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# THE DIMENSION-FREE SYMMETRIC GAUSSIAN ISOPERIMETRIC PROFILE: FINITE ATTAINMENT AND CRITICAL RANK

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Abstract. We determine the infimum of Gaussian perimeter over centrally symmetric sets when the ambient dimension is allowed to vary. The answer is the lower envelope of the perimeters of centered balls and their complements. At every mass other than one half the infimum is attained in finite dimension, proving the finite-attainment conjecture stated by Heilman. Every attaining ball rank is at most $1/(2|a-1/2|^2)$. At mass one half the value is $1/\sqrt{\pi}$ and no finite-dimensional centrally symmetric set attains it. Every optimal ball rank satisfies $k|a-1/2|^2 \to 1/(9\pi)$ as $a \to 1/2$. The profile has expansion $\pi^{-1/2} - (3\sqrt{\pi}/2)|a-1/2|^2 + O(|a-1/2|^4)$, and it yields the sharp even Gaussian Cheeger inequality. The geometric input is Viana's classification of isoperimetric regions in real projective spaces. The additional step is a stability estimate that controls the smaller of the two spherical ranks away from half volume and identifies the only regime in which both ranks can escape to infinity.

## 1. The profile and the main results

The symmetric Gaussian isoperimetric problem asks how central symmetry changes the least perimeter at a prescribed Gaussian volume. In this paper the volume is fixed but the ambient dimension may vary. We determine the resulting lower envelope, prove that a finite-dimensional ball or its complement attains it away from half volume, and describe the diverging optimal ball rank as the volume approaches one half.

Let $\gamma_d$ be standard Gaussian probability measure on $\mathbb{R}^d$, with density

$$\phi_d(x) = (2\pi)^{-d/2} e^{-|x|^2/2}.$$

For a set of locally finite perimeter put

$$P_{\gamma_d}(E) = \int_{\partial^* E} \phi_d(x)\, d\mathcal{H}^{d-1}(x).$$

The perimeter is allowed to be infinite and is defined to be infinite for measurable sets which do not have locally finite perimeter. Symmetry always means central symmetry, $E = -E$ up to a null set; neither convexity nor invariance under the separate coordinate reflections is assumed. Define

$$I_d(a) = \inf\{P_{\gamma_d}(E) : E = -E,\ \gamma_d(E) = a\}, \tag{1.1}$$

$$I_\infty(a) = \inf_{d \geq 1} I_d(a), \qquad 0 < a < 1. \tag{1.2}$$

Our perimeter uses the ambient probability density on the boundary. The surface-area convention in [6, 7] is $\sqrt{2\pi}\, P_{\gamma_d}$.

For $k \geq 1$ let

$$\rho_k(r) = \frac{2^{1-k/2}}{\Gamma(k/2)} r^{k-1} e^{-r^2/2}, \qquad F_k(r) = \int_0^r \rho_k(t)\, dt, \qquad r > 0. \tag{1.3}$$

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Write $R_k(a) = F_k^{-1}(a)$ and

$$j_k(a) = \rho_k(R_k(a)). \tag{1.4}$$

Thus $j_k(a)$ is the Gaussian perimeter of the centered ball of mass $a$ in $\mathbb{R}^k$. A product with a Gaussian Euclidean factor does not change either its mass or its perimeter. An *attaining ball rank at mass* $a$ is an integer $k$ for which $j_k(\max\{a, 1-a\}) = I_\infty(a)$. This is the dimension of the ball factor, not the ambient dimension of an arbitrary minimizing set.

**Background and relation to previous work.** The symmetric question originates in Barthe's work [3]. His theorem for unconditional sets concerns an anisotropic perimeter; the present paper uses the usual Euclidean Gaussian perimeter. For that perimeter, Barchiesi and Julin [2] proved that strips minimize at volumes sufficiently close to one, with a threshold independent of dimension. Heilman [6, 7] obtained partial classification results through Gaussian variations. La Manna's local stability threshold for balls [8] will enter our quantitative rank bound through an explicit second-variation calculation.

The question of determining (1.2) is stated explicitly in [7, Section 1.3], together with the conjecture that its value is realized in a finite dimension at every mass different from $1/2$. The half-mass lower bound $1/\sqrt{\pi}$ is proposed in the introductory discussion of [6]. The results below address these dimension-free questions. Recent work of Barchiesi, Cagnetti, and Julin [1] develops a double-arc rearrangement for unconditional sets and discusses a route through high-dimensional spheres. Milman [10] resolves the fixed-dimensional symmetric problem in the plane. Our result concerns the infimum over all finite dimensions and does not require a classification in each prescribed dimension.

**Theorem 1.1** (Exact profile and finite attainment)**.** *For $0 < a < 1$, put $m = \max\{a, 1-a\}$. Then*

$$I_\infty(a) = \inf_{k\geq 1} \min\{j_k(a), j_k(1-a)\} = \inf_{k\geq 1} j_k(m). \tag{1.5}$$

*If $a \neq 1/2$, these infima are minima. A centered ball of mass $a$ when $a > 1/2$, and the complement of a centered ball of mass $1-a$ when $a < 1/2$, realizes $I_\infty(a)$ in a finite dimension. Every integer $k$ with $j_k(m) = I_\infty(a)$ satisfies*

$$k \leq \frac{1}{2|a-1/2|^2}. \tag{1.6}$$

*Consequently,*

$$I_d(a) = I_\infty(a) \quad \text{if} \quad d \geq \left\lceil \frac{1}{2|a-1/2|^2} \right\rceil, \tag{1.7}$$

*and a round cylinder or its complement realizes this value. At the exceptional mass,*

$$I_\infty(1/2) = \frac{1}{\sqrt{\pi}}. \tag{1.8}$$

*This value is approached by centered half-mass balls as their dimension tends to infinity, but it is not realized by any finite-dimensional centrally symmetric set.*

