with $n \geq 6$, are already covered by Theorem 4.5 of Part I. Relative to Part I, Corollary 13.2 adds precisely the three order-32 groups displayed above. Whether a smaller order suffices is open (Problem (P3) of Section 12).

For a vertex set X write $d_G(X)$ for the size of its edge cut. A graph with at least one vertex is 4-edge-connected here if $d_G(X) \geq 4$ for every nonempty proper vertex set X; parallel edges and loops are permitted.

Theorem 13.3 (Four-edge-connected binary flows)**.** *Let G be a finite 4-edge-connected multigraph. For every $m \geq 3$, the graph $F(G,\mathbb{F}_2^m)$ is nonempty and connected. Moreover, every nowhere-zero $\mathbb{F}_2^m$-flow is joined to one omitting a nonzero group value.*

Theorem 13.3 is proved independently of the machinery of Part I in Section 19. The classical tree-packing theorem of Nash-Williams and Tutte is its principal external input. For comparison, [CLSWX, Theorem 6.1] proves connectivity on 4-edge-connected graphs for every finite abelian group of order at least 5,251,339. Its Theorem 4.3 already treats Eulerian graphs for every even-order abelian group. Thus the additional coverage here, relative to those statements and Corollary 13.2, is for non-Eulerian 4-edge-connected graphs at binary orders 8 and 16. This comparison is based on the cited results.

The proof of Theorem 13.1 uses the extraction, frame and averaging machinery of Part I. Section 14 states the exact inputs, with their locations, so that this dependency is explicit. The new local arguments and the global induction are given below. Section 30 is independent of those inputs: it proves a component correspondence for arbitrary finite abelian groups. Its family is restricted by a fixed subgroup and edge set; it need not have the same components as the unrestricted flow graph.

The extension allowing five designated star cuts is Theorem 20.1, proved in Section 20. Its support consequence, Corollary 20.3, is a sufficient order-sixteen test; it does not show that the required support exists. Section 21 proves the conditional four-cut reduction of Theorem 21.1, keeping its structural and smaller-graph assumptions explicit. The independent cycle-order argument in Section 22 gives Theorem 22.1: for a binary group of order $q \geq 8$, each component may have up to $3 \cdot 2^{q/2} - 4$ odd-degree vertices. Here q is group order, not rank. Its values 44 and 764 improve the earlier bounds 28 and 508 of [Go, Theorem 1.4];

the cases $q \geq 32$ are already covered by Corollary 13.2. That general proof does not use Theorem 21.1. The special order-sixteen argument in Section 23 does use it and proves Theorem 23.1, improving 764 to 1020 per component. We do not know whether these bounds are optimal; universal connectivity at orders eight and sixteen remains open.

Theorem 24.8 gives a further order-sixteen class: finite simple connected bridgeless cubic graphs with every cyclic edge cut of size at least eleven. Here cyclic means that each shore contains a cycle. Section 24 proves the required edge-and-claw factor theorem in full, with avoidance of a prescribed edge and a prescribed single-edge component when one endpoint has degree two. The resulting factor gives an even support through any chosen flow-value class with independent uncovered vertices. A partition count and Corollary 19.5 then supply the quotient flow argument. Together with the small-cut reductions, this forces a cyclic cut of size five through ten in a least order-sixteen core, that is, in a connected simple cubic counterexample with the least number of odd-degree vertices, as supplied by Lemma 22.4 if a counterexample exists. It does not settle those remaining cuts or universal connectivity.

Theorem 13.4 (Order-sixteen bound from small cyclic shores)**.** *If each connected component of a finite bridgeless multigraph H has at most* 3036 *odd-degree vertices, then $F(H, \mathbb{F}_2^4)$ is nonempty and connected.*

Section 25 proves this theorem. The proof uses [Go, Theorem 1.2(c)], an unpublished manuscript, through Lemma 22.4. A connected cyclic shore with at most ten boundary edges in a cubic graph of girth at least eighteen has at least 998 vertices; at most five boundary edges force at least 1011. If the rounded order-eight criterion of Corollary 19.7 fails for the quotient G/W, where W is the set of edges outside an edge-and-claw factor, then Lemma 25.2 gives either four disjoint shores of the first kind or three of the second kind together with five single vertices. Thus a least order-sixteen core has at least 3038 vertices. This improves the intermediate 1020 bound and retains its seventeen-cycle and four-cut dependencies. We do not know whether this bound is optimal.

Two later arguments enlarge the number 3036. Theorem 26.4 in Section 26 replaces it by 3996: a sharper count (Lemma 26.2) gives at least 3998 vertices when four shores of the first kind occur, and in a core with at most 3996 vertices the decomposition into three shores of the second kind together with five single vertices is unique (Lemma 26.3), whereas two suitably chosen flow values would give two different ones. Theorem 28.6 in Section 28 replaces it by 19022, by a different route that does not use Theorem 26.4. Take a flow f of a least core G that is not joined to a flow omitting a value, a value b of f, and an even edge set that contains $f^{-1}(b)$, and form the quotient of G by the cycles of that set. Section 27 shows that if, after some sequence of contractions of good sets (defined before Lemma 19.4) and of induced cycles with at most seven edges in this quotient, the flow graph of the last quotient over a subgroup of order eight that does not contain b is connected, then f is joined to a flow that omits b (Theorem 27.4). So in a least core that flow graph is never connected, and the last quotient of a sequence that cannot be continued is simple, has girth at least eight and has no good set (Lemma 27.6). Section 28 takes the even edge set from an edge-and-claw factor and counts the vertices of G inside the nodes of this last quotient. The result is Theorem 28.6: if each connected component of a finite bridgeless multigraph H has at most 19022 odd-degree vertices, then $F(H, \mathbb{F}_2^4)$ is nonempty and connected. Thus a least order-sixteen core has at least 19024 vertices. We do not know whether this bound is optimal; universal connectivity at order sixteen remains open. Lemma 22.4 uses Theorem 1.2(c) of the unpublished manuscript [Go], and so, through it, do Theorems 22.1, 23.1, 13.4, 26.4 and 28.6

and Corollaries 22.2, 24.9 and 27.5; the order-32 results, Theorem 13.1 and Corollary 13.2, do not use it.

Section 29 concerns existence with a prescribed boundary, not moves. Let H be a finite connected bridgeless multigraph and let $\beta : V(H) \to \mathbb{F}_2^3$ have total zero. Theorem 29.2 states that if β is nonzero at no more than seven vertices, then there is a map $E(H) \to \mathbb{F}_2^3 \setminus \{0\}$ whose sum over the edges at each vertex v is $\beta(v)$. A cycle of eight vertices, with a suitable β that is nonzero at all of them, shows that seven cannot be raised. Theorem 29.9 gives such a map for every β with total zero when the graph has no loop and every cut has at least three edges, even if on each of at most eight edges one further value is forbidden. The proofs use Jaeger's theorem and Edmonds' matroid intersection theorem. No other section uses Theorem 29.2 or Theorem 29.9.

14. Elementary facts and the inherited structural inputs

Write $\overline{x}$ for the image of x under the quotient map $\Gamma \to A$. For an edge map f, the notation $\overline{f}$ is applied edge by edge. A nonzero quotient value protects an edge from becoming zero under any B-valued change.

Lemma 14.1 (Cycle decomposition and quotient lifting)**.** *For any abelian group, every flow is a sum of signed cycle flows. A flow supported in a subgraph can be decomposed into cycles of that subgraph. Every A-flow lifts to a Γ-flow, without an assumed homomorphic section $A \to \Gamma$.*

Proof. The decompositions are Lemma 1.1(a), applied to the graph or to the subgraph. To lift an A-flow, lift each coefficient of such a decomposition separately and sum the resulting signed Γ-cycle flows. □

We also use the existence theorem of Seymour and Tutte, Theorem 0.1: a finite bridgeless multigraph has a nowhere-zero flow over every finite abelian group of order at least six.

14.1. **Extraction and frames.** Good subgraphs, of the two kinds (CYC) and (SUB), are as in Definition 2.1. In the kind (SUB) the paths that replace the edges of H_0 are the *threads* (Definition 2.6); their interiors are pairwise disjoint. Reversing a reference orientation negates its flow values and preserves the reconfiguration graph.

We use the following results of Part I. A loopless multigraph on $n \geq 3$ vertices with more than $4n/3-2$ edges contains a good subgraph (Lemma 2.2). Every subdivided good subgraph has a frame (Definition 2.9, by Lemmas 2.5 and A.5). Every frame has the properties listed in Proposition 14.2 below (Lemmas 2.7 and 2.11, and the stage properties (G1)–(G4) stated after Definition 2.9). These are inherited theorems, not consequences of the binary counting argument of Part II.

We use the notation of Part I for frames: the truncated cubic graphs Q_b with disjoint union Q, the original and expansion edges, the threads P_t with lengths $\ell(t)$, and the map π on flows of Q (Definition 2.6); the red and blue edges, the nodes $1, \dots, s$, their operands and stages, and the pairs $N_i = \{e_i, f_i\}$, numbered so that each node follows every node in its two operands (Definitions 2.8 and 2.9); for a stage U, the set V_U and the graph Q_U (the notation after Definition 2.9); and the protected and free threads, the union H_P of the protected threads, the subgraph T^e and the official cycles C_i (Definition 2.10). Choose one edge $g_i \in N_i$ at every node. Let T^g consist of all red edges and the selected edges g_i.

Proposition 14.2 (Inherited frame properties)**.** *Every subdivided good subgraph has a frame, and every such frame satisfies:*

(i) *π maps flows of Q onto flows of H, over any abelian group. Every flow of H is constant on each thread. A signed cycle vector of Q maps to a sum of signed vectors of pairwise edge-disjoint cycles of H.*
(ii) *For every selection g, T^g is a spanning tree in each component Q_b and, on restriction, in the graph Q_U of every stage.*
(iii) *C_i contains e_i, f_i. If $l < i$, then C_l contains no edge of N_i. Every flow of Q has a unique expansion in the signed official cycles.*
(iv) *H_P is connected and contains every vertex of H except the interiors of free threads. For each free thread P_ε there is a cycle D_ε containing it and otherwise only protected threads. Direct this cycle along the free thread.*
(v) *Let H_g consist of the vertices of H_0 and the threads of all original edges in T^g. It is connected. Its missing vertices are precisely interiors of unselected original blue threads. A path in H_g uses, from each pair, only the selected thread.*

The last assertion follows by contracting expansion edges in the trees of (ii), restoring original threads, and using the connected block structure of H_0; it is also step (1) of the proof of Lemma 3.9 in Part I. Images of paths under these contractions are connected subgraphs, not necessarily paths: a simple path is then chosen in the image when needed.

Following Definition 2.14, for a subdivided good H, a nowhere-zero Γ-flow f of a containing graph is *H-good* if $u = \overline{f}$ is nonzero on every protected edge. For a short-cycle good H, it is H-good if u is nonzero throughout H. In the subdivision case define a list on each blue original edge by

$$\Lambda_f(t) = \{-u(a) : a \in E(P_t)\},$$

and give expansion edges empty lists. In general, for lists Λ on blue edges, let $P(\Lambda)$ be the set of A-flows Y of Q with $Y(t) \notin \Lambda(t)$ on every blue edge. Thus

$$Y \in P(\Lambda_f) \quad \Longleftrightarrow \quad u + \pi(Y) \text{ is nonzero on all protected edges.}$$

Here $\pi(Y)$ is extended by zero outside H. We have $|\Lambda_f(t)| \le \ell(t)$, and $0 \in P(\Lambda_f)$ when f is H-good.

14.2. **The finite-round averaging input.** Remark 4.2 in Part I, from the proof of Lemma 4.1, gives the following quantitative fact. If Γ is the product of two nontrivial finite abelian groups, B is any subgroup, and all nonzero cycle coefficients are allowed, its random legal rounds give, after $6j$ rounds, a nowhere-zero flow reachable from any initial nowhere-zero flow such that each edge has quotient zero with probability at most

$$r + 2(1-\delta)^j, \qquad r = \frac{|B|-1}{|\Gamma|-1}, \quad \delta = \frac{|\Gamma|-1}{|\Gamma|^3}. \tag{1}$$

Every finite outcome is reachable by finitely many legal moves. The subgroup B need not be either product factor. This estimate precedes the strict ratio restriction in step (6) of that proof; we use the estimate, not that strictly stated conclusion, at equality.

Lemma 14.3 (Averaging with an integer threshold)**.** *Under the preceding product hypothesis, every nowhere-zero flow on an m-edge graph is joined to one with at most $\lfloor rm \rfloor$ quotient-zero edges. In particular, $r \le 1/9$ gives at most $m/9$ quotient-zero edges.*

Proof. For $m = 0$ there is nothing to prove. Let Z_j count the quotient-zero edges after $6j$ rounds. By (1) and linearity of expectation,

$$\mathbb{E}Z_j \le rm + 2m(1-\delta)^j.$$

Set $k = \lfloor rm \rfloor$. Choose a finite j with $2m(1-\delta)^j < k + 1 - rm$, whose right side is strictly positive even when rm is an integer. Then $\mathbb{E}Z_j < k + 1$, so some reachable outcome has $Z_j \le k$. No independence between different edges or infinite sequence of moves is used. □

In a loopless 3-edge-connected graph with $n \ge 3$ vertices and m edges, the resulting nonzero quotient support has at least

$$\frac{8m}{9} \ge \frac{4n}{3} > \frac{4n}{3} - 2$$

edges. Extraction therefore applies, including at $r = 1/9$. If $n = 2$, then $m \ge 3$ and the support has more than two edges, so it contains a two-edge cycle. In both cases we obtain a good subgraph entirely in nonzero quotient support.

15. A binary list argument

In this section A is a finite elementary abelian 2-group. Cycle vectors with coefficients in A can be written as unsigned indicators; this convention is not applied to Γ-valued moves.

Lemma 15.1 (Repeated values)**.** *Let W be a finite set with $|W| < |A|$, let $v : W \to A$, and put $V = v(W)$. Suppose $R_1, \dots, R_r \subseteq W$, $r \ge 2$, have empty common intersection. For every $\sigma \in A$, some j satisfies*

$$V \cup (\sigma + v(R_j)) \ne A.$$

Proof. Put $M = A \setminus V$. If all the displayed unions were A, every value of $\sigma + M$ would lie in every $v(R_j)$. A value with a unique preimage in W cannot do so, because that preimage would belong to all R_j. Thus each of the $|M|$ values in $\sigma + M$ has at least two preimages. The number of repeated values is at most $|W| - |V|$, whereas $|M| = |A| - |V| > |W| - |V|$, a contradiction. □

An edge set D of Q is *even* if every vertex has even degree in D. For $\sigma \ne 0$, the map $\sigma\mathbf{1}_D$ is an A-flow precisely when D is even. Every nonempty even set decomposes into edge-disjoint cycles. Call $\sigma\mathbf{1}_D$ a *general step*. It is not required to be supported on just one or two official cycles.

Lemma 15.2 (Binary list walk)**.** *For a frame and a list assignment Λ, suppose*

$$|\Lambda(e_i)| + |\Lambda(f_i)| < |A| \quad \text{at every node } i.$$

Any two flows in $P(\Lambda)$ are joined inside $P(\Lambda)$ by a sequence of general steps.

Proof. Let Λ_i keep only the lists at nodes $i, i+1, \dots, s$. Prove the assertion for Λ_i by downward induction on i. With no lists, the official cycle expansion gives the assertion. Suppose it holds for $i+1$, put $\chi = \mathbf{1}_{C_i}$, and set

$$X_e(Z) = Z(e_i) + \Lambda(e_i), \quad X_f(Z) = Z(f_i) + \Lambda(f_i), \quad X(Z) = X_e(Z) \cup X_f(Z).$$

The official-cycle properties imply that for $Z \in P(\Lambda_{i+1})$, adding $c\chi$ leaves all later lists satisfied, and

$$Z + c\chi \in P(\Lambda_i) \iff c \notin X(Z). \tag{2}$$

Take a walk $Y = Y_0, \dots, Y_n = Y'$ in $P(\Lambda_{i+1})$, each step $Y_r - Y_{r-1} = \sigma\mathbf{1}_D$, supplied by induction. We follow it with compensated states $Y_r + c_r\chi$ satisfying (2); start with $c_0 = 0$.

For one step write $X_e = X_e(Y_{r-1})$, $X_f = X_f(Y_{r-1})$, and let $k_e, k_f \in \{0,1\}$ indicate whether e_i, f_i lie in D. Then

$$X(Y_r) = (X_e + k_e\sigma) \cup (X_f + k_f\sigma).$$

We seek $c \notin X_e \cup X_f$ such that either

$$\text{(I) } c \notin X(Y_r), \qquad \text{or} \qquad \text{(II) } c + \sigma \notin X(Y_r).$$

For $(k_e, k_f) = (0,0)$ the current c_{r-1} satisfies (I); for $(1,1)$ it satisfies (II). For $(1,0)$ we need c outside one of

$$(X_e \cup X_f) \cup (\sigma + X_e), \qquad (X_e \cup X_f) \cup (\sigma + X_f).$$

Apply Lemma 15.1 to disjoint tagged copies of X_e and X_f, with the value map forgetting the tag. There are fewer than $|A|$ positions, and the two candidate subsets are disjoint. This gives c. The case $(0,1)$ exchanges the two labels.

First change c_{r-1} to c along C_i if necessary; both compensated endpoints are legal. In case (I) follow the original general step and set $c_r = c$. In case (II) set $c_r = c + \sigma$ and use

$$\sigma\mathbf{1}_D + \sigma\mathbf{1}_{C_i} = \sigma\mathbf{1}_{D \triangle E(C_i)}.$$

The symmetric difference is even; if empty, omit the step. Thus this too is an allowed general step. At the end, change c_n to zero along C_i; the final state Y' satisfies its lists. This proves the induction. Allowing arbitrary even supports is essential to closure under the displayed symmetric difference. □

16. Signed realization inside a good subgraph

Throughout this and the next section, B and A are binary, $|B| \geq 4$, and $|A| \geq 8$, as in Theorem 13.1. We use any frame; there are no additional counting restrictions on it.

Lemma 16.1 (Free-thread preparation)**.** *Let f be H-good on a subdivided good subgraph and put $u = \overline{f}$. For selected free threads, let $T_\varepsilon \subsetneq B$ contain*

$$W_\varepsilon = \{-f(a) : a \in E(P_\varepsilon),\ u(a) = 0\}.$$

One can translate each selected free thread by any chosen $y_\varepsilon \in B \setminus T_\varepsilon$, using a legal cycle move on D_ε (or no move for $y_\varepsilon = 0$). The quotient is unchanged, other free threads are unchanged, and the resulting flow is H-good.

Proof. On its own free thread $\chi_{D_\varepsilon} = 1$. Avoiding W_ε protects all old quotient-zero edges of that thread. Every other edge of the cycle is protected by its nonzero quotient. Distinct preparations affect no other free thread, so perform them successively. Each nonzero coefficient is an allowed move. □

Lemma 16.2 (One auxiliary cycle)**.** *Let f be H-good and K a cycle of Q. Suppose $0 \neq \sigma \in A$ and $u + \pi(\sigma\chi_K)$ is nonzero on every protected edge. Then f is joined by moves supported in H to an H-good flow with that quotient.*

Proof. By Proposition 14.2(i), $\pi(\chi_K)$ is a sum of signed vectors of edge-disjoint cycles $O_1, \dots, O_j$ of H. Choose a lift γ of σ. On a free thread each O_l either contains the entire thread or misses it, and no two O_l use the same thread. For the prospective signed addition $\gamma\pi(\chi_K)$, on each free thread forbid $-f(a)$ when $u(a) = 0$, and forbid $-f(a) - \gamma\pi(\chi_K)(a)$ when the new quotient is zero and that edge changes. These forbidden values lie in B. A changed edge cannot have quotient zero both before and after, since $\sigma \neq 0$. Hence at most one value is forbidden per thread edge, giving at most three forbidden values.

Use Lemma 16.1, then add $\gamma\chi_{O_l}$ successively. Protected edges and nonzero final quotients are safe. The preparations protect every newly zero quotient on a free thread, as well as every unchanged zero quotient. Because the cycles are edge-disjoint, during these cycle additions every edge has its prepared value or its final value. This proves the assertion. The coefficient γ need not have order two; all additions in this proof are signed. □

This is the signed one-cycle mechanism of Lemma 3.7 in Part I, reproduced here for the all-coefficients setting.

Lemma 16.3 (Internal quotient adjustment)**.** *Let f be H-good on a subdivided good subgraph. For every $Y \in P(\Lambda_f)$ there is an H-good flow joined to f by moves on H, equal to f outside H, with quotient $u + \pi(Y)$.*

Proof. At a node the two lists have total size at most six, which is less than $|A|$. Lemma 15.2 joins 0 to Y by general steps in $P(\Lambda_f)$. Decompose each even support into edge-disjoint cycles of Q. For each partial sum, every auxiliary edge has its value from the old or the final state of this general step, so the lists remain satisfied. Apply Lemma 16.2 to each auxiliary cycle, starting each time from the currently reached full flow. That lemma requires only current H-goodness and nonzero final protected quotients; it does not require the original full values or the original lists. Thus the successive realizations give the stated quotient and leave the exterior unchanged. □

17. Lifting a contracted move

Contract all vertices of a connected good subgraph H to a single vertex, delete its edges, and retain every other edge, including new loops. Denote the resulting graph by G/H.

Lemma 17.1 (Local lifting)**.** *Let f be an H-good Γ-flow of G and let f^* be its restriction to G/H. Every legal move*

$$g^* = f^* + \lambda\chi_{C^*}$$

can be lifted to a path from f to an H-good flow g whose restriction is g^*.

Proof. If the edges of C^* already form a cycle in G, perform that move directly. Otherwise they form a path with distinct ends $p, p' \in V(H)$, all other vertices outside H. It suffices to find a path $J \subseteq H$ between them and prepare inside H before adding $\lambda\chi_C$, where $C = C^* \cup J$ is directed to agree with C^* outside H. Write $\sigma = \overline{\lambda}$ and $u = \overline{f}$. Call the move a *quotient step* if $\sigma \neq 0$, and a *kernel step* if $\sigma = 0$, that is, if $\lambda \in B$.

A short-cycle good subgraph. Choose an arc J of H with at most two edges. If $\sigma = 0$, the quotient protects all edges of H and no preparation is needed. Otherwise, a quotient addition $c\chi_H$ has at most five forbidden values of c for maintaining a nonzero quotient on H, and the further addition $\sigma\chi_C$ on J forbids at most two more. As $|A| \geq 8$, choose c avoiding all seven. If $c \neq 0$, add any lift of c on the signed cycle H; the final quotient is nonzero throughout H, so this is legal. Now perform $\lambda\chi_C$. The flow is still H-good and has the prescribed restriction.

A subdivision and a nonzero quotient step. Suppose $\sigma \neq 0$. We construct $Y \in P(\Lambda_f)$ and a path J such that

$$\begin{aligned} &u_1 = u + \pi(Y) \text{ is nonzero on protected edges, and} \\ &u_1(a) + \sigma \neq 0 \text{ on protected edges of } J. \end{aligned} \tag{3}$$

Let W_i be the edge positions on the original threads of e_i, f_i; thus $|W_i| \le 6$. Choose official coefficients $x_i \in A$ in decreasing order of i. At stage i put

$$Z_i = \sum_{l>i} x_l \mathbf{1}_{C_l}, \qquad v_i(a) = Z_i(t) + u(a) \quad (a \in E(P_t),\ t \in N_i), \qquad V_i = v_i(W_i).$$

For $Y = \sum_i x_i \mathbf{1}_{C_i}$ the triangular official-cycle properties give

$$(u + \pi(Y))(a) = v_i(a) + x_i \qquad (a \in W_i). \tag{4}$$

At the same time choose a permitted position set $R_i \subseteq W_i$ and ensure

$$x_i \notin V_i \cup (\sigma + v_i(R_i)). \tag{5}$$

The later coefficients already determine v_i, so all these choices are well-defined in this order.

If both p, p' are interior to the same thread, it has length three and they are joined by its middle edge. Take that edge as J. At its node, if protected, let R_i be this single edge; take all other R_l empty. At most $6 + 1 < 8$ values are excluded at any node. If the thread is free all R_l are empty; its middle edge is handled by free-thread preparation below.

Otherwise classify each node by the interiors containing the endpoints. There are three possibilities.

(i) *Neither endpoint is inside its pair.* The two candidate sets are the entire thread edge sets of e_i and f_i, empty for an expansion edge. They are disjoint. Lemma 15.1 chooses one, $R_i = E(P_{g_i})$, and a coefficient satisfying (5).

(ii) *Exactly one endpoint, say p_i, is inside its pair.* Let its thread be P_{t_i}, with ends z, z', and write $S_z, S_{z'}$ for the edge sets of the two pieces from p_i to these ends. Let T' be the other thread edge set in the pair. The three candidate sets

$$S_z \cup T', \qquad S_{z'} \cup T', \qquad S_z \cup S_{z'}$$

have empty common intersection. Apply Lemma 15.1. For the first or second choice select the other edge of the pair as g_i and attach the corresponding piece from p_i to z or z'. For the third choose $g_i = t_i$ and attach no extra piece; the endpoint then lies in H_g itself.

(iii) *The two endpoints are inside different threads of this pair.* Each thread has one end on each operand side of node i. The two candidate sets consist of the two endpoint pieces going to the first side, or the two pieces going to the second side. They are disjoint. Lemma 15.1 selects one side and a coefficient satisfying (5). Choose any g_i to complete the tree selection; only the selected endpoint pieces will be used at this node.

There is either one node of the third type and all others have the first type, or at most two nodes of the second type and all others have the first type. The types depend only on the fixed endpoints and frame. All applications of Lemma 15.1 use $|W_i| \le 6 < |A|$.

We now build the path from these independent permissions. If there is no node of the third type, each endpoint lies in H_g or has an attached piece ending in H_g. Join the two resulting ends by a path in connected H_g and attach those pieces. The interiors of attached pieces lie on unselected threads, outside H_g, and two attached pieces are on different threads. Hence the union is a path. The selected thread and attached piece rules ensure that it uses only R_i at each pair.

If node i has the third type, use the spanning tree of its chosen operand stage to join the corresponding ends of its two pair edges. Neither pair edge is in that operand stage. Contract expansion edges and restore original threads. The image is a connected subgraph L containing both required thread ends, using neither thread of N_i, and using only selected

threads at other nodes. Adjoin the chosen endpoint pieces and choose a simple p–p' path in this connected union. Again it uses only R_i at each node. We do not require the image of the auxiliary tree path itself to be a simple path after contraction.

Equations (4)–(5) prove (3). Use Lemma 16.3 to reach a full flow f_1 with quotient u_1. For every free thread meeting J, forbid

$$W_\varepsilon = \{-f_1(a) : a \in E(P_\varepsilon),\ u_1(a) = 0\},$$
$$T_\varepsilon = W_\varepsilon \cup \big\{-f_1(a) - \lambda\chi_C(a) : a \in E(P_\varepsilon) \cap E(J),\ u_1(a) + \sigma\chi_C(a) = 0\big\}.$$

These are elements of B. Because $\sigma \neq 0$, each thread edge contributes at most one forbidden value; thus $|T_\varepsilon| \leq 3 < |B|$. Lemma 16.1 gives a flow f_2 with all these obstructions avoided. Now $f_2 + \lambda\chi_C$ is legal and H-good. Protected edges of J are safe by (3), new quotient zeros on free threads are safe by the second forbidden set, and the first forbidden set preserves old quotient zeros throughout each prepared thread. The signs in $\lambda\chi_C$ and in the forbidden set are retained even though $\sigma = -\sigma$.

A subdivision and a kernel step. Suppose $\sigma = 0$. Then $\lambda \in B$ and $\lambda = -\lambda$. In this part, and only for these B-coefficients, signs disappear. On a free thread put

$$Z_\varepsilon = \{a \in E(P_\varepsilon) : u(a) = 0\}, \qquad F_\varepsilon = \{f(a) : a \in Z_\varepsilon\} \subseteq B \setminus \{0\}.$$

For any thread edge set S, define

$$T_\varepsilon(S) = F_\varepsilon \cup \{f(a) + \lambda : a \in Z_\varepsilon \cap S\}.$$

Choosing $y \in B \setminus T_\varepsilon(S)$ allows preparation on D_ε, protects the entire thread before the move, and protects the edges of S after the further λ addition.

For two disjoint sets S_1, S_2, at least one $T_\varepsilon(S_j)$ is proper in B. Otherwise each contains zero, forcing two distinct positions of value λ, one in each piece. Since $|Z_\varepsilon| \leq 3$, this gives $|F_\varepsilon| \leq 2$. Each equality $T_\varepsilon(S_j) = B$, with $|B| \geq 4$, then requires at least two positions in $Z_\varepsilon \cap S_j$. The two disjoint pieces would require at least four positions, a contradiction.

If p, p' lie in the same free thread, choose between its middle edge and its two outer edges, which are disjoint candidate sets. For the second choice join the distinct thread ends by a path in H_P. A preparation outside the corresponding proper forbidden set makes the chosen path safe. Otherwise, for each endpoint interior to a free thread choose one of its two disjoint pieces to a thread end with a proper forbidden set, and prepare that thread. An endpoint not interior to a free thread is already in H_P. Join the resulting endpoints by a path in H_P and attach the selected pieces. The resulting union is a path: H_P contains no interior vertex of a free thread, and any two pieces are on different threads. Preparations on different free threads leave each other's full values unchanged.

Finally add $\lambda\chi_C$. On protected edges the unchanged quotient is nonzero; on the free pieces the selected preparation prevents zero; elsewhere in H nothing changes. Outside H the resulting flow is g^*. This completes all cases. □

18. Universal connectivity

Proof of Theorem 13.1. Every $x \in \Gamma$ satisfies $2x \in B$, hence $4x = 0$. Since $|\Gamma| \geq 32$, the finite abelian structure theorem shows that Γ is a product of two nontrivial groups. This product is used for averaging only; it need not split B or the quotient map onto A. Lemma 14.3 and

the assumed ratio supply a good subgraph in nonzero quotient support in every loopless 3-edge-connected graph on at least two vertices.

Call a Γ-flow a *target* if its A-coordinate is nowhere zero. Every bridgeless graph has a target: use the order-at-least-six existence theorem for A, then Lemma 14.1 to lift it. The lift is automatically nowhere zero. We prove that every nowhere-zero flow is joined to every target on a connected bridgeless graph. Induct on the number of vertices, first proving the loopless case at each order and then allowing loops.

A connected loopless graph on one vertex has only its empty flow. If loops are present, apply the loopless result to their deletion, then adjust each loop separately to its target value with a single nonzero cycle coefficient, omitting changes already achieved.

Suppose the loopless graph has a cut of size two, with edges a, b. Contract a. The resulting graph is bridgeless and has fewer vertices, though it may have loops. Every flow on it lifts uniquely to a flow on the original graph by conservation at the two ends of a. Conservation across the two-edge cut forces the lifted value of a to be the value of b or its negative. Thus restriction preserves, in both directions, being nowhere zero and being a target. A cycle of the contracted graph lifts to a cycle either directly or by adjoining a to the path between its ends. The unique flow lift therefore takes every legal contracted move to a legal original move. Induction applies.

It remains to treat a loopless 3-edge-connected graph. By averaging and extraction join the initial flow f to an H-good flow f' for some good H with at least two vertices. The graph G/H is bridgeless and has fewer vertices: a cycle of G through a retained edge projects to a closed trail containing that edge, hence to a cycle through it. By induction the restriction of f' is joined to the restriction h^* of the chosen target h. Lift this path successively by Lemma 17.1. We reach an H-good flow f_0 equal to h outside H.

If H is a short cycle, the quotient discrepancy $\overline{h} - \overline{f_0}$ is $c\chi_H$. If $c \neq 0$, add a lift of c on H; the final quotient is the nowhere-zero target quotient, so the move is legal. The remaining discrepancy is $y\chi_H$ with $y \in B$ and is corrected in one move if nonzero, protected by the quotient.

If H is a subdivision, the quotient discrepancy is a flow supported on H. By Proposition 14.2(i) it is $\pi(Y)$ for an A-flow Y on Q. Since the target quotient is nonzero everywhere, $Y \in P(\Lambda_{f_0})$. Lemma 16.3 aligns the quotient on H. The remaining full discrepancy is a B-valued flow supported in H. Decompose it into signed cycles of H and add them successively. Every changed edge keeps its nonzero target quotient, so all moves are legal. We reach h.

This completes the induction. For a disconnected graph apply it component by component. Existence of a target proves nonemptiness as well as connectivity, including the graph with no edges and its single empty flow. □

Proof of Corollary 13.2. Write $\Gamma = \mathbb{Z}_4^a \times \mathbb{F}_2^b$ with $2a + b \geq 5$. The binary subgroups 2Γ and $\Gamma[2]$ have dimensions a and $a + b$, respectively. Set $t = \max(a, 2)$ and choose

$$2\Gamma \subseteq B \subseteq \Gamma[2], \qquad \dim_{\mathbb{F}_2} B = t.$$

Such a subgroup exists since $a + b \geq t$. The quotient is binary and has dimension $d = 2a + b - t \geq 3$. If $q = |A| \geq 16$, the required ratio holds, since it is equivalent to

$$(9 - q)|B| \leq 8.$$

If $d = 3$, the possibilities for (a, b, t) are

$$(0, 5, 2), \quad (1, 3, 2), \quad (2, 1, 2), \quad (3, 0, 3).$$

The first three give ratio $3/31 < 1/9$. The last, $\Gamma = \mathbb{Z}_4^3$ with $B = 2\Gamma$, gives $7/63 = 1/9$. Apply Theorem 13.1 in every case. For the binary specialization one can simply take a subgroup of order four in $\mathbb{F}_2^n$; the quotient has order at least eight and the ratio is $3/(2^n - 1) \le 3/31$. □

Remark 18.1 (Where the hypotheses enter)**.** The quotient's exponent two is used in the symmetric-difference compensation of Lemma 15.2 and in the path permissions. The bounds $6 < 8$ and $7 < 8$ control paired lists and the same-thread or short-cycle cases. The kernel's exponent two is used in the disjoint-piece argument; $3 < 4$ permits preparation. The ratio is used only for averaging and extraction. A quotient of order four, a kernel of order two, or a cyclic kernel of order four is not covered by these local proofs.

19. Four-edge-connected binary flows

This section proves Theorem 13.3. Put $\Gamma = \mathbb{F}_2^m$, $m \ge 3$. All coefficients in this section are binary, so a cycle move adds $\lambda \mathbf{1}_C$ for $\lambda \ne 0$ and is legal exactly when no edge of C currently has value λ. Applying the same move again reverses it. Cycles are simple, with a parallel pair allowed as a two-edge cycle. We first work without loops and restore them at the end.

For an edge map on a graph H, its *boundary* at a vertex is the sum of incident values. A prescribed boundary is a map $\beta : V(H) \to \Gamma$ with total sum zero. In the lemmas involving β, all edge maps are nowhere zero and have that same boundary. Their differences have boundary zero, and cycle additions preserve the boundary. A zero-boundary flow is called *main* if it is joined to a flow omitting a nonzero value. The definition alone does not imply that all main flows on an arbitrary graph belong to one component.

We use the Nash-Williams–Tutte tree-packing criterion [NW1, Tu2]: a finite multigraph contains k edge-disjoint spanning trees if and only if every partition into q nonempty vertex sets has at least $k(q-1)$ edges between parts. In particular a 4-edge-connected graph has two edge-disjoint spanning trees, since summing its part cuts gives at least $2q$ crossing edges when $q \ge 2$.

19.1. Linear changes and two spanning trees.

Lemma 19.1 (Legal linear changes)**.** *Let j have boundary zero, let $d \in \Gamma \setminus \{0\}$, and let $\ell : \Gamma \to \mathbb{F}_2$ be linear with $\ell(d) = 0$. Applying $x \mapsto x + \ell(x)d$ to every value of j gives a flow joined to j. Consequently a flow joined to one omitting a value v is joined to one omitting any specified nonzero value b.*

Proof. The edges with $\ell(j(e)) = 1$ form an even set, and none has value d. Decompose this set into edge-disjoint cycles and add d on them successively. These legal moves implement the stated map. Such transvections generate $\mathrm{GL}(m, 2)$: elementary row additions generate the invertible binary matrices, with row interchanges obtained by three row additions. An invertible linear map can send v to b, proving the final assertion. □

Lemma 19.2 (Two-tree boundary adjustment)**.** *Let T be a spanning tree of H such that the spanning graph with edge set $R = E(H) \setminus E(T)$ is connected. Fix $b \ne 0$ and a boundary β.*

(i) *Any two boundary-β edge maps having no value b on T are joined by cycle moves in H, through maps with that property.*

(ii) *For every basis $x_1, x_2, x_3 = b, x_4, \dots, x_m$ and every spanning tree T_2 in R, there is such a map g omitting b on all of H, with only the values $x_1, x_1 + x_2$ on T_2.*

Proof. Write u for the quotient of the current map modulo $\langle b \rangle$. Its values on T are nonzero. We use two operations.

First, for an edge $e \in R$ with $u(e) \neq 0$, add b on its fundamental cycle in $T + e$. This changes no other R-edge and preserves every quotient value. It is legal because e and all tree edges on that cycle have nonzero quotient. Call this a *bit correction*.

Second, let $t \in T$ and choose λ whose quotient τ is neither zero nor $u(t)$. A simple R-path P joins the ends of t. At each edge of P currently having full value λ, use a bit correction to change it to $\lambda + b$. Distinct corrections leave each other's R-edges unchanged and preserve all tree quotients. Now add λ on $t + P$. This is legal; among tree quotients only $u(t)$ changes, becoming the nonzero value $u(t) + \tau$.

Given maps j, j' in (i), perform the second operation on each tree edge where their quotient values differ, using $\tau = u(t) + u'(t)$. It is nonzero and different from $u(t)$. After these operations the quotient discrepancy is a zero-boundary flow supported in R. Decompose it into quotient cycle flows within R. Lift and perform these cycle additions one at a time, before each addition correcting its currently dangerous edges by bit corrections. The tree quotients remain fixed throughout, although their full values can change by b. Edges of R may acquire the full value b, which is permitted. At the end the quotient maps agree everywhere.

The remaining discrepancy is $b\mathbf{1}_D$ for an even set D. Any R-edge of D has nonzero quotient: where the common quotient is zero, both full values are b. Correct each such R-edge using its bit correction. Their symmetric difference agrees with D off T; it therefore equals D, because an even edge set supported on a tree is empty. This reaches j' and proves (i).

For (ii), set $T_1 = T$ and write β_i for the coordinates of β in the chosen basis. There is a unique binary edge set D_1 with boundary β_1 containing $E(H) \setminus E(T_1)$: solve the remaining boundary on T_1 by successively removing leaves. Define D_2 similarly using T_2, β_2. For $i \geq 3$, let D_i have boundary β_i and be supported on T_1. Put

$$g = \sum_{i=1}^{m} x_i \mathbf{1}_{D_i}.$$

Every edge is outside T_1 or outside T_2, so it belongs to D_1 or D_2. Hence g is nowhere zero and never equals $b = x_3$. An edge of T_2 belongs to D_1 and to no D_i with $i \geq 3$; its value is x_1 or $x_1 + x_2$. The construction has boundary β, as required. □

19.2. **Splitting and contraction.**

Lemma 19.3 (Balanced degree-four splitting)**.** *Suppose s has degree four with edges $e_i = su_i$ to distinct vertices, and $j(e_1) = j(e_2)$, hence $j(e_3) = j(e_4)$. Delete s and replace its two pairs by edges u_1u_2, u_3u_4 with their common values, obtaining U' and j'. If j' is main, then j is main.*

Proof. Lift a cycle of U' by replacing each new edge by its two-edge path through s. The lift is one cycle or decomposes into two edge-disjoint cycles meeting only at s. A legal coefficient downstairs is absent on both lifted cycles. Add it successively on them. The second is unchanged by the first, and after both moves the values in each pair are equal again. Pair equality is not needed at the intermediate state. This lifts every downstairs step. At the final endpoint, omitting a value downstairs omits it on the two restored paths as well. □

For a flow j on U and $b \neq 0$, call a vertex set X, $|X| \geq 2$, *good for* (j, b) if $U[X]$ has a spanning tree T_X without value b whose edge complement in $U[X]$ is connected and spanning. In U/X identify the vertices of X and delete all edges of $U[X]$, retaining every other edge.

Lemma 19.4 (Contraction of a good set)**.** *If X is good for (j, b) and the restriction of j to U/X is main, then j is main.*

Proof. By Lemma 19.1, a downstairs path reaches a flow omitting b. Lift it while keeping T_X free of b. A downstairs cycle already forming a cycle of U is handled directly. Otherwise it is an exterior path between distinct vertices $p, p' \in X$.

If the step coefficient is b, close this path along T_X. Every edge on the resulting cycle avoids b, and each changed tree value remains nonzero and different from b.

If the step coefficient is $k \neq b$, use the current boundary of the restriction to $U[X]$. By Lemma 19.2, move inside $U[X]$ to its canonical map for a basis with $x_2 = k, x_3 = b$ and a spanning tree T_2 of $U[X] \setminus E(T_X)$. Its T_2-values are $x_1, x_1 + k$, both different from k. Close the exterior path through T_2 and add k. This final move leaves T_X unchanged. Preparations keep the exterior fixed and preserve the required tree condition.

Once the exterior omits b, apply Lemma 19.2 once more inside X to reach a map omitting b there too. This is a full flow omitting b. □

19.3. **Proof of the four-edge-connected theorem.**

Proof of Theorem 13.3. First suppose G is loopless. A graph on one vertex has only its empty flow. On a graph with at least two vertices, the tree-packing criterion gives two edge-disjoint spanning trees. Lemma 19.2(ii), with boundary zero, constructs a nowhere-zero flow, proving nonemptiness.

For a fixed $b \neq 0$, Lemma 19.2(i) connects all flows omitting b. Every main flow is joined to one omitting b by Lemma 19.1. Thus it remains to prove that every flow is main.

Fix $m \geq 3$ and suppose a counterexample (U, j) exists with the least number of vertices. No flow joined to j is main. Contracting any vertex set preserves the cut condition, since every cut of the contraction is a cut of U.

