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SHEIL-SMALL'S (1974) COVERING PROBLEM HAS A POSITIVE ANSWER, AND OTHER MISCELLANEA

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Abstract. T. Sheil-Small asked whether every self-inversive polynomial maps the unit disc onto a domain that contains a disc whose radius is the largest modulus of the polynomial's coefficients (Problem 4.24 in Hayman and Lingham's problem book). We show that the answer is yes. More generally, every non-constant polynomial whose coefficients have the same moduli when read in reverse order maps the unit disc onto a domain that contains an open disc of this radius. This class contains both the self-inversive polynomials, whose zeros are symmetric with respect to the unit circle, and the polynomials whose zeros, with multiplicity, are symmetric under taking reciprocals (the literal reading of the problem as printed), so the theorem answers the question under either reading. The radius cannot be increased. For a polynomial with all its zeros on the unit circle, the largest disc in the image has radius exactly half the maximum modulus of the polynomial on the circle. For these polynomials the question is therefore equivalent to asking whether every coefficient has modulus at most half this maximum. O'Hara and Rodriguez and, independently, Saff and Sheil-Small proved this in 1974 for every coefficient except the middle one, and Kristiansen proved the middle case in 1979. The new case is that of polynomials with zeros off the unit circle. We treat this case with the Schur–Cohn Hermitian form of the polynomial minus a value that it does not take in the disc. This expanded version adds a description by rays of the image of a self-inversive polynomial, two examples, and the trigonometric form of the middle case.[1] The results of Sections 2–5 are also proved in Lean 4, with the coefficient bound taken as given.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc, and write $D(c, r) = \{w \in \mathbb{C} : |w - c| < r\}$ for the open disc with centre c and radius $r > 0$. For a polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ of degree $n \geq 1$ we put

$$A(P) = \max_{0 \leq k \leq n} |a_k|, \qquad M(P) = \max_{|z|=1} |P(z)|, \qquad P^*(z) = z^n \, \overline{P(1/\bar{z})} = \sum_{k=0}^{n} \overline{a_{n-k}} \, z^k.$$

When P is fixed we write $A = A(P)$ and $M = M(P)$. The polynomial P is called *self-inversive* if $P^* = \lambda P$ for some constant λ with $|\lambda| = 1$. When $P(0) \neq 0$, this says exactly that the zeros of P are symmetric with respect to the unit circle: together with each zero ζ, the point $1/\bar{\zeta}$ is a zero of the same multiplicity (Lemma 2.1 below). This is the definition used by Saff and Sheil-Small [15, p. 21]. Sheil-Small's book [17, p. 228] requires $\lambda = 1$; every self-inversive polynomial satisfies this after multiplication by a suitable constant of modulus 1. T. Sheil-Small posed the following problem in the problem list of the 1973 Canterbury symposium on complex analysis, published in 1974 [5]; it is Problem 4.24 in Hayman and Lingham's problem book [6] (see its Tables 1 and 2).

> **Problem** (Sheil-Small). Let $P(z) = \sum_0^n a_k z^k$ be a self-inversive polynomial, i.e. if ζ is a zero of $P(\zeta)$ with multiplicity m, then $1/\zeta$ is also a zero with multiplicity

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Lean 4 proof of Sections 2–5: github.com/david44hello/paper_sheil-small-1974-covering/tree/v3.20/lean
[1]This is an expanded version of the following focused version [3]: https://hexagonmath.org/2611.00154

> m. Is it true that $w = P(z)$ maps $\mathbb{D}$ onto a domain containing a disc of radius $A = \max_{0\le k\le n} |a_k|$?

The update in [6] reads: "No progress on this problem has been reported to us." We answer the question in the affirmative. The table and Figure 1 compare our results with what was known.

New result	**Best result before this paper**
Every non-constant P with $\lvert a_k\rvert = \lvert a_{n-k}\rvert$ for all k (in particular every self-inversive P) maps $\mathbb{D}$ onto a domain containing an open disc of radius A (Theorem 4.1).	Open: Problem 4.24 of [6], with no progress reported.
If all zeros of P lie on the unit circle, the largest disc in $P(\mathbb{D})$ has radius exactly $M/2$, so here the question is equivalent to $M \ge 2A$ (Theorem 3.1).	We have not found the disc of radius $M/2$ stated, but it follows from de Bruijn's theorem (1947; see Lemma 3.2), applied at a point ζ of maximum modulus, where $\zeta P'(\zeta) = \frac{n}{2}P(\zeta)$ (an identity implicit in Saff and Sheil-Small, 1974). The inequality $M \ge 2A$ (asked by Hayman, 1967) was proved by O'Hara–Rodriguez and Saff–Sheil-Small (1974) except for the middle coefficient, and in full by Kristiansen (1979; a gap in one lemma was closed in the 1980 correction).

- *Theorem 4.1 (the covering theorem).* Let $n \ge 1$, and let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree at most n, not constant, with $|a_k| = |a_{n-k}|$ for $0 \le k \le n$. Then $P(\mathbb{D})$ contains an open disc of radius $A = \max_k |a_k|$. This answers Sheil-Small's question [6, Problem 4.24], under either reading of the definition of self-inversive (the quotation has $1/\zeta$, our definition $1/\bar{\zeta}$; see below), and the radius A cannot be increased.
- *Theorem 3.1 (all zeros on the unit circle).* If P has degree $n \ge 1$ and all its zeros on the unit circle, and $|P(\zeta)| = M(P)$ with $|\zeta| = 1$, then $P(\mathbb{D})$ contains $D\big(P(\zeta)/2, M(P)/2\big)$ and no open disc of larger radius.

We make two remarks on Theorem 4.1. First, the problem as printed in [6] pairs a zero ζ with $1/\zeta$, not with $1/\bar{\zeta}$. Read literally, it concerns the polynomials with $z^n P(1/z) = \pm P(z)$, that is, $a_{n-k} = \pm a_k$. With complex coefficients this is a different class (for example $1 + 3iz + z^2$ is of this kind but is not self-inversive). Both classes satisfy $|a_k| = |a_{n-k}|$ (Lemma 2.1), so Theorem 4.1 answers the question under either reading. Second, the radius A cannot be increased, and the disc must be taken open: for $P(z) = 1 + z^n$ we have $A = 1$ and $P(\mathbb{D}) = D(1, 1)$. Section 5 and Figure 2 give examples in which A is large compared with $|a_0|$ and the radius A is still best possible.