**Theorem 1.2** (Critical rank)**.** *Let $\delta > 0$ tend to zero and choose any attaining ball rank $k = k(\delta)$ in $I_\infty(1/2+\delta) = j_k(1/2+\delta)$. Then*

$$k(\delta)\delta^2 \longrightarrow \frac{1}{9\pi}. \tag{1.9}$$

*Moreover,*

$$I_\infty(1/2+\delta) = \frac{1}{\sqrt{\pi}} - \frac{3\sqrt{\pi}}{2}\delta^2 + O(\delta^4). \tag{1.10}$$

*By symmetry the same expansion holds on the other side of $1/2$. In particular, $I_\infty$ is differentiable at $1/2$, with derivative zero.*

**Corollary 1.3** (Sharp even Gaussian Cheeger inequality)**.** *Every centrally symmetric set of finite Gaussian perimeter satisfies*

$$P_{\gamma_d}(E) \geq \frac{2}{\sqrt{\pi}} \min\{\gamma_d(E), 1-\gamma_d(E)\}. \tag{1.11}$$

*For every even Lipschitz function* $f : \mathbb{R}^d \to \mathbb{R}$*,*

$$\inf_{c\in\mathbb{R}} \int |f-c|\, d\gamma_d \leq \frac{\sqrt{\pi}}{2} \int |\nabla f|\, d\gamma_d. \tag{1.12}$$

*Both constants are optimal when all finite dimensions are considered. The constant* $c$ *on the left of* (1.12) *can be any median of* $f$*.*

*Remark* 1.4 (Scope). Theorem 1.1 determines the profile with the dimension unrestricted and proves finite attainment away from half mass. It does not assert the formula

$$I_d(a) = \min_{1\leq k\leq d} \min\{j_k(a), j_k(1-a)\}$$

for every prescribed smaller dimension $d$, and it does not classify every equality case. Those fixed-dimension assertions are stronger. The results here also do not assert that every intermediate integer occurs as an optimal rank. In particular, (1.9) describes the ranks in the ball representation of the dimension-free profile, not a classification of all minimizers or a formula for their least possible ambient dimension.

**Geometric input and proof strategy.** Viana's theorem [11, Corollary 1.3] classifies antipodally invariant isoperimetric regions on a sphere as thresholds of a squared coordinate projection. Their masses are beta probabilities. We first use their second variation to locate the threshold within one unit of the first block dimension. A two-block central limit argument then shows that both block dimensions can diverge only at mass one half. Fixed ranks converge to Gaussian balls, while the escaping ranks have perimeter tending to $1/\sqrt{\pi}$. We transfer this profile limit to arbitrary Gaussian sets using the exact perimeter formula for a spherical lift.

The related Poincaré-limit strategy and the role of the smaller spherical rank are discussed in [1, Appendix B, Remark 8]. The rank estimate proved below is what makes that limiting argument compact away from half mass. The self-shrinker classification from [7, Theorem 6.1] is used only in the last step excluding finite-dimensional attainment at half mass; it is not needed to compute the profile. More precisely, Sections 2–4 determine the profile and prove finite attainment. Section 5 proves the quantitative bound. Section 6 obtains the critical expansion without assuming nonattainment. Its differentiability consequence then supplies the zero-multiplier hypothesis needed in Section 7.

## 2. Spherical profiles and stability

Throughout the spherical argument $N \geq 4$ is even. This subsequence is sufficient for the Gaussian limit and permits an exactly balanced test at half mass. Let $\sigma_N$ be normalized surface measure on $S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$. Spherical perimeter is the boundary area divided by the total area of this sphere, with the metric of radius $\sqrt{N}$. Denote the antipodal isoperimetric profile by

$$\mathcal{I}_N(a) = \inf\{P_{\sigma_N}(A) : A = -A,\ \sigma_N(A) = a\}.$$

**Proposition 2.1** (Viana's classification in probability coordinates)**.** *For* $1 \leq r \leq N-1$*, write* $s = N-r$ *and let* $t = t_{N,r}(a)$ *be the* $a$*-quantile of* $\mathrm{Beta}(r/2, s/2)$*. Then*

$$\mathcal{I}_N(a) = \min_{1\leq r\leq N-1} J_{N,r}(a), \qquad J_{N,r}(a) = \frac{2}{\sqrt{N}} \sqrt{t(1-t)}\, \frac{t^{r/2-1}(1-t)^{s/2-1}}{B(r/2, s/2)}. \tag{2.1}$$

*The corresponding region is, after rotation,*

$$A_{N,r,t} = \left\{ x \in S^{N-1}(\sqrt{N}) : \frac{x_1^2 + \cdots + x_r^2}{N} \le t \right\}. \tag{2.2}$$

*Proof.* The quotient map to real projective space is a two-sheeted local isometry. An antipodally invariant finite-perimeter set descends to the quotient; its volume, boundary area, and the total ambient volume are all divided by two. Thus the normalized isoperimetric problems agree. Apply [11, Corollary 1.3 and Section 4.1]; the latter section includes finite-perimeter regions and the possible singular set. Rescaling the metric to radius $\sqrt{N}$ preserves the classification and rescales normalized perimeter by $N^{-1/2}$. Complements can be put in the form (2.2) by interchanging the two coordinate blocks. For a uniform point on the sphere the variable $T = (x_1^2 + \cdots + x_r^2)/N$ has the indicated beta distribution. Its tangent gradient has length $2\sqrt{T(1-T)}/\sqrt{N}$. Coarea gives (2.1). □

**Lemma 2.2** (Uniform continuity of the spherical profiles)**.** *The functions $\mathcal{I}_N$ are concave and symmetric about $1/2$, and*

$$0 \le \mathcal{I}_N(a) \le \sqrt{2/\pi}. \tag{2.3}$$

*They have a common Lipschitz bound on every compact subinterval of $(0,1)$.*

*Proof.* Put $\theta = \arcsin\sqrt{T}$. Its density on $(0, \pi/2)$ is

$$g(\theta) = \frac{2}{B(r/2, s/2)} \sin^{r-1}\theta \cos^{s-1}\theta.$$

The function $\log g$ is concave. If $G$ is its distribution function, then $a \mapsto g(G^{-1}(a))$ is concave, since its second derivative is $(\log g)''(G^{-1}(a))/g(G^{-1}(a))$. This function divided by $\sqrt{N}$ is $J_{N,r}$. The infimum of concave functions is concave, so (2.1) proves concavity of $\mathcal{I}_N$. Complementation gives its symmetry.

