A LOWER BOUND 0.11096505 FOR ROMANOFF'S CONSTANT

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Abstract. Let d be the lower density of the integers of the form $p + 2^a$ with p prime and $a \geq 1$. We prove $d \geq 0.11096505$, unconditionally. The previous best unconditional bound is $d \geq 0.110825$, due to Tomar. The proof is Tomar's method, with the same analytic inputs, applied to a different modulus of the same period 27720; the bound is certified in exact rational arithmetic by Tomar's checking program, with changes that only make the computation faster and that we verify against the original program.

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1. Result

For a positive integer n let $r(n)$ be the number of pairs (p, a) with p prime, $a \geq 1$ and $n = p + 2^a$, and

$$d = \liminf_{N\to\infty} \frac{1}{N} \#\{n \leq N : \ r(n) > 0\}.$$

Romanoff proved $d > 0$ [6]. Explicit lower bounds are 0.0868 (Chen and Sun [1]), 0.0936275 (Habsieger and Sivak-Fischler [3]), 0.107648 (Elsholtz and Schlage-Puchta [2], recomputed as 0.10788 by Johnston and Trudgian [4]) and 0.110825 (Tomar [8]). In Tao's list of optimization constants [7] this is the lower bound for C_{45}.

Theorem 1. *We have* $d \geq 0.11096505$.

The theorem is unconditional: it uses exactly the analytic inputs of Theorem 1.1 of [8], namely the per-class sieve bound with the constant 8 quoted by Johnston and Trudgian from [2, (28)], the prime number theorem in arithmetic progressions to a fixed modulus, Pintz's bound for the series R_0 [5], and Pintz's inequality for nonnegative integers. It does not use the conditional input of Theorem 1.2 of [8].

2. The method

We recall the method of [8], which refines [2]; we refer to [8] for the proofs. Let l be an odd squarefree integer and P the order of 2 modulo l. For a residue class $k \bmod l$ let $U_k = \{r \in \mathbb{Z}/P\mathbb{Z} : (k - 2^r, l) = 1\}$ and $s_k = |U_k|$. In each class the first moment of $r(n)$ is given by the prime number theorem in progressions, and the second moment is bounded by the per-class sieve bound, applied to the pairs $p_1 + 2^{a_1} = p_2 + 2^{a_2}$ grouped by $(a_1 \bmod P, a_2 \bmod P)$, together with upper bounds B_g $(g \mid P)$ for the singular series. The part of the singular series that is not computed is bounded only through its total mass ϑ, and the worst distribution of this mass over the divisors of P is taken. With Pintz's inequality in each class this gives ([8, Proposition 4.2])

$$d \ \geq \ \min_{\beta} \ \frac{\nu}{l} \sum_{k:\, s_k>0} \left(\frac{2s_k}{j_k+1} - \frac{8\kappa}{j_k(j_k+1)} \Big(\sum_{g\mid P} B_g S_{k,g} + \sum_{g\mid P} g\, \beta_g\, S_{k,g} \Big) \right),$$

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where $S_{k,g} = \sum_{a \bmod g} |U_k \cap (a + g\mathbb{Z})|^2$, κ and ν are explicit constants depending on l, the $j_k \geq 1$ are arbitrary integers, and β runs through the vertices of the simplex $\{\beta_g \geq 0,\ \sum_g \beta_g \leq \vartheta\}$.

The bound depends on the modulus l. Tomar uses

$$l_3 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 73 \cdot 241,$$

with period $P = 27720$, and obtains 0.110825. We use

$$l_E = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 73 \cdot 89 \cdot 127,$$

that is, l_3 without 241 and with 89 and 127, for which $\mathrm{ord}_{89}(2) = 11$ and $\mathrm{ord}_{127}(2) = 7$. The period is again $P = 27720$.

