A NARROWER ADMISSIBLE 234 360 000-TUPLE AND $H_5 \le 4\,891\,341\,734$

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Abstract. Let $H_m = \liminf_{n\to\infty}(p_{n+m}-p_n)$. Kapila proved DHL[234 360 000, 6] and deduced $H_5 \le 5\,040\,082\,836$, the diameter of the tuple formed by the 234 360 000 smallest primes larger than 234 360 000. We construct an admissible 234 360 000-tuple of diameter 4 891 341 734, which gives $H_5 \le 4\,891\,341\,734$ with Kapila's theorem. The tuple consists of ± 1 and of $\pm q$ for primes q in a symmetric window, with 1 746 831 primes removed; its admissibility is checked by two independent programs.

hexagon:2611.00516v1

1. Result

For $m \ge 1$ let $H_m = \liminf_{n\to\infty}(p_{n+m} - p_n)$, where p_n is the n-th prime. A set $\mathcal{H}$ of k integers is *admissible* if for every prime p it misses at least one residue class modulo p; it suffices to check the primes $p \le k$. The statement DHL$[k, m+1]$ asserts that every admissible k-tuple has infinitely many translates containing at least $m+1$ primes, and it implies $H_m \le \operatorname{diam}\mathcal{H}$ for every admissible k-tuple $\mathcal{H}$. Following Maynard [2], the Polymath project [3] obtained $H_5 \le 80\,550\,202\,480$, Stadlmann [6] improved this to 77 510 685 234, and Roeling [5] to 14 541 349 288. Kapila [1, Theorem 1.1] proved DHL[234 360 000, 6] and obtained $H_5 \le 5\,040\,082\,836$. This number is the diameter of the tuple of the $k = 234\,360\,000$ smallest primes larger than k [1, Table 3], which is admissible because none of its elements is divisible by a prime $p \le k$; we recomputed this diameter exactly (ancillary file `previous_record_width.py`).

Theorem 1. *There is an admissible* 234 360 000*-tuple of diameter* 4 891 341 734.

Corollary 1. *Assume* DHL[234 360 000, 6]*, which is proved in* [1]*. Then* $H_5 \le 4\,891\,341\,734$.

The only new ingredient is the tuple: the bound relies on the statement DHL[234 360 000, 6] of [1], which we use as stated.

2. The tuple

Let $P_0 = 50\,100$, $a = 2\,445\,670\,867$ and let R be the set of the 1 746 831 primes listed as successive differences in two ancillary files, to be read one after the other: `removed_gaps_1.csv`, then `removed_gaps_2.csv`; they all lie in $(P_0,\ 37\,847\,053]$. Put

$$A = \{1\} \cup \{q \text{ prime}:\ P_0 < q \le a\} \setminus R, \qquad \mathcal{H} = \{\pm m:\ m \in A\}.$$

Then $|\mathcal{H}| = 234\,360\,000$, and a is prime and lies in A, so $\operatorname{diam}\mathcal{H} = 2a = 4\,891\,341\,734$.

Lemma 1. *$\mathcal{H}$ is admissible if, for every prime $p \le 234\,360\,000$ with $p > P_0$ and $p \notin R$, there is an integer c with $1 \le c \le 20\,000$, $2c < p$, such that no $m \in A$ satisfies $m \equiv \pm c \pmod p$.*

Proof. If $p \le P_0$, no element of $\mathcal{H}$ is divisible by p, since the elements are ± 1 and $\pm q$ with q prime and $q > P_0$. If $p > P_0$ and $p \in R$, the only element of $\{1\} \cup \{\text{primes}\}$ divisible by p is p itself, which is not in A; so the class 0 is missed. Otherwise, by hypothesis no element $\pm m$ of $\mathcal{H}$ lies

Date: 15 November 2026.

2020 *Mathematics Subject Classification.* 11N05, 11N35, 11Y16.

Key words and phrases. gaps between primes, admissible tuples, Maynard–Tao sieve.

in the class $c \bmod p$, because $m \equiv c$ or $-m \equiv c$ would give $m \equiv \pm c$. For $p > 234\,360\,000 = |\mathcal{H}|$ some class is missed trivially. □

How R was chosen. We started from $\{1\} \cup \{q \text{ prime} : P_0 < q \leq 2.5 \cdot 10^9\}$ and treated the primes $p \leq k$ with $p > P_0$ in decreasing order: we searched the smallest $c \leq 20\,000$ with both classes $\pm c \bmod p$ empty, and if there was none we removed p. Removing elements keeps the classes found earlier empty, so the final set satisfies the hypothesis of Lemma 1; $\mathcal{H}$ consists of the elements of absolute value at most a. This is a variant of the sieve of Hensley and Richards, which sieves a symmetric interval $[-x, x]$ by the classes $0 \bmod p$ for increasing primes p (see [4]); the difference is that for most primes $p \leq k$ we do not sieve the class 0 but exhibit an empty pair of classes $\pm c$, so that these primes $\pm p$ stay in the tuple. The removed primes are those for which the expected number of elements per class, about $|\mathcal{H}|/p$, is too large for an empty class to exist or to be found within the search budget; they are all smaller than $3.8 \cdot 10^7$.

3. Verification

- `verify_tuple.py` (a C program embedded in a Python script, compiled with `gcc`) builds A with a sieve of Eratosthenes on bytes, removes R, checks $|\mathcal{H}| = 234\,360\,000$ and $a \in A$, and then, for every prime $p \leq 234\,360\,000$, scans the class 0 directly (all multiples of p up to a) and, if it is occupied, searches a pair $\pm c$ with $c \leq 20\,000$ that is empty, again by direct scanning. It reports 12 868 673 primes, the class 0 missed for 1 751 974 of them and a pair $\pm c$ for the other 11 116 699, and the diameter 4 891 341 734. It needs about 3 GB of memory and three minutes.
- `build_tuple.py` reproduces the construction and writes the list of the classes c found for each prime (about 178 MB, not included). `verify_classes_numpy.py`, written independently in Python with `numpy`, rebuilds A with its own sieve and checks this list: every prime $P_0 < p \leq k$ has exactly one entry, every listed pair $\pm c$ is empty, and the cardinality and the diameter are as stated.
- For $k = 111\,300$ the same construction gives a tuple of diameter 1 449 286; both programs accept it, and a direct computation of all residues modulo every prime $p \leq k$ confirms its admissibility. `verify_tuple.py` rejects the tuple in which one removed prime is put back and another element removed (no class is then empty modulo 1021, by exhaustive search), and `verify_classes_numpy.py` rejects a certificate in which one class is altered, both for $k = 111\,300$ and for $k = 234\,360\,000$ in the second case.

Remark 1. The diameter is about $k(\log k + 1.60)$. For the values of k used in [1] for H_2, H_3, H_4 (2 355, 111 300, 5 129 500) the same construction gives the diameters 25 562, 1 449 286 and 90 027 902, larger than those of [1]; so we do not improve these bounds.

Acknowledgements and use of AI

The mathematics, the computations and the writing were produced with Claude (Anthropic) under the author's direction. The bound depends on the statement DHL[234 360 000, 6] of [1]. The result has not been examined by a professional mathematician.

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