hexagon:2611.00513v3

The Hibi–Li (2016) conjecture is fully resolved, and other miscellanea

Dave Goel

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Abstract

We prove the equality part of the Hibi–Li (2016) conjecture, its last open part: for every n-element poset P and every $2 \le k \le n-1$, if the chain polytope $\mathcal{C}(P)$ and the order polytope $\mathcal{O}(P)$ have the same number of k-dimensional faces, then they are unimodularly equivalent. Its inequality part, that $\mathcal{C}(P)$ has at least as many k-dimensional faces as $\mathcal{O}(P)$, was first proved by Fourier; for edges the equality part fails, since the two polytopes always have the same number of edges. So the conjecture is now fully resolved.[1] The second version of Fourier's preprint gives an independent proof of the equality part. Both parts follow from two theorems. First, the face numbers of a bounded marked chain-order polytope weakly increase when its chain part grows; the partition is arbitrary, and the marking may be real and degenerate. Its admissible case is also proved, independently, in the second version of Fourier's preprint. Second, for an ordinary n-element poset P whose order and chain polytopes are not unimodularly equivalent, we prove the sharp inequalities $f_k(\mathcal{C}(P)) - f_k(\mathcal{O}(P)) \ge \binom{n-3}{k-2}$ for $2 \le k \le n-1$. We characterize the posets for which all these bounds are attained at once, and the set of indices k at which the bound is attained. For posets whose order and chain polytopes differ by one in the number of 2-faces, we obtain a sharper bound that depends on the difference of the facet numbers. For nested chain parts $C \subseteq C'$ in the bounded marked setting, equality of facet numbers is equivalent to affine equivalence; integral markings give lattice equivalence. We prove that regular marked posets have irredundant canonical facet descriptions for every partition, including when maximal elements are unmarked. In that unbounded setting, a switch, which adds one element to the chain part, weakly increases every face number if the element has an upper endpoint, and weakly decreases every face number otherwise. A nonzero change has its nonzero coefficients exactly in an interval of degrees that ends one below the dimension. Its quotient by $(1+t)^2$ for an increase, and by $1+t$ for a decrease, has coefficients of the sign of the change. A connected example shows that increases and decreases can cancel along a nested sequence. For bounded marked posets and several pairwise incomparable switched order elements, the alternating sum of face polynomials is a higher-order subdivision defect. It obeys a sign rule under mutual transversality, and for layered posets whose distinct lower and upper neighbourhoods form Berge-acyclic families; without these hypotheses the sign rule can fail, and the defect can have coefficients of both signs. When all these neighbourhoods have two elements, the defect is, up to sign, a power of $2+t$ times the chromatic polynomials of the lower and the upper neighbourhood graph, evaluated at $-t$; the sign rule then holds exactly when both graphs are forests. For neighbourhoods of any size, the defect of a layered poset is, up to sign and a power of $2+t$, a product of two alternating sums of chromatic polynomials of hypergraphs at $-t$; for neighbourhoods with three or more elements, Berge-acyclicity is not necessary for the sign rule. The arguments combine fixed-height lifts and clipped markings with general transverse-subdivision results proved in the appendices.

[1] This is an expanded version of the following focused version [16]: https://hexagonmath.org/2611.00512

Table 1: The three main results and the best results known before this paper.

Best result before this paper	New result
Part (b) was known for $k = 2$ [14, Theorem 43] and for $k = n-1$ [18], and was open for $3 \le k \le n-2$ [20, Section 5]; for $k = 1$ it fails, since the numbers of edges always agree [19]. Part (a), the inequality, was proved by Fourier [10].	*The Hibi–Li (2016) conjecture is fully resolved:* we prove its last open part, the equality part (part (b) of [18, Conjecture 2.4]), for every $2 \le k \le n-1$. For an n-element poset R, equal numbers of k-dimensional faces, $f_k(\mathcal{O}(R)) = f_k(\mathcal{C}(R))$, force $\mathcal{O}(R)$ and $\mathcal{C}(R)$ to be unimodularly equivalent (Corollary 1.3). Proved independently in the second version of Fourier's preprint [11, Theorem 8.2].
A gap of at least 1 for $k = 2$ and $k = n-1$, where $\binom{n-3}{k-2} = 1$ (from the results in the row above); no positive lower bound for $3 \le k \le n-2$ [20, Section 5]. A formula for the difference of the facet numbers [12, Theorem 2.1].	*Sharp gap:* $f_k(\mathcal{C}(R)) - f_k(\mathcal{O}(R)) \ge \binom{n-3}{k-2}$ for $2 \le k \le n-1$ whenever the two polytopes are not unimodularly equivalent; for every $n \ge 5$ some R attains all these bounds (Theorem 1.2).
The admissible, unmarked case of [6, Conjecture 5.3], proved in [10]; earlier, for maximal ranked posets, the unmarked case in which C consists of the elements of rank at most j [1, Theorem 4.1]. The vertex case of [6, Conjecture 5.1] was open [6, Remark 5.2(3)].	*Bounded marked monotonicity:* $F_{Q_C(P,\lambda)} \le F_{Q_{C'}(P,\lambda)}$ for all $C \subseteq C'$, for every partition and every order-preserving real, possibly degenerate, marking (Theorem 1.1). This contains [6, Conjectures 5.1 and 5.3], and it proves the bounded part of [7, Conjecture 7.10], whose version for unbounded polyhedra fails (Example 8.3). In particular, it contains part (a) of the Hibi–Li (2016) conjecture, $f_k(\mathcal{O}(R)) \le f_k(\mathcal{C}(R))$, first proved by Fourier [10]. The admissible case (order ideals $C \subseteq C'$), which includes [6, Conjectures 5.1 and 5.3] for order-preserving markings, is proved independently in the second version of Fourier's preprint [11, Theorem 8.1]; non-admissible partitions are new here.

This is the expanded version of the paper "The Hibi–Li (2016) *conjecture is fully resolved" [16], at* https://doi.org/10.5281/zenodo.23704734

1 Results and conventions

Table 1 compares the three main results with the best results known before this paper.
The new results of the paper are listed below; the notation is fixed after the list and in the sections cited.

- *The Hibi–Li (2016) conjecture is fully resolved* (Theorem 1.2 and Corollary 1.3). We prove its equality part, which was open for $3 \le k \le n-2$ [20, Section 5]: for every n-element poset R and every $2 \le k \le n-1$, $f_k(\mathcal{O}(R)) = f_k(\mathcal{C}(R))$ only if $\mathcal{O}(R)$ and $\mathcal{C}(R)$ are unimodularly equivalent. Its inequality part, $f_k(\mathcal{O}(R)) \le f_k(\mathcal{C}(R))$, was first proved by Fourier [10] and is also contained in Theorem 1.1 (next item). For $k = 1$ the equality part fails, since the numbers of edges always agree [19]. The second version of Fourier's preprint [11, Theorem 8.2] gives an independent proof of the equality part.[2]

[2]The first version of this paper, with the equality part, was posted on October 24, 2026, and shared with Prof. Fourier. Later, on October 29, 2026, the second version of Fourier's preprint [11] was posted; it proves the equality part independently. However, unlike Fourier's later preprint, our earlier proof also gives a sharp lower bound

- *Bounded marked monotonicity* (Theorem 1.1). The face numbers of a bounded marked chain-order polytope weakly increase when its chain part grows. The partition is arbitrary, and the marking may be real and degenerate. This contains Conjectures 5.1 and 5.3 of [6] for order-preserving markings, including the vertex case of Conjecture 5.1, and proves the bounded part of [7, Conjecture 7.10]. The admissible case, which contains both conjectures, is also proved, independently, in the second version of Fourier's preprint [11, Theorem 8.1]; non-admissible partitions are new here.

- *Boundedness is needed* (Example 8.3 and Remark 8.4). For polyhedra with only the minimal elements marked, Conjecture 7.10 of [7] fails as stated: a cone over a square becomes an orthant, and the numbers of rays and of facets both decrease. The same example shows that the extension of [7, Proposition 2.8] in a footnote of that preprint fails.

- *The sharp unmarked gap* (Theorem 1.2 and Corollary 1.3). For an n-element poset R whose order polytope and chain polytope are not unimodularly equivalent, $f_k(\mathcal{C}(R)) - f_k(\mathcal{O}(R)) \geq \binom{n-3}{k-2}$ for $2 \leq k \leq n-1$, and for every $n \geq 5$ some R attains all these bounds. So equality of a single face number f_k, $2 \leq k \leq n-1$, forces unimodular equivalence. This proves part (b) of [18, Conjecture 2.4] for all these k; the cases $3 \leq k \leq n-2$ were open.

- *Equality cases and tight indices* (Theorems 6.1, 6.2, 6.4 and 6.9, and Corollary 6.10). For these posets we characterize equality in the whole sharp bound by four poset conditions; equality in one coefficient f_k, $3 \leq k \leq n-2$, already forces it. We characterize the posets whose order and chain polytopes differ by one in the number of 2-faces. Call $k \in \{2, \ldots, n-1\}$ *tight* for R if $f_k(\mathcal{C}(R)) - f_k(\mathcal{O}(R)) = \binom{n-3}{k-2}$. For every n-element poset R whose order polytope and chain polytope are not unimodularly equivalent, the set of tight indices is $\{2, \ldots, n-1\}$, $\{2, n-1\}$, $\{n-1\}$, $\{2\}$ or empty, and Corollary 6.10 says which.

- *A prescribed facet difference* (Theorem 6.12 and Corollary 6.13). For such a poset, if the numbers of 2-faces differ by one and the numbers of facets by m, then $1 \leq m \leq n-4$ and $F_{\mathcal{C}(R)} - F_{\mathcal{O}(R)} \geq t^2(1+t)^{n-m-2}\big((1+t)^m - t^m\big)$. The bound is attained for all such n and m, its equality cases are characterized, and equality in one coefficient of degree k, for some $3 \leq k \leq n-2$, forces equality in the whole bound.

- *Marked equality* (Theorem 5.1 and Corollary 5.3). For nested chain parts $C \subseteq C'$ in the bounded marked setting, with polytopes of positive dimension, equality of facet numbers, equality of all face numbers and affine equivalence are equivalent; the equivalence can be chosen as an ambient map with linear part in $\mathrm{GL}_{|P \setminus A|}(\mathbb{Z})$, and integral markings give lattice equivalence. For one switch, unless the two polytopes are affinely equivalent, the face polynomial grows by at least $t^{e-2}(1+t)^2$, where $e+1$ is the dimension of the two polytopes. For $C = \varnothing$ and $C' = P \setminus A$, in dimension at least 3, the equivalence of equal facet numbers and affine equivalence also follows from [11, Theorem 8.2], obtained independently.

- *Regularity implies tameness* (Theorem 5.7). If all minimal elements are marked and the marked poset is regular, the canonical rows give an irredundant facet description for every partition, also when maximal elements are unmarked; so the marked poset is tame in the sense of [7, Definition 7.1].

(Theorem 1.2), already in that first version, and the results of the short paper [16] are also formalized in Lean 4. The first version is at https://doi.org/10.5281/zenodo.23601976

- *Comparison without marked maxima* (Theorem 7.2, Corollaries 7.4 and 7.5, Theorem 7.7 and Example 7.9). When all minimal elements are marked, a switch weakly increases every face number if the switched element has an upper endpoint chain (the *increasing regime*, Section 7), and weakly decreases every face number otherwise (the *decreasing regime*). A nonzero change has its nonzero coefficients exactly in the degrees $r, \dots, N-1$ for some r, where N is the dimension, and its quotient by $(1+t)^2$ in the increasing regime, and by $1+t$ in the decreasing regime, has coefficients of the sign of the change. For one switch we give an exact facet formula, and in positive dimension equality of facet numbers is equivalent to affine equivalence. A connected example shows that along a nested sequence the facet changes of the two regimes can cancel although the polyhedra are not affinely equivalent.

- *Several switches* (Theorems 9.4 and 9.7, Corollary 9.8 and Proposition 9.9). For bounded marked posets and several pairwise incomparable switched order elements, the alternating sum of face polynomials is a higher-order subdivision defect. Under mutual transversality it obeys a sign rule and a binomial lower bound; Proposition 9.9 gives a poset criterion for mutual transversality.

- *Layered posets* (Theorems 9.10, 9.13, 9.15 and 9.16). For layered posets the alternating sum is a product of a lower and an upper factor, and the sign rule holds when the distinct lower and the distinct upper neighbourhoods form Berge-acyclic families. When all neighbourhoods have two elements, the alternating sum is, up to sign, a power of $2+t$ times the chromatic polynomials of the lower and the upper neighbourhood graph at $-t$, and the sign rule holds exactly when both graphs are forests. For neighbourhoods of any size it is, up to sign and a power of $2+t$, a product of two alternating sums of chromatic polynomials of hypergraphs at $-t$.

- *Neighbourhoods with three or more elements* (Theorems 9.27, 9.31, 9.42 and 9.47, Corollary 9.43, and Examples 9.17 and 9.20). Berge-acyclicity is then not necessary for the sign rule. We give sufficient conditions on the groups of neighbourhoods and the leading term at $t=-1$. When the groups are single sets, two-member groups whose members are not contained in each other, or three-member core groups, we determine the sign of every coefficient; each three-member core group flips the sign. If the distinct lower neighbourhoods, or the distinct upper ones, form one four-member group as in Theorem 9.47, the conclusion of Theorem 9.13 fails for them unless their factor is zero. The proofs of Theorems 9.42 and 9.47 use exact computer checks; the programs and their outputs are ancillary files of this paper. In Example 9.20 the alternating sum has coefficients of both signs.

- *Comparable switches* (Theorems 9.50 and 9.54). For two comparable switches joined by a bridge, or with chain elements between them as in Section 9.7, the mixed difference lies in $(1+t)^3\mathbb{Z}_{\geq 0}[t]$, and it is zero when the lower envelope of the lower switch or the upper envelope of the upper switch is affine.

- *Transverse polyhedral subdivisions* (Theorem G.2, Corollary G.3 and Theorem H.3). For m mutually transverse polyhedral subdivisions of a polyhedron, the face-count defect and its part at each face of the polyhedron are $(1+t)^m$ times a polynomial with nonnegative coefficients, and a nonzero defect is bounded below by $t^r(1+t)^{d-r}$, where r is its lowest degree and d the dimension of the polyhedron. No regularity or boundedness is assumed.

Let P be a finite poset, let $A \subseteq P$ contain all its minimal and maximal elements, and let $\lambda : A \to \mathbb{R}$ be order-preserving. Partition the unmarked elements as $P \setminus A = C \sqcup O$. We call C

the *chain part* and O the *order part*. In the standard endpoint definition, the marked chain-order polytope $Q_C(P,\lambda)$ consists of the vectors $x \in \mathbb{R}^P$ with $x_a = \lambda(a)$ for $a \in A$, $x_c \geq 0$ for $c \in C$, and

$$\sum_{i=1}^{r} x_{c_i} \leq x_b - x_a \qquad (a < c_1 < \cdots < c_r < b), \tag{1}$$

where $a, b \in A \cup O$ and all $c_i \in C$. Here $r \geq 0$. The inequalities (1) and the nonnegativities $x_c \geq 0$ are the *rows* of this description. For $r = 0$ the row reads $x_a \leq x_b$. For $q \in O$, the *switch* of q moves q from the order part to the chain part. It replaces $Q_C(P,\lambda)$ by $Q_{C\cup\{q\}}(P,\lambda)$. We usually suppress the fixed marked coordinates. This is the vertex-partition construction of [7, 8], not an edge-partition construction. Transfer maps and marking decompositions are established tools [7, 8]. The hypothesis that A contains all minimal and maximal elements makes $Q_C(P,\lambda)$ bounded; it is in force until expressly relaxed: Section 5.7 permits unmarked maxima for tameness, and Section 7 proves both weak comparison directions and single-switch equality for general minimum-marked polyhedra. A partition is *admissible* when C is an ideal of the induced unmarked poset; we do not impose that condition.

For a polytope Q, let $f_k(Q)$ count its k-dimensional faces, with $f_k(Q) = 0$ for $k < 0$ and for $k > \dim Q$. We include the whole polytope and exclude the empty face in

$$F_Q(t) = \sum_{k\geq 0} f_k(Q) t^k.$$

All comparisons of polynomials below are coefficientwise. Dimensions and relative interiors are taken in the relevant affine hulls.

Theorem 1.1 (Bounded marked monotonicity)**.** *For every $C \subseteq C' \subseteq P \setminus A$,*

$$F_{Q_C(P,\lambda)}(t) \leq F_{Q_{C'}(P,\lambda)}(t).$$

No regularity, integrality, distinct-marking, or admissibility hypothesis is required.

For order ideals $C \subseteq C'$ (the admissible case), including degenerate markings, this is also proved, independently, in the second version of Fourier's preprint [11, Theorem 8.1]; this case contains [6, Conjectures 5.1 and 5.3].

For order-preserving markings this contains Conjecture 5.3 of Fang and Fourier [6], whose marked posets have all extrema marked, and proves the bounded part of Conjecture 7.10 of [7]. That conjecture is stated there for possibly unbounded marked chain-order polyhedra with all minimal elements marked. Example 8.3 shows that it fails in that generality: a cone over a square becomes an orthant, and the numbers of rays and of facets both decrease. Theorem 7.2 gives the corresponding two weak directions without marked maxima. For order-preserving markings, Theorem 1.1 with $C = \varnothing$ and $C' = P \setminus A$ gives [6, Conjecture 5.1], including the vertex case $k = 0$, which is recorded as open in [6, Remark 5.2(3)]. The admissible, unmarked case of [6, Conjecture 5.3], which Theorem 1.1 contains, was proved by Fourier [10]; earlier, Ahmad, Fourier and Joswig [1, Section 1 and Theorem 4.1] had shown, for maximal ranked posets (ordinal sums of antichains), that the face numbers weakly increase when the elements of rank at most j receive chain coordinates and j grows.

Organization. The general comparison, support and quotient results used below are proved in Appendices B–H. Section 2 restates those for one and two subdivisions. Section 9 cites the results for m subdivisions (Proposition E.3, Corollary G.3, Remark G.6, Theorem H.3) directly. Our task

here is to establish the required poset geometry and its consequences. The fixed-height separation argument is in Section 3, and the ordinary sharp-gap construction is in Section 4. Section 5 treats marked equality, affine maps and regularization. Section 6 gives the ordinary equality classifications and refinements. Section 7 treats the endpoint alternatives and the limits of nested comparisons without marked maxima. Section 8 gives examples and shows the scope of the comparison. Section 9 switches several elements at once (pairwise incomparable ones, and in Section 9.7 two comparable ones) and studies the sign of the resulting alternating sum.

For an ordinary poset R, write

$$\mathcal{O}(R) = \{v \in [0,1]^R : v_p \le v_q \text{ if } p \le q\}, \qquad \mathcal{C}(R) = \{x \ge 0 : \sum_{p \in K} x_p \le 1 \text{ for every chain } K\}.$$

These are obtained by adjoining a marked minimum of value 0 and a marked maximum of value 1. The order polytope defined with the opposite convention for order coordinates is affinely equivalent to $\mathcal{O}(R)$. Let X be the five-element poset $a, b < c < g, h$ of Figure 1 (left), where a and b are incomparable, and so are g and h. A poset R *contains* X if some five elements of R form an induced subposet isomorphic to X; these five elements need not form a convex subset of R. Otherwise R is *X-free*. Write $\downarrow v = \{x : x \le v\}$ and $\uparrow v = \{x : v \le x\}$; these sets include v.

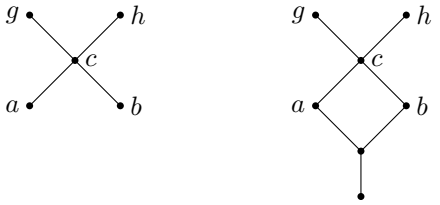


Figure 1: The poset X (left) and the seven-element poset obtained from X by adjoining a new minimum two times in succession (right). For every $n \ge 5$, adjoining $n-5$ new minima to X, one after another, gives an n-element poset that attains the bound of Theorem 1.2 (Section 4.2).

Theorem 1.2 (Sharp unmarked gap)**.** *If R has n elements and contains X, then*

$$F_{\mathcal{C}(R)}(t) - F_{\mathcal{O}(R)}(t) \ge t^2(1+t)^{n-3}. \tag{2}$$

The inequality is coefficientwise. For every $n \ge 5$, equality holds for some R. The simultaneous equality cases are characterized in Theorem 6.1.

Corollary 1.3. *For $2 \le k \le n-1$, equality $f_k(\mathcal{O}(R)) = f_k(\mathcal{C}(R))$ holds if and only if R is X-free, equivalently, if and only if the two polytopes are unimodularly equivalent.*

For X-free posets the two polytopes are unimodularly equivalent, and so have equal face numbers; this is due to Hibi and Li [18]. Theorem 1.2 gives the converse: equality of f_k for one k with $2 \le k \le n-1$ forces R to be X-free. The equality characterization was already known for $k = 2$ by Freij-Hollanti, Lundström and Mori [14, Theorem 43] (Theorem 6.1 in the arXiv version), and for $k = n-1$ by Hibi and Li [18]. Earlier, Mori [22, Theorem 3.6] had shown, for maximal ranked posets, that the numbers of triangular 2-faces agree exactly when the poset is X-free. Lee [20, Section 5] records that the equality statement remained open for $3 \le k \le n-2$ (see also [21, Section 3.3]); Corollary 1.3 settles it. The second version of Fourier's preprint [11, Theorem 8.2] gives an independent proof of this statement; it shows that, for posets that contain X, the face-number inequalities are strict for $2 \le k \le n-1$, and it states no explicit lower bound for the differences. Corollary 1.3 proves part (b) of [18, Conjecture 2.4] for every $2 \le k \le n-1$. Part (a) is contained in Theorem 1.1: for an ordinary poset R, adjoin marked extrema of values 0 and 1 and take $C = \varnothing$ and $C' = R$. So the paper proves the whole conjecture for these k. Part (a) was

first proved in general by Fourier [10]; special cases had been proved earlier for maximal ranked posets [1, Corollary 4.2] and for the posets built from X-free posets by disjoint unions and ordinal sums [13, Theorem 1]. For $k = 1$, part (b) fails as stated, since the two polytopes always have the same number of edges [19, Theorem 2.1, arXiv version], as noted by Mori [22, Section 1] and by Lundström [21, Remark 3.3.1]. In degree n both face polynomials have coefficient one (the polytope itself), so equality always holds there. Figure 2 shows the equality part before and after this paper.