Parallel edges are impossible. If x, y have two parallel edges e, e', then $X = \{x, y\}$ is good for (j, b) for any nonzero $b \neq j(e)$: take $T_X = \{e\}$ and use e' in its connected complement. The contraction has fewer vertices, so its restricted flow is main. Lemma 19.4 gives a contradiction. Hence U is simple.

At least two pairings at a degree-four vertex are admissible. Suppose s has incident edges $e_i = su_i$, $1 \leq i \leq 4$, with distinct neighbors, and put $p_i = j(e_i)$. Their sum is zero. A pairing ρ of the four edges gives a split graph U_ρ by removing s and joining the two endpoints in each pair.

For $\varnothing \neq X \subsetneq V(U) \setminus \{s\}$, let t count the neighbors u_i in X and let σ count the pairs of ρ with exactly one endpoint in X. Then

$$d_{U_\rho}(X) = d_U(X) - t + \sigma, \qquad d_U(X) \geq 4, \qquad d_U(X) + 4 - 2t = d_U(X \cup \{s\}) \geq 4.$$

For $t = 0, 1, 3, 4$ these inequalities give a split cut of size at least four. For $t = 2$ they do so unless $\sigma = 0$ and $d_U(X) \leq 5$. Call such a set a *blocker* of the pairing: it contains exactly the two neighbors in one pair. Its complement in $V(U) \setminus \{s\}$ is also a blocker of the same cut size.

Two different pairings cannot both have blockers. After relabeling and choosing complementary blockers if necessary, suppose X_1 contains u_1, u_2 only, X_2 contains u_1, u_3 only, and both cuts have size at most five. Write $e_U(P, Q)$ for the number of edges joining disjoint

vertex sets P, Q. In U,

$$d_U(X_1) + d_U(X_2) = d_U(X_1 \setminus X_2) + d_U(X_2 \setminus X_1) + 2e_U\big(X_1 \cap X_2, V(U) \setminus (X_1 \cup X_2)\big).$$

The first two terms on the right are at least four, and the final edge count includes su_1. All bounds must therefore be equalities: both left cuts have size five, and each difference cut has size four. Writing $I = X_1 \cap X_2$, the identity

$$d_U(X_1) = d_U(X_1 \setminus X_2) + d_U(I) - 2e_U(X_1 \setminus X_2, I)$$

shows that $d_U(I)$ is odd, hence at least five. Cut submodularity gives $d_U(X_1 \cup X_2) \leq 5$. But this union contains exactly three neighbors of s, so

$$d_U(X_1 \cup X_2 \cup \{s\}) = d_U(X_1 \cup X_2) + 4 - 6 \leq 3,$$

contradicting the cut condition, with u_4 still outside. Thus at least two of the three pairings preserve 4-edge-connectivity. Call them 12|34 and 13|24. Their split graphs have fewer vertices, so all their flows are main.

The flow values rule out degree four. For pairing 12|34, equality $p_1 = p_2$ would allow Lemma 19.3, contradicting non-mainness. Thus

$$\lambda_1 = p_1 + p_2 = p_3 + p_4 \neq 0.$$

None of the four incident values equals λ_1, since its partner would then be zero. If a cycle through s using one edge from each pair avoided λ_1, adding λ_1 would make the values in both pairs equal. Lemma 19.3 would again make the reached flow main.

Consequently, in $U - s$ no path avoiding λ_1 joins $\{u_1, u_2\}$ to $\{u_3, u_4\}$. Grouping components after deletion of the λ_1-edges gives a partition A_1, B_1 of $V(U) \setminus \{s\}$, with $u_1, u_2 \in A_1$, $u_3, u_4 \in B_1$, and only λ_1-edges crossing it. Pairing 13|24 similarly gives $\lambda_2 = p_1 + p_3$ and a partition A_2, B_2 separating $\{u_1, u_3\}$ from $\{u_2, u_4\}$.

The set $Q = A_1 \cap A_2$ contains u_1 and no other neighbor of s. Its cut consists of e_1 and edges with values among λ_1, λ_2. Conservation across this cut forces

$$p_1 \in \{0, \lambda_1, \lambda_2, \lambda_1 + \lambda_2\}.$$

These alternatives give, respectively, $p_1 = 0$, $p_2 = 0$, $p_3 = 0$, or $p_4 = 0$, a contradiction. This also covers coincident λ_1, λ_2. No vertex of degree four exists in U.

The density contradiction. Every vertex now has degree at least five. Write $n = |V(U)|$, $e = |E(U)|$, so $e \geq 5n/2$. Choose a least-used nonzero value b and let N be the edges not carrying it. Since $2^m - 1 \geq 7$,

$$|N| \geq \frac{6}{7}e \geq \frac{15}{7}n \geq 2(n-1).$$

Choose a vertex set S of least size among those with $|S| \geq 2$ and $e_N(S) \geq 2|S| - 2$, where $e_N(S)$ counts internal edges of N. Such a set exists because the full vertex set qualifies. Any proper part of S with at least two vertices has fewer than twice its order minus two internal N-edges; a singleton has zero. For every partition of S into $q \geq 2$ parts, the number of crossing N-edges is therefore at least

$$(2|S| - 2) - (2|S| - 2q) = 2(q-1).$$

The tree-packing criterion supplies two edge-disjoint spanning trees in $N[S]$. One is a b-free tree, and the other is in its connected spanning complement in $U[S]$. Thus S is good for

(j, b). Contracting S reduces the vertex count and preserves the cut condition. Minimality and Lemma 19.4 force j to be main, the final contradiction.

This proves the loopless result. With loops, first apply it after deleting the loops. Once the nonloop flow omits a value b, change any loop of value b to a different nonzero value by a move on that loop. Loops contribute zero boundary and can also be adjusted independently when joining two flows. Thus the theorem, including nonemptiness and reachability of a value-omitting state, holds with loops. If the graph with no vertices is included, its unique empty flow supplies the additional vacuous case. □

19.4. **Consequences and the rank-two boundary.**

Corollary 19.5 (Partition density)**.** *For $\Gamma = \mathbb{F}_2^m$, $m \geq 3$, suppose every partition of $V(U)$ into q nonempty parts has at least*

$$\frac{2(2^m - 1)}{2^m - 2}(q - 1)$$

crossing edges. Then $F(U, \Gamma)$ is nonempty and connected, and every flow is main. At order eight the coefficient is $7/3$.

Proof. Loops do not affect the partition condition. Delete them first and apply the following argument to the resulting loopless graph; afterward restore their values independently as in the proof of Theorem 13.3. The hypothesis is preserved by contraction and implies the two-tree criterion. Its singleton partition gives enough total edges that deleting a least-used value leaves at least $2(n - 1)$ edges. The minimal dense-set argument and good-set contraction in the preceding proof then suffice without the degree-four step. Nonemptiness and connection of main flows follow from the same two-tree lemmas. □

Corollary 19.6 (All qualifying trees and values, at fixed boundary)**.** *Let H have two edge-disjoint spanning trees, and fix a boundary β in $\mathbb{F}_2^m$, $m \geq 3$. All nowhere-zero boundary-β maps that omit some nonzero value on some spanning tree with connected spanning complement lie in one reconfiguration component.*

Proof. For a fixed value and tree, use Lemma 19.2(i). A map omitting b everywhere belongs to the family for every qualifying tree, so these families connect for fixed b. The canonical map in part (ii) omits both $b = x_3$ and $x_1 + b$: on T_1 its second coordinate is one, and off T_1 its third coordinate is zero. Given distinct b, b', choose $x_1 = b + b'$ and complete x_1, b to the specified basis. The canonical map omits both values and joins their families. This uses no global linear change at nonzero boundary. □

Corollary 19.7 (Rounded partition density at order eight)**.** *Let U be a finite multigraph with at least one vertex. Suppose every partition of $V(U)$ into r nonempty parts, with c crossing edges, satisfies*

$$6c \geq 14r - 20.$$

Then $F(U, \mathbb{F}_2^3)$ is nonempty and connected, and every flow is main.

Proof. First delete all loops, which affect neither the hypothesis nor conservation on the other edges. The loopless one-vertex graph has its unique empty flow, which omits every nonzero value. Suppose $n = |V(U)| \geq 2$. For every partition with $r \geq 2$,

$$14r - 20 > 6(2r - 3).$$

Since the crossing count is integral, it is at least $2r-2$. Thus U has two edge-disjoint spanning trees. Lemma 19.2 gives a nowhere-zero flow, and Corollary 19.6 joins all value-omitting flows at boundary zero. It remains to prove that every flow reaches one.

Suppose the contrary. Among all loopless multigraphs that satisfy the partition hypothesis and carry a nowhere-zero flow that is not main, choose one with the least number of vertices; call it again U, with n vertices, and let j be a flow on it that is not main. Then $n \geq 2$ by the one-vertex case. The partition hypothesis is preserved by contraction: a partition of the contracted graph lifts to one with the same number of parts and crossing edges. Write $e = |E(U)|$. Each edge belongs to the complements of exactly six of the seven nonzero value classes of j. If all seven complements had at most $2n-3$ edges, then

$$6e \leq 7(2n-3) = 14n-21,$$

contrary to the hypothesis on the partition into single vertices. Some nonzero value b therefore has a complement N with at least $2n-2$ edges. Any such value serves; it need not be a least-used one.

Choose $S \subseteq V(U)$ of least size with $|S| \geq 2$ and $e_N(S) \geq 2|S|-2$. For a partition of S into $q \geq 2$ parts, minimality bounds the sum of the internal N-edge counts in those parts by $2|S|-2q$. Hence at least $2(q-1)$ edges of $N[S]$ cross that partition. The tree-packing criterion gives two edge-disjoint spanning trees in $N[S]$. Both are b-free, as all of N is. Take one of them as the tree; the other lies in its edge complement in $U[S]$, which is therefore connected and spanning. So S is good for (j, b). The graph U/S is loopless, has fewer vertices than U and satisfies the partition hypothesis, so the restriction of j to it is main by the choice of U. Lemma 19.4 then makes j main, a contradiction. Hence every flow is main. Together with the common component of value-omitting flows, this proves connectedness.

Restore the loops independently. Once the nonloop flow omits a value b, any loop carrying b can be changed to another nonzero value by a move on that loop. Loop values can also be joined independently, so nonemptiness, connectedness and mainness all persist. This includes a single vertex with loops. □

Corollary 19.8 (A matching criterion at order sixteen)**.** *Let G be cubic and 3-edge-connected, with every three-edge cut the cut at a single vertex. Let f be a nowhere-zero $\mathbb{F}_2^4$-flow. If some perfect matching omits some nonzero value of f, then f is joined to a flow omitting a nonzero value on all of G.*

Proof. Let the matching be M, let its missing value be b, and choose a hyperplane K not containing b. Put $W = \{e : f(e) \notin K\}$ and $W' = E(G) \setminus M$. Both are even sets and both contain every b-edge. Hence $W \triangle W'$ is even and contains no b-edge. Adding b on its edge-disjoint cycles gives a legal path to a flow with quotient support W' modulo K.

The set W' is a 2-factor. Contract each of its cycles, retaining loops. A nontrivial cut in the quotient lifts to a cut in G whose two sides are unions of whole W'-cycles. Neither side is a single vertex, so its size cannot be three; it is at least four. The restriction to the matching is a nowhere-zero K-flow on this quotient. By Theorem 13.3, it is joined to one missing some $v \in K \setminus \{0\}$.

Lift each quotient cycle by paths in its visited W'-cycles. The resulting K-move is safe on those paths because their full values lie outside K, and safe on the matching by its prescribed downstairs endpoints. A quotient loop lifts to a chord and an arc of its W'-cycle. Thus the path lifts while all W'-values stay outside K. Its endpoint omits v both on the matching and on W', proving the claim. In particular, in any component containing no main flow, every perfect matching carries all fifteen values at every state of that component. □

Remark 19.9 (Mainness at rank two)**.** The omitted-value conclusion fails for $\mathbb{F}_2^2$. Let its nonzero values be $a, b, c = a + b$. On K_6, alternate b, c around a Hamiltonian six-cycle and put a on its other nine edges. Each vertex has sum $3a + b + c = 0$, giving a nowhere-zero flow. Every nontrivial cut has size $s(6 - s) \geq 5$. But omitting a value would leave at most two independent nonzero values, each occurring an even number of times at every vertex by conservation, contrary to degree five. Thus no flow on K_6 omits a nonzero value. This example concerns only the omitted-value conclusion, not the connectivity of $F(K_6, \mathbb{F}_2^2)$. The proof above needs three coordinates in the canonical two-tree construction and at least seven nonzero values in its density count.

20. At most five exceptional star cuts

Degrees in this section are measured after deleting loops. Let $\mathcal{C}_5$ consist of finite multigraphs with at least one vertex satisfying the following condition: either the graph is a singleton, or it has a designated set T of at most five degree-three vertices such that every edge cut determined by a nonempty proper vertex set has size at least four, except cuts of the form $\delta(\{t\})$ for $t \in T$. Complementary shores determine the same edge cut. In particular, every nonsingleton graph in $\mathcal{C}_5$ is connected and has minimum cut at least three.

Theorem 20.1 (Five exceptional stars)**.** *For $U \in \mathcal{C}_5$ and $m \geq 3$, the graph $F(U, \mathbb{F}_2^m)$ is nonempty and connected. Every flow is joined to a flow whose values lie among the three nonzero elements of a two-dimensional subspace.*

The designation concerns cuts, not a literal count of all degree-three vertices: the three-edge dipole belongs to $\mathcal{C}_5$ with just one designated vertex. For a connected graph on at least three vertices, different degree-three vertices have different star cuts, since equality of two such edge sets would make them an entire dipole component. Consequently every degree-three vertex is then designated.

20.1. **Contraction, two trees, and a short cycle.** The class $\mathcal{C}_5$ is closed under contraction of a vertex set, with internal edges deleted. A surviving three-edge cut lifts to an original designated star cut. Its designated singleton is an unswallowed vertex, or no such cut survives the contraction; a singleton contraction is the identity. The surviving designated vertices retain degree three and witness the quotient condition. Their number does not increase.

Every member has two edge-disjoint spanning trees. For a partition into $q \geq 3$ parts, only a designated singleton part can have a cut of size three: the complement of a singleton cannot be a part of a partition with at least three parts. The sum of part-cut sizes is at least $4q - 5$; being even, it is at least $4q - 4$. Thus there are at least $2(q - 1)$ crossing edges. For $q = 2$ there are at least three, and the tree-packing criterion applies.

These trees also give a three-value anchor. For disjoint spanning trees T_1, T_2, let D_i be the unique even set containing $E(U) \setminus E(T_i)$, obtained by solving the remaining binary boundary on T_i. If x_1, x_2 are independent, then

$$g = x_1 \mathbf{1}_{D_1} + x_2 \mathbf{1}_{D_2}$$

is nowhere zero: every edge is outside at least one tree. Its values lie in $\{x_1, x_2, x_1 + x_2\}$. By Lemmas 19.1 and 19.2, all main flows on a member of $\mathcal{C}_5$ lie in one component. It is therefore enough to prove that every flow is main.

We will also use the following binary short-cycle lifting fact. It is the needed direction of the contraction criterion of [CLSWX, Theorem 5.9, Theorem 5.11 and Corollary 5.14]; we include its elementary proof.

Lemma 20.2 (Lifting across a short induced cycle)**.** *Let A be elementary abelian of exponent two, and let C be an induced cycle of length $r < |A|$ in a loopless graph H. If $F(H/C, A)$ is nonempty and connected, then so is $F(H, A)$.*

Proof. For any quotient flow the prescribed boundary on C has total zero, so it has a solution on C. Adding a constant around C changes that solution; at most r constants produce a zero edge. Hence every quotient flow has a nowhere-zero extension. Two such extensions of one quotient flow differ by a constant on C, and are adjacent or equal.

Consider a legal quotient cycle move of coefficient λ. If it is already a cycle upstairs, perform it there. Otherwise its lift off C is a path P between distinct vertices of C, avoiding λ. Let S, S' be the two complementary arcs of C, and let V be the set of current values on C. There exists $c \notin V$ for which at least one of

$$c \notin \lambda + f(S), \qquad c \notin \lambda + f(S')$$

holds. Indeed, if both tests failed for every $c \in M = A \setminus V$, each value in $\lambda + M$ would occur on both disjoint arcs. The number of distinct repeated values among the r edges is at most $r - |V|$, whereas $|M| = |A| - |V| > r - |V|$, a contradiction.

Add c around C, doing nothing if $c = 0$. The cycle stays nowhere zero. Now add λ on P together with the permitted arc. This is legal and realizes the quotient move. The connected fibres adjust the final extension, proving the assertion. □

20.2. **Proof of the five-star theorem.**

Proof of Theorem 20.1. Remove loops first; they can later be changed independently to anchor values. Fix $m \geq 3$ and choose a nonmain flow j in this class on a graph U with the least possible number n of vertices. The singleton is immediate. A pair of vertices with parallel edges is a good set for (j, b) for some b different from the value on one of these edges. Contraction, minimality and Lemma 19.4 give a contradiction. This removes the dipole case before counting designated vertices. Thus U is simple. A shortest cycle of length at most seven is induced and shorter than 2^m; contraction stays in $\mathcal{C}_5$, and Lemma 20.2 and minimality give a contradiction. Hence U has minimum degree three and girth at least eight.

Degree-four vertices must meet degree-three vertices. Suppose s has degree four and none of its distinct neighbors $u_1, \ldots, u_4$ has degree three. None of these neighbors is designated. For a pairing ρ of the incident edges and a nontrivial shore $X \subset V(U) \setminus \{s\}$, let t be the number of neighbors in X and σ the number of paired edges crossing X. Then

$$d_{U_\rho}(X) = d_U(X) - t + \sigma, \qquad d_U(X \cup \{s\}) = d_U(X) + 4 - 2t.$$

For $t = 0$, a cut retains its size and any exceptional singleton remains allowed. For $t = 1, 2$, the shore contains a nondesignated neighbor and its complement contains several vertices, so $d_U(X) \geq 4$. For $t = 3$, the complement of $X \cup \{s\}$ contains a nondesignated neighbor, so $d_U(X) - 2 \geq 4$. For $t = 4$, the split cut is $d_U(X \cup \{s\})$, at least four unless its complement is a designated singleton, whose star remains an allowed exception. The only possible new forbidden cuts therefore have $t = 2$, $\sigma = 0$, and $d_U(X) \leq 5$.

Two different pairings cannot both have such blockers. Choose shores X_1, X_2 containing respectively just u_1, u_2 and just u_1, u_3. The difference shores contain nondesignated neighbors u_2, u_3, and their cuts have size at least four. The uncrossing identity used in the proof of Theorem 13.3, with su_1 joining the intersection to the outside union, forces

$$d_U(X_1) = d_U(X_2) = 5, \qquad d_U(X_1 \setminus X_2) = d_U(X_2 \setminus X_1) = 4.$$

The intersection cut has odd size and is at least four, hence at least five. Submodularity gives $d_U(X_1 \cup X_2) \leq 5$, and adding s lowers this by two. The resulting cut has size at most three, with several vertices on one shore and nondesignated u_4 on the other. This violates the class condition. At least two pairings therefore preserve $\mathcal{C}_5$.

Their split graphs are smaller. A balanced preserved pairing would make j main by Lemma 19.3. For an unbalanced pairing $12|34$, put $p_i = j(su_i)$ and $\lambda_1 = p_1+p_2 = p_3+p_4 \neq 0$. A crossing cycle through s avoiding λ_1 would balance this pairing and again give mainness. Thus, in $U - s$, the pairs are separated by a cut of λ_1-edges. The other preserved pairing $13|24$ gives a cut of $\lambda_2 = p_1 + p_3$-edges. The intersection of the shores containing u_1 has precisely that neighbor of s, so conservation gives

$$p_1 \in \{0, \lambda_1, \lambda_2, \lambda_1 + \lambda_2\}.$$

Every alternative makes one of p_1, p_2, p_3, p_4 zero, exactly as in the degree-four step of Theorem 13.3. This contradiction proves the claim about neighbors.

Let k be the actual number of degree-three vertices, a the number of degree-four vertices, and $h = n - k - a$ the remaining number. We have $k \leq 5$ and $a \leq 3k$. Writing $e = |E(U)|$,

$$2e \geq 3k + 4a + 5h = 5n - 2k - a \geq 5n - 5k. \tag{6}$$

An integer density bound. For any nonzero b, let N_b be the set of edges not carrying b. If $|N_b| \geq 2(n-1)$, the minimal dense-set argument in Theorem 13.3 supplies a vertex set with two edge-disjoint spanning trees in N_b. It is good for (j, b), and its contraction contradicts minimality via Lemma 19.4. Hence $|N_b| \leq 2n - 3$ for every b. For a least-used value, $|N_b| \geq 6e/7$, so

$$5n - 5k \leq 2e \leq \frac{7}{3}(2n - 3), \qquad n \leq 15k - 21 \leq 54.$$

The girth count. Root a search at the two ends u, v of an edge, without traversing that edge again, and grow three levels on each side. Girth at least eight makes all counted vertices distinct: a collision within a side creates a cycle of length at most six, and a collision across sides one of length at most seven. Every intermediate vertex has at least two forward neighbors. Thus

$$n \geq 2 + (d(u) + d(v) - 2)(1 + 2 + 4) = 7(d(u) + d(v)) - 12. \tag{7}$$

In particular $n \geq 30$. Two adjacent vertices of degree at least five would give $n \geq 58$, contradicting $n \leq 54$. The h vertices of degree at least five are consequently independent. Counting their incident edges gives

$$5h \leq 3k + 4a \leq 15k, \qquad n = k + a + h \leq 7k \leq 35.$$

If $h > 0$, an edge from a vertex of degree at least five has degree sum at least eight and gives $n \geq 44$, again impossible. Hence $h = 0$, and $n = k + a \leq 4k \leq 20 < 30$. This final contradiction shows every flow is main. The two-tree argument connects all flows to the three-value anchor. Restoring loops, and independently assigning them any of the anchor values, proves the theorem as stated. □

Corollary 20.3 (A sufficient support condition at order sixteen)**.** *Let G be cubic and 3-edge-connected, with every three-edge cut a vertex star, and let f be a nowhere-zero $\mathbb{F}_2^4$-flow. If an even edge set W' contains all edges of some nonzero value b and leaves at most five vertices uncovered, then f is main. Equivalently, it suffices that $G - f^{-1}(b)$ have a spanning subgraph with every degree odd and at most five vertices of degree three.*

Proof. Choose a hyperplane K avoiding b, and let $W = \{e : f(e) \notin K\}$. The even set $W \triangle W'$ has no b-edge. Adding b on its cycle components gives a flow whose outside-K support is W'. Since G is cubic, W' is a union of disjoint cycles. Contract them, retaining loops. The remaining edges form a nowhere-zero K-flow. Every three-edge cut of the quotient lifts to a vertex star of G; its singleton shore must be a vertex outside W'. These at most five vertices designate the exceptional cuts. In particular their degrees, after deleting loops, remain three. The quotient belongs to $\mathcal{C}_5$.

Theorem 20.1 joins this quotient flow to one omitting a nonzero K-value. Lift each quotient cycle with arcs in the contracted cycles of W', as in Corollary 19.8. Those arcs stay outside K, so the lifted K-moves are safe. A quotient loop lifts to a chord and an arc. The endpoint omits the chosen value everywhere. Finally, the complement of W' has degree one or three at every vertex, with degree three exactly at uncovered vertices, and contains no b-edge. Taking complements proves the equivalent formulation. We do not show that this spanning subgraph exists. □

21. A conditional four-edge-cut reduction at order sixteen

In this section $\Gamma = \mathbb{F}_2^4$, and all graphs are finite and loopless; parallel pairs count as cycles. A cut is always determined by a nonempty proper vertex set. All statements of joining refer to the unrestricted flow graph: a constructed path may leave the boundary types being compared.

Theorem 21.1 (Conditional four-cut reduction)**.** *Suppose that G satisfies the following hypotheses.*

(H1) *G is cubic and 3-edge-connected, and every three-edge cut is a vertex star.*
(H2) *For every bridgeless graph G' with fewer odd-degree vertices than G, any two nowhere-zero Γ-flows of G' are joined.*
(H3) *Some vertex set X has $d_G(X) = 4$, $|X| \geq 4$, and $|V(G) \setminus X| \geq 4$.*

Then $F(G, \Gamma)$ is nonempty and connected.

Only six smaller graphs will use (H2): the three wire closures of each side of the cut. They are cubic and bridgeless, with at most $|V(G)| - 4$ vertices, hence fewer odd-degree vertices. They need not inherit (H1). For existence we use Jaeger's theorem that every bridgeless multigraph has a nowhere-zero binary rank-three flow [Ja1, Ja2]; an isomorphism gives a flow in any specified order-eight subgroup of Γ.

21.1. **Sides, linear transport, and classes.** In a cubic 3-edge-connected graph, $G - v$ is connected: different components would each require at least three edges to v, exceeding its degree. Consequently a cut equal to the star of v has one shore $\{v\}$. This explains the singleton-shore interpretation of (H1).

Write the four cut edges as $e_1, \ldots, e_4$, and the two sides as Z_1, Z_2. For either side Z, put $D = G[Z]$ and let z_i be the endpoint of e_i in Z. The terminals are distinct. A vertex incident with three cut edges would leave a one-edge cut on deleting it from its shore. A vertex incident with two would leave a three-edge cut with at least three and five vertices on its shores, violating (H1). The graph D is connected, since two components would need at least six cut edges. It is also bridgeless: an internal bridge separating t terminals from $4 - t$ gives cuts in G of sizes $t + 1$ and $5 - t$. These force $t = 2$, and then give a three-edge cut with neither shore a singleton, a contradiction.

A *side flow* assigns nonzero values to $E(D)$ and the four cut edges, satisfying conservation at each vertex of Z. Its ordered boundary $b = (b_1, \dots, b_4)$ has sum zero. A full flow is a pair of side flows with identical boundaries. An *inner move* is a cycle move entirely in D. Write $f \equiv g$ when side flows with the same boundary are joined by inner moves. Reversibility makes this an equivalence relation.

There are five boundary types: Q, all four entries equal; P_π, two distinct equal pairs for one of the three pairings π; and D_4, all four entries distinct. In type D_4 any three entries are independent, since a dependence would make the fourth entry zero.

We repeatedly use the following even-set observation. If nowhere-zero endpoint maps of the same boundary differ by $d\mathbf{1}_E$ on an internal even set, they are inner-equivalent: every edge of E avoids d at the first endpoint, since otherwise the second endpoint would be zero. Its edge-disjoint cycle decomposition gives legal successive moves. This also holds for full flows.

Lemma 21.2 (Internal stabilizers and linear transport)**.** *Let f be a side flow with boundary b.*

(i) *An invertible linear map fixing every b_i sends f to an inner-equivalent flow.*

(ii) *Let $I \subseteq \{1,2,3,4\}$ have size two or four. Suppose $R \subseteq E(D)$ has odd degree exactly at z_i for $i \in I$, avoids $\lambda \neq 0$, and $b_i \neq \lambda$ for $i \in I$. Let f' add λ on R and those cut edges. If a functional ℓ satisfies*

$$\ell(b_i) = \mathbf{1}_I(i), \qquad \ell(\lambda) = 0,$$

then $f' \equiv t(f)$ for $t(x) = x + \ell(x)\lambda$.

Proof. Put $U = \langle b_1, \dots, b_4 \rangle$. In a basis extending one of U, its pointwise stabilizer has matrices

$$\begin{pmatrix} I & A \\ 0 & B \end{pmatrix}, \qquad B \text{ invertible.}$$

Row additions on the complementary coordinates generate B; the remaining maps add vectors of U to complementary basis vectors. Thus this stabilizer is generated by transvections $x \mapsto x + \ell(x)d$ with $\ell|_U = 0$ and $\ell(d) = 0$. For a side flow, the support of $\ell(f)$ is internal and even, and avoids d. Each generator therefore acts by inner moves, proving (i).

For (ii), the internal set $S = \{e : \ell(f(e)) = 1\}$ has the same odd terminals as R. The states $t(f)$ and f' have identical boundaries and differ by λ on the internal even set $R \triangle S$. The even-set observation proves the claim. □

For side flows of the same type, define $f \sim g$ if $L(f) \equiv g$ for an invertible linear map L taking the boundary of f to that of g in order. Linear maps preserve inner paths, so this is an equivalence relation; call its equivalence classes *side classes*. Every side class has a representative over every boundary of its type, by transitivity on one, two, or three independent vectors. By Lemma 21.2(i), two representatives of one class at the same boundary are inner-equivalent.

In particular, full flows of the same boundary type with the same two side classes are joined. First apply a global invertible linear map to align their boundaries, using Lemma 19.1; then use inner paths on the two sides. The global path may pass through other boundary types. Lemma 21.2(ii) also shows that any step meeting its functional conditions preserves the side class.

21.2. **Two walls and a direct move to equal pairs.** For $\lambda \neq 0$, call $Y \subseteq Z$ a *λ-wall* if every edge of D leaving Y has value λ. Summing the side-flow equations on the component reachable from z_i in the graph without λ-edges shows that its terminal-value sum is 0 or λ.

Lemma 21.3 (Wall intersection)**.** *If Y is a λ-wall and Y' a λ'-wall, then*

$$\sum_{i:z_i \in Y \cap Y'} b_i \in \langle \lambda, \lambda' \rangle.$$

Proof. Sum conservation over $Y \cap Y'$. Internal edges cancel. Every remaining internal edge leaves Y or Y', so has value λ or λ'. The cut edges contribute precisely the displayed sum. The empty intersection gives zero as well. □

For a type-D_4 boundary and a pairing $\pi = \{i,k\} \mid \{j,l\}$, put $d_\pi = b_i + b_k = b_j + b_l$. It is nonzero and equals no boundary entry. A *π-wall* is a d_π-wall separating the two terminal pairs. Either all terminals are joined by d_π-free paths, or a π-wall exists: the reachable-terminal count excludes sizes one and three, and a reachable pair containing i must be $\{i,k\}$.

A side flow of type D_4 has a wall for at most one pairing. Otherwise complement two walls so both contain terminal 1, with other terminals k, k', where $k \neq k'$. Their terminal intersection is just 1. Lemma 21.3 forces

$$b_1 \in \{0, b_1 + b_k, b_1 + b_{k'}, b_k + b_{k'}\}.$$

The first three possibilities make a boundary entry zero; the last makes the remaining fourth entry zero. All are impossible.

Across the two sides at most two of the three pairings have a wall. Choose a pairing free on both sides. Their d_π-free paths from z_i to z_j, together with e_i, e_j, form one legal cycle of G. Adding d_π changes b_i to b_k and b_j to b_l. Thus every full flow of type D_4 reaches type P_π in one move. No smaller-graph hypothesis is used here.

21.3. **Wire closures and their class graphs.** For a side Z and pairing $\pi = \{i,k\} \mid \{j,l\}$, form $H(Z,\pi)$ by adding to D the two *wire* edges $w = z_i z_k$ and $w' = z_j z_l$. This is a loopless cubic bridgeless graph on $|Z| \leq |V(G)| - 4$ vertices: D is connected and bridgeless, and a wire can be deleted without disconnecting the graph. Parallel edges are permitted. Hypothesis (H2) makes its flow graph connected. Its flows are exactly the side flows of types Q and P_π, identifying each wire value with the two corresponding boundary entries.

Both types occur. Take a nowhere-zero flow of $H(Z,\pi)$ in an order-eight subgroup K. If the wire values differ, this already gives type P_π. If they agree, add a value outside K on the cycle consisting of w and a path in D between its ends. This avoids w', and all changed values lie outside K, so the two wire values are now different.

For type Q with any prescribed $\beta \neq 0$, choose an order-eight subgroup K avoiding β and a nowhere-zero K-flow h on D. Choose an edge set J of D with odd degree exactly at its four terminals, for example the symmetric difference of two paths pairing them. Put

$$p(e) = h(e) + \beta \mathbf{1}_J(e) \quad (e \in E(D)), \qquad p(e_i) = \beta \quad (1 \leq i \leq 4).$$

Off J the internal values are nonzero in K; on J they lie in the nonzero coset $\beta + K$. The extra boundary at each terminal is balanced by its cut edge. Hence p is a nowhere-zero side flow of type Q.

Let $B(Z,\pi)$ be the bipartite graph whose vertices are side classes of types Q and P_π, with an edge when representatives are related by a cycle move in $H(Z,\pi)$ using exactly one wire and changing the type. Edges are undirected because every move reverses.

Every cycle move of $H(Z,\pi)$ either keeps its side class or follows an edge of $B(Z,\pi)$. With no wire it is inner. With both wires it preserves the type: for coefficient λ, a functional equal to one on both wire values and zero on λ exists. For equal wire values this uses $\lambda \neq \beta$; for distinct values a, c, the only elements of their span where this functional is one are a, c, both excluded by legality. Extend the functional if λ is outside that span, and apply Lemma 21.2(ii).

With just one wire, a Q state goes to P_π. From a P_π state with wire values a, c, the coefficient $a + c$ gives type Q. Every other legal coefficient preserves P_π: choose $\ell(a) = 1$, $\ell(c) = 0$, $\ell(\lambda) = 0$. The only obstructions in the span are a and $a + c$, already excluded. The same transport lemma applies. Since $F(H(Z,\pi),\Gamma)$ is connected and both types occur, $B(Z,\pi)$ is connected, has both parts nonempty, and has no isolated vertex.

21.4. **Synchronizing the two sides.** Two facts at a type-Q boundary $(\beta,\beta,\beta,\beta)$ control which wire an exit uses. Fix $\lambda \neq 0, \beta$. Every component after deleting the λ-edges contains an even number of terminals, by the terminal-sum count. Consequently, if one pair of π is joined by a λ-free path, so is its complementary pair.

Moreover the two choices lead to the same side class. Let P and P' be λ-free paths for the two complementary pairs, and let g, g' be the two resulting side flows after changing the chosen path and its two cut edges. Choose $\ell(\beta) = 1$, $\ell(\lambda) = 0$, and $t(x) = x + \ell(x)\lambda$. This exchanges β and $\beta + \lambda$. If $S = \{e \in E(D) : \ell(f(e)) = 1\}$ for the starting flow, then on internal edges

$$t(g) = f + \lambda \mathbf{1}_{P \triangle S}, \qquad g' = f + \lambda \mathbf{1}_{P'}.$$

On the first pair of cut edges $t(g)$ has value β, and on the complementary pair it has value $\beta + \lambda$, exactly as g'. Their internal difference is λ on the even set $P \triangle S \triangle P'$, so $t(g) \equiv g'$. Thus $g \sim g'$. Two paths for the same pair similarly give inner-equivalent endpoints by their even symmetric difference.

Lemma 21.4 (Paired class transitions)**.** *Fix π. On each side let a class q_s of type Q be adjacent to a class p_s of type P_π in $B(Z_s,\pi)$. Any full flow with side classes (q_1, q_2) is joined to any full flow with side classes (p_1, p_2).*

Proof. Let the starting common boundary be β. Transport each edge witness linearly to this boundary. The fixed-boundary class property gives inner paths from the starting sides to these witnesses. Let their exit coefficients be λ_1, λ_2, each nonzero and different from β.

An invertible linear map fixing β and taking λ_2 to λ_1 exists by extending independent vectors to bases. Lemma 21.2(i) implements it internally on side two, without changing either prescribed class. If its witness uses the complementary wire, the preceding two facts supply a path for the same wire used by side one, still reaching p_2. We now have two simple internal paths with the same terminal pair and coefficient. Together with the two corresponding cut edges they form a legal full cycle, reaching the prescribed pair of classes. Global boundary alignment followed by inner paths joins this endpoint to any requested representative. Reversibility gives the assertion in the opposite direction as well. □

For fixed π, consider pairs of vertices from $B(Z_1,\pi)$ and $B(Z_2,\pi)$ with matching types. Join two pairs when both coordinates are adjacent. This graph is connected. Between two pairs of type Q, the two coordinate walks have even lengths; pad the shorter by going to a neighbor and back until their lengths agree. The simultaneous walk has matching types at every step. Every pair of type P_π has a neighbor of type Q, since neither factor has an isolated vertex.

Every such pair has a full-flow representative: the two classes admit a common ordered boundary. Full flows representing one pair are joined by boundary alignment and inner moves, and adjacent pairs lift by Lemma 21.4. Hence all full flows of types Q and P_π lie in one component of the unrestricted flow graph. The argument uses paths in the whole flow graph, not only in the subgraph induced by those boundary types.

Completion of Theorem 21.1. The type-Q constructions on the two sides with one common β give a full nowhere-zero flow. For each of the three pairings the preceding argument puts all Q and P_π flows into one component. They share the nonempty type-Q family, so these components agree. The direct wall argument brings every type-D_4 flow into this component. The boundary types exhaust all possibilities, proving nonemptiness and connectedness. □

Corollary 21.5 (Conditional least-core consequence)**.** *Let G be a bridgeless counterexample to $\mathbb{F}_2^4$-flow connectivity with the minimum number of odd-degree vertices among all bridgeless counterexamples. If G satisfies* (H1)*, then every four-edge cut is the cut around two vertices joined by one edge. Every cut with a cycle on both sides has size at least five.*

Proof. Minimality gives (H2), so (H3) is impossible. In a cubic graph a shore of a four-cut has even order, since $3|X| - 2|E(G[X])| = 4$. If its order is less than four, it is two and has exactly one internal edge, hence no cycle. Hypothesis (H1) excludes smaller cuts with cycles on both sides. These observations prove both assertions. □

The normal-form interface is separate from this four-cut proof. Lemma 22.4 in Section 22 supplies a connected simple cubic representative at the same minimum odd count, satisfying (H1) and the wide hypothesis (H2). Section 22 justifies the general binary interface and gives order-sixteen girth at least seventeen and core size at least 766, independently of the four-cut theorem. Section 23 then uses Corollary 21.5 to exclude seventeen-cycles, giving girth at least eighteen and at least 1022 vertices. Section 25 later strengthens the size bound to 3038, Section 26 to 3998, and Section 28 to 19024. The resulting representative is cyclically five-edge-connected. These statements do not show that every minimum-odd-count representative is cubic, and they do not settle connectivity across a five-edge cut.

22. A cycle-order improvement of the odd-vertex bound

Write $A = \mathbb{F}_2^m$ and $q = |A| = 2^m$, with $m \geq 3$. In particular, q denotes the group *order*, not its rank. Graphs are finite multigraphs; loops contribute two to degree and zero to binary conservation. As throughout, a cycle is a connected 2-regular subgraph, including a loop or a digon, and a legal move adds a nonzero coefficient absent from that cycle. In this section and in Section 23, Σ denotes a set of group values, not a step system as in Definition 3.1; in Section 28 it denotes a sum of degrees.

Theorem 22.1 (Improved odd-vertex bound)**.** *Let $A = \mathbb{F}_2^m$ have order $q \geq 8$. If each connected component of a finite bridgeless multigraph G has at most*

$$B_*(q) = 3 \cdot 2^{q/2} - 4$$

odd-degree vertices, then $F(G, A)$ is nonempty and connected.

Corollary 22.2. *The conclusion holds for $A = \mathbb{F}_2^3$ when each component has at most* 44 *odd-degree vertices, and for $A = \mathbb{F}_2^4$ when each has at most* 764 *odd-degree vertices.*

The proof uses [Go, Theorem 1.2(c)], an unpublished manuscript, through Lemma 22.4. The cycle-index argument below requires $q \geq 8$, so the theorem is stated only for $q \geq 8$. For $q \geq 32$, Corollary 13.2 already gives the stronger universal result. The proof of Theorem 22.1 uses no four-cut reduction. Its new step handles a cycle of length *equal* to the group order, rather than only shorter cycles. The order-sixteen value 764 remains a consequence of this general formula; Theorem 23.1 later improves it using the four-cut theorem.

22.1. **The general least-odd-count interface.** We first state the local inputs precisely. The two-odd-vertex case is [Go, Theorem 1.2(c)]: a bridgeless graph in which each component has at most two odd-degree vertices has a connected flow graph over every finite abelian group of order at least four. The other local inputs, balanced splitting of two equal-valued nonloop incident edges, the degree-four local entry and family joining, and triangle extension and projection, are written out and proved below; they follow the corresponding lemmas of [Go, Part 2] (Lemmas 17.1, 17.2, 18.1 and 18.2, Proposition 18.3 and Lemma 21.1; for exponent two, also Lemma 69.2 of its Part 6), which we cite for credit, and are local statements, not an invocation of a bounded theorem beyond its range. The small-cut gluing is proved below as Lemma 22.3. We explain the use of these inputs, including the rank requirements, to preserve the quantifiers of minimality.

Balanced splitting replaces edges uv, vw of equal outward value by an edge uw, retaining v; $u = w$ is permitted and gives a loop. It decreases edge count, preserves degree parity and does not increase maximum degree. A cycle move downstairs lifts through the two-edge path, possibly decomposing into more than one cycle upstairs; the common coefficient is safe on every lifted edge. Thus connectedness downstairs joins the corresponding balanced family in the full flow graph.

At a loopless degree-four vertex, every binary flow reaches a balanced pair in at most one move, and if every nonempty balanced-pair family is joined then all flows are joined. This is [Go, Lemma 69.2]; we include its short proof, which needs no rank-four assumption. With four distinct incident values a, b, c, d, their sum is zero and they form a coset of $K = \langle a + b, a + c \rangle$, disjoint from K. The nonzero quotient support modulo K is bridgeless: a bridge in that support would violate conservation across its cut. A cycle through the vertex in this support permits a safe K-increment that matches its two used values to the two unused values. If all four values can become equal, the three pairing families intersect. Otherwise a balanced state has values a, a, b, b with $a \neq b$; the support modulo $\langle a + b \rangle$ has a cycle through the vertex, and its safe $(a + b)$-move links this family to a different one. The graph of nonempty pairing families has at most three vertices and no isolated vertex, so is connected. No rank-four assumption or bound on the number of odd vertices occurs in this local reasoning.

For the triangle step, remove loops and put

$$\Psi(G) = \sum_{v \in V(G)} \max\{\deg(v) - 4, 0\}.$$

Replace an even-degree vertex of degree $d \geq 6$ by a triangle with external groups $2, 2, d - 4$, giving degrees $4, 4, d - 2$. For odd $d \geq 5$, use groups $1, 2, d - 3$, giving degrees $3, 4, d - 1$. The odd count is preserved and Ψ decreases. Every prescribed exterior flow extends over the triangle: its three edge values have the form of one parameter plus fixed offsets, so at most three parameters are forbidden, fewer than q. An upstairs cycle projects to an even support downstairs; decompose it into edge-disjoint cycles and retain its coefficient. Each retained

edge remains old or final during this path, so no zero is created. A projected step may be stationary. These are the extension and projection properties of [Go, Lemma 21.1].