Hayman asked in 1967 [4, p. 24] whether every polynomial P with all its zeros on the unit circle satisfies $M(P) \ge 2A(P)$; there the inequality is misprinted as $M \le 2A$ [6, Update 4.2]. O'Hara and Rodriguez [12, Theorem 3] and, independently, Saff and Sheil-Small [15, Theorem 3] proved $|a_k| \le M/2$ for every k except the middle index $k = n/2$ (n even). Kristiansen [8, 9] proved the middle case; see [6, Update 4.2], and [13] for an account of his proof. (Some later papers, for example [7, 14], still describe the middle case as open.)

The disc of radius $M/2$ in Theorem 3.1 comes from a covering theorem of de Bruijn [2] (Lemma 3.2 is the form we use), applied at a point ζ of maximum modulus, where $\zeta P'(\zeta) = \frac{n}{2}P(\zeta)$. Each ingredient appears in one of the two 1974 proofs that $\max_{|z|=1} |P'| = \frac{n}{2}M(P)$ for self-inversive P: Saff and Sheil-Small [15, Theorem 7] work at a point of maximum modulus, and O'Hara and Rodriguez [12, Theorem 1] apply de Bruijn's theorem at a boundary point. In Theorem 3.1, that no open disc of radius larger than $M/2$ lies in $P(\mathbb{D})$ is elementary. Thus the case of Sheil-Small's question in which all zeros lie on the unit circle can be assembled from results in print since 1980, although we have not found it stated. By Theorem 3.1, a positive answer for self-inversive polynomials also implies the inequality $M \ge 2A$ of O'Hara–Rodriguez, Saff–Sheil-Small and Kristiansen for polynomials with all zeros on the unit circle, so any proof of this answer also

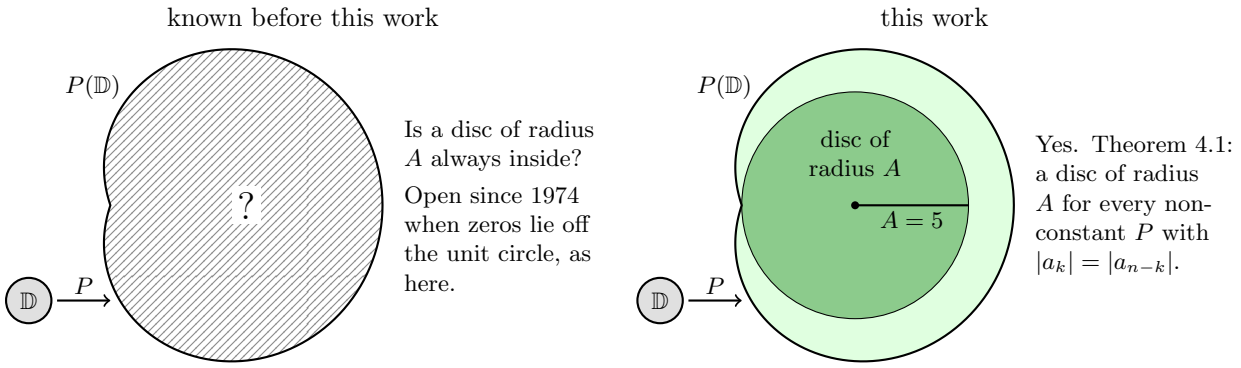


FIGURE 1. The image of the unit disc $\mathbb{D}$ under the self-inversive polynomial $P(z) = 1 + z + 5z^2 + z^3 + z^4$, drawn to scale. Here $A = 5$, and the zeros of P lie off the unit circle. The class of Theorem 4.1 contains the self-inversive polynomials and the polynomials with $a_{n-k} = \pm a_k$ (the literal reading of Problem 4.24), and the polynomials with all zeros on the unit circle are self-inversive (Lemma 2.1). Left, known before this work: the question was listed as open, whichever reading is meant; when all zeros lie on the unit circle, a disc of radius A follows from results in print: the inequality $M \geq 2A$ (1974, 1979) with de Bruijn's theorem. Right, this work: Theorem 4.1 gives a disc of radius A for every non-constant member of the class, and A is the best radius that holds for all of them (Section 5); Theorem 3.1 gives the exact largest radius $M/2$ when all zeros lie on the unit circle.

proves that inequality (see Remark 4.2). The new part of this paper is the case in which P has zeros off the unit circle. For self-inversive P the disc of radius $M/2$ still exists (Proposition 3.3), but $M/2$ may be smaller than A: for $1 + 4z + z^2$ we have $M/2 = 3$ and $A = 4$ (Figure 2, right). We therefore reduce to the circle case by another route, which we now sketch.

The idea of the proof. Suppose that w is a value omitted by P on $\mathbb{D}$. Then $H = P - w$ has no zero in $\mathbb{D}$, and two classical facts about such a polynomial do most of the work. First, on the unit circle $|zH'(z)| \leq |nH(z) - zH'(z)|$. Second, write $\big(H(z)\overline{H(\eta)} - H^*(z)\overline{H^*(\eta)}\big)/(1 - z\bar{\eta}) = \sum_{i,j} K_{ij} z^i \bar{\eta}^j$; the Hermitian form of Schur and Cohn, given by the matrix (K_{ij}), is positive semidefinite, and its entries are explicit in the coefficients of H (Lemma 2.2). In the proof of Theorem 3.1, the first fact, applied at a point ζ where $|P|$ is largest, places every omitted value at distance at least $M/2$ from $P(\zeta)/2$. For Theorem 4.1, let P have degree at most n with $|a_k| = |a_{n-k}|$ for all k. If $a_0 = 0$, then $P(0) = 0$, so every omitted value w is non-zero, and a 2×2 minor of (K_{ij}) gives $|w| \geq A$; hence $D(0, A) \subset P(\mathbb{D})$. If $a_0 \neq 0$, we normalize so that $a_0 = a_n = 1$. Then the diagonal entries K_{jj} all equal $|1 - w|^2 - 1 \geq 0$, so $|1 - w| \geq 1$, which gives $D(1, 1) \subset P(\mathbb{D})$ and settles the case $A = 1$. When $A > 1$, let j be an index with $|a_j| = A$ and put $E = a_{n-j}/\overline{a_j}$, so that $|E| = 1$ and E does not depend on w; the 2×2 minor on the indices 0 and j gives $|w - (1 + E)| \geq A$ unless $w = 1 - E$. Consequently every value in $D(1 + E, A)$ is taken, except possibly the single value $w_* = 1 - E$ (Figure 3). If w_* is indeed omitted, then $P - w_*$ has all its zeros on the unit circle and the same A. Theorem 3.1, together with the inequality $M \geq 2A$ (Theorem 3.4), then applies to it.