For $r = 1$, the maximum of $g$ is $2/B(1/2, (N-1)/2)$. The gamma-ratio estimate

$$\frac{\Gamma(x + 1/2)}{\Gamma(x)} \le \sqrt{x}, \qquad x > 0,$$

follows from Cauchy–Schwarz applied to a gamma random variable. It yields

$$\mathcal{I}_N(a) \le \frac{2\Gamma(N/2)}{\sqrt{\pi N}\,\Gamma((N-1)/2)} < \sqrt{2/\pi}.$$

The lower envelope tends to zero at both endpoints: use $r = N - 1$ at zero and $r = 1$ at one. A nonnegative concave function bounded by a fixed constant and vanishing at the endpoints has a uniformly bounded slope on $[\varepsilon, 1-\varepsilon]$. This proves the last assertion. □

**Lemma 2.3** (Stable thresholds)**.** *Suppose* (2.2) *minimizes the antipodal perimeter problem and $r, s \ge 2$. Then*

$$\frac{r-1}{N} \le t \le \frac{r+1}{N}. \tag{2.4}$$

*Proof.* Write $\Sigma = \partial A_{N,r,t}$. The boundary is $S^{r-1}(\sqrt{Nt}) \times S^{s-1}(\sqrt{N(1-t)})$. Up to the common choice of orientation, its principal curvatures are $\sqrt{(1-t)/(Nt)}$ with multiplicity $r-1$, and $-\sqrt{t/(N(1-t))}$ with multiplicity $s-1$. Since the ambient Ricci curvature in a unit direction is $(N-2)/N$, the Jacobi potential is

$$|A|^2 + \mathrm{Ric}(\nu,\nu) = \frac{1}{N}\left(\frac{r-1}{t} + \frac{s-1}{1-t}\right). \tag{2.5}$$

For an even smooth normal speed $f$ with $\int_\Sigma f = 0$, stability gives

$$Q_\Sigma(f) = \int_\Sigma \left(|\nabla_\Sigma f|^2 - (|A|^2 + \mathrm{Ric}(\nu,\nu))f^2\right) d\mathcal{H}^{N-2} \geq 0.$$

Indeed, $\nu(-x) = -\nu(x)$, so an even normal speed produces an antipodal normal graph. Adding a constant normal displacement of order two enforces the volume constraint exactly: the derivative of volume in the constant direction is the positive boundary area, so the implicit function theorem applies. This correction does not alter the constrained quadratic form above.

Take a nonzero degree-two spherical harmonic on the first factor and make it constant on the second. It is even and has zero mean on the product, and its Laplace eigenvalue is $2r/(Nt)$. Substitution in $Q_\Sigma(f) \geq 0$ gives

$$\frac{r-1}{t} + \frac{s-1}{1-t} \leq \frac{2r}{t},$$

which is $Nt \leq r+1$. A degree-two harmonic in the second block gives $Nt \geq r-1$. These harmonics exist precisely in the range needed here, $r, s \geq 2$. No pinching claim is made when a block has dimension one. □

**Lemma 2.4** (Escape of both ranks forces half mass)**.** *Let $r_N + s_N = N$ and $r_N, s_N \to \infty$. If $|Nt_N - r_N| \leq 1$, then*

$$\mathbb{P}\{\mathrm{Beta}(r_N/2, s_N/2) \leq t_N\} \longrightarrow \frac{1}{2}. \tag{2.6}$$

*Consequently, at each fixed mass $a \neq 1/2$, the smaller rank of any spherical isoperimetric minimizer stays bounded as $N \to \infty$.*

*Proof.* Suppress the subscript $N$. Let $U$ and $V$ be independent chi-square variables of degrees $r$ and $s$. Then $T = U/(U+V)$ has the beta law. Set $D = \sqrt{2rsN}$ and

$$W = \frac{s(U-r) - r(V-s)}{D} = \sqrt{\frac{s}{N}}\frac{U-r}{\sqrt{2r}} - \sqrt{\frac{r}{N}}\frac{V-s}{\sqrt{2s}}.$$

The two standardized variables converge to independent standard normals. Along any subsequence the two coefficients have a further convergent subsequence, and their squared sum is one. Thus $W$ converges to a standard normal without a restriction on $r/N$. Also

$$\mathbb{E}\frac{U+V}{D} = \sqrt{\frac{1}{2r} + \frac{1}{2s}} \longrightarrow 0.$$

For $c = \pm 1$ there is the exact equivalence

$$T \leq \frac{r+c}{N} \quad \Longleftrightarrow \quad W \leq c\frac{U+V}{D}.$$

The probabilities at both endpoints tend to $1/2$, by convergence in distribution of $W$ and convergence in probability of the right-hand threshold to zero. Squeezing proves (2.6).

If the smaller minimizing rank were unbounded at a fixed nonhalf mass, a subsequence would have both ranks tending to infinity. Lemma 2.3 would then contradict (2.6). A rank equal to one already has bounded smaller rank and needs no stability estimate. □

## 3. The two profile limits

**Lemma 3.1** (Fixed-rank and escaping-rank limits)**.** *For every fixed $k \geq 1$,*

$$J_{N,k}(a) \longrightarrow j_k(a) \tag{3.1}$$

*locally uniformly for $a \in (0,1)$. If $r + s = N$, both $r$ and $s$ tend to infinity, and $|Nt - r| \leq 1$, then the perimeter of (2.2) tends to $1/\sqrt{\pi}$.*

*Proof.* For fixed $k$, the density of $X = NT$, with $T \sim \mathrm{Beta}(k/2, (N-k)/2)$, is

$$\frac{\Gamma(N/2)N^{-k/2}}{\Gamma(k/2)\Gamma((N-k)/2)} x^{k/2-1}(1-x/N)^{(N-k)/2-1}, \qquad 0 < x < N.$$