Lemma 22.3 (Pinning and gluing at two or three edges)**.** *For $A = \mathbb{F}_2^m$, $m \geq 3$, two joined flows agreeing on the star of a loopless vertex of degree two or three are joined by moves avoiding that vertex. Consequently a bridgeless graph with a two- or three-edge cut and connected shores has connected flow graph whenever both capped shores have connected flow graphs.*

Proof. Normalize a path between the two given flows, one move at a time. At a degree-two vertex the star values are a, a; a through-step of coefficient t excludes $0, a$, so choose $\ell(a) = 1$, $\ell(t) = 0$. At degree three the used star values are a, b and the unused one is $a + b$. Legality excludes $t = a, b$, so $\ell(a) = \ell(b) = 1$, $\ell(t) = 0$ is consistent: on their plane the remaining allowed nonzero value is $a + b$, where ℓ is zero, and outside that plane the functional extends.

The transvection $T(x) = x + \ell(x)t$ restores the changed star. If the move was on C, then

$$T(f + t\mathbf{1}_C) = f + t(\ell(f) + \mathbf{1}_C).$$

The difference support is even and avoids the star. Its two endpoints are nowhere zero, so the even-set argument realizes it by moves avoiding the vertex. Along the path retain the accumulated invertible linear map. At the last endpoint it fixes the span of the original star values pointwise. In a basis starting with that span, the pointwise stabilizer has block form $\left(\begin{smallmatrix} I & B \\ 0 & C \end{smallmatrix}\right)$. Row additions generate C, and shears into the fixed span generate B. Each generator has its functional zero on the star span and acts by safe cycles avoiding the star. Undoing the accumulated map completes the pinned path. This factorization works in rank three, and indeed already in rank two.

Now cap each shore of the small cut by a new vertex. Project two full flows to the capped sides and join the first pair of side projections. A step avoiding its cap lifts directly. For a step through two cap edges, the functional just constructed on the other side has support of degree two at its cap, on exactly the corresponding edges, and avoids the coefficient. A cycle decomposition supplies a cycle through these edges. Splicing the two cap-deleted paths gives a legal full-graph cycle; a zero-length side path can give a permitted digon. Thus the first side path lifts while the second stays in its ordinary component. At its end the second-side star agrees with the target, so the pinned assertion finishes on that side with the first side fixed. □

Lemma 22.4 (Wide minimal core)**.** *If a bridgeless graph has disconnected A-flow graph, let p be the minimum total number of odd-degree vertices among all such graphs. There is a counterexample with the* same *p which is connected, simple, cubic,* 3*-edge-connected, has only vertex-star three-cuts, and has girth at least q. Every bridgeless graph with fewer than p odd vertices has connected flow graph, with no restriction on its degrees.*

This lemma, and every result below that uses it, depends on Theorem 1.2(c) of the unpublished manuscript [Go].

Proof. The final statement is the original minimization, made before imposing any structural restrictions. Existence for all bridgeless graphs follows by embedding an order-eight flow, including independent loop values. The two-odd-vertex theorem [Go, Theorem 1.2(c)] gives $p \geq 4$.

If all subquartic graphs at this odd count were connected, induction on Ψ using the triangle extension and projection would give connectivity for arbitrary degrees at the same

count. Hence a subquartic counterexample at odd count p exists. Choose one with the fewest edges. Independent loops, isolated vertices, component products and suppression of degree-two vertices reduce to a connected loopless graph of degrees three or four. A bridge is impossible in a graph with a nowhere-zero flow. A nonempty balanced family at a degree-four vertex splits to fewer edges, preserving the odd count and the subquartic property. The split graph has a nowhere-zero flow from that family, so it is bridgeless and edge minimality applies. Its smaller flow space is connected, and the balanced lift joins that family. The local family-joining result then excludes degree four. Thus the graph is cubic on exactly p vertices.

A parallel pair is an induced digon unless it forms the whole triple-edge dipole; the latter has only two odd vertices and is already excluded. Contracting the digon reduces odd count by two. More generally a shortest cycle of length $\ell < q$ is induced, and its contraction leaves

$$p - \ell + (\ell \bmod 2) \leq p - 2$$

odd vertices. The quotient may have high degree, but it is bridgeless and the wide minimality applies. Lemma 20.2 lifts its connectivity, a contradiction. The core is therefore simple and has girth at least q.

For a cut of size two or three, each shore is connected: every shore component needs at least two cut edges in a bridgeless graph. Capping each shore gives a nowhere-zero flow by restriction of an original flow, so these capped graphs are bridgeless. For a two-cut its shore has even order and the cap has even degree, giving strictly fewer than p odd vertices on each side. For a nonstar three-cut both shores have odd order at least three; adding a cubic cap gives at most $p - 2$ odd vertices on each side. The wide minimality and Lemma 22.3 contradict disconnectedness in both cases. Only star three-cuts remain.

Every reduction above uses a local statement valid for $q \geq 8$, and preserves the outer minimum odd count where required. In particular, no use was made of the bound $2^{q/2+1} - 4$ of [Go, Theorem 1.4] as a theorem at larger odd counts. Apart from the existence input of Section 14, and the two-odd-vertex theorem [Go, Theorem 1.2(c)], which is used for $p \geq 4$ and for the triple-edge dipole, every step is proved above, including at $q = 8$. □

22.2. **The index graph of admissible pairs.**

Lemma 22.5. *On a cycle of length $r \geq 8$, form a graph whose vertices are pairs of nonincident cycle edges. Join two such pairs when their union is a matching of three cycle edges. This graph is connected.*

Proof. Index the edges by $0, \ldots, r-1$ modulo r. All pairs containing 0 are connected: $\{0,2\}$ is adjacent to $\{0,j\}$ for $4 \leq j \leq r-2$, and $\{0,3\}$ is reached via $\{0,5\}$, using the matching triples $\{0,2,5\}$ and $\{0,3,5\}$.

Take a pair $\{a,b\}$ without 0. If neither index neighbors 0, add 0 to obtain a matching triple and a link to a pair containing 0. If exactly one, say a, neighbors 0, the closed cycle neighborhoods of $0, a$ together have four indices and that of b adds at most three. Since $r \geq 8$, choose c outside their union. Then

$$\{a,b\} \longrightarrow \{b,c\} \longrightarrow \{0,b\}$$

are valid links. If both indices neighbor 0, the pair is $\{1, r-1\}$; the matching triple $\{1, r-1, 3\}$ joins it to $\{1,3\}$, which has exactly one neighbor of 0. These cases exhaust all admissible pairs. □

22.3. **Excluding a cycle whose length is the group order.**

Lemma 22.6 (Cycle-order exclusion)**.** *The core of Lemma 22.4 has no cycle of length exactly q.*

Proof. Suppose it has such a cycle C, with vertices $v_0, \dots, v_{q-1}$ and edges $e_i = v_i v_{i+1}$, indices modulo q. A chord would give a shorter cycle, so each third edge p_i at v_i has its other end outside C. Write $n = |V(G)| = p$.

Two-arc test graphs and their exact fibres. Choose nonincident e_a, e_b. Deleting them leaves a connected graph by 3-edge-connectivity. A bridge after deletion would give a cut contained in $\{e_a, e_b, f\}$; it must be a three-star and would make e_a, e_b incident. Thus the deletion is bridgeless. Contract its two remaining cycle arcs, deleting their internal edges, to obtain $Y(a,b)$. The arcs have vertex sets

$$v_{a+1}, \dots, v_b \quad \text{and} \quad v_{b+1}, \dots, v_a$$

in cyclic order. Its edges are exactly the edges of G outside C, and its two new vertices have degrees adding to q. It is bridgeless by contraction and has at most $n - q + 2 < n$ odd vertices. Thus $F(Y(a,b), A)$ is nonempty and connected by wide minimality and existence.

For a flow θ on $Y(a,b)$, let σ_i be the sum of its spoke values from p_{a+1} through p_i in cyclic order, with $\sigma_a = 0$. Conservation at the two contracted vertices gives $\sigma_b = 0$ and consistency after a full turn. All conserved extensions to G are

$$\phi_c(e_i) = c + \sigma_i, \qquad \phi_c|_{E(G)\setminus E(C)} = \theta, \qquad c \in A.$$

Put $\Sigma = \{\sigma_i : 0 \le i < q\}$ and $M = A \setminus \Sigma$. The repeated entries $\sigma_a = \sigma_b$ give $|\Sigma| \le q - 1$, so $M \neq \varnothing$. Exactly the constants in M give nowhere-zero extensions. Two distinct allowed constants are joined by adding their sum around C: a forbidden zero would require the other constant to lie in Σ. Each exact fibre is therefore a clique.

Conversely, a full flow has equal values on e_a, e_b if and only if its spoke sum on each arc is zero, which says its exterior assignment is a flow of $Y(a,b)$. Let $\mathcal{E}(a,b)$ denote this full-flow family.

Lifting a move and the repeated-anchor count. Let a downstairs move add $\lambda \neq 0$ on a cycle D of $Y(a,b)$. Each contracted vertex has D-degree zero or two. Let χ_i be the parity of the number of selected spokes among $p_{a+1}, \dots, p_i$. Then $\chi_a = \chi_b = 0$. Define $I = \{e_i : \chi_i = 1\}$ and $O = E(C) \setminus I$. Both $D \cup I$ and $D \cup O$, read as edge sets of G, are even: the change of χ at v_i equals whether $p_i \in D$. In particular $e_a, e_b \in O$.

The exterior edges of these supports already avoid λ by legality downstairs. For a chosen extension ϕ_c, its edges in $J = I$ or O avoid λ exactly when

$$c + \lambda \notin \Sigma_J, \qquad \Sigma_J = \{\sigma_i : e_i \in J\}.$$

Such a choice of $c \in M$ and J always exists. Otherwise, writing $s = |M| \ge 1$, all s distinct translates $c + \lambda$ would lie in $\Sigma_I \cap \Sigma_O$. But the two selected edges are in O with the same partial sum, so

$$|\Sigma_I| + |\Sigma_O| \le |I| + |O| - 1 = q - 1.$$

It would follow that

$$q - s = |\Sigma_I \cup \Sigma_O| \le |\Sigma_I| + |\Sigma_O| - s \le q - 1 - s,$$

a contradiction. Move within the fibre to the chosen c, then add λ on an edge-disjoint cycle decomposition of $D \cup J$. Each used edge remains old or final during the sequence, so every

cycle move is legal. The endpoint is an extension of the new exterior flow. Since e_a, e_b both lie in O, their equality holds again at the endpoint whichever part was chosen. If D visits neither contracted vertex, I is empty and the lift is simply D.

Every path downstairs consequently lifts to an ambient path connecting the appropriate fibres. Thus all flows of $\mathcal{E}(a,b)$ lie in one component of $F(G,A)$. Intermediate individual moves need not preserve their selected-edge equality; the argument does not need the induced subgraph on $\mathcal{E}(a,b)$ to be connected.

Triple intersections without a four-cut hypothesis. Choose three pairwise nonincident cycle edges e_a, e_b, e_d. Their deletion G_3 is connected: a separating cut among them would have size three and be a star, impossible for a matching. Contract its three remaining cycle arcs to form $Y(a,b,d)$. A *surviving* edge f of this graph is outside C. If it were a bridge, it would also be a bridge of G_3, since a path avoiding f projects to a walk avoiding it. One shore would give a cut B of G with

$$f \in B \subseteq \{f, e_a, e_b, e_d\}.$$

A cycle meets a cut evenly, and $f \notin E(C)$. Hence B contains zero or two of the selected cycle edges. Zero makes f a bridge of G; two makes a three-star containing two nonincident cycle edges. Both are impossible. This proves $Y(a,b,d)$ bridgeless. We do not need G_3 itself to be bridgeless: a remaining edge of C could be a bridge there and is contracted.

The three new vertex degrees add to the even number q, so at most two are odd. Thus the test graph has at most $n - q + 2 < n$ odd vertices. A nowhere-zero flow on it has zero spoke sum on each of the three arcs. As an exterior assignment it is also a flow on each associated two-arc test graph. Its partial sums satisfy $\sigma_a = \sigma_b = \sigma_d = 0$, so any allowed extension from the preceding construction gives one full flow with

$$\phi(e_a) = \phi(e_b) = \phi(e_d).$$

It is a common member of $\mathcal{E}(a,b)$, $\mathcal{E}(a,d)$ and $\mathcal{E}(b,d)$.

Coverage and completion. A full flow assigns only $q-1$ possible nonzero values to the q cycle edges, so some pair repeats a value. The repeated edges are nonincident: equal values at two edges of a cubic vertex would force its third value to be zero. Thus the admissible families $\mathcal{E}(a,b)$ cover all full flows. Lemma 22.5 connects their indices, and every index adjacency has a common full flow by the triple-intersection argument. Since each family lies in one ambient component, all full flows lie in that component. This contradicts the choice of the core and proves the exclusion. □

22.4. **Counting the resulting core and taking component products.**

Proof of Theorem 22.1 and Corollary 22.2. If a counterexample exists, Lemmas 22.4 and 22.6 give a simple cubic representative at the same minimum odd count p, with girth at least $q+1$. Put $r = q/2$. The breadth-first tree expansion from one vertex through depth r has distinct vertices: a collision of two nonbacktracking walks of length at most r would create a cycle of length at most $2r = q$. Hence

$$p = |V(G)| \geq 1 + 3\sum_{i=0}^{r-1} 2^i = 3 \cdot 2^{q/2} - 2 = B_*(q) + 2.$$

The count uses only the distinctness of the tree-expansion vertices. Edges between outermost vertices may create cycles of length $q+1$ and do not invalidate this count.

Consequently every bridgeless graph with at most $B_*(q)$ odd vertices in total has connected flow graph; the number of odd vertices is even. Apply this conclusion to each connected component separately. Its flow space is nonempty by the order-eight existence theorem embedded in A. Products over components, and independent adjustments of loop values, give the stated per-component theorem even when the whole graph has more than $B_*(q)$ odd vertices. At $q = 8, 16$ the formula gives $44, 764$, respectively. □

This general argument gives order-sixteen girth at least seventeen and at least 766 core vertices, and order-eight girth at least nine and at least 46 core vertices. It does not itself exclude a cycle longer than q. The next section gives a further special order-sixteen exclusion using the four-cut theorem. These remain restrictions on possible counterexamples; we do not know whether they are optimal.

23. A special order-sixteen bound from seventeen-cycle exclusion

We now fix $\Gamma = \mathbb{F}_2^4$. This section proves Theorem 23.1. Unlike the general bound in Section 22, its family-joining argument uses the four-cut reduction. All additions in this section are binary. Loops count twice in degree and contribute zero to conservation.

Theorem 23.1 (Special order-sixteen bound)**.** *If each connected component of a finite bridgeless multigraph H has at most* 1020 *odd-degree vertices, then $F(H, \mathbb{F}_2^4)$ is nonempty and connected.*

The proof uses [Go, Theorem 1.2(c)], an unpublished manuscript, through Lemma 22.4.

Suppose a counterexample exists, and minimize the total number p of odd-degree vertices among *all* bridgeless counterexamples. Lemma 22.4 supplies a connected simple cubic representative G with $n = |V(G)| = p$, three-edge connectivity and only vertex-star three-cuts. Every bridgeless multigraph with fewer than p odd vertices has connected flow graph, including disconnected graphs and graphs with high or even degrees. The existence input makes these smaller flow spaces nonempty. Lemmas 22.4 and 22.6 give girth at least seventeen, while Corollary 21.5 says that every four-edge cut of G is the boundary of two adjacent vertices.

Lemma 23.2 (Seventeen-cycle exclusion)**.** *The order-sixteen least core just described has no cycle of length seventeen.*

We prove the lemma in the following subsections. Suppose C is such a cycle, with vertices $v_0, \dots, v_{16}$, edges $e_i = v_i v_{i+1}$ and spokes p_i at v_i, indices modulo 17. A chord would give a cycle of length at most nine, so every spoke has its other end outside C. The exclusion proof needs only this chordlessness from the girth input; the preceding sixteen-cycle exclusion is also needed at the end to deduce girth eighteen. Two equal-valued cycle edges cannot meet, since their third incident edge would have value zero.

23.1. **Block families and their test graphs.** A *pattern* is a list of distinct cycle edges in cyclic order, in either direction,

$$\Pi = (f_1, \dots, f_r; g_1, \dots, g_t), \qquad (r, t) \in \{(3,0), (4,0), (2,2), (3,2)\}.$$

Edges within each block $B_1 = \{f_1, \dots, f_r\}$ and $B_2 = \{g_1, g_2\}$, when present, are pairwise nonincident; edges of different blocks may meet. Let $\mathcal{E}(\Pi)$ be the flows constant on each block. The two block values need not differ. Put $d = r - 1$ if $t = 0$, and $d = r + t - 2$ otherwise, so $d = 2, 3, 2, 3$ for the four shapes.

Deleting the selected cycle edges leaves arcs, possibly consisting of one vertex. The arcs between f_i, f_{i+1}, and between g_1, g_2 when present, are *inner*. The remaining arc, or the two remaining arcs together, is *outer*. Define $Y(\Pi)$ by retaining all edges outside C, replacing each inner arc by one vertex and identifying all outer-arc vertices into a single vertex z_0. Outside vertices remain distinct. The graph has no loops because C is chordless, but need not be assumed connected. At most four vertices replace seventeen odd vertices, so it has at most $n - 13 < p$ odd vertices. If it is bridgeless, wide minimality and existence make its flow graph nonempty and connected.

Lemma 23.3 (Exact fibres and move lifting)**.** *If $Y(\Pi)$ is bridgeless, then $\mathcal{E}(\Pi)$ is nonempty and all its flows lie in one component of $F(G, \Gamma)$. Conversely, if $\mathcal{E}(\Pi)$ is nonempty, then $Y(\Pi)$ is bridgeless.*

Proof. Conservation gives $\phi(e_i) = \phi(e_{i-1}) + \phi(p_i)$. Thus the selected block equalities are exactly zero spoke sum on every inner arc; total spoke sum zero then gives conservation at z_0. Consequently a full flow lies in $\mathcal{E}(\Pi)$ exactly when its exterior restriction is a flow of $Y(\Pi)$. This proves the converse, since a bridge cannot carry a nonzero flow value.

For a flow θ of $Y(\Pi)$, start at $f_1 = e_a$ and let σ_i be the successive spoke partial sums, with $\sigma_a = 0$ after a full turn. All conserved extensions have the form

$$\phi_c(e_i) = c + \sigma_i, \qquad \phi_c|_{E(G)\setminus E(C)} = \theta, \qquad c \in \Gamma.$$

The partial sums are zero on B_1 and constant on B_2. Writing $\Sigma = \{\sigma_i\}$, the d forced repetitions give $|\Sigma| \le 17 - d \le 15$. Exactly the constants in $M = \Gamma \setminus \Sigma$ give nowhere-zero extensions, so every fibre is nonempty. Two distinct extensions are joined by the whole-cycle move with coefficient $c + c'$: a zero would require $\sigma_i = c'$, impossible. Each exact fibre is a clique.

Consider a legal λ-move on a cycle D of $Y(\Pi)$. Let χ_i be the parity prefix of its selected spokes from $a + 1$ to i. Each inner arc, and all arcs together, have even selected-spoke count. Hence χ is zero on B_1 and constant on B_2. Put

$$I = \{e_i : \chi_i = 1\}, \quad O = E(C) \setminus I, \quad \Sigma_J = \{\sigma_i : e_i \in J\}.$$

Both $D \cup I$ and $D \cup O$, read in G, are even edge sets: at v_i, membership of p_i in D equals the change of χ. Their exterior edges avoid λ. For $c \in M$, the part on $J = I$ or O also avoids λ exactly when $c + \lambda \notin \Sigma_J$.

Each block lies wholly in one parity part, so all d repetitions remain within those parts and

$$|\Sigma_I| + |\Sigma_O| \le 17 - d \le 15.$$

If every allowed c failed for both parts, the $m = |M| \ge 1$ distinct values $c + \lambda$ would lie in their intersection. Then

$$16 - m = |\Sigma_I \cup \Sigma_O| \le |\Sigma_I| + |\Sigma_O| - m \le 15 - m,$$

impossible. Move within the original fibre to a safe c and add λ along an edge-disjoint cycle decomposition of the safe even set. Each edge retains its original value until its own cycle is used; all steps are legal. The endpoint lies over the new exterior flow and again respects the block equalities. Lifting a path in the connected smaller flow graph and using the fibre cliques proves the assertion. Intermediate moves may leave $\mathcal{E}(\Pi)$; only connectivity in the whole flow graph is shown. □

Lemma 23.4 (Bridge classification)**.** *A bridge f of $Y(\Pi)$ pulls back to a cut $Q \cup \{f\}$ of G, where f is outside C and Q consists of four selected edges. For shape $(4,0)$, $Q = B_1$; for $(2,2)$, $Q = B_1 \cup B_2$; for $(3,2)$, Q is two edges of B_1 together with B_2. Shape $(3,0)$ is always bridgeless.*

Proof. Pull back one bridge shore, also when Y is disconnected. Its cut contains f and no other edge outside C, and no unselected cycle edge. The shores immediately before and after each block agree: its outer arcs are identified at z_0. Thus each block contributes an even number of cut edges. Zero selected edges would make f a bridge of G. Two selected edges would be from one block and, with f, form a vertex-star three-cut, contradicting nonincidence within that block. The only remaining cases are the four-edge selections listed. □

Call a triple or quadruple of cycle edges *admissible* if its edges are pairwise nonincident. For an admissible triple t, its equal-value family $\mathcal{E}(t)$ is nonempty and joined by the two lemmas. Call an admissible quadruple Q *blocked* if $Q \cup \{f\}$ is a cut for some f outside C. If it is not blocked, there is a flow equal on all four edges. If it is blocked, no such flow exists, since its cut sum would be $\phi(f) \neq 0$.

23.2. **One component for triples and noncrossing pairs.**

Lemma 23.5 (Blocked-anchor window)**.** *For a fixed admissible triple t, at most three fourth edges extend it to blocked admissible quadruples.*

Proof. For two blocked extensions p, p', write their cuts as $t \cup \{p, f\}$ and $t \cup \{p', f'\}$. If $f = f'$, their symmetric difference is the forbidden two-cut $\{p, p'\}$. Otherwise it is the four-cut $\{p, p', f, f'\}$, which bounds adjacent vertices u, w by Corollary 21.5. If both are on C, chordlessness puts uw on C, and p, p' are its two neighbouring edges. If only one is on C, p, p' meet at it. If neither is on C, no cycle edge leaves them, impossible. Thus the indices of p, p' differ by one or two modulo 17.

A set of indices pairwise at distance at most two has size at most three. Relative to index 0, the others lie in $\{-2, -1, 1, 2\}$; the only compatible pairs among these are $\{-2, -1\}, \{-1, 1\}, \{1, 2\}$, with no compatible triple. □

Fix nonincident a, b and let P be the possible third anchors. The closed cycle neighbourhoods of a, b exclude five or six positions, so $|P| = 12$ or 11. Join $p, p' \in P$ when $\{a, b, p, p'\}$ is admissible and not blocked. Each p has at most two incident alternatives and three blocked alternatives, hence degree at least $|P| - 6$. Two nonadjacent vertices have a common neighbour because

$$2(|P| - 6) > |P| - 2.$$

An adjacency supplies a flow common to $\mathcal{E}(a, b, p)$ and $\mathcal{E}(a, b, p')$. Thus all triple families containing a, b lie in one component, denoted $S(a, b)$.

Given any two admissible pairs $\{a, b\}, \{a', b'\}$, choose d avoiding the closed neighbourhoods of their four edges, then d' avoiding those and d. At most twelve and fifteen of seventeen positions are excluded, respectively. The matching triples

$$\{a, b, d\}, \quad \{a, d, d'\}, \quad \{d, d', a'\}, \quad \{d', a', b'\}$$

have nonempty families and link $S(a, b)$ to $S(a', b')$. This works also if the pairs share an edge. Hence one component S_T contains every admissible triple family, and therefore every flow with a value occurring at least three times on C.

We need one more use of the four-cut classification. If h_1, h_2 are the two cycle edges at v, and e_1, e_2 are two other nonincident cycle edges, then

$$\{h_1, h_2, e_1, e_2, f\} \text{ is not a cut for any } f \notin E(C). \tag{8}$$

Indeed, take symmetric difference with the star $\{h_1, h_2, p_v\}$. If $f = p_v$ it is a two-cut. Otherwise the four-cut $\{e_1, e_2, f, p_v\}$ bounds adjacent u, w. If only one is on C, e_1, e_2 meet, impossible. If neither is on C, no cycle edge leaves. If both are on C, their joining edge is on C and their boundary spokes include p_v, so one is v. Then one of h_1, h_2 must leave $\{u, w\}$ and be e_1 or e_2, again impossible.

Lemma 23.6 (Noncrossing pairs)**.** *Every flow with two noncrossing equal-value pairs on C lies in S_T.*

Proof. Choose the pairs in cyclic order a, b, c, d, with $\phi(a) = \phi(b)$ and $\phi(c) = \phi(d)$. Their shape-$(2, 2)$ family is nonempty, so Lemma 23.3 joins the whole family. It suffices to find one member in S_T. Within-pair meetings are impossible. The only possible cross-pair meetings are b, c and d, a.

If there is no meeting, the quadruple is admissible and unblocked: a cut consisting of these four edges and f would have nonzero sum $\phi(f)$. A four-equal flow exists and joins the family to S_T.

Here is a common tool for the other cases. Take an edge e, nonincident with all four anchors, on the open a–b arc or the open d–a arc. The patterns $(a, e, b; c, d)$ and $(e, a, b; c, d)$ are of shape $(3, 2)$. A flow in either is a triple flow in the original two-pair family. By Lemma 23.4, the only bridge obstructions are

$$\{a, b, c, d, f\}, \quad \{b, e, c, d, f\}, \quad \{a, e, c, d, f\}.$$

The first is excluded by the given flow. If b, c meet, the second is excluded by (8); if d, a meet, the third is excluded too.

If both meetings occur, the two inner arcs contain thirteen edges in total. Rotate (a, b, c, d) to (c, d, a, b) if necessary so the a–b arc has at least seven edges. An edge other than its first or last meets none of the anchors. Both remaining obstructions are excluded, so its test graph is bridgeless and supplies the required triple flow.

If exactly one meeting occurs, rotate the names so it is b, c. Let the open a–b, c–d, d–a arcs have lengths $p', q', r' \geq 1$, with $p' + q' + r' = 13$. Put

$$A = \max(p' - 2, 0) + \max(r' - 2, 0), \qquad B = \max(q' - 2, 0) + \max(r' - 2, 0).$$

At least one is at least four. If $r' \leq 2$, one of p', q' is at least six. If $r' \geq 3$, then $A + B \geq r' + 5 \geq 8$. Reversing the names to (d, c, b, a) exchanges A, B and keeps the same meeting pattern, so assume $A \geq 4$.

These A choices are exactly the eligible e on the two arcs of the common tool. The meeting at b, c excludes the second bridge obstruction. For the third, $\{a, c, d\}$ is one fixed admissible triple across *both* candidate arcs. Lemma 23.5 blocks at most three choices. Some choice therefore has a bridgeless test graph and a triple flow in the original family. This finishes all cases. □

23.3. **Crossing configurations, a switch, and ears.** Call a flow of *type T* if some value occurs at least three times on C, and of *type N* if it is not of type T and has two noncrossing pairs. All other flows have *type X*: every value occurs at most twice and all repeated pairs

cross pairwise. If k values repeat, there are $2k$ *anchors*, $17-2k$ singleton edges and $17-k$ distinct used values. Thus

$$2 \le k \le 8, \qquad \text{the number of missing nonzero values is } k-2.$$

Write X_k for this type. Call an X_k flow *shut* if no single legal move takes it to type T or N. The arguments below analyze this condition only.

For an X_k flow, the anchors have cyclic values

$$r_1, \dots, r_k, r_1, \dots, r_k. \tag{9}$$

Indeed, each side of one repeated pair contains exactly one anchor of each other pair, so the partner of the jth anchor is the $(j+k)$th. The open intervals between consecutive anchors, possibly empty, are *anchor arcs*; each borders two repeated labels.

Lemma 23.7 (Two-pair switch)**.** *In an X_k flow, let x, y be distinct repeated values and $s = x + y$. If a cycle avoiding s contains a nonempty proper subset of their four anchors, adding s on it gives type T or N.*

Proof. The four anchors occur with values x, y, x, y. The move swaps x, y on its selected anchors. Selecting one or three produces a triple. Selecting two of the same old value makes all four equal. Selecting one of each changes the crossing pairing to one of the two noncrossing pairings. Other changed edges cannot erase these selected equalities. In particular, for a shut flow such a cycle contains all four anchors or none. □

For a nonzero linear functional $\ell : \Gamma \to \mathbb{F}_2$, the edges whose values have ℓ-value one form a vertex-disjoint union of cycles: conservation and cubicity give degree zero or two. Denote this support by S_ℓ. Let L_1, L_0 be the repeated values with functional value one and zero. A nonzero kernel value outside L_0 occurs on C at most once. A *zero gap* is a maximal nonempty run of ℓ-zero cycle edges. At each gap endpoint the spoke has ℓ-value one.

Lemma 23.8 (Ear construction)**.** *Let ϕ be shut, and suppose $|L_1| \ge 2$. Form a graph on L_1 joining w, w' when $w + w' \notin L_0$, and suppose it is connected. Then:*

(i) *One cycle D of S_ℓ contains every ℓ-one edge of C.*

(ii) *Every maximal path Q of D outside C is an* ear*: its endpoints are distinct vertices of C, its interior avoids C, and all its values have ℓ-value one. Of the two endpoint paths on C, one, J', contains all L_1 anchors; the other, J, contains none.*

(iii) *If a singleton ℓ-one value u admits $w \in L_1$ such that $u + w$ is absent from J or absent from J', its edge lies on J'. In particular it suffices that $u + w \notin L_0$.*

(iv) *If the preceding condition holds for every ear and every singleton ℓ-one value, every ear closes a whole zero gap and every zero gap J has its closing ear Q_J. The union $Z_J = J \cup Q_J$ is a cycle. Every nonzero $\lambda \in \ker \ell$ absent from J gives a legal move on Z_J, changing precisely the edges of J on C.*

Proof. Choose a joined pair $x, y \in L_1$ and the cycle D of S_ℓ through an x-anchor. It avoids $s = x + y$, so Lemma 23.7 puts both x- and both y-anchors on D. For any other ℓ-one value $u \ne x$ occurring on C, the coefficient $u + x$ is a nonzero kernel value. Its legal move on D sends the two x-anchors to u. An original u-edge outside D would create a triple. Hence every such occurrence is on D, proving (i).

If C has no ℓ-zero edge, $D = C$ and there are no ears or gaps. Otherwise, at an endpoint of a zero gap, D leaves C along the spoke. An internal ear vertex cannot lie on C: it would

require two edges outside C at a cubic cycle vertex. The endpoints are distinct because the ear is a proper path of D. This remains valid for a chord, which is a one-edge ear.

The value s is in the kernel and outside L_0, so it occurs on C at most once. One endpoint path P of an ear therefore avoids it, as does the ear itself. The switch on $Q \cup P$ puts all four base anchors on one endpoint path J' and none on the other J. Let $W \subseteq L_1$ be the labels with both anchors on J'. It contains x, y. If $w \in W$ is joined to $u \in L_1$, the coefficient $u + w$ again has a free endpoint path. If that path is J', the two w-anchors become u and any u-anchor on J would give a triple. If it is J, that anchor becomes w while the original two w-anchors stay fixed. Thus $u \in W$. Connectedness propagates this to all of L_1, proving (ii).

The same two tests apply to a singleton u under the free-path condition in (iii): either its edge on J stays u while two w-anchors become u, or it becomes w while those two anchors remain. Hence it must be on J'.

Under (iv), J contains no ℓ-one edge. At both endpoints the continuation on J' is ℓ-one, so J is a maximal zero gap. Conversely, D leaves C at any endpoint of a gap, starting an ear. That ear closes a gap at the same endpoint, which can only be the original gap. Finally its ear values are outside the kernel and its gap avoids λ, proving legality and the exact changed portion of C. □

23.4. **Exactly two crossing pairs.**

Lemma 23.9. *Every X_2 flow has a one-move exit to type T or N.*

Proof. Suppose it is shut. Its four anchors have cyclic values x, y, x, y, and all thirteen other nonzero values occur once. Put $s = x + y$, $W = \langle x, y \rangle$ and $g = \eta(s)$, where $\eta(v)$ denotes the unique edge of a singleton value. The four anchor arcs are A_1, A_2, A_3, A_4 in order. A pair of other edges crosses both original pairs exactly when its edges lie in opposite arcs: A_1, A_3 or A_2, A_4.

Take any of the four functionals with $\ell(x) = \ell(y) = 1$. Here $L_1 = \{x, y\}$ and $L_0 = \varnothing$, so all hypotheses of Lemma 23.8 hold. The seven nonzero kernel values occur once each on the zero gaps. Let X_J be the values of a gap J. If $\lambda \in \ker \ell \setminus \{0\}$ is absent from J and $v \in X_J$ has $w = v + \lambda \notin X_J$, the gap move creates the pair $\{\eta(v), \eta(w)\}$. Both original pairs remain unchanged, so the new pair must cross both. Its edges lie in opposite anchor arcs.

For two different gaps choose v, w on them and $\lambda = v + w$. Their singleton value sets are disjoint, so λ is absent from at least one of the two gaps. Apply the preceding move there. Thus any two gaps lie in opposite arcs, and there cannot be three gaps. If A_g is the arc containing g, all zero edges lie in A_g and its opposite A'_g, in one interval in each; the interval in A_g contains g.

For $w \notin W$, let $Z(w)$ be the set of these four functionals vanishing at w. It has size two. To see both its size and its multiplicities, write the four functionals as $\ell_0 + m$, with m in the two-dimensional annihilator of W. Evaluation at $w \notin W$ takes each binary value twice. Equality $Z(w) = Z(w')$ fixes both the evaluation functional on this annihilator and $\ell_0(w)$; hence $w' \in w + W$ and $\ell_0(w') = \ell_0(w)$. This leaves exactly $w' = w$ or $w + s$. Each zero set therefore belongs to just two values.

Every one of the twelve outside-W singleton edges lies in A_g or A'_g, because it is zero for some functional. Together with g these are all thirteen singleton edges. On either side of g in A_g, the sets $Z(w)$ are nested as one moves away from g: a zero interval containing a farther edge and g contains all intermediate edges. Since the sets all have size two, they are equal along that side. At most two edges occur on each side, so $|A_g| \leq 5$.

In A'_g, group consecutive equal zero sets into blocks. Each block has at most two edges. Each functional occurs in a single interval, so every change of block introduces a functional not used in any previous block: a vanished one cannot reappear. Starting with two of four functionals allows at most two changes, hence three blocks and $|A'_g| \le 6$. The other two arcs are empty. The cycle has at most $4 + 5 + 6 = 15$ edges, contradicting seventeen. □

23.5. **One exceptional repeated value.**

Lemma 23.10 (One exception)**.** *Let a shut X_k flow, $k \ge 3$, admit a functional zero on one repeated value z and one on all the others. For $k = 3$ assume additionally that z is not the sum of the other two. There are exactly two zero gaps, each containing one z-anchor and no other anchor. Every singleton edge whose value is mapped to zero by the functional therefore lies in an anchor arc bordering z.*

Proof. Here $L_0 = \{z\}$. For $k = 3$ the safe-sum graph on the two one-labels is connected by hypothesis. For $k \ge 4$, any missing adjacency $w + w' = z$ is replaced by a path through a third label w'', since neither $w + w''$ nor $w' + w''$ equals z. For any singleton one-value u, its sums with two distinct one-labels differ, so one is not z. Lemma 23.8 applies in full.

The two z-anchors are separated on both sides by one-support anchors, so belong to different gaps J_a, J_b. A third gap J would contain only singleton values. Choose v there. If $v + z$ is absent from J, its gap move creates a third z-edge. Otherwise $v + z$ is a singleton on J and is absent from both J_a, J_b. Use it separately on these two gaps. Each test makes a new v-pair with the same untouched edge $\eta(v)$ and preserves every one-support pair. Fix one such pair. The two z-anchors lie on opposite sides of it, so $\eta(v)$ cannot be opposite both. One test produces noncrossing pairs, a contradiction. Hence only the two anchor gaps exist, proving the last assertion too. □

Lemma 23.11 (Three crossing pairs)**.** *An X_3 flow reaches type T or N in at most two moves, and in one move if its three repeated values are independent.*

Proof. Let the repeated values be x, y, z. First assume independence and shutness. For each exceptional label z_0 among them, there are two functionals taking zero on z_0 and one on the others w_1, w_2. Both are one at precisely

$$w_1, \quad w_2, \quad w_1 + z_0, \quad w_2 + z_0.$$

Indeed their difference has kernel $\langle x, y, z\rangle$, and these are its four elements where either prescribed functional is one. Lemma 23.10 applies. A singleton whose value is neither $w_1 + z_0$ nor $w_2 + z_0$ must lie in an anchor arc bordering z_0. A singleton outside $\{x + y, x + z, y + z\}$ would therefore have to border all three labels, impossible. Eleven distinct singleton values cannot lie in a set of size three.

If the labels are dependent, their distinctness and nonzeroness give $x + y + z = 0$. There is exactly one missing nonzero value c. The legal whole-cycle c-move preserves every equality class; its new repeated values are distinct, nonzero and sum to $c \ne 0$, hence independent. The first case supplies one further move. □

Lemma 23.12 (Four labels with nonzero sum)**.** *An X_4 flow whose four repeated values have nonzero sum has a one-move exit to type T or N.*

Proof. Suppose it is shut. Four distinct nonzero binary values have rank four, or rank three with a unique relation of size three or four. The nonzero-sum hypothesis excludes the last case.

In rank four, index the basis $r_1, \dots, r_4$ by its repeated cyclic order. Let $\ell_i(r_i) = 0$ and $\ell_i(r_j) = 1$ for $j \neq i$. A singleton in the arc bordering r_i, r_{i+1} can have zero under ℓ_j only for $j = i, i+1$, by Lemma 23.10. Write its value as the sum over a subset U of the basis. Weight one is a repeated value, and weight zero is prohibited. At weight two the zero indices are exactly the two complementary indices, which must be cyclically adjacent; four values qualify. At weight three there are three zero indices, impossible. At weight four one value qualifies. Thus at most five distinct singleton values are possible, against nine.

In rank three, write $R = \{y_1, y_2, y_3, t\}$ with $y_1 + y_2 + y_3 = 0$ and t outside their plane, and let $H = \langle R \rangle$. For each exceptional $z_0 \in \{y_1, y_2, y_3\}$, prescribing zero there and one on the other three is consistent on H. Its two extensions to Γ differ by the functional with kernel H. Every value outside H is zero for one extension for each of the three choices of z_0. Its singleton edge would have to border three labels, impossible. But there are eight outside-H values and only two missing values, so at least six such singleton edges occur. This is a contradiction. □

Lemma 23.13 (Five labels with a five-term relation)**.** *An X_5 flow with rank-four repeated values whose unique relation is the sum of all five has a one-move exit to type T or N.*

Proof. Suppose it is shut, with labels $r_1, \dots, r_5$. For each i, the functional zero at r_i and one at the four others exists: its assigned values respect the sole relation. The ten sums $r_a + r_b$, $a < b$, are distinct and are not repeated values, since any contrary equality would give a relation of size at most four. They exhaust the ten remaining nonzero values. Such a sum is zero under the three exceptional-label functionals with indices outside $\{a, b\}$. Lemma 23.10 would require its singleton arc to border three labels. Thus no singleton could exist, whereas there are seven. □

23.6. **Two exceptional repeated values and the remaining small cases.**

Lemma 23.14 (Direct gap swap)**.** *Let an X_k flow, $k \geq 4$, have exactly two repeated values z, z' in $\ker \ell$. Suppose every zero gap has a closing ear in S_ℓ, with no cycle edge on that ear. Then a single move gives type T or N.*

Proof. The nonzero kernel value $w = z + z'$ is different from both exceptions, so is not a repeated value and occurs on C at most once. Between the two occurrences of either exceptional label on each side there is a one-support anchor, by (9) and $k \geq 4$. Thus a gap contains at most one anchor of each exceptional value. The four exceptional anchors occupy at least two gaps, and at most one of those gaps contains an edge valued w.

Choose an anchor-containing gap avoiding w. Its ear also avoids w, so the gap cycle admits a legal w-move. If the gap contains one exceptional anchor, it changes to the other value, whose two original anchors remain: a triple. If it contains one anchor of each, their values swap while their partners stay fixed. In cyclic order these four were h, h', g, g', with h, g equal and h', g' equal. The new pairs are $\{h', g\}$ and $\{h, g'\}$, which do not cross. Other changed cycle edges cannot erase these equalities. □

Lemma 23.15 (Four labels with zero sum)**.** *Every X_4 flow whose repeated values sum to zero has a one-move exit to type T or N.*

Proof. Suppose it is shut. Index the repeated values by cyclic order. The first three are independent, since their sum is the distinct nonzero fourth value. Choose $\ell(r_1) = \ell(r_3) = 1$, $\ell(r_2) = 0$; then $\ell(r_4) = 0$. The safe-sum graph on $\{r_1, r_3\}$ is connected because $r_1 + r_3 = r_2 + r_4$ differs from both exceptions. Parts (i),(ii) of Lemma 23.8 apply.

For an ear, its path J without one-support anchors lies between consecutive anchors of r_1, r_3. It contains at most one occurrence of r_2 or r_4, and therefore omits at least one of these *values* entirely. For a singleton one-value u, if either $u + r_1$ or $u + r_3$ is outside $\{r_2, r_4\}$, it has a free endpoint path. Otherwise these two distinct sums are exactly the two exceptions, one absent from J. Thus part (iii) applies even in this latter case. Part (iv) gives every gap its ear, and Lemma 23.14 contradicts shutness. □

Lemma 23.16 (All five-pair cases)**.** *Every X_5 flow reaches type T or N in at most two moves.*

Proof. Its five distinct nonzero labels have rank three or four. In rank four there is exactly one nontrivial relation, of size three, four or five. The size-five case is Lemma 23.13. In the other rank-four cases suppose the flow is shut.