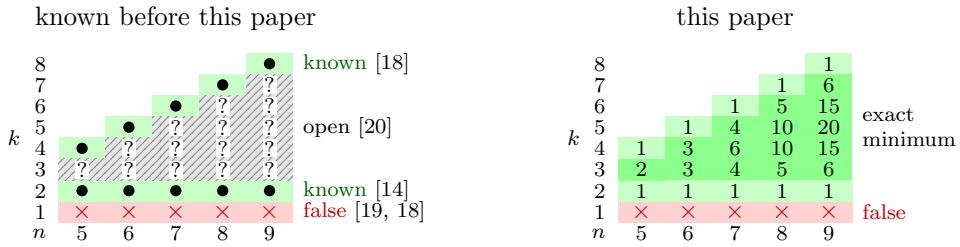


Figure 2: Green: the equality part of the Hibi–Li (2016) conjecture (part (b) of [18, Conjecture 2.4]) is known to hold (darker green: it was open before this paper); red: it is known to fail; grey, hatched: unknown. Columns: posets with n elements; rows: faces of dimension k; for $n \le 4$ no poset contains X. Left, before this paper (sources in Table 1): false for $k = 1$ ($\times$), known for $k = 2$ and for $k = n - 1$ (dots), open otherwise (?). Right, this paper: if R contains X, then $\mathcal{C}(R)$ has at least $\binom{n-3}{k-2}$ more k-dimensional faces than $\mathcal{O}(R)$ (the number at each point; this is the exact minimum), and one n-element poset attains all these bounds at once (Theorem 1.2); so, for $2 \le k \le n - 1$, equality holds exactly when R is X-free (Corollary 1.3).

2 General comparison results used below

All results in this section are taken from Appendices B–H; their proofs are given there. We include the hypotheses and conventions needed for the applications. Let $\mathcal{S}$ be a finite polyhedral subdivision of a polyhedron B. Its cells include all nonempty faces of its cells, and each geometric cell is counted once. We put $F_{\mathcal{S}}(t) = \sum_k s(k)t^k$, where $s(k)$ is the number of k-dimensional cells of $\mathcal{S}$. The carrier of a cell is the smallest face of B containing it. No faces at infinity are included. Dimensions are actual affine dimensions. Two subdivisions $\mathcal{S}, \mathcal{T}$ are transverse relative to every face if, for every nonempty face E of B and every $x \in \operatorname{relint}(E)$, the minimal cells S_x of $\mathcal{S}|_E$ and T_x of $\mathcal{T}|_E$ that contain x satisfy

$$\dim(S_x \cap T_x) = \dim S_x + \dim T_x - \dim E.$$

These two cells contain x in their relative interiors. We call this property *relative transversality*. For a nonempty face E of B, $\operatorname{dir}(E)$ is the linear space parallel to $\operatorname{aff}(E)$, and the conormal space of a cell S at $x \in \operatorname{relint}(S)$ is the annihilator of $\operatorname{dir}(S)$ in $\operatorname{dir}(E)^*$. Let $\mathcal{R} = \mathcal{S} \vee \mathcal{T}$ be their common refinement, whose cells are the nonempty sets $S \cap T$ with $S \in \mathcal{S}$ and $T \in \mathcal{T}$. For $e = \dim E$, write $s_E(k)$, $t_E(k)$ and $r_E(k)$ for the numbers of k-dimensional cells of $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{R}$ with carrier E, and

$$\delta_{E,k} = r_E(k) - s_E(k) - t_E(k) + \mathbf{1}_{k=e}, \qquad \delta_E(t) = \sum_k \delta_{E,k} t^k.$$

The effective affinity subdivision of a finite maximum L of affine functions on B is the complex formed by the maximal convex subsets of B of dimension $\dim B$ on which L is affine, together with

all their nonempty faces. Such subdivisions are called regular. A finite minimum is treated after negation.

Lemma 2.1 (Two-envelope comparison; Lemma B.2 and its proof)**.** *Let B be a nonempty polyhedron, L a finite maximum of affine functions, and U a finite minimum of affine functions, with $L \le U$. Let $\mathcal{S}, \mathcal{T}$ be their effective affinity subdivisions and put*

$$Q = \{(u, z) : u \in B,\ L(u) \le z \le U(u)\}, \quad Q' = \{(u, w) : u \in B,\ 0 \le w \le U(u) - L(u)\}.$$

Then

$$F_{Q'} - F_Q = F_B + F_{\mathcal{R}} - F_{\mathcal{S}} - F_{\mathcal{T}}. \tag{3}$$

Under relative transversality, every carrier coefficient satisfies

$$\delta_{E,k} \ge 0. \tag{4}$$

The identity holds also when some fibres have zero width, over all of B or over some of its faces.

The carrier-counting part of the proof of Lemma B.2 uses neither $L \le U$ nor regularity; it applies to finite transverse subdivisions. The condition $L \le U$ is needed only for the *fibre identity* (3). The underlying comparison arguments follow the proofs in [10, Lemmas 4.1 and 4.3]; the general polyhedral proof is given in Appendix B.

Lemma 2.2 (Ordinary crossing surplus; Lemma C.2)**.** *In the setting of Lemma 2.1, assume relative transversality on every face. If in* $\operatorname{relint}(E)$ *the two subdivisions have an ordinary crossing, with one effective wall from each, independent normals and all four full-dimensional sign regions, then*

$$\delta_E(t) \ge t^{e-2}(1+t)^2, \qquad e = \dim E.$$

Lemma 2.3 (Coface count; Lemma C.1)**.** *A nonempty e-face of a d-dimensional polyhedron has at least $\binom{d-e}{j}$ cofaces of dimension $e + j$, for $0 \le j \le d - e$. The cases $j = 0$ and $j = d - e$ are included, and the ambient polyhedron need not be simple or pointed.*

Theorem 2.4 (Terminal support for a transverse pair; Theorem C.4 and Remark H.5)**.** *For two finite polyhedral subdivisions of a nonempty d-polyhedron, transverse relative to every nonempty face,*

$$\Delta = F_B + F_{\mathcal{S}\vee\mathcal{T}} - F_{\mathcal{S}} - F_{\mathcal{T}}$$

is either zero or has strictly positive coefficients exactly in degrees $r, r+1, \dots, d$, with $0 \le r \le d-2$. The base may be unbounded and may have lineality.

Lemma 2.5 (Weighted carrier Euler; Lemma E.1)**.** *For any two finite polyhedral subdivisions transverse relative to every nonempty face, each $\delta_E(t)$ and their total are divisible by $(1+t)^2$ in $\mathbb{Z}[t]$. No regularity or boundedness is needed.*

Theorem 2.6 (Nonnegative quotients; Corollary F.4)**.** *Under the hypotheses of Lemma 2.5, each $\delta_E(t)$ and their total belong to $(1+t)^2\mathbb{Z}_{\ge 0}[t]$.*

Proposition 2.7 (A single-subdivision factor; Proposition F.5)**.** *For every finite polyhedral subdivision $\mathcal{S}$ of a nonempty polyhedron B, the carrier excess $\sum_k (s_E(k) - \mathbf{1}_{k=\dim E})t^k$ and the total $F_{\mathcal{S}} - F_B$ belong to $(1+t)\mathbb{Z}_{\ge 0}[t]$. No regularity or transversality is assumed.*

For a function L on B, put $\operatorname{epi}_B(L) = \{(u, z) : u \in B,\ z \ge L(u)\}$.

Lemma 2.8 (Epigraph face counts; Lemma B.3)**.** *For a finite affine maximum L on a nonempty polyhedron B, with effective affinity subdivision $\mathcal{S}$,*

$$F_{\mathrm{epi}_B(L)} = F_{\mathcal{S}} + tF_B, \qquad F_{B\times[0,\infty)} = (1+t)F_B.$$

Theorem 2.9 (One-envelope support; Theorem C.6)**.** *For $B, L, \mathcal{S}$ as in Lemma 2.8, with $d = \dim B$, the difference $F_{\mathcal{S}} - F_B$ is zero if and only if L is affine on B. Otherwise its strictly positive coefficients occur exactly in degrees $r, \dots, d$, where $0 \le r \le d-1$.*

3 Proof of bounded marked monotonicity

This section proves Theorem 1.1, one switch at a time.

3.1 The base of one switch

In (1) the chain $a < c_1 < \cdots < c_r < b$ is arbitrary, but requiring it to be saturated gives the same polytope. Indeed, refine a given chain to a saturated one, split it at every marked or order element that it passes, and add the resulting inequalities. The endpoint values telescope, and dropping the added nonnegative chain coordinates gives the original inequality.

The known partial transfer [7, Theorem 2.1] maps an *order extension* of λ, that is, an order-preserving map $v\colon P \to \mathbb{R}$ with $v_a = \lambda(a)$ for $a \in A$, to

$$(T_C v)_c = v_c - \max_{p \lessdot c} v_p \quad (c \in C), \qquad (T_C v)_o = v_o \quad (o \in O), \tag{5}$$

retaining marks. Here $p \lessdot c$ means that c covers p. The inverse of T_C recursively adds the predecessor maximum at chain coordinates, leaving marked and order coordinates fixed. It applies to arbitrary partitions and real, possibly degenerate, markings. In particular our polytopes are nonempty and compact: marked order extensions exist and are bounded between marked extrema, and the transfer is continuous.

Fix $q \in O$, delete it from P, keeping the induced order on $P\setminus\{q\}$, and let $B(\lambda) = Q_C(P\setminus\{q\}, \lambda)$ be the *deleted base* of the switch of q. For its coordinates u put

$$L(u) = \max_{a<K<q} \big(y_a + \sum_{c\in K} u_c\big), \qquad\qquad U(u) = \min_{q<H<b} \big(y_b - \sum_{c\in H} u_c\big).$$

Here K, H are chains in C, possibly empty, endpoints are in $A \cup (O\setminus\{q\})$, and y_a is a marked value or an order coordinate. Marked extrema make both indexing sets nonempty. Every lower/upper pair concatenates into a chain of the deleted poset, so $L \le U$ on B. The nonconstant parts of L and U are supported strictly below and strictly above q, respectively.

Chains of P avoiding q are exactly chains of the induced deletion, although their cover relations may differ. The old inequalities listing q have it as an endpoint, and give $L \le z_q \le U$. The new inequalities listing q have it in their chain interior and give $0 \le w_q \le U - L$. Conversely, every pair of a lower and an upper chain gives such an inequality. Thus $Q_C(P,\lambda)$ and $Q_{C\cup\{q\}}(P,\lambda)$ project onto B with exactly the fibres of Q and Q' in Lemma 2.1. We call them the *old* and the *new* polytope of the switch, and L and U its lower and upper *envelopes*.

3.2 A fixed-height lift and its support faces

Let $x \in \operatorname{relint}(E)$ for a face E of B, and suppose $L(x) < U(x)$. Fix a real number h with $L(x) < h < U(x)$. For y near x in E, lift it to the *old full polytope* with q-coordinate h, then invert (5) on P. Write the order extension as $v(y)$. Since q is an order element ($q \in O$), $v_q(y) = h$.

Clip the full vector, including the marks, by

$$v_p^- = \min(v_p, h), \quad v_p^+ = (v_p - h)_+, \qquad \lambda_a^- = \min(\lambda_a, h), \quad \lambda_a^+ = (\lambda_a - h)_+.$$

Transfer both pieces with the same C and their respective markings, and only then project away q, obtaining y^-, y^+. Monotone clipping commutes with predecessor maxima, and

$$[\min(s,h) - \min(m,h)] + [(s-h)_+ - (m-h)_+] = s - m.$$

Consequently $y = y^- + y^+$, with $y^\pm \in B(\lambda^\pm)$. The construction permits coincident clipped marks and dimension drops. The clipping method follows [10, Lemma 5.1] and the order-polyhedron decomposition in [8, Proposition 15]; the calculation above proves the required transferred split directly.

Adding the jointly linear inequalities gives the standard inclusion [8, Lemma 11]

$$B(\lambda^-) + B(\lambda^+) \subseteq B(\lambda).$$

Choose a linear functional c whose maximum M on $B(\lambda)$ is attained exactly on E, and let $M^\pm$ be its maxima on $B(\lambda^\pm)$. Splitting x gives

$$M = c(x^-) + c(x^+) \le M^- + M^+ \le M.$$

Thus equality holds, all nearby clipped points lie in fixed maximizing faces E^-, E^+, and $E^- + E^+ \subseteq E$. Translating E^- by one fixed point of E^+ shows

$$a := y^- - x^- \in \operatorname{dir}(E). \tag{6}$$

3.3 Separation on every carrier

The condition $v_q = h$ implies $v_p \le h$ below q and $v_r \ge h$ above q. Below q, lower clipping preserves an order coordinate and also each chain-coordinate difference in (5). Above q, an order coordinate clips to h. For a chain coordinate $r > q$, a saturated q–r chain supplies a predecessor of r with value at least h; both v_r and its predecessor maximum clip to h, so its lower transferred coordinate is zero. Hence the vector a in (6) copies $y - x$ on every lower coordinate and vanishes on every upper coordinate, regardless of their types.

Put $V = \operatorname{dir}(E)$. If a lower-supported linear form ℓ and an upper-supported form m agree on V, take $y = x + \varepsilon d$ for arbitrary $d \in V$ and sufficiently small positive ε. Then $\varepsilon\ell(d) = \ell(a) = m(a) = 0$. Their common restriction is therefore zero. The conormal spaces of the minimal affinity cells at x are spanned by linear parts of active effective-branch differences. Those for L have lower supports and those for U have upper supports, so the two conormal spaces in V^* have zero intersection. As x lies in the relative interior of E, this gives the dimension formula of relative transversality at x.

If $L(x) = U(x)$, then the concave nonnegative function $U - L$ vanishes at the relative-interior point x, hence on all of E, as every segment of E that ends at x extends beyond x inside E. The common restriction of L, U is affine, so both subdivisions on E are trivial. Thus relative transversality holds on all carriers. Lemma 2.1 proves nonnegativity for this switch. Adding the elements of $C' \setminus C$ one by one proves Theorem 1.1.

4 The sharp ordinary gap

This section proves Theorem 1.2.

4.1 A four-dimensional face

Let R contain X, with five elements a, b, c, g, h named as in the definition of X. We call c a *centre* of X in R. Choose $q \geq c$ maximal among elements whose strict upper set is not a chain. Such an element has at least two upper covers, which we again call g, h. Indeed, a unique upper cover would lie below every successor, and its own strict upper set would still contain an incomparable pair, contradicting maximality. The original incomparable $a, b < c$ remain below q.

Take the order ideal $I = R \setminus {\uparrow}q$ as the chain part. Use marked extrema $\hat{0}, \hat{1}$ of values $0, 1$, and consider the switch of q. In its deleted base B, retain the chain coordinates $\alpha = x_a, \beta = x_b$ and the order coordinates $\gamma = y_g, \eta = y_h$. Set every other chain coordinate to zero and every other order coordinate to one. These supporting equalities define a face E_0, described exactly by

$$0 \leq \alpha, \beta \leq \gamma, \eta \leq 1. \tag{7}$$

A chain contains at most one of a, b. Every order element is above q, and no other such element is below g or h, since these cover q. These observations show that (7) describes E_0 exactly. In particular E_0 has dimension four.

Write z for the remaining coordinates, and z_0 for their values on E_0. Let u_0 be the point $(1/4, 1/4, 3/4, 3/4; z_0)$ of E_0. At u_0, every nontrivial defining inequality involving one of these four coordinates is strict. All active inequalities involve only z. Hence, for any coface $E \supseteq E_0$, there are neighbourhoods V_4 of $(1/4, 1/4, 3/4, 3/4)$ and V_z of z_0, and a set $Q_E \subseteq V_z$, such that

$$E \cap (V_4 \times V_z) = V_4 \times Q_E.$$

To justify this simultaneously for all cofaces, retain a neighbourhood where every previously strict inequality stays strict. Facets defining E contain u_0, so their equations involve only z as well. In this local product, $\operatorname{relint}(E) \cap (V_4 \times V_z) = V_4 \times \operatorname{relint}(Q_E)$. Figure 3 shows a two-dimensional slice of E_0 through u_0.

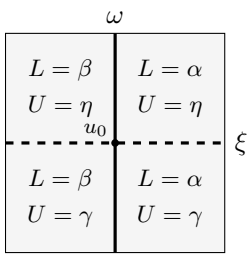


Figure 3: The face E_0 of Section 4.1, sliced by the plane $\{\alpha + \beta = \frac{1}{2},\ \gamma + \eta = \frac{3}{2}\}$ through u_0, in the coordinates $\xi = \alpha - \beta$ and $\omega = \gamma - \eta$ ($|\xi|, |\omega| \leq \frac{1}{2}$). On E_0 we have $L = \max(\alpha, \beta)$ and $U = \min(\gamma, \eta)$. The function L is not affine on any neighbourhood of a point with $\xi = 0$ (solid line), and U is not affine on any neighbourhood of a point with $\omega = 0$ (dashed line). These two walls cross at u_0 with independent normals, and all four sign regions occur: this is the ordinary crossing of Lemma 2.2, and Lemma 4.1 carries it to every coface of E_0. For $R = X$ we have $B = E_0$, and the point u_0, the four half-axes and the four quadrants are the slices of the crossing cell, the four wall pieces and the four full cells of the common refinement counted in Section 4.2.

Lemma 4.1. *Every coface $E \supseteq E_0$ contains an ordinary crossing of the lower and upper affinity subdivisions in its relative interior.*

Proof. Near u_0, a maximizing lower chain contains exactly one of a, b: chains containing neither have sums near zero, whereas the singleton sums are near $1/4$. Hence locally

$$L = \max(\alpha + \mu_a(z), \beta + \mu_b(z)), \qquad U = \min(\gamma, \eta),$$

where μ_a and μ_b are maxima of residual chain forms, both zero at z_0. The other upper coordinates remain near one. Choose $z \in \operatorname{relint}(Q_E)$ near z_0 outside every proper agreement hyperplane between these residual forms restricted to $\operatorname{aff}(Q_E)$, identifying forms whose restrictions to $\operatorname{aff}(Q_E)$ coincide. Then μ_a and μ_b are affine on a relative neighbourhood of z; this holds trivially if Q_E is a point.

Set

$$\alpha = \tfrac{1}{4} + \tfrac{\mu_b(z) - \mu_a(z)}{2}, \qquad \beta = \tfrac{1}{4} - \tfrac{\mu_b(z) - \mu_a(z)}{2}, \qquad \gamma = \eta = \tfrac{3}{4}.$$

For z close enough to z_0, this point belongs to $V_4 \times \operatorname{relint}(Q_E)$. The two lower branches tie, as do the two upper branches. Their normals have four-coordinate parts $(1, -1, 0, 0)$ and $(0, 0, 1, -1)$, so they are independent on $\operatorname{dir}(E)$. All four sign regions occur. No additional effective wall is present because the residual branches were chosen on open chambers and all other branches have strict slack. Branches agreeing on an open part of a face agree on its affine hull, so identical restrictions introduce no artificial cells in the restricted subdivision. □

4.2 Counting cofaces and attaining the bound

Write $d = \dim B = n-1$. Lemma 2.3 gives at least $\binom{d-4}{e-4}$ cofaces of E_0 of dimension e, for $4 \le e \le d$. When $d = 4$, this is the single face $E_0 = B$.

Combining Lemmas 2.2 and 4.1, with nonnegative contributions from all other carriers, gives for this switch

$$\begin{aligned} F_{\text{new}} - F_{\text{old}} &\ge \sum_{e=4}^{d} \binom{d-4}{e-4} t^{e-2}(1+t)^2 \\ &= t^2(1+t)^{d-2} = t^2(1+t)^{n-3}. \end{aligned}$$

Switch the elements of I first, then q, then the rest. Every switch is nonnegative by Theorem 1.1, which needs no admissibility, so telescoping proves (2).

For sharpness start with $R = X$ and switch its elements in the order a, b, c, g, h of a linear extension. The two lower switches have a constant lower envelope, and the two upper switches have a constant upper envelope; their differences vanish in (3). For the central switch, $I = \{a, b\}$ and $B = E_0$. Each proper carrier lies in one of the facets

$$\alpha = 0,\ \beta = 0,\ \gamma = 1,\ \eta = 1, \qquad \alpha = \gamma,\ \alpha = \eta,\ \beta = \gamma,\ \beta = \eta.$$

On the first four, one function is affine; on the other four $L = U$. Thus proper carriers contribute zero. On the full carrier, each subdivision has two full cells and one interior wall, and the refinement has four full cells, four wall pieces and one crossing cell (Figure 3). The exact gap is $t^2 + 2t^3 + t^4$.

Adjoining a new minimum (Figure 1, right) makes both ordinary polytopes pyramids over the old ones. In order coordinates, fixing the coordinate of the new minimum at s gives the section $s\mathbf{1} + (1-s)\mathcal{O}(R)$; in chain coordinates the section is $(1-s)\mathcal{C}(R)$. Since $F_{\text{pyr}(Q)} = 1 + (1+t)F_Q$, differences multiply by $1+t$. Repeating this operation $n-5$ times attains (2). This completes the proof of Theorem 1.2.

4.3 Support at intermediate ordinary partitions

Corollary 4.2 (Ordinary switches with arbitrary chain parts)**.** *Let P be an ordinary n-element poset, adjoin marked extrema of values $0, 1$, and choose any chain part $C \subseteq P$ and $q \in P \setminus C$. Write $\widehat{P}$ for the augmented marked poset, λ for this marking, and $Q_C = Q_C(\widehat{P}, \lambda)$. No ideal condition is imposed on C. Then*

$$F_{Q_{C\cup\{q\}}}(t) - F_{Q_C}(t)$$

is zero or has positive coefficients exactly in degrees $r, \ldots, n-1$, where $2 \leq r \leq n-3$.

Proof. The deleted base $B(\lambda) = Q_C(\widehat{P} \setminus \{q\}, \lambda)$ is a bounded polytope of dimension $n-1$: it is a partial-transfer image of the full-dimensional order polytope on the remaining $n-1$ elements. Section 3 proves relative transversality on every face of this base for arbitrary chain parts. The identity (3) and Theorem 2.4 therefore give the terminal interval with $0 \leq r \leq n-3$.