Denote this density by $f_{N,k}$. The gamma-ratio asymptotic and $(1-x/N)^{(N-k)/2-1} \to e^{-x/2}$ show that it converges locally uniformly on $(0,\infty)$ to the chi-square density $f_k$. For convergence in distribution, write $X = U/((U+V)/N)$ with independent $U \sim \chi^2_k$ and $V \sim \chi^2_{N-k}$. The denominator tends to one in probability, so $X \Rightarrow \chi^2_k$. The limiting distribution function is continuous and strictly increasing; hence its interior quantiles converge, uniformly on each compact mass interval. In particular, if $x_{N,k}(a) = Nt_{N,k}(a)$, then

$$x_{N,k}(a) \longrightarrow R_k(a)^2, \qquad J_{N,k}(a) = 2\sqrt{x_{N,k}(a)}\sqrt{1 - x_{N,k}(a)/N}\, f_{N,k}(x_{N,k}(a)).$$

The quantiles stay in a compact subset of $(0,\infty)$ on such a mass interval. Passing to the limit in this formula, and using $2r f_k(r^2) = \rho_k(r)$, proves (3.1) with the asserted uniformity.

For the second assertion put $p = r/N$. At $t = p$ the perimeter equals

$$\frac{2}{\sqrt{N}} \frac{p^{(r-1)/2}(1-p)^{(s-1)/2}}{B(r/2, s/2)}. \tag{3.2}$$

Stirling's formula gives, uniformly whenever both $r$ and $s$ tend to infinity,

$$B(r/2, s/2) = 2\sqrt{\pi}\, \frac{r^{(r-1)/2} s^{(s-1)/2}}{N^{(N-1)/2}} \exp\bigl(O(1/r + 1/s)\bigr).$$

All powers therefore cancel in (3.2), leaving $\pi^{-1/2}\exp(O(1/r+1/s))$. For completeness, the derivative of the logarithm of the numerator, at $p$, is

$$\frac{N}{2}\left(\frac{1}{s} - \frac{1}{r}\right),$$

and its second derivative between $p$ and a point at distance at most $1/N$ is $O(N^2(1/r+1/s))$. Moving $t$ within the pinching interval therefore changes the logarithm by $O(1/r+1/s)$. These estimates remain valid when $p \to 0$ or $p \to 1$, since the relative shift in each block is at most the reciprocal of its rank. □

**Lemma 3.2** (A strict half-mass gap in every finite rank)**.** *For every $k \geq 1$,*

$$j_k(1/2) > \frac{e^{1/(18k)}}{\sqrt{\pi}} > \frac{1}{\sqrt{\pi}}, \qquad \lim_{k\to\infty} j_k(1/2) = \frac{1}{\sqrt{\pi}}. \tag{3.3}$$

*Moreover, as $k \to \infty$,*

$$R_k(1/2)^2 = k - \frac{2}{3} + O(k^{-1}), \qquad j_k(1/2) = \frac{1}{\sqrt{\pi}}\left(1 + \frac{1}{18k} + O(k^{-2})\right). \tag{3.4}$$

*Proof.* Let $M(x)$ be the median of a gamma variable of shape $x$ and scale one. The Chen–Rubin lower bound [5] and [4, Proposition 3.6] give

$$x - \frac{1}{3} < M(x) < x\exp\left(-\frac{1}{3x}\right), \qquad x > 0.$$

For the upper estimate, the cited proposition in the published version of [4] states that $x\log(x/M(x))$ decreases to $1/3$, which yields the displayed strict bound. Hence, writing $u_k = R_k(1/2)^2$,

$$k - \frac{2}{3} < u_k < ke^{-2/(3k)} =: v_k. \tag{3.5}$$

The chi density decreases above its mode, whose square is $k-1$; this includes $k=1$, for which the mode is zero. Both quantities in (3.5) lie above that mode. The upper Stirling bound

$$\Gamma(x) < \sqrt{2\pi}\, x^{x-1/2} e^{-x+1/(12x)}$$

therefore gives

$$\begin{aligned}\log\bigl(\sqrt{\pi}\, j_k(1/2)\bigr) &> \log\bigl(\sqrt{\pi}\, \rho_k(\sqrt{v_k})\bigr) \\ &> -\frac{k-1}{3k} + \frac{k}{2}(1 - e^{-2/(3k)}) - \frac{1}{6k} \\ &> \frac{1}{18k}.\end{aligned}$$

The last step uses $1 - e^{-x} > x - x^2/2$ for $x > 0$. The two median bounds also imply the first expansion in (3.4). Put $b = u_k - k = -2/3 + O(k^{-1})$. The density formula and Stirling's expansion give explicitly

$$\begin{aligned}\log\bigl(\sqrt{\pi}\, j_k(1/2)\bigr) &= -\frac{1}{6k} + \frac{k-1}{2}\log(1 + b/k) - \frac{b}{2} + O(k^{-3}) \\ &= \frac{1}{18k} + O(k^{-2}),\end{aligned}$$

which proves the remaining assertions. □

Define for now

$$C(a) = \inf_{k \geq 1} \min\{j_k(a), j_k(1-a)\}. \tag{3.6}$$

We set $C(0) = C(1) = 0$. This gives continuity at the endpoints, since $0 \leq C(a) \leq j_1(1-a)$ near zero and $0 \leq C(a) \leq j_1(a)$ near one. Interior continuity will follow from the proposition below.

**Proposition 3.3** (Limit of the spherical profiles)**.** *As even $N \to \infty$,*

$$\mathcal{I}_N(a) \longrightarrow C(a) \tag{3.7}$$

*locally uniformly on $(0,1)$. The function $C$ is concave and symmetric. If $a \neq 1/2$, the infimum in (3.6) is attained by some finite rank. At $a = 1/2$, $C(a) = 1/\sqrt{\pi}$.*

*Proof.* Fix $a \neq 1/2$ and choose a minimizing rank for each sphere. Lemma 2.4 bounds its smaller rank. On a subsequence this rank is a fixed integer $k$, and one of the two block orientations is fixed. Lemma 3.1 then identifies the limiting perimeter as $j_k(a)$ or $j_k(1-a)$, which is at least $C(a)$. Conversely, testing any fixed rank in either orientation gives

$$\limsup_{N\to\infty} \mathcal{I}_N(a) \leq \inf_{k \geq 1} \min\{j_k(a), j_k(1-a)\} = C(a).$$

Every subsequential limit is therefore $C(a)$, and a fixed rank on a subsequence realizes this value.