If four labels sum to zero, name them z, z', x_1, x_2, and name the fifth r_e, outside their span. Assign functional values zero to z, z' and one to x_1, x_2, r_e. The assignment respects the sole relation. The safe-sum graph is connected: $x_1 + x_2 = z + z'$ is neither exception, while $r_e + x_1$ and $r_e + x_2$ are outside the four-label span. For a singleton one-value u, the three distinct sums with x_1, x_2, r_e cannot all lie in the two-element exception set. Thus all hypotheses of Lemma 23.8 hold, and Lemma 23.14 gives a contradiction.

If three labels satisfy $x_1 + x_2 + z = 0$, call the other two q_1, q_2. The values x_1, x_2, q_1, q_2 are independent. Choose the functional one on x_1, x_2, q_2 and zero on q_1; it is also zero on z. The safe-sum graph contains the path x_1, q_2, x_2: either forbidden sum would give a second relation or identify q_2 with a plane label. Again the three candidate sums for a singleton cannot all be the two exceptions. The ear lemma and direct gap swap contradict shutness.

Finally suppose the rank is three. Let U be the order-eight span and m_1, m_2 its two nonzero elements not in the repeated set. There are three missing values on C, at most two in U, so choose a missing $c \notin U$. The whole-cycle c-move preserves the equality pattern. Every odd-size sum of the five translated labels lies in $c + U$ and is nonzero, and a two-term sum is nonzero by distinctness. The sum of the original five labels is $m_1 + m_2$. Exactly one four-label subset sums to zero: the one omitting $m_1 + m_2$, which belongs to the repeated set because it is nonzero and differs from m_1, m_2. Translation preserves four-term sums. Thus the translated set has exactly one relation, of size four, and has rank four. The previous case gives a second move to T or N. □

23.7. **Removing a pair when six or more values repeat.** An anchor is *flanked* if both neighbouring edges on C are anchors. The singleton edges can prevent at most $2(17 - 2k)$ anchors from being flanked. Hence there are at least

$$2k - 2(17 - 2k) = 6k - 34 \tag{10}$$

flanked anchors: at least two, eight and fourteen when $k = 6, 7, 8$.

Lemma 23.17 (Pair removal)**.** *Let a shut X_k flow, $k \geq 3$, have a flanked anchor e of value z, with neighbouring anchor values x, y. Suppose $x + y$ is not a repeated value, and some missing value c' differs from both $x + z$ and $y + z$. Then one move gives type X_{k-1}.*

Proof. The labels x, y, z are distinct by (9). There is a functional with

$$\ell(x) = \ell(y) = 1, \qquad \ell(z) = \ell(c') = 0.$$

Indeed, consistency could fail only if a relation among these four values contained exactly one of x, y. This would say that x or y belongs to $\{0, z, c', z + c'\}$. Nonzeroness, distinctness and missingness exclude the first three possibilities; the two exclusions on c' exclude the fourth.

Let u, u' be the ends of e, and let e^- of value x meet e at u, while e^+ of value y meets it at u'. In the one-support cycle D through e^-, the edge e is absent and the spoke at u is present. Follow that spoke along D to its first subsequent vertex v' on C. This gives a path Q with no cycle edge or internal cycle vertex; $v' \neq u$. All its values are one-support.

Let J be the endpoint path on C starting from u with e, and J' the other, starting with e^-. The nonzero value $s = x + y$ is in the kernel and occurs on C at most once. One of $Q \cup J$, $Q \cup J'$ avoids it. The switch lemma puts all four x/y anchors on one endpoint path and none on the other. Since $e^- \in J'$, the path J contains none, in particular not e^+. But e^+ is the very next edge after e, so $J = \{e\}$ and $v' = u'$.

The cycle $\{e\} \cup Q$ avoids $\lambda = z + c'$: its ear has functional value one, whereas $\ell(\lambda) = 0$, and $\lambda \neq z, 0$. Add λ there. Exactly e changes on C, from z to the previously missing c'. The other z-edge is now a singleton, all other pairs are untouched, and the remaining $k - 1$ pairs still cross pairwise. □

Lemma 23.18 (Descent for six through eight pairs)**.** *An X_k flow with $k = 6, 7, 8$ reaches type T, N or X_{k-1} in at most two moves.*

Proof. Let R be the set of its k repeated values and let A_0 contain zero and all missing values. Thus $|A_0| = k - 1$ and $A_0 \cap R = \varnothing$. Choose a flanked anchor by (10), with neighbouring values x, y, and put $s = x + y$. There is $\kappa \in A_0$ with $s + \kappa \notin R$. Otherwise the $k - 1$ elements of $s + A_0$ would lie in R while excluding both x and y: their preimages are y and x, outside A_0. They cannot fit in the other $k - 2$ elements.

If $\kappa = 0$ do nothing; otherwise add κ on the whole cycle, a legal move because it is missing. The new flow ψ has the same equality classes and flanked anchors, repeated set $R + \kappa$, and zero together with its missing values equal to $A_0 + \kappa$. Its neighbour sum remains s and is not repeated because $s + \kappa \notin R$. At least $k - 2 \geq 4$ missing values remain, so one avoids the two excluded sums with the central anchor in Lemma 23.17. If ψ is not shut it has a one-move T/N exit; if it is shut the pair-removal lemma gives X_{k-1}. Including the optional whole-cycle move gives the bound. □

23.8. **Completion and the per-component bound.**

Completion of the proof of Lemma 23.2. Every X_k flow joins a T or N flow by induction on k. The base cases $k = 2, 3, 4, 5$ are Lemmas 23.9, 23.11, 23.12, 23.15, 23.13 and 23.16. For $k = 6, 7, 8$, Lemma 23.18 either exits directly or decreases k. Every flow on C is of type T, N or X_k with $2 \leq k \leq 8$. The family argument and Lemma 23.6 put all T/N flows in the same nonempty component S_T, and the local exits put every remaining flow there too. This contradicts the core's disconnected flow graph and excludes the seventeen-cycle. □

Proof of Theorem 23.1. If any bridgeless counterexample exists, choose the wide minimum odd count p and its simple cubic representative. Lemmas 22.4 and 22.6 at $q = 16$ and Lemma 23.2 give girth at least eighteen. Root nonbacktracking binary expansions at both ends of an edge uv, excluding that edge initially, to depth eight on each side. A collision within one expansion gives a cycle of length at most sixteen; a collision between sides gives one of length at most seventeen using uv. Hence all expansion vertices are distinct and

$$p = |V(G)| \geq 2 \sum_{i=0}^{8} 2^i = 2 \cdot 511 = 1022.$$

Only distinctness of these vertices is required. Additional edges between outer vertices may produce eighteen-cycles and are allowed.

Thus every bridgeless multigraph with at most 1020 odd vertices in total has connected flow graph: the number of odd vertices is even, so this is the effective bound strictly below 1022. Its flow graph is nonempty by the existence input. Apply this total-count conclusion separately to each connected component of the graph in the theorem. A cycle move belongs to one graph component, so the full flow graph is the Cartesian product of the component flow graphs. Each loop has an independent nonzero value, and any two such values are joined by a legal move on that loop; loops do not change degree parity. These products prove the stated nonempty, connected conclusion for *each-component* odd count at most 1020, allowing both loops and parallel edges. □

Theorem 23.1 improves the order-sixteen consequence of Corollary 22.2; the order-eight bound remains 44. Its four-cut dependency occurs in the family-joining argument, not in the local crossing exits. It supplies no exclusion of an eighteen-cycle and no universal order-sixteen conclusion.

24. Edge-and-claw factors and cyclic edge cuts

A *cyclic edge cut* is an edge cut whose two shores each contain a cycle. This section proves connectivity at order sixteen for simple bridgeless cubic graphs with no cyclic edge cut of size at most ten. Its graph-theoretic ingredient is a spanning factor whose components are edges and claws. Such a factor is called a *perfect pseudo-matching* in [FBK]. We give the complete proof needed here, without a priority claim for the factor theorem.

24.1. **The factor theorem and its parity base.** Let $\mathcal{Q}$ be the class of finite simple connected bridgeless graphs of even order whose degrees belong to $\{2,3\}$. An *edge-and-claw factor* is a spanning subgraph with every component isomorphic to K_2 or $K_{1,3}$. A vertex of degree one in the factor is *light*, and a vertex of degree three is a *centre*.

Theorem 24.1 (An edge-and-claw factor with a prescribed edge condition)**.** *For every $G \in \mathcal{Q}$ the following hold:*

(i) *For every edge e, there is an edge-and-claw factor avoiding e.*
(ii) *For every edge e with a degree-two endpoint in G, there is an edge-and-claw factor in which e is a K_2 component.*

We call these demands (A) and (K), respectively. The latter can be false for an edge with two degree-three endpoints: the edge between the degree-three vertices of K_4 minus an edge cannot be a K_2 component of such a factor.

Lemma 24.2 (Parity designs)**.** *Let L be a connected multigraph with edges designated odd or even. Suppose $\pi : V(L) \to \mathbb{F}_2$ has sum congruent modulo two to the number of odd edges. One can assign each odd edge a head and choose a set U of even edges so that, at every vertex v,*

$$\#\{\text{odd edges with head } v\} + d_U(v) \equiv \pi(v) \pmod 2.$$

If a designated even edge g is not a bridge, the choice may satisfy $g \notin U$.

Proof. First, a connected multigraph admits an orientation with prescribed indegree parities exactly when their sum agrees with its edge count modulo two. Start with any orientation. The set of vertices with wrong parity has even size; pair them by paths and reverse these paths, taking the symmetric difference of their edge sets if they overlap.

Let L_i be the components of the spanning odd-edge subgraph, including isolated vertices. Put

$$\tau_i = \sum_{v \in V(L_i)} \pi(v) + |E(L_i)| \pmod 2.$$

The sum of the τ_i is zero. Contract the L_i and, if required, delete g first. The resulting graph is connected, since g is not a bridge. Solve the degree-parity demands τ_i on a spanning tree to obtain U, using only even edges and avoiding g. Within each L_i, the desired indegree parities $\pi(v) + d_U(v)$ now have sum $|E(L_i)|$ modulo two. The orientation observation supplies the heads. Loops in the contracted graph are unnecessary and cause no exception. □

Lemma 24.3 (No adjacent degree-three vertices)**.** *Theorem 24.1 holds when no edge of $G \in \mathcal{Q}$ joins two degree-three vertices.*

Proof. If every degree is two, G is an even cycle and its two perfect matchings give both demands. Otherwise suppress all degree-two vertices to obtain a connected bridgeless cubic multigraph H. It has no loop: a loop would leave the third edge at its vertex as a bridge. Also $H - p$ is connected for every vertex p, since each of two components would need at least two edges to p to avoid a bridge.

Every edge $h = pq$ of H corresponds to a run

$$p, z_1, \ldots, z_{\ell(h)}, q$$

in G, where $\ell(h) \geq 1$. Call h odd or even according to $\ell(h)$, let O be the set of odd edges, and put $n = |V(H)|$. Even order of G gives $n \equiv |O| \pmod 2$. On each run, a factor's edges alternate. For odd h, exactly one terminal edge is selected; assign its end as the head of h. For even h, either both terminal edges or neither are selected; let U consist of the former edges. Thus a design with odd count at each hub gives degree one or three there. Every neighbour of a hub in the factor is a light run vertex. Consequently the resulting subgraph is an edge-and-claw factor.

Two patterns suffice to control any chosen edge on the run of $h = pq$. If h is even, pattern N leaves both terminal edges unselected. It follows from Lemma 24.2 on H with $\pi = 1$ and $g = h$.

Pattern P at p selects the terminal edge of h at p and leaves the other two incidences at p unselected, so p is light. For odd h assign its head to p; for even h put $h \in U$. Every other odd edge at p has its head at its other end, and every other even edge at p is outside U. On $H - p$ use parity demand $1 + c(v)$, where $c(v)$ counts these already selected incidences at v. Its total is

$$n - 1 + \mathbf{1}_{h \text{ even}} + d_O(p) - \mathbf{1}_{h \text{ odd}} \equiv |O| - d_O(p) \pmod 2.$$

This is the number of odd edges of $H - p$ modulo two, so Lemma 24.2 applies. This also covers the case that $H - p$ has just one vertex.

For an odd run, patterns P at its two ends give its two alternating selections. For an even run, P and N give them. Hence any specified run edge can be avoided. A selected internal run edge is automatically a K_2 component, since both endpoints have degree two in G. For a selected terminal edge, use P at its hub to make that hub light. This proves (A) and (K). □

If e joins two degree-three vertices and $G - e$ is bridgeless, then $G - e \in \mathcal{Q}$. Any factor of $G - e$ avoids e, and either demand for another edge transfers from $G - e$ to G. In particular an original degree-two endpoint in demand (K) is not touched by deleting e. This is the secondary induction step on edge count.

24.2. **Two-edge cuts and partial factors.** Suppose $\{\varepsilon, \varepsilon'\}$ is a two-edge cut with connected shores X, Y of size at least three, where $\varepsilon = xy$ and $\varepsilon' = x'y'$, with $x, x' \in X$. The terminals x, x' are distinct, as are y, y': otherwise the remaining part of a shore would attach to the common terminal through a bridge.

A partial factor on X is a spanning subgraph of $G[X]$ whose components are edges, claws, and the following possible exceptions: an isolated terminal, or a two-edge path centred at a terminal. A terminal has state E, C, H when its partial degree is $0, 1, 2$, respectively. In state H its two neighbours are leaves; it has degree three in G. A terminal in state C may be a leaf of the other terminal's two-edge path.

The following sufficient gluing rule will be used throughout. Include a cut edge only with endpoint states E, E, E, H or H, E; omit it only with endpoint states C, C. The included edge either makes a K_2 or supplies the third leaf of a claw. The union is then an edge-and-claw factor. An existing internal K_2 remains a component. In particular one must never join H to H.

Close the shore X by one of three caps:

(i) $G_X(1)$ adds a new path x, z, x'.
(ii) $G_X(2)$ adds a new path x, z_1, z_2, x'.
(iii) $G_X(D)$ adds new vertices u, u', a, b and edges

$$xu, \quad x'u', \quad ua, \quad ub, \quad u'a, \quad u'b.$$

All three graphs are simple and connected, preserve the old degrees, and are bridgeless. Indeed, an internal bridge of $G[X]$ must separate the terminals, and each cap reconnects them; its own edges lie on cycles. Their orders are $|X| + 1$, $|X| + 2$, $|X| + 4$. Restricting a factor to X gives the following partial states:

cap	possible pairs of states
1	$(E \text{ or } H, C)$, $(C, E \text{ or } H)$
2	(C, C), $\{E, H\} \times \{E, H\}$
D	(C, C), $(E, E \text{ or } H)$, $(E \text{ or } H, E)$.

For cap 1, the new degree-two vertex selects one edge. For cap 2, selecting $z_1 z_2$ gives C, C; otherwise both terminal edges are selected. For cap D, each of a, b selects one neighbour. Different choices give C, C. Equal choices make one of u, u' a claw centre and force state E at the corresponding old terminal. In particular cap D never supplies H, H.

Lemma 24.4 (States supplied by a smaller shore closure)**.** *Suppose Theorem 24.1 holds for all members of $\mathcal{Q}$ with fewer vertices than G. In the preceding cut setting:*

(i) *If $|Y|$ is odd, Y supplies states (E, C) and (C, E).*
(ii) *If $|Y|$ is even, Y supplies (C, C), (E, t') and (t, E) for some $t, t' \in \{E, H\}$. If y has degree two in G, it supplies (E, E).*

Proof. For odd $|Y|$ use $G_Y(1)$, and apply (K) to either terminal edge of its cap. For even $|Y|$ use $G_Y(2)$: demand (K) for the middle edge gives C, C, and for either terminal edge it gives E, t' or t, E. If y has degree two, $t \neq H$, yielding E, E. The chosen closure has even order and fewer vertices than G, since $|X| \geq 3$. □

Lemma 24.5 (Reduction across a two-edge cut)**.** *Under the same induction assumption, a demand (A) or (K) for an edge e_0 of G holds in each of these cases:*

(i) *e_0 is a cut edge;*
(ii) *both shores have odd order;*

(iii) $e_0 \in E(G[X])$, $|Y|$ *is even, and* Y *supplies* (E, E)*;*
(iv) $e_0 \in E(G[X])$ *and* $|Y|$ *is even and at least six;*
(v) $e_0 \in E(G[X])$, $G[Y]$ *is the four-cycle* y, a, y', b, y*, and* x, x' *are not adjacent.*

Proof. For (i), take $e_0 = \varepsilon$. To avoid it, use C, C on both even shores and no cut edge; on odd shores use C, E on both and include only ε'. For demand (K), suppose y has degree two. On odd shores use E, C on both and only ε. On even shores use E, t' on X and E, E on Y, including both cut edges. The edge ε has states E, E and is a K_2 component.

For (ii), (i) handles a cut edge, so by symmetry take $e_0 \in E(G[X])$. In the smaller graph $G_X(1)$ choose a factor satisfying its demand. If its new vertex is joined to x, the states on X are $E/H, C$. Pair them with E, C on Y and include ε. If joined to x', use the symmetric construction. The internal demand survives.

For (iii), use a factor of $G_X(2)$ satisfying the demand. Its C, C state pairs with C, C on Y using no cut edges. Any state in $\{E, H\}^2$ pairs with E, E on Y using both cut edges.

For (iv), the closure $G_X(D)$ is strictly smaller than G. Choose a factor satisfying the demand. Its C, C state is handled as before. Pair a state E, s' with t, E on Y, or s, E with E, t' on Y, using both cut edges. These pairs are legal even when s, s', t, t' are H.

For (v), replace Y by the edge xx' to obtain a smaller graph $G^* \in \mathcal{Q}$. Simplicity uses the nonadjacency assumption; bridgelessness follows because the new edge reconnects the terminals. The old degrees are preserved, and $e_0 \neq xx'$. Take a factor of G^* satisfying its demand. If xx' is absent, its restriction has states C, C; pair with the two edges $ya, y'b$ on Y, using no cut edges. If xx' is a K_2 component, the states are E, E; on Y use the path a, y, b and isolated y', and include both cut edges. If xx' belongs to a claw centred at x, the states are H, E; on Y use isolated y and the path a, y', b, again with both cut edges. The case with centre x' is symmetric. All gluing obeys the stated rule and preserves any internal K_2 component. □

24.3. **Completion of the factor induction.**

Proof of Theorem 24.1. Induct lexicographically on $(|V(G)|, |E(G)|)$, and fix the edge e_0 and its demand. Lemma 24.3 handles the case without adjacent degree-three vertices. Otherwise choose an edge $e = xy$ joining two such vertices. If $G - e$ is bridgeless, use the secondary induction step described above. If it has a bridge $\varepsilon' = x'y'$, then $\{e, \varepsilon'\}$ is a two-edge cut with connected shores X, Y. Each shore has at least three vertices: the endpoint of e has two distinct neighbours on its shore. The terminals on each shore are distinct.

Lemma 24.5(i),(ii) leaves only an internal demand with both shores even. By symmetry take $e_0 \in E(G[X])$. Part (iv) handles $|Y| \geq 6$, so $Y = \{y, y', a, b\}$ has order four. If a, b are adjacent, their edge with y, y' isolated supplies E, E, and part (iii) applies. Otherwise each of a, b must be adjacent to both y, y', since it has degree at least two. The vertex y already has its three neighbours x, a, b, so yy' is absent. Thus $G[Y]$ is precisely the four-cycle in part (v), which applies unless x, x' are adjacent.

It remains to treat that adjacency. Write the neighbours of x as y, x', w. The vertex x' has degree three: if it had degree two, the edge xw would be a bridge to the remaining vertices of X. Write its third neighbour as $w' \in X$. In fact $w \neq w'$. If they were equal, a third edge at this common vertex would be a bridge to the rest of X; without such an edge, $X = \{x, x', w\}$ would have odd order.

Put $X^- = X \setminus \{x, x'\}$ and $Y^+ = Y \cup \{x, x'\}$. The edges $xw, x'w'$ form a two-edge cut, $|Y^+| = 6$, and $|X^-|$ is even and at least two. The shore X^- is connected, since otherwise

one of its components would attach through just one of these two edges, a bridge. The shore Y^+ is visibly connected. Three cases finish the proof.

(i) If $e_0 = xx'$, only (A) is required, since both its endpoints have degree three. On Y^+ take the edge xy and the claw centred at y' with leaves a, b, x'. On X^- take the single edge ww' if it has order two. If it has order at least four, Lemma 24.4 at the new cut supplies a C, C partial factor, which is a complete factor of $G[X^-]$. Their union avoids xx'.

(ii) If $|X^-| \geq 4$ and e_0 is xw, $x'w'$, or lies inside X^-, apply Lemma 24.5(i) or (iv) to the new cut. The opposite shore has order six.

(iii) If $X^- = \{w, w'\}$, its two vertices are adjacent and each has degree two in G. One factor consists of the edges ww', xy and the claw centred at y' with leaves a, b, x'. A second consists of the four edges

$$xw, \quad x'w', \quad ya, \quad y'b.$$

The first has ww' as a component and avoids $xw, x'w'$; the second has $xw, x'w'$ as components and avoids ww'. These give both demands for every remaining edge e_0.

Every shore closure used has fewer vertices; the only induction at fixed vertex count was deletion of e, which decreases edges. This completes both demands simultaneously. □

24.4. **The partition count and the flow consequence.**

Lemma 24.6 (An even support with independent uncovered vertices)**.** *Let G be simple cubic and let W be a nonempty even edge set. Suppose the vertices T not incident with W are independent in G, and every cyclic edge cut of G has size at least eleven. Then every partition of $V(G/W)$ into r nonempty parts has at least $7(r-1)/3$ crossing edges. Consequently $F(G/W, \mathbb{F}_2^3)$ is nonempty and connected, and every flow on this quotient is main.*

Proof. The nontrivial components of W are disjoint cycles. Retain all other edges and loops when contracting them. Refine any part of a partition consisting only of vertices of T into singleton parts. Independence of T makes the crossing count unchanged, whereas the required lower bound can only increase. It therefore suffices to consider the refined partition. Let a be its singleton T-parts and β its remaining parts. Each remaining part contains a contracted W-cycle; it may also contain vertices of T. Since $W \neq \varnothing$, $\beta \geq 1$.

Let m be the number of crossing edges. If $\beta = 1$, all three edges at each singleton go to the other part, so $m = 3a \geq 7a/3$. If $\beta \geq 2$, the preimage of each of the β parts and its complement both contain a cycle. Their boundaries are at least eleven, giving

$$2m \geq 3a + 11\beta, \qquad m \geq 3a.$$

If $m < 7(a + \beta - 1)/3$, these inequalities would give

$$19\beta + 14 < 5a, \qquad 2a < 7(\beta - 1).$$

Twice the first inequality and five times the second imply $3\beta + 63 < 0$, a contradiction. Thus the density bound holds for every partition. Apply Corollary 19.5, which includes loops and the assertion that every quotient flow is main. □

Lemma 24.7 (Value-omitting flows in a common component)**.** *Let G be loopless and 3-edge-connected. In $F(G, \mathbb{F}_2^4)$ all flows omitting at least one nonzero value belong to one component, if any such flow exists.*

Proof. For a one-vertex graph the unique empty flow proves the assertion. Assume henceforth that G has at least two vertices. Fix $b \neq 0$ and put $B = \langle b \rangle$, $A = \mathbb{F}_2^4/B$. Regard G as a good subgraph of itself, with every original thread of length one. Take a frame as in Proposition 14.2. The proofs of Lemmas 16.1–16.3 apply here with $|B| = 2$, $|A| = 8$: each free thread forbids at most one kernel value, strictly fewer than two, and the two lists at a node have total size at most two, strictly fewer than eight. These are precisely the numerical uses of the larger groups in those proofs.

If f, h both omit b, they are G-good and their quotient discrepancy lifts to a flow Y on the frame by Proposition 14.2(i). Its endpoint has nonzero quotient everywhere, so $Y \in P(\Lambda_f)$. The specialized internal adjustment joins f to a flow with quotient equal to that of h. The remaining difference is $b\mathbf{1}_D$ on an even edge set D. Add b on its edge-disjoint cycles; all edges are protected by their nonzero common quotient. This joins f to h in the full flow graph. The intermediate path need not omit b.

Finally Lemma 19.1 joins a flow omitting any nonzero value to one omitting this fixed b. Thus all the stated flows lie in the same component. □

Theorem 24.8 (Cubic graphs with large cyclic edge cuts)**.** *Let G be a finite simple connected bridgeless cubic graph. If every cyclic edge cut has size at least eleven, then $F(G, \mathbb{F}_2^4)$ is nonempty and connected. Every such flow is joined to a flow omitting a nonzero value.*

Proof. Nonemptiness follows from the existence input in Section 14. Start with any nowhere-zero flow f and choose a nonzero value b occurring on an edge. Its value class $R = f^{-1}(b)$ is a nonempty matching, since two equal values at a cubic vertex would force the third to be zero. On $G - R$, the quotient of f modulo $\langle b \rangle$ is a nowhere-zero flow, so every component is bridgeless. For a component vertex set S, the cut lies in R; conservation implies that its size is even. The cubic degree identity

$$3|S| = 2|E(G[S])| + d_G(S)$$

then implies that $|S|$ is even. Every component of $G - R$ belongs to $\mathcal{Q}$.

Choose an edge-and-claw factor in each component by Theorem 24.1, and let J be their union. Put $W = E(G) \setminus E(J)$. It is even and contains R, so is nonempty. Its uncovered vertices are exactly the claw centres. They are independent in G: a centre uses all three of its incident G-edges in J, and two adjacent centres would belong to the same factor component, which is impossible. Lemma 24.6 now applies to G/W.

Choose a hyperplane $K \leq \mathbb{F}_2^4$ not containing b, and let $W_0 = \{e : f(e) \notin K\}$. Both W_0 and W are even and contain R. Their symmetric difference therefore avoids b. Add b on its edge-disjoint cycles. Each move is legal, and the resulting flow has outside-K support exactly W. Restriction to the edges outside W gives a nowhere-zero K-flow on G/W. By Lemma 24.6, it is joined to a flow omitting some $v \in K \setminus \{0\}$.

Lift each quotient cycle through the contracted W-cycles by arcs joining its two incidences. The lifted object is a cycle; a quotient loop lifts to a chord and an arc in its contracted cycle. At uncontracted vertices no routing is needed. Its coefficient lies in K, so every added arc stays outside K and is protected from zero. All other changed edges have their prescribed nonzero downstairs endpoints. Hence the entire path lifts. The final flow omits v both off W and on W, so every starting flow is main.

It remains to join these main flows. A cut of size at most two in a cubic graph has a cycle in each shore: if a nonempty shore S were a forest with c components, then

$$d_G(S) = 3|S| - 2(|S| - c) = |S| + 2c \geq 3.$$

Thus the cyclic-cut hypothesis makes G 3-edge-connected. Lemma 24.7 joins all the value-omitting endpoints. This proves connectedness. □

Corollary 24.9 (A remaining cyclic cut in a least core)**.** *If a least order-sixteen counterexample exists in the sense of Lemma 22.4, its simple cubic representative has a cyclic edge cut of size between five and ten.*

The proof uses [Go, Theorem 1.2(c)], an unpublished manuscript, through Lemma 22.4.

Proof. Theorem 24.8 gives a cyclic cut of size at most ten. The small-cut conclusions of Lemma 22.4 and Corollary 21.5 exclude cyclic cuts of sizes at most four. Indeed, the possible three-edge stars and the four-edge cuts at adjacent vertex pairs have an acyclic shore. □

This narrows the remaining least-core structure; it does not prove universal order-sixteen connectivity. The next section uses the factor construction to improve the order-sixteen odd-vertex bound further.

25. Small cyclic shores and the bound 3036

We combine a local size argument with the rounded density criterion to prove Theorem 13.4. The argument uses the girth-eighteen and cyclic-cut conclusions already established for a least order-sixteen core; in particular it retains the seventeen-cycle and four-cut dependencies.

Lemma 25.1 (Size of a small cyclic shore)**.** *Let G be a finite simple cubic graph of girth at least eighteen. Suppose $G[X]$ is connected and contains a cycle. If $d_G(X) \le 10$, then $|X| \ge 998$. If $d_G(X) \le 5$, then $|X| \ge 1011$.*

Proof. Repeatedly remove a vertex of degree one from the current induced subgraph. Removing such a vertex deletes two old boundary edges and exposes its unique internal edge, decreasing the boundary size by one. The remaining induced subgraph stays connected and retains a cycle. The process therefore ends at a nonempty induced subgraph $F = G[X']$ of minimum degree two. Its degrees are two or three. Write $N = |X'|$ and let D count its degree-two vertices. Each such vertex has exactly one boundary edge, so

$$D = d_G(X') \le d_G(X) \le 10, \qquad |E(F)| = (3N - D)/2.$$

The graph F is not a cycle: otherwise $D = N \ge 18$. Thus it has a degree-three vertex.

We first prove $N \ge 36$, to exclude the small branch of a later quadratic inequality. Root the distance levels of F at a degree-three vertex and write n_i for the size of level i. Girth at least eighteen ensures unique parents through level eight and no edges within levels one through seven. Every vertex in levels one through seven consequently has one or two distinct children, according to its degree, and different parents have different children. Hence $n_1 = 3$ and $n_i \ge 3$ for $1 \le i \le 8$. Let a and c be the numbers of degree-two and degree-three vertices in levels one through seven. Then

$$a + c \ge 21, \qquad a \le 10, \qquad c \ge 11.$$

The level recurrence gives $n_8 = 3 + c$, so $N \ge 1 + (a + c) + (3 + c) \ge 36$.

Now root an expansion at an edge uv of F, with $S_0 = \{u, v\}$ and S_i the vertices at distance i from this two-vertex set. A collision of the rooted paths through depth eight would give a cycle of length at most seventeen, using uv if the paths have different roots. An edge within a level S_i with $1 \le i \le 7$ would give a cycle of length at most sixteen. Indeed, the rooted paths of its two ends descend one level at each step, so together with that edge, and with uv when their roots differ, they form a closed walk of at most $2i + 2 \le 16$ edges that

traverses the level edge exactly once. Removing that edge leaves a walk between its ends, which contains a path, and this path closes with the level edge to a cycle. Thus, writing a_i for the number of degree-two vertices in S_i,

$$|S_0| = 2, \qquad |S_{i+1}| = 2|S_i| - a_i \quad (0 \le i \le 7).$$

At a root, the central edge uv takes the place of its parent edge. Edges between vertices in the outer level S_8 are allowed and do not change this recurrence or the number of vertices counted. Unrolling the recurrence gives

$$N \ge \sum_{i=0}^{8} |S_i| = 1022 - L(uv), \qquad L(uv) = \sum_{i=0}^{7} \bigl(2^{8-i} - 1\bigr) a_i. \tag{11}$$

In particular a degree-two root has weight 255.

Average $L(uv)$ over all edges of F. Fix a degree-two vertex w. Exactly two root edges have distance zero from w. For $i \ge 1$, at most 2^i vertices have distance i from w: the root has degree two and subsequent branching is at most two. A root edge at distance i has a nearest endpoint at that distance and cannot be a predecessor edge of that endpoint. There are therefore at most 2^{i+1} such root edges. Counting an edge more than once only enlarges this upper bound. The total contribution of w to all the losses is at most

$$2 \cdot 255 + \sum_{i=1}^{7} 2^{i+1} \bigl(2^{8-i} - 1\bigr) = 510 + 7 \cdot 512 - 508 = 3586.$$

Consequently some root edge satisfies

$$L(uv) \le \frac{7172D}{3N - D} \le \frac{71720}{3N - 10}.$$

The second inequality follows because $D/(3N - D)$ increases with D. Equation (11) now implies

$$3N^2 - 3076N + 81940 \ge 0.$$

This convex quadratic is negative at $N = 28$ and $N = 997$, with values -1836 and -2805. It is therefore negative throughout the interval between them. Since $N \ge 36$, we obtain $N \ge 998$, and hence $|X| \ge 998$.

If the original boundary has size at most five, then $D \le 5$, while the preceding conclusion still gives $N \ge 998$. Some integer loss is then at most

$$\frac{35860}{3N - 5} \le \frac{35860}{2989} < 12,$$

so that loss is at most eleven. Equation (11) gives $N \ge 1022 - 11 = 1011$. The original shore has at least as many vertices as its pruned core. □

Lemma 25.2 (A rounded failing-partition alternative)**.** *Let G be finite, simple, connected and cubic, with every cyclic edge cut of size at least five. Let J be an edge-and-claw factor, and suppose $W = E(G) \setminus E(J)$ is nonempty. If some partition of $V(G/W)$ fails the condition of Corollary 19.7, then either:*

(i) *there are at least four pairwise disjoint connected cyclic shores, each with cyclic complement and boundary of size at most ten; or*

(ii) *$V(G)$ is partitioned into three connected cyclic shores, each with boundary of size exactly five, and five single claw centres. There are no edges between the three cyclic shores.*

Proof. Refine the failing partition into connected parts in G/W. Its crossing count does not change, while the number of parts can only increase, so failure persists. Each part lifts to a connected part of G, with every W-cycle wholly contained in one part. The vertices not incident with W are the claw centres and are independent. A part containing no W-cycle can therefore consist only of a single centre.

Write s for the number of these singleton parts, t for the number of other parts, and ε for the number of edges between those t parts. Every edge at a singleton centre goes to one of the other parts. If c is the crossing count and $d_1, \dots, d_t$ are the boundaries of the other parts, then

$$r = s + t, \qquad c = 3s + \varepsilon, \qquad \sum_{j=1}^{t} d_j = 3s + 2\varepsilon.$$

Set $\delta = 7(r-1) - 3c = 7(t-1) - 2s - 3\varepsilon$. Rounded failure means $6c < 14r - 20$, and hence $\delta \geq 4$ by integrality. In particular

$$2s + 3\varepsilon \leq 7t - 11. \tag{12}$$

Since W is nonempty, $t \geq 1$; inequality (12) excludes $t = 1$. Each of the t parts and its complement now contain a W-cycle, so $d_j \geq 5$. Combining $3s + 2\varepsilon \geq 5t$ with (12) gives

$$11t \geq 33 + 5\varepsilon,$$

and therefore $t \geq 3$.

Let k count the d_j that are at most ten. The remaining boundaries are at least eleven, and the exact sum above gives

$$11t - 6k \leq \sum_{j=1}^{t} d_j = \frac{21}{2}(t-1) - \frac{3}{2}\delta - \frac{5}{2}\varepsilon.$$

Thus

$$12k \geq t + 21 + 3\delta + 5\varepsilon \geq t + 33 + 5\varepsilon.$$

If $k \geq 4$, the corresponding disjoint parts give the first outcome. If $k \leq 3$, the inequalities force $k = t = 3$, $\varepsilon = 0$ and $\delta = 4$. Then $14 - 2s = 4$, so $s = 5$, and the three boundaries sum to fifteen. Each is at least five, so all are exactly five. The parts cover all of $V(G)$, giving the second outcome. □

Proof of Theorem 13.4. Suppose a bridgeless graph has disconnected $\mathbb{F}_2^4$-flow graph, and let p be the minimum total number of odd-degree vertices among all such graphs. Lemma 22.4 supplies a connected simple cubic representative G on p vertices, with three-edge connectivity and only vertex-star three-cuts. Lemmas 22.4, 22.6 and 23.2 give girth at least eighteen, and Corollary 21.5 gives cyclic edge-cut size at least five. These conclusions use wide minimality, not a restriction to smaller cubic graphs.

All value-omitting flows of G lie in one component by Lemma 24.7. Since the flow graph is nonempty and disconnected, some flow f is not main. Choose a nonzero value b occurring in f. As in the proof of Theorem 24.8, its class $R = f^{-1}(b)$ is a matching, and every component of $G - R$ is bridgeless, has even order and has all degrees two or three. Theorem 24.1 gives an edge-and-claw factor in each component. Their union J has an even complement W containing R, so W is nonempty.

The quotient G/W cannot satisfy Corollary 19.7. If it did, choose a hyperplane K avoiding b, align the outside-K support of f with W by legal b-cycle moves, and restrict to a nowhere-zero K-flow on G/W. The corollary would join it to a flow omitting a nonzero K-value.

Lift each quotient move through arcs of W-cycles, whose values stay outside K, exactly as in Theorem 24.8. The endpoint would omit that value on all of G, so f would be main, contradicting its choice.

Apply Lemma 25.2 to a failing partition. In its first outcome, Lemma 25.1 gives at least $4 \cdot 998 = 3992$ vertices in four disjoint shores. In its second outcome, the three five-boundary shores and the five singleton centres give

$$p \geq 3 \cdot 1011 + 5 = 3038.$$

Thus every least counterexample has at least 3038 odd-degree vertices. No bridgeless graph with at most 3036 odd-degree vertices can have disconnected flow graph. The number of odd-degree vertices is even, so this is the even threshold immediately below 3038.

Finally apply this conclusion to each connected component of H. Nowhere-zero flow existence is the input of Section 14. Every cycle move stays within one component, and the flow graph is the Cartesian product of the component flow graphs, hence is nonempty and connected. Loops have independent nonzero values, can be changed by single-loop moves, and contribute two to degree, so they cause no exception. □

The proof gives a lower bound of 3038 on the order of a least order-sixteen core. It neither excludes eighteen-cycles nor constructs an example attaining this order, and it does not establish universal order-sixteen connectivity.

26. A unique exceptional decomposition and the bound 3996

This section uses only results of the preceding sections. Throughout, G is a least order-sixteen core, as in the proof of Theorem 13.4: connected, simple, cubic, of girth at least eighteen, and every cut with a cycle on both sides has at least five edges (Corollary 21.5; a cut is the set of edges with exactly one end in a nonempty proper vertex set S, connected or not). We call this last property (C5). We write $n = |V(G)|$ and $d(S)$ for the size of that cut, the number of edges with exactly one end in S. For disjoint S, T we have $d(S \cup T) = d(S) + d(T) - 2e(S, T)$; for any S, T we have $d(S \cap T) + d(S \cup T) \leq d(S) + d(T)$ and $d(S \setminus T) + d(T \setminus S) \leq d(S) + d(T)$. If $G[S]$ is a forest with $\mathrm{comp}(S)$ components, then $d(S) = |S| + 2\,\mathrm{comp}(S)$, also when S is empty.

A *shore* is a set X such that $G[X]$ is connected and contains a cycle, and $V \setminus X$ contains a cycle; so $d(X) \geq 5$ by (C5). An *exceptional decomposition* $\mathcal{D} = (X_1, X_2, X_3; Z)$ is a partition of V into three shores X_1, X_2, X_3 with $d(X_i) = 5$ and a set Z of five vertices, the *connectors*, such that no edge joins two different X_i and every edge that leaves some X_i ends in Z. The second outcome of Lemma 25.2 gives such a decomposition, with Z the five single parts, which are claw centres of J: a claw centre of J is a vertex with all three edges in J, and the proof of Lemma 25.2 shows that a part containing no cycle of W is a single claw centre. The definition does not ask that Z consist of claw centres.

Lemma 26.1 (The exceptional case is rigid)**.** *In an exceptional decomposition $(X_1, X_2, X_3; Z)$:*

(a) *every $z \in Z$ has exactly one neighbour in each X_i;*
(b) *the five edges that leave X_i end at five different vertices of X_i, and no vertex of X_i has two or more neighbours in Z;*
(c) *Z is independent, and X_1, X_2, X_3 are exactly the components of $G - Z$;*
(d) *every vertex has at most one neighbour in Z.*

Proof. (a) The fifteen edges that leave X_1, X_2, X_3 all end in Z, and the degree sum of Z is $15 = 15 + 2e(Z)$. So $e(Z) = 0$: Z is independent, and all three edges of each $z \in Z$ go to

the sets X_i. If z had two or three neighbours in X_1, then $d(X_1 \cup \{z\}) = 5+3-4 = 4$ or $5+3-6 = 2$, while both sides contain a cycle ($V \setminus (X_1 \cup \{z\})$ contains X_2), against (C5). So z has one neighbour in each X_i.

(b) Let $v \in X_i$ have exactly two neighbours in Z. Its third neighbour is in X_i, so $X_i \setminus \{v\}$ is not empty and $d(X_i \setminus \{v\}) = 5-2+1 = 4$. As $V \setminus (X_i \setminus \{v\})$ contains X_j, (C5) makes $X_i \setminus \{v\}$ a forest, with $|X_i \setminus \{v\}| + 2\,\mathrm{comp} = 4$; so $|X_i| \le 3$. But three or fewer vertices contain no cycle, as the girth is at least eighteen, and X_i contains one. If $v \in X_i$ has three neighbours in Z, it has no neighbour in X_i; as $G[X_i]$ is connected, $X_i = \{v\}$, which contains no cycle.

(c) follows from the definition and (a). (d) holds for the vertices of the X_i by (b), and for the vertices of Z by (c). □

Lemma 26.2 (A stronger form of Lemma 25.2)**.** *Assume the setting of Lemma 25.2, with the refined failing partition of its proof: s single parts (single claw centres), t other parts with boundaries $d_1, \dots, d_t$, ε edges between these t parts, and k the number of indices j with $d_j \le 10$. If moreover G has girth at least eighteen, then either $k \ge 4$, there are four pairwise disjoint shores with boundary at most ten and at least six single claw centres outside them, and $n \ge 4 \cdot 998 + 6 = 3998$; or $k \le 3$, and the second outcome of Lemma 25.2 holds.*

Proof. The proof of Lemma 25.2 shows: $2s + 3\varepsilon \le 7t - 11$ (inequality (12)); $t \ge 3$; $3s + 2\varepsilon = d_1 + \cdots + d_t \ge 5t$, since each of the t parts contains a cycle of W and so does its complement; and $k \le 3$ forces $k = t = 3$, $\varepsilon = 0$, $s = 5$ and the second outcome. Let $k \ge 4$. Three times the third relation minus two times the first gives $5s \ge t + 22 \ge k + 22 \ge 26$, so $s \ge 6$. The k parts with $d_j \le 10$ are connected and contain a cycle of W, and so do their complements; so they are pairwise disjoint shores, each with at least 998 vertices by Lemma 25.1. The s single parts are other parts of the partition. So $n \ge 4 \cdot 998 + 6 = 3998$. □

Lemma 26.3 (At most one exceptional decomposition)**.** *Let $n \le 3996$, and let $\mathcal{D}_1 = (X_1, X_2, X_3; Z)$ and $\mathcal{D}_2 = (Y_1, Y_2, Y_3; Z')$ be exceptional decompositions. Then $Z' = Z$. So, by Lemma 26.1(c), $\mathcal{D}_2$ is $\mathcal{D}_1$ up to the order of the three shores.*

Proof. Every X_i and every Y_j has at least 1011 vertices by Lemma 25.1, so each has at most $n - 5 - 2 \cdot 1011 \le 1969$ vertices.