Put the specified switch in a nested sequence from the empty chain part to all of P, adding the elements of C first, then q, and then the rest. Every step is nonnegative by Theorem 1.1, without a linear-extension restriction. The order and chain polytopes at the two ends of the sequence have equal vertex numbers, by their descriptions using filters and antichains [25], and equal edge numbers by [19, Theorem 2.1, arXiv version]. The nonnegative increments in each of degrees zero and one sum to zero, so every step has zero coefficients there. Hence a nonzero switch has $r \geq 2$. □

For admissible ordinary switches, the required transversality also follows from [10, Proposition 5.2], together with [10, Lemmas 4.1 and 4.3] for the comparison. The arbitrary-partition conclusion above instead uses the relative-transversality proof in Section 3. The lower bound $r \geq 2$ uses the equal vertex and edge numbers of the ordinary order and chain polytopes, so we prove it for ordinary polytopes.

5 Marked equality and support

We retain the bounded marked hypotheses of Theorem 1.1. Fix $q \in P\setminus(A\cup C)$, and let B, L, U, Q, Q' be the deleted base, the envelopes and the old and new fibre polytopes from Section 3. All dimensions in this section are actual affine dimensions. Only Section 5.7 relaxes the marked-maxima hypothesis. The bounded results also supply the tameness input for the later unbounded extension in Section 7.4.

Theorem 5.1. *The following conditions are equivalent:*

(1) Q and Q' have the same face numbers in every dimension;

(2) Q and Q' are affinely equivalent;

(3) L is affine on B, or U is affine on B.

If neither function is affine, then $e = \dim B \geq 2$, $\dim Q = \dim Q' = e+1$, and

$$F_{Q'}(t) - F_Q(t) \geq t^{e-2}(1+t)^2. \tag{8}$$

In particular, for positive-dimensional Q, equality of facet numbers is another equivalent condition.

The question of when the marked order polytope and the marked chain polytope are unimodularly equivalent was treated in [9]. For admissible mixed partitions, regularization, facet counts and star-based equivalences appear in [6, Propositions 4.4–4.5, Theorem 4.7 and Corollary 4.8].

For arbitrary partitions, elementary equivalences for non-chain-order-star switches are given in [7, Proposition 2.8]. The facet and equivalence criterion for *tame* marked posets, whose prescribed inequalities define facets for every partition, is given in [7, Proposition 7.11]. The common reductions of [7, Propositions 6.1–6.2 and Remark 6.3] already handle constant marked intervals and redundant covers. Consequently, the tame conclusion also applies to inputs whose common regularization is tame. So the reduction of degenerate markings and the elementary affine maps are known tools. The result here imposes no tameness hypothesis and gives a direct affine-envelope criterion together with (8). Theorem 5.7 below also proves that every regular marked poset is tame. Hence the earlier common reductions and the tame criterion [7, Proposition 7.11] give a second route to the applications (Corollary 5.9). This route is not independent of the direct one: the proof of Theorem 5.7 uses Proposition 5.2. The direct envelope proof also gives the exact facet formula of Proposition 5.2.

5.1 Fixed-height slices and compression

This subsection and the next prove Theorem 5.1. If L is affine, the map $(u, z) \mapsto (u, z - L(u))$ is an affine equivalence from Q onto Q'; if U is affine, the map $(u, z) \mapsto (u, U(u) - z)$ is one. It remains to prove (8) when both functions are nonaffine. Put

$$\alpha = \max_{a \in A,\ a<q} \lambda(a), \qquad \beta = \min_{b \in A,\ q<b} \lambda(b).$$

Marked extrema make these sets nonempty, and $\alpha \le L \le U \le \beta$. Since L is not affine, $\alpha < \beta$: if $\alpha = \beta$, then $L = \alpha$ is constant. Also $U - L$ is not identically zero: otherwise $L = U$ would be both convex and concave, hence affine. Some fibre has positive length, so $\dim Q = \dim Q' = e + 1$.

The range of the order coordinate q on Q is exactly $[\alpha, \beta]$. Indeed, any prescribed value in this interval extends to a marked order extension, and the partial transfer fixes this coordinate. Choose h with $\alpha < h < \beta$. A linear map sends the relative interior of a convex set onto the relative interior of its image. Hence the slice $q = h$ meets $\operatorname{relint}(Q)$, and it has dimension e. Its projection, bijective after deleting the fixed coordinate, is

$$B_h = \{u \in B : L(u) \le h \le U(u)\}.$$

Thus $\operatorname{aff}(B_h) = \operatorname{aff}(B)$, and their projections onto any chosen coordinate set also have the same affine hull. This fact will justify the relative-interior perturbations below, even for degenerate markings.

Let $D = \{p : p < q\}$, a predecessor-closed set. Invert the standard partial transfer on D to obtain order values v for any feasible lower-coordinate vector. For $0 < \tau \le 1$, replace them by

$$\phi_\tau(v_p) = \min(v_p, \alpha) + \tau(v_p - \alpha)_+.$$

Every marked value in D is at most α, so it is fixed. Order is preserved, and only values above α decrease. Hence external upper marked bounds remain valid; an external marked element below an element of D would itself belong to D. For small enough τ, all predecessor values are below h. The prescription together with $q = h$ therefore extends to a full marked order extension.

Transfer the two clipped pieces on D, with their split marks, to vectors u^-, u^+. Commutation of clipping with predecessor maxima gives the compressed mixed vector $u_\tau = u^- + \tau u^+$, and its lower maximum is $\alpha + \tau(L(u) - \alpha)$. For each endpoint-chain form ℓ_i, its two deficits satisfy

$$g_i^- = \alpha - \ell_i(u^-) \ge 0, \qquad g_i^+ = L(u) - \alpha - \ell_i^+(u^+) \ge 0.$$

Here each form uses the marking appropriate to its piece. The original deficit is $g_i^- + g_i^+$, while the compressed deficit is $g_i^- + \tau g_i^+$. For $\tau > 0$ they vanish for exactly the same forms. Thus compression preserves the entire active-chain set and makes $L < h$.

Choose points in the relative interiors of two full-dimensional affinity regions of the lower projection on which L has different effective formulas. Their active-form sets are disjoint: a common form active on open pieces would identify the formulas. After compression these sets remain disjoint in the lower projection of B_h. A finite maximum affine on a polytope has one defining form active everywhere: at a relative-interior point, a vanishing nonnegative affine deficit vanishes on the whole polytope. Consequently L remains nonaffine on the projected slice. Equal affine hulls and strict inactive-form gaps permit moving the two compressed points into its relative interior while retaining their regions and $L < h$.

Dualizing the poset and negating order values and marks gives the upper transfer

$$x_c = \min_{r \succ c} v_r - v_c \quad (c \in C),$$

with order coordinates retained. Its image is the same endpoint-defined mixed polytope. On $F = \{p : p > q\}$, the dual compression is

$$\psi_\tau(s) = \max(s, \beta) + \tau(\min(s, \beta) - \beta).$$

It fixes upper marks, preserves external lower marked bounds and all active upper-chain sets, and moves the upper minimum to $\beta + \tau(U - \beta) > h$. Thus two different upper affinity regions persist in the relative interior of the upper projection of B_h, inside $U > h$.

5.2 Gluing the two projections

Let v, w be full marked order extensions with $v_q = w_q = h$. Set

$$z_p = \begin{cases} v_p, & p \le q, \\ \max\{\min(v_p, h), w_p\}, & p \not\le q. \end{cases}$$

No element outside the down-set $\{p : p \le q\}$ lies below an element of it. For a comparison $p < r$ with $p \le q$ and $r \not\le q$, $v_p \le h$ and $v_p \le v_r$ give $v_p \le \min(v_r, h) \le z_r$. On the complement, z is the pointwise maximum of the two order-preserving functions $\min(v, h)$ and w, hence order-preserving. All marks are unchanged. Hence z is a marked order extension.

Below q, every predecessor remains in the fixed down-set, so $T_C z$ has the lower mixed coordinates of $T_C v$. Above q, $z_r = w_r \ge h$. For a chain coordinate $r > q$, a predecessor on a saturated q–r path has value at least h, unchanged by gluing. Every other predecessor with w-value at least h is also unchanged; a predecessor strictly below q cannot cover r, and q itself is fixed at h. Any changed predecessor stays at most h. Thus the predecessor maximum is unchanged, and all upper mixed coordinates agree with $T_C w$.

It follows that the projection of B_h onto the lower and upper coordinate sets is the Cartesian product of the separate projections. Coordinates incomparable with q may adjust; B_h itself need not be a product. To use the upper-compressed data, first extend it using upper transfer, then invert standard transfer on that full mixed point to obtain w. These two order parametrizations need not coincide, but both retain $q = h$.

In the relatively open convex lower region $L < h$, a generic segment between points in the two effective affinity regions crosses a codimension-one wall and avoids cells of codimension at least two. Choose such an ordinary wall point. Do the same for the upper region $U > h$. The two wall points

lie in the relative interior of the product projection. Since a linear map sends the relative interior of a convex set onto the relative interior of its image, the pair lifts to $x \in \operatorname{relint}(B_h) \subseteq \operatorname{relint}(B)$.

At x each active-difference space has rank one on its projected affine hull. Pullback retains this rank because the projected affine hulls agree with those of the full base. The two nonzero normal lines are independent by the product projection, or by the proved marked relative transversality. An interior ball in $\operatorname{aff}(B)$ realizes all four sign regions. Thus x is an ordinary full-carrier crossing and $e \geq 2$. Lemma 2.2 gives $\delta_B(t) \geq t^{e-2}(1+t)^2$; nonnegativity on other carriers proves (8). The coefficient of t^e in $t^{e-2}(1+t)^2$ is positive, so the facet numbers differ; this proves the remaining equivalences in Theorem 5.1.

5.3 An exact facet difference

Proposition 5.2. *Let s_L, s_U be the numbers of full-dimensional effective affinity cells of L, U on B, respectively. If some fibre has positive length, then, for $e = \dim B$,*

$$f_e(Q') - f_e(Q) = (s_L - 1)(s_U - 1).$$

Here e is the facet degree of Q, Q'. The counts use distinct effective regions, not formally listed endpoint chains.

Proof. Positive width implies $\alpha < \beta$ and $\dim Q = e + 1$, even if one envelope is affine. Thus the fixed-height affine-hull argument above still applies. Choose any full lower affinity cell and a point in its relative interior also lying in $\operatorname{relint}(B)$. Projection onto the lower coordinates gives strict gaps to all other effective forms, after identifying forms whose restrictions agree.

Compression preserves this particular active-form set and moves its witness into $L < h$. Equal projected affine hulls permit perturbation into the relative interior of the lower projection of B_h without losing the strict gaps. Do the same for any full upper cell inside $U > h$. The product projection lifts the chosen pair to $\operatorname{relint}(B_h) \subseteq \operatorname{relint}(B)$. A relative neighbourhood lies in both chosen affinity cells, so their intersection is full-dimensional. Convexity makes it exactly one full refinement cell. Distinct pairs give distinct cells, and every full refinement cell identifies its unique pair. Hence there are exactly $s_L s_U$ full refinement cells.

At degree e only the full base can be a carrier. The fibre identity (3) therefore gives $1 + s_L s_U - s_L - s_U = (s_L - 1)(s_U - 1)$. This proves the claim. □

If every fibre collapses, $L = U$ is affine, $s_L = s_U = 1$, and the two polytopes are affinely equivalent. Their facet difference is zero whenever their actual dimension is positive (a point has no facets to compare). The proof of Proposition 5.2 shows that every full lower affinity cell meets every full upper one in exactly one full refinement cell. In the known tame settings the facet formula is consistent with the chain-count formula of [7, Proposition 7.11].

5.4 Nested partitions and ambient affine maps

Corollary 5.3. *Let $C \subseteq C'$, and let d be the common actual dimension of $Q_C(P, \lambda)$ and $Q_{C'}(P, \lambda)$. For $d \geq 1$, equality of facet numbers, equality of all face numbers, and affine equivalence are equivalent. When they hold, there is an ambient map $x \mapsto Mx + b$ taking one polytope to the other, with $M \in \mathrm{GL}_{|P \setminus A|}(\mathbb{Z})$ and every coordinate of b in the additive integer span of the marks. For integral marks this is a lattice equivalence. If $d = 0$, there is again such a map $x \mapsto Mx + b$.*

Proof. Partial transfer and its inverse are finite piecewise-affine maps between each mixed polytope and the same marked order polytope. On each of its finitely many polyhedral pieces an affine map

cannot raise dimension. Since the image is convex, its relative interior cannot be covered by finitely many affine images of smaller dimension, so its dimension is at most the largest dimension of those image pieces. Applying this to the map and its inverse shows that the polytopes of all intermediate partitions have the same actual dimension. Fixed coordinates and coincident marks are allowed, so this dimension need not equal $|P \setminus A|$. Choose any ordering of $C' \setminus C$. By Theorem 1.1, the successive facet increments are nonnegative. If their sum is zero, each is zero, and Theorem 5.1 supplies an affine map at every step. Composition gives affine equivalence. The reverse implications are immediate. No ideal condition is needed on intermediate chain parts.

An affine envelope agrees throughout its base with one original endpoint-chain form, by the nonnegative-affine-deficit argument above. Such a form has integer coefficients on the other unmarked coordinates, and its constant term is a marked value or zero. The selected shear or reflection therefore extends to the whole ambient space as an integer matrix equal to the identity except in row q, whose q entry is 1 or -1. Its determinant is respectively 1 or -1, and its translation belongs coordinatewise to the integer span of the marks. Composition preserves these properties. This argument also works on a singleton base; when $d = 0$, apply it at every step without assigning facet counts. □

Different switch orderings can give different affine maps. For real marks the translation need not be integral. For one unmarked element between marks $\sqrt{2}$ and $\sqrt{2}+1$, the marked order polytope is $[\sqrt{2}, \sqrt{2}+1]$ and the marked chain polytope is $[0, 1]$. This is why the lattice statement is restricted to integral markings.

5.5 Support for nested marked comparisons

Corollary 5.4 (Terminal support for bounded marked comparisons)**.** *Let P be finite, let A contain all its minimal and maximal elements, and let $\lambda : A \to \mathbb{R}$ be order-preserving. For any $C \subseteq C' \subseteq P \setminus A$, let N be the common actual dimension of $Q_C(P, \lambda)$ and $Q_{C'}(P, \lambda)$. Either their face-number polynomials are equal, or the positive coefficients of*

$$F_{Q_{C'}(P,\lambda)}(t) - F_{Q_C(P,\lambda)}(t)$$

occur exactly in degrees $r, r+1, \ldots, N-1$ for some $0 \le r \le N-3$. In particular, the polynomials are equal when $N \le 2$. No integrality, distinct-marking or admissibility hypothesis is required.

Proof. As in the proof of Corollary 5.3, the polytopes of all intermediate partitions have the same actual dimension N, which need not equal $|P \setminus A|$.

Consider one switch, with deleted base B, finite envelopes L, U, and $e = \dim B$. Put $H = U - L \ge 0$. If H vanishes identically, then $L = U$ is both convex and concave on B, hence affine. The old polytope is its graph over B, and the new one is $B \times \{0\}$. Their defect vanishes; in this case $N = e$, not $e + 1$.

Otherwise $H > 0$ on $\operatorname{relint}(B)$. Indeed, H is concave and nonnegative, so a zero in $\operatorname{relint}(B)$ would force $H = 0$ on all of B: extend a short segment beyond the zero. A relative-interior base neighbourhood therefore has positive-length fibres, giving $N = e+1$ for both old and new polytopes.

Work in $\operatorname{aff}(B)$, suppressing fixed ambient directions. The marked comparison proof supplies finite affinity subdivisions transverse relative to every face of B, and Lemma 2.1 identifies their subdivision defect with the switch defect. Theorem 2.4 then says that the defect of a nonzero switch has positive support exactly $r, \ldots, e$ with $0 \le r \le e-2$, that is, $r, \ldots, N-1$ with $r \le N-3$. Zero-width fibres on proper base faces cause no additional case: their restricted subdivisions are trivial and are already accounted for in the fibre identity (3).

Choose any ordering of $C' \setminus C$ and telescope the switch defects. They are coefficientwise nonnegative by Theorem 1.1. If all vanish, the conclusion is immediate. Otherwise, by the previous paragraph, the jth nonzero increment has positive coefficients exactly in degrees $r_j, \dots, N-1$, for some $r_j \le N-3$. The union of these intervals is exactly $\{\min_j r_j, \dots, N-1\}$, and nonnegativity prevents cancellation. The coefficient at degree N vanishes because both whole-polytope counts are one. This proves the result, including the dimension-zero and empty-sequence cases. □

Remark 5.5. The starting degree need not be at least two for marked polytopes. Example 8.1 has actual dimension three and defect $1+2t+t^2$, so $r=0$, even with integral marks. Corollary 5.4 assumes that all extrema are marked. Corollary 7.5 treats sequences with one regime throughout when marked maxima are not assumed.

5.6 A first-switch test from marked intervals

For an unmarked p, put

$$\underline{\lambda}_p = \max_{a\in A,\ a<p} \lambda(a), \qquad \overline{\lambda}_p = \min_{b\in A,\ p<b} \lambda(b).$$

For $C=\varnothing$, let $D_{\max}$ be the maximal unmarked elements strictly below q, and $F_{\min}$ the minimal unmarked elements strictly above q. With α, β as above, set

$$D_* = \{p \in D_{\max} : \overline{\lambda}_p > \alpha\}, \qquad F_* = \{r \in F_{\min} : \underline{\lambda}_r < \beta\}.$$

Then L is affine exactly when D_* is empty, or $D_* = \{p\}$ with $\underline{\lambda}_p = \alpha$. Dually, U is affine exactly when F_* is empty, or $F_* = \{r\}$ with $\overline{\lambda}_r = \beta$. Together with Theorem 5.1, these criteria give an exact test, using only the poset and the marking, for equality of all face numbers at the first switch: equality holds exactly when L or U is affine.

Indeed, $L = \max(\alpha, \max_{p\in D_*} v_p)$: nonmaximal unmarked elements below q are dominated by maximal ones, and $\overline{\lambda}_p \le \alpha$ prevents a variable from beating the constant. A sole effective predecessor competes with the constant precisely when $\underline{\lambda}_p < \alpha$. If at least two remain, they form an antichain and their values can be prescribed independently in a small common interval above α and below all their $\overline{\lambda}_p$. Each can be the unique largest value; compatible prescriptions extend to a marked order extension. Hence the maximum is nonaffine. The upper assertion is dual. If $\alpha = \beta$, both effective sets are empty. This test includes repeated marks and is stated for $C=\varnothing$.

A combinatorial test for general partitions. For C arbitrary, Corollary 5.9 below gives a test after one common regularization. The elementary interval test above avoids that reduction in the first-switch case.

5.7 Regularity and canonical facet descriptions

In this subsection only, we allow unmarked maximal elements. Let P be finite, let A contain all minimal elements, and let $\lambda : A \to \mathbb{R}$ be order-preserving. We use the same endpoint definition for the possibly unbounded marked chain-order polyhedron $Q_C(P,\lambda)$, with $P \setminus A = C \sqcup O$.

A cover $p \lessdot r$ is *nonredundant* if every pair of marked elements a, b satisfying $a \le r$ and $p \le b$ has either $a = b$ or $\lambda(a) < \lambda(b)$. Such a and b are the *lower* and *upper witnesses* of the cover. We call (P, A, λ) *regular* when all its covers are nonredundant. This is the corrected definition of regularity given in [23, Definition 3.19], also used in [7, Section 6]; it does not require distinct values at incomparable marked elements.

By the *canonical rows* for a partition we mean

$$x_c \geq 0 \quad (c \in C), \qquad x_a + \sum_{i=1}^{r} x_{p_i} \leq x_b \quad (a \lessdot p_1 \lessdot \cdots \lessdot p_r \lessdot b),$$

where $a, b \in A \cup O$, all $p_i \in C$, and $r \geq 0$. Marked coordinates are replaced by their fixed values. Saturation is in the original poset P. The fixed-mark equations are not facet rows; we omit constant rows with $r = 0$ and both endpoints marked, as permitted in [7, Proposition 1.3]. These rows define $Q_C(P, \lambda)$ for every partition, irrespective of their redundancy.

Lemma 5.6. *Suppose all extrema are marked, $N = |P \setminus A| \geq 1$, $Q_C(P, \lambda)$ is full-dimensional in $\mathbb{R}^N$, and its canonical description is irredundant, with one row per facet. For every $q \in O$, the same holds once q is moved to the chain part.*

Proof. Let k count the canonical chains ending at q, let ℓ count those starting at q, and let v count all other canonical rows, including the existing chain nonnegativities. Marked extrema imply $k, \ell \geq 1$. The old polytope has $v + k + \ell$ facets. Full dimension gives $\dim B = N - 1$ and some fibre of positive length. Moreover $U - L > 0$ on $\operatorname{relint}(B)$: otherwise the concave nonnegative function $U - L$ would vanish on all of B, contradicting full dimension.

A row ending at q has q-coefficient -1, when its inequality is written with zero on the right; a row starting at q has coefficient 1. Thus these rows support graph facets, not vertical facets. Deleting q is injective on each such graph hyperplane, so its $(N-1)$-dimensional facet projects onto a full-dimensional lower or upper affinity cell of the deleted base B. Distinct irredundant rows give distinct graph facets and distinct effective full affinity cells. Conversely, the graph of an effective full lower affinity cell is a supporting face of dimension $N - 1$, hence a facet. Some canonical row supports it. That row cannot be vertical, since its projection has full dimension in B. It cannot be an upper endpoint row either: then $U = L$ on the full cell, contrary to the strict width on $\operatorname{relint}(B)$. It must therefore be a lower endpoint row. The upper case is symmetric. Consequently the two effective region counts in Proposition 5.2 are exactly k and ℓ. The possibly redundant all-chain expressions used earlier give the same envelopes and do not create further effective regions.

The new polytope also has dimension N, by the finite piecewise-affine transfer and its inverse. Proposition 5.2 therefore yields

$$f_{N-1}(Q_{C \cup \{q\}}) = v + k + \ell + (k-1)(\ell - 1) = v + 1 + k\ell.$$

We now count the new canonical rows, without assuming their irredundancy. Exactly the $k + \ell$ chains with endpoint q disappear. A new chain involving q has it in its interior and splits uniquely there into one old lower chain and one old upper chain. Conversely any such pair concatenates to a saturated chain: the lower vertices are strictly below q and the upper vertices strictly above it, and all successive pairs are original covers. Thus precisely $k\ell$ indexed chain rows appear, together with $x_q \geq 0$. All other chains and nonnegativities remain. The new system has $v + 1 + k\ell$ rows; its new chain rows are nonconstant because they contain x_q.