Section 2 relates the two readings to the condition $|a_k| = |a_{n-k}|$ and collects the facts about omitted values. Section 3 proves Theorem 3.1, Section 4 proves Theorem 4.1, and Section 5 discusses sharpness. This paper is an expanded version of [3]. Sections 6–8 add a description of $P(\mathbb{D})$ by rays for self-inversive P, which shows that $P(\mathbb{D}) \cup \{0\}$ is star-shaped; two examples; and, following O'Hara and Rodriguez [12], the trigonometric form of the middle case.

2. Self-inversive polynomials and omitted values

Throughout, $n \geq 1$ is fixed, and for a polynomial $H(z) = \sum_{k=0}^{n} h_k z^k$ of degree *at most* n we write $H^*(z) = z^n\,\overline{H(1/\bar{z})} = \sum_{k=0}^{n} \overline{h_{n-k}}\, z^k$ (the coefficient h_n may vanish). Then $(H^*)^* = H$, and $|H^*(z)| = |H(z)|$ when $|z| = 1$. We first record how the two readings of "self-inversive" relate to the condition $|a_k| = |a_{n-k}|$.

Lemma 2.1. *Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with $P(0) \neq 0$.*

(a) *The zeros of P are symmetric under $\zeta \mapsto 1/\bar{\zeta}$, with multiplicities, if and only if $P^* = \lambda P$ for some $|\lambda| = 1$. In that case $\overline{a_{n-k}} = \lambda a_k$ for every k.*

(b) *The zeros of P are symmetric under $\zeta \mapsto 1/\zeta$, with multiplicities, if and only if there is $\varepsilon \in \{1, -1\}$ such that $a_{n-k} = \varepsilon a_k$ for every k.*

In both cases $|a_k| = |a_{n-k}|$ for every k. Every polynomial of degree $n \geq 1$ with all its zeros on the unit circle is self-inversive.

Proof. (a) The zeros of P^* are the points $1/\bar{\zeta}$, where ζ runs through the zeros of P, with the same multiplicities, and P^* has degree n because its leading coefficient is $\overline{a_0} \neq 0$. If the zeros are symmetric, P and P^* have the same zeros and the same degree, so $P^* = \lambda P$ for a constant $\lambda \neq 0$. Since $P^{**} = P$, we get $P = \bar{\lambda} P^* = |\lambda|^2 P$, so $|\lambda| = 1$. The converse is clear, and comparing coefficients in $P^* = \lambda P$ gives $\overline{a_{n-k}} = \lambda a_k$. (b) The same argument with $\widetilde{P}(z) = z^n P(1/z) = \sum_k a_{n-k} z^k$ in place of P^* gives $\widetilde{P} = \varepsilon P$, and applying the map $P \mapsto \widetilde{P}$ twice gives $\varepsilon^2 = 1$. If all zeros $\zeta_1, \ldots, \zeta_n$ lie on the unit circle, then $1/\bar{\zeta_l} = \zeta_l$ and $P(0) \neq 0$, so (a) applies. □

The next lemma is the source of all our inequalities. Part (a) is a classical inequality (on $|z| = 1$ it says $|H'(z)| \leq |(H^*)'(z)|$, the key step in the usual proof of the Erdős–Lax inequality [10]), and part (b), the positivity of the Schur–Cohn Hermitian form, is the easy half of the Schur–Cohn criterion for the location of zeros [16, 1]; see also [11]. We include the short proofs.

Lemma 2.2. *Let $H(z) = \sum_{k=0}^{n} h_k z^k$ be a polynomial of degree at most n with no zero in $\mathbb{D}$. Then there are numbers $\beta_1, \ldots, \beta_n$ with $|\beta_l| \leq 1$ such that*

$$H(z) = h_0 \prod_{l=1}^{n} (1 - \bar{\beta}_l z), \qquad H^*(z) = \overline{h_0} \prod_{l=1}^{n} (z - \beta_l). \tag{1}$$

Consequently:

(a) $|zH'(z)| \leq |nH(z) - zH'(z)|$ *for every z with $|z| = 1$;*

(b) *the function*

$$K(z, \eta) = \frac{H(z)\overline{H(\eta)} - H^*(z)\overline{H^*(\eta)}}{1 - z\bar{\eta}} = \sum_{i,j=0}^{n-1} K_{ij}\, z^i\, \bar{\eta}^j$$

is a polynomial in z and $\bar{\eta}$, and its coefficient matrix $(K_{ij})_{0 \leq i,j \leq n-1}$ is positive semidefinite;

(c) *for $0 \leq j \leq n-1$,*

$$K_{jj} = \sum_{r=0}^{j} \bigl(|h_r|^2 - |h_{n-r}|^2\bigr), \qquad K_{0j} = h_0 \overline{h_j} - \overline{h_n}\, h_{n-j}. \tag{2}$$

Proof. Since $0 \in \mathbb{D}$, $h_0 = H(0) \neq 0$. Let $m \leq n$ be the degree of H and $z_1, \ldots, z_m$ its zeros, repeated according to multiplicity. By hypothesis $|z_l| \geq 1$. Put $\beta_l = 1/\bar{z}_l$ for $l \leq m$ and $\beta_l = 0$ for $m < l \leq n$. Then $H(z) = h_m \prod_{l \leq m} (z - z_l) = h_0 \prod_{l \leq m} (1 - z/z_l)$, which is the first formula in (1) because $1/z_l = \bar{\beta}_l$. The second follows from the definition of H^*: $H^*(z) = z^n\, \overline{h_0} \prod_{l=1}^{n} (1 - \beta_l/z) = \overline{h_0} \prod_{l=1}^{n} (z - \beta_l)$.