Step 1: matching the shores. Fix i, j and put $\mathrm{Cap} = X_i \cap Y_j$ and $\mathrm{Cup} = X_i \cup Y_j$. The set $V \setminus \mathrm{Cap}$ contains the other two Y's, so it contains a cycle, and Cup contains a cycle. The set $V \setminus \mathrm{Cup}$ contains $(Y_k \cup Y_l) \setminus X_i$, for the other indices k, l, which has at least $2022 - 1969 = 53$ vertices; if $V \setminus \mathrm{Cup}$ were a forest, then $d(\mathrm{Cup}) = d(V \setminus \mathrm{Cup}) \ge 53 + 2$, but $d(\mathrm{Cup}) \le d(X_i) + d(Y_j) = 10$. So $V \setminus \mathrm{Cup}$ contains a cycle and $d(\mathrm{Cup}) \ge 5$ by (C5). By submodularity, $d(\mathrm{Cap}) \le 10 - d(\mathrm{Cup}) \le 5$. If Cap contains a cycle, then $d(\mathrm{Cap}) \ge 5$ by (C5), so $d(\mathrm{Cap}) = d(\mathrm{Cup}) = 5$. Then Cap is connected: its component with a cycle has boundary at least five alone (C5), and every other component R would add $d(R) \ge 3$ (if R contains a cycle, $d(R) \ge 5$ by (C5); if R is a tree, $d(R) = |R| + 2$). So Cap is a shore with $d(\mathrm{Cap}) = 5$, and $|\mathrm{Cap}| \ge 1011$. If Cap is a forest, then $|\mathrm{Cap}| + 2\,\mathrm{comp}(\mathrm{Cap}) \le 5$, so $|\mathrm{Cap}| \le 3$, also when Cap is empty. The set Y_j lies in $X_1 \cup X_2 \cup X_3 \cup Z$ and has at least $1011 > 3 + 3 + 3 + 5$ vertices, so some $X_i \cap Y_j$ contains a cycle; two of them would give $|Y_j| \ge 2022 > 1969$. So there is exactly one such $i = \sigma(j)$, and σ is a bijection, since two indices j with one i would give $|X_i| \ge 2022$. We renumber so that σ is the identity. So for each i the set $X_i \cap Y_i$ is a shore with boundary five, and $|X_i \cap Y_j| \le 3$ for all $j \ne i$.

Step 2: the leftovers. Put $P_i = X_i \setminus Y_i$, $Q_i = Y_i \setminus X_i$, $m_i = |Z' \cap X_i|$ and $m_i' = |Z \cap Y_i|$. The set P_i lies in $(X_i \cap Y_j) \cup (X_i \cap Y_k) \cup (Z' \cap X_i)$, so $|P_i| \le 3 + 3 + 5 = 11 < 18$,

and $G[P_i]$ is a forest by the girth. As X_i is the disjoint union of $X_i \cap Y_i$ and P_i, we get $5 = d(X_i) = d(X_i \cap Y_i) + d(P_i) - 2e(P_i, X_i \cap Y_i) = 5 + d(P_i) - 2e(P_i, X_i \cap Y_i)$, so $d(P_i) = 2e(P_i, X_i \cap Y_i)$. An edge from P_i into $X_i \cap Y_i \subseteq Y_i$ starts at a vertex of Z': a vertex of P_i outside Z' lies in Y_j or Y_k, and no edge joins two different Y's. Every vertex of Z' has exactly one neighbour in Y_i (Lemma 26.1(a) for $\mathcal{D}_2$). So $e(P_i, X_i \cap Y_i) \leq m_i$; note that $Z' \cap X_i \subseteq P_i$, as Z' misses Y_i. So $|P_i| + 2\operatorname{comp}(P_i) = d(P_i) \leq 2m_i$, and with $|P_i| = m_i + q_i$, where q_i counts the vertices of P_i outside Z', we get $q_i + 2\operatorname{comp}(P_i) \leq m_i$. If $m_i \geq 1$, then $q_i \geq 1$: a vertex $y \in Z' \cap X_i$ has at most one neighbour in Z (Lemma 26.1(d) for $\mathcal{D}_1$), so at least two neighbours in X_i; its three neighbours lie one in each of Y_1, Y_2, Y_3 (Lemma 26.1(a) for $\mathcal{D}_2$); so a neighbour of y in X_i lies in Y_j or Y_k, hence in P_i, and it is not in Z', which is independent (Lemma 26.1(c) for $\mathcal{D}_2$). So if P_i is not empty, then $\operatorname{comp}(P_i) \geq 1$, $m_i = 0$ is impossible, and $m_i \geq q_i + 2 \geq 3$. The same holds with the roles of $\mathcal{D}_1$ and $\mathcal{D}_2$ exchanged: if Q_i is not empty, then $m'_i \geq 3$. (Here $d(Q_i) = 2e(Q_i, X_i \cap Y_i)$; an edge from Q_i into $X_i \cap Y_i$ starts in Z; every vertex of Z has one neighbour in X_i (Lemma 26.1(a) for $\mathcal{D}_1$); a vertex of $Z \cap Y_i$ has at most one neighbour in Z' (Lemma 26.1(d) for $\mathcal{D}_2$), so at least two neighbours in Y_i, at most one of them in X_i (Lemma 26.1(a) for $\mathcal{D}_1$), so one in X_j or X_k, hence in Q_i, and not in Z (Lemma 26.1(c) for $\mathcal{D}_1$).)

Step 3. Suppose $Z' \neq Z$. Some vertex of Z' is not in Z, so it lies in some X_i; then $m_i \geq 1$, so P_i is not empty and $m_i \geq 3$. As $m_1 + m_2 + m_3 \leq 5$, this holds for one index i_0 only; for the other two indices $m_i = 0$, so P_i is empty, that is, $X_i \subseteq Y_i$. Some vertex of Z is not in Z', so it lies in some Y_i; then Q_i is not empty and $m'_i \geq 3$, for one index i_1 only; for the other two indices Q_i is empty, that is, $Y_i \subseteq X_i$. Take an index i that is neither i_0 nor i_1; then $X_i = Y_i$. Every $z \in Z$ has a neighbour in $X_i = Y_i$ and is not in Y_i, as it is not in X_i; since Y_i is a component of $G - Z'$ (Lemma 26.1(c) for $\mathcal{D}_2$), z is in Z'. So $Z \subseteq Z'$, and $Z = Z'$, as both have five vertices: a contradiction. □

Theorem 26.4 (The bound 3996)**.** *If each connected component of a finite bridgeless multigraph H has at most* 3996 *odd-degree vertices, then $F(H, \mathbb{F}_2^4)$ is nonempty and connected. A least order-sixteen core has at least* 3998 *vertices.*

The proof uses [Go, Theorem 1.2(c)], an unpublished manuscript, through Lemma 22.4.

Proof. As in the proof of Theorem 13.4, take the least core G on $p = n$ vertices, suppose $n \leq 3996$, and take a flow f that is not main, that is, not joined to a flow that omits a value; it exists, since all value-omitting flows lie in one component by Lemma 24.7 and the flow graph is not connected. Let b be any value that occurs in f. Every component of $G - f^{-1}(b)$ is bridgeless, of even order, with all degrees two or three (the proof of Theorem 24.8 shows this for any value that occurs), so it has an edge-and-claw factor by Theorem 24.1. For any such factor J of $G - f^{-1}(b)$, the quotient G/W, with $W = E(G) \setminus E(J)$, fails Corollary 19.7: otherwise f is main, exactly as in the proof of Theorem 13.4, whose argument uses of J only that W is even and contains $f^{-1}(b)$ and that the claw centres are independent, which holds for every edge-and-claw factor. So Lemmas 25.2 and 26.2 apply to a failing partition. The case $k \geq 4$ gives $n \geq 3998$ by Lemma 26.2, which is impossible. So $k \leq 3$, and the second outcome holds: an exceptional decomposition whose five connectors are claw centres of J.

Take one such decomposition $\mathcal{D} = (X_1, X_2, X_3; Z)$, for some value b_0 and factor J_0. By Lemma 26.3 it is the only exceptional decomposition of G, up to the order of the three shores. Now let $z_1 \in Z$, let x_1 be its neighbour in X_1, and let $b = f(z_1x_1)$; this value is not 0 and occurs in f. Take any factor J of $G - f^{-1}(b)$. The edge z_1x_1 has value b, so it lies in $W = E(G) \setminus E(J)$, and z_1 lies on W: it is not a claw centre of J. The second outcome

for (b, J) gives an exceptional decomposition whose connectors are claw centres of J, so its set of connectors is not Z, as $z_1 \in Z$. This is a second exceptional decomposition with another set of connectors, against Lemma 26.3. This contradiction shows that no such G exists with $n \leq 3996$. So $n \geq 3997$, and n is even as G is cubic, so $n \geq 3998$. Hence no finite bridgeless multigraph each of whose components has at most 3996 odd-degree vertices has a disconnected flow graph; the passage to components, loops and nonemptiness is the last paragraph of the proof of Theorem 13.4. □

The value 3998 is what this argument gives; we do not know whether it is best possible.

27. Chains of good sets and short cycles

In this section K is a subgroup of order eight of $\mathbb{F}_2^4$, with a basis x_1, x_2, x_3; so $K \cong \mathbb{F}_2^3$. A flow is a nowhere-zero flow, and $f \sim g$ means that f and g are joined by moves. For a multigraph U and a set X of at least two vertices, U/X identifies the vertices of X and deletes all edges of $U[X]$.

27.1. **Three spanning trees through a linear forest.** We say that a graph G has *property* (H) if G is cubic, simple and connected, no cut of G has fewer than three edges, every cut of three edges is the set of edges at one vertex, and every cut of four edges is the set of the four edges that leave two adjacent vertices. A least order-sixteen core, in the sense of Lemma 22.4 for the group $\mathbb{F}_2^4$, has property (H) by Lemma 22.4 and Corollary 21.5.

Lemma 27.1. *Let G have property* (H)*, and let F be a set of edges of G with no cycle and with at most two edges at each vertex. Then G has spanning trees T^1, T^2, T^3 such that $F \subseteq T^3$ and no edge lies in all three trees.*

Proof. The matroid form. Put $E_0 = E(G) \setminus F$ and $n = |V(G)|$. Take two copies of each edge of E_0 and one copy of each edge of F; this is the ground set Ω of a multigraph G'. Let $M_1 = M_2$ be the cycle matroid of G'. In M_3 a set I of copies of edges of E_0 is independent if $I \cup F$ has no cycle, and the edges of F are loops; its rank is $n - 1 - |F|$. If M_1, M_2, M_3 have pairwise disjoint bases B_1, B_2, B_3, then $T^1 = B_1$, $T^2 = B_2$ and $T^3 = F \cup B_3$, with the copies forgotten, are spanning trees. An edge of E_0 has two copies, so it is in at most two of the trees; an edge of F is one element, so it is in at most one of T^1, T^2. So no edge is in all three. By the matroid union theorem [NW2], such bases exist if and only if, for every $A \subseteq \Omega$,

$$|\Omega \setminus A| + r_1(A) + r_2(A) + r_3(A) \geq 3(n-1) - |F|.$$

Reduction to partitions. Adding the second copy of an edge to A changes no rank and does not raise $|\Omega \setminus A|$. Adding to A an edge whose two ends are in one component of the graph of A changes no rank, also not r_3, since $r_3(A) = r_G(A \cup F) - |F|$, and does not raise $|\Omega \setminus A|$. So it suffices to take A to be all copies of all edges inside the parts of a partition of $V(G)$ into r parts, each part connected. Let m be the number of edges of G between different parts and m_F the number of them in F. Let Φ be the multigraph on the parts whose edges are these m_F edges, and ν its cyclomatic number (edges minus nodes plus components). Then $|\Omega \setminus A| = 2(m - m_F) + m_F$, $r_1(A) = r_2(A) = n - r$ and $r_3(A) = \big(n - (r - (m_F - \nu))\big) - |F|$, and the inequality becomes

$$\nu \leq 2m - 3r + 3. \tag{13}$$

Proof of (13). For $r = 1$ we have $m = 0$, $\nu = 0$ and the right side is 0. Let $r \geq 2$. For a part k let $d_k \geq 3$ be the number of edges that leave it and δ_k the number of them in F; then $\sum_k d_k = 2m$.

Case A: some part X has $d(X) \le 4$ and is neither one vertex nor two adjacent vertices. By (H), the cut of X is the cut of one vertex or of two adjacent vertices; in a connected graph two vertex sets with the same cut are equal or complementary, so $V(G) \setminus X$ is one vertex or two adjacent vertices, and $r \le 3$. If $r = 2$ and the rest is one vertex, then $m = 3$, the right side is 3, $m_F \le 2$ and $\nu \le 1$. If $r = 2$ and the rest is two vertices in one part, then $m = 4$, the right side is 5, $m_F \le 4$ and $\nu \le 3$. If $r = 3$, the two vertices being two parts, then $m = 5$, the right side is 4, $m_F \le 4$ and $\nu \le 3$. Here we use that $\nu = 0$ if $m_F = 0$, and that $\nu \le m_F - 1$ if $m_F \ge 1$, because Φ then has an edge. So (13) holds.

Case B: every part with $d_k \le 4$ is one vertex ($d_k = 3$) or two adjacent vertices ($d_k = 4$). For a part with $\delta_k \ge 1$ put $t_k = d_k - 2 - \delta_k/2$. For one vertex, $\delta_k \le 2$ by the hypothesis on F, so $t_k \ge 0$, with equality only if $\delta_k = 2$. For two adjacent vertices, $\delta_k \le 4$, so $t_k \ge 0$, with equality only if all four leaving edges are in F. For $d_k \ge 5$, $t_k \ge d_k/2 - 2 \ge \frac{1}{2}$. Let Φ_i be a component of Φ with an edge, N_i its set of parts and ν_i its cyclomatic number. Then $\nu_i = 1 + \sum_{k \in N_i} (\delta_k/2 - 1)$, so

$$\sum_{k \in N_i} t_k = \sum_{k \in N_i} (d_k - 3) - \nu_i + 1,$$

an integer that is at least 0. If it is 0, every part of N_i is one vertex with two leaving edges in F or two adjacent vertices with all four leaving edges in F. Then every vertex of these parts is on exactly two edges of Φ_i, and the other ends are again in parts of N_i. So the edges of Φ_i form a graph in which every vertex has degree two; it has a cycle, which is a cycle of G inside F. This is impossible. So the sum is at least 1, that is, $\nu_i \le \sum_{k \in N_i} (d_k - 3)$. The sets N_i are disjoint and $d_k - 3 \ge 0$ for all parts, so $\nu \le \sum_k (d_k - 3) = 2m - 3r$, and (13) holds. □

27.2. **The flow of three trees.**

Lemma 27.2. *Let G be connected and let T^1, T^2, T^3 be spanning trees with no edge in all three. Let D_i be the even edge set that contains $E(G) \setminus T^i$; it exists and is unique, being the sum of the fundamental cycles of the edges outside T^i. Put $h = x_1 \mathbf{1}_{D_1} + x_2 \mathbf{1}_{D_2} + x_3 \mathbf{1}_{D_3}$. Then h is a nowhere-zero flow with values in K, and $h(e) \ne x_3$ for every edge e of T^3.*

Proof. h is a sum of flows. Every edge e is outside some T^i, so $e \in D_i$ and $h(e) \ne 0$. Let $e \in T^3$. Then e is outside T^1 or outside T^2, so e is in D_1 or in D_2; hence $h(e)$ has a coordinate x_1 or x_2, and $h(e) \ne x_3$. □

27.3. **A good set, for two flows.**

Lemma 27.3. *Let U be a finite multigraph without loops, X a set of at least two vertices, and T a spanning tree of $U[X]$ whose edge complement in $U[X]$ is connected and spanning. Let j and y be flows of U with values in K, and let v, v' be nonzero elements of K such that j does not take the value v on T and y does not take the value v' on T. If the restrictions of j and y to U/X are joined in $F(U/X, K)$, then $j \sim y$ in $F(U, K)$.*

Proof. First lift the walk of U/X step by step, as in the proof of Lemma 19.4. A step on a cycle of U/X that is a cycle of U is done as it is. A step on an exterior path between two different vertices of X is closed through T if its coefficient is v, and through a spanning tree T_2 of the complement, after the preparation inside $U[X]$ by Lemma 19.2, if its coefficient is another value. That proof lifts every step of any walk and keeps the property "no value v on T"; it does not use where the walk ends. The end is a flow $j'' \sim j$ with $j'' = y$ on all edges outside $U[X]$ and with no value v on T.

Now j'' and y are nowhere zero on $U[X]$ and have the same boundary there. The graph $U[X]$ has two edge-disjoint spanning trees, T and a tree of the complement. The map j'' omits v on T and y omits v' on T. By Corollary 19.6 the two maps are joined by moves inside $U[X]$, and these are moves of U. □

27.4. **Chains.** *The setting.* Let G have property (H), let f be an $\mathbb{F}_2^4$-flow of G, b a value of f, and W' an even edge set that contains $f^{-1}(b)$. Put $J' = E(G) \setminus W'$. The graph G/W' contracts every cycle of W' to one node and keeps the vertices off W'. An edge of J' with both ends on one cycle of W', a *chord*, would be a loop; throughout, U denotes G/W' with these loops deleted. So $E(U)$ is J' without the chords. Let K be a subgroup of order eight that does not contain b. Then $f \sim f_1$ for a flow f_1 with all values on W' outside K and all values on J' in K: the set of edges with a value of f outside K and the set W' are two even sets that both contain $f^{-1}(b)$; their symmetric difference is a set of disjoint cycles with no edge of value b; add b on these cycles. Let j be f_1 on the edges of U; it is a flow of U with values in K.

Chains. A *chain* for j is a sequence $U_0 = U, U_1, \dots, U_k$ in which each U_i arises from U_{i-1} by one of two steps.

(G) A *good-set step*: a set S_i of at least two vertices of U_{i-1}, a spanning tree T_i of $U_{i-1}[S_i]$ whose edge complement in $U_{i-1}[S_i]$ is connected and spanning, and a nonzero element v_i of K that j does not take on T_i; and $U_i = U_{i-1}/S_i$. So S_i is good for (j, v_i) in U_{i-1}, in the sense defined before Lemma 19.4.

(C) A *short-cycle step*: an induced cycle C of U_{i-1} with at most seven edges; and $U_i = U_{i-1}/V(C)$. A cycle with two edges is induced if its two vertices are joined by exactly two edges.

No graph U_i has a loop: a good-set step deletes all edges inside the set, and an induced cycle has no chord.

Theorem 27.4. *If for some chain the flow graph $F(U_k, K)$ is connected, then f is joined to a flow of G that omits the value b.*

Proof. Let F be the union of the edge sets of the trees T_i of the good-set steps. These are edges of J', so edges of G.

(a) *F has at most two edges at each vertex of G.* A vertex on W' has one edge of J'. Let c be a vertex off W'; it is a vertex of U of degree three. If c is in no merged set, no edge at c is in F. Otherwise let i be the first step whose set contains c. A tree of an earlier step has no edge at c, since its edges have both ends in its own set. If step i is a good-set step, the complement of T_i in $U_{i-1}[S_i]$ is connected and spanning and S_i has at least two vertices, so it has an edge at c; this edge is not in T_i, and it is deleted in U_i, so it is in no later tree. If step i is a short-cycle step, the two edges of C at c are deleted and are in no tree. In both cases at most two edges at c are in F.

(b) *F has no cycle in G.* First, the union of the trees of the steps $i, \dots, k$ has no cycle in U_{i-1}, by induction down from $i = k$. Let Z be a cycle of U_{i-1} inside this union. If step i is a good-set step, Z is not inside the tree T_i; its edges outside T_i are edges of U_i, because they lie in later trees, and after the contraction of S_i they form a nonempty edge set with even degree at each vertex of U_i, which contains a cycle inside the later trees. If step i is a short-cycle step, Z has no edge of C, since the edges of C are not edges of U_i, and no chord of C; so its image in U_i is not empty, has even degrees and no loop, and it contains a cycle of U_i inside the later trees. Both are impossible. So F has no cycle in U. A cycle of G inside

F has two edges of J' at each of its vertices, so all its vertices are off W', and it is a cycle of U. This is impossible.

(c) *The target.* By (a), (b) and Lemma 27.1 there are spanning trees T^1, T^2, T^3 of G with $F \subseteq T^3$ and no common edge. By Lemma 27.2 there is a flow h of G with values in K, nowhere zero, with $h(e) \neq x_3$ for every edge e of F. Let y be h on the edges of U; it is a flow of U with values in K, and its restriction is a flow of each U_i.

(d) *Down the chain.* We claim that $j \sim y$ on U_i for $i = k, k-1, \dots, 0$, where both maps are restricted to U_i. The same two flows j and y are used at every level; only the relation $\sim$ passes down the chain, not a walk. For $i = k$ this holds because $F(U_k, K)$ is connected. From i to $i-1$ at a good-set step, apply Lemma 27.3 in U_{i-1} with $X = S_i$, $T = T_i$, $v = v_i$ and $v' = x_3$; the map y does not take the value x_3 on T_i because $T_i \subseteq F$. From i to $i-1$ at a short-cycle step, the proof of Lemma 20.2 lifts every move of the quotient across the cycle, which has fewer than eight edges, and shows that two extensions of one quotient flow are equal or adjacent; so $j \sim y$ on $U_{i-1}/V(C)$ gives $j \sim y$ on U_{i-1}, with no condition on y. For $i = 0$ we get $j \sim y$ in $F(U, K)$.

(e) *The end.* Lift the moves of U through arcs of the cycles of W'; a coefficient in K is safe there, because the values on W' stay outside K. Then set every chord to its value under h by one move on the chord and an arc of its cycle; the coefficient is in K and is not the value of the chord. This gives $f_1 \sim f_2$ with $f_2 = h$ on J' and all values on W' outside K. Put $g = h$ on J' and $g = h + b$ on W'. Then g is a flow, because W' is even; it is nowhere zero; and it omits b, because h is nowhere zero on W' and the values on J' are in K. The difference $f_2 - g$ is 0 on J' and has values in K on W', so it is a constant of K on each cycle of W'; adding this constant on the cycle is a move. So $f \sim g$. □

27.5. **The consequence for a least core.**

Corollary 27.5. *Let G be a least order-sixteen core and f a flow of G that is not joined to a flow omitting a value. Then for every b, W', K as above and every chain, the flow graph $F(U_k, K)$ is not connected, and U_k has at least 46 vertices of odd degree. Moreover, U_k has a cut of at most three edges, and every such cut is the set of the three edges at a vertex off W' that lies in no merged set; so in every chain some vertex off W' is left out.*

The proof uses [Go, Theorem 1.2(c)], an unpublished manuscript, through Lemma 22.4.

Proof. The first statement is Theorem 27.4. The graph U_k is a finite bridgeless multigraph, because G has no bridge and neither a contraction nor the deletion of a loop makes one; so Corollary 22.2 gives more than 44 vertices of odd degree, and their number is even. By Theorem 13.3, U_k is not 4-edge-connected; it has at least two vertices, so it has a cut of at most three edges. A cut of U_k is a cut of G all of whose edges are in J'. By (H) it is the set of the three edges at one vertex c; so c is off W', c is in no merged set, and c is a vertex of degree three of U_k. □

A chain is *complete* if no further step is possible.

Lemma 27.6. *The last graph U_k of a complete chain is simple, has girth at least eight, and has no good set: no set of at least two vertices of U_k is good for (j, v) for a nonzero element v of K.*

Proof. A set that is good for (j, v), with v a nonzero element of K, would allow a further good-set step. Two vertices joined by two or more parallel edges form such a set, with one of these edges as the tree and any nonzero element of K other than its value; so U_k is simple.

A cycle with the fewest edges of a simple graph is induced; so a cycle with at most seven edges would allow a further short-cycle step. □

28. The final quotient of a complete chain

Throughout this section G is a least order-sixteen core, as in the proof of Theorem 13.4: a connected simple cubic graph on p vertices whose $\mathbb{F}_2^4$-flow graph is not connected, 3-edge-connected with only vertex-star three-cuts (Lemma 22.4), of girth at least eighteen (Lemmas 22.6 and 23.2), and with every cut that has a cycle on both sides of size at least five (Corollary 21.5). We write $n = |V(G)| = p$; n is even. All value-omitting flows of G lie in one component (Lemma 24.7), so some flow f of G is not joined to a flow omitting a value.

28.1. **The setting.** Fix a value b of f. As in the proof of Theorem 13.4, the graph $G - f^{-1}(b)$ has an edge-and-claw factor J (Theorem 24.1 and the proof of Theorem 24.8), and $W = E(G) \setminus E(J)$ is a nonempty even edge set that contains $f^{-1}(b)$. The vertices not on W are the claw centres; no two of them are adjacent, since an edge between two claw centres would lie in one claw, and a claw has one centre. Fix a subgroup K of order eight with $b \notin K$. Let f_1, U and j be as in the setting of Theorem 27.4 with $W' = W$: we have $f \sim f_1$ by adding b on the cycles of the symmetric difference of W and the set of edges with an f-value outside K, so the values of f_1 outside K are exactly those on W; $U = G/W$ with its loops (the chords of the cycles of W) deleted; and j is the restriction of f_1 to $E(U)$, a nowhere-zero K-flow of U. Take a complete chain for j (Section 27) and let U^* be its last graph, with n^* vertices and m^* edges. We call the vertices of U^* *nodes*.

Lemma 28.1. (Q1) *U^* is simple, has girth at least eight, and has no good set for j.*
(Q2) *$F(U^*, K)$ is not connected, and U^* has at least 46 vertices of odd degree; in particular $n^* \geq 46$.*
(Q3) *$6m^* \leq 14n^* - 21$.*

Proof. (Q1) is Lemma 27.6. (Q2) is Corollary 27.5 for the complete chain; it applies because U^* is connected (as G is), loopless (the loops are deleted at the start, and every later step deletes the edges inside the merged set), and bridgeless (a cut of U^* is a cut of G with the same edges, and G is 3-edge-connected).

(Q3) follows the argument of Corollary 19.7. Fix a nonzero value v of K and let N be the set of edges of U^* whose j-value is not v. Suppose $|N| \geq 2n^* - 2$. As $n^* \geq 2$ by (Q2), the whole node set has at least two nodes and at least $2n^* - 2$ edges of N inside; take a set S of least size with $|S| \geq 2$ and at least $2|S| - 2$ edges of N inside. For a partition of S into $q \geq 2$ parts, a part P with at least two nodes has at most $2|P| - 3$ edges of N inside, since it is smaller than S, and a part of one node has none, since U^* has no loop. So at least $(2|S| - 2) - (2|S| - 2q) = 2(q - 1)$ edges of $N[S]$ cross the partition. By the tree-packing theorem of Nash-Williams and Tutte [NW1, Tu2], $N[S]$ has two edge-disjoint spanning trees. One of them has no value v; the other lies in its complement in $U^*[S]$, which is therefore connected and spanning. So S is a good set for (j, v), against (Q1). Hence $|N| \leq 2n^* - 3$ for each of the seven values v. Every edge lies in N for six of the seven values (all but its own value), so $6m^* \leq 7(2n^* - 3) = 14n^* - 21$. □

Nodes. A *single centre* is a node of U^* that is one claw centre; a *lump* is any other node. Every lump contains a cycle of W: each step of the chain merges a connected set (a good set has a spanning tree, and a cycle is connected), and claw centres are pairwise not adjacent in U, so every merged set contains a node that contains a cycle of W. A single centre has

degree three in U^* and three different neighbours, by (Q1), and all of them are lumps. We write s for the number of single centres, t for the number of lumps, and ε for the number of edges between two lumps. For a lump L, we also write L for the set of vertices of G that it contains; then $G[L]$ is connected, and the degree $d(L)$ of L in U^* is the number of edges of G with exactly one end in L.

28.2. **Small shores of boundary at most fourteen.**

Lemma 28.2. *Let G be a finite simple cubic graph of girth at least eighteen, and let X be a set of vertices such that $G[X]$ is connected and contains a cycle. If $d(X) \le 14$, then $|X| \ge 988$.*

Proof. Prune vertices of degree one as in the proof of Lemma 25.1: this gives a connected core $F = G[X']$ of minimum degree two, with $D = d(X') \le d(X) \le 14$ vertices of degree two and $N = |X'|$ vertices. F is not a cycle, since a cycle would have $D = N \ge 18$.

First bound: $N \ge 44$. (The first bound in the proof of Lemma 25.1 gives only $N \ge 32$ when $D \le 14$, which lies below the smaller root 38.7 of the quadratic below; so a finer count is needed.) Root the distance levels of F at a vertex of degree three. Girth at least eighteen gives unique parents through level eight and no edge inside levels one to seven. Write n_i for the size of level i and c_i for the number of vertices of degree three in level i; then $n_1 = 3$ and $n_{i+1} = n_i + c_i$. At most fourteen vertices of levels one to seven have degree two, so

$$21 + 5c_1 + 4c_2 + 3c_3 + 2c_4 + c_5 - c_7 \le 14, \quad \text{that is} \quad c_7 \ge 7 + 5c_1 + 4c_2 + 3c_3 + 2c_4 + c_5.$$

With $c_7 \le n_7 = 3 + c_1 + \cdots + c_6$ this gives $c_6 \ge 4 + 4c_1 + 3c_2 + 2c_3 + c_4$, and with $c_6 \le n_6$ it gives $c_5 \ge 1 + 3c_1 + 2c_2 + c_3$. So $c_5 \ge 1$, $c_6 \ge 4$, $c_7 \ge 8$, and the root with levels one to eight has $25 + (7c_1 + 6c_2 + \cdots + 1c_7) \ge 25 + 3 + 8 + 8 = 44$ vertices.

The averaging. The averaging in the proof of Lemma 25.1 (equation (11) and the bound 3586 for each vertex of degree two) is used here for $D \le 14$. That proof uses $D \le 10$ in three places only, and none of them blocks $D \le 14$: "F is not a cycle" needs $D < 18$; its first bound $N \ge 36$ is replaced by $N \ge 44$ above; and its last step puts the bound on D into $D/(3N - D)$, which grows with D. It gives a root edge with loss at most

$$\frac{7172D}{3N - D} \le \frac{100408}{3N - 14},$$

so $N \ge 1022 - 100408/(3N - 14)$, that is,

$$3N^2 - 3080N + 114716 \ge 0.$$

This convex quadratic has the values 2008, -841, -2737 and 108 at $N = 38$, 39, 987 and 988, so it is negative exactly for $39 \le N \le 987$. With $N \ge 44$ we get $N \ge 988$, and $|X| \ge N$. □

28.3. **Linear and triangle-free claw triples.**

Lemma 28.3. (i) *Two single centres have at most one common neighbour. Hence $s \le t(t-1)/6$.*

(ii) *Let L be a lump with x single-centre neighbours $z_1, \ldots, z_x$, and let N_i be the set of the two neighbours of z_i other than L. A single centre that is not adjacent to L has at most one neighbour in $N = N_1 \cup \cdots \cup N_x$. Hence, for the lump L with the most single-centre neighbours,*

$$s \le x + \binom{t - 1 - 2x}{2}, \qquad \frac{3s}{t} \le x \le \frac{t-1}{2}, \tag{14}$$

where $\binom{m}{2} = m(m-1)/2$, *and* $\binom{m}{2} = 0$ *for* $m \le 1$.

Proof. (i) Two common neighbours would give a cycle of four edges, against girth eight. So the s neighbour triples of the single centres cover $3s$ different pairs of lumps, and $3s \le \binom{t}{2}$.

(ii) By (i) the sets N_i are pairwise disjoint, so N has $2x$ lumps, and $L \notin N$. Let A, B be two neighbours in N of a single centre z not adjacent to L. If both lie in one N_i, then z and z_i share A and B, against (i). If $A \in N_i$ and $B \in N_j$ with $i \ne j$, then $z_i\, L\, z_j\, B\, z\, A\, z_i$ is a cycle of six edges, against girth eight. Its six nodes are different: the centres z_i, z_j are different, and z is neither of them, as z is not adjacent to L; the lumps A, B are different, as N_i and N_j are disjoint, and L is neither of them, as $L \notin N$; and a centre is never a lump. So each of the $s - x$ single centres not adjacent to L has at least two neighbours in the set R of the lumps other than L and those in N, and $|R| = t - 1 - 2x$. Choose one such pair in R for each of them; by (i) the pairs are different. This gives the first inequality of (14). For the lump with the most single-centre neighbours, $x \ge 3s/t$ (the average), and $2x \le t - 1$, since L and N are $2x + 1$ different lumps. □

For $t \ge 2$ let $S(t)$ be the largest whole $s \le t(t-1)/6$ for which some whole x with $3s/t \le x \le (t-1)/2$ satisfies (14). For $t = 15, \dots, 30$,

$$S(t) = 19,\ 21,\ 22,\ 26,\ 31,\ 33,\ 35,\ 42,\ 46,\ 48,\ 52,\ 60,\ 63,\ 65,\ 74,\ 80.$$

For example, at $t = 19$ the value $s = 32$ forces $x \ge 6$, and $6 + \binom{6}{2} = 21$, $7 + \binom{4}{2} = 13$, $8 + \binom{2}{2} = 9$ and $9 + 0 = 9$ are all below 32; the value $s = 31$ has $x = 5$ with $5 + \binom{8}{2} = 33$.

28.4. **A ball around the best edge.** We use the setting and the notation of Section 28.1, Lemma 28.1 and Lemma 28.3, and the count of the lumps: with $\Sigma = 3s + 2\varepsilon$ the sum of the degrees of the lumps and a the number of lumps L with $d(L) \le 14$,

$$2s + 3\varepsilon \le 7t - 11, \qquad \Sigma \ge 5a + 15(t - a), \qquad n \ge 988a + 18(t - a) + s, \tag{15}$$

by (Q3) (as $m^* = 3s + \varepsilon$ and $n^* = s + t$, and both sides are whole numbers), by $d(L) \ge 5$ for every lump (since $t \ge 2$, a lump and its complement both contain a cycle), and by Lemma 28.2 for the lumps with $d(L) \le 14$ and girth eighteen for the others, as the lumps are pairwise disjoint vertex sets of G and each single centre is one more vertex. Here $t \ge 2$: $t \ge 1$ as W is not empty, and $t = 1$ would force $s = 0$, as U^* is simple, and $n^* = 1$, against (Q2). Moreover $s \ge 1$: from (15), $5t \le \Sigma \le (5s + 14t - 22)/3$, that is, $5s \ge t + 22$.

Lemma 28.4. *Let L be a lump with the most single-centre neighbours, x of them, and $d = d(L)$. Then $s + t \ge 28 + 11(d - 1)$, where $d \ge 5$ and $d \ge x \ge 3s/t$. Hence*

$$s + t \ge \max\Big(72,\ 17 + \frac{33s}{t}\Big). \tag{16}$$

Proof. The $3s$ edges at the single centres end at lumps, so the average number of single-centre neighbours of a lump is $3s/t$, and $x \ge 3s/t > 0$ as $s \ge 1$. Let c be a single centre adjacent to L, and for a node v let $r(v)$ be its distance to the set $\{c, L\}$. If a node v with $r(v) \le 3$ had two different shortest paths from $\{c, L\}$, these paths, with the edge cL if they start at different ends, would contain a cycle of at most $2r(v) + 1 \le 7$ edges; and if a node v with $r(v) \le 2$ had a neighbour w with $r(w) \le r(v)$ other than its parent (the parent of c is L, and the parent of L is c), the shortest paths to v and to w with the edge vw, and with cL if needed, would contain a cycle of at most $2 + 2 + 1 + 1 = 6$ edges. Both are excluded by girth eight (Q1). So the nodes with $r \le 3$ are reached along a tree that contains the edge cL:

every node v with $r(v) \le 2$ has all its neighbours other than its parent at distance $r(v)+1$, and two different nodes with $r \le 2$ have no such neighbour in common.

On the side of c: c has two other neighbours, which are lumps; each has at least four further neighbours, as its degree is at least five, which gives at least eight nodes at distance two; each of these has at least two further neighbours, which gives at least sixteen nodes at distance three. Together at least 26. On the side of L: L has $d-1$ other neighbours. If such a neighbour is a single centre, it has two further neighbours, which are lumps with at least four further neighbours each: $1+2+8=11$ nodes. If it is a lump, it has at least four further neighbours with at least two further neighbours each: $1+4+8=13$ nodes. So each of the $d-1$ branches has at least eleven nodes, and with c and L we get $s+t=n^* \ge 2+26+11(d-1)$. Finally $d \ge 5$ gives 72, and $d \ge x \ge 3s/t$ gives $28+11(3s/t-1)=17+33s/t$. □

For $t \le 32$, (16) says $s(33-t) \le t(t-17)$; for $t \ge 33$ its second part holds for every s.

Lemma 28.5. *Let $t \ge 3$ and $s > (t-1)/2$. If (14) holds for some whole x with $3s/t \le x \le (t-1)/2$, then*

$$Q_t(s) = 36s^2 - 2t(7t-12)s + t^2(t-1)(t-2) \ge 0 \qquad \text{and} \qquad s \le \frac{t(t-3)}{6} \le \frac{t(7t-12)}{36},$$

the last number being the vertex of Q_t.

Proof. Put $m = t-1-2x$. If $m \le 1$, then $\binom{m}{2} = 0$ and $s \le x \le (t-1)/2$, which is excluded; so $m \ge 2$. The function $g(y) = y + (t-1-2y)(t-2-2y)/2$ has derivative $2-2(t-1-2y)$, which is negative while $t-1-2y > 1$, so on the whole interval $[3s/t, x]$, where $t-1-2y \ge m \ge 2$. Hence $s \le x + \binom{m}{2} = g(x) \le g(3s/t)$. With $u = 6s/t$, so that $3s/t = u/2$ and $s = ut/6$, this reads $ut/6 \le u/2 + (t-1-u)(t-2-u)/2$, that is, $u^2 - \frac{7t-12}{3}u + (t-1)(t-2) \ge 0$; multiplied by t^2 this is $Q_t(s) \ge 0$. And $u \le 2x \le t-3$, as $m \ge 2$, so $s \le t(t-3)/6$, which is at most $t(7t-12)/36$ since $6t-18 \le 7t-12$. □

As Q_t is convex and s lies at or below its vertex, s lies at or below the smaller root of Q_t. For $t = 31, \dots, 38$ the largest such whole s is

$$S'(t) = 87,\ 93,\ 99,\ 105,\ 111,\ 118,\ 125,\ 132,$$

as $Q_t(S'(t)) \ge 0 > Q_t(S'(t)+1)$, both points below the vertex: the pairs of values are $(2784, -3626)$, $(1860, -4976)$, $(2178, -5112)$, $(3996, -3776)$, $(7596, -686)$, $(4464, -4284)$, $(2690, -6552)$, $(2544, -7220)$.

28.5. **The count and the bound.**

Theorem 28.6. *If each connected component of a finite bridgeless multigraph H has at most* 19022 *odd-degree vertices, then $F(H, \mathbb{F}_2^4)$ is nonempty and connected. A least order-sixteen core has at least* 19024 *vertices.*

The proof uses [Go, Theorem 1.2(c)], an unpublished manuscript, through Lemma 22.4.

Proof. We show $n \ge 19023$; as n is even, then $n \ge 19024$.

(1) *No case with $t \le 26$.* By (16), $s \ge 72-t \ge 46$ and $s(33-t) \le t(t-17)$. For $t \le 17$ the right side is at most 0, against $s \ge 1$ (and against $s \ge 46$). For $18 \le t \le 26$, $t(t-17)/(33-t)$ grows with t and is $234/7 < 34$ at $t = 26$, against $s \ge 46$.

(2) *The rule for one row.* From (15), $\Sigma \le (5s+14t-22)/3$ and $\Sigma \ge 15t-10a$, so $a \ge (31t-5s+22)/30$, and $n \ge 970a + 18t + s$. Let $\mathrm{lo} = 72-t$, by (16), and let hi be the largest allowed s: $S(t)$ for $t = 29, 30$; $S'(t)$ for $t = 31, \dots, 36$ (Lemma 28.5 applies,

as $s \geq \mathrm{lo} \geq 36 > (t-1)/2$); (16) for $t = 27, 28$; and $s \leq (7t-11)/2$ for $t = 37, 38$. Let a_0 be the least whole number at least $(31t - 5\,\mathrm{hi} + 22)/30$; then $a \geq a_0$. If $a \geq a_0 + 1$, then $n \geq 970(a_0+1) + 18t + \mathrm{lo}$. If $a = a_0$, then $\Sigma \geq 15t - 10a_0$; but $\Sigma \leq M(s) = 3s + 2\lfloor (7t - 11 - 2s)/3 \rfloor$, and M grows with s (by one or three at each step). Let s_0 be the least $s \geq \mathrm{lo}$ with $M(s) \geq 15t - 10a_0$; then $s \geq s_0$ and $n \geq 970a_0 + 18t + s_0$.

(3) *The table.*

t	lo..hi	a_0	$15t - 10a_0$	s_0 ($M(s_0)$, $M(s_0-1)$)	least n
27	45..45	22	185	45 (193; $s_0 = \mathrm{lo}$)	21871
28	44..61	20	220	58 (220, 217)	19962
29	43..74	19	245	71 (245, 244)	19023
30	42..80	19	260	77 (261, 258)	19047
31	41..87	19	275	83 (275, 274)	19071
32	40..93	19	290	90 (292, 289)	19096
33	39..99	19	305	95 (305, 302)	19119
34	38..105	19	320	102 (320, 319)	19144
35	37..111	19	335	108 (336, 333)	19168
36	36..118	19	350	114 (350, 349)	19192
37	35..124	19	365	121 (367, 364)	19217
38	34..127	19	380	126 (380, 377)	19240

In each row the last entry is the smaller of $970a_0 + 18t + s_0$ and $970(a_0+1) + 18t + \mathrm{lo}$; the second is always the larger one (it is 22841, 20918, 19965, 19982, 19999, 20016, 20033, 20050, 20067, 20084, 20101, 20118).