At $a = 1/2$, the test $r = s = N/2$ has quantile exactly $t = 1/2$ and its perimeter tends to $1/\sqrt{\pi}$ by Stirling's formula. A bounded smaller minimizing rank on any subsequence would instead have a limit $j_k(1/2) > 1/\sqrt{\pi}$, by Lemma 3.2. Thus both minimizing ranks tend to infinity. Lemmas 2.3 and 3.1 give the matching limit $1/\sqrt{\pi}$. This is also $C(1/2)$ by Lemma 3.2.

Finally Lemma 2.2 supplies equicontinuity on each compact mass interval. The pointwise limit is consequently continuous and the convergence is locally uniform. Concavity and symmetry pass to the limit. □

## 4. Transfer to arbitrary Gaussian sets

**Lemma 4.1** (Exact perimeter of a spherical lift)**.** *Let $E \subset \mathbb{R}^d$ be bounded with smooth boundary, and let*

$$\widehat{E}_N = \{x \in S^{N-1}(\sqrt{N}) : (x_1, \dots, x_d) \in E\}.$$

*For all sufficiently large $N$,*

$$\sigma_N(\widehat{E}_N) = \int_E w_{N,d}(x)\, dx, \tag{4.1}$$

$$P_{\sigma_N}(\widehat{E}_N) = \int_{\partial E} w_{N,d}(x) \sqrt{1 - \frac{\langle x, \nu_E(x)\rangle^2}{N}}\, d\mathcal{H}^{d-1}(x), \tag{4.2}$$

*where*

$$w_{N,d}(x) = \frac{\Gamma(N/2)}{\Gamma((N-d)/2)(\pi N)^{d/2}} \left(1 - \frac{|x|^2}{N}\right)^{(N-d-2)/2}, \qquad |x| < \sqrt{N}. \tag{4.3}$$

*In particular, the mass and perimeter in* (4.1)– (4.2) *converge to $\gamma_d(E)$ and $P_{\gamma_d}(E)$.*

*Proof.* The projection of normalized spherical measure onto the first $d$ coordinates has density (4.3), which gives (4.1). Let $g$ be a signed distance function to $\partial E$ near the boundary. On the sphere its pullback has tangent gradient of squared length

$$|\nabla g|^2 - \frac{\langle x, \nabla g\rangle^2}{N} = 1 - \frac{\langle x, \nu_E\rangle^2}{N}.$$

For clarity, let $\eta_\varepsilon$ be an approximate identity on the real line, supported near zero, and let $\pi_d$ be projection onto the first $d$ coordinates. Coarea on the sphere expresses the perimeter of the lift as the limit of

$$\int_{S^{N-1}(\sqrt{N})} \eta_\varepsilon(g(\pi_d y)) \big|\nabla_S (g \circ \pi_d)(y)\big|\, d\sigma_N(y).$$

Disintegration replaces this integral by the integral over $\mathbb{R}^d$ against $w_{N,d}(x)\, dx$, with the same square-root gradient factor. Euclidean coarea for the signed distance $g$ then gives (4.2) as $\varepsilon \downarrow 0$. Since $E$ is bounded and $N$ is large, the projection is a submersion near this boundary and has no additional boundary contribution there. On each fixed compact set, $w_{N,d} \to \phi_d$ uniformly and the square-root factor tends uniformly to one. The asserted limits follow. □

**Lemma 4.2** (Smooth approximation with central symmetry)**.** *If $E \subset \mathbb{R}^d$ is centrally symmetric and $P_{\gamma_d}(E) < \infty$, there are bounded open centrally symmetric sets $E_j$ with smooth boundary such that*

$$\gamma_d(E_j \triangle E) \longrightarrow 0, \qquad P_{\gamma_d}(E_j) \longrightarrow P_{\gamma_d}(E).$$

*The approximating volumes need not equal $\gamma_d(E)$ exactly.*

*Proof.* We give the weighted and symmetric details of the usual strict finite-perimeter approximation; see [9] for the underlying $BV$ approximation and coarea theorems. For almost every $R > 0$,

$$P_{\gamma_d}(E \cap B_R) \le P_{\gamma_d}(E; B_R) + P_{\gamma_d}(B_R).$$

The last term is a constant times $R^{d-1}e^{-R^2/2}$, hence tends to zero. Together with weighted lower semicontinuity, this proves strict convergence of the symmetric truncations as $R \to \infty$. It remains to approximate a bounded symmetric set $A$.

Let $u_\varepsilon = \mathbf{1}_A * \zeta_\varepsilon$, where $\zeta_\varepsilon$ is nonnegative, smooth, compactly supported, even, and has integral one. Then $0 \le u_\varepsilon \le 1$, $u_\varepsilon(-x) = u_\varepsilon(x)$, and its support is bounded. Convolution and Fubini yield

$$\int \phi_d |\nabla u_\varepsilon|\, dx \le \int (\phi_d * \zeta_\varepsilon)\, d|D\mathbf{1}_A| \longrightarrow P_{\gamma_d}(A).$$

Here $A$ has finite ordinary perimeter because the weight is bounded away from zero on its compact support. Weighted lower semicontinuity gives the reverse limiting inequality, so the displayed weighted variations converge.