(a) Let $|z| = 1$ with $H(z) \neq 0$. By (1), $zH'(z)/H(z) = \sum_l \big(1 - 1/(1-\tau_l)\big)$ with $\tau_l = \bar\beta_l z$, $|\tau_l| \le 1$. For $|\tau| \le 1$, $\tau \neq 1$, we have $\operatorname{Re}\frac{1}{1-\tau} = \frac{1-\operatorname{Re}\tau}{|1-\tau|^2} \ge \frac12$, because $2(1-\operatorname{Re}\tau) - |1-\tau|^2 = 1 - |\tau|^2 \ge 0$. Hence $\operatorname{Re}\big(zH'(z)/H(z)\big) \le n/2$, and

$$|nH(z) - zH'(z)|^2 - |zH'(z)|^2 = n|H(z)|^2\Big(n - 2\operatorname{Re}\frac{zH'(z)}{H(z)}\Big) \ge 0.$$

At a zero of H on the circle the inequality in (a) is an equality.

(b) For $|\beta| \le 1$ and all $z, \eta \in \mathbb{C}$ one checks by multiplying out that

$$(1-\bar\beta z)(1-\beta\bar\eta) - (z-\beta)(\bar\eta - \bar\beta) = (1-|\beta|^2)(1 - z\bar\eta). \tag{3}$$

Put $u_l = (1-\bar\beta_l z)(1-\beta_l\bar\eta)$ and $v_l = (z-\beta_l)(\bar\eta - \bar\beta_l)$. Then $H(z)\overline{H(\eta)} = |h_0|^2 \prod_l u_l$ and $H^*(z)\overline{H^*(\eta)} = |h_0|^2 \prod_l v_l$, and the telescoping identity $\prod_l u_l - \prod_l v_l = \sum_l \big(\prod_{r<l} v_r\big)(u_l - v_l)\big(\prod_{r>l} u_r\big)$ together with (3) gives

$$K(z,\eta) = |h_0|^2 \sum_{l=1}^{n} (1-|\beta_l|^2)\, f_l(z)\, \overline{f_l(\eta)}, \qquad f_l(z) = \prod_{r<l}(z-\beta_r)\prod_{r>l}(1-\bar\beta_r z).$$

Each f_l is a polynomial of degree at most $n-1$. If $c_l \in \mathbb{C}^n$ is its vector of coefficients, then $(K_{ij}) = |h_0|^2 \sum_l (1-|\beta_l|^2)\, c_l\, c_l^*$ (where c_l^* is the conjugate transpose), a sum of positive semidefinite matrices.

(c) Compare the coefficients of $z^i\bar\eta^j$ on both sides of $(1-z\bar\eta)K(z,\eta) = H(z)\overline{H(\eta)} - H^*(z)\overline{H^*(\eta)}$. Since the k-th coefficient of H^* is $\overline{h_{n-k}}$, this gives $K_{ij} - K_{i-1,j-1} = h_i\overline{h_j} - \overline{h_{n-i}}\, h_{n-j}$ (with $K_{ij} = 0$ when an index is negative). Summing along the diagonal gives the formula for K_{jj}, and the case $i = 0$ gives K_{0j}. □

3. Polynomials with all zeros on the unit circle

In this section we prove the following theorem.

Theorem 3.1. *Let P be a polynomial of degree $n \ge 1$ with all its zeros on the unit circle, and let $|\zeta| = 1$ with $|P(\zeta)| = M(P)$. Then $P(\mathbb{D})$ contains the disc $D\big(P(\zeta)/2,\, M(P)/2\big)$, and it contains no open disc of larger radius. Consequently $P(\mathbb{D})$ contains an open disc of radius $A(P)$ if and only if $M(P) \ge 2A(P)$.*

The first step is a covering lemma: a single boundary point z_0 forces a whole disc into the image. It is closely related to a theorem of de Bruijn [2], which is quoted in [12, p. 332] for open and closed circular regions. To reach the lemma from it, take $K = \mathbb{D}$ in the notation there, let u run over $\mathbb{D}$, and let $z \in \mathbb{D}$ tend to z_0; this gives the lemma for polynomials of degree exactly n. We give a short proof based on Lemma 2.2(a); it also covers polynomials of degree less than n.

Lemma 3.2. *Let f be a polynomial of degree at most n, and let $|z_0| = 1$ with $f'(z_0) \neq 0$. Then $f(\mathbb{D})$ contains the open disc with centre $f(z_0) - z_0 f'(z_0)/n$ and radius $|f'(z_0)|/n$.*

Proof. Let $w \notin f(\mathbb{D})$. Then $H = f - w$ has degree at most n and no zero in $\mathbb{D}$, and $H' = f'$. Lemma 2.2(a) at z_0 gives $|f'(z_0)| \le \big|n\big(f(z_0) - w\big) - z_0 f'(z_0)\big|$, that is, $\big|w - \big(f(z_0) - z_0 f'(z_0)/n\big)\big| \ge |f'(z_0)|/n$. Therefore no omitted value lies in the disc. □

Proposition 3.3. *Let F be a polynomial of degree at most n, not constant, with $F^* = \lambda F$ for some $|\lambda| = 1$. Let $M = M(F)$ and let $|\zeta| = 1$ with $|F(\zeta)| = M$. Then $\zeta F'(\zeta) = \frac{n}{2}F(\zeta)$, and $D\big(F(\zeta)/2,\, M/2\big) \subset F(\mathbb{D})$.*

Proof. For real t put $g(t) = |F(e^{it})|^2$. Since $1/\bar z = z$ on the unit circle, $F^* = \lambda F$ gives $\overline{F(e^{it})} = \lambda e^{-int} F(e^{it})$, so $g(t) = \lambda e^{-int} F(e^{it})^2$. Near t_0 with $e^{it_0} = \zeta$ we have $F(e^{it}) \neq 0$ (as $M > 0$), and

logarithmic differentiation gives

$$\frac{g'(t)}{g(t)} = -in + 2i\,\frac{zF'(z)}{F(z)}\Big|_{z=e^{it}}.$$

The left side is real, and it vanishes at t_0, where g is largest. Hence $\zeta F'(\zeta) = \frac{n}{2}F(\zeta)$. In particular $F'(\zeta) \neq 0$, and Lemma 3.2 with $z_0 = \zeta$ gives the disc with centre $F(\zeta) - \frac{1}{2}F(\zeta) = F(\zeta)/2$ and radius $\frac{1}{n}\cdot\frac{n}{2}|F(\zeta)| = M/2$. □