(4) $t \geq 39$. $s \leq (7t-11)/2$ gives $\Sigma \leq 10.5t - 16.5$, so $a \geq (4.5t + 16.5)/10 > 19$, $a \geq 20$, and $n \geq 970a + 18t \geq 97(4.5t + 16.5) + 18t = 454.5t + 1600.5 \geq 19326$.

So $n \geq 19023$ in every case, the least row being $t = 29$, $s = 71$, $a = 19$. The passage to components, loops and nonemptiness is the last paragraph of the proof of Theorem 13.4. □

The proof does not show that 19022 is best possible.

29. A boundary theorem with at most seven terminals

This section concerns existence only; it uses no moves. Throughout it $K = \mathbb{F}_2^3$. Its two external inputs are Jaeger's theorem [Ja1], that every bridgeless multigraph has a nowhere-zero K-flow, and Edmonds' matroid intersection theorem [Ed]: two matroids M_A, M_B on one ground set E have a common independent set with t elements if and only if $r_A(S) + r_B(E \setminus S) \geq t$ for every $S \subseteq E$.

29.1. **Instances and the statement.**

Definition 29.1. An *instance* is a pair (H, β), where H is a finite connected bridgeless multigraph with at least one vertex (loops and parallel edges are allowed) and $\beta : V(H) \to K$ satisfies $\sum_v \beta(v) = 0$. The vertices with $\beta(v) \neq 0$ are the *terminals*. A *solution* is a map $f : E(H) \to K \setminus \{0\}$ such that, for every vertex v, the sum of f over the edges at v equals $\beta(v)$; a loop at v is counted twice in this sum, so it contributes 0.

Every edge is counted twice in the sum of all the vertex sums; this is why the total of β must be 0. For a vertex set X we write $d_H(X)$ for the number of edges with exactly one end in X, and H/X for the graph in which X becomes one vertex and the edges inside X are deleted.

Theorem 29.2 (Seven terminals)**.** *Every instance with at most seven terminals has a solution.*

The number seven cannot be raised. Let H be a cycle $v_1 \cdots v_8$ whose eight vertices are terminals and whose partial sums $s_i = \beta(v_1) + \cdots + \beta(v_i)$ are the eight elements of K. A map with boundary β has the form $p + s_i$ on the edge $v_i v_{i+1}$, and it takes the value 0 on some edge. The hypothesis that H is bridgeless cannot be dropped either: the value on a bridge is the sum of β over one of its sides, which can be 0.

For a cubic graph the theorem takes the following form, in which the values on the edges leaving a vertex set are prescribed.

Corollary 29.3. *Let G be a cubic graph and $L \subseteq V(G)$ a set such that $G[L]$ is connected and bridgeless, with at most seven edges leaving L, all at different vertices of L. Then every assignment of nonzero values of K with sum 0 to the leaving edges extends to a map $E(G[L]) \to K \setminus \{0\}$ whose sum at every vertex of L, the leaving edges included, is 0.*

Proof. Put $H = G[L]$ and let $\beta(v)$ be the prescribed value of the leaving edge at v, and $\beta(v) = 0$ if no edge leaves L at v. Then (H, β) is an instance with at most seven terminals, and a solution is the required map. □

29.2. **Cutting off a set.**

Lemma 29.4. *Let (H, β) be an instance with at most seven terminals, and suppose Theorem 29.2 holds for all instances with fewer edges than H. Let $X \subsetneq V(H)$ have at least two vertices, with $H[X]$ connected and bridgeless, and suppose that the number of terminals in X plus $d_H(X)$ is at most seven. Then (H, β) has a solution.*

Proof. Let $H_1 = H/X$, with x the vertex that replaces X, and put $\beta_1(x) = \sum_{v \in X} \beta(v)$ and $\beta_1 = \beta$ elsewhere. The graph H_1 is connected, is bridgeless because a cut of H_1 is a cut of H, and has fewer edges because $H[X]$ has an edge. It has no more terminals than H, since x is a terminal only if X contains one. Take a solution f_1. On $H[X]$ let $\beta_2(v)$ be $\beta(v)$ plus the sum of f_1 over the edges that leave X at v. Its total is $\beta(X) + \beta_1(x) = 0$, and it has at most (terminals in X) $+ d_H(X) \le 7$ terminals. The graph $H[X]$ is connected and bridgeless, and it has fewer edges than H because an edge leaves X. Take a solution f_2 of $(H[X], \beta_2)$. Then f_1 outside $H[X]$ and f_2 inside form a solution of (H, β). □

Lemma 29.5. *Let (H, β) be an instance with at most seven terminals, and suppose Theorem 29.2 holds for all instances with fewer edges than H. Let X have at least two vertices, with $H[X]$ connected, $d_H(X) = 2$, and at most one terminal in X. Then (H, β) has a solution. Here $H[X]$ may have bridges.*

Proof. First solve H/X, which has fewer edges and no more terminals. The vertex x has two edges, with values p and $p + s$, where $s = \beta(X)$; both are nonzero. Let u_a, u_b be the ends in X of the two leaving edges and y the terminal of X, if there is one; some of these vertices may coincide. The new boundary on $H[X]$ is nonzero at most at these three vertices. Its total is 0, so it is nonzero at exactly zero, two or three vertices: at two vertices the two values are equal, and at three vertices they are $c, d, c + d$ with c, d different and nonzero.

If $u_a \neq u_b$, the new boundary is nonzero at both. Indeed, at u_a it is the value of f_1 on the leaving edge at u_a if u_a is not the terminal; if u_a is the terminal, then $s = \beta(y)$ and the new boundary at u_a is this value plus $\beta(y)$, which is the value of f_1 on the other leaving edge. The same holds at u_b. So the case of zero vertices occurs only with $u_a = u_b$, and in the other two cases with $u_a \neq u_b$ both u_a and u_b are among the vertices with nonzero new boundary.

Close $H[X]$: with nothing in the case of zero vertices, with one new edge between the two vertices, or with one new vertex joined to the three vertices. The closed graph is bridgeless. A bridge of $H[X]$ separates u_a from u_b, because it is not a bridge of H, and the new edge or the new vertex joins u_a to u_b; if $u_a = u_b$, then $H[X]$ has no bridge at all; and a new edge is not a bridge because $H[X]$ is connected. By Jaeger's theorem the closed graph has a nowhere-zero K-flow. A linear automorphism of K carries it to a flow with the wanted values on the new edges: for one new edge, any nonzero value goes to any nonzero value; for a new vertex of degree three, any two different nonzero values go to any two different nonzero values, and the third value is their sum. Its restriction to $H[X]$ is a solution for the new boundary, and together with f_1 it solves (H, β). □

29.3. **A least counterexample.** Let (H, β) be a counterexample to Theorem 29.2 with the fewest edges, and let $k \leq 7$ be the number of its terminals.

Lemma 29.6. *The following hold.*

(M0) *H has a vertex of degree at least three.*

(M1) *H has no loop, no vertex of degree two with $\beta = 0$, and no vertex of degree two both of whose edges go to one vertex.*

(M2) *If $X \subsetneq V(H)$ has at least two vertices and $H[X]$ is connected and bridgeless, then the number of terminals in X plus $d_H(X)$ is at least eight.*

Let G' be the graph obtained from H by suppressing all vertices of degree two; its vertices are the vertices of H of degree at least three. Then:

(M3) *G' has no loop and at least two vertices, and every cut of G' has at least three edges.*

(M4) *Every edge of G' carries at most one vertex of degree two of H, and such a vertex is a terminal.*

Proof. (M0). Otherwise H is connected and bridgeless with all degrees at most two. If it is one vertex without an edge, then $\beta = 0$ and the empty map is a solution. Otherwise it is a cycle $v_1 \cdots v_n$ (a loop is a cycle with one edge). Put $f = p + s_i$ on the edge $v_i v_{i+1}$, where $s_i = \beta(v_1) + \cdots + \beta(v_i)$. The s_i take at most $\max(1, k) \leq 7$ values, so some p is none of them, and f is a solution.

(M1). Suppose a vertex of degree two has $\beta = 0$. Its two edges are different: a vertex whose only edge is a loop would be all of H, since H is connected, against (M0). Suppress the vertex, so that its two edges become one edge. The new graph is connected and bridgeless, has one edge fewer and the same terminals; a solution of it gives both old edges the value of the new edge. Suppose next that H has a loop. Delete it; the graph stays connected and bridgeless. Solve it and give the loop any nonzero value; the loop adds 0 at its vertex. Finally suppose a vertex y of degree two has both its edges to one vertex v. Then $H - y$, with $\beta(v) + \beta(y)$ at v, is an instance with fewer edges and no more terminals: it is connected, and a bridge of $H - y$ would be a bridge of H. Take a solution, and give the two edges at y the values p and $p + \beta(y)$ with $p \notin \{0, \beta(y)\}$.

(M2) is Lemma 29.4.

(M3), first part. By (M0) the graph G' is defined. A loop of G' is a cycle of H with exactly one vertex v of degree at least three and r vertices of degree two. The case $r = 0$ is a loop of H and the case $r = 1$ is a vertex of degree two with both edges to v; both are excluded by (M1). For $r \geq 2$ let X be the set of all other vertices. Then $H[X]$ is connected, a bridge of it would be a bridge of H, and X has at least two vertices, because v has a third edge, which by (M1) is not a loop and does not end on the cycle. By (M1) the r vertices of degree

two are terminals, so the number of terminals in X plus $d_H(X)$ is at most $(k-r)+2 \leq 7$, against (M2). So G' has no loop. A vertex of degree at least three without a loop has a neighbour in G', so G' has at least two vertices.

(M3), the cuts. G' has no cut of zero or one edges, because H is connected and bridgeless. Suppose G' has a cut of two edges and take an inclusion-least side X' of such a cut. Then $G'[X']$ is connected, since a component of it is a side of a cut of at most two edges and so is all of X'. It is bridgeless, since a bridge would give a smaller side of a cut of two edges. And X' has at least two vertices, since one vertex of G' has degree at least three and no loop. Let X be X' together with the vertices of degree two on the edges of G' inside X'. Then $H[X]$ is connected and bridgeless, X has at least two vertices and $d_H(X) = 2$; by (M2), X contains at least six terminals. Let $Z = V(H) \setminus X$; it contains the vertices of degree two on the two cut edges, and it contains at most one terminal. The graph $H[Z]$ is connected: the other side of the cut is connected in G', because a component of a side has at least two leaving edges (G' has no cut of zero or one edges) and only two edges leave the side; and the vertices on the cut edges hang on it as end paths. The set Z has at least two vertices, since the other side is not a single vertex of G' (such a vertex has degree at least three and no loop). And $d_H(Z) = 2$. Lemma 29.5 for Z, which does not ask that $H[Z]$ be bridgeless, gives a solution: a contradiction.

(M4). A vertex of degree two is a terminal by (M1). Let $r \geq 2$ be the number of vertices of degree two on an edge e of G'; e is not a loop. Let X be the set of all other vertices of H. Then $H[X]$ is connected, because $G' - e$ is connected by (M3). It is bridgeless, because a bridge g of $G' - e$ would give the cut $\{e, g\}$ of G', against (M3). The set X has at least two vertices, the ends of e. And the number of terminals in X plus $d_H(X)$ is $(k-r)+2 \leq 7$, against (M2). □

Thus the vertices of H are the vertices of G' together with at most one vertex of degree two on each edge of G', each of them a terminal. At most seven edges of G' carry a terminal, and the other terminals are vertices of G'.

29.4. **Three trees and a flag.** In this subsection and the next, G' is any multigraph without loops, with at least two vertices, in which every cut has at least three edges. Different *marked* edges $e_1, \dots, e_j$ carry values $z_1, \dots, z_j \in K \setminus \{0\}$, and $b : V(G') \to K$ has total 0. We want a map $a : E(G') \to K$ with boundary b, that is, with sum $b(v)$ over the edges at each vertex v, such that $a(e) \neq 0$ for every edge and $a(e_i) \neq z_i$ for every marked edge.

An instance as at the end of the last subsection gives this setting. The marked edges are the edges of G' that carry a terminal y_i of degree two, and $z_i = \beta(y_i)$. Choose one end v_i of e_i and let $b(v)$ be $\beta(v)$ plus the sum of the z_i with $v_i = v$; its total is the total of β, which is 0. A wanted map a gives a solution f: put $f = a(e)$ on an edge without a terminal; on e_i put $a(e_i)$ on the half away from v_i and $a(e_i) + z_i$ on the half at v_i.

A *flag* is a line $\ell = \{0, x\}$ inside a plane P of K. The *colour* of a nonzero value z is 1 if $z \in \ell$, is 2 if $z \in P \setminus \ell$, and is 3 if $z \notin P$. Take a basis $\varphi_1, \varphi_2, \varphi_3$ of the linear forms with $P = \ker \varphi_3$ and $\ell = \ker \varphi_2 \cap \ker \varphi_3$. Write $c_m = \varphi_m \circ a$ and $q(z) = (\varphi_2(z), \varphi_3(z))$. Then colour 1 means $q(z) = (0,0)$ and $\varphi_1(z) = 1$; colour 2 means $q(z) = (1,0)$; and colour 3 means $\varphi_3(z) = 1$.

Three spanning trees T_1, T_2, T_3 of G' satisfy *the rule* if

(i) no edge lies in all three trees, and
(ii) a marked edge of colour m does not lie in both trees other than T_m.

Lemma 29.7. *If three spanning trees satisfying the rule exist for some flag, then the wanted map a exists, for every b.*

Proof. For each m: once c_m is given on the edges outside T_m, exactly one choice on the edges of T_m gives c_m the boundary $\varphi_m \circ b$, because T_m is a spanning tree and the total of $\varphi_m \circ b$ is 0. We choose the coordinates in three stages.

Stage 3. Put $c_3(e) = 1$ on every edge outside T_3. Now c_3 is known on all edges.

Stage 2. On an edge e outside T_2: if e is marked with value z, $c_3(e) = 1$ and $\varphi_3(z) = 1$, put $c_2(e) = \varphi_2(z) + 1$; in all other cases put $c_2(e) = 1$. Now c_2 is known on all edges; write $q(e) = (c_2(e), c_3(e))$.

Stage 1. On every edge e outside T_1 choose $c_1(e)$ so that $a(e) \neq 0$ and, if e is marked with value z, $a(e) \neq z$.

We show that stage 1 is possible and that the edges of T_1 are also good. There are three kinds of edges: (A) $e \notin T_3$; (B) $e \in T_3$ and $e \notin T_2$; (C) $e \in T_2 \cap T_3$, and then $e \notin T_1$ by rule (i).

Fact 1: in cases (A) and (B), $q(e) \neq (0, 0)$. In (A), $c_3(e) = 1$. In (B), if $c_3(e) = 0$, then stage 2 gave $c_2(e) = 1$.

Fact 2: in cases (A) and (B), if e is marked with value z and $q(e) = q(z)$, then $e \notin T_1$. In (A), $c_3(e) = 1$, so $\varphi_3(z) = 1$ and z has colour 3. If e were outside T_2, stage 2 would give $c_2(e) = \varphi_2(z) + 1$ and $q(e) \neq q(z)$. So $e \in T_2$, and by rule (ii) for colour 3, $e \notin T_1$. In (B), if $c_3(e) = 1$, then $\varphi_3(z) = 1$ and stage 2 gave $c_2(e) = \varphi_2(z) + 1$, which is impossible. So $c_3(e) = 0 = \varphi_3(z)$ and $c_2(e) = 1 = \varphi_2(z)$: colour 2. As $e \in T_3$, rule (ii) for colour 2 gives $e \notin T_1$.

The edges in T_1 are of kinds (A) and (B). By Fact 1, $q(e) \neq (0, 0)$, so $a(e) \neq 0$; by Fact 2, $q(e) \neq q(z)$ if e is marked, so $a(e) \neq z$; whatever $c_1(e)$ is.

The edges outside T_1. The value $c_1(e) = 0$ is forbidden only if $q(e) = (0, 0)$, and the value $c_1(e) = \varphi_1(z)$ is forbidden only if $q(e) = q(z)$. In (A) and (B), $q(e) \neq (0, 0)$, so at most one value is forbidden. In (C), both values are forbidden only if $q(z) = (0, 0)$, that is, colour 1; but by rule (ii) an edge of colour 1 is not in T_2 and T_3 together. So stage 1 always has a value. □

29.5. **The trees exist.**

Lemma 29.8. *Let G' be as above, with at most eight marked edges, each with a colour* 1, 2 *or* 3 (*any colours*). *Then three spanning trees satisfying the rule exist. The same holds with at most eleven marked edges, at most eight of each colour.*

Proof. The ground set consists of the pairs (e, m) with $e \in E(G')$ and $m \in \{1, 2, 3\}$, read as "tree m uses e". In the matroid M_A a set I is independent if, for each m, the edges e with $(e, m) \in I$ form a forest. In the matroid M_B a set I is independent if for every edge e the set $I_e = \{m : (e, m) \in I\}$ is allowed: for an unmarked edge, at most two elements; for a marked edge of colour m, not both elements other than m. Thus M_B is a direct sum of matroids of rank two on the three elements $(e, 1), (e, 2), (e, 3)$: the uniform matroid for an unmarked edge; for colour m, the two elements other than m are parallel and m is free. The rank $r_e(T)$ of $T \subseteq \{1, 2, 3\}$ is $\min(|T|, 2)$ for an unmarked edge, and $[m \in T] + [T \not\subseteq \{m\}]$ for a marked edge of colour m.

Let $n = |V(G')|$. The graph G' is connected, since every cut has at least three edges, so a forest with $n - 1$ edges is a spanning tree. A common independent set with $3(n - 1)$ elements is the same as three spanning trees satisfying the rule, because each of the three forests has

at most $n-1$ edges and hence exactly $n-1$. By Edmonds' theorem such a set exists if and only if $r_A(S) + r_B(S^c) \geq 3(n-1)$ for every subset S of the ground set. For such an S let R_m be the set of edges e with $(e,m) \notin S$, let p_m be the number of components of $G' - R_m$, and let $T_e = \{m : e \in R_m\}$. Then $r_A(S) = (n-p_1) + (n-p_2) + (n-p_3)$, and the condition becomes

$$(p_1 - 1) + (p_2 - 1) + (p_3 - 1) \leq \sum_{e \in E(G')} r_e(T_e). \tag{17}$$

To prove (17), let $A = \{m : p_m \geq 2\}$. Making R_m empty for every $m \notin A$ does not change the left side, where $p_m = 1$ already, and does not increase the right side, which is a rank. So assume $R_m = \emptyset$ for $m \notin A$. For $m \in A$, every component of $G' - R_m$ has at least three leaving edges, all of them in R_m, and an edge is counted at most twice; so $|R_m| \geq 3p_m/2$ and $p_m - 1 \leq \frac{2}{3}|R_m| - 1$. With $N = |R_1| + |R_2| + |R_3| = \sum_e |T_e|$, the left side of (17) is at most $\frac{2}{3}N - |A|$.

For every edge, $r_e(T) \geq \frac{2}{3}|T|$, with one exception: a marked edge of colour m with T equal to the two elements other than m, where $r_e(T) = 1 = \frac{2}{3}|T| - \frac{1}{3}$. Indeed, for an unmarked edge, $|T| = 1, 2, 3$ gives $r_e = 1, 2, 2$ against $\frac{2}{3}, \frac{4}{3}, 2$; for colour m, the set $\{m\}$ and a single other element have rank 1, the sets $\{m, m'\}$ and $\{1, 2, 3\}$ have rank 2, and the two other elements together have rank 1 against $\frac{4}{3}$. Let D be the number of exceptional edges. The right side of (17) is at least $\frac{2}{3}N - \frac{1}{3}D$, so the right side minus the left side is at least $|A| - \frac{1}{3}D$.

If $|A| \leq 1$, every T_e has at most one element and $D = 0$. If $|A| = 2$, every exceptional edge has the colour outside A, so $D \leq 8$ and $|A| - \frac{1}{3}D \geq -\frac{2}{3}$. If $|A| = 3$, then $D \leq 11$ and $|A| - \frac{1}{3}D \geq -\frac{2}{3}$. Both sides of (17) are integers, so the inequality holds. □

29.6. **Proofs of the theorems.**

Proof of Theorem 29.2. Take a counterexample with the fewest edges. By Lemma 29.6, G' has no loop and at least two vertices, every cut of G' has at least three edges, and every edge of G' carries at most one vertex of degree two of H, which is a terminal. So we are in the setting of the last two subsections with $j \leq 7$ marked edges: each marked edge holds its own terminal of degree two, and there are at most seven terminals. Take any flag. Lemma 29.8 gives three spanning trees satisfying the rule, Lemma 29.7 gives the wanted map a, and a gives a solution of (H, β): a contradiction. □

The two lemmas give more when no induction is needed.

Theorem 29.9. *Let G' be a multigraph without loops, with at least two vertices, in which every cut has at least three edges. Let at most eight different edges e_i be marked, with any nonzero values $z_i \in K$, and let $b : V(G') \to K$ be any map with total 0. Then there is a map $a : E(G') \to K$ with boundary b such that $a(e) \neq 0$ for every edge and $a(e_i) \neq z_i$ for every marked edge. The same holds with at most eleven marked edges if some flag gives no colour to more than eight of them.*

Proof. Lemma 29.8 and Lemma 29.7. □

30. A component correspondence for free edge sets

This section allows any finite abelian Γ, any subgroup B with $|B| \geq 2$, and $A = \Gamma/B$. Neither group is assumed binary and the quotient need not split. All moves are signed.

Subgroup-valued corrections on fundamental cycles already appear in the proof of [EHLMNS, Theorem 6.3], where tree edges are protected modulo a subgroup while cotree edges are corrected. A complementary-subgroup preparation appears in [CLSWX, Lemma 6.12], under different hypotheses and with a different conclusion. The statement below concerns exact fibres and connected components of a specified induced family; kernel cycle correction itself is not new.

Let G be connected and let $R \subseteq E(G)$ satisfy that $G - R$ is connected as a spanning graph. Contract each component of $(V(G), R)$ and delete the edges of R, retaining all other edges, including loops; call the result G/R. Let $\mathcal{K}_R$ be the induced subgraph of $F(G, \Gamma)$ on states satisfying

$$\overline{f(e)} \neq 0 \qquad (e \notin R).$$

Conservation summed over each R-component shows that restriction defines a map

$$p : V(\mathcal{K}_R) \longrightarrow V(F(G/R, A)), \qquad p(f) = \overline{f}\big|_{E(G)\setminus R}.$$

Theorem 30.1 (Free-set component correspondence)**.** *The map p is surjective, every exact fibre is connected in $\mathcal{K}_R$, and p induces a bijection between the connected components of $\mathcal{K}_R$ and those of $F(G/R, A)$.*

This is a correspondence for the *restricted induced graph* $\mathcal{K}_R$. An edge upstairs may project to a path of several moves downstairs; p need not be a graph homomorphism. The two vertex sets are empty together when no downstairs flow exists.

Proof. For every $e \in R$, connectedness of $G - R$ gives a cycle D_e containing e and no other edge of R: use a simple path in $G - R$ between its ends, or take the loop itself. Direct it so that $\chi_{D_e}(e) = 1$. A correction on D_e changes no other R-edge. If its coefficient lies in B, it preserves all outside-R quotients.

Surjectivity on states. Let h be a nowhere-zero A-flow on G/R. Extend it to an A-flow u on G. Inside each R-component the prescribed vertex imbalances sum to zero by conservation at the contracted vertex, so solve them on a spanning tree of that R-component; assign zero to its other R-edges, keeping every prescribed outside-R value unchanged. Lemma 14.1 lifts u to a Γ-flow F, which may have full zeros only on quotient-zero edges of R.

For every edge e of R whose quotient is zero, choose $z_e \in B \setminus \{0\}$ and set $\beta_e = z_e - F(e) \in B$. Then

$$f = F + \sum_{\substack{e \in R \\ u(e)=0}} \beta_e \chi_{D_e}$$

has value z_e at each such edge, changes no other R-edge, and preserves the nonzero quotient on every outside edge. Thus f is a nowhere-zero state with $p(f) = h$. This constructs a state, not a legal path starting from the possibly zero flow F.

Realizing a quotient cycle change. Let $f \in V(\mathcal{K}_R)$ have quotient u. Suppose C is a cycle of G, $0 \neq \sigma \in A$, and $u' = u + \sigma\chi_C$ is nonzero outside R. Choose a lift $\gamma \in \Gamma$ of σ. Every newly zero quotient edge lies in $C \cap R$ and currently has nonzero quotient. For each such edge e, choose $\beta_e \in B$ avoiding the single value

$$-f(e) - \gamma\chi_C(e).$$

There is a choice since $|B| \geq 2$. If $\beta_e \neq 0$, add $\beta_e\chi_{D_e}$. This is legal: e has nonzero quotient before and during preparation, and all other changed edges lie outside R with nonzero quotient. Preparations leave other R-edge data unchanged.

Now add $\gamma\chi_C$. Newly zero quotients have nonzero full values by construction. An edge with quotient zero both before and after cannot lie on C, since $\sigma \neq 0$; it belongs to R and was untouched by the preparations. All other edges are protected by their final quotients. The result is a state with quotient u', reached within $\mathcal{K}_R$.

Connected exact fibres. Suppose $p(f) = p(g)$. The quotient difference $\overline{g} - \overline{f}$ is supported on R. Decompose it into signed cycle flows within R, using Lemma 14.1, and realize the nonzero terms one at a time by the preceding construction. Outside-R quotients never change, so the path stays in the exact fibre. Now the quotients agree everywhere.

At each quotient-zero edge e, both the current full value and $g(e)$ are nonzero elements of B. Add their difference on D_e, skipping a zero coefficient. This changes e directly to the nonzero target value, leaves every other R-edge unchanged, and changes only quotient-protected edges outside R. After these legal alignments the remaining difference is a B-valued flow supported on the subgraph of nonzero quotient edges. Decompose it into signed cycles *of that subgraph* and add them successively. Every intermediate value is protected by its unchanged quotient. This reaches g in the exact fibre. The explicit full-value alignment at quotient-zero edges is necessary when B has more than two elements.

Lifting paths from the contracted graph. A directed cycle of G/R lifts to a directed cycle of G by joining its entry and exit vertices with a simple R-path inside each visited contracted component. Distinct visited components are disjoint. A quotient loop uses its single outside edge and an internal R-path, or that edge alone if already a loop. The lifted cycle's signed restriction is the prescribed downstairs vector. The quotient-cycle construction therefore realizes any downstairs step starting from any lift. Connected exact fibres then allow any prescribed terminal lift. Consequently downstairs-connected states have all their lifts in one component of $\mathcal{K}_R$.

Projecting paths to the contracted graph. Consider a legal step $f \to f + \gamma\chi_C$ within $\mathcal{K}_R$. If $\overline{\gamma} = 0$, its projection is unchanged. Otherwise set $\sigma = \overline{\gamma}$. Restrict the *integer* signed vector χ_C to outside-R edges after contraction. It is an integer circulation z with coefficients in $\{0, 1, -1\}$, each used edge occurring once. Orient its used edges according to these signs. The resulting directed graph is balanced at each vertex, and so decomposes into edge-disjoint directed cycles, including loops. There is no cancellation and every edge keeps its sign in z.

Add σ on these directed quotient cycles successively. Each edge has either its old or its final value, both nonzero because the two full endpoint states lie in $\mathcal{K}_R$. All intermediate quotient flows are therefore nowhere zero. If the projected support is empty, no move is needed. This is an integer signed decomposition, not an even-support decomposition modulo two.

Thus an upstairs path projects to a downstairs path. Together with state surjectivity and the converse path implication, this proves the bijection on components. □

Corollary 30.2 (A common ambient component for fixed B)**.** *Suppose G has a nowhere-zero A-flow. For each edge set R with $G - R$ connected and $F(G/R, A)$ connected, the family $\mathcal{K}_R$ is nonempty and connected. All such families lie in one component of $F(G, \Gamma)$.*

Proof. Lift a nowhere-zero A-flow on G by Lemma 14.1. The lift has nonzero quotient on every edge, hence belongs to every such family. Theorem 30.1 makes each family connected. □

The assumption $|B| \geq 2$ cannot simply be dropped. Take three parallel edges between two vertices, let R be one edge, and let $\Gamma = A = \mathbb{Z}_2$ with trivial B. The graph $G - R$ is

connected and G/R consists of two loops, which has a nowhere-zero A-flow. The original cubic graph has none: three values equal to one do not conserve at a vertex. Thus state surjectivity already fails.

Distinct components of $\mathcal{K}_R$ can in principle merge along paths leaving the family. A disconnected downstairs graph therefore does not furnish a counterexample to unrestricted Γ-flow connectivity. Corollary 30.2 also fixes B; it makes no identification across different subgroup choices.

31. Outlook and limitations

Universal connectivity for $\mathbb{F}_2^3$ or $\mathbb{F}_2^4$ on all bridgeless graphs remains open. Theorems 22.1 and 28.6 give the respective bounds 44 and 19022 on odd-degree vertices per component. A least core has at least 46 vertices at order eight and 19024 at order sixteen. The general formula and its intermediate order-sixteen bound 764 remain valid without the four-cut theorem. The order-sixteen bounds 1020, 3036, 3996 and 19022 use the four-cut reduction and seventeen-cycle exclusion; the last three also use rounded partition density, the small-shore count and the edge-and-claw factor of Section 24. Lemma 22.4 uses Theorem 1.2(c) of the unpublished manuscript [Go], and so, through it, do Theorems 22.1, 23.1, 13.4, 26.4 and 28.6 and Corollaries 22.2, 24.9 and 27.5; the order-32 results, Theorem 13.1 and Corollary 13.2, do not use it. The bounds 3036, 3996 and 19022 also use Lemma 24.7, whose proof takes a frame as in Proposition 14.2; Theorem 24.8 uses Lemma 24.7 but not Lemma 22.4. Whether cycles of length eighteen can be excluded, and whether these bounds are optimal, is open. Theorem 20.1 permits at most five designated exceptional star cuts; it does not cover arbitrary collections of five small cuts. In the cubic setting, Corollaries 19.8 and 20.3 give sufficient matching and support tests, but no construction meeting either test for every flow is supplied. Theorem 21.1 removes a four-cut only under its explicit structural and smaller-graph hypotheses; it does not remove a five-cut. Another possible route would be to find, after legal reconfiguration, an edge set R with connected complement for which the appropriate quotient flow graph of G/R is connected. Theorem 30.1 would then control that restricted family. To prove a universal conclusion this route still needs an accessibility theorem and a way to place all starting states in a common ambient component. The component correspondence alone does not give these.

The bounds for threads of up to three edges do not permit the same proof with order 32 replaced by order 16: a paired list can have six entries, exceeding a quotient of order four, and a free thread can forbid three kernel values, exceeding a kernel of order two. Lemma 24.7 uses the separate one-edge-thread case. The short-cycle contraction families of [CLSWX, Theorem 5.9 and Corollary 5.14] are not new here.

Theorem 24.8 supplies a different sufficient support construction for cubic graphs with all cyclic cuts of size at least eleven. Corollary 24.9 shows that a remaining least core has a cyclic cut of size five through ten. The factor theorem places no upper bound on the number of uncovered vertices or support cycles, so it does not by itself settle these smaller cyclic cuts. Sections 26 and 28 apply the same factor to every least core; they bound its size from below and do not exclude it. Theorem 29.2 is an existence statement and enters no connectivity proof here. We do not know whether the factor theorem is new.

Dependencies and status. The all-bridgeless binary-extension proof is an extension of Part I, with extraction and frame existence inherited from Section 2 and Appendix A, and the finite-round marginal estimate inherited from the proof of Lemma 4.1. Their

statements and uses are explicit in Section 14. The local arguments are written here in the all-nonzero-coefficients setting; the signed global induction is included in full. The separate four-edge-connected proof in Section 19 uses the classical tree-packing criterion, elementary boundary solving on a tree, and binary cycle decomposition; its other lemmas are proved there. Section 20 builds on those proved lemmas and includes the binary short-cycle lift used in its argument. Section 21 proves the conditional four-cut theorem, with Jaeger's existence theorem as an external input. Section 22 proves the wide least-odd-count interface, with the existence input of Section 14 and the two-odd-vertex theorem [Go, Theorem 1.2(c)] as its outside inputs, and supplies the complete cycle-order lifting and family-intersection proof. It is independent of the four-cut theorem and does not use the bound $2^{q/2+1}-4$ of [Go, Theorem 1.4] beyond its range. The independent free-set correspondence uses only elementary cycle decomposition and connectedness of $G-R$. Section 23 gives the full special order-sixteen argument. Its core input is Lemma 22.4; its family joining uses Corollary 21.5; its local crossing exits use only cubicity and the fixed cycle length and group order. Lemma 22.6 supplies the exclusion of sixteen-cycles, which the proof of Theorem 23.1 combines with Lemma 23.2 to obtain girth at least eighteen. The component-product argument is explicit in the proof of Theorem 23.1.

Section 24 proves the factor theorem in full; the proof uses no enumeration. The partition count and support transfer are written out there. Lemma 24.7 specializes the earlier frame proofs to single-edge threads, explicitly checking the smaller kernel bound; Theorem 24.8 also uses the nowhere-zero existence input from Section 14. Within that section, only the least-core corollary uses the four-cut reduction.

Corollary 19.7 gives the stronger integer criterion used in Section 25. The proofs of Section 25 use no graph enumeration.

Section 26 uses the properties of a least core (Lemmas 22.4, 22.6 and 23.2, Corollary 21.5), Sections 24 and 25 and Corollary 19.7. Section 27 uses Lemmas 19.2, 19.4 and 20.2, Corollary 19.6, Theorem 13.3 and Corollary 22.2, and, for a least core, Lemma 22.4 and Corollary 21.5, with the matroid union theorem [NW2] as an external input. Section 28 uses Section 27, the properties of a least core, Section 24, the argument of Corollary 19.7, the small-shore count of Lemma 25.1, the last paragraph of the proof of Theorem 13.4 and the tree-packing theorem [NW1, Tu2]; its finite arithmetic is a table of twelve rows and the values of two functions at twenty-four arguments, all checkable by hand; no graph enumeration is used. Section 29 is independent of all other sections; its external inputs are Jaeger's theorem [Ja1] and Edmonds' matroid intersection theorem [Ed].

Part III. Recolouring 2-degenerate graphs with four colours

32. Recolouring: introduction

In Part III all graphs are finite and simple. Let G be a graph and k a positive integer. The *k-recolouring graph* $\mathcal{R}_k(G)$ has as vertices the proper k-colourings of G; two colourings are adjacent if they differ at exactly one vertex. A graph is *d-degenerate* if every subgraph has a vertex of degree at most d. A d-degenerate graph has a proper $(d+1)$-colouring, and for $k \geq d+2$ the graph $\mathcal{R}_k(G)$ is connected (Dyer, Flaxman, Frieze and Vigoda [DFFV]; see also [Ce, Thm. 2.7]). For $k = d+1$ it can be disconnected ($G = K_{d+1}$). For $d = 2$ every component of $\mathcal{R}_3(G)$ has diameter $O(n^2)$ [CvdHJ], but for every $d \geq 3$ there are d-degenerate graphs G for which $\mathcal{R}_{d+1}(G)$ has a component of superpolynomial diameter, at least $2^{c\sqrt{n}}$

for some $c > 0$ [BC, Thm. 1.4] (these graphs have degeneracy d; see [Ce, p. 109]). Cereceda asked how large the diameter is for $k \geq d+2$.

Conjecture 32.1 (Cereceda [Ce])**.** *For every $d \geq 1$ and every d-degenerate graph G on n vertices,* $\operatorname{diam} \mathcal{R}_{d+2}(G) = O(n^2)$.

The conjecture holds for $d = 1$ (forests): Cereceda proved $\operatorname{diam} \mathcal{R}_k(G) = O(n^2)$ for $k \geq 2d+1$ [Ce, Thm. 5.22], and $2d+1 = d+2$ when $d = 1$; see also [BJLPP, Thms. 2 and 6]. Bousquet and Heinrich [BH] proved $\operatorname{diam} \mathcal{R}_k(G) = O(n^{d+1})$ for every d-degenerate G and $k \geq d+2$, the first polynomial bound for $k = d+2$; for $d = 2$ this is $O(n^3)$. They also proved the quadratic bound once $k \geq \frac{3}{2}(d+1)$, which for $d = 2$ means $k \geq 5$. Before their work no polynomial bound was known for $\mathcal{R}_4(G)$ over all 2-degenerate graphs G. For $d = 2$ the following partial results are known ($k = 4$ unless stated otherwise):

- sparse graphs, measured by the maximum average degree $\operatorname{mad}(G)$: Feghali [Feg], improving the polynomial bound of Bousquet and Perarnau [BP, Thm. 2], proved $\operatorname{diam} \mathcal{R}_k(G) = O(n(\log n)^{d-1})$ when $\operatorname{mad}(G) \leq d - \varepsilon$ and $k \geq d+1$ (for $k = 4$: $\operatorname{mad}(G) \leq 3 - \varepsilon$ for a fixed $\varepsilon > 0$, and the bound $O(n \log^2 n)$), and Zheng and Lu [ZL] sharpened the exponent of the logarithm; with lists of size 4, Cranston [Cr, Thm. 3] proved that each vertex need be recoloured at most 14 times when $\operatorname{mad}(G) < 22/9$. Pan, Wang and Liu [PWL, Thm. 1.4] extended the density range to $\operatorname{mad}(G) < 5/2$, with at most 18 recolourings per vertex, including planar graphs of girth at least 10;
- 2-degenerate graphs of maximum degree at most 3: $\operatorname{diam} \mathcal{R}_4(G) = O(n^2)$ by Feghali, Johnson and Paulusma [FJP, Thm. 5] (for $\Delta = 3$, by [FJP, Thm. 2]: for every connected G with $\Delta \geq 3$, $\mathcal{R}_{\Delta+1}(G)$ has at most one component that is not a single vertex, and its diameter is $O(n^2)$; for 2-degenerate G the graph $\mathcal{R}_4(G)$ is connected);
- chordal bipartite graphs (not all of them 2-degenerate): at most $2n^2$ moves for every $k \geq 3$ [BJLPP, Thms. 2, 8, 10];
- planar graphs of large girth (see Table 1 of [BBFHMP]): for 4 colours, n times a power of $\log n$ at girth 7 and more [Feg], and $O(n^3)$ at girth 6 [BH]; at girth 5 (planar graphs of girth 5 need not be 2-degenerate) the graph $\mathcal{R}_4$ is connected [BBFHMP];
- graphs of treewidth at most 2: $2n(n+1)$ for $k \geq 4$ by Bonamy and Bousquet [BB, Thm. 2] (who prove $2(n^2+n)$ for $k \geq \operatorname{tw}+2$), linear for $k \geq 5$ [BBH], while $k = 4$ needs order n^2 moves [BJLPP, Thm. 12, $k = 3$]; and d-degenerate chordal graphs, linear for $k \geq 2d+1$ [WL].

A 2-degenerate graph can have maximum average degree arbitrarily close to 4, for example $K_{2,m}$. Known linear subfamilies already have unbounded treewidth and maximum average degree at least 3: graphs obtained by subdividing every edge once have Grundy number at most 3, so Bonamy and Bousquet's theorem [BB, Thm. 1] gives four-colour diameter at most $8n$. Here the Grundy number is the largest number of colours used by a greedy colouring over all vertex orders (in this paragraph greedy colours are numbered $1, 2, \dots$). Indeed, a subdivision vertex has degree 2; if an original vertex first received greedy colour at least 4, its earlier colour-3 neighbour would be a subdivision vertex with at most one earlier neighbour, a contradiction. Once-subdivided K_m has treewidth $m-1$ and average degree $4(m-1)/(m+1)$, which is at least 3 for $m \geq 7$. Our result covers the entire class of bipartite 2-degenerate graphs.

Our results.

Theorem 32.2. *For every bipartite graph G on n vertices of degeneracy at most 2,* $\operatorname{diam}\mathcal{R}_4(G) \le 2n(n+1)$.

The class contains all trees, even cycles, square grids, every graph obtained by subdividing each edge of a graph once, and graphs of every treewidth.

Proposition 32.3. *The order n^2 in Theorem 32.2 cannot be improved: there are bipartite planar graphs H_n of degeneracy 2 on $N = n+2$ vertices, for $n \ge 5$, with* $\operatorname{diam}\mathcal{R}_4(H_n) \ge (N-7)^2/8$, *and bipartite graphs J_n of degeneracy 2 and girth 6 on $N = n+4$ vertices, for $n \ge 29$, with* $\operatorname{diam}\mathcal{R}_4(J_n) \ge (N-33)^2/128$. *($J_n$ has girth 6, so no bound of lower order than n^2 holds for 2-degenerate graphs of girth at least 5 or 6.)*

Quadratic lower bounds for four colours were known before, already for graphs of treewidth 2: the fan (a path together with a vertex joined to all its vertices) has $\operatorname{diam}\mathcal{R}_4 = \Theta(n^2)$ [BJLPP, Thm. 12, $k=3$] (see also [BBH]); the point of H_n and J_n is that they lie in the class of Theorem 32.2: they are bipartite, and J_n even has girth 6.

Theorem 32.4 (computer-assisted)**.** *For every graph G on n vertices of cone width at most 2 (Section 36),* $\operatorname{diam}\mathcal{R}_4(G) \le 4n(n+1)$. *The class contains every graph of treewidth at most 2 and graphs of treewidth 3 with triangles, and it is closed under subgraphs (Proposition 36.3).*

Theorem 32.5. *Let G be a graph on n vertices with an independent set Z of vertices of degree at most 2 such that $G-Z$ is bipartite and 2-degenerate. Then* $\operatorname{diam}\mathcal{R}_4(G) \le 5n(n+1)$. *(A more general form is Theorem 37.2.)*

All upper bounds go through one reduction (Lemma 33.5, the *sweep*): a quadratic bound for G follows from a *linear* bound for the instances on the cones, in which the colour of one vertex (the sink) is prescribed and every other vertex is greedy. In every instance we use, each vertex is recoloured a bounded number of times (two for bipartite instances).