Fix $0 < \eta < 1/2$. For every $t \in [\eta, 1-\eta]$,

$$\gamma_d(\{u_\varepsilon > t\} \triangle A) \le \eta^{-1} \|u_\varepsilon - \mathbf{1}_A\|_{L^1(\gamma_d)}.$$

Weighted coarea and Sard's theorem allow a regular value $t_\varepsilon \in [\eta, 1-\eta]$ with

$$P_{\gamma_d}(\{u_\varepsilon > t_\varepsilon\}) \le \frac{\int \phi_d |\nabla u_\varepsilon| \, dx}{1-2\eta} + \varepsilon.$$

These superlevel sets are smooth, bounded, and centrally symmetric. Let $\eta$ decrease to zero, choose $\varepsilon$ so that the $L^1$ error is $o(\eta)$, and use lower semicontinuity again. This produces strict approximation of $A$. A final diagonal choice over the truncations proves the lemma. □

**Proposition 4.3** (Gaussian lower bound and exact profile)**.** *For every centrally symmetric Gaussian finite-perimeter set,*

$$P_{\gamma_d}(E) \ge C(\gamma_d(E)). \tag{4.4}$$

*Consequently* $I_\infty = C$.

*Proof.* The assertion at mass zero or one is immediate. For interior mass, first suppose $E$ is bounded and smooth. Its lift is antipodally invariant, so

$$P_{\sigma_N}(\widehat{E}_N) \ge \mathcal{I}_N(\sigma_N(\widehat{E}_N)).$$

Write $a_N = \sigma_N(\widehat{E}_N)$ and $a = \gamma_d(E)$. By Lemma 4.1, $a_N \to a$, so these masses belong to a fixed compact interval $K \subset (0,1)$ for all large $N$. In particular,

$$|\mathcal{I}_N(a_N) - C(a)| \le \sup_{b \in K} |\mathcal{I}_N(b) - C(b)| + |C(a_N) - C(a)| \longrightarrow 0$$

by Proposition 3.3. Taking the perimeter limit now proves (4.4) for smooth bounded sets. For a general set of finite Gaussian perimeter, apply Lemma 4.2 and continuity of $C$. This also explains why neither the spherical lift nor the smooth approximation has to preserve the mass exactly. Infinite perimeter causes no difficulty.

Thus $I_\infty(a) \ge C(a)$. Centered balls of mass $a$ in every rank, and complements of centered balls of mass $1-a$, give the reverse inequality in (3.6). At half mass their limiting value is provided by Lemma 3.2. Hence $I_\infty = C$ at all interior masses. □

## 5. Orientation and a quantitative rank bound

**Lemma 5.1** (The larger-mass ball is preferred)**.** *For $k \ge 1$ and $0 < a < 1/2$,*

$$j_k(a) > j_k(1-a). \tag{5.1}$$

*Proof.* For $k = 1$, the density $\rho_1$ is strictly decreasing. Suppose $k > 1$ and put

$$V(r) = \frac{r^2}{2} - (k-1)\log r.$$

It is strictly convex, has its minimum at $r_0 = \sqrt{k-1}$, and $V''$ is strictly decreasing. For $v > V(r_0)$, let $r_-(v) < r_0 < r_+(v)$ be the two inverse branches. Define $p_-(v) = -V'(r_-(v))$ and $p_+(v) = V'(r_+(v))$. Both vanish at the minimum level, and

$$\frac{d}{dv} p_\pm(v)^2 = 2V''(r_\pm(v)).$$

It follows that $p_-(v) > p_+(v)$ above the minimum. At points of equal density, the left and right tail probabilities are proportional to

$$\int_v^\infty \frac{e^{-w}}{p_-(w)}\,dw \quad \text{and} \quad \int_v^\infty \frac{e^{-w}}{p_+(w)}\,dw,$$

respectively. The right tail is strictly larger.

If $R_k(a) \geq r_0$, both relevant quantiles lie on the decreasing branch and (5.1) is immediate. Otherwise let $y > r_0$ have the same density as $R_k(a)$. The right-tail comparison gives $1-F_k(y) > a$, whence $y < R_k(1-a)$. The density is decreasing on this branch, so again (5.1) follows. □

**Lemma 5.2** (A uniform density bound)**.** *For every $k \geq 1$,*

$$\sup_{r>0} \rho_k(r) \leq \sqrt{2/\pi}. \tag{5.2}$$

*Proof.* Equality holds for the maximum of $\rho_1$. For $k \geq 2$, the lower Stirling bound for $\Gamma(k/2)$ gives

$$\rho_k(\sqrt{k-1}) < \frac{e^{1/2}}{\sqrt{\pi}} \left(1 - \frac{1}{k}\right)^{(k-1)/2} \leq \sqrt{\frac{e}{2\pi}} < \sqrt{\frac{2}{\pi}}.$$

Here $(1 + 1/(k-1))^{k-1} \geq 2$ by the binomial formula, and $e < 4$. □

**Proposition 5.3** (Every active rank is bounded)**.** *Let $m > 1/2$ and suppose $j_k(m) = I_\infty(m)$. If $k \geq 2$, then*

$$R_k(m)^2 \leq k + 1 \tag{5.3}$$

*and*

$$k \leq \frac{25}{18\pi(m-1/2)^2} < \frac{1}{2(m-1/2)^2}. \tag{5.4}$$

*The last, weaker bound also holds for $k = 1$.*

*Proof.* We include the second-variation argument for the necessary radius bound; the same threshold appears in La Manna's local minimality and instability theorem [8]. The centered $k$-ball of radius $R = R_k(m)$ is a symmetric Gaussian perimeter minimizer in $\mathbb{R}^k$, by Proposition 4.3. For an even mean-zero normal speed $f$ on its boundary, Gaussian second variation gives

$$\int_{\partial B_R} \left( |\nabla f|^2 - \left(1 + \frac{k-1}{R^2}\right) f^2 \right) \phi_k \, d\mathcal{H}^{k-1} \geq 0.$$

Take $f$ to be a nonzero degree-two spherical harmonic, of Laplace eigenvalue $2k/R^2$. Its symmetry and zero mean make it admissible; an even constant correction of second order enforces exact volume. The displayed inequality becomes $((k+1)/R^2 - 1) \int f^2 \phi_k \geq 0$, proving (5.3).