Proof of Theorem 3.1. Write $M = M(P)$. By Lemma 2.1, P is self-inversive, so Proposition 3.3 gives $D(P(\zeta)/2, M/2) \subset P(\mathbb{D})$. Now let $D(c,r) \subset P(\mathbb{D})$; we show that $r \leq M/2$. Since P has no zero in $\mathbb{D}$, the point 0 is not in $D(c,r)$, so $|c| \geq r > 0$. By the maximum principle $|P| < M$ in $\mathbb{D}$, so $D(c,r) \subset D(0,M)$. Since $D(c,r)$ contains the points $c + tc/|c|$ with $0 \leq t < r$, we get $|c| + r \leq M$. Together with $|c| \geq r$, this gives $2r \leq |c| + r \leq M$. The largest radius is therefore exactly $M/2$, and a disc of radius A lies in $P(\mathbb{D})$ if and only if $A \leq M/2$ (Figure 2, left). □

To compare $A(P)$ with $M(P)$ we use the following theorem.

Theorem 3.4 ([8, 9, 12, 15])**.** *Let $P(z) = \sum_{k=0}^{n} a_k z^k$ have degree $n \geq 1$ and all its zeros on the unit circle. Then $|a_k| \leq M(P)/2$ for $0 \leq k \leq n$.*

For $k \neq n/2$ this is [12, Theorem 3] and [15, Theorem 3]. For $k = n/2$ (n even) it is due to Kristiansen [8, Section 2], who works with real trigonometric polynomials whose zeros are all real. To pass to his setting, put $m = n/2$. By Lemma 2.1, $P^* = \lambda P$ with $|\lambda| = 1$, so $Q(\theta) = e^{-im\theta}P(e^{i\theta})$ satisfies $\overline{Q} = \lambda Q$, and $T = cQ$ is real when $c^2 = \lambda$. Then T is a real trigonometric polynomial of exact degree m (its top coefficient is $c\,a_n \neq 0$) whose $2m$ zeros in a period are all real, $\max|T| = M(P)$, and the constant term of T is $c\,a_{n/2}$. So the case $k = n/2$ says that the constant term t_0 of such a T satisfies $|t_0| \leq \frac{1}{2}\max|T|$; this is Conjecture 4 of [15], applied to T and to $-T$. A gap in the proof of one of Kristiansen's lemmas was closed by T. Sheil-Small in the correction [9].

Corollary 3.5. *If P has degree $n \geq 1$ and all its zeros on the unit circle, then $P(\mathbb{D})$ contains an open disc of radius $A(P)$.*

Proof. By Theorem 3.4, $A(P) \leq M(P)/2$; now apply Theorem 3.1. □

4. The covering theorem

In this section we prove the main theorem, using Lemma 2.2 for the omitted values and Corollary 3.5 for the polynomials with all zeros on the unit circle.

Theorem 4.1. *Let $n \geq 1$, and let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree at most n that is not constant. Suppose that*

$$|a_k| = |a_{n-k}| \qquad (0 \leq k \leq n). \tag{4}$$

Then $P(\mathbb{D})$ contains an open disc of radius $A = \max_k |a_k|$.

Proof. Since P is not constant, $A > 0$. We distinguish two cases.

Case 1: $a_0 = 0$. Then $a_n = 0$ as well, so the maximum A is attained at an index j with $1 \leq j \leq n-1$ (in particular $n \geq 2$). Moreover $P(0) = 0$. Let $w \notin P(\mathbb{D})$; then $w \neq 0$. Apply Lemma 2.2 to $H = P - w$, of degree at most n, whose coefficients are $h_0 = -w$, $h_n = 0$ and $h_k = a_k$ for $0 < k < n$. In the sum (2) for K_{jj} the terms with $1 \leq r \leq j$ vanish by (4): for these r both r and $n - r$ lie between 1 and $n - 1$, so $h_r = a_r$ and $h_{n-r} = a_{n-r}$. Hence $K_{jj} = |h_0|^2 - |h_n|^2 = |w|^2$, and $K_{0j} = -w\,\overline{a_j}$. The 2×2 principal minor of the positive semidefinite matrix (K_{pq}) on the indices 0 and j gives $|K_{0j}|^2 \leq K_{00}K_{jj}$, that is, $|w|^2A^2 \leq |w|^4$. Hence $|w| \geq A$ for every omitted value w, and $D(0,A) \subset P(\mathbb{D})$.

Case 2: $a_0 \neq 0$. Then $a_n \neq 0$, so P has degree n. Choose α with $|\alpha| = 1$ and $\alpha^n = a_0/a_n$, and replace P by $P(\alpha z)/a_0$. This multiplies $P(\mathbb{D})$ by $1/a_0$, hence every radius by $1/|a_0|$; it multiplies A by the same factor and preserves (4). We may therefore assume $a_0 = a_n = 1$, and then $A \geq 1$.

Let $w \notin P(\mathbb{D})$ and apply Lemma 2.2 to $H = P - w$: now $h_0 = 1 - w$, $h_n = 1$ and $h_k = a_k$ for $0 < k < n$. As in Case 1, the terms with $1 \leq r \leq j$ in (2) cancel, so all diagonal entries are equal:

$$K_{jj} = |1 - w|^2 - 1 =: d \qquad (0 \leq j \leq n - 1).$$

Positive semidefiniteness gives $d \geq 0$, that is, $|w - 1| \geq 1$. If $A = 1$, this shows that every value in $D(1, 1)$ is taken, and we are done.