3. The computation

The computation uses the structure of [8, Appendix A] and the certificate format `r45a-u3f` of its checking program. The base primes are $3, 5, 7, 11, 13, 17, 31$ (period 120; their nonempty patterns fall into 822 rotation orbits). The core primes are $41, 19, 73, 37$ (period 360). The remaining primes are treated as independent components: $43, 29, 127$ for the factor 7 of the period (orders $14, 28, 7$) and $23, 89$ for the factor 11 (orders $11, 11$). The program checks that the orders of 2 modulo these primes come from proven complete factorizations, that this decomposition is valid, that the multiplicities of the combined patterns (380 of them) add up to the product of the independent primes, and that the total weight equals l_E.

The singular series is bounded as in [8, Section 5], from the complete factorizations of $\Phi_t(2)$ for $t \leq 300$ and the known factorizations for $301 \leq t \leq 1200$. The program gives $T(0) \leq 1.179578218$ and a tail mass $\vartheta \leq 1.9583 \cdot 10^{-4}$. The Pintz indices j_k range over $1, \ldots, 43$. The value at the vertex $\beta = 0$ is $0.1110554\ldots$; the minimum over the 97 vertices is at $\beta = \vartheta e_{27720}$, and the program checks in exact rational arithmetic that it is at least 0.11096505. This proves Theorem 1.

Changes to the checking program. Tomar's program [8] would need about 9.6 hours on two cores for this certificate. We changed three things, all inside the `u3f` part of the program, and an ancillary file lists them exactly.

(1) In each base orbit, identical rows of the core enumeration (the same mask on $\mathbb{Z}/360\mathbb{Z}$) are merged and their weights added. Every quantity computed from a row depends on the row only through its mask and is linear in its weight, so the sums do not change; the program checks that the total weight is unchanged.
(2) The weighted sums over the rows are computed in 64-bit integers, with an asserted bound against overflow, instead of in floating point; the original bound that makes the computed $S_{k,g}$ exact is kept.
(3) The base orbits can be processed in slices, with the exact partial sums written to a file after each orbit, and the slices are recombined exactly; the recombination repeats all the checks of the original program and checks that the slices cover every orbit exactly once.

With these changes the computation took about three hours on two cores.

Checks.

- On Tomar's test certificate `test_cert_t1_u3f`, the changed program gives 0.107200250, the value of Tomar's log, both in one run and in two slices recombined.
- Tomar's original program, changed only so that it processes a slice of orbits and writes its exact sums, and the changed program were run on the certificate of this note for the slices of orbits 0–1 (the most expensive), 410–413 and 815–821 (the least expensive). The exact sums agree in all 2717, 1747 and 292 integers respectively.

- The original program accepts the certificate of this note: all its checks up to the evaluation of the orbits pass.

The original program has not been run end to end on this certificate, and the computation has not been formalized in Lean (Tomar's bound 0.110825 is also proved in Lean [8]).

4. Ancillary files

The directory `anc/` contains the following files.

- The certificate `cert_E_27720.json`.
- The changed program `check_r45u3f_patched.py` and the list of its changes.
- The exact partial sums `partial_sums_0_822.json`, and the logs of the run and of the recombination.
- Tomar's factorization data `fac300_41e4cdd6.json` and `fac1200_fdb_af082adc.json`.
- For the comparison with the original program: `chk_orig_tranche.py`, its list of changes, the slice sums and `cmp.py`.
- The Apache License 2.0, under which Tomar's program is distributed.

To reproduce, with Python 3, `python-flint` and `numpy`:

```
TRANCHE=0:822 SORTIE=s.json python3 -I check_r45u3f_patched.py \
  fac300_41e4cdd6.json cert_E_27720.json 3000000 \
  fac1200_fdb_af082adc.json 1200 --workers 2
RECOMBINE=s.json python3 -I check_r45u3f_patched.py \
  fac300_41e4cdd6.json cert_E_27720.json 3000000 \
  fac1200_fdb_af082adc.json 1200 --claim 0.11096505
```

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References

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[8] T. S. Tomar, *New lower bounds for Romanoff's constant*, version 11 (2026), doi:10.5281/zenodo.23715955; programs at `https://github.com/tejasvi/romanoff-constant`, tag v11.

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