Put $\delta = m - 1/2$. The median lower bound (3.5) and Lemma 5.2 give

$$\begin{aligned} \delta &= \int_{R_k(1/2)}^{R_k(m)} \rho_k(r)\,dr \\ &\leq \sqrt{\frac{2}{\pi}} \left( \sqrt{k+1} - \sqrt{k-2/3} \right) \leq \frac{5}{6}\sqrt{\frac{2}{\pi}}\, k^{-1/2}. \end{aligned}$$

For the last inequality, $\sqrt{k+1} + \sqrt{k-2/3} \geq 2\sqrt{k}$ when $k \geq 2$. Squaring reduces this fact to

$$(k+1)(k-2/3) - (k-1/6)^2 = \frac{2k}{3} - \frac{25}{36} > 0.$$

This proves (5.4); the comparison with $1/2$ uses $\pi > 3$. For $k = 1$, the weaker bound is automatic since $0 < \delta < 1/2$. □

Propositions 3.3 and 4.3, together with Lemma 5.1, prove (1.5), finite attainment away from 1/2, and the asserted half-mass value. Proposition 5.3 proves (1.6). Taking the product of an attaining ball or its complement with $\mathbb{R}^{d-k}$ proves (1.7). The nonattainment assertion at half mass is proved in Section 7.

## 6. The critical-rank asymptotics

**Lemma 6.1** (A uniform expansion near half mass)**.** *Put $u = k^{-1/2}$ and $c_0 = \pi^{-1/2}$. There are constants $k_0 \geq 1$, $t_0 > 0$, and $C < \infty$, independent of $k$ and $t$, such that, for $k \geq k_0$ and $|t| \leq t_0$,*

$$j_k(1/2+t) = c_0 + \frac{u^2}{18\sqrt{\pi}} - \frac{tu}{3} - \sqrt{\pi}\, t^2 + O((u+|t|)^4). \tag{6.1}$$

*Here the absolute value of the error is at most $C(u+|t|)^4$.*

*Proof.* Differentiation of the quantile and density gives the exact identities

$$R_k' = \frac{1}{j_k}, \qquad j_k' = \frac{k-1}{R_k} - R_k, \qquad j_k'' = -\frac{1+(k-1)/R_k^2}{j_k}. \tag{6.2}$$

Lemma 3.2 consequently yields

$$j_k(1/2) = c_0 + \frac{u^2}{18\sqrt{\pi}} + O(u^4), \tag{6.3}$$

$$j_k'(1/2) = -\frac{u}{3} + O(u^3), \tag{6.4}$$

$$j_k''(1/2) = -2\sqrt{\pi} + O(u^2). \tag{6.5}$$

We justify a remainder uniform in both parameters. Let

$$Z(t) = R_k(1/2+t) - \sqrt{k}, \qquad J(t) = j_k(1/2+t).$$

Then

$$Z' = \frac{1}{J}, \qquad J' = A(u,Z) := -2Z + u\frac{Z^2-1}{1+uZ}. \tag{6.6}$$

The initial conditions are $Z(0) = -u/3 + O(u^3)$ and (6.3). On one fixed small interval $|t| \leq t_0$, and for all sufficiently small $u$, these ordinary differential equations give $J \geq c_0/2$ and $Z = O(u+|t|)$, with uniform constants. For example, these bounds follow by continuation in a compact rectangle about $(Z,J) = (0,c_0)$, since the right-hand sides are uniformly smooth there. Explicitly,

$$A_Z = -1 - \frac{1-u^2}{(1+uZ)^2}, \qquad A_{ZZ} = \frac{2u(1-u^2)}{(1+uZ)^3}.$$

It follows from (6.6) that

$$J''' = \frac{A_{ZZ} - A_Z A}{J^2} = O(u+|t|).$$

Taylor's formula with integral remainder, combined with (6.3)–(6.5), now has error bounded by

$$C(u^4 + |t|u^3 + t^2u^2 + |t|^3u + t^4),$$

which proves (6.1). □

*Proof of Theorem 1.2.* Every attaining rank tends to infinity as $\delta \downarrow 0$. Otherwise a bounded-rank subsequence, continuity of $I_\infty = C$, and (3.3) would contradict $I_\infty(1/2) = c_0$.

Choose an integer $k$ nearest to $(9\pi\delta^2)^{-1}$. For this test, $u = k^{-1/2} = 3\sqrt{\pi}\,\delta + O(\delta^3)$. The $O(\delta^3)$ error follows from the bounded integer-rounding error and the derivative of $x \mapsto x^{-1/2}$ at $x \asymp \delta^{-2}$. Lemma 6.1 gives

$$I_\infty(1/2+\delta) \leq c_0 - \frac{3\sqrt{\pi}}{2}\delta^2 + O(\delta^4). \tag{6.7}$$

For an attaining rank put $u = k(\delta)^{-1/2}$. One first has $u = O(\delta)$, uniformly over all choices of attaining ranks. If this uniform bound failed, we could choose a sequence of attaining ranks with $u/\delta \to \infty$. Along that sequence, $u \to 0$ and (6.1) gives $j_k(1/2+\delta) = c_0 + u^2/(18\sqrt{\pi}) + o(u^2) > c_0$, contrary to (6.7).

The expansion is therefore uniform with remainder $O(\delta^4)$ at every attaining rank. Completing the square gives

$$I_\infty(1/2+\delta) = c_0 - \frac{3\sqrt{\pi}}{2}\delta^2 + \frac{(u - 3\sqrt{\pi}\,\delta)^2}{18\sqrt{\pi}} + O(\delta^4). \tag{6.8}$$

Comparison with (6.7) yields $u = 3\sqrt{\pi}\,\delta + O(\delta^2)$, hence (1.9). The nonnegative square in (6.8) and (6.7) prove (1.10). Symmetry gives the expansion on the left of half mass and differentiability at that point. No differentiability of the lower envelope in a whole neighborhood of that point is used. □

## 7. Half mass is not attained in finite dimension

**Proposition 7.1.** *If $E \subset \mathbb{R}^d$ is centrally symmetric and $\gamma_d(E) = 1/2$, then*

$$P_{\gamma_d}(E) > \frac{1}{\sqrt{\pi}}. \tag{7.1}$$

*Proof.* Only the equality case needs to be excluded. Suppose equality holds. Then $E$ is a symmetric perimeter minimizer in its ambient dimension; indeed it realizes the infimum over all dimensions. A product with $\mathbb{R}$ retains this property, so we may assume the ambient dimension is at least two.