Suppose $A > 1$. Then A is attained at some j with $1 \leq j \leq n - 1$. Put $E = a_{n-j}/\overline{a_j}$ and $u = w - 1$. By (4), $|E| = 1$. By (2), $K_{0j} = (1 - w)\overline{a_j} - a_{n-j} = -\overline{a_j}\,(u + E)$, and the 2×2 minor on the indices 0 and j gives

$$A\,|u + E| \leq d = |u|^2 - 1. \tag{5}$$

Since $|E| = 1$, we also have $|u|^2 - 1 = |u^2| - |E^2| \leq |u^2 - E^2| = |u + E|\,|u - E|$, so (5) gives $A\,|u+E| \leq |u+E|\,|u-E|$. If $u \neq -E$, dividing by $|u+E|$ gives $|u-E| \geq A$, that is, $|w-(1+E)| \geq A$. We have shown that every omitted value other than $w_* = 1-E$ lies outside $D(1+E, A)$ (Figure 3):

$$D(1 + E, A) \setminus \{w_*\} \subset P(\mathbb{D}). \tag{6}$$

If $w_* \in P(\mathbb{D})$, then (6) gives $D(1 + E, A) \subset P(\mathbb{D})$, and we are done. Suppose $w_* \notin P(\mathbb{D})$, and let $S = P - w_*$. Its constant term is E and its leading coefficient is 1. It has no zero in $\mathbb{D}$, so its n zeros have modulus at least 1, while their product has modulus $|E| = 1$. Hence all zeros of S lie on the unit circle. The coefficients of S are E, $a_1, \ldots, a_{n-1}$ and 1, so $A(S) = A$. By Corollary 3.5, $S(\mathbb{D})$ contains an open disc of radius A, and so does $P(\mathbb{D}) = S(\mathbb{D}) + w_*$. □

Remark 4.2. Since $|w_* - (1 + E)| = 2$, the point $w_* = 1 - E$ lies in $D(1 + E, A)$ only when $A > 2$ (Figure 3, right), that is, only when $A > 2|a_0|$ before the normalization. Thus Theorem 3.4 is needed only when $A > 2|a_0|$ and w_* is omitted; otherwise (6) already gives $D(1 + E, A) \subset P(\mathbb{D})$. The threshold is the same as in [12, Section 2, Remark C], where the middle case of Theorem 3.4 is proved when $|a_{n/2}| \leq 2|a_n|$. Moreover, if A is attained at some index $j \neq n/2$, then $A(S) = |a_j|$ with $j \neq n/2$, and [12, Theorem 3] or [15, Theorem 3] suffices. Thus the middle case of Theorem 3.4, due to Kristiansen, is used only when n is even, $|a_{n/2}|$ is strictly larger than every other $|a_k|$ and than $2|a_0|$, and w_* is omitted. The middle case cannot be bypassed: by Theorem 3.1, for polynomials with all zeros on the unit circle Theorem 4.1 is equivalent to Theorem 3.4, so every proof of Theorem 4.1 also proves the middle case.

5. Sharpness

We show that the radius A in Theorem 4.1 cannot be increased, and that this remains true when A is large compared with $|a_0|$.

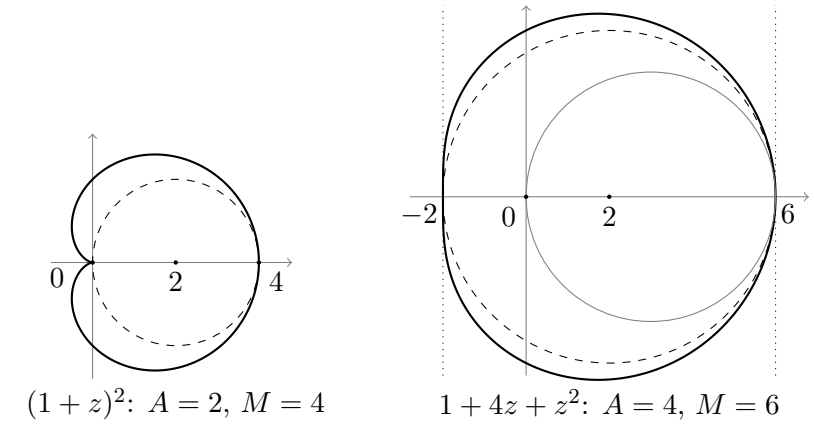
Example 5.1. For $P(z) = 1 + z^n$ we have $A = 1$ and $P(\mathbb{D}) = D(1, 1)$, which contains no open disc of radius larger than 1 and no closed disc of radius 1. Thus in Theorem 4.1 the radius cannot be increased, and "open" cannot be replaced by "closed".

The next example shows that the radius stays sharp when A is large compared with $|a_0|$.

Example 5.2. Let $a \geq 4$, $m \geq 1$, and $P(z) = 1 + az^m + z^{2m}$. With $z^m = e^{i\theta}$ we get $P = e^{i\theta}(a + 2\cos\theta)$, so

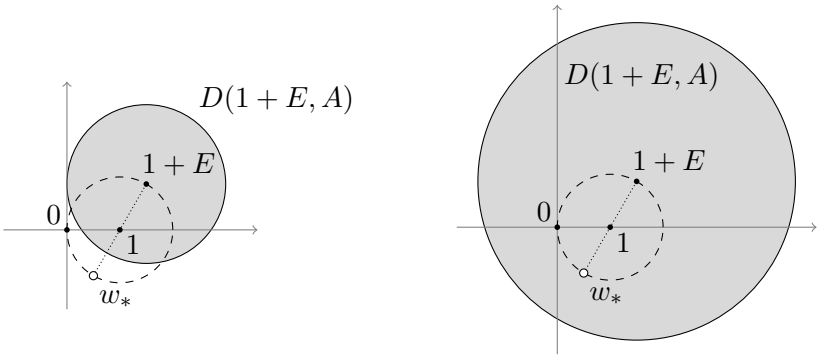
$$\operatorname{Re} P = a\cos\theta + 2\cos^2\theta \qquad (|z| = 1).$$

As a function of $x = \cos\theta \in [-1, 1]$, the right-hand side has derivative $a + 4x \geq 0$, so it takes values in $[2 - a, 2 + a]$. By the maximum principle for the harmonic function $\operatorname{Re} P$, the set $P(\mathbb{D})$ lies in the strip $2 - a < \operatorname{Re} w < 2 + a$, of width $2a$. Every disc in $P(\mathbb{D})$ therefore has radius at most $a = A$.



$(1+z)^2$: $A=2$, $M=4$ $\qquad$ $1+4z+z^2$: $A=4$, $M=6$

FIGURE 2. Two images of $\mathbb{D}$, in polar coordinates $w=\rho e^{i\psi}$. Left: the image under $(1+z)^2$ is bounded by the cardioid $\rho=2+2\cos\psi$; all zeros lie on the unit circle, and the largest disc inside it is $D(2,2)$, of radius $M/2=A$ (Theorem 3.1). Right: the image under $1+4z+z^2$ is bounded by the limaçon $\rho=4+2\cos\psi$; it lies in the strip $-2<\operatorname{Re} w<6$ (dotted), and the dashed disc $D(2,4)$ of Theorem 4.1 has the largest possible radius $A=4$ (Example 5.2); the grey disc is $D(3,3)$, of radius $M/2$.



