AN ADMISSIBLE 232 465 000-TUPLE WITH FEW LONG PAIRS, AND $H_5 \le 4\,817\,211\,684$

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ABSTRACT. Let $H_m = \liminf_{n\to\infty}(p_{n+m} - p_n)$. Version 2 of Kapila's preprint hexagon:2610.00419 proves $H_5 \le 4\,830\,355\,416$ by a pair-penalty criterion: with certified constants, an admissible 232 465 000-tuple with at most 4 672 314 960 pairs of elements further apart than D gives $H_5 \le D$. We construct an admissible 232 465 000-tuple with 4 672 314 941 pairs further apart than 4 817 211 684, which gives $H_5 \le 4\,817\,211\,684$ with the certificates of that preprint, unchanged. The tuple is obtained from the one of the preprint by putting back 375 314 pairs $\pm p$, removing 158 382 elements and adding ± 1; its admissibility and its pair count are checked by two independent programs.

hexagon:2611.00516v2

1. Result

For $m \ge 1$ let $H_m = \liminf_{n\to\infty}(p_{n+m} - p_n)$, where p_n is the n-th prime. A set of integers is *admissible* if for every prime p it misses at least one residue class modulo p; for a set of K elements it suffices to check the primes $p \le K$. For a finite set $\mathcal{H}$ and $D > 0$ let $B_D(\mathcal{H})$ be the number of pairs $\{h, h'\} \subset \mathcal{H}$ with $|h - h'| > D$.

Version 2 of [1] (6 November 2026) proves $H_2 \le 19\,844$, $H_3 \le 1\,357\,780$, $H_4 \le 82\,234\,788$ and $H_5 \le 4\,830\,355\,416$. For $m = 5$ its final step [1, Proposition 10.2 and Section 11] reads: an admissible tuple $\mathcal{H}$ of $K = 232\,465\,000$ elements gives $H_5 \le D$ if

$$\rho\,\frac{K}{k}\,q_- - 5(1-\beta_-) - B_D(\mathcal{H})\,C_+ > 0, \tag{1}$$

with $k = 232\,500\,000$, $\rho = 151\,320\,927\,303\,522\,335/2^{59}$ and the certified endpoints $q_- = 19.04804762745624$, $\beta_- = 0.00014210693394371$, $C_+ = 1.491020798040 \cdot 10^{-14}$ printed in [1, Section 11.1]. With these exact rationals, (1) holds if and only if $B_D(\mathcal{H}) \le 4\,672\,314\,960$. The certificates of [1] are used as stated; only the tuple is new. The same certificates were applied to other tuples in [2], which gives $H_5 \le 4\,827\,554\,952$ (and smaller tuples for $m = 2, 3, 4$).

Theorem 1. *There is an admissible tuple $\mathcal{H}$ of 232 465 000 elements, contained in $[-2\,427\,786\,701,\ 2\,391\,510\,637]$, with $B_D(\mathcal{H}) = 4\,672\,314\,941$ for $D = 4\,817\,211\,684$.*

Corollary 1. *With the certificates of* [1]*, $H_5 \le 4\,817\,211\,684$.*

Proof. Theorem 1 and (1): the left-hand side equals $2.958\ldots \cdot 10^{-13} > 0$ for $B_D = 4\,672\,314\,941$. □

Version 1 of the present note gave $H_5 \le 4\,891\,341\,734$ with the statement DHL[234 360 000, 6] of version 1 of [1]; that bound had already been improved by version 2 of [1].

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2. The tuple

Let $L = 2\,432\,107\,459$. The certificate of [1, Section 12.3] gives, for every prime $10^7 < p \leq 234\,360\,000$, a residue r_p modulo p; the parent set of [1] is $\{\pm q : \ q \text{ prime}, \ 10^7 < q \leq L\}$ from which the pairs $\pm p$ with $r_p = 0$ (664 149 of them) are deleted. We change this as follows (ancillary files):

(1) r_p is replaced by the residue given in `h5_patches_1.csv`, `h5_patches_2.csv` for 375 483 primes: for 375 314 of them $r_p = 0$ in [1] and the new residue is nonzero, so the pair $\pm p$ is put back; for 169 others a nonzero residue is replaced by another one. After this, 288 835 residues are 0, all for primes $p \leq 20\,894\,023$.
(2) the 158 382 signed elements listed in `h5_removed.csv` are removed;
(3) 1 and -1 are added;
(4) $\mathcal{H}$ is the set of the remaining elements x with $-2\,427\,786\,701 \leq x \leq 2\,391\,510\,637$.