Choose the usual finite-perimeter representative of $E$. The regularity and first-variation statements for symmetric Gaussian minimizers, in [7, Lemmas 4.1 and 4.3], give a smooth reduced boundary $\Sigma$ and a constant $\lambda$ such that

$$H - \langle x, \nu \rangle = \lambda \tag{7.2}$$

there, where $\nu$ is the outward unit normal and $H = \operatorname{div}_\Sigma \nu$. We use only compactly supported variations on the regular part; the classification invoked below includes the possible singular set. Choose a regular point $x \neq 0$ and a small regular patch $U$ such that $U$ and $-U$ are disjoint. Such a point exists because the positive perimeter is carried by the reduced boundary. Choose a nonzero, nonnegative, even smooth function $f$ supported on $U \cup (-U)$. Since $\nu(-x) = -\nu(x)$, the normal vector field $f\nu$ is odd. Extend it to a compactly supported smooth ambient field and antisymmetrize the extension to obtain $X(-x) = -X(x)$ with $X \cdot \nu = f$ on $\Sigma$. The flow of $X$ commutes with $x \mapsto -x$ and thus produces centrally symmetric sets $E_s$ for both signs of small $s$. Writing $a(s) = \gamma_d(E_s)$ and $p(s) = P_{\gamma_d}(E_s)$, we have

$$a'(0) = \int_\Sigma f \phi_d \, d\mathcal{H}^{d-1} > 0, \qquad p'(0) = \lambda a'(0).$$

By the already proved profile inequality, $p(s) \geq I_\infty(a(s))$, with equality at $s = 0$. Theorem 1.2 shows that $I_\infty$ is differentiable at $1/2$ with derivative zero. Consequently $s \mapsto p(s) - I_\infty(a(s))$ is differentiable at zero and has a local minimum there, so $p'(0) = 0$. The first-variation identity and $a'(0) > 0$ imply $\lambda = 0$.

We now apply the existing self-shrinker classification [7, Theorem 6.1]. Its hypotheses are precisely minimization in the centrally symmetric class at a fixed Gaussian volume and $H = \langle x, \nu \rangle$ on the reduced boundary; both have just been verified. Compactness and convexity are not extra assumptions in that theorem. It says that the boundary is a hyperplane through zero or a self-shrinking round cylinder. In the latter case its active dimension $k \geq 2$ has radius $\sqrt{k-1}$. The indicator of a finite-perimeter set is constant almost everywhere on each connected component where its distributional derivative vanishes. The two components of the complement of this cylinder therefore force $E$ to be its interior or exterior, up to null sets; the two constant choices would have

mass zero or one. But (3.5) gives $R_k(1/2)^2 > k - 2/3 > k - 1$, so neither side has mass $1/2$. A hyperplane through zero exchanges its two sides under $x \mapsto -x$ and cannot bound a nontrivial centrally symmetric set. Both possibilities contradict the assumed mass. This proves (7.1) and completes Theorem 1.1. □

## 8. The sharp even Gaussian Cheeger inequality

*Proof of Corollary 1.3.* The concave symmetric profile $C = I_\infty$, extended by zero at the endpoints, satisfies $C(1/2) = 1/\sqrt{\pi}$. Hence

$$C(a) \geq \frac{2}{\sqrt{\pi}} \min\{a, 1-a\}.$$

Together with Proposition 4.3, this proves (1.11).

Let $c$ be a median of an even Lipschitz function $f$. The sets $\{f > c + t\}$ and $\{f < c - t\}$ are centrally symmetric and have mass at most $1/2$ for $t > 0$. Layer cake and (1.11), followed by Gaussian coarea, give

$$\begin{aligned}\int |f - c| \, d\gamma_d &= \int_0^\infty \left[\gamma_d(f > c + t) + \gamma_d(f < c - t)\right] dt \\ &\leq \frac{\sqrt{\pi}}{2} \int_0^\infty \left[P_{\gamma_d}(\{f > c + t\}) + P_{\gamma_d}(\{f < c - t\})\right] dt \\ &= \frac{\sqrt{\pi}}{2} \int |\nabla f| \, d\gamma_d.\end{aligned}$$

A median minimizes the integral over constants, proving (1.12). Finally, centered half-mass balls have perimeter tending to $1/\sqrt{\pi}$ by Lemma 3.2. Their indicators, approximated by even radial Lipschitz functions, prove sharpness in both formulations. More explicitly, for a fixed half-mass ball $B$, radial transition layers give functions $0 \leq f_\varepsilon \leq 1$ with

$$f_\varepsilon \longrightarrow \mathbf{1}_B \quad \text{in } L^1(\gamma_k), \qquad \int |\nabla f_\varepsilon| \, d\gamma_k \longrightarrow P_{\gamma_k}(B).$$

The map $f \mapsto \inf_c \|f - c\|_{L^1(\gamma_k)}$ is 1-Lipschitz in $L^1$, and its value at $\mathbf{1}_B$ is $1/2$. First letting $\varepsilon \downarrow 0$ and then $k \to \infty$ shows that the functional constant cannot be decreased. □

*Remark* 8.1 (Strictness and the centering constant). The set inequality is strict in each finite dimension whenever the mass is nontrivial. At half mass this is Proposition 7.1. If $0 < a < 1/2$, choose an attaining rank $k$ for $C(a) = j_k(1 - a)$. The concavity of $q \mapsto j_k(1 - q)$, its zero value at $q = 0$, and Lemma 3.2 give

$$C(a) \geq 2a \, j_k(1/2) > \frac{2a}{\sqrt{\pi}}.$$

Complementation handles $a > 1/2$. The functional statement is centered at a median, equivalently at the best constant in $L^1$; no replacement of this constant by the mean is asserted.

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