$A=3/2$ $\qquad$ $A=3$

FIGURE 3. The discs in Case 2 of the proof of Theorem 4.1, after the normalization $a_0=a_n=1$, drawn for $E=e^{i\pi/3}$. The image $P(\mathbb{D})$ itself is not drawn. The dashed circle is $|w-1|=1$, the boundary of $D(1,1)$. The grey disc is $D(1+E,A)$, with its centre $1+E$, and the open dot is the point $w_*=1-E$, joined to $1+E$ by a dotted segment. The left panel has $A=3/2$, the right panel $A=3$. See the proof of Theorem 4.1 and Remark 4.2.

By Theorem 4.1 a disc of radius A does lie in $P(\mathbb{D})$: here $E=1$, and $w_*=0$ is taken because $P(z)=0$ when z^m is the root of t^2+at+1 in $(-1,0)$ (the roots are negative with product 1), so the disc of the proof is $D(2,a)$ (Figure 2, right, for $a=4$, $m=1$).

6. The ray criterion and the star shape of the image

For a polynomial F with $F^*=F$, the image $F(\mathbb{D})$ can be read off from the boundary curve $t\mapsto F(e^{it})$, ray by ray. We first show how a single point of the curve forces a whole segment of a ray into the image.

Lemma 6.1 (Ray lemma)**.** *Let F be a polynomial of degree at most n with $F^* = F$. Let $w \neq 0$, and let ξ be a point of the unit circle with $\xi^n = w/\bar{w}$. If $\operatorname{Re}\big(F(\xi)/w\big) > 1$, then $w \in F(\mathbb{D})$.*

Proof. Suppose that $w \notin F(\mathbb{D})$. Then $H = F - w$ has no zero in $\mathbb{D}$, and $H^* = F^* - \bar{w}z^n = F - \bar{w}z^n$. Setting $\eta = z$ in (3) gives $|1 - \bar{\beta}z|^2 - |z - \beta|^2 = (1 - |\beta|^2)(1 - |z|^2) \geq 0$ for $z \in \mathbb{D}$, so by (1) $|H^*(z)| \leq |H(z)|$ in $\mathbb{D}$. For $0 < r < 1$ the point $z = r\xi$ lies in $\mathbb{D}$, and $\bar{w}z^n = r^n\bar{w}\,\xi^n = r^n w$; therefore $|F(r\xi) - r^n w| \leq |F(r\xi) - w|$. Squaring both sides and cancelling $|F(r\xi)|^2$, we get $2(1 - r^n)\operatorname{Re}\big(F(r\xi)\bar{w}\big) \leq (1 - r^{2n})|w|^2$, that is,

$$\operatorname{Re}\frac{F(r\xi)}{w} \leq \frac{1 + r^n}{2}.$$

Letting $r \to 1$ gives $\operatorname{Re}\big(F(\xi)/w\big) \leq 1$, contrary to the hypothesis. □

Theorem 6.2. *Let F be a polynomial of degree $n \geq 1$ with $F^* = F$, and let $w \neq 0$. Then $w \in F(\mathbb{D})$ if and only if there are a point ξ of the unit circle and a number $s > 1$ with $F(\xi) = sw$. Consequently $F(\mathbb{D}) \cup \{0\}$ is star-shaped with respect to 0.*

Proof. Let $\Xi = \{\xi : \xi^n = w/\bar{w}\}$. It consists of n distinct points of the unit circle. For $|\xi| = 1$ the identity $F^* = F$ gives $F(\xi) = \xi^n\,\overline{F(\xi)}$. If $F(\xi) = sw$ with $s > 0$, this gives $\xi^n = w/\bar{w}$, so $\xi \in \Xi$. For every $\xi \in \Xi$ the same identity shows that $F(\xi)/w$ is real.

If $F(\xi) = sw$ with $s > 1$, then $\xi \in \Xi$ and $\operatorname{Re}(F(\xi)/w) = s > 1$, so $w \in F(\mathbb{D})$ by Lemma 6.1.

Conversely, suppose that $F(\xi)/w \leq 1$ for every $\xi \in \Xi$; we show that $w \notin F(\mathbb{D})$. Consider the rational function

$$\Phi(z) = \frac{F(z) - w}{w - \bar{w}z^n}.$$

Its denominator vanishes exactly on Ξ, and nowhere in $\mathbb{D}$. At $\xi \in \Xi$ the derivative of the denominator is $-n\bar{w}\xi^{n-1} = -nw/\xi$, so Φ has at most a simple pole at ξ, with residue $-2c_\xi\,\xi$, where $c_\xi = \big(F(\xi)/w - 1\big)/(2n)$ is real and ≤ 0. The function $(\xi + z)/(\xi - z)$ has residue -2ξ at ξ. Therefore

$$G(z) = \Phi(z) - \sum_{\xi \in \Xi} c_\xi \frac{\xi + z}{\xi - z}$$

is a rational function without poles in $\mathbb{C}$. It is bounded at infinity, because $\deg(F - w) \leq n$, so G is constant. On the unit circle outside Ξ we have $H = F - w \neq 0$, because a point of the circle where $F = w$ lies in Ξ (by the first paragraph of the proof, with $s = 1$). On this set $H^* = F - \bar{w}z^n$ and $|H^*| = |H|$, so $\Phi = H/(H^* - H) = 1/(B - 1)$ with $B = H^*/H$ of modulus 1; hence $\operatorname{Re}\Phi = -1/2$ there. Also $\operatorname{Re}\big((\xi + z)/(\xi - z)\big) = (1 - |z|^2)/|\xi - z|^2$, which vanishes on the circle. So $\operatorname{Re} G = -1/2$, and for $z \in \mathbb{D}$

$$\operatorname{Re}\Phi(z) = -\frac{1}{2} + \sum_{\xi \in \Xi} c_\xi \frac{1 - |z|^2}{|\xi - z|^2} \leq -\frac{1}{2}.$$

In particular Φ has no zero in $\mathbb{D}$, so neither has $F - w$, and $w \notin F(\mathbb{D})$.