33. Setting: orders, cones, instances, and the sweep

Colours are $0 < 1 < 2 < 3$. Let G have an order $v_1, \dots, v_n$ in which every vertex has at most two *later* neighbours, its *out-neighbours*; such an order exists exactly when G is 2-degenerate (take a degeneracy order). An arc goes from a vertex to each of its out-neighbours.

Definition 33.1 (cones)**.** For $1 \le t \le n$, the *cone* C_t is the set of vertices v_j, $j \le t$, from which v_t can be reached along arcs inside $\{v_1, \dots, v_t\}$. Note $v_t \in C_t$ and $\sum_t |C_t| \le 1 + 2 + \cdots + n$.

Definition 33.2 (instances)**.** An *instance* has vertices in a *creation order*, the *sink* s first; every vertex v has a set $N^+(v)$ of at most two *out-neighbours* among the vertices before it, the sink has none, and the edges are the pairs $\{v,u\}$, $u \in N^+(v)$. A vertex other than the sink without out-neighbours is a *root vertex*. Every vertex has a list $L(v) \subseteq \{0,1,2,3\}$ with $|L(v)| \ge |N^+(v)| + 2$; the sink has a list $\{g,x\}$ with $0 \le g < x \le 3$ (the *main case* is $g = 0$). A list is *tight* if $|L(v)| = |N^+(v)| + 2$ (the sink's list always is). The colourings α and β are *greedy*: $\alpha(s) = x$, $\beta(s) = g$, and every other vertex, in creation order, takes the least colour of its list that none of its out-neighbours has (so α and β depend only on the out-neighbours and the lists, not on the creation order). A *sequence* recolours one vertex at a time, to a colour of its list that none of its neighbours has at that time. Whether an instance lies in one of the classes below depends only on its graph and its arcs, not on the lists or the colours.

Lemma 33.3. *In an instance whose sink has the list $\{g, x\}$ with $0 < g < x$ (α has the sink on x, β on g, all other vertices greedy) there is a sequence from α to β of at most $2t$ moves, t the number of vertices.*

We use heights on trees. The *residue* of the colours $1, 2, 3$ is $1, 2, 0$. Let T be a tree and c a proper colouring of T with colours in $\{1, 2, 3\}$. A *height* of c is an integer function h on $V(T)$ with $h(v) \equiv \operatorname{res} c(v) \pmod 3$ and $|h(u) - h(v)| = 1$ on every edge; it exists (adjacent residues differ by ± 1 mod 3) and is unique up to an additive constant in $3\mathbb{Z}$.

Lemma 33.4 (one tree)**.** *Let T be a tree with root r; every vertex other than r may use the colours $1, 2, 3$, and r may use a set $P(r) \subseteq \{1, 2, 3\}$. Let c, c' be proper colourings of T within these sets, h a height of c, and E a function to the even integers such that $h + E$ is a height of c' and $|E(r)| \le 2$ if $|P(r)| \le 2$. Then c can be changed into c' inside T by exactly $\frac{1}{2}\sum_v |E(v)|$ moves, every intermediate colouring proper and within the allowed colours.*

Proof. Induction on $\sum |E|$. If $E = 0$ then $c' = c$. Suppose some $E(v) > 0$, and choose such v with $h(v)$ least. If a neighbour u had $h(u) = h(v) - 1$, then $(h + E)(u) \ge (h + E)(v) - 1 \ge h(v) + 1 = h(u) + 2$, so $E(u) \ge 2$ and $h(u) < h(v)$, against the choice of v. So every neighbour of v has height $h(v) + 1$ and hence the colour of residue $h(v) + 1$. Recolour v with the colour y of residue $h(v) + 2$: the colouring stays proper. y is allowed at v: clear unless $v = r$ and $|P(r)| \le 2$; then $E(r) = 2$, so $\operatorname{res} c'(r) = h(r) + 2$ and $y = c'(r)$. The new colouring has the height $h + 2[v]$, where $[v]$ is the indicator function of v, and $E - 2[v]$ satisfies the hypotheses with $\sum |E|$ smaller by 2. If no $E(v) > 0$, argue symmetrically with $E(v) < 0$ and $h(v)$ largest. □

Proof of Lemma 33.3. The sink is never on 0, so by induction along the creation order a vertex $v \ne s$ has $\alpha(v) = 0$ if and only if $0 \in L(v)$ and no out-neighbour has the colour 0, and the same for β; hence $K = \alpha^{-1}(0) = \beta^{-1}(0)$. Let F be the set of the other vertices. (F1) Every $v \in F$ has at most one out-neighbour in F: if $0 \in L(v)$, some out-neighbour is on 0; if not, $|L(v)| \le 3$ and v has at most one out-neighbour. So F induces a forest in which each vertex has at most one *parent* (an out-neighbour in F); all neighbours of a vertex of F other than its parent and its children lie in K (so every neighbour of a vertex of a component T of F that lies outside T is in K). (F2) A vertex of F with a parent has $\{1, 2, 3\} \subseteq L(v)$ (if $0 \in L(v)$ it has two out-neighbours and $|L(v)| = 4$; else $L(v) = \{1, 2, 3\}$). A root $r \ne s$ of F has all its out-neighbours on 0, so $\alpha(r) = \beta(r) = \min(L(r) \setminus \{0\})$; a vertex of F with a parent p takes the least colour of $\{1, 2, 3\}$ other than the colour of p, in α and in β. Hence $\alpha = \beta$ outside the tree T of F that contains s, and in a greedy colouring c the child of a vertex on 1 is on 2 and the child of a vertex on 2 or 3 is on 1. For a height h of c on T this gives, for $v \in T \setminus \{s\}$: $h(v) - h(s) \in \{0, 1\}$ if $c(s) = 1$, $\in \{-1, 0\}$ if $c(s) = 2$, and $\in \{1, 2\}$ if $c(s) = 3$. Let h_α, h_β be heights of $\alpha|_T, \beta|_T$ with $E = h_\beta - h_\alpha$ and $E(s) \in \{-2, 2\}$ (possible since $\alpha(s) \ne \beta(s)$). Then $E(v) - E(s) = (h_\beta(v) - h_\beta(s)) - (h_\alpha(v) - h_\alpha(s)) \in [-3, 3]$, and $E(v)$ is even (every height has $h(v) - h(s) \equiv \operatorname{dist}_T(v, s) \pmod 2$), so $|E(v)| \le 4$. Apply Lemma 33.4 on T with root s and $P(s) = L(s) = \{g, x\}$; every other vertex of T has a parent in T, so by (F2) the colours $1, 2, 3$ are in its list. While K stays on 0 (no vertex of F uses 0), this turns α into β in $\frac{1}{2}\sum_{v \in T} |E(v)| \le 2|T| \le 2t$ moves. □

Lemma 33.5 (the sweep)**.** *Let $\mathcal{C}$ be a class of instances (closed under changing the lists) and $C \ge 2$ such that every instance I of $\mathcal{C}$ in the main case has a sequence from α to β of at most $C|V(I)|$ moves. Let G have an order $v_1, \dots, v_n$ in which every vertex has at most 2*

later neighbours and in which, for every t, the instance on C_t with sink v_t defined below lies in $\mathcal{C}$. Then

$$\operatorname{diam} \mathcal{R}_4(G) \le 2C\,(|C_1| + \cdots + |C_n|) \le Cn(n+1).$$

Proof. A vertex v is *greedy* in a colouring c of G if $c(v)$ is the least colour that no out-neighbour of v has. There is exactly one colouring γ in which all vertices are greedy (colour $v_n, v_{n-1}, \dots, v_1$ in turn). We show that from any colouring γ is reached in at most $C\sum_t |C_t|$ moves; two colourings are then joined through γ.

The sweep has steps $t = 1, \dots, n$; after step t the vertices $v_1, \dots, v_t$ are greedy. At step t let c be the current colouring, $x = c(v_t)$, and g the least colour that no out-neighbour of v_t has. If $x = g$ nothing is done. Otherwise take the instance on C_t with sink v_t, out-neighbours of each $v \in C_t$ its out-neighbours in G that lie in C_t, creation order v_t followed by the other vertices of C_t by decreasing index (so every vertex comes after its out-neighbours), $L(v_t) = \{g, x\}$ (we show below that $g < x$), and for every other $v \in C_t$ the list $L(v) = \{0,1,2,3\}$ minus the colours of the out-neighbours of v outside C_t. Then $|L(v)| \ge 4 - |N^+(v) \setminus C_t| \ge |N^+(v) \cap C_t| + 2$. Every neighbour of a vertex $v \in C_t$ that lies outside C_t is an out-neighbour of v: an in-neighbour of v is earlier than v, hence among $v_1, \dots, v_t$, and it reaches v_t through v, so it lies in C_t. Hence a sequence of the instance is a sequence in G: the neighbours outside C_t do not move and their colours are not in the lists. The colouring c restricted to C_t is α: the vertices of C_t other than v_t are greedy before the step, and the greedy colour of a vertex in G is the least colour of its list that no out-neighbour inside C_t has (by induction along the creation order). Likewise β makes all of C_t greedy. A vertex v_j, $j < t$, outside C_t has no out-neighbour in C_t (else it would lie in C_t), so it stays greedy.

If $g = 0$ the instance lies in $\mathcal{C}$ and has the list $\{0, x\}$ at the sink: at most $C|C_t|$ moves. If $g \ne 0$, then x is also a colour no out-neighbour of v_t has, so $0 < g < x$, and Lemma 33.3 gives at most $2|C_t| \le C|C_t|$ moves. □

Remark 33.6. In a cone every vertex other than the sink has an out-neighbour inside the cone, so the instances of the sweep have no root vertices. Membership of an instance in the classes below does not depend on the lists or on the colouring.

34. Bipartite graphs: proof of Theorem 32.2

Theorem 34.1. *In every instance without root vertices whose graph is bipartite there is a sequence from α to β in which the sink moves exactly once and every other vertex at most twice. In particular* $\operatorname{dist}(\alpha, \beta) \le 2t - 1$.

Throughout this section the instance has no root vertices, so every vertex other than the sink has one or two out-neighbours, and a vertex with two out-neighbours has the list $\{0,1,2,3\}$.

If the sink has the list $\{g, x\}$ with $g \ne 0$, the sequence of Lemma 33.3 already has the required form: in its proof, Lemma 33.4 moves every vertex v of the tree T exactly $|E(v)|/2 \le 2$ times and the sink exactly once ($|E(s)| = 2$), and no other vertex moves. So from now on $g = 0$ (the main case).

Lemma 34.2. *For every vertex $v \ne s$, $\alpha(v), \beta(v) \in \{0,1,2\}$.*

Proof. At most one colour of $L(v)$ is 3, so at least $|N^+(v)| + 1$ colours of $L(v)$ lie in $\{0,1,2\}$; the out-neighbours occupy at most $|N^+(v)|$ of them, and the greedy colour is the least free colour of the list. □

The *pattern* of v is the pair $(\alpha(v), \beta(v))$; the pattern of the sink is $(x, 0)$. A vertex is *fixed* if $\alpha(v) = \beta(v)$, a *low swapper* if its pattern is $(0, 1)$ or $(1, 0)$, and a *high mover* if it is not fixed and has the colour 2 at one end. By Lemma 34.2 every vertex other than the sink is of exactly one of these kinds; the sink is a low swapper if $x = 1$, a high mover if $x = 2$, and none of the three if $x = 3$. A *leaver* has a pattern $(0, b)$ with $b \neq 0$, a *joiner* a pattern $(a, 0)$ with $a \neq 0$; the sink is a joiner.

Lemma 34.3. *Every leaver has an out-neighbour that is a joiner, and every joiner other than s has an out-neighbour that is a leaver.*

Proof. A leaver v has $0 \in L(v)$ and is not on 0 in β, so some out-neighbour u has $\beta(u) = 0$; and $\alpha(u) \neq 0$ because $\alpha(v) = 0$. A joiner $v \neq s$ has $0 \in L(v)$ and is not on 0 in α, so some out-neighbour u has $\alpha(u) = 0$, and $\beta(u) \neq 0$; $u \neq s$ because $\alpha(s) = x \neq 0$. □

Lemma 34.4 (upper middle)**.** *Let μ be a proper colouring of an instance with $\mu(v) \in L(v)$, $\mu(v) \geq \alpha(v)$ and $\mu(v) \geq \beta(v)$ for every v, and $\mu(s) = x$. Recolour first every vertex from $\alpha(v)$ to $\mu(v)$, the vertices in falling order of μ, and then every vertex from $\mu(v)$ to $\beta(v)$, in rising order of μ (ties in any order; no move when the colours agree). This is a sequence from α to β in which every vertex moves at most twice and the sink once.*

Proof. Let v, w be adjacent, so $\mu(v) \neq \mu(w)$; say $\mu(w) > \mu(v)$. In the first part w moves first, while v is on $\alpha(v) \leq \mu(v) < \mu(w)$; when v moves, w is on $\mu(w) \neq \mu(v)$. In the second part v moves first, to $\beta(v)$, while w is on $\mu(w) > \mu(v) \geq \beta(v)$; when w moves to $\beta(w)$, v is on $\beta(v) \neq \beta(w)$. All colours used lie in the lists. The sink starts on $x = \mu(s)$ and moves only in the second part. □

So Theorem 34.1 follows once we find such a μ. For the rest of the section the instance is bipartite; it is connected (every vertex reaches s), S_0 denotes the side of s and S_1 the other side, and the out-neighbours of a vertex lie on the other side.

Lemma 34.5 (sides)**.** *In a bipartite instance without root vertices, a vertex of $S_0 \setminus \{s\}$ has one of the patterns $(0,0)$, $(1,1)$, $(2,2)$, $(1,0)$, $(2,0)$, $(1,2)$, and a vertex of S_1 one of $(0,0)$, $(1,1)$, $(2,2)$, $(0,1)$, $(0,2)$, $(2,1)$. Equivalently: every joiner lies on S_0 and every leaver on S_1; and a vertex that is not fixed and avoids 0 at both ends has the pattern $(1,2)$ on S_0 and $(2,1)$ on S_1.*

Proof. Induction along the creation order; the statement is used only for out-neighbours of the new vertex v, which are earlier. If v is a leaver, it has a joiner out-neighbour (Lemma 34.3), which lies on S_0 (by induction, or it is s); so $v \in S_1$. If $v \neq s$ is a joiner, it has a leaver out-neighbour, which lies on S_1; so $v \in S_0$. Suppose $v \in S_0$ has the pattern $(2, 1)$. Then $1 \in L(v)$, and 1 is not free in α: some out-neighbour p has $\alpha(p) = 1$, and $\beta(p) \neq 1$ since β is proper. But $p \in S_1$, and the only pattern of S_1 with first entry 1 is $(1, 1)$; a contradiction. Suppose $v \in S_1$ has the pattern $(1, 2)$. Then $1 \in L(v)$ and some out-neighbour p has $\beta(p) = 1 \neq \alpha(p)$. But $p \in S_0$, and the only pattern there with second entry 1 is $(1, 1)$ (the sink has second entry 0); a contradiction. The remaining patterns are fixed. □

Definition 34.6 (roles)**.** Each vertex gets a role D (direct), P_2 (parked on 2) or P_3 (parked on 3), in creation order. A colour $r \in \{2, 3\}$ is *free for parking* at v if $r \in L(v)$, no out-neighbour of v is parked on r, and, if $r = 2$, no out-neighbour of v is fixed on 2. The sink gets D if $x = 1$, P_2 if $x = 2$, P_3 if $x = 3$. Every other vertex v gets: (a) D if v is fixed; (b) if v is a low swapper: D if no out-neighbour of v is a low swapper with role D, and otherwise P_r for a colour r free for parking; (c) if v is a high mover: P_r for a colour r free for parking.

Proposition 34.7. *In a bipartite instance without root vertices a colour free for parking exists whenever the rule asks for one, and the roles satisfy: (p1) a vertex in P_r has r in its list, no two adjacent vertices share a role P_2 or a role P_3, and the sink is in D, P_2, P_3 for $x = 1, 2, 3$ (and $x \in L(s)$); (p2) every vertex of D is fixed or a low swapper; (p3) no vertex of D fixed on the colour 2 has a neighbour in P_2; (p4) no two low swappers in D are adjacent.*

Proof. Every edge joins a vertex to an out-neighbour, so (p1), (p3), (p4) can be checked when the later end of an edge gets its role; (p2) holds by the rule. For (p3) with the vertex of P_2 as the later end, the clause "$r = 2$" of the rule applies; the case where the vertex fixed on 2 is the later end is case (a) below. Let v be the new vertex.

(a) v fixed. Only (p3) can fail, if v is fixed on 2 and has an out-neighbour in P_2. We show that every out-neighbour p of a vertex v fixed on 2 is fixed (hence in D). First, p never has the colour 2, so p is not the sink with $x = 2$. Nor is p the sink with $x = 3$: then s is on 3 in α, and one of $0, 1$ lies in $L(v)$ and is free in α (with two out-neighbours $L(v) = \{0, 1, 2, 3\}$ and the other one takes at most one of $0, 1$; with one out-neighbour $L(v)$ has 3 or 4 colours, so it contains 0 or 1), so $\alpha(v) \in \{0, 1\}$. Hence if p is not fixed, it is a low swapper. If $p = (1, 0)$ (so $p \in S_0$, $v \in S_1$): $\alpha(v) = 2$ forces $0 \notin L(v)$ or a second out-neighbour p' with $\alpha(p') = 0$, and $\beta(v) = 2$ forces $1 \notin L(v)$ or $\beta(p') = 1$. With one out-neighbour $L(v)$ would miss 0 and 1, too small; with two, $L(v) = \{0, 1, 2, 3\}$, so p' has the pattern $(0, 1)$ and lies on S_0, against Lemma 34.5. If $p = (0, 1)$ (so $p \in S_1$, $v \in S_0$) the same argument gives a second out-neighbour with pattern $(1, 0)$ on S_1, again impossible.

(b) v a low swapper with an out-neighbour p that is a low swapper in D. Then $0, 1 \in L(v)$. If p is the only out-neighbour, $L(v)$ has a third colour, 2 or 3, and p blocks neither. If there is a second out-neighbour p', then $L(v) = \{0, 1, 2, 3\}$ and p' blocks at most one of $2, 3$ (it is parked on one colour, or fixed on 2, or neither).

(c) v a high mover. Then $2, 3 \in L(v)$: with two out-neighbours $L(v) = \{0, 1, 2, 3\}$; with one, the greedy colour is the least or second least colour of the list and v has the colour 2 at one end, so $L(v)$ is $\{0, 2, 3\}$ or $\{1, 2, 3\}$. No out-neighbour is fixed on 2, since v has the colour 2 at one end. So the rule can fail only if v has two out-neighbours parked on different colours. But a high mover with two out-neighbours has a fixed out-neighbour: at the end where v has the colour 2, its out-neighbours have the colours 0 and 1, and by Lemma 34.5: for $v = (0, 2) \in S_1$, the out-neighbour on 1 in β has the pattern $(1, 1)$; for $v = (2, 1) \in S_1$, the out-neighbour on 0 in α has $(0, 0)$; for $v = (2, 0) \in S_0$, the out-neighbour on 1 in α has $(1, 1)$; for $v = (1, 2) \in S_0$, the out-neighbour on 0 in β has $(0, 0)$. So at most one out-neighbour is parked and the other parking colour is free. The choice of r gives (p1), and (p3) for an out-neighbour fixed on 2; (p4) holds by (b). □

Proof of Theorem 34.1. Take the roles of Proposition 34.7 and put $\mu(v) = 3$ on P_3, $\mu(v) = 2$ on P_2, $\mu(v) = \alpha(v) = \beta(v)$ on a fixed vertex of D, and $\mu(v) = 1$ on a low swapper of D. By (p2) every vertex gets a value. $\mu(v) \in L(v)$ by (p1), and because 1 is an end colour of a low swapper. $\mu \geq \alpha, \beta$ by Lemma 34.2 (for the sink: $\mu(s) = x$ by the rule, and $x \geq \alpha(s), \beta(s)$). μ is proper: two parked neighbours have 3 and 2 by (p1); a vertex of D has a value in $\{0, 1, 2\}$, so it differs from a neighbour in P_3, and from a neighbour in P_2 by (p3) (a low swapper has 1); two low swappers of D are not adjacent (p4); a vertex fixed on 1 is not adjacent to a low swapper (the swapper is on 1 in α or in β); two fixed neighbours differ because α is proper. Lemma 34.4 finishes the proof. □

Proof of Theorem 32.2. Apply Lemma 33.5 with $\mathcal{C}$ = the bipartite instances without root vertices and $C = 2$. A cone of a bipartite graph induces a bipartite graph, and every vertex

of a cone other than its sink has an out-neighbour inside the cone; Theorem 34.1 holds for all lists of the sizes required, not only tight ones. □

Corollary 34.8 (every cone bipartite)**.** *If G has an order with at most 2 later neighbours per vertex in which every cone induces a bipartite graph, then* $\operatorname{diam}\mathcal{R}_4(G) \le 4\sum_t |C_t| \le 2n(n+1)$. *Such graphs need not be bipartite (the 5-cycle with a suitable order), but they are triangle-free.*

Proof. The proof of Theorem 32.2 uses bipartiteness only for the cones. □

Remark 34.9. Two moves per vertex do not suffice for every instance with odd cycles. Take the sink v_0 with the list $\{0,1\}$ and $x=1$, and $v_1,\dots,v_7$ in this creation order, with out-neighbours and lists v_1,v_2: $\{v_0\}$, $\{0,1,2\}$; v_3: $\{v_1\}$, $\{0,2,3\}$; v_4: $\{v_2,v_3\}$, $\{0,1,2,3\}$; v_5: $\{v_4\}$, $\{0,1,2\}$; v_6: $\{v_4\}$, $\{1,2,3\}$; v_7: $\{v_3,v_6\}$, $\{0,1,2,3\}$. Its graph is a 5-cycle $v_0v_1v_3v_4v_2$ and a 4-cycle $v_3v_4v_6v_7$ sharing the edge v_3v_4, plus the pendant vertex v_5. On $(v_0,\dots,v_7)$, $\alpha=(1,0,0,2,1,0,2,0)$ and $\beta=(0,1,1,0,2,0,1,2)$. The sequence $v_2{\to}2$, $v_3{\to}3$, $v_1{\to}2$, $v_0{\to}0$, $v_1{\to}1$, $v_3{\to}2$, $v_4{\to}3$, $v_2{\to}1$, $v_6{\to}1$, $v_7{\to}3$, $v_3{\to}0$, $v_4{\to}2$, $v_7{\to}2$ moves v_3 three times and every other vertex at most twice. No sequence moves every vertex at most twice. Indeed, when v_0 moves, v_1 and v_2 are on 2; so v_2 goes $0,2,1$, and v_3 goes $2,3,0$, reaching 3 before v_1 reaches 2. The vertex v_4 must leave 1 before v_2 returns to 1 and can enter 2 only after v_2 has left 2, so it does not go to 2 directly; through 0 it would cost v_5 three moves; so v_4 goes $1,3,2$, entering 3 after v_3 has left 3 (if v_4 were on 3 while v_3 is still on 2, neither could move on). Hence v_3 enters 0 while v_4 is on 1, and v_7 must leave 0 before that. The vertex v_6 cannot use 3 (it would have to leave 3 before v_4 enters 3, for 1, which v_4 holds until then, or enter 3 after v_4 has left it, but v_4 enters 2 only after v_6 has left 2), so v_6 stays on 2 until it moves to 1; it cannot be on 1 while v_4 is on 1, so it is still on 2 when v_7 leaves 0, and v_7 cannot go to 2 directly. The vertex v_7 cannot use 3 either: before v_3 enters 3 it would block v_3, which can leave 2 only for 3, while v_7 could leave 3 only for its end colour 2; afterwards v_3 holds 3 until it enters 0. So v_7 goes to 1, and then needs v_6 off 2, while v_6 needs v_7 off 1. An exhaustive computer search over the instances without root vertices and with $g=0$ shows that every triangle-free one with at most 7 vertices has a sequence with at most two moves per vertex, and that among all 800,613,300 triangle-free ones with 8 vertices and an odd cycle exactly 21 (counted with their creation orders and lists) have none, all on this graph. (For $g\neq 0$, the sequence built in the proof of Lemma 33.3 moves every vertex at most twice, see the start of Section 34; and an instance with root vertices reduces to a smaller one without them: delete the vertices that do not reach the sink, which have $\alpha=\beta$, and remove their colours from the lists of their in-neighbours.) So 8 is the least number of vertices of a triangle-free instance with this property.

35. Lower bounds: proof of Proposition 32.3

For $n\ge 2$ let H_n be the path $v_1v_2\cdots v_n$ together with a vertex a joined to every v_i with i odd and a vertex b joined to every v_i with i even. It has $N=n+2$ vertices and $2n-1$ edges; it is bipartite (sides: odd path vertices and b; even path vertices and a), planar, and 2-degenerate (in the order $v_1,\dots,v_n,a,b$ every vertex has at most two later neighbours).

Proposition 35.1. *For $n\ge 5$ there are proper 4-colourings σ^+,σ^- of H_n at distance at least $(n-5)^2/8$ in $\mathcal{R}_4(H_n)$. Hence* $\operatorname{diam}\mathcal{R}_4(H_n)\ge (N-7)^2/8$.

Let $\sigma^{\pm}$ give a and b the colour 3 and v_i the colour $\pm(i-1) \bmod 3$. Call a colouring *plain* if a and b have the colour 3. In a plain colouring the path uses only $0,1,2$, and the *step* $s_i=\operatorname{col}(v_{i+1})-\operatorname{col}(v_i)\in\{+1,-1\} \pmod 3$ is defined for $1\le i<n$. A *height function* is

$h\colon \{1,\dots,n\} \to \mathbb{Z}$ with $h(i+1) - h(i) = s_i$; the *winding* is $W = h(n) - h(1)$. For σ^+ all steps are $+1$; for σ^- all are -1.

Lemma 35.2. *A move of a path vertex between two plain colourings changes one value of a height function by exactly* 2 *(and keeps the others).*

Proof. Let v_i move, $1 < i < n$. Its new colour is not 3 (it is joined to a or b), not its old colour, and not the colours of v_{i-1}, v_{i+1}, which also differ from its old colour. As only $0, 1, 2$ are available, v_{i-1} and v_{i+1} have the same colour, so $s_{i-1} = -s_i$, and after the move both steps change sign: changing $h(i)$ by 2 gives a height function of the new colouring. For $i = 1$ (or n) only s_1 (or s_{n-1}) changes sign, and $h(1)$ (or $h(n)$) changes by 2. □

Lemma 35.3 (flat)**.** *If a plain colouring allows a move of a or of b, then $|W| \le 4$.*

Proof. Let a move to $c \in \{0,1,2\}$; then no odd path vertex has the colour c, and every odd path vertex has one of the other two colours c_1, c_2. For consecutive odd vertices $u = v_{2k-1}$, $u' = v_{2k+1}$ with middle vertex $e = v_{2k}$: if u, u' have the same colour, $h(2k+1) = h(2k-1)$. Otherwise e has neither c_1 nor c_2 nor 3, so it has c; then u, e, u' have three different colours, both steps have the same sign, and $h(2k+1) - h(2k-1) = \pm 2$, where the sign is $+$ exactly when $c = \mathrm{col}(u) + 1$. Since $c = c_1 + 1$ if and only if $c = c_2 - 1$, every change from c_1 to c_2 along the odd vertices moves the height by 2η and every change from c_2 to c_1 by -2η, for one sign η; these changes alternate. So the heights of the odd vertices take at most two values, 2 apart; the first odd vertex is v_1 and the last is v_n or v_{n-1}, so $|h(n) - h(1)| \le 2 + 1 = 3$. For b the same argument works on the even vertices (when two consecutive even vertices differ, the odd vertex between them has c, because it is joined to a, which is on 3); the first even vertex is v_2 and the last is v_n or v_{n-1}, so $|h(n) - h(1)| \le 1 + 2 + 1 = 4$. □

Lemma 35.4 (cost)**.** *Let h, g be functions on $\{1,\dots,n\}$ with all steps ± 1, all steps of h equal to $+1$, and let $q \ge 0$ with $g(n) - g(1) \le (n-1) - q$ and $q/4 + 1 \le n/2$. Then $\sum_i |g(i) - h(i)| \ge q^2/8$. The same holds with all signs reversed (all steps of h equal to -1 and $g(n) - g(1) \ge -(n-1) + q$).*

Proof. Put $f = g - h$. The steps of f are 0 or -2, so f does not increase, and $f(1) - f(n) \ge q$. For $i \le n/2$ and $i' = n + 1 - i$, f falls by at most $2(i-1)$ on $[1, i]$ and on $[i', n]$, so $|f(i)| + |f(i')| \ge f(i) - f(i') \ge q - 4(i-1)$. With $m = \lfloor q/4 \rfloor$, the indices $i = 1, \dots, m+1$ are at most $n/2$, the pairs $\{i, i'\}$ are disjoint, and $\sum_i |f(i)| \ge \sum_{j=0}^{m} (q - 4j) = (m+1)(q-2m) \ge (q/4)(q/2) = q^2/8$. □

Proof of Proposition 35.1. Consider a sequence from σ^+ to σ^-. If a and b never move, all colourings are plain and Lemma 35.2 carries one height function h_t along the sequence, from $h_0(i) = i - 1$ to a function g with all steps -1. Then $f = g - h_0$ satisfies $|f(i)| + |f(n+1-i)| \ge 2(n+1-2i)$, so $\sum |f| \ge 2\lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor) \ge (n^2-1)/2$; each move changes $\sum_i |h_t(i) - h_0(i)|$ by at most 2, so there are at least $(n^2-1)/4 > (n-5)^2/8$ moves. Otherwise let $T_1 < T_2$ be the first and the last move of a or b (whichever leaves 3 must return). Before T_1 all colourings are plain, and the colouring just before T_1 allows a move of a or b, so by Lemma 35.3 its winding is at most $4 = (n-1) - q$ with $q = n - 5$, and by Lemma 35.4 its height function is at ℓ^1-distance at least $(n-5)^2/8$ from h_0: at least $(n-5)^2/16$ moves precede T_1. After T_2 neither a nor b moves and σ^- is plain, so all colourings after T_2 are plain, and the colouring just after T_2 allows the move T_2 to be undone; reading the tail backwards from σ^- and using the mirror form of Lemma 35.4, at least $(n-5)^2/16$ moves follow T_2. □

Remark 35.5. The exact distance from σ^+ to σ^- for $n = 10, \dots, 13$ (breadth-first search) is 24, 28, 33, 39, about $n^2/4$. We have not found H_n in the literature. It is the bipartite analogue of the fan of [BJLPP, Thm. 12], obtained by splitting its apex into two vertices. Height functions of 3-colourings are used for recolouring in Cereceda's thesis [Ce, Ch. 3 and Thm. 4.13], following Goldberg, Martin and Paterson [GMP]; see also the weighted sum in [BJLPP, Thm. 11]. Since H_n is planar of girth 4 and $\mathcal{R}_4(H_n)$ is connected (Theorem 32.2), it gives a quadratic lower bound for four colours on planar graphs of girth 4 (compare the remark after Table 1 of [BBFHMP] on lower bounds for planar graphs of given girth).

Girth six. For $n \geq 5$ let J_n be the path $v_1 \cdots v_n$ together with four vertices $a_0, \dots, a_3$, where a_j is joined to every v_i with $i \equiv j \pmod 4$. It has $N = n+4$ vertices; it is 2-degenerate (order $v_1, \dots, v_n, a_0, \dots, a_3$), bipartite (odd path vertices with a_0, a_2 against even path vertices with a_1, a_3), and has girth 6. Indeed the path is acyclic and the a_j are pairwise non-adjacent; a cycle through exactly one a_j is $a_j v_i \cdots v_l a_j$ with $l - i \geq 4$, of length at least 6; on a cycle through two or more a_j, between two consecutive ones lies a path segment of at least two vertices (each path vertex is joined to exactly one a_j), and there are at least two such segments, so the length is at least $2 + 2 + 2 = 6$. Finally $a_1 v_1 v_2 v_3 v_4 v_5$ is a 6-cycle. So J_n is a 2-degenerate graph of girth 6.

Proposition 35.6. *For $n \geq 29$, σ^+ and σ^- (all a_j on 3, v_i on $\pm(i-1) \bmod 3$) are at distance at least $(n-29)^2/128$ in $\mathcal{R}_4(J_n)$. Hence* $\operatorname{diam} \mathcal{R}_4(J_n) \geq (N-33)^2/128$.

Plain colourings (all a_j on 3), steps, heights and windings are as before, and Lemma 35.2 holds with the same proof.

Lemma 35.7. *If a plain colouring of J_n allows a move of some a_j, then $|W| \leq 3(n-1)/4+7$.*

Proof. We may assume $n \geq 4$. The move goes to a colour $c \in \{0,1,2\}$ that no neighbour of a_j has; write the other two colours as c_1 and $c_2 = c_1+1 \pmod 3$. Let $p_0 < \cdots < p_r$ be the indices of the path neighbours of a_j ($p_{k+1} = p_k + 4$, $p_0 \leq 4$, $p_r \geq n-3$), and $D_k = h(p_{k+1}) - h(p_k)$, a sum of four steps, so $D_k \in \{-4, -2, 0, 2, 4\}$ and $D_k \equiv \operatorname{col}(v_{p_{k+1}}) - \operatorname{col}(v_{p_k}) \pmod 3$. If the two colours agree, $D_k = 0$; a change from c_1 to c_2 gives $D_k \in \{4, -2\}$, and from c_2 to c_1 gives $D_k \in \{2, -4\}$. The two kinds of change alternate, so at most $\lceil r/2 \rceil$ of the D_k equal 4 and the others are at most 2: $\sum_k D_k \leq 3r + 1$, and symmetrically $\geq -(3r+1)$. At most three steps lie before p_0 and after p_r each, and $r \leq (n-1)/4$, so $|W| \leq (3r+1)+6 \leq 3(n-1)/4+7$. □

Proof of Proposition 35.6. Put $q = (n-29)/4 \geq 0$, so $3(n-1)/4 + 7 = (n-1) - q$ and $q/4 + 1 \leq n/2$. If no a_j moves, at least $(n^2-1)/4$ moves are needed as for H_n. Otherwise let $T_1 < T_2$ be the first and last moves among ALL apex vertices a_j (possibly different apices; any apex that leaves 3 must return). Before T_1 and after T_2 all colourings are plain and only path vertices move; Lemmas 35.7 and 35.4 give at least $q^2/16$ moves before T_1 and, reading backwards from σ^-, at least $q^2/16$ after T_2; these are different moves, so in total at least $q^2/8 = (n-29)^2/128$. □

Remark 35.8. With $k \geq 4$ apex vertices and $n \geq k+2$ path vertices, the analogous graph has girth 6 ($a_j v_i v_{i+1} a_{j+1} v_{i+k+1} v_{i+k}$), so this family gives nothing for girth 7 or more. Whether the diameter can be of order n^2 for 2-degenerate graphs of girth ≥ 7 is open; for planar graphs of girth ≥ 7 it cannot (their maximum average degree is below 14/5, and [Feg] gives n times a power of $\log n$). The exact distances from σ^+ to σ^- in $\mathcal{R}_4(J_n)$ for $n = 5, \dots, 18$ (breadth-first search) are $6, 8, 8, 9, 11, 14, 15, 17, 20, 24, 26, 29, 31, 33$, all below $2n$: the square appears late.

36. Cone width two

Cones and their width. An *orientation* of G, in this section, is an acyclic orientation in which every vertex has at most 2 out-neighbours. An order as in Section 33 gives one (every edge points to its later end), and every orientation comes from such an order (any order that puts every vertex before its out-neighbours). The *cone* of a vertex x is x together with all vertices from which x can be reached along arcs; x is its *top*. A vertex that reaches v_t along arcs is earlier than v_t, and so is every vertex on the way; hence the set C_t of Section 33 is the cone of v_t, and it depends only on the orientation. The instance on C_t (Lemma 33.5) gives each $v \in C_t$ its out-neighbours inside C_t; so its graph is the graph that G induces on the cone (an edge of G between two vertices of the cone is an arc from the earlier to the later one), its sink is the top, and it has no root vertex.

An *elimination order* of a graph H is an order of $V(H)$ in which the vertices are removed. The *neighbours left* of v are the vertices u after v that are adjacent to v or joined to v by a path all of whose inner vertices are before v. The *width* of the order is the largest number of neighbours left. An elimination order of a graph with arcs (a cone, or the graph of an instance) is *topological* if every vertex comes before its out-neighbours (so the top of a cone comes last). The *width of a cone* is the least width of a topological elimination order of the graph it induces, and the *cone width* $\mathrm{cw}(G)$ is the least, over all orientations of G, of the largest width of a cone. A graph with no orientation (one that is not 2-degenerate) has no cone width and lies in no class $\{\mathrm{cw} \le w\}$.

Lemma 36.1 (two forms)**.** *Remove the vertices of H in an elimination order, and when a vertex is removed join its neighbours of that time to each other. Then the neighbours of v at the time of its removal are exactly its neighbours left.*

Proof. We show by induction on the number of removed vertices that two vertices u, w that are left are adjacent at that time exactly when they are adjacent in H or joined in H by a path whose inner vertices are all removed. Let z be removed next. A new edge uw appears exactly when u and w are neighbours of z; by induction each is joined to z through removed vertices, and the two together give a walk from u to w through removed vertices and z, which contains such a path. Conversely, on such a path from u to w let y be the inner vertex removed last; just before y is removed, the two parts of the path make u and w neighbours of y (induction), so uw is an edge after y goes, and edges stay. (See also [RTL].) □

Theorem 36.2 (Theorem 32.4; computer-assisted)**.** *If $\mathrm{cw}(G) \le 2$, then $\operatorname{diam} \mathcal{R}_4(G) \le 8\sum_t |C_t| \le 4n(n+1)$, where C_t are the cones of any orientation in which every cone has width at most 2.*

Proposition 36.3. *(a) Every graph of treewidth at most 2 has cone width at most 2. More generally, a graph with an elimination order of width at most w in which every vertex has at most 2 later neighbours has cone width at most w. (b) If $\mathrm{cw}(G) \le w$, then $\mathrm{cw}(G') \le w$ for every subgraph G' of G. (c) The class $\{\mathrm{cw} \le 2\}$ contains graphs of treewidth 3 with triangles, for instance the graph W below.*

Proof. (a) Take an elimination order of width at most w and orient every edge to its later end. The out-neighbours of a vertex are its later neighbours: at most 2 by hypothesis (for $w = 2$ this follows, since they are among its at most 2 neighbours left). The order cut down to a cone is topological, and cutting an elimination order down to an induced subgraph does not raise its width (a path through removed vertices of the subgraph is such a path of G). A

graph of treewidth at most 2 has an elimination order of width at most 2 (see [Bo]). (b) Keep the orientation. The new cone of x lies inside the old one; cut a topological elimination order of the old cone down to the new cone. Every vertex still comes before its out-neighbours, and its neighbours left are among the old ones (a path of G' through removed vertices of the new cone is such a path of the old cone). (c) Let W have the vertices a, b, c, d, a vertex xy for each pair $\{x, y\} \subseteq \{a, b, c, d\}$ joined to x and y (K_4 with every edge subdivided once), a triangle $t_0t_1t_2$ with three further vertices s_{01}, s_{12}, s_{20}, where s_{ij} is joined to t_i and t_j, and one more edge at_0 (16 vertices, 22 edges). Its treewidth is at least 3 (it contains a subdivided K_4) and at most 3 (the order below, as an elimination order of all of W, has width 3). Take the order $ab, ac, ad, bc, bd, cd, b, c, d, a, s_{01}, s_{12}, s_{20}, t_0, t_1, t_2$ and orient every edge to its later end; every vertex has at most 2 later neighbours. The cones of $ab, \dots, cd, s_{01}, s_{12}, s_{20}$ are single vertices; the cones of b, c, d and a are stars (width 1). The cone of t_2 consists of $ab, ac, ad, a, s_{01}, s_{12}, s_{20}, t_0, t_1, t_2$; in this order it is topological, and the neighbours left are $\{a\}$ for ab, ac, ad; $\{t_0\}$ for a; $\{t_0, t_1\}$, $\{t_1, t_2\}$, $\{t_2, t_0\}$ for s_{01}, s_{12}, s_{20}; $\{t_1, t_2\}$ for t_0; $\{t_2\}$ for t_1: width 2. The cones of t_0 and t_1 lie inside it; cut this order down to them, as in (a). □

Remark 36.4. Treewidth is not bounded by cone width: for $m \geq 2$, K_m with every edge subdivided once has cone width 1 (orient each subdividing vertex to its two ends; every cone is a star) and treewidth $m - 1$. Among the graphs on at most 8 vertices that have an orientation, a census by two independent programs found 21 minimal graphs outside $\{\mathrm{cw} \leq 2\}$ ($1, 1, 3, 16$ with $5, 6, 7, 8$ vertices), all of cone width 3 and treewidth 3; the smallest is K_4 with one edge subdivided (5 vertices). For treewidth at most 2 the bound $2n(n + 1)$ with four colours is known [BB, Thm. 2]; Theorem 36.2 has the weaker constant 4 but covers the larger class. Neither of the classes of Theorems 32.2 and 32.4 contains the other: the triangle has cone width 2 and is not bipartite, and the bipartite 2-degenerate graph on $\{0, \dots, 7\}$ with the edges $04, 05, 07, 14, 16, 25, 26, 27, 35, 36, 37$ has cone width 3 (by two independent programs: none of its 130 orientations has all cones of width at most 2).

Instances of bag width two. Let I be an instance (Section 33). Sets of at most two vertices become *available* as vertices are created: at the start $\{s\}$ is available; every vertex $v \neq s$ is *created on* an available set $B(v) \supseteq N^+(v)$; if $B(v) = \{s\}$ the set $\{s, v\}$ becomes available, and if $B(v) = \{a, b\}$ the sets $\{a, v\}$ and $\{b, v\}$ become available; available sets stay available. The instance is in BAG(2) if some creation order (the sink first, every vertex after its out-neighbours) admits such sets $B(v)$. Membership does not depend on the lists.

Lemma 36.5. *Let $s = u_0, u_1, \dots, u_{m-1}$ be an order of the vertices of an instance in which every vertex comes after its out-neighbours. Sets $B(v)$ as above exist for this order if and only if the reversed order $u_{m-1}, \dots, u_1, s$ is an elimination order of width at most 2 of the graph of the instance. Hence an instance lies in* BAG(2) *if and only if its graph has a topological elimination order of width at most 2 that ends with the sink; and the instance on C_t lies in* BAG(2) *if and only if the cone C_t has width at most 2.*

Proof. "Before" and "after" refer to the creation order; the elimination removes u_{m-1} first, so the neighbours left of v are the vertices u before v that are adjacent to v or joined to v by a path whose inner vertices are all after v.