Every facet of a full-dimensional polytope given by finitely many rows must be supported by a row, and one row supports at most one facet. Equality of the number of facets and indexed rows therefore forces every row to support a distinct facet. In particular no algebraic duplicate or redundant row remains. This proves the lemma. □

Theorem 5.7 (Regularity implies tameness)**.** *Let P be finite with all minimal elements marked, and suppose (P, A, λ) is regular in the corrected sense above. For every partition $P \setminus A = C \sqcup O$,*

the canonical rows give an irredundant facet description of $Q_C(P,\lambda)$ in its unmarked coordinates. In particular, the marked poset is tame in the sense of [7, Definition 7.1].

Proof. First suppose all maximal elements are marked. Regularity implies strictness of the marking on comparable marked elements and excludes covers between marked elements [23, Proposition 3.20]. A strict marking extends to a strictly order-preserving map on P [23, Proposition 2.3]. The marked order polytope is therefore full-dimensional in its $N = |P \setminus A|$ free coordinates. Its facets correspond bijectively to the covers of P [23, Theorem 3.22]. Since there are no marked-to-marked covers, these are exactly the canonical rows for $C = \varnothing$.

If $N = 0$ there is no unmarked switch and the assertion is vacuous. Otherwise move the elements of the desired set C one at a time, in any order. By Lemma 5.6, full dimension and the irredundancy of the canonical description persist at every step. This proves the bounded assertion without imposing admissibility on the partition.

Now suppose some maximal elements of the full poset P are unmarked. Adjoin one new marked element b_∞ with value

$$H > \max_{a \in A} \lambda(a),$$

and add the relations $m < b_\infty$ for every old maximal element $m \in P \setminus A$, along with their transitive consequences. Here *maximal* means maximal in the full old poset, not merely in its unmarked induced subposet. For nonempty P, the set A is nonempty because it contains all minima; the empty case is vacuous.

The old induced order and all old covers are unchanged. The new covers are exactly $m \lessdot b_\infty$ for these unmarked maxima: any other old element below b_∞ lies below such an m. Old marked maxima remain maximal, so b_∞ is an additional maximal element and need not be a greatest element. All extrema of the enlarged poset P_H are now marked, and its unmarked coordinate set is unchanged.

Regularity is preserved. For an old cover $p \lessdot r$, the new marked element cannot be a lower witness $a \le r$. The only new case has upper witness b_∞, and then $\lambda(a) < H$ for every old lower marked witness. For a new cover $m \lessdot b_\infty$, the only marked element at or above m is b_∞, because m was an unmarked maximal element of P. A marked lower witness is either b_∞ itself or an old marked element of value less than H. Hence every new cover is nonredundant as well.

Fix a partition of the unchanged unmarked set. The bounded assertion gives a full-dimensional capped polytope $Q_H = Q_C(P_H, \lambda_H) \subseteq \mathbb{R}^N$ with irredundant canonical rows. Every old canonical row appears unchanged: its old saturated chain remains saturated, and the new marked element cannot be an interior chain vertex. The additional rows are endpoint chains ending at b_∞. After suppressing its fixed coordinate,

$$Q_H \subseteq Q_C(P,\lambda) = Q$$

in the same ambient space.

For any old row, its tight face in Q_H is a facet of dimension $N-1$. The tight face of the identical valid row in Q contains this facet. Since Q contains the full-dimensional Q_H and the row is nonconstant and proper, its tight face in Q has dimension exactly $N-1$; it is a facet. Two old rows cannot support the same facet of Q, since their supporting hyperplanes would then coincide and would also give duplicate facets of Q_H. Thus all old rows remain distinct facet rows. This proves the assertion for the arbitrary chosen partition. □

Remark 5.8. This proves the regularity-to-tameness direction of [7, Conjecture 7.2]. No limit as H tends to infinity is needed. The theorem gives sufficiency. For regular inputs the constant marked-cover rows do not exist, so their optional omission has no effect on the theorem.

The converse is noted in [7] directly after that conjecture: without regularity, a constant interval or a redundant cover gives a row that is not a facet. If the constant rows of covers between marked elements are kept, as in that argument, this settles [7, Conjecture 7.2]. If they are omitted, as permitted by [7, Proposition 1.3], the converse fails. For example, take $a < b$ and $a < p < c$ with a, b, c marked $0, 1, 1$: the cover $a \lessdot b$ is redundant, yet every partition has irredundant rows. With [7, Proposition 7.9], the generic descriptions of [7, Definition 1.2] are irredundant for every regular marked poset as well.

An ordinary poset with adjoined extrema marked 0 and 1 is regular, so for every partition its canonical rows define exactly the facets; for ranked posets this is [7, Proposition 7.6]. The result concerns irredundancy only: it does not extend bounded face-number monotonicity to unbounded polyhedra, which fail that comparison in Example 8.3.

5.8 A common regularization gives a combinatorial test

Return to the bounded hypotheses, with all extrema of P marked. Choose the following reduction once for the marked poset, independently of the chain part. Contract all constant-interval blocks, and then successively remove redundant covers until the resulting marked poset (R, A_R, λ_R) is regular. Redundancy is recomputed after each deletion; deleting all initially redundant covers at once is not valid [23, Remark 3.25]. These are the transformations of [7, Propositions 6.1–6.2].

Let $J \subseteq P \setminus A$ be the unmarked elements in contracted blocks and let $V = (P \setminus A) \setminus J$ be the surviving unmarked singletons. We identify V with the unmarked set of R. Equally marked incomparable elements are not identified merely because their values agree. For a chain part C, put $D = C \cap V$.

Corollary 5.9. *Fix $q \notin A \cup C$. If $q \in J$, the switch from C to $C \cup \{q\}$ preserves every face number and is an ambient translation. If $q \in V$, count the saturated chains in the final poset R that end at q, start in $A_R \cup (V \setminus (D \cup \{q\}))$, and have all internal elements in D; call their number k. Define ℓ dually for chains starting at q and ending in this same other-endpoint set. Chains consisting of a single cover are included. Then $k, \ell \geq 1$ and, with $d = |V|$ the common actual dimension,*

$$f_{d-1}(Q_{C\cup\{q\}}(P, \lambda)) - f_{d-1}(Q_C(P, \lambda)) = (k-1)(\ell-1).$$

Consequently all face numbers remain unchanged exactly when $k = 1$ or $\ell = 1$. In that event the two polytopes are affinely equivalent, and are lattice-equivalent for integral marks. The product and its vanishing are independent of the chosen valid common reduction.

Proof. For $p \in J$, write b_p for the value of its contracted marked block. The coordinate description in the proof of [7, Proposition 6.1], followed by Proposition 6.2 there, gives an affine isomorphism, the *insertion map* Γ_C, from $Q_D(R, \lambda_R)$ onto $Q_C(P, \lambda)$:

$$(\Gamma_C y)_p = \begin{cases} y_p, & p \in V, \\ b_p, & p \in J \setminus C, \\ 0, & p \in J \cap C. \end{cases}$$

Its inverse is projection to V. Cover deletion changes no coordinates. The reduced order polyhedron is bounded, since it is affinely isomorphic to the original one. If R had an unmarked minimal element, decreasing only its coordinate would give an unbounded ray; an unmarked maximal element similarly gives an increasing ray. Hence all extrema of R are marked. Regularity and transfer give dimension $d = |V|$ for every partition, and therefore for the original polytopes as well.

When $q \in J$, the reduced chain part is unchanged. The insertion maps differ only at q, so the ambient translation $x \mapsto x - b_q e_q$ takes the old polytope to the new one. This also covers the case $d = 0$.

When $q \in V$, the two insertion maps agree and the reduced chain parts differ exactly at the surviving element q. All extrema of R are marked, so lower and upper endpoint chains exist. By Theorem 5.7, their canonical rows are distinct facets. Moving q deletes $k + \ell$ endpoint rows and adds $k\ell$ concatenated rows and one nonnegativity row, as in Lemma 5.6. This proves the stated facet gap. The zero criterion and affine assertions follow from Corollary 5.3 for the original bounded polytopes. Finally, the product equals an intrinsic facet difference, which proves its independence of the reduction choices. □

Remark 5.10. Raw chain counts before reduction can be wrong. Let the marked minima a, b both have value zero and the marked maxima c, d both have value one, with covers $a, b \lessdot q \lessdot c, d$. Both mixed polytopes are $[0, 1]$. The unreduced counts are two and two, whereas a valid sequential reduction retains one lower and one upper cover. The reduced counts are one and one, giving the correct zero gap. All four original covers are individually redundant, but removing them all at once leaves q unconstrained. The test uses saturation in the final R and the same reduction for both partitions.

5.9 Face-polynomial consequences

Corollary 5.11 (An explicit face-polynomial coefficient)**.** *Let P be finite, let A contain all its minimal and maximal elements, and let $\lambda : A \to \mathbb{R}$ be order-preserving. The polytopes $Q_C(P, \lambda)$, over all $C \subseteq P \setminus A$, have a common actual dimension N. For any two chain parts C, C', whether nested or not,*

$$(1 + t)^2 \;\mid\; F_{Q_{C'}(P,\lambda)}(t) - F_{Q_C(P,\lambda)}(t).$$

For a nonempty convex N-polytope Q, define the polynomial

$$H_Q(u) = (1 - u)^{N+1} \left[1 + \frac{u}{1 - u} F_Q\left(\frac{u}{1 - u} \right) \right]. \tag{9}$$

Then $[u^N] H_{Q_C(P,\lambda)}(u)$ is independent of C. No integrality or distinct-marking assumption is needed.

Proof. As in the proof of Corollary 5.3, the polytopes have a common actual dimension N; it may be smaller than the number of unmarked coordinates, due to fixed coordinates or coincident marks.

For a single switch the fibre identity (3) of Lemma 2.1 and the relative transversality proved on all carriers in Section 3 (the proof of Theorem 1.1) identify its face-polynomial difference with a subdivision defect covered by Lemma 2.5. If the width vanishes identically, the old polytope is an affine graph over its base and the new one is the zero-height copy of that base, so the defect is zero directly. In this case N is the base dimension; otherwise the dimension is one larger. Both cases give the stated divisibility for a single switch. Telescope from the empty chain part to each of C and C', then subtract. This proves divisibility for arbitrary C, C', without requiring them to be comparable.

The expression in (9) is a polynomial, since a k-face contributes $u^{k+1}(1-u)^{N-k}$ and $0 \le k \le N$. Assume first $N \ge 1$, and put $D(t) = F_{Q_{C'}}(t) - F_{Q_C}(t) = \sum_{k=0}^{N} D_k t^k$. The empty-face term cancels between the two transforms, giving

$$H_{Q_{C'}}(u) - H_{Q_C}(u) = \sum_{k=0}^{N} D_k u^{k+1} (1 - u)^{N-k}.$$

The $k = N$ term contributes nothing in degree N. Extracting that coefficient from the remaining terms gives

$$[u^N]\big(H_{Q_{C'}} - H_{Q_C}\big) = \sum_{k=0}^{N-1}(N-k)(-1)^{N-k-1}D_k$$
$$= (-1)^{N-1}\big(ND(-1) + D'(-1)\big) = 0.$$

The last equality follows from the divisibility already proved. In dimension zero every nonempty polytope is a point and its displayed transform is 1, so the conclusion holds directly as well. □

Remark 5.12. The transform in (9) is defined directly from the face numbers, including its displayed empty-face term. Its coefficient invariance holds for arbitrary chain parts; the nonnegative quotient statement below requires nested chain parts.

Corollary 5.13 (Nonnegative bounded marked quotients)**.** *Under the bounded marked hypotheses of Theorem 1.1, for every $C \subseteq C'$ one has*

$$F_{Q_{C'}(P,\lambda)}(t) - F_{Q_C(P,\lambda)}(t) \in (1+t)^2\mathbb{Z}_{\geq 0}[t].$$

Proof. For each single switch the deleted base is a polytope, and its two affinity subdivisions are transverse relative to every face by Section 3. Lemma 2.1 identifies the face difference with their subdivision defect, so Theorem 2.6 applies. Telescope along single switches from C to C'. Globally zero-width switches contribute zero and need no separate dimension assumption. □

6 Equality in the sharp bound

We now characterize the posets for which equality holds in (2) in every degree at once. This is different from equality of the order and chain face numbers in Corollary 1.3.

For $v \in R$, let $d(v)$ count maximal chains from the adjoined minimum to v, and let $r(v)$ count upper covers in the augmented poset. An element v is a *star* if $d(v) \geq 2$ and $r(v) \geq 2$; it then has *type* $(d(v), r(v))$. We use the known formula [12, Theorem 2.1]

$$f_{n-1}(\mathcal{C}(R)) - f_{n-1}(\mathcal{O}(R)) = \sum_{v\in R}(d(v)-1)(r(v)-1). \tag{10}$$

Write $D_R(t) = F_{\mathcal{C}(R)}(t) - F_{\mathcal{O}(R)}(t) = \sum_k D_k t^k$. We call D_{n-1} the *facet gap* and D_2 the *two-face gap* of R. We write $H_n(t) = t^2(1+t)^{n-3}$ for the right side of (2). As in Section 1, an index $k \in \{2, \ldots, n-1\}$ is *tight* if $D_k = \binom{n-3}{k-2}$. For posets P_1 and P_2, the *ordinal sum* $P_1 \oplus P_2$ is their disjoint union, in which every element of P_1 lies below every element of P_2. By (10), facet gap one means exactly one star q, with $d(q) = r(q) = 2$. The facet-gap-one class was previously characterized as the X-orchids in [3, Theorem 14]. We impose further conditions to characterize equality in the full sharp polynomial. A switch of q with chain ideal I is *admissible* when $I \cup \{q\}$ is an ideal; the late switch has $I = R \setminus {\uparrow}q$, the *late ideal* of q.

Theorem 6.1. *Suppose R contains X, and put $n = |R|$. Then*

$$F_{\mathcal{C}(R)} - F_{\mathcal{O}(R)} = t^2(1+t)^{n-3}$$

if and only if the following conditions hold.

(C1) There is exactly one star q, and $d(q) = r(q) = 2$.

(C2) There are chains B_0, S, T_0, where S is nonempty with least element c and greatest element q, and incomparable pairs d_1, d_2 and u_1, u_2 such that

$$\downarrow q = B_0 \oplus \{d_1, d_2\} \oplus S, \qquad \uparrow c = S \oplus \{u_1, u_2\} \oplus T_0.$$

Here B_0, T_0 may be empty.

(C3) Every element outside $K = (\downarrow q) \cup (\uparrow c)$ is above both d_1 and d_2, or below both u_1 and u_2.

(C4) No outside element below both u_1 and u_2 lies below an outside element above both d_1 and d_2.

The next three subsections prove Theorem 6.1.

6.1 Two cuts and an ascent observation

Assume first that the unique star is q with $d(q) = r(q) = 2$. Let u_1, u_2 be the two upper covers of q. Use the switch with $I = R \setminus \uparrow q$. All other switches in a compatible linear extension have zero defect by [6, Theorem 4.7]. This also follows from the fibre identity: at a non-star, either the lower maximal chain is unique and L is affine, or the upper cover is unique and U is affine. For the switch of q, the lower envelope is the maximum of the sums on the two maximal chains below q, and the upper envelope is $\min(y_{u_1}, y_{u_2})$. Thus each affinity subdivision is a single hyperplane cut of its base $\mathcal{B}$.

Call a face E of $\mathcal{B}$ *transverse* if both cuts meet in its relative interior and both functions are nonaffine there. This property of a face is not the relative transversality of Section 2. The relative transversality already proved makes their wall normals independent. If $e = \dim E$, the refinement has four full cells, four wall pieces and one crossing cell with this carrier. The two original cuts each have two full cells and one wall. The resulting contribution is $t^{e-2}(1+t)^2$. If only one function is nonaffine, the contribution vanishes. If both cuts occur but do not meet in the relative interior, there are three full cells and two walls, again giving zero. Convexity makes all these pieces connected cells; boundary pieces have smaller carriers. Consequently

$$F_{\mathcal{C}(R)} - F_{\mathcal{O}(R)} = (1+t)^2 \sum_{E \text{ transverse}} t^{\dim E - 2}. \tag{11}$$

Vertex numbers of the ordinary endpoint polytopes agree by [25, Corollary 1.3 and Theorem 2.2]; their edge numbers agree by [19, Theorem 2.1, arXiv version]. Since the summands are nonnegative, every transverse face has dimension at least four. In particular, the two-face gap counts transverse four-faces exactly.

We will also need the following *ascent observation*, which uses the notation of condition (C2). If $z > d_1$ and $z > d_2$, and the strict upper set of z is not a chain, then $z \leq q$. Indeed, choose $v \geq z$ maximal with a nonchain strict upper set. A unique upper cover would lie below every strict successor and would still have a nonchain strict upper set, contradicting maximality. Hence $r(v) \geq 2$. The incomparable $d_1, d_2 < v$ give two distinct maximal lower chains, so v is a star. Uniqueness gives $v = q$, proving the assertion. In particular, an element outside K cannot be both above the two d_i and below the two u_i. Here the star q is unique, but a centre of X in R need not be: every element of S can be a centre.

6.2 Necessity

Suppose equality holds. Comparing the coefficients of t^{n-1} and using (10) gives condition (C1). Each of the two maximal chains of $\downarrow q$ has a nonempty *exclusive part*, the set of its elements that are not on the other chain. An element exclusive to one chain cannot be comparable with an element exclusive to the other: splicing at such a comparison would give a third maximal chain. If one exclusive part contains two elements a, a' and the other contains b, the construction in Section 4 gives distinct transverse four-faces with free lower coordinate pairs (a, b) and (a', b). They are distinct because each fixes to zero a coordinate free in the other. Formula (11) would give a two-face gap of at least two. Thus both exclusive parts are singletons d_1, d_2, which are incomparable. Every common element of the two chains lies below both or above both. Their first common successor c covers both.

Apply this argument to the opposite poset. Coordinate complementation identifies its order polytope with the original one, and its chain polytope is unchanged. In the opposite poset, c has upper covers d_1, d_2 and at least two maximal lower chains, from the original upper chains through q, u_1 and q, u_2. It is therefore the unique dual star of type $(2, 2)$. The same exclusive-part argument shows that the two chains of $\uparrow c$ differ only at u_1, u_2. All their common lower, middle and upper parts are chains. This proves condition (C2), without using conditions (C3) and (C4).

Put $\alpha = x_{d_1}, \beta = x_{d_2}, \gamma = y_{u_1}, \eta = y_{u_2}$. Let E_0 be the four-face obtained by setting all other chain coordinates to zero and all T_0 coordinates to one. Its inequalities are (7), and the walls are $\alpha = \beta$ and $\gamma = \eta$. Every outside element is incomparable with S and belongs to I. Suppose an outside z satisfies neither alternative in condition (C3). Relabel so that $d_2 \not< z$, and put $\varepsilon = \mathbf{1}_{d_1 < z}$. The element z is below at most one u_i. Set $R_z = \gamma$ if $z < u_1$, $R_z = \eta$ if $z < u_2$, and $R_z = 1$ otherwise. Keep x_z in addition to the four core coordinates, fix the other coordinates as for E_0, and make the valid row

$$x_z + \varepsilon\alpha = R_z$$

tight. All other chains through z give weaker bounds. Eliminating x_z leaves exactly (7); in particular $R_z - \varepsilon\alpha \geq 0$ there. At $(\alpha, \beta, \gamma, \eta) = (1/4, 1/4, 3/4, 3/4)$, its x_z is positive. This is a second transverse four-face, contradicting the two-face coefficient. Condition (C3) follows.

Call the outside types $\mathcal{A}_+$ (above both d_i) and $\mathcal{A}_-$ (below both u_i); the ascent observation makes them disjoint. Suppose $w \in \mathcal{A}_-$, $z \in \mathcal{A}_+$ and $w < z$. The same observation implies that w is above at most one d_i and z below at most one u_i. Relabel so that any $d_i < w$ is d_1, and $z \not< u_2$. Put $\varepsilon = \mathbf{1}_{d_1 < w}$ and set $R_z = \gamma$ if $z < u_1$, otherwise $R_z = 1$. Keep x_w, x_z and the four core coordinates, fix the others as before, and impose the valid tight row

$$x_w + x_z + \varepsilon\alpha = R_z.$$

At $\alpha = \beta = 1/20$, $x_w = 1/5$, $\gamma = \eta = 3/4$, take $x_z = R_z - 1/5 - \varepsilon/20 > 0$. The other effective rows involving w or z are

$$x_w + \varepsilon\alpha \leq \gamma, \eta, \qquad x_z + \alpha \leq R_z, \quad x_z + \beta \leq R_z,$$

or weaker rows. They and the core inequalities are all strict at the witness. Thus there is one independent equation in six free coordinates: this is a five-face. Eliminating x_z leaves the four core coordinates freely variable near the witness, so the walls cross transversely. This face does not contain E_0, where the chosen row is strict at its point with $(\alpha, \beta, \gamma, \eta) = (1/4, 1/4, 3/4, 3/4)$. Cofaces of E_0 already give the sharp lower bound, and this extra face contributes $t^3(1+t)^2$ in (11). Equality is impossible. This proves condition (C4).

6.3 Sufficiency

Assume conditions (C1)–(C4). Both outside types have chain coordinates in the chosen ideal I; the remaining order coordinates are u_1, u_2, T_0. An outside element is incomparable with S. A chain cannot contain both outside types: an element of $\mathcal{A}_-$ below an element of $\mathcal{A}_+$ is forbidden by condition (C4), and an element of $\mathcal{A}_+$ below an element of $\mathcal{A}_-$ would put an outside element above both d_i and below both u_i, contrary to the ascent observation. Therefore a chain containing an $\mathcal{A}_+$ element uses only B_0, at most one d_i, and $\mathcal{A}_+$ elements. A chain containing none lies below both u_i.

Consider the endpoint inequalities defining $\mathcal{B}$. No order element lies below an element of I, so a nonempty chain has the adjoined minimum as lower endpoint. If such a chain has no $\mathcal{A}_+$ element, its sum is bounded by both γ and η. Tightness at one of these endpoints forces a sign for $\gamma - \eta$; tightness at a T_0 endpoint or at the adjoined maximum $\hat{1}$ forces $\gamma = \eta$. If the chain contains an $\mathcal{A}_+$ element and one d_i, replace that d_i by the other one to obtain an equally valid chain with the same endpoint. Tightness forces a sign for $\alpha - \beta$. If it contains neither d_i, inserting either is possible and tightness forces $\alpha = \beta = 0$.