For the last statement, let $w \in F(\mathbb{D})$, $w \neq 0$, and choose ξ and $s > 1$ with $F(\xi) = sw$. For $0 < \sigma \leq 1$ we have $F(\xi) = (s/\sigma)(\sigma w)$ with $s/\sigma > 1$, so $\sigma w \in F(\mathbb{D})$. □

If $F^* = \lambda F$ with $|\lambda| = 1$, choose μ with $\mu^2 = \lambda$; then $(\mu F)^* = \bar{\mu}\lambda F = \mu F$. So Theorem 6.2 applies to μF, and $F(\mathbb{D}) \cup \{0\}$ is star-shaped with respect to 0 for every self-inversive F. When F has a zero in $\mathbb{D}$, the image $F(\mathbb{D})$ itself is star-shaped with respect to 0.

7. Two examples

The first example shows that the disc of radius $M/2$ in Proposition 3.3 needs self-inversivity: the weaker condition (4) is not enough.

Example 7.1. Let $c = (1-2i)/\sqrt{5}$, so $|c| = 1$, and $Q(z) = z^3 - z^2 - z + c$. All four coefficients have modulus 1, so Q satisfies (4) and $A(Q) = 1$. Put $w_* = c - 1$ and $F(z) = Q(z) - w_* = z^3 - z^2 - z + 1 = (z-1)^2(z+1)$. The zeros of F lie on the unit circle, so $w_* \notin Q(\mathbb{D})$. For $z = e^{it}$ and $x = \cos t$,

$$|F(z)|^2 = |z-1|^4\,|z+1|^2 = 8(1-x)^2(1+x),$$

and $(1-x)^2(1+x)$ has derivative $-(1-x)(1+3x)$, so its maximum on $[-1,1]$ is $32/27$, at $x = -1/3$. Hence $M(F) = 16/(3\sqrt{3})$, and by the maximum principle $Q(\mathbb{D}) \subset D(w_*, M(F))$. As in the proof of Theorem 3.1, every disc in $Q(\mathbb{D})$ has radius at most $M(F)/2 = 8/(3\sqrt{3}) \approx 1.540$. On the other hand, $Q(i) = 1 - 2i + c = (1 + 1/\sqrt{5})(1-2i)$, so $M(Q) \geq |Q(i)| = 1 + \sqrt{5}$ and $M(Q)/2 \geq (1+\sqrt{5})/2 \approx 1.618$. So $Q(\mathbb{D})$ contains no disc of radius $M(Q)/2$, although it contains one of radius $A(Q) = 1$ by Theorem 4.1.

The second example shows that the exceptional value w_* in the proof of Theorem 4.1 can indeed be omitted.

Example 7.2. Let $P(z) = (1+z)^3 = 1 + 3z + 3z^2 + z^3$. Here $a_0 = a_3 = 1$ and $A = 3$, attained at $j = 1$, so $E = a_2/\overline{a_1} = 1$, the disc of (6) is $D(2,3)$, and $w_* = 0$. Since the only zero of P is -1, the value 0 is omitted (Figure 4; compare Figure 3, right), although it lies in $D(2,3)$. So $D(2,3) \not\subset P(\mathbb{D})$, and the proof needs its last case, in which w_* is omitted: all zeros of P lie on the unit circle, $M(P) = 8$, and Theorem 3.1 gives the disc $D(4,4)$, of the largest possible radius $M(P)/2 = 4$.

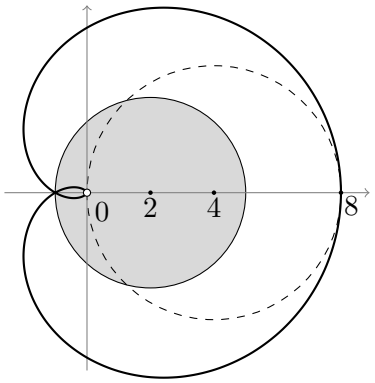


FIGURE 4. Example 7.2, $P(z) = (1+z)^3$. The solid curve is the image $P(e^{it})$, $0 \leq t \leq 2\pi$, of the unit circle. The grey disc is $D(2,3)$, the disc of (6) for this P, and the open dot is $w_* = 0$, which P omits although it lies in $D(2,3)$. The dashed disc is $D(4,4)$, given by Theorem 3.1.

8. The middle coefficient

Theorem 3.1 shows that, for polynomials with all zeros on the unit circle, our covering statement is equivalent to Theorem 3.4; in particular it implies the middle case $k = n/2$. That case has a natural trigonometric form, observed by O'Hara and Rodriguez [12, Section 2, Remark D]; Kristiansen [8] works in this trigonometric setting. Let $n = 2m$ and let P have all zeros on the unit circle; after multiplying P by a unimodular constant we may assume $P^* = P$ (Lemma 2.1 and the remark after Theorem 6.2). Then $R(t) = e^{-imt}P(e^{it}) = \sum_{k=0}^{2m} a_k e^{i(k-m)t}$ is a real trigonometric polynomial of degree m (the polynomial T introduced after Theorem 3.4, with $c = 1$), with mean value a_m and maximum modulus $M(P)$. Its $2m$ zeros in a period correspond, with multiplicity, to those of P through $z = e^{it}$, and every real trigonometric polynomial of degree m with $2m$ real zeros in a period arises in this way. So the middle case of Theorem 3.4 says:

> a real trigonometric polynomial of degree m with $2m$ real zeros in a period (counted with multiplicity) has a mean value whose absolute value is at most half its maximum modulus.

When $R \geq 0$, every zero of R has even multiplicity, and by the Fejér–Riesz theorem $R = |q(e^{it})|^2$ for a polynomial q of degree m with all its zeros on the unit circle; then the mean of R is $\sum_k |q_k|^2$ and the maximum of R is $M(q)^2$. For such q, O'Hara and Rodriguez proved $2\sum_k |q_k|^2 \leq M(q)^2$ [12, Corollary 1]; hence $a_m \leq M(P)/2$, and the middle case holds when $R \geq 0$. Here $P = \lambda q^2$ with $|\lambda| = 1$, so this is [12, Section 2, Remark A] read in the other direction: there the middle coefficient of q^2 is shown to have modulus $\sum_k |q_k|^2$.

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