Then $\mathcal{H}$ has 117 071 452 negative and 115 393 548 positive elements, 232 465 000 in all. Every $x \in \mathcal{H}$ is ± 1 or $\pm q$ with q a prime larger than 10^7.

Lemma 1. *$\mathcal{H}$ is admissible provided that, for every prime $10^7 < p \leq 232\,465\,000$, some class $c_p \neq 0$ modulo p contains no element of $\mathcal{H}$ whenever $\pm p \in \mathcal{H}$.*

Proof. If $p \leq 10^7$, no element of $\mathcal{H}$ is divisible by p. If $10^7 < p$ and $\pm p \notin \mathcal{H}$, the class 0 is missed, because the only elements of the form $\pm q$ divisible by p are $\pm p$. Otherwise c_p is missed. The primes $p > |\mathcal{H}|$ need no check. □

The classes c_p are the residues r_p after step 1, except for the 1 917 107 primes for which $r_p \in \{1, p-1\}$ (this class contains 1 or -1); for these, c_p is the smallest $c \geq 2$ whose class contains no element of $\mathcal{H}$, computed by the verification program. Admissibility is only required for $\mathcal{H}$ itself, not for the whole parent set, which leaves more classes empty than in [1]; the residues of step 1 were chosen among the classes empty in $\mathcal{H}$, and an element was removed in step 2 when it was the only element of such a class.

3. Verification

`h5_verify.py` reads the data files of the certificate bundle of [1] (its residue file and three patch files; none of its programs is run), checks the SHA-256 of [1, Section 12.3], applies the changes above, computes the classes c_p for the primes with $r_p \in \{1, p-1\}$ (all 1 917 107 are found), and runs two verifiers.

- Verifier A (Python, `numpy`) rebuilds $\mathcal{H}$ with its own sieve, checks its cardinality, checks for each of the 11 915 259 primes $10^7 < p \leq 234\,360\,000$ with $c_p \neq 0$ that no element of $\mathcal{H}$ lies in the class c_p (along the whole progression, on both sides), checks that $\pm p \notin \mathcal{H}$ when $r_p = 0$, and counts the pairs further apart than D by binary search.
- Verifier B (C, `h5_verifier_B.py`, compiled by `h5_verify.py`) rebuilds $\mathcal{H}$ with a sieve of one bit per integer, and for every one of the 12 770 635 primes $p \leq 232\,465\,000$ scans the class 0 (for $p \leq 10^7$) or the class c_p along all its elements in $[-2\,427\,786\,701, \ 2\,391\,510\,637]$; it counts the pairs further apart than D with two cursors.

Both report 0 failures, $B_D(\mathcal{H}) = 4\,672\,314\,941$ for $D = 4\,817\,211\,684$, and 4 672 326 852 pairs further apart than $D - 1$ (log `h5_verify_run.log`; about three minutes and 5 GB). Both verifiers reject the data when one residue is replaced by the class of an element of $\mathcal{H}$ (log `h5_verify_trap_run.log`).

Acknowledgements and use of AI

The computations and the writing were produced with Claude (Anthropic) under the author's direction. The bound depends on the certificates and theorems of [1], which are used as stated and were not checked here, except for the finite data. No part of this contribution has been examined by a professional mathematician.

References

[1] A. Kapila, *Finite prime gaps from weighted Selberg sieves and pair penalties*, Hexagon preprint, hexagon:2610.00419v2 (6 November 2026); certificate bundle `https://github.com/a-kapila/maths-proofs/tree/main/prime-gaps`.

[2] *Smaller $H_2, \ldots, H_5$ from the certificates of hexagon:2610.00419v2*, repository `https://github.com/pruzze/bounded-gaps-witnesses`, directory `kapila_h2_h5` (17 November 2026).

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