Tightness of $\alpha \geq 0$ or $\beta \geq 0$ also forces one lower wall sign. A tight comparison from u_i to a T_0 coordinate or to one forces one upper wall sign, since the other u_j has the same upper bound. Tightness of an order-coordinate lower bound forces either $\alpha = \beta = 0$ or $\gamma = \eta = 0$. These cases include empty-chain rows and exhaust the defining inequalities except the noncore chain nonnegativities, the comparisons within T_0, and their upper bound 1.

Every remaining row is tight on E_0. Thus the active rows of any transverse face all contain E_0, so that face is a coface of E_0. Conversely every coface is transverse by Lemma 4.1. At the point of E_0 with $(\alpha, \beta, \gamma, \eta) = (1/4, 1/4, 3/4, 3/4)$, an irredundant active list consists of $x_i \geq 0$ for $i \in I \setminus \{d_1, d_2\}$ and, writing $T_0 = \{t_1 < \cdots < t_m\}$, the rows

$$y_{t_1} \leq \cdots \leq y_{t_m} \leq 1.$$

There are exactly $n - 5$ independent rows. Their slack coordinates x_i, $y_{t_{j+1}} - y_{t_j}$ and $1 - y_{t_m}$ form an orthant; the T_0 list is empty when $m = 0$. The coface interval above E_0 is therefore Boolean, with $\binom{n-5}{e-4}$ cofaces of dimension e. Substitution in (11) gives

$$(1+t)^2 \sum_{e=4}^{n-1} \binom{n-5}{e-4} t^{e-2} = t^2(1+t)^{n-3},$$

completing the proof.

6.4 Equality at one middle coefficient

Theorem 6.2. *For an n-element poset R containing X and $3 \leq k \leq n-2$,*

$$f_k(\mathcal{C}(R)) - f_k(\mathcal{O}(R)) = \binom{n-3}{k-2}$$

if and only if $D_R = H_n$, equivalently, if and only if the four conditions in Theorem 6.1 hold. If the facet gap is one, the set of tight indices in $\{2, \ldots, n-1\}$ is one of $\{2, \ldots, n-1\}$, $\{2, n-1\}$, or $\{n-1\}$.

Proof of Theorem 6.2. We first record two elementary observations about faces. In the one-star setting of (11), transverse faces are closed under taking cofaces. To see this, use the affine map

$\phi = (\ell_1 - \ell_2, y_{u_1} - y_{u_2})$ defined by the two wall differences, where ℓ_1 and ℓ_2 are the sums on the two maximal chains below q. If E is transverse, its wall normals are independent, so $\phi(\operatorname{relint} E)$ is an open neighbourhood of zero in $\mathbb{R}^2$. For a coface E', the convex, relatively open set $V = \phi(\operatorname{relint} E')$ has closure containing that neighbourhood. It is therefore full-dimensional and open. The identity $\operatorname{int}(\overline{V}) = V$ for a nonempty open convex set implies that the neighbourhood, and hence zero, lies in V. Thus E' is transverse as well.

Second, if E is a proper face of a polytope B and H is a face not contained in E, then for every j with $\dim E \le j \le \dim B - 1$ there is a j-face containing E but not H. In the face figure of E, the join of E and H corresponds to a nonempty face G. Choose a facet not containing G, or any proper-face flag if G is the whole figure. A flag in that facet gives all required dimensions. If the face figure is a point, the empty face gives the case $j = \dim E$.

The case of facet gap one. Assume there is a unique star q of type $(2, 2)$. Choose any incomparable $a, b < q$ and the two upper covers u_1, u_2, and use the four-face E_0 from Section 4 with $I = R \setminus \uparrow q$. Every $x > q$ inherits two maximal lower chains, and is not a star. Thus it has exactly one upper cover in the augmented poset. The strict upper set together with the adjoined top is a rooted tree, oriented toward that top.

At the point u_0 of Section 4.1, an irredundant list of active rows consists of the lower bounds for other chain coordinates and one outgoing order comparison for each element of $(\uparrow q) \setminus \{q, u_1, u_2\}$. Rows starting at the free covers are strict, as are all chain rows. Each other order coordinate has its parent among those coordinates or at the adjoined top; the parent cannot be a cover of q. The parent differences form an independent triangular system ending at one. Together with the chain-coordinate lower bounds, they give exactly $n - 5$ independent orthant coordinates. Consequently the coface interval above E_0 is Boolean even without conditions (C2)–(C4) of Theorem 6.1.

Call a transverse face *extra* if it does not contain E_0. The cofaces of E_0 account for H_n exactly in (11), so

$$D_R - H_n = (1 + t)^2 \sum_{E \text{ extra}} t^{\dim E - 2}. \tag{12}$$

If extra faces exist, let e_0 be their least dimension. The preceding two observations supply extra faces of every dimension from e_0 through $n - 2$, and no extra face has dimension $n - 1$ because the whole base contains E_0. Thus the excess is positive exactly in degrees $e_0 - 2, \dots, n - 2$.

The necessity proof of Theorem 6.1 shows that failure of (C2) or of (C3) gives an extra four-face; failure of (C4), when (C2) and (C3) hold, gives an extra five-face. If the second identity of (C2), the one for $\uparrow c$, fails, apply the dual argument to obtain $D_2 \ge 2$, then use (11) in the original base to obtain its extra four-face. If all four conditions hold, Theorem 6.1 gives $D_R = H_n$, and (12) gives no extra faces. All transverse faces have dimension at least four. Therefore, if extra faces exist, $e_0 \in \{4, 5\}$; otherwise $D_R = H_n$. This proves the claimed three possible tight-index sets in the facet-gap-one case.

The case of facet gap at least two. The late-ideal four-face construction and coface-crossing proof in Section 4 apply to any star p, any incomparable $a, b < p$, and any two upper covers g, h of p. Maximality was used there only to obtain the covers. We show that facet gap at least two gives strict inequality in every degree $3, \dots, n - 1$. Here $n \ge 6$: with five elements, containment of X forces $R = X$, whose facet gap is one.

Suppose first that some star p has $d(p) \ge 3$ or $r(p) \ge 3$; other stars are allowed. If $d(p) \ge 3$, the poset $\{x \in R : x < p\}$ has at least two distinct incomparable pairs: a poset with exactly one incomparable pair is an ordinal sum of a chain, that pair and a chain, and has just two maximal

chains. If $r(p) \geq 3$, choose two distinct pairs of upper covers instead. The late switch of p therefore has two distinct crossing four-faces E_0, E_0'. Each has all its cofaces crossing. The cofaces of E_0 give at least H_n; the face-figure observation applied to E_0' and E_0 gives additional crossing carriers of every dimension $4, \ldots, n-2$. They force excess in degrees $2, \ldots, n-2$, and facet gap at least two gives excess at $n-1$. Other switches are nonnegative.

Otherwise all stars have type $(2,2)$ and there are at least two. If two stars satisfy $p < p'$, the late ideals $R \setminus \uparrow p \subset R \setminus \uparrow p'$ are nested. Order their difference compatibly with R, switching p immediately after its late ideal and p' immediately after its late ideal. Both switches give at least H_n, so $D_R \geq 2H_n$.

It remains to treat incomparable stars p, p'. Start with

$$I' = R \setminus (\uparrow p \cup \uparrow p'),$$

then switch p', then the other elements of $(\uparrow p') \setminus \uparrow p$ in a compatible order, then p. These sets are ideals: a predecessor of an element outside $\uparrow p$ cannot belong to $\uparrow p$. The later switch of p has exactly its late ideal and contributes at least H_n.

For the earlier switch of p', choose incomparable $a', b' < p'$ and upper covers g, h of p'. Its base has a five-face as follows. Set all chain coordinates except a', b' to zero, set order coordinates in $(\uparrow p') \setminus \{p', g, h\}$ to one, and impose

$$y_v = y_p \quad (v \in K), \qquad K = (\uparrow p) \setminus \uparrow p'.$$

These are supporting equalities since $p \leq v$ for every $v \in K$. The free coordinates are

$$\alpha = x_{a'}, \quad \beta = x_{b'}, \quad \gamma = y_g, \quad \eta = y_h, \quad \zeta = y_p.$$

Besides $0 \leq \alpha, \beta \leq \gamma, \eta \leq 1$ and $0 \leq \zeta \leq 1$, the only effective inequalities are optional comparisons $\alpha \leq \zeta$ or $\beta \leq \zeta$ when the corresponding lower element is below some member of K, and $\zeta \leq \gamma$ or $\zeta \leq \eta$ when some member of K is below the corresponding upper cover. Common upper elements of the two stars have value $1, \gamma$, or η, not ζ. No fixed-one coordinate is below K, since that would force its target into $\uparrow p'$, and none is below g or h, since these cover p'. These facts exclude further nontrivial rows.

At $(\alpha, \beta, \gamma, \eta, \zeta) = (1/4, 1/4, 3/4, 3/4, 1/2)$ every surviving inequality is strict. Thus this face F has dimension five, with $L = \max(\alpha, \beta)$ and $U = \min(\gamma, \eta)$. Using coordinates $w_v = y_v - y_p$ for $v \in K \setminus \{p\}$, all active equalities at this point involve only the other coordinates. The local product argument of Section 4.1 therefore applies to every coface of F. Nearby maximizing lower chains contain a' or b', giving $\max(\alpha + \mu_a(z), \beta + \mu_b(z))$; the only minimizing upper branches are γ, η. The same generic residual-form argument as in Lemma 4.1 gives an ordinary crossing on every such coface.

A coface flag supplies a positive carrier contribution in every dimension $5, \ldots, n-1$. Hence this earlier switch contributes at least

$$(1+t)^2(t^3 + \cdots + t^{n-3}),$$

which is positive in every degree $3, \ldots, n-1$. The later switch of p already supplies H_n, proving strictness in all those degrees.

Since R contains X, it has a star and its facet gap is at least one. If one middle coefficient is tight, the facet gap is therefore one. Its tight-index alternatives already proved then force $D_R = H_n$. The converse is immediate. □

The coefficient of t^2 behaves differently. The preceding arguments for a star p with $d(p) \geq 3$ or $r(p) \geq 3$ and for comparable stars show that $D_2 = 1$ requires all stars to have type $(2,2)$ and to be pairwise incomparable. The posets attaining the two-face bound, that is, those with $D_2 = 1$, are characterized below: Theorem 6.4 treats facet gap one, and Theorem 6.9 treats any number of stars.

6.5 The two-face bound when the facet gap is one

We now distinguish the three tight-index alternatives in Theorem 6.2. Throughout this subsection R is an ordinary poset, with only the adjoined marked minimum and maximum of values $0, 1$. We do not allow additional fixed interior marks in the deformation argument below.

Lemma 6.3 (Deforming order levels)**.** *Let I be an order ideal of an ordinary poset R, let $q \notin I$, and suppose $I \cup \{q\}$ is an order ideal. Let $\mathcal{B}$ be the deleted base of the switch of q at I. Take $x \in \operatorname{relint}(E)$ for a face E of $\mathcal{B}$, and let $v = T_I^{-1}(x)$ be its standard inverse-transfer order vector on the augmented poset $R \setminus \{q\}$. If v has m distinct values strictly between 0 and 1, then independently varying these values a little, while preserving their order and equality groups and fixing $0, 1$, gives an m-dimensional affine patch in $\operatorname{relint}(E)$ after transfer.*

Proof. The transfer bijection and inverse are those of [7, Theorem 2.1], applied to the deleted poset itself. All covers and predecessor maxima in this proof belong to that augmented deleted poset. For a chain coordinate p,

$$x_p = v_p - \max_{r \lessdot p} v_r.$$

An increasing reassignment of the finite set of order levels preserves the maximizing predecessor level and whether this difference is zero.

Consider any endpoint-chain inequality with nonempty interior, $a < p_1 < \cdots < p_\ell < b$, and put $p_0 = a$ and $\mu_i = \max_{r \lessdot p_i} v_r$. Its slack is

$$x_b - x_a - \sum_{i=1}^{\ell} x_{p_i} = v_b - v_{p_\ell} + \sum_{i=1}^{\ell} (\mu_i - v_{p_{i-1}}). \tag{13}$$

Every term on the right is nonnegative. Indeed, a predecessor on a saturated path from p_{i-1} to p_i has value at least $v_{p_{i-1}}$, even when the chosen chain step is not a cover. Changing levels strictly increasingly preserves whether each term is zero or positive, and hence whether the inequality is tight. The same holds for empty-interior order rows and for chain-coordinate nonnegativities. Thus the entire defining active-row set is preserved, including redundant rows, so the new mixed point stays in $\operatorname{relint}(E)$.

Write the interior levels as $0 < \rho_1 < \cdots < \rho_m < 1$. On this fixed weak ordering, each predecessor maximum is a specified level, even when several predecessors tie. The map from the nearby level parameters to mixed coordinates is therefore affine. It is injective by bijectivity of transfer, since every level is the value v_p of some element p of the deleted poset. Its image has dimension m. □

Theorem 6.4. *Let R be an ordinary n-element poset containing X, with facet gap one. Then*

$$f_2(\mathcal{C}(R)) - f_2(\mathcal{O}(R)) = 1$$

if and only if conditions (C2) and (C3) of Theorem 6.1 hold. Condition (C4) is not required. More precisely, define the set of tight indices by

$$\mathcal{I}(R) = \left\{ k \in \{2, \ldots, n-1\} : f_k(\mathcal{C}(R)) - f_k(\mathcal{O}(R)) = \binom{n-3}{k-2} \right\}.$$

Then

$$\mathcal{I}(R) = \begin{cases} \{2, \dots, n-1\}, & \textit{if conditions (C2)–(C4) hold,} \\ \{2, n-1\}, & \textit{if conditions (C2), (C3) hold and (C4) fails,} \\ \{n-1\}, & \textit{otherwise.} \end{cases}$$

Here condition (C1) is automatic from the facet-gap assumption.

Proof. The necessity arguments for conditions (C2) and (C3) in Theorem 6.1 use only the unique star and a two-face gap of one. If the second identity of (C2), the one for $\uparrow c$, fails, the dual argument gives $D_2(R) = D_2(R^{\mathrm{op}}) \geq 2$; the one-star formula in the original base then gives an extra transverse four-face there. It remains to prove sufficiency of conditions (C2) and (C3), without assuming condition (C4).

Use their core $B_0 \oplus \{d_1, d_2\} \oplus S \oplus \{u_1, u_2\} \oplus T_0$, and the notation

$$\alpha = x_{d_1}, \quad \beta = x_{d_2}, \quad \gamma = y_{u_1}, \quad \eta = y_{u_2}.$$

The two walls are $H_S = \{\alpha = \beta\}$ and $H_T = \{\gamma = \eta\}$. The outside attachment types $\mathcal{A}_-$ and $\mathcal{A}_+$ are as above: respectively below both u_i and above both d_i. The ascent observation makes them disjoint and excludes an element of $\mathcal{A}_+$ below an element of $\mathcal{A}_-$. Both types are incomparable with S. No outside element can be below a d_i or above a u_i, since it would then belong to the core.

Three order levels force dimension at least five. Let E be transverse and choose $x \in \operatorname{relint}(E) \cap H_S \cap H_T$. In its inverse order vector v, the two d_i have the same predecessor maximum: the order value at the top of B_0, or zero if B_0 is empty. This uses the exact principal down-set in condition (C2). Hence

$$\alpha - \beta = v_{d_1} - v_{d_2}, \qquad \gamma - \eta = v_{u_1} - v_{u_2}.$$

Denote the common values at this point by $D = v_{d_1} = v_{d_2}$ and $G = v_{u_1} = v_{u_2}$. They satisfy $0 < D < G < 1$. The common value $\alpha = \beta$ is positive: otherwise both nonnegativities would be tight on E and its lower wall could not cut. Thus $D \geq \alpha > 0$. The full prefix $B_0 \cup \{d_i\}$ has sum D and is bounded by both γ and η; equality $D = G$ would make both rows tight on E, forcing $\gamma = \eta$ throughout E. Similarly, $G = 1$ would make both upper bounds tight and destroy the upper cut. We use here the fact that a defining row tight at a relative-interior point of a face is tight on the whole face.

Every order-level deformation in Lemma 6.3 preserves the equalities $v_{d_1} = v_{d_2}$ and $v_{u_1} = v_{u_2}$, and thus both walls. Their restricted normals on $\operatorname{aff}(E)$ are independent by relative transversality. Consequently their intersection with $\operatorname{relint}(E)$ has dimension $\dim(E) - 2$: it is relatively open in the intersection of two independent affine hyperplanes and contains x. If v has three distinct interior levels, the lemma supplies a three-dimensional patch in that intersection, so $\dim(E) \geq 5$.

A tight chain using both attachment types supplies a third level. Suppose an endpoint-chain row involving both types is tight at x. Its lower endpoint is the adjoined minimum, since I is an ideal. Let w be the last $\mathcal{A}_-$ element of the chain. After w there is a nonempty suffix of $\mathcal{A}_+$ elements: no $\mathcal{A}_+$ element can precede an $\mathcal{A}_-$ element, and no core chain element can lie above the outside element w. Write P_w for the prefix sum through w, and H for the suffix sum. The tight row is

$$P_w + H = y_b,$$

where b is an order endpoint or the adjoined marked maximum.

Replacing the entire prefix by $B_0 \cup \{d_i\}$ gives a valid chain with the same suffix and endpoint, since every suffix element is above both d_i. Its prefix sum is $v_{d_i} = D$. Thus $D + H \le y_b$, and $P_w \ge D$ at x. If $P_w = D$, both replacement inequalities are tight on E; subtracting them gives $\alpha = \beta$ throughout E, a contradiction. Therefore $P_w > D$. The original prefix is below both u_i, since it ends at $w \in \mathcal{A}_-$. Hence $P_w \le \min(\gamma, \eta)$. Equality $P_w = G$ would make both inequalities tight on E, forcing $\gamma = \eta$ there. Consequently $P_w < G$.

Every unmarked predecessor of w is in the chain ideal I. The standard inverse recurrence therefore identifies v_w as the maximum chain sum from the minimum through w, so $P_w \le v_w$. A maximizing chain through w can be concatenated with the same suffix and endpoint, giving $v_w + H \le y_b = P_w + H$. Thus $v_w = P_w$, and

$$0 < D < v_w < G < 1.$$

These three distinct interior order levels force $\dim(E) \ge 5$. The deformation preserves all active rows simultaneously; this argument does not assume that only one mixed chain row is tight.

Exhausting transverse four-faces. The earlier sufficiency proof used condition (C4) only to exclude chains containing both types. Its other tight-row cases remain valid: a chain with no $\mathcal{A}_+$ element is below both u_i, so its tightness forces an upper-wall sign or equality. A chain containing an element of $\mathcal{A}_+$ but none of $\mathcal{A}_-$ uses only B_0, at most one d_i, and $\mathcal{A}_+$ elements. Swapping the d_i, or inserting either when absent, forces a lower-wall sign or equality. Tight nonnegativity at a d_i, an order comparison from a u_i to T_0 or to $\hat{1}$, or an order-coordinate lower bound likewise destroys a crossing. These include the empty-interior endpoint rows.

The remaining rows are the nonnegativities of chain coordinates other than d_1, d_2, the comparisons within T_0, and their upper bound 1. Each is tight on the standard four-face E_0. A transverse four-face cannot have a tight row using both attachment types by the preceding paragraph. Hence every defining row tight on it is tight on E_0. It contains E_0 and, having the same dimension, equals E_0. Formula (11) now shows that the two-face gap is one.

Finally, if conditions (C2) and (C3) hold but (C4) fails, the previously constructed extra five-face exists, while no extra four-face exists by this proof. Thus the least extra dimension is exactly five. Failure of (C2), or of (C3) once (C2) holds, gives least extra dimension four. When all conditions hold there are no extra faces. Applying (12) and its support calculation in Section 6.4 gives the three stated tight-index sets. The next subsection treats two-face equality with multiple stars. □

6.6 The two-face bound with several stars

We now classify equality in the two-face bound for every ordinary poset R containing X. As in Section 6.5, the only marks are the adjoined extrema of values $0, 1$.

Switching the elements in the order of a linear extension switches each element once, admissibly, and every admissible switch occurs in some linear extension: switch the elements of I first, then q, then the rest. Every switch defect is coefficientwise nonnegative by Theorem 1.1 (equivalently, (4) with the relative transversality of Section 3), and a switch at a non-star has zero defect at every admissible ideal, by [6, Theorem 4.7] and the fibre identity of Section 6.1.

We say that a star q has the *core shape* if there are chains $B_0(q), T_0(q)$, a nonempty chain $S(q)$ from an element $c(q)$ to q, and incomparable pairs $d_1(q), d_2(q)$ and $u_1(q), u_2(q)$ with

$$\downarrow q = B_0(q) \oplus \{d_1(q), d_2(q)\} \oplus S(q), \qquad \uparrow c(q) = S(q) \oplus \{u_1(q), u_2(q)\} \oplus T_0(q).$$

This is condition (C2) of Theorem 6.1 at q. Every element of $K(q) = \downarrow q \cup \uparrow c(q)$ is comparable with q. Let $\mathcal{A}_+(q)$ and $\mathcal{A}_-(q)$ consist of the elements outside $K(q)$ that lie above both $d_i(q)$,

respectively below both $u_i(q)$. With several stars these sets may meet; an element of $\mathcal{A}_+(q) \cap \mathcal{A}_-(q)$ is *sandwiched* for q. The star q satisfies the *attachment condition* if every element outside $K(q)$ lies in $\mathcal{A}_+(q) \cup \mathcal{A}_-(q)$; this is condition (C3) of Theorem 6.1 at q. Two stars with the core shape are *twins* if they have the same pair $\{d_1, d_2\}$ and the same pair $\{u_1, u_2\}$. For a star q with the core shape, put $\alpha = x_{d_1(q)}$, $\beta = x_{d_2(q)}$, $\gamma = y_{u_1(q)}$ and $\eta = y_{u_2(q)}$. The two walls are $H_S = \{\alpha = \beta\}$ and $H_T = \{\gamma = \eta\}$, and $E_0 = E_0(q)$ is the four-face of the late switch of q obtained by setting all other chain coordinates to zero and all $T_0(q)$ coordinates to one.

Lemma 6.5. *Let q be a star of type $(2,2)$, and consider an admissible switch of q with chain ideal I and deleted base $\mathcal{B}$.*

(1) Its defect equals $(1+t)^2 \sum_E t^{\dim E - 2}$, summed over the transverse faces E of $\mathcal{B}$. Every transverse face has dimension at least four, and the degree-two coefficient of the defect is the number of transverse four-faces.