Sets to width. Let T be the graph whose edges are the available sets of size 2. Such a set is made when its later vertex v is created, with its other vertex in $B(v)$, and every $\{c, v\}$, $c \in B(v)$, is made. So the neighbours of v in T before v are exactly the vertices of $B(v)$: at most two, and adjacent in T if there are two. Removing the vertices of T last-created

first, the graph at the time v is removed is T on v and the vertices before it (induction: the neighbours of v at that time form $B(v)$, which is already a clique, so no edge is added). By Lemma 36.1 every vertex has at most 2 neighbours left in T. The graph of the instance is a subgraph of T on the same vertices ($N^+(v) \subseteq B(v)$), and a path in it is a path in T.

Width to sets. Let $H(v)$ be the set of neighbours left of v; $|H(v)| \leq 2$ and $N^+(v) \subseteq H(v)$. If $a, b \in H(v)$ with a before b, then $a \in H(b)$ (go from b to v and on to a: a walk whose inner vertices are v or after v, hence after b; a path from b to a lies inside it). Create the vertices in order, maintaining: (i) after v is created, $\{a, v\}$ is available for every $a \in H(v)$. If $H(v) = \{a, b\}$ with a before b: $a \in H(b)$, so $\{a, b\}$ is available by (i) for b; take $B(v) = \{a, b\}$. If $H(v) = \{a\}$: take $B(v) = \{s\}$ if $a = s$, and otherwise any available set $\{c, a\}$ of size 2 (one was made when a was created). If $H(v) = \emptyset$, then v is a root vertex; take $B(v) = \{s\}$. In each case $N^+(v) \subseteq H(v) \subseteq B(v)$ and (i) holds for v.

The last sentence: as said at the start of this section, the instance on C_t has the graph that G induces on the cone, with the top as sink; a topological elimination order of the cone, read backwards, has the sink first and every vertex after its out-neighbours, so it is a creation order, and conversely. □

Four moves per vertex. Fix $C \geq 1$. A *word* is a sequence over $\{0, 1, 2, 3\}$ with between 1 and $C + 1$ letters and consecutive letters different (the colours a vertex takes in a sequence; at most C moves). A *state* of two vertices (places 0 and 1) consists of their two words and an *order*: a word over $\{0, 1\}$, with as many digits p as place p has moves, saying which place makes the first, second, ... of their moves (the state with the places exchanged is the same state). A *new vertex* for a state is a set $P \subseteq \{0, 1\}$ of places (its out-neighbours) with a tight list L: $|L| = 2$ if $P = \emptyset$, $|L| = 3$ if $|P| = 1$, and $L = \{0, 1, 2, 3\}$ if $P = \{0, 1\}$ ($6 + 4 + 4 + 1 = 15$ new vertices; for the sink alone $P \subseteq \{0\}$, 10 new vertices). An *answer* to a new vertex (P, L) is a word $w = w_0 \dots w_m$ and a word π over $\{0, 1, 2\}$ with (A1) $m \leq C$, all $w_i \in L$, $w_i \neq w_{i+1}$; (A2) w_0 is the least colour of L that no place of P has as its first colour, and w_m the least colour of L that no place of P has as its last colour; (A3) π has exactly m digits 2, and deleting them leaves the order of the state; (A4) replaying π from the first colours, at a digit 2 the new vertex takes its next colour, which no place of P has at that time, and at a digit $p \in P$ the place p takes its next colour, which the new vertex does not have at that time. The two *states after* the answer are (place 1, new vertex) and (place 0, new vertex), with π restricted to the corresponding digits (renamed 0, 1 in the order old place, new vertex). For the sink alone (place 0, its order $0 \dots 0$), an answer is defined in the same way with π over $\{0, 2\}$, and there is one state after it, (sink, new vertex). A *closed core* for C is a set K of states and, for each $x \in \{1, 2, 3\}$, a word of the sink from x to 0 inside $\{0, x\}$ with at most C moves, such that (K1) every state of K has, for each of its 15 new vertices, an answer whose two states after it lie in K, and (K2) the word of the sink has, for each of its 10 new vertices, an answer whose state after it lies in K.

Proposition 36.6. *If a closed core for C exists, then every instance of* BAG(2) *in the main case with tight lists has a sequence from α to β in which every vertex moves at most C times.*

Proof. Go through the instance in a creation order with sets $B(v)$, keeping one list M of moves with: (I1) the moves of M, applied from α on the vertices created so far, form a sequence that ends in β there, with at most C moves per vertex; (I2) for every available set $\{a, b\}$ the words of a and b in M and the order of their moves in M form a state of K, and the word of s in M is the core's word for x. Start with the moves of the sink's word. When v is created on $B(v) = \{a, b\}$, v is a new vertex of the state of $B(v)$ (P the places of

its out-neighbours, L its tight list); take the answer given by (K1), give v the word w, and insert the moves of v into M as π prescribes (between the same two moves of a and b; the order of the old moves is unchanged). By (A2) and greediness, w runs from $\alpha(v)$ to $\beta(v)$; by (A1) it stays in the list with at most C moves. Neighbours are a vertex and one of its out-neighbours; pairs of old vertices are unaffected, and for $u \in N^+(v) \subseteq B(v)$ condition (A4) is exactly properness on the edge uv (initially $\alpha(v) \neq \alpha(u)$ by greediness), because the colours of u and v at any time depend only on the moves of a, b, v, whose order in M is π. The old available sets keep their words and orders (the order of the old moves is unchanged), and the new ones $\{a, v\}, \{b, v\}$ carry the two states after the answer, which lie in K. Creation on $\{s\}$ is the same with (K2). At the end M is the required sequence. □

Lemma 36.7 (tight lists suffice)**.** *Let a class of instances be such that membership does not depend on the lists. If every instance of the class in the main case with tight lists has a sequence with at most C moves per vertex, so does every instance of the class in the main case.*

Proof. Replace each list $L(v)$ by a tight list $L'(v) \subseteq L(v)$ containing $\alpha(v)$ and $\beta(v)$ (the sink's list $\{0, x\}$ is unchanged; for a vertex with two out-neighbours $L(v)$ is already tight; a root vertex has $\alpha(v) = \beta(v)$). By induction along the creation order, greedy colouring with the lists L' gives the same α and β: a smaller colour of $L'(v)$ free of the out-neighbours' colours would be a smaller such colour of $L(v)$. A sequence for the lists L' stays inside the lists L. □

Theorem 36.8 (computer-assisted)**.** *A closed core for $C = 4$ exists. Hence, by Proposition 36.6 and Lemma 36.7, every instance of* BAG(2) *in the main case has a sequence from α to β in which every vertex moves at most 4 times; in particular* $\mathrm{dist}(\alpha, \beta) \leq 4t$*. The 4 cannot be lowered: there are instances of* BAG(2) *without root vertices, with 27, 30 and 33 vertices ($x = 1, 2, 3$), with no sequence of at most 3 moves per vertex; so no closed core exists for $C \leq 3$ (a closed core for $C \leq 3$ is one for $C = 3$, and it would give such a sequence).*

About the computation. For $C = 4$ there are 484 words and 5,261,242 states. Starting from all states and deleting, until nothing changes, every state that has a new vertex without an answer whose states after it remain, leaves 3,258,618 states; the sink word $x0$ satisfies (K2) for $x = 1, 2, 3$. From it a closed core of 17,485 states with 262,275 answers for (K1) (15 per state), together with the three sink words and their 30 answers for (K2) (ten for each), was extracted and written to a file. The file is verified line by line, without search, against (A1)–(A4), (K1), (K2) by three checkers: one written with the search (separate code), and two written independently from the definitions alone. The two independent checkers reject damaged copies and reject the file when the cap is 3; the first rejects a copy with one changed answer. One independent run also built, for every state of the core, a real instance using all 15 new vertices and replayed it, so every answer line was used. An end-to-end test built the sequences of Proposition 36.6 from the file for 1,800 random instances of BAG(2) (up to 600 vertices) and replayed them move by move: all were legal, ended in β, and moved every vertex at most 4 times. For $C \leq 3$, after the same deletion no word of the sink satisfies (K2), for $x = 1, 2, 3$; the three explicit instances (which prove this by themselves, as stated) were confirmed by several SAT models from two independent implementations and by a search that uses no solver. The core file `core_C4.txt` (17,485 states, 262,275 answers and 30 answers at the sink; SHA-256 beginning `24881187`), the three checkers, the three instances with 27, 30 and 33 vertices, the search without a solver, the end-to-end test and a README are ancillary files of this paper. □

Proof of Theorem 36.2. Take an order inducing an orientation in which every cone has width at most 2. By Lemma 36.5 every instance of the sweep lies in BAG(2), a class that does not depend on lists or colourings. By Theorem 36.8, every instance of BAG(2) in the main case has a sequence of at most $4t$ moves. Lemma 33.5 with $C = 4$ gives the bound. □

37. The terminal extension

In an instance, an *in-neighbour* of v is a vertex that has v as an out-neighbour.

Lemma 37.1 (terminal vertices)**.** *Let I be an instance without root vertices on t vertices, and Z a set of vertices other than the sink without in-neighbours such that $I - Z$ is bipartite. Then there is a sequence from α to β of at most $2(t-|Z|)-1+\sum_{z\in Z}(2\deg_I(z)+1) \le 5t-1$ moves, where $\deg_I$ is the degree in I.*

Proof. Z is independent (on an edge between two members, the later-created end is an in-neighbour of the other), and every neighbour of $z \in Z$ is an out-neighbour, so $\deg_I(z) \in \{1,2\}$ and $|L(z)| \ge \deg_I(z)+2$. A vertex outside Z has no out-neighbour in Z, so $I - Z$ is an instance without root vertices, with the same lists and sink, whose greedy colourings are the restrictions of α, β. By Theorem 34.1 it has a sequence in which the sink moves once and every other vertex at most twice. Follow this sequence; immediately before a vertex $p \notin Z$ moves from a colour c_{old} to a colour c_{new}, every $z \in Z$ adjacent to p and on c_{new} first moves to a colour of $L(z)$ other than c_{old}, c_{new} and the colours of its other neighbours; at most $\deg_I(z)+1$ colours are excluded, so such a colour exists, and the move stays proper after p moves. Members of Z are pairwise non-adjacent, so these moves do not interfere. At the end every z moves to $\beta(z)$; its neighbours are then on their β colours, so this is proper. A vertex z moves at most once per move of a neighbour and once at the end: at most $2\deg_I(z)+1 \le 5$ times. □

Theorem 37.2. *Let G have an order with at most 2 later neighbours per vertex such that, for every t, the cone C_t becomes bipartite when the vertices of C_t other than v_t without an earlier neighbour (in the order of G) in C_t are removed. Then* $\operatorname{diam}\mathcal{R}_4(G) \le 10\sum_t |C_t| \le 5n(n+1)$.

Proof. The creation order of the instance on C_t is the reverse of the order of G, so the in-neighbours of a vertex in the instance are its earlier neighbours in C_t in the order of G, and it has no root vertices (Remark 33.6); apply Lemma 33.5 with $C = 5$ and Lemma 37.1. □

Corollary 37.3. *Let G be a graph on n vertices with an independent set Z of vertices of degree at most 2 such that $G-Z$ is bipartite and 2-degenerate. Then* $\operatorname{diam}\mathcal{R}_4(G) \le 5n(n+1)$.

Proof. Order Z first, then $G - Z$ in an order with at most 2 later neighbours. Every $z \in Z$ has at most two later neighbours (all its neighbours), and no vertex of $G - Z$ gains a later neighbour. A cone C_t with $v_t \in Z$ is $\{v_t\}$ (Z is independent and comes first). For $v_t \notin Z$, the members of Z in C_t have no earlier neighbour, and $C_t \setminus Z$ induces a subgraph of $G - Z$, which is bipartite; removing the vertices of C_t other than v_t without an earlier neighbour in C_t removes all of $Z \cap C_t$ (as $v_t \notin Z$) and leaves a subset of $C_t \setminus Z$. Apply Theorem 37.2. (A vertex of Z that is isolated in G is harmless: its cone is itself.) □

Remark 37.4. The class of Corollary 37.3 contains graphs with triangles (a triangle: $Z =$ one vertex), so it is not contained in the class of Corollary 34.8, whose graphs are triangle-free; both contain the bipartite 2-degenerate graphs ($Z = \emptyset$). For example, add to a square grid any number of new vertices of degree 2, each joined to two grid vertices at odd distance (for instance the two ends of an edge); every new vertex closes odd cycles (a triangle in that case), and the graph satisfies the hypothesis of Theorem 32.5 with Z the set of new vertices.

38. Open problems

(1) Cereceda's conjecture for $d = 2$ remains open. By Lemma 33.5 it would follow from a bound $O(t)$ on the length of a shortest sequence for every instance in the main case, in particular from a constant bound on the number of moves per vertex.
(2) Girth. The graphs J_n show that $\operatorname{diam} \mathcal{R}_4$ can be of order n^2 for 2-degenerate graphs of girth 6, and Remark 34.9 shows that two moves per vertex no longer suffice once odd cycles are allowed (its instance has girth 4). Is $\operatorname{diam} \mathcal{R}_4(G) = O(n^2)$ for every 2-degenerate graph G of girth at least 5? Is there a constant C such that every instance of girth at least 5 has a sequence moving every vertex at most C times?
(3) The constant for bipartite graphs: Theorem 32.2 and Proposition 32.3 leave a factor of 16 between the leading terms $2n^2$ and $n^2/8$.
(4) Cone width: does Theorem 32.4 extend to cone width 3, for instance by a finite certificate as in Theorem 36.8?
(5) Larger d: is $\operatorname{diam} \mathcal{R}_{d+2}(G) = O(n^2)$ for bipartite d-degenerate graphs? (Bousquet and Heinrich [BH] proved this for planar bipartite graphs, which are 3-degenerate, with 5 colours; Cranston and Mahmoud [CM] extended it to planar graphs with no cycles of length 3 or 5, with at most $4n^2$ moves.) The per-vertex form of Theorem 34.1 does not extend to $d \geq 3$ (lists in $\{0, \ldots, d+1\}$, at most d out-neighbours): the instance with creation order $s, v_1, \ldots, v_5$, out-neighbours $N^+(v_1) = \{s\}$, $N^+(v_2) = N^+(v_3) = \{v_1\}$, $N^+(v_4) = \{v_2, v_3\}$, $N^+(v_5) = \{v_1, v_4\}$ and lists $L(s) = \{0, 1\}$, $L(v_1) = \{0, 1, 2\}$, $L(v_2) = \{0, 2, 3\}$, $L(v_3) = \{1, 2, 4\}$, $L(v_4) = \{1, 2, 3, 4\}$, $L(v_5) = \{0, 1, 2, 3\}$ has no root vertex and a bipartite graph ($K_{2,3}$ with the sink as a pendant vertex), but no sequence from α to β in which every vertex moves at most twice; its shortest sequences still have $11 = 2t - 1$ moves, and one of them moves every vertex at most three times.

Theorem numbers for [BB], [BJLPP], [BC], [BP], [Cr] and [FJP] refer to the open versions named in the entries.

Appendix A. The closure lemma of Jaeger, Linial, Payan and Tarsi

This appendix proves the two facts on truncated cubic 3-edge-connected graphs that Section 2 uses: every such graph has a well colouring (Lemma A.5), and a well colouring gives a closure sequence of cycles (Lemmas A.6 and A.7). Both are results of Jaeger, Linial, Payan and Tarsi [JLPT]: the generation and the colouring are [JLPT, Lemmas 3.2 and 3.3, pp. 175 and 177], and the closure sequence is the observation at the start of the proof of their Theorem 3.2 [JLPT, p. 179]. The proofs are written out in full. This appendix contains no new result.

Definitions and counts. TC3 is the class of the truncated cubic 3-edge-connected graphs of Definition 2.4: the graphs $Q = Q^+ - v$, where Q^+ is a cubic 3-edge-connected graph on at least four vertices and v is a vertex of Q^+. The words 2-join, 2-constructible, 2-construction, stage, node and operand are those of Definition 2.8, and 2C is the class of the 2-constructible graphs. Graphs can have parallel edges (a 2-constructible graph often has them).

Definition A.1 (Well colouring)**.** Let Q be a graph in which every edge is blue or red. A *red component* of Q is a component of the graph with the vertex set $V(Q)$ and the red edges. Q/red is the graph whose vertices are the red components of Q and whose edges are the blue

edges of Q; the ends of a blue edge in Q/red are the red components of its two ends. The colouring is a *well colouring* if

(i) the blue edges form a spanning tree of Q, and

(ii) Q/red is 2-constructible.

This is the notion of Definition 2.9 (F1).

Lemma A.2 (Counts)**.** (a) *Let $Q = Q^+ - v$ be in* TC3*, with n vertices. Then Q^+ and Q are simple graphs, Q is connected, n is odd, Q has $3(n-1)/2$ edges, exactly three vertices of Q have degree 2 (the neighbours of v), and all other vertices of Q have degree 3. The smallest graph of* TC3 *is the triangle ($Q^+ = K_4$).*

(b) *Let U_0 be a graph on j vertices with a 2-construction. Then U_0 is connected, has no loop, and has $2(j-1)$ edges. Two stages are disjoint, or one is a subgraph of the other. Every edge of U_0 is an edge of exactly one node. Every stage U is an induced subgraph of U_0, and the edges of U are the edges of the nodes that are subgraphs of U.*

(c) *In a well coloured graph no blue edge has both ends in one red component. If a graph of* TC3 *with n vertices is well coloured and has j red components, then $2(j-1) = n-1$, and the red edges form a forest.*

Proof. (a) A loop at a vertex x of the cubic graph Q^+ makes the third edge at x a bridge. Two parallel edges with the ends x, y leave at most 2 edges between $\{x, y\}$ and the other vertices, and there are other vertices because Q^+ has at least four. Both are impossible in a 3-edge-connected graph. So Q^+ is simple, and Q is simple. A cubic graph has an even number of vertices, so $n = |V(Q^+)| - 1$ is odd, and Q has $3(n+1)/2 - 3 = 3(n-1)/2$ edges. The three neighbours of v are different; they lose one edge, and no other vertex does. If Q had more than one component, one of them would get at most one of the three edges at v; this edge alone, or no edge, would separate it from the rest of Q^+. So Q is connected.

(b) Induction on the 2-construction. For one vertex all is clear. Let U_0 be the 2-join of U_0' and U_0'', with j' and j'' vertices. The two graphs are connected and disjoint, and a new edge joins them; so U_0 is connected. A new edge has its ends in two different graphs, so it is no loop. The number of edges is $2(j'-1) + 2(j''-1) + 2 = 2(j-1)$. The stages other than U_0 are the stages of U_0' and the stages of U_0''; the first are subgraphs of U_0' and the second are subgraphs of U_0''. This gives the statement on two stages, and the statements on the edges. Let $U \neq U_0$ be a stage, say a stage of U_0', and let t be an edge of U_0 with both ends in $V(U)$. Then t is not a new edge and not an edge of U_0''. So t is an edge of U_0', and by induction it is an edge of U.

(c) A blue edge with both ends in one red component is a loop of Q/red; by (b) there is none. Let Q be in TC3. By (b), Q/red has $2(j-1)$ edges. They are the blue edges, and the blue spanning tree has $n-1$ edges. So $2(j-1) = n-1$. By (a) there are $3(n-1)/2 - (n-1) = (n-1)/2 = n-j$ red edges. A graph with n vertices, j components and $n-j$ edges is a forest. □

The graphs of the stages. Let Q be a graph with a well colouring and with a 2-construction of Q/red. For a stage U let V_U be the union of the red components that are the vertices of U, and let Q_U be the subgraph of Q induced on V_U. We call Q_U the *graph of* U. For $U = Q$/red it is Q.

Lemma A.3 (The graphs of the stages)**.** *Let Q be a graph with a well colouring and with a 2-construction of Q/red, and let U be a stage.*

(a) *The edges of Q_U are the red edges with both ends in V_U and the edges of the nodes that are subgraphs of U.*

(b) *Q_U is connected.*

(c) *If U is a node with the operands U', U'' and the two edges t_1, t_2, then V_U is the disjoint union of $V_{U'}$ and $V_{U''}$, each of t_1, t_2 has one end in $V_{U'}$ and one end in $V_{U''}$, and $E(Q_U) = E(Q_{U'}) \cup E(Q_{U''}) \cup \{t_1, t_2\}$.*

Proof. (a) A red edge has both ends in one red component. A blue edge with both ends in V_U is an edge of Q/red whose two ends are vertices of U. By Lemma A.2 (b), U is an induced subgraph of Q/red, so this edge is an edge of U, and the edges of U are the edges of the nodes that are subgraphs of U.

(c) follows from (a) and Lemma A.2 (b).

(b) Induction on the 2-construction. If U has one vertex, Q_U is one red component with its red edges. If U is a node, then $Q_{U'}$ and $Q_{U''}$ are connected, and t_1 joins them. □

Two operations. Let Q' and Q'' be disjoint graphs, and let d be 2 or 3.

- A *d-sticking* of Q' and Q'': z is a vertex of Q' of degree d with the edges $t_1, \dots, t_d$, and $z_1, \dots, z_d$ are d different vertices of degree 2 of Q''. The result is the graph $(Q' - z) \cup Q''$ in which every t_i keeps its end in $Q' - z$ and gets z_i as its other end.
- A *growing* of Q' at an edge y_1y_2 and a vertex x' of degree 2: put a new vertex y on the edge y_1y_2 (the edge becomes the path y_1, y, y_2), and add a new vertex x with two edges, to x' and to y.

Generation.

Lemma A.4 (Generation)**.** *Every graph Q of* TC3 *other than the triangle is a 3-sticking or a 2-sticking of two graphs of* TC3 *with fewer vertices, or a growing of a graph of* TC3 *with fewer vertices.*

Proof. Let $Q = Q^+ - v$, and let Q be not the triangle. Then $|V(Q^+)| \geq 6$.

Case 1: $V(Q^+)$ is the disjoint union of two sets V_1, V_2 with $|V_1|, |V_2| \geq 2$ and with exactly three edges between V_1 and V_2.

(1a) The three edges are pairwise disjoint. Suppose that a vertex x of V_1 meets two or three of them (the same for V_2). Move x to the other side. Then 2 or 0 edges join the two sides, and both sides are non-empty because $|V_1| \geq 2$. This is impossible.

(1b) $|V_i|$ is odd: three times $|V_i|$ is twice the number of edges inside V_i, plus 3. So $|V_i| \geq 3$.

(1c) Let Q^+/V_2 be the graph obtained from Q^+ by the identification of all vertices of V_2 to one new vertex and the deletion of the edges inside V_2. The new vertex has degree 3 (the three edges between V_1 and V_2), and every other vertex keeps its degree 3. Put $Q_1^+ = Q^+/V_2$ and $Q_2^+ = Q^+/V_1$. Both graphs are cubic. They are 3-edge-connected: a partition of the vertex set of Q_1^+ into two non-empty sets gives such a partition of $V(Q^+)$ with the same edges between the two sets (put V_2 on the side of the new vertex); the same for Q_2^+. They have $|V_i| + 1 \geq 4$ vertices.

By symmetry let v be in V_1, and let z be the vertex of Q_1^+ that comes from V_2. Put $Q' = Q_1^+ - v$, and let Q'' be the subgraph of Q^+ induced on V_2; this is Q_2^+ minus the vertex that comes from V_1. Both are in TC3. The vertices of degree 2 of Q'' are the three ends in V_2 of the edges between V_1 and V_2. By (1a), at most one of these three edges is at v.

- No one is at v: then z has degree 3 in Q', and Q is the 3-sticking of Q' and Q'' at z.

– One of them is at v; let y' be its end in V_2: then z has degree 2 in Q', and Q is the 2-sticking of Q' and Q'' at z. The two edges of z go to two of the three vertices of degree 2 of Q''; the third one is y'.

Q' has $|V_1|$ vertices and Q'' has $|V_2|$ vertices. Q has $|V_1| + |V_2| - 1$ vertices; by (1b) this is more than each of the two numbers.

Case 2: there is no such pair V_1, V_2. Then Q^+ has no triangle: exactly three edges leave a triangle of a simple cubic graph, and $|V(Q^+)| \geq 6$, so a triangle would give a pair as in Case 1. Let x be a neighbour of v, and xy an edge of Q. In Q^+ the neighbours of x are v, y, x', and the neighbours of y are x, y_1, y_2. Because Q^+ is simple and has no triangle, v, x', y_1, y_2 are four different vertices, and they are different from x and y. Let $\widehat{Q}^+$ be the graph obtained from Q^+ by the deletion of the edge xy and the suppression of x and y: the path v, x, x' becomes a new edge vx', and the path y_1, y, y_2 becomes a new edge y_1y_2.

Claim: $\widehat{Q}^+$ is cubic and 3-edge-connected, and it has $|V(Q^+)| - 2 \geq 4$ vertices.

Proof of the claim. $\widehat{Q}^+$ is cubic by its construction. Suppose that $V(\widehat{Q}^+)$ is the disjoint union of two non-empty sets V', V'' such that the set E^* of the edges between them has at most 2 elements (E^* can be empty). Put x back: on the side of v and x' if they are on the same side; on any side if the edge vx' is in E^* (then exactly one of the edges vx, xx' joins the two sides, in the place of vx'). Do the same for y with y_1, y_2. This is a partition of $V(Q^+)$ into two non-empty sets, and the number of edges of Q^+ between them is $|E^*|$ if x and y are on the same side, and $|E^*| + 1$ if they are not.

If vx' or y_1y_2 is in E^*, we can choose the sides such that x and y are together. Then at most 2 edges of Q^+ join the two sides: impossible. In the other case x and y have no choice. If they are on the same side, again at most 2 edges join the sides: impossible. If they are on different sides and $|E^*| \leq 1$, at most 2 edges join the sides: impossible. If they are on different sides and $|E^*| = 2$, exactly three edges join the sides, and the sides contain $\{x, v, x'\}$ and $\{y, y_1, y_2\}$: a pair as in Case 1, which does not exist.

So $\widehat{Q}^+$ is connected (E^* empty) and 3-edge-connected. This proves the claim.

The vertex v is a vertex of $\widehat{Q}^+$, so $\widehat{Q} = \widehat{Q}^+ - v$ is in TC3. The vertex x' is a neighbour of v in $\widehat{Q}^+$, so it has degree 2 in $\widehat{Q}$. And y_1y_2 is an edge of $\widehat{Q}$, because y_1 and y_2 are not v. The graph Q is the growing of $\widehat{Q}$ at the edge y_1y_2 and the vertex x', with the new vertices y and x. $\widehat{Q}$ has two vertices fewer than Q. □

Well colouring.

Lemma A.5 (Well colouring)**.** *Every graph of* TC3 *has a well colouring.*

Proof. Induction on the number of vertices. By Lemma A.4 it is enough to show three things: the triangle has a well colouring; a graph of TC3 that is a sticking of two well coloured graphs of TC3 has a well colouring; a graph of TC3 that is a growing of a well coloured graph of TC3 has a well colouring.

Triangle. Colour two edges blue and one edge red. The blue edges form a spanning tree. There are two red components, and the two blue edges join them: Q/red is the 2-join of two single vertices.

Sticking. Let Q' and Q'' in TC3 be well coloured, with the blue trees T' and T''. Let Q be the d-sticking of Q' and Q'' at z, with the edges $t_1, \ldots, t_d$ of z and their new ends $z_1, \ldots, z_d$. Keep all colours.

(i) The blue edges of Q are the edges of T' and of T''; the edges of T' at z have a new end. Their number is $(|V(Q')|-1)+(|V(Q'')|-1)=|V(Q)|-1$. The tree T'' connects $V(Q'')$. Every component of $T'-z$ contains a neighbour of z in T', and the edge from this neighbour to z ends now in $V(Q'')$. So the blue edges form a connected spanning subgraph of Q with $|V(Q)|-1$ edges: a spanning tree.

(ii) Let Ω be the red component of z in Q'; it is a vertex of Q'/red. By Lemma A.2 (c) the red edges of Q' form a forest. So Ω without z is the disjoint union of trees Ω_i, one for every *red* edge t_i at z; the end of t_i other than z is in Ω_i. (If no edge at z is red, then $\Omega=\{z\}$ and there is no Ω_i.) In Q, a red edge t_i joins Ω_i to the red component of z_i in Q'', and a blue edge t_i is a blue edge at the red component of z_i. All other red components of Q' and of Q'' are not changed. So the red components of Q are: the red components of Q' other than Ω; and the red components of Q'', some of them enlarged by trees Ω_i. An edge of Q'/red at Ω is a blue edge of Q' with one end in Ω. This end is z or a vertex of some Ω_i. In both cases the edge ends now in a red component that comes from Q''. Its other end is a vertex of Q'/red other than Ω, because Q'/red has no loop. So Q/red is obtained in this way: take Q'/red without the vertex Ω, and Q''/red, and every edge of Q'/red at Ω, with its end Ω moved to some vertex of Q''/red.

It remains to show that such a graph is 2-constructible. Claim: let U_a and U_b be disjoint 2-constructible graphs and ω a vertex of U_a; let U be obtained from U_a without ω, from U_b, and from every edge of U_a at ω, with its end ω moved to some vertex of U_b. Then U is 2-constructible. Proof by induction on a 2-construction of U_a. If U_a has only the vertex ω, then $U=U_b$. If U_a is the 2-join of U_a' and U_a'' and ω is in U_a', make the same change in U_a'; by induction the result U' is 2-constructible. Each of the two new edges of the 2-join has one end in U_a'' and one end in U_a'; if this end was ω, it is now a vertex of U_b, which is in U'. So U is the 2-join of U' and U_a''. This proves the claim. So Q is well coloured.

Growing. Let Q' in TC3 be well coloured with the blue tree T', and let Q be the growing of Q' at the edge y_1y_2 and the vertex x', with the new vertices y and x. Colour the edges xx' and xy blue. Keep the colours of the old edges other than y_1y_2.

The edge y_1y_2 is red. Colour y_1y and yy_2 red. The blue edges are those of T', and xx', and xy: a spanning tree of Q (x hangs on x', and y hangs on x). The vertex y is now in the red component of y_1 and y_2, and x is a new red component with one vertex. So Q/red is Q'/red and one new vertex x with two edges, to the red components of x' and of y: a 2-join of Q'/red and a single vertex.

The edge y_1y_2 is blue. T' without the edge y_1y_2 has two components, T_1' with y_1 and T_2' with y_2. If x' is in T_1', colour yy_2 blue and y_1y red. If x' is in T_2', colour y_1y blue and yy_2 red. Look at the first case; the second one is the same with y_1 and y_2 exchanged. The number of blue edges is $(|V(Q')|-1)-1+3=|V(Q)|-1$, and the blue edges connect $V(Q)$: T_1', then the path x', x, y, y_2, then T_2'. So they form a spanning tree. The vertex y is in the red component of its red neighbour y_1. The vertices y_1 and y_2 are in different red components of Q', by Lemma A.2 (c). So the blue edge yy_2 joins the same two red components that y_1y_2 joined. And x is a new red component with one vertex and with two blue edges. Again Q/red is a 2-join of Q'/red and a single vertex.

In all cases Q is well coloured. □

From a 2-construction to a closure sequence.

Lemma A.6 (Closure sequence)**.** *Let Q be a graph with a well colouring and with a 2-construction of Q/red. Number the nodes $1,\dots,s$ such that every node comes after all nodes*

that are subgraphs of its two operands. Then there are cycles $C_1, \dots, C_s$ of Q such that, for every i, the cycle C_i contains the two edges of the node i, and all other edges of C_i are red edges or edges of nodes with a smaller number. So, with the set R of the red edges,

$$E(Q) = R \cup E(C_1) \cup \cdots \cup E(C_s),$$

and each C_i has at most two edges outside $R \cup E(C_1) \cup \cdots \cup E(C_{i-1})$. (Here it is not used that the red edges form a forest.)

Proof. Let i be a node with the operands U', U'' and the edges t_1, t_2. By Lemma A.3 (b), (c) the graphs $Q_{U'}$ and $Q_{U''}$ are connected and disjoint, and each of t_1, t_2 has one end in each of them. Take a path of $Q_{U'}$ between the ends of t_1 and t_2 in $V_{U'}$, and a path of $Q_{U''}$ between their ends in $V_{U''}$; a path can have length 0. The two paths with t_1 and t_2 form a cycle C_i. (If both paths have length 0, it is the cycle of length 2 with the edges t_1, t_2.) By Lemma A.3 (a), an edge of $Q_{U'}$ or of $Q_{U''}$ is red, or it is an edge of a node that is a subgraph of U' or of U''; such a node has a number $h < i$, and its edges are on C_h. By Lemma A.2 (b) every blue edge is an edge of a node, so it is on one of the cycles. □

Lemma A.7 (The closure lemma)**.** *Let Q be a truncated cubic 3-edge-connected graph. Then Q has a spanning tree T_0 and cycles $C_1, \dots, C_s$ such that, with $R = E(Q) \setminus E(T_0)$,*

$$E(Q) = R \cup E(C_1) \cup \cdots \cup E(C_s),$$

and each C_i has at most two edges outside $R \cup E(C_1) \cup \cdots \cup E(C_{i-1})$.

Proof. Lemma A.5 gives a well colouring of Q. Let T_0 be the tree of the blue edges; then R is the set of the red edges. Take a 2-construction of Q/red and number its nodes as in Lemma A.6. This lemma gives the cycles. □

Remark A.8 (How the closure sequence is used)**.** In the words of [JLPT], Lemma A.7 says that Q is the 2-closure of the spanning subgraph with the edge set R. For $i = 1, \dots, s$ let N_i be the set of the edges of C_i outside $R \cup E(C_1) \cup \cdots \cup E(C_{i-1})$. The sets N_i are disjoint, each has at most two edges, and their union is $E(T_0)$. For a group A and elements $x_1, \dots, x_s$ of A, the value of the A-flow $x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$ on an edge of N_i does not depend on $x_1, \dots, x_{i-1}$. So the coefficients can be chosen in the order $x_s, \dots, x_1$: when x_i is chosen, the values on N_i are final, and every value that is forbidden on an edge of N_i forbids one value of x_i. This is the reverse choice.

For a frame (Definition 2.9), the official cycles of Lemma 2.11 are a sequence as in Lemma A.6, with $N_i = \{e_i, f_i\}$; they are fixed by the frame, and they have the additional property (O4).

Acknowledgments

This work was done overwhelmingly by AI models and supervised by the author. The models were Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

References

[BBFHMP] V. Bartier, N. Bousquet, C. Feghali, M. Heinrich, B. Moore, T. Pierron, Recoloring planar graphs of girth at least five, SIAM J. Discrete Math. 37 (2023) 332–350; arXiv:2112.00631.

[BBH] V. Bartier, N. Bousquet, M. Heinrich, Recoloring graphs of treewidth 2, Discrete Math. 344 (2021) 112553; arXiv:2012.11459.

[Bo] H. L. Bodlaender, A partial k-arboretum of graphs with bounded treewidth, Theoret. Comput. Sci. 209 (1998) 1–45.

[BB] M. Bonamy, N. Bousquet, Recoloring graphs via tree decompositions, European J. Combin. 69 (2018) 200–213; arXiv:1403.6386.

[BJLPP] M. Bonamy, M. Johnson, I. Lignos, V. Patel, D. Paulusma, Reconfiguration graphs for vertex colourings of chordal and chordal bipartite graphs, J. Comb. Optim. 27 (2014) 132–143; author version HAL lirmm-00782882.

[BC] P. Bonsma, L. Cereceda, Finding paths between graph colourings: PSPACE-completeness and superpolynomial distances, Theoret. Comput. Sci. 410 (2009) 5215–5226; preprint of 18 May 2007, TU Berlin DepositOnce.

[BH] N. Bousquet, M. Heinrich, A polynomial version of Cereceda's conjecture, J. Combin. Theory Ser. B 155 (2022) 1–16; arXiv:1903.05619.

[BP] N. Bousquet, G. Perarnau, Fast recoloring of sparse graphs, European J. Combin. 52 (2016) 1–11; arXiv:1411.6997.

[Ca] A. L. Cauchy, *Recherches sur les nombres*, J. École Polytech. 9 (1813), 99–116.

[Ce] L. Cereceda, Mixing graph colourings, PhD thesis, London School of Economics, 2007.

[CvdHJ] L. Cereceda, J. van den Heuvel, M. Johnson, Finding paths between 3-colorings, J. Graph Theory 67 (2011) 69–82.

[Cr] D. W. Cranston, List-recoloring of sparse graphs, European J. Combin. 105 (2022) 103562; arXiv:2201.05133.

[CLSWX] D. W. Cranston, J. Li, B. Su, Z. Wang, and N. Xu, *Reconfiguration of nowhere-zero flows*, arXiv:2606.24685v1, 2026.

[CM] D. W. Cranston, R. Mahmoud, 5-coloring reconfiguration of planar graphs with no short odd cycles, J. Graph Theory 105 (2024) 670–679; arXiv:2208.02228.

[Da] H. Davenport, *On the addition of residue classes*, J. London Math. Soc. 10 (1935), 30–32.

[DFFV] M. Dyer, A. D. Flaxman, A. M. Frieze, E. Vigoda, Randomly coloring sparse random graphs with fewer colors than the maximum degree, Random Structures Algorithms 29 (2006) 450–465.

[Ed] J. Edmonds, *Submodular functions, matroids, and certain polyhedra*, in: *Combinatorial Structures and their Applications* (Proc. Calgary Internat. Conf., Calgary, 1969), Gordon and Breach, New York, 1970, pp. 69–87; reprinted in: *Combinatorial Optimization – Eureka, You Shrink!*, Lecture Notes in Comput. Sci. 2570, Springer, Berlin, 2003, pp. 11–26, doi:10.1007/3-540-36478-1_2.

[EHLMNS] L. Esperet, K. Hendrey, A. Lagoutte, M. Marseloo, S. Norin, and R. Steiner, *Nowhere-zero flow reconfiguration*, arXiv:2512.17342v4, July 3, 2026.

[Feg] C. Feghali, Reconfiguring colorings of graphs with bounded maximum average degree, J. Combin. Theory Ser. B 147 (2021) 133–138; arXiv:1904.12698.

[FJP] C. Feghali, M. Johnson, D. Paulusma, A reconfigurations analogue of Brooks' theorem and its consequences, J. Graph Theory 83 (2016) 340–358; arXiv:1501.05800.

[Fe] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, 3rd ed., Wiley, New York, 1968.

[FBK] H. Fleischner, B. Bagheri Gh. and B. Klocker, *Perfect pseudo-matchings in cubic graphs*, Graphs Combin. 40 (2024), Article 118, doi:10.1007/s00373-024-02844-y.

[GR] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics 207, Springer, New York, 2001.

[Go] D. Goel, *Eulerian graphs are flow-connected: a proof of a conjecture of Cranston, Li, Su, Wang and Xu*, unpublished manuscript, version 2.00, October 8, 2026.

[GMP] L. A. Goldberg, R. Martin, M. Paterson, Random sampling of 3-colorings in $\mathbb{Z}^2$, Random Structures Algorithms 24 (2004) 279–302.

[HW] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, Oxford, 2008.

[Ho] A. J. Hoffman, *Some recent applications of the theory of linear inequalities to extremal combinatorial analysis*, Proc. Sympos. Appl. Math. 10 (1960), 113–127.

[Ja1] F. Jaeger, *Flows and generalized coloring theorems in graphs*, J. Combin. Theory Ser. B 26 (1979), 205–216, doi:10.1016/0095-8956(79)90057-1.

[Ja2] F. Jaeger, *Nowhere-zero flow problems*, in: L. W. Beineke and R. J. Wilson (eds.), *Selected Topics in Graph Theory 3*, Academic Press, London, 1988, pp. 71–95.

[JLPT] F. Jaeger, N. Linial, C. Payan and M. Tarsi, *Group connectivity of graphs—a nonhomogeneous analogue of nowhere-zero flow properties*, J. Combin. Theory Ser. B 56 (1992), 165–182, doi:10.1016/0095-8956(92)90016-Q.

[Ma] W. Mader, *Über den Zusammenhang symmetrischer Graphen*, Arch. Math. (Basel) 21 (1970), 331–336, doi:10.1007/BF01220924.

[NW1] C. St. J. A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. London Math. Soc. 36 (1961), 445–450, doi:10.1112/jlms/s1-36.1.445.

[NW2] C. St. J. A. Nash-Williams, *An application of matroids to graph theory*, in: P. Rosenstiehl (ed.), *Theory of Graphs* (International Symposium, Rome, 1966), Gordon and Breach, New York, 1967, pp. 263–265; J. Edmonds, *Matroid partition*, in: G. B. Dantzig and A. F. Veinott, Jr. (eds.), *Mathematics of the Decision Sciences, Part 1*, Lectures in Applied Mathematics 11, Amer. Math. Soc., Providence, RI, 1968, pp. 335–345.

[PWL] C. Pan, W. Wang, R. Liu, List-recoloring of two classes of planar graphs, arXiv:2510.17628 (2025).

[RTL] D. J. Rose, R. E. Tarjan, G. S. Lueker, Algorithmic aspects of vertex elimination on graphs, SIAM J. Comput. 5 (1976) 266–283.

[Se] P. D. Seymour, *Nowhere-zero 6-flows*, J. Combin. Theory Ser. B 30 (1981), 130–135.

[Tu1] W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. 6 (1954), 80–91.

[Tu2] W. T. Tutte, *On the problem of decomposing a graph into n connected factors*, J. London Math. Soc. 36 (1961), 221–230, doi:10.1112/jlms/s1-36.1.221.

[Vo] A. G. Vosper, *The critical pairs of subsets of a group of prime order*, J. London Math. Soc. 31 (1956), 200–205.

[WL] Y. Wang, M. Lu, Linear recoloring diameter of degenerate chordal graphs and bounded treewidth graphs, arXiv:2509.15456 (2025).

[Wa] M. E. Watkins, *Connectivity of transitive graphs*, J. Combin. Theory 8 (1970), 23–29.

[ZL] R. Zheng, J. Lu, The diameter of recoloring graphs under a maximum average degree bound, arXiv:2609.32688 (2026).