(2) Suppose that q has the core shape. Let E be transverse, let $x \in \operatorname{relint}(E) \cap H_S \cap H_T$, and let v be the inverse order vector of x. Then $D = v_{d_1} = v_{d_2}$ and $G = v_{u_1} = v_{u_2}$ satisfy $0 < D < G < 1$, and $\dim E \geq 2 + m$, where m is the number of distinct values of v in $(0,1)$. In particular, if $\dim E = 4$, every value of v lies in $\{0, D, G, 1\}$.

Proof. Since I and $I \cup \{q\}$ are ideals, every element strictly below q is a chain coordinate and every element strictly above q is an order coordinate. Hence the lower envelope is the maximum of the sums on the two maximal chains below q, and the upper envelope is $\min(y_{u_1}, y_{u_2})$, as for the late switch in Section 6.1. Each affinity subdivision is one hyperplane cut, and the carrier count leading to (11) applies to this single switch; uniqueness of the star was used there only to identify one switch with D_R. Take a linear extension containing this switch. Its switch defects are coefficientwise nonnegative and sum to D_R, which vanishes in degrees zero and one by the vertex and edge equalities quoted after (11). So this defect vanishes in those degrees, and there are no transverse faces of dimension two or three.

For part (2), the two d_i have the same predecessor maximum, the top of $B_0(q)$ or zero, so $\alpha - \beta = v_{d_1} - v_{d_2}$. Also $\gamma - \eta = v_{u_1} - v_{u_2}$, since u_1, u_2 are order coordinates. These identities hold at an arbitrary admissible ideal. Hence every order-level deformation of Lemma 6.3 preserves both walls. By relative transversality the intersection of $\operatorname{relint}(E)$ with both walls has dimension $\dim E - 2$, and it contains the m-dimensional patch of that lemma. The bounds $0 < D < G < 1$ follow as in the proof of Theorem 6.4. The rows used there are the nonnegativities of α, β, the endpoint rows from $B_0(q) \cup \{d_i\}$ to u_j, and the upper bounds of y_{u_j}, and these are rows of $\mathcal{B}$ at every admissible ideal. □

Lemma 6.6. *Let q be a star with the core shape and the attachment condition. Then E_0 is the only transverse four-face of the late switch of q, even when $\mathcal{A}_+(q)$ and $\mathcal{A}_-(q)$ meet.*

Proof. Write d_i, u_i, B_0, S, T_0 for the data of q. The order coordinates are u_1, u_2 and those of T_0. As in the sufficiency proof of Theorem 6.1, the nonnegativities other than at d_1, d_2, the comparisons within T_0 and their upper bound 1 are tight on E_0. A tight nonnegativity at a d_i, a tight comparison from a u_i to T_0 or to $\hat{1}$, or a tight order lower bound destroys a crossing. Consider a nonempty chain row; its lower endpoint is the adjoined minimum, since I is an ideal. Let p be the top element of its chain. The chain coordinates in $K(q)$ are those of $\downarrow q \setminus \{q\}$.

If p is below both u_i, which holds when $p \in \downarrow q \setminus \{q\}$ or $p \in \mathcal{A}_-(q)$, then the whole chain is below both u_i. Tightness forces a sign of $\gamma - \eta$ at the endpoint u_j, and $\gamma = \eta$ at an endpoint in T_0 or at $\hat{1}$.

Otherwise p is outside $K(q)$ and lies in $\mathcal{A}_+(q)$. If the chain contains no element of $\mathcal{A}_-(q)$, it contains no element of S (else p would be above c) and at most one d_i, and its outside elements lie in $\mathcal{A}_+(q)$. Swapping or inserting a d_i forces a sign of $\alpha - \beta$ or $\alpha = \beta = 0$.

If the chain contains an element of $\mathcal{A}_-(q)$, let w be the last one. The elements after w lie above an outside element, hence outside $K(q)$; they are not in $\mathcal{A}_-(q)$, so they lie in $\mathcal{A}_+(q)$, and p is one of them. The argument in the proof of Theorem 6.4 that a tight chain using both attachment types supplies a third level applies verbatim to this w. It gives $0 < D < v_w < G < 1$ at a crossing point, so $\dim E \geq 5$ by Lemma 6.5. This holds also when w is sandwiched.

Hence every row tight on a transverse four-face is tight on E_0. The face contains E_0 and therefore equals it. □

Lemma 6.7. *Let q be a star with the core shape and the attachment condition, and consider an admissible switch of q with order part O. The switch has a transverse four-face if and only if no element of $O \setminus {\uparrow}q$ is sandwiched for q.*

Proof. Suppose that $z \in O \setminus {\uparrow}q$ is sandwiched, and let x be a crossing point of a transverse four-face E. The endpoint rows from $B_0 \cup \{d_i\}$ to z and the comparisons $z < u_j$ give $D \leq y_z \leq G$, so $y_z \in \{D, G\}$ by Lemma 6.5. If $y_z = D$, both endpoint rows from $B_0 \cup \{d_i\}$ to z are tight at x, hence on E, and subtracting them gives $\alpha = \beta$ on E. If $y_z = G$, both comparisons $y_z \leq y_{u_j}$ are tight on E, giving $\gamma = \eta$ there. Either way E is not transverse.

Conversely, suppose that no element of $O' = O \setminus {\uparrow}q$ is sandwiched. An element of O' is incomparable with q, since ${\downarrow}q \setminus \{q\} \subseteq I$, and so lies outside $K(q)$. By the attachment condition it has exactly one of six types: P_j, above both d_i and below u_j only ($j = 1, 2$); P_0, above both d_i and below no u_i; M_i, below both u_i and above d_i only ($i = 1, 2$); M_0, below both u_i and above no d_i. Write $\alpha_1 = \alpha$ and $\alpha_2 = \beta$. For parameters near $(\alpha, \beta, \gamma, \eta) = (1/4, 1/4, 3/4, 3/4)$, put $x_{d_1} = \alpha$, $x_{d_2} = \beta$, all other chain coordinates zero, $y_{u_1} = \gamma$, $y_{u_2} = \eta$, all T_0 coordinates one, and give $z \in O'$ the value $\gamma, \eta, 1, \alpha, \beta, 0$ according as its type is $P_1, P_2, P_0, M_1, M_2, M_0$.

If $z < z'$ in O', the d_i below z lie below z' and the u_i above z' lie above z. Hence, with the type of z written first, none of the following pairs of types occurs: the types P_1 and P_2, in either order; the types M_1 and M_2, in either order; a type P_j or P_0 followed by a type M_i or M_0; the type P_0 followed by a type P_j; and a type M_i followed by the type M_0. So the assigned values weakly increase. No element of O' lies above a u_i, since it would be above q, or below a d_i. A nonempty chain row starts at the adjoined minimum and has sum 0, α or β. If its sum is α_i, it contains d_i, so its endpoint lies above d_i. Either that endpoint has type M_i, and then its value is α_i, or its value is one of γ, η and 1. Thus every defining row holds, and none has slack $\pm(\alpha - \beta)$ or $\pm(\gamma - \eta)$. Every other slack is identically zero or a difference of two of the numbers $0, \alpha, \beta, \gamma, \eta, 1$, which is positive near the base point, since $0 < \alpha, \beta < \gamma, \eta < 1$ there.

The following supporting rows are tight on this patch: the nonnegativities of all chain coordinates other than d_1, d_2; the upper bounds of the T_0 and P_0 coordinates; the comparisons $z < u_j$ for z of type P_j; the endpoint rows from $B_0 \cup \{d_i\}$ to z for z of type M_i; and the lower bounds of the M_0 coordinates. They fix every coordinate other than $\alpha, \beta, \gamma, \eta$ as an affine function of these four, so the face that they cut out has dimension at most four. It contains the open four-parameter patch, on which every row keeps its tight or strict status. So it is a four-face, and the crossing point lies in its relative interior, where both walls vary independently. The face is transverse. □

Lemma 6.8. *Let q be a star with the core shape, and let z be sandwiched for q. Following unique upper covers from z reaches a star $z^* \geq z$ that is sandwiched for q. If z^* also has the core shape, then z^* and q are twins.*

Proof. Let w be strictly below both u_i with a unique upper cover w'. Then both u_i lie in $\uparrow w'$, and w' is neither u_1 nor u_2, since u_1 and u_2 are incomparable. So the walk stays strictly below both u_i and stops at some z^* with $r(z^*) \geq 2$. As z^* is above the incomparable d_1, d_2, also $d(z^*) \geq 2$, so z^* is a star. It is not in $\downarrow q$, since z is not. It is not above q, since it would then be at or above some u_i. Thus z^* is incomparable with q, lies outside $K(q)$, and is sandwiched for q. Under the core shape of z^*, the only incomparable pair in $\downarrow z^*$ is $\{d_1(z^*), d_2(z^*)\}$ and the only one in $\uparrow z^*$ is $\{u_1(z^*), u_2(z^*)\}$. The pairs $\{d_1, d_2\}$ and $\{u_1, u_2\}$ of q are such pairs. Hence z^* and q are twins. □

Theorem 6.9. *Let R be an ordinary poset containing X. Then $D_2 = 1$ if and only if there are incomparable d_1, d_2 and incomparable u_1, u_2 such that the following hold.*

(1) Every star q has the core shape with $d_i(q) = d_i$ and $u_i(q) = u_i$.

(2) Every element outside $K = \bigcup_q K(q)$ is above both d_i or below both u_i.

In that case $B_0 = \downarrow d_1 \setminus \{d_1\}$ and $T_0 = \uparrow u_1 \setminus \{u_1\}$ are the same for all stars. The stars $q_1, \ldots, q_m$ have type $(2,2)$ and are pairwise incomparable, and the facet gap is m. The chains $S(q_j)$ are pairwise disjoint and pairwise incomparable, so that

$$K = B_0 \oplus \{d_1, d_2\} \oplus \big(S(q_1) + \cdots + S(q_m)\big) \oplus \{u_1, u_2\} \oplus T_0,$$

where $+$ denotes disjoint union. No element outside K is sandwiched. For $m = 1$, conditions (1) and (2) are conditions (C2) and (C3) of Theorem 6.1.

Proof. Suppose that $D_2 = 1$. If the facet gap is one, (10) gives a unique star of type $(2,2)$. Otherwise the facet-gap-at-least-two argument of Section 6.4 shows that all stars have type $(2,2)$ and are pairwise incomparable. Fix a star q and a linear extension that switches q late. By nonnegativity and Lemma 6.5, the late switch of q has at most one transverse four-face, and E_0 is one. The exclusive-part argument in the necessity proof of Theorem 6.1 therefore gives $\downarrow q = B_0(q) \oplus \{d_1(q), d_2(q)\} \oplus S(q)$.

For the upper part, pass to R^{op}, which has the same D_2. There $c(q)$ has the two upper covers $d_1(q), d_2(q)$ and at least two maximal lower chains, through $q, u_1(q)$ and through $q, u_2(q)$. So it is a star of R^{op}, of type $(2,2)$ by the first step applied to R^{op}. The lower exclusive-part argument at $c(q)$ in R^{op} gives $\uparrow c(q) = S(q) \oplus \{u_1(q), u_2(q)\} \oplus T_0(q)$. Thus q has the core shape. The attachment construction in the necessity proof of Theorem 6.1 uses only the core of q and its late switch, so it gives the attachment condition at q.

Now let q, q' be stars that are not twins. Switch the elements of $R \setminus (\uparrow q \cup \uparrow q')$ first, then q, then the rest of $R \setminus \uparrow q'$, then q' late, then the rest. These switches are admissible, since no predecessor of q lies above q'. At the switch of q, every unswitched element not above q lies above q'. If one of them were sandwiched for q, Lemma 6.8 would give a twin z^* of q above q'; as stars are pairwise incomparable, $z^* = q'$, a contradiction. So Lemma 6.7 gives a transverse four-face at the switch of q, and the late switch of q' has $E_0(q')$. By nonnegativity, $D_2 \geq 2$. Hence all stars are twins, with common d_i and u_i. Then $B_0(q) = \downarrow d_1 \setminus \{d_1\}$ and $T_0(q) = \uparrow u_1 \setminus \{u_1\}$ do not depend on q, which gives condition 1. Condition 2 is the attachment condition at any star.

Conversely, assume conditions (1) and (2). If $q < q'$ for stars, then $q' \in \uparrow q \setminus \{q\} = \{u_1, u_2\} \oplus T_0$, while condition 1 puts q' strictly below both u_i. So the stars are pairwise incomparable, and they have type $(2,2)$. For a star q, an element outside $K(q)$ lies in some $S(q')$ with $q' \neq q$, where it is sandwiched, or outside K. So q satisfies the attachment condition. Fix a star p and a linear extension that switches p late. Switches at non-stars contribute zero. The late switch of p contributes exactly one in degree two, by Lemmas 6.5 and 6.6. Every other star q is switched

while p is an order coordinate. Since p is sandwiched for q and is not above q, Lemma 6.7 gives no transverse four-face there. Hence $D_2 = 1$.

For the structure, the elements of $S(q)$ strictly below q have a unique upper cover by condition 1. Let $s \in S(q)$ and $s' \in S(q')$ with $s \leq s'$. Then $s \in {\downarrow}q'$ lies above both d_i, hence in $S(q')$. The unique upward path from s reaches its first branching element, which is both q and q'. A sandwiched element outside K would lie below a star by Lemma 6.8, hence in K. The facet gap is m by (10). □

Corollary 6.10. *Let R be an ordinary n-element poset containing X, and let $\mathcal{I}(R)$ be its set of tight indices, as in Theorem 6.4. If the facet gap is one, $\mathcal{I}(R)$ is $\{2, \ldots, n-1\}$, $\{2, n-1\}$ or $\{n-1\}$, as described there. If the facet gap is at least two, then $\mathcal{I}(R) = \{2\}$ when the conditions of Theorem 6.9 hold, and $\mathcal{I}(R) = \varnothing$ otherwise.*

Proof. A facet gap of at least two gives strict inequality in every degree $3, \ldots, n-1$ (Section 6.4). The degree-two case is Theorem 6.9. □

Example 6.11. Write A_r for an r-element antichain. In $A_2 \oplus A_m \oplus A_2$ with $m \geq 2$, the middle elements are twin stars, so $D_2 = 1$ while the facet gap is m. For $m = 2$, the switch of the first middle element after the bottom layer contributes $t^3(1+t)^2$, and the late switch of the second contributes $t^2(1+t)^3$. In contrast, take the poset given by $a, b < p < g, h$ and $c, d < p' < g, h$. The stars p, p' are not twins, so $D_2 \geq 2$. Indeed, at the switch of p' with $I = \{a, b, c, d\}$, the face with $y_p = x_a = x_b = 0$ and with x_c, x_d, y_g, y_h free is a transverse four-face. ◇

6.7 A fixed-facet refinement in the two-face equality class

For an ordinary n-element poset R, write $D_R(t) = F_{\mathcal{C}(R)}(t) - F_{\mathcal{O}(R)}(t)$ as above. We refine the sharp bound within the class of posets with $f_2(\mathcal{C}(R)) - f_2(\mathcal{O}(R)) = 1$, classified in Theorem 6.9.

Theorem 6.12. *Suppose R contains X and $f_2(\mathcal{C}(R)) - f_2(\mathcal{O}(R)) = 1$. Let $m = f_{n-1}(\mathcal{C}(R)) - f_{n-1}(\mathcal{O}(R))$. Then $1 \leq m \leq n-4$ and, coefficientwise,*

$$D_R(t) \;\geq\; B_{n,m}(t) := t^2(1+t)^{n-m-2}\big((1+t)^m - t^m\big). \tag{14}$$

Use the common pairs d_1, d_2 and u_1, u_2, stars $q_1, \ldots, q_m$, and the union K of the cores from Theorem 6.9. Put

$$\mathcal{A}_- = \{z \in R \setminus K : z < u_1, u_2\}, \qquad \mathcal{A}_+ = \{w \in R \setminus K : d_1, d_2 < w\}.$$

Equality holds in (14) if and only if there are no $z \in \mathcal{A}_-$ and $w \in \mathcal{A}_+$ with $z < w$. For every n, m in the stated range the bound is attained.

We call the equality condition of Theorem 6.12, that no element of $\mathcal{A}_-$ lies below an element of $\mathcal{A}_+$, the *outside condition*.

Proof. The classification gives m pairwise incomparable twin stars, each of type $(2, 2)$, with common chains B_0, T_0 and nonempty, pairwise incomparable middle chains $S(q_j)$. The latter end at their respective stars. In particular there are at least $m + 4$ elements. Every element outside K belongs to exactly one of $\mathcal{A}_-, \mathcal{A}_+$, since no outside element lies above both d_i and below both u_i. An outside element is incomparable with every middle-chain element: lying below such an element puts it in a principal down-set of a star, and lying above one puts it in the upper set of the first element of that middle chain. Either would put it in K.

A common switching order. Switch all elements of

$$I_0 = R \setminus \big(\{q_1, \dots, q_m, u_1, u_2\} \cup T_0\big)$$

first, in a compatible order, then $q_1, \dots, q_m$, and finally u_1, u_2, T_0 in a compatible order. The displayed complement is an up-set: every strict successor of any star lies in $\{u_1, u_2\} \cup T_0$. Thus I_0 and all the intermediate chain parts are ideals. Non-star switches have zero defect.

At the switch of q_j, let $a = m - j$ be the number of other stars not yet switched. The remaining order coordinates in its deleted base are u_1, u_2, those of T_0 and those of these a stars. Write $z_1, \dots, z_a$ for the coordinates of these stars, and put $\alpha = x_{d_1}$, $\beta = x_{d_2}$, $\gamma = y_{u_1}$, $\eta = y_{u_2}$. The two maximal chains below q_j share B_0 and its proper middle prefix, and differ only at d_1, d_2. Their sums therefore differ by $\alpha - \beta$ on the whole base. The upper envelope is $\min(\gamma, \eta)$. By Lemma 6.5, this switch has defect

$$(1+t)^2 \sum_{F \text{ transverse}} t^{\dim F - 2}, \tag{15}$$

with two global cuts $\alpha = \beta$ and $\gamma = \eta$.

The active-row certificate. In this base, set every chain coordinate other than d_1, d_2 to zero, and every coordinate of T_0 to one. These are supporting conditions and define a face E_j. The remaining effective inequalities are exactly

$$0 \le \alpha, \beta \le \gamma, \eta \le 1, \qquad \alpha, \beta \le z_s \le \gamma, \eta \quad (1 \le s \le a).$$

Indeed, each chain sum is 0, α, or β; an unswitched-star endpoint is above both d_i; and the only other order endpoints are u_1, u_2, T_0. The stars have no mutual comparisons, and outside elements are incomparable with them. All displayed inequalities are strict at

$$\alpha = \beta = \tfrac{1}{4}, \qquad z_s = \tfrac{1}{2}, \qquad \gamma = \eta = \tfrac{3}{4}.$$

Thus $\dim E_j = 4 + a$ and the two walls cross there.

Let $T_0 = \{t_1 < \cdots < t_h\}$. At this witness every nonempty chain row and every order comparison involving a free order coordinate is strict. An irredundant active list consists precisely of the nonnegativities of the other chain coordinates and

$$y_{t_1} \le \cdots \le y_{t_h} \le 1.$$

The latter list is empty if $h = 0$. The corresponding slacks are the other chain coordinates, $y_{t_{i+1}} - y_{t_i}$, and $1 - y_{t_h}$. There are $n - a - h - 5$ of the first kind and h of the second kind, hence $n - a - 5$ independent slack coordinates. Together with the free coordinates they form an affine coordinate system. All other rows are locally strict or redundant consequences of these active rows. The normal cone is therefore simplicial, and the coface interval of E_j is Boolean of rank $n - a - 5$.

Every coface F of E_j is transverse. For the linear map $\pi = (\alpha - \beta, \gamma - \eta)$, the image $\pi(E_j)$ contains a neighbourhood of zero in $\mathbb{R}^2$. So does $\pi(F)$, and $\pi(\operatorname{relint} F) = \operatorname{relint} \pi(F)$ gives a crossing in $\operatorname{relint} F$. The map has rank two on $\operatorname{dir}(F)$. Consequently the cofaces of E_j contribute exactly

$$(1+t)^2 t^{a+2} (1+t)^{n-a-5} = t^{a+2}(1+t)^{n-a-3}$$

to (15). Other transverse faces only add nonnegative terms. Summing over $a = 0, \dots, m-1$ gives

$$t^2(1+t)^{n-m-2} \sum_{a=0}^{m-1} t^a (1+t)^{m-1-a} = t^2(1+t)^{n-m-2}\big((1+t)^m - t^m\big),$$

which proves the lower bound.

Sufficiency of the outside condition. Suppose no element of $\mathcal{A}_-$ lies below an element of $\mathcal{A}_+$. Fix a transverse face F at step j, and choose a crossing point in its relative interior. A valid affine inequality tight at this point is tight on all of F. In particular, if a tight row forces one sign of either wall difference, that difference vanishes on F, contradicting transversality.

No order element lies below a chain element of the ideal, so a nonempty chain row starts at the adjoined minimum. We exhaust its possible upper endpoints. If it ends at an unswitched star, its vertices lie in B_0, at most one d_i, and that star's proper middle prefix. When d_i occurs, swapping it for the other one gives a valid row with the same endpoint and common summands. Tightness forces one lower-wall sign. When neither occurs, either can be inserted; tightness then forces $\alpha = \beta = 0$. The proper middle prefix of that star consists of chain coordinates. It contributes the same summands to both rows, so it does not affect these comparisons.

Now suppose the upper endpoint is u_i, an element of T_0, or the adjoined maximum. If the top chain element is below both u_i, the entire sum is bounded by both γ and η. Tightness forces an upper-wall sign at a u_i endpoint, or equality at a T_0 endpoint or at $\hat{1}$. Otherwise the top is an outside $\mathcal{A}_+$ element. No $\mathcal{A}_-$ element can precede it by assumption, and no middle-chain element can precede it because it would then lie in a core. Thus the chain uses only B_0, at most one d_i, and $\mathcal{A}_+$ elements. Swapping or inserting d_i again excludes tightness. These are all-chain rows, so the replacement chains need not be saturated.

For empty-interior rows, a comparison from an unswitched star to u_i, to T_0 or to $\hat{1}$ forces an upper-wall sign or equality, since the star is below both u_i. The same applies to comparisons from u_i to T_0 or to $\hat{1}$. Distinct stars are incomparable. A tight nonnegativity at a d_i excludes a lower crossing. At a crossing the d_i coordinates are positive, so the lower bound zero of any remaining order coordinate is also strict: all are above the lower pair.

The only remaining rows are the chain nonnegativities away from d_1, d_2 and the comparisons and upper bound within T_0. They are all tight on E_j. Therefore every defining row tight on F is tight on E_j, so F contains E_j and is a coface of E_j. The transverse faces are exactly those cofaces, proving equality at every step.

Necessity of the outside condition. Suppose $z \in \mathcal{A}_-$, $w \in \mathcal{A}_+$ and $z < w$. Use the last star switch, which is a late switch. Every other star, every proper middle prefix, and every outside element now lies in the chain part. Since z is not above both d_i, relabel the lower pair so any $d_i < z$ is d_1. Since w is not below both u_i, relabel the upper pair so $w \not< u_2$. Put $\varepsilon = \mathbf{1}_{d_1 < z}$ and set $R_w = \gamma$ if $w < u_1$, and $R_w = 1$ otherwise. Keep x_z, x_w, α, β, γ and η free, set the other chain coordinates to zero and the coordinates of T_0 to one, and make the valid endpoint row

$$x_z + x_w + \varepsilon\alpha = R_w$$

tight. This is the extra-five-face construction in the necessity proof of Theorem 6.1. Its strict witness is

$$\alpha = \beta = \tfrac{1}{20}, \qquad x_z = \tfrac{1}{5}, \qquad \gamma = \eta = \tfrac{3}{4}, \qquad x_w = R_w - \tfrac{1}{5} - \tfrac{\varepsilon}{20} > 0.$$

The other effective bounds involving these outside coordinates are

$$x_z + \varepsilon\alpha \le \gamma, \eta, \qquad x_w + \alpha \le R_w, \qquad x_w + \beta \le R_w,$$

or weaker rows, and all are strict at the witness. Neither outside element is comparable with any middle branch, so no chain can combine one of them with an additional twin branch. Those zeroed branches supply only the usual core bounds, which are strict as well. Thus there is exactly one independent equation in six free coordinates. Eliminating x_w leaves the four core coordinates free near the witness, so this is a transverse five-face.

Its mixed endpoint row is strict at the usual interior point of E_m, where $x_z = x_w = 0$. Hence it does not contain E_m and is not one of the cofaces counted in the lower bound $B_{n,m}$. It contributes an additional $t^3(1+t)^2$ to the last switch. All other carrier and switch contributions are nonnegative, so equality is impossible.

Finally, for any $1 \le m \le n-4$, take the ordinal sum of a chain with $n-m-4$ elements and $A_2 \oplus A_m \oplus A_2$. This poset has exactly the m middle stars, satisfies the conditions of Theorem 6.9, and has no outside elements. The equality case just proved therefore attains the bound. □

Corollary 6.13. *Under the hypotheses of Theorem 6.12, if the outside condition there fails, then $D_R(t) - B_{n,m}(t)$ has positive coefficients exactly in degrees $3, \dots, n-2$. In particular, equality at any one coefficient in that range is equivalent to equality in the whole bound.*

Proof. At each star switch, subtracting the Boolean coface contribution leaves the sum in (15) over transverse faces that do not contain E_j. The necessity proof supplies such a five-face G at the last switch. Since G does not contain E_m, some facet containing G does not contain E_m. A face flag from G to that facet gives extra faces of every dimension $5, \dots, n-2$. They are all transverse: the image of G under the two wall differences contains a neighbourhood of zero, and the same relative-interior image argument applies to every coface of G. Their contributions give positivity in every degree $3, \dots, n-2$.

There is no extra face of dimension $n-1$, because the whole deleted base contains E_j. Thus the surplus has degree at most $n-2$. Its degree-two coefficient is zero since both D_R and $B_{n,m}$ have that coefficient equal to one; degrees zero and one vanish as well. All remaining contributions are nonnegative. The equality alternative follows from Theorem 6.12. □

7 Comparison, support and equality without marked maxima

In this section we allow unmarked maximal elements. All minimal elements remain marked, and the marking may be real and degenerate. For a polyhedron, F_Q counts its ordinary nonempty faces, including Q itself; no faces at infinity are added. For a finite polyhedral complex we count each nonempty geometric cell once, including all its faces. All dimensions and relative interiors are actual affine dimensions and relative interiors.

7.1 Clipping on an unbounded supporting face

For an unbounded base, the following elementary lemma replaces the comparison of maxima in Section 3.2. It needs neither compactness nor an optimization theorem.

Lemma 7.1 (A split point on a supporting face)**.** *Let B, B^-, B^+ be nonempty polyhedra with $B^- + B^+ \subseteq B$. Suppose a functional c exposes a nonempty face E of B at finite value M, and $x = x^- + x^+ \in E$ with $x^\pm \in B^\pm$. Then $x^\pm$ maximize c on $B^\pm$. Every split $y = y^- + y^+ \in E$ with $y^\pm \in B^\pm$ lies in these same two maximizing faces. In particular, $y^- - x^- \in \operatorname{dir}(E)$.*

Proof. For every $a \in B^-$, inclusion gives $c(a) + c(x^+) \le M = c(x^-) + c(x^+)$, so $c(a) \le c(x^-)$. The other maximum follows in the same way. The two inequalities for a split $y \in E$ must both be equalities because their sum is M. Thus $y^- + x^+$ and $x^- + x^+$ both belong to E, which proves the direction assertion. □

Now let all minima of P be marked, fix $q \in O$, and assume that some nontrivial upper endpoint chain runs from q to another marked or order element and has only chain elements in its interior. Here nontrivial means that the endpoints are distinct: the interior may be empty, so direct covers

are included. We call the set of all such chains the *upper endpoint family* of q. Use the induced deletion and the same all-chain endpoint expressions B, L, U as in Section 3. The lower family is nonempty because all minima are marked. Both families are finite and nonempty, so L, U are finite-valued. Concatenating any lower and upper chain gives a base inequality and proves $L \le U$ throughout B. Chains avoiding q are exactly those of the induced deletion, even when its cover relations differ. Hence the old and new projections are onto B, with fibres $[L, U]$ and $[0, U - L]$ respectively.

The arguments of Sections 3.2 and 3.3 now prove relative transversality over this unbounded base, with two changes. First, the lift goes to the old full polyhedron, through the full partial transfer (5) [7, Theorem 2.1]; as there, the clipping works for arbitrary real marks. Second, Lemma 7.1, applied to the split of x and a functional exposing E, replaces the comparison of maxima in Section 3.2; it gives $a = y^- - x^- \in \operatorname{dir}(E)$. For the whole face $E = B$ one may take the zero functional.

7.2 The epigraph alternative

If the upper endpoint family is empty, the same all-chain argument gives the old polyhedron as $\operatorname{epi}_B(L)$. The new one is exactly $B \times [0, \infty)$: after the switch, any chain inequality involving q would require an upper marked or order endpoint, and the only remaining condition on that coordinate is nonnegativity. The general counts used for these models are the identities in Lemma 2.8.

Theorem 7.2 (Two weak comparison directions)**.** *Let P be finite with all minimal elements marked, let λ be real and order-preserving, and let $P \setminus A = C \sqcup O$. For the switch of $q \in O$:*

(1) If a nontrivial upper endpoint chain from q to another element of $A \cup O$ exists, with all internal elements in C (possibly none), then

$$F_{Q_C(P,\lambda)} \le F_{Q_{C\cup\{q\}}(P,\lambda)}.$$

(2) If no such upper chain exists, then

$$F_{Q_{C\cup\{q\}}(P,\lambda)} - F_{Q_C(P,\lambda)} = F_B - F_{\mathcal{S}} \le 0,$$

where B is the deleted base and $\mathcal{S}$ is the effective affinity subdivision of the lower envelope of q.

Both inequalities are coefficientwise and count ordinary nonempty faces, including the whole polyhedron.

Proof. In the first case the fibres are finite intervals, and the clipping argument above proves relative transversality. Lemma 2.1 gives the inequality of case (1). Lemma 2.8 and Proposition 2.7 prove the second case. □

We say that the switch of q is in the *increasing regime* if an upper endpoint chain as in Theorem 7.2(1) exists, and in the *decreasing regime* otherwise. An upper endpoint exists exactly when some marked or order element lies strictly above q: along a saturated path to one, stop at the first such element. Thus in the decreasing regime q need not be a maximal element of the full poset; its entire strict upper set may already consist of chain elements. In particular, if every element of a nested switch set lies strictly below a marked element, an upper endpoint exists at every intermediate step. Every step is then in the increasing regime, and summing gives the weak increase for that nested pair.

Remark 7.3. These alternatives specify weak comparison directions, not strict increase versus strict decrease. Equality is possible in either regime. This weak-direction theorem alone does not classify equality. Section 7.4 treats single-switch equality (Theorem 7.7) and full-region incidence (Corollary 7.8) by a separate regularization argument.

7.3 Signed support and quotient factors

Corollary 7.4 (Support for a single marked switch)**.** *Let P be finite, let A contain all its minimal elements, let $\lambda : A \to \mathbb{R}$ be order-preserving, and write $P \setminus A = C \sqcup O$. For $q \in O$, put*

$$\Delta_q = F_{Q_{C\cup\{q\}}(P,\lambda)} - F_{Q_C(P,\lambda)}, \qquad N = \dim Q_C(P,\lambda).$$

Both polyhedra have actual dimension N.

(1) If the upper endpoint family of q is nonempty, then $\Delta_q \in (1+t)^2\mathbb{Z}_{\geq 0}[t]$. Either $\Delta_q = 0$, or its positive coefficients occur exactly in degrees $r, \ldots, N-1$ with $0 \leq r \leq N-3$.

(2) If the upper endpoint family is empty, then $-\Delta_q \in (1+t)\mathbb{Z}_{\geq 0}[t]$. Either $\Delta_q = 0$, or its negative coefficients occur exactly in degrees $r, \ldots, N-1$ with $0 \leq r \leq N-2$.

In the first case a nonzero change is strict in the top three proper-face degrees; in the second it is strict in the top two. In particular the first case has zero change for $N \leq 2$, and the second for $N \leq 1$.

Proof. The dimension argument in the proof of Corollary 5.3 uses only convexity, so it applies to these polyhedra: they have equal actual dimensions, even with fixed coordinates or coincident marks.

In the increasing regime the deleted base B has finite envelopes $L \leq U$ and the old and new fibres are respectively $[L, U]$ and $[0, U-L]$. Put $e = \dim B$ and $H = U - L$. If $H \equiv 0$, then $L = U$ is both convex and concave, hence affine on B. The old graph and the new $B \times \{0\}$ are affinely equivalent; here $N = e$ and $\Delta_q = 0$. Otherwise the nonnegative concave function H is strictly positive on $\operatorname{relint}(B)$. Indeed, a zero at a relative-interior point and a short segment extension beyond it would force H to vanish at every point of B. Thus both polyhedra have positive-length fibres over a relative-interior open set and $N = e + 1$. The supporting-face clipping argument of Lemma 7.1 and its subsequent conormal calculation give relative transversality over B, in its actual affine hull. Lemma 2.1 and Theorem 2.4 therefore give either zero change or positive support $r, \ldots, e$ with $0 \leq r \leq e - 2 = N - 3$. Fibres that collapse only over proper base faces are already included in that fibre identity. The same identity and Theorem 2.6 give the nonnegative quotient by $(1+t)^2$.

In the decreasing regime the old polyhedron is $\operatorname{epi}_B(L)$ and the new one is $B \times [0, \infty)$. Every fibre is a ray, so $N = e + 1$ without a zero-width exception. The epigraph identities give $-\Delta_q = F_{\mathcal{S}} - F_B$. Theorem 2.9 gives either zero or positive support $r, \ldots, e$ with $0 \leq r \leq e-1 = N-2$. Proposition 2.7 gives the nonnegative quotient of $-\Delta_q$ by $1 + t$. This proves the second assertion. □

Corollary 7.5 (Nested comparisons with one regime throughout)**.** *Under the same minimum-marked hypotheses, let $C \subseteq C' \subseteq P \setminus A$, and fix an ordering $q_1, \ldots, q_m$ of $C' \setminus C$. Put $C_j = C \cup \{q_1, \ldots, q_j\}$ and let N be the common actual dimension of all $Q_{C_j}(P,\lambda)$.*

(1) If at every step the switched element q_j has a nonempty upper endpoint family for the chain part C_{j-1}, then the total difference $F_{Q_{C'}} - F_{Q_C}$ is zero or has positive support exactly $r, \ldots, N-1$, where $0 \leq r \leq N-3$. The total difference belongs to $(1+t)^2\mathbb{Z}_{\geq 0}[t]$.

(2) If at every step that upper endpoint family is empty, then the total difference is zero or has negative support exactly $r, \dots, N-1$, where $0 \le r \le N-2$. The negative of the total difference belongs to $(1+t)\mathbb{Z}_{\ge 0}[t]$.

Proof. The dimension argument in the proof of Corollary 5.3 applies to the polyhedra of every intermediate partition. Telescope the single-switch differences. In the first case they are all nonnegative and in the second they are all nonpositive. Their nonnegative quotient polynomials add in the respective factorizations from Corollary 7.4. If some are nonzero, each has an interval of strict support ending at the same degree $N-1$. The union is the interval beginning at the least of their starting degrees; changes of one sign cannot cancel. If all changes are zero, including the empty-sequence case, the total is zero. The coefficient in degree N is zero because every whole-polyhedron count is one. □

Remark 7.6. In a sequence containing both types of switch, changes of opposite sign can cancel, as Example 7.9 shows. Also, in the marked setting the starting degree r can be smaller than two. For an example in the decreasing regime, take marked minima m, a with marks $0, 1$, unmarked p, q, and covers $m \lessdot p \lessdot q$ and $a \lessdot q$. The base is a ray in the coordinate x of p, and $L(x) = \max(x, 1)$. The old and new face vectors are respectively $(2, 3, 1)$ and $(1, 2, 1)$, so $\Delta_q = -1 - t$ and $r = 0$. For Theorem 2.9 itself, $B = \mathbb{R}^d$ and $L(x) = \max(0, x_1)$ give $F_{\mathcal{S}} - F_B = t^{d-1} + t^d$ for every $d \ge 1$. Thus its latest possible starting degree is attained even on a base with lineality.

7.4 Single-switch equality without marked maxima

We retain the minimum-marked hypotheses of Theorem 7.2. The common regularization of Section 5.8 is available for these polyhedra by [7, Propositions 6.1–6.2]: contract constant-interval blocks, then successively remove currently redundant covers. Choose one such reduction independently of the partition, and write (R, A_R, λ_R) for its regular output. Let J be the original unmarked elements in contracted blocks and $V = (P \setminus A) \setminus J$ the surviving singleton labels. For $C \subseteq P \setminus A$, put $D = C \cap V$.

The reduced minima are still marked. Indeed, every coordinate of the original marked order polyhedron has a finite lower bound from a marked minimum. The coordinate insertion map of the common reduction retains every coordinate in V identically, so the reduced order polyhedron has those same lower bounds. An unmarked minimal element of R would permit its coordinate to decrease without bound with all other coordinates fixed, a contradiction. Strictness of the reduced marking and partial transfer give common actual dimension $d = |V|$ for all reduced and original mixed polyhedra.

Theorem 7.7 (Single-switch facet formula and equality)**.** *Fix $q \in P \setminus (A \cup C)$ and suppress the fixed marked coordinates. If $q \in J$, switching q is the ambient translation $x \mapsto x - b_q e_q$, where b_q is its contracted block value. In particular all face numbers are unchanged.*

Suppose instead that $q \in V$. Let k count the nontrivial saturated chains in the final poset R ending at q, starting in $A_R \cup (V \setminus (D \cup \{q\}))$, with all internal elements in D. Define ℓ dually for chains starting at q and ending in that same other-endpoint set. Direct covers are included. Then $k \ge 1$, $\ell \ge 0$, $d \ge 1$, and

$$f_{d-1}(Q_{C\cup\{q\}}(P, \lambda)) - f_{d-1}(Q_C(P, \lambda)) = (k-1)(\ell-1).$$

The number k is the effective full-region count of the original lower envelope. The value $\ell = 0$ means that the upper envelope is absent; otherwise ℓ is its effective full-region count. In particular these numbers are independent of the chosen valid common regularization.

For a positive-dimensional single-switch pair, equality of facet numbers, equality of all face numbers, and affine equivalence are equivalent. If $q \in V$, they hold exactly when $k = 1$ or $\ell = 1$. Whenever they hold, there is an ambient map $x \mapsto Mx + b$ with $M \in \mathrm{GL}_{|P\setminus A|}(\mathbb{Z})$ and translation in the additive integer span of the marks. For integral marks this is a lattice equivalence. If the common dimension is zero, the two polyhedra are points that differ by such a translation.

Proof. For each $p \in J$, let b_p be its block value. The common reduction identifies $Q_D(R, \lambda_R)$ with $Q_C(P, \lambda)$ by the insertion map

$$(\Gamma_C y)_p = \begin{cases} y_p, & p \in V, \\ b_p, & p \in J \setminus C, \\ 0, & p \in J \cap C. \end{cases}$$

Its inverse on the image is projection to V; subsequent cover deletion changes no coordinates. If $q \in J$, the reduced chain part is unchanged, and the two insertion maps differ by $-b_q e_q$. This proves the first assertion, including $d = 0$.

Now assume $q \in V$. The insertion maps for the two partitions are identical and preserve the q coordinate. Their projection away from q is an affine isomorphism Γ_B between the actual reduced and original bases. In particular the entire q-fibres agree. If an upper endpoint family is nonempty, its finite collection of affine forms gives a finite upper bound at each base point. If the family is empty, the fibre is unbounded above. Equality of the fibres therefore preserves presence or absence of the upper envelope. It also gives

$$L_{\text{original}} \circ \Gamma_B = L_{\text{reduced}}, \qquad U_{\text{original}} \circ \Gamma_B = U_{\text{reduced}}$$

when the second function exists.

All minima of R are marked, so at least one lower endpoint chain exists and $k \geq 1$. An upper chain need not exist. Theorem 5.7 applies to this regular minimum-marked poset, for every partition. Thus its canonical rows, with fixed equations and constant marked-cover rows excluded, are distinct facets of a full-dimensional polyhedron. We use the canonical chain-row count of [7, Proposition 7.11]; the case of an empty upper family is included below. Let v count rows not starting or ending at q. The old facet count is $v + k + \ell$.

On switching q, these $k + \ell$ endpoint rows disappear. Every new endpoint chain involving q has it in its interior and splits uniquely into a lower and upper old chain. Conversely each such pair concatenates to a saturated chain in the same final R. Hence $k\ell$ chain rows and one nonnegativity row appear, giving $v + 1 + k\ell$ new facets. If $\ell = 0$, there are no concatenated rows and this count is $v + 1$, and the facet change is $1 - k$, which may be negative. Subtraction proves the formula in both cases. Affine insertion transfers the counts to the original pair.

A lower endpoint row has nonzero q coefficient, so its facet is a graph over a full-dimensional lower affinity region of the base. Conversely, the graph of an effective full lower region is a supporting face of dimension $d - 1$, hence a facet. A supporting canonical row cannot be vertical, since the projection is full-dimensional. Nor can it be an upper endpoint row: if upper rows exist, that would force $U = L$ on a full cell, and the concave nonnegative function $U - L$ would vanish everywhere, contrary to full dimension. If the upper family is empty, there is no upper row to consider. Thus the supporting row is lower, as in Lemma 5.6. Distinct canonical facets give distinct regions. This identifies k with the lower-region count; the upper argument applies when $\ell > 0$. The identities through Γ_B transfer these counts to the original envelopes and prove their independence of the reduction.

Since $k \geq 1$ and $\ell \geq 0$, the facet difference vanishes exactly when $k = 1$ or $\ell = 1$. In the first case the lower envelope is affine on the whole actual base, and $(u, z) \mapsto (u, z - L(u))$ is an affine

equivalence, whether the upper fibres are finite or infinite. In the second case the upper envelope is finite and affine, and the map is $(u, z) \mapsto (u, U(u) - z)$. These maps are in the original coordinates, since the envelopes were identified through Γ_B.

An envelope that is affine on the actual base agrees there with one of its original endpoint forms. To verify this, choose an active form at a relative-interior point; its nonnegative affine deficit vanishes at that point and therefore on the whole affine hull. Such an endpoint form has integer coefficients on the other unmarked coordinates and its constant term is an original mark or zero. Each of these maps thus extends to all $|P \setminus A|$ unmarked coordinates by changing only row q of the identity, with q coefficient 1 or -1. Its determinant is correspondingly 1 or -1, and its translation belongs to the stated mark span. The fixed-coordinate translation has the same property. Affine equivalence preserves all faces and hence facets, which completes the equivalences. □

Corollary 7.8 (Full-region incidence)**.** *In the setting of Theorem 7.7, suppose that $q \in V$ and $\ell > 0$. Every effective full lower affinity region meets every effective full upper affinity region in full dimension in the common actual base. There are exactly $k\ell$ full cells in their common refinement.*

Proof. The reduced old polyhedron is full-dimensional in $\mathbb{R}^d$. Consequently its base has dimension $e = d - 1$, and some fibre has positive width. Let s be the number of full refinement cells. The degree-e coefficient of (3) is the facet difference $1 + s - k - \ell$. The canonical count in Theorem 7.7 identifies this with $(k - 1)(\ell - 1)$, so $s = k\ell$. A pair of full convex affinity regions has at most one full-dimensional intersection, and every full refinement cell is such an intersection. There are only $k\ell$ pairs, so every pair occurs. The base isomorphism transfers this conclusion to the original envelopes. □

The argument is not circular: the proof of Theorem 5.7 uses no result of this section, only earlier results applied to bounded polytopes, results of [23] and the added marked maximum b_∞; the results of this subsection then rest on that theorem.

Example 7.9 (Nested facet changes can cancel)**.** Let P have one marked minimum m of value zero and eight unmarked elements. Specify the unmarked relations by

$$a, b < q < r, s, \qquad c, d < t,$$

where a, b are incomparable, r, s are incomparable, and c, d are incomparable. Put $m < a, b, c, d$, with no additional relations except transitive consequences. The full marked poset is connected. Its free-coordinate polyhedron is a product of the two displayed components because their only common coordinate is the fixed mark $m = 0$.

Start with $C = \varnothing$ and compare it with $C' = \{q, t\}$. In the first component the old cone is

$$Q_1 = \{(a, b, q, r, s) : \ a, b \geq 0; \ q \geq a, b; \ r, s \geq q\}.$$

Subtracting q from r, s identifies Q_1 with $K \times \mathbb{R}^2_{\geq 0}$, where

$$K = \{(a, b, q) : 0 \leq a, b \leq q\}$$

is the three-dimensional cone over a square. It has four facets and four rays, so Q_1 has six facets and six rays.

After switching q, write its new coordinate as w. The inequalities are $a, b, w \geq 0$ and $a + w, b + w \leq r, s$. Subtracting w from r, s identifies this cone with $K' \times \mathbb{R}_{\geq 0}$, where

$$K' = \{(a, b, r, s) : \ a, b \geq 0; \ r \geq a, b; \ s \geq a, b\}.$$

The six displayed inequalities of K' are distinct facets. For example, $(0,1,2,2)$ makes only $a \geq 0$ tight, and $(2,1,2,3)$ makes only $r \geq a$ tight; the symmetries $a \leftrightarrow b$ and $r \leftrightarrow s$ handle the rest. Its five extreme rays, in the displayed coordinate order, are

$$(0,0,1,0), \quad (0,0,0,1), \quad (1,0,1,1), \quad (0,1,1,1), \quad (1,1,1,1).$$

They generate K': subtract $\min(a,b)$ times the last ray, then the excess of a or b using the corresponding ray, and finally the nonnegative excesses of r and s using the first two rays. Each listed ray is extreme, since its tight inequalities have rank three. Thus the switched first component has seven facets and six rays.

The second component before switching is $\{(c,d,t) : 0 \leq c,d \leq t\}$, another square cone, with four facets and four rays. Switching t, which has no upper endpoint chain, makes it the three-dimensional nonnegative orthant, with three facets and three rays. Products of pointed cones add facet counts and extreme-ray counts. Therefore the original and final eight-dimensional polyhedra both have ten facets, but their ray counts are respectively ten and nine. They cannot have the same face numbers or be affinely equivalent.

Here the switch in the increasing regime adds one facet and the switch in the decreasing regime removes one. The example shows that the single-switch equality theorem (Theorem 7.7) has no unrestricted nested analogue. The same two cones also arise from the admissible nested chain parts $C_0 = \{a,b,c,d\}$ and $C_0' = C_0 \cup \{q,t\}$: each of those four bottom coordinates has lower envelope zero, so switching it is the identity. Thus the cancellation persists even along admissible partitions of this connected marked poset. $\diamond$

For a nested sequence retaining an upper endpoint at every step, suppose first that the common actual dimension is positive. Theorem 7.2 makes all facet increments nonnegative. A zero total increment forces each increment to vanish, and the ambient maps from Theorem 7.7 compose. If the common dimension is zero, compose the fixed-coordinate translations instead. Thus the usual nested equality conclusion remains valid for such a sequence. This restricted conclusion does not apply to Example 7.9.

8 Examples and the scope of the comparison

Example 8.1 (Marked vertex numbers can increase)**.** Let marked elements $m_0 < m_1 < m_2 < m_3$ have values $0,1,2,3$. Add unmarked p,q,r with

$$m_0 < p < q < r < m_3, \qquad m_1 < q < m_2,$$

and no additional comparabilities beyond their consequences. Initially take $C = \{p\}$ and switch q. The old polytope is

$$0 \leq p \leq q, \qquad 1 \leq q \leq 2, \qquad q \leq r \leq 3.$$

It is simple with eight vertices, twelve edges and six facets. The new polytope, with $w = q - \max(p,1)$ and coordinates (p,w,r), has the seven facet inequalities

$$p, w \geq 0, \quad r \leq 3, \quad w + p \leq r, \quad w + p \leq 2, \quad w \leq r - 1, \quad w \leq 1.$$

Its vertices are

$$(0,0,1), (1,0,1), (2,0,2), (2,0,3), (0,0,3),$$
$$(0,1,3), (1,1,3), (0,1,2), (1,1,2).$$

The facets are five quadrilaterals, one pentagon and one triangle, so it has fourteen edges and seven facets. Thus the two full face vectors are $(8, 12, 6, 1)$ and $(9, 14, 7, 1)$.

The transfer does not map vertices to vertices: the old vertex $(1, 1, 3)$ goes to $(1, 0, 3)$, the midpoint of a new edge. The proof of monotonicity compares subdivision cells, not images of individual vertices. ◇

Example 8.2 (Merged marks and a nonadmissible switch)**.** Keep the value 0 of m_0 in Example 8.1, and replace the values $1, 2, 3$ of m_1, m_2, m_3 by α, β, H, where $0 \le \alpha \le \beta \le H$. The old inequalities become $0 \le p \le q$, $\alpha \le q \le \beta$, $q \le r \le H$. The face vectors in actual affine dimensions are as follows.

Marks	Dimension	Old	New
$0 < \alpha < \beta < H$	3	$(8, 12, 6, 1)$	$(9, 14, 7, 1)$
$\alpha = 0 < \beta < H$	3	$(6, 9, 5, 1)$	$(6, 9, 5, 1)$
$0 < \alpha < \beta = H$	3	$(6, 9, 5, 1)$	$(6, 9, 5, 1)$
$\alpha = 0 < \beta = H$	3	$(4, 6, 4, 1)$	$(4, 6, 4, 1)$
$0 < \alpha = \beta < H$	2	$(4, 4, 1)$	$(4, 4, 1)$
$\alpha = \beta = 0 < H$	1	$(2, 1)$	$(2, 1)$
$0 < \alpha = \beta = H$	1	$(2, 1)$	$(2, 1)$
$\alpha = \beta = H = 0$	0	(1)	(1)

Here the base has $L = \max(p, \alpha)$ and $U = \min(r, \beta)$. When $\alpha = 0$, L is affine; when $\beta = H$, U is affine. When $\alpha = \beta = c$, the fibre collapses and the remaining polytope is $[0, c] \times [c, H]$. For $0 < \alpha < \beta < H$, the nine new vertices are

$$(0, 0, \alpha), (\alpha, 0, \alpha), (\beta, 0, \beta), (\beta, 0, H), (0, 0, H),$$
$$(0, \beta - \alpha, H), (\alpha, \beta - \alpha, H), (0, \beta - \alpha, \beta), (\alpha, \beta - \alpha, \beta).$$

These explicit vertices and the preceding boundary descriptions give the table without a limiting argument.

On the same poset take $C = \{r\}$, and again switch q. These chain parts are not ideals. With s the chain coordinate at r, the old inequalities are $0 \le p \le q$, $\alpha \le q \le \beta$, $s \ge 0$, $q + s \le H$. The new inequalities are

$$p, w, s \ge 0, \quad p + w \le \beta, \quad w \le \beta - \alpha, \quad p + w + s \le H, \quad w + s \le H - \alpha.$$

The affine substitution $r = H - s$ identifies both old and new polytopes with the preceding family, so the entire table also supplies nonadmissible examples. ◇

Example 8.3 (Boundedness cannot be omitted)**.** Take $m < a, b < q$ with a, b incomparable and mark only m, with value zero. The all-order polyhedron is the cone $0 \le a, b \le q$. Its section $q = 1$ is a square, and its face vector is $(1, 4, 4, 1)$. After switching q alone, the polyhedron is the nonnegative orthant, with face vector $(1, 3, 3, 1)$. Both ray and facet counts decrease. The same two polyhedra arise for the admissible chain parts $\{a, b\}$ and $\{a, b, q\}$, since a and b have lower envelope zero, so switching them first changes nothing. Thus Theorem 1.1 has no unrestricted unbounded version with the same endpoint definition. ◇

Remark 8.4. The footnote in [7] (preprint version) extends [7, Proposition 2.8] to polyhedra with only the minimal elements marked. Example 8.3 shows that this extension fails when, in the notation of Theorem 7.7, $\ell = 0$: the switched element q has $k = 2$ and $\ell = 0$, so it is not a chain-order star element in the sense of [7, Definition 2.7], and the marked poset is regular; yet

the face vectors $(1,4,4,1)$ and $(1,3,3,1)$ differ. The correct condition is $k = 1$ or $\ell = 1$, as in Theorem 7.7; for $\ell = 0$ and $k \geq 2$ the facet count in the proof of [7, Proposition 7.11] gives the change $(k-1)(\ell-1) = 1-k$, which is negative.

Remark 8.5. The strict increase of the face numbers given by Theorem 1.2 need not hold at each intermediate switch. For the ordinal sum $\{a,b\} \oplus \{p,q\} \oplus \{g,h\}$ of three 2-element antichains, switch q after switching only a and b. Its base is $0 \leq a,b \leq p \leq g,h \leq 1$, and its exact difference is $t^3(1+t)^2$, with no two-face increase. The late ideal used in Section 4 is what supplies the four-dimensional face.

9 Several switches

The proof of Theorem 1.1 adds the elements of $C' \setminus C$ one at a time. We now switch several pairwise incomparable order elements together and form the alternating sum of the resulting face polynomials. It equals the alternating sum defined as in Proposition E.3 for the affinity subdivisions of the lower and upper envelopes of the switched elements, and this gives the subdivision sums a poset meaning. Each coincidence among the envelope subdivisions reverses the sign, so the sum can be negative. Under mutual transversality, its sign and a power of $1+t$ dividing it are determined; without transversality both signs can occur in one polynomial. For layered posets the same conclusion holds whenever the distinct lower and the distinct upper neighbourhoods form Berge-acyclic families (Theorem 9.13; Proposition 9.14 gives the exact value). For two-element neighbourhoods the sum is, up to sign and a power of $2+t$, a product of chromatic polynomials at $-t$ (Theorem 9.15), and for neighbourhoods of any size each side (lower or upper) contributes a signed sum of chromatic polynomials of hypergraphs at $-t$ (Theorem 9.16).

Section 9.5 treats added order relations between lower and upper elements of different switches (Proposition 9.23) and matches the values in Examples 9.20 and 9.21 layer by layer (Example 9.24). Section 9.6 treats neighbourhoods with three or more elements. When every group has at most two members, it gives an exact value under a pendant-pair condition (Theorem 9.27) and the leading term in powers of $1+t$; when a signed count is nonzero, this term fixes the order of vanishing at $t = -1$ (Theorem 9.31). If the distinct sets of a side form one three-member core group, the conclusion of Theorem 9.13 fails in every nonzero coefficient (Theorem 9.36). This remains so with one-member groups attached (Proposition 9.37), and the leading term extends to such groups (Theorem 9.38). When every group is a one-member group, a two-member group neither of whose members contains the other, or a three-member core group, each core group reverses the sign (Theorem 9.42). Finally, suppose that the distinct sets of a side form one four-member group, that every element of their union lies in at least two of them, and that none of them is the union of the others inside it. Then the conclusion of Theorem 9.13 fails on that side unless its factor is zero (Theorem 9.47). Section 9.7 treats two comparable switches, in three cases. In the first, they lie across a bridge cover (Theorem 9.50). In the second, only chain elements lie between them, and every element above the lower one or below the upper one is comparable with both; then the sum lies in $(1+t)^3\mathbb{Z}_{\geq 0}[t]$ (Theorem 9.54). In the third, an order element lies strictly between them on every maximal chain; then Theorem 9.4 carries over, and the sum can be negative (Proposition 9.56 and Example 9.57).

9.1 Mixed differences and envelopes

Keep the bounded hypotheses of Section 1. Let $Y = \{q_1, \dots, q_r\} \subseteq O$ be an antichain, $r \geq 1$. For $J \subseteq [r] = \{1, \dots, r\}$ put $C_J = C \cup \{q_j : j \in J\}$, and define the *mixed difference*

$$\mathcal{M}_C(Y) = \sum_{J \subseteq [r]} (-1)^{r-|J|} F_{Q_{C_J}(P,\lambda)}.$$

For $r = 1$ this is the difference $F_{Q'} - F_Q$ of Section 3.

Every unmarked element lies strictly between two marked ones, so the induced poset $P \setminus Y$, with the same marking and the partition $C \sqcup (O \setminus Y)$, again has all its extrema marked. Put $B = Q_C(P \setminus Y, \lambda)$. For each i define L_i and U_i on B by the formulas of Section 3.1, with $q = q_i$, chains K, H in C, and endpoints in $A \cup (O \setminus Y)$. No other element of Y is comparable with q_i, so these are the envelopes of q_i in $P \setminus (Y \setminus \{q_i\})$, whose deleted base is B; in particular $L_i \leq U_i$ on B. We call L_i, U_i the envelopes of the switch of q_i. Let $\mathcal{S}_i, \mathcal{T}_i$ be their effective affinity subdivisions of B, and put $Z_i = \{u \in B : L_i(u) = U_i(u)\}$.

Lemma 9.1. *For $J \subseteq [r]$, after reordering coordinates, $Q_{C_J}(P, \lambda)$ is the set of $(u, z) \in B \times \mathbb{R}^r$ with*

$$L_i(u) \leq z_i \leq U_i(u) \quad (i \notin J), \qquad 0 \leq z_i \leq U_i(u) - L_i(u) \quad (i \in J).$$

Proof. A chain of P meets the antichain Y at most once. The inequalities (1) whose chains avoid Y are those of B. If a chain meets Y in q_i, its other members lie in $P \setminus Y$, and the argument of Section 3.1 applies verbatim: for $i \notin J$ the element q_i is an endpoint and these inequalities give $L_i \leq z_i \leq U_i$; for $i \in J$ it is a chain element and they give $0 \leq z_i \leq U_i - L_i$. Conversely, every lower and every upper chain of q_i, and every concatenation of the two, gives such an inequality. □

For finite polyhedral subdivisions $\mathcal{G}_1, \dots, \mathcal{G}_m$ of B, each including all nonempty faces, write $\mathcal{G}_J = \bigvee_{j \in J} \mathcal{G}_j$, with $\mathcal{G}_\varnothing$ the face complex of B, and

$$\Delta(\mathcal{G}_1, \dots, \mathcal{G}_m) = \sum_{J \subseteq [m]} (-1)^{m-|J|} F_{\mathcal{G}_J} = \sum_E \Delta_E(\mathcal{G}_1, \dots, \mathcal{G}_m),$$

where Δ_E is the same alternating sum restricted to the cells with carrier E, as in Proposition E.3. For $m = 2$ this is the defect $F_B + F_{\mathcal{S} \vee \mathcal{T}} - F_{\mathcal{S}} - F_{\mathcal{T}}$ of Section 2. A subdivision is *trivial on* a face E if its restriction to E is the face complex of E, and *trivial* if it is trivial on B.

Lemma 9.2. *Let E be a face of B.*

(1) If a finite maximum, or a finite minimum, g of affine functions is affine on E, then its affinity subdivision is trivial on E.

(2) If a subdivision $\mathcal{G}$ is trivial on E, then for every finite polyhedral subdivision $\mathcal{X}$ of B including all faces, $\mathcal{X}$ and $\mathcal{X} \vee \mathcal{G}$ have the same cells contained in E.

Proof. (1) The sets $\{u \in E : g(u) = \ell(u)\}$, for the finitely many affine pieces ℓ of g, are closed, convex and cover E, so one of them has nonempty relative interior in E. Since $g|_E$ is affine, that piece agrees with g on all of E. The set $D = \{u \in B : g(u) = \ell(u)\}$ is the projection of the face of the epigraph (for a maximum) or hypograph (for a minimum) on which the valid inequality given by ℓ is tight. That face lies in the graph of g, so D is a cell of the affinity subdivision by Lemma 2.8 and the proof of Lemma B.3. Moreover E is a face of D, cut out by a hyperplane supporting B along E. So E is a cell, and the cells contained in E are its faces. (2) A cell $X \cap G$ of $\mathcal{X} \vee \mathcal{G}$ contained in E equals $X \cap (G \cap E)$, where $G \cap E$ is a face of E and hence of B; so it is a face of X, hence a cell of $\mathcal{X}$. Conversely a cell $X \subseteq E$ of $\mathcal{X}$ equals $X \cap E$, and $E \in \mathcal{G}$. □

Lemma 9.3 (Faces of interval fibrations)**.** *For $i \in [r]$ let $a_i \le b_i$ be functions on B, a_i a finite maximum and b_i a finite minimum of affine functions, with affinity subdivisions $\mathcal{A}_i, \mathcal{B}_i$, and put $Z_i' = \{u \in B : a_i(u) = b_i(u)\}$. Let*

$$\Omega = \{(u, z) \in B \times \mathbb{R}^r : \ a_i(u) \le z_i \le b_i(u) \ (i \in [r])\}.$$

A type *is a partition $[r] = I_- \sqcup I_+ \sqcup I_0$, empty parts allowed. For a type put $\mathcal{R} = \bigvee_{i \in I_-} \mathcal{A}_i \vee \bigvee_{i \in I_+} \mathcal{B}_i$, the face complex of B if $I_- \cup I_+ = \varnothing$. For $\sigma \in \mathcal{R}$ let $G(\sigma)$ be the set of $(u, z) \in \Omega$ with $u \in \sigma$, $z_i = a_i(u)$ for $i \in I_-$, and $z_i = b_i(u)$ for $i \in I_+$. Then G is a bijection from the pairs consisting of a type and a cell $\sigma \in \mathcal{R}$ with $\sigma \not\subseteq Z_i'$ for all $i \in I_+ \cup I_0$, onto the nonempty faces of Ω. Moreover $\dim G(\sigma) = \dim \sigma + |I_0|$.*

Proof. Since B is a polytope, so is Ω, and every nonempty face of Ω is exposed by a linear functional. Let $\psi(u, z) = c(u) + \sum_i \alpha_i z_i$ be a linear functional. For fixed $u \in B$, ψ is maximized on the fibre exactly when $z_i = b_i(u)$ for $\alpha_i > 0$ and $z_i = a_i(u)$ for $\alpha_i < 0$, the other coordinates being arbitrary in their intervals. The maximum is

$$\varphi(u) = c(u) + \sum_{\alpha_i > 0} \alpha_i b_i(u) + \sum_{\alpha_i < 0} \alpha_i a_i(u),$$

a concave piecewise-affine function. By the Jensen argument in the proof of Lemma B.2, a sum of concave functions is affine on a convex set exactly when every summand is; hence the affinity subdivision of φ is $\mathcal{R}$ for the type with $I_\pm = \{i : \pm\alpha_i > 0\}$. Write (u, y) for the points of the hypograph of φ. By Lemma 2.8 and the proof of Lemma B.3, applied to $-\varphi$, the faces of the hypograph of φ over B that lie in its graph are the graphs of φ over the cells of $\mathcal{R}$, and they are exactly the faces exposed by functionals $\ell(u) + sy$ with $s > 0$.

Every face is some $G(\sigma)$. If ψ exposes a nonempty face of Ω, that face is $G(\sigma)$ for the type above and $\sigma = \operatorname{argmax}_B \varphi$. This is the projection of the face of the hypograph exposed by y, so $\sigma \in \mathcal{R}$.

Every $G(\sigma)$ is a face. Given a type and $\sigma \in \mathcal{R}$, put $\varphi_0 = \sum_{i \in I_+} b_i - \sum_{i \in I_-} a_i$, with affinity subdivision $\mathcal{R}$, and choose ℓ with $\sigma = \operatorname{argmax}_B(\varphi_0 + \ell)$. Then $\ell(u) + \sum_{i \in I_+} z_i - \sum_{i \in I_-} z_i$ exposes $G(\sigma)$.

Normal form and dimension. As in the proof of Lemma B.2, each Z_i' is a union of faces of B, on each of which $a_i = b_i$ is affine. If $\sigma \subseteq Z_i'$, a relative-interior point of σ lies in Z_i' and in the relative interior of the carrier E of σ, so $E \subseteq Z_i'$. By Lemma 9.2, σ remains a cell when i is moved to I_-, and $G(\sigma)$ does not change, because the fibre in coordinate i is a point. So every face has a representation with $\sigma \not\subseteq Z_i'$ for $i \in I_+ \cup I_0$. In such a representation σ is the projection of the face, and $a_i < b_i$ on $\operatorname{relint}(\sigma)$ for $i \in I_+ \cup I_0$, by the argument just given. Hence i lies in I_-, in I_+ or in I_0 according as the face lies in $\{z_i = a_i(u)\}$, lies in $\{z_i = b_i(u)\}$ but not in $\{z_i = a_i(u)\}$, or lies in neither; the representation is unique. Finally, a_i for $i \in I_-$ and b_i for $i \in I_+$ are affine on σ, so $G(\sigma)$ is affinely isomorphic to $\{(u, z_{I_0}) : u \in \sigma, \ a_i(u) \le z_i \le b_i(u) \ (i \in I_0)\}$, of dimension $\dim \sigma + |I_0|$. □

Theorem 9.4 (Mixed differences as higher-order defects)**.** *For every antichain $Y = \{q_1, \dots, q_r\}$ in the order part O,*

$$\mathcal{M}_C(Y) = \Delta(\mathcal{S}_1, \mathcal{T}_1, \dots, \mathcal{S}_r, \mathcal{T}_r).$$

No transversality is assumed.

Proof. By Lemma 9.1, $Q_{C_J}(P, \lambda)$ is the set Ω of Lemma 9.3 with $(a_i, b_i) = (L_i, U_i)$ for $i \notin J$ and $(a_i, b_i) = (0, U_i - L_i)$ for $i \in J$. In the second case $U_i - L_i$ is the minimum, over pairs of

an upper and a lower branch, of their differences, hence a finite minimum of affine functions as Lemma 9.3 requires; $\mathcal{A}_i$ is the face complex of B, and $\mathcal{B}_i = \mathcal{S}_i \vee \mathcal{T}_i$ by the Jensen argument. In both cases $Z_i' = Z_i$. So Lemma 9.3 expresses $\mathcal{M}_C(Y)$ as a signed sum over J, types and cells, and the contribution of each index i depends only on its state (order, sign -1; chain, sign $+1$) and its type.

For $i \in I_0$, both states add nothing to the refinement, impose $\sigma \not\subseteq Z_i$, and add one to the dimension; so the terms with $I_0 \neq \varnothing$ cancel in pairs. For $i \in I_-$, the order state adds $\mathcal{S}_i$ with sign -1 and the chain state adds nothing with sign $+1$. For $i \in I_+$, the order state adds $\mathcal{T}_i$ with sign -1 and the chain state adds $\mathcal{S}_i \vee \mathcal{T}_i$ with sign $+1$, both under the condition $\sigma \not\subseteq Z_i$. Without these conditions, the four choices for each i are the four subsets of $\{\mathcal{S}_i, \mathcal{T}_i\}$, with the signs they have in Δ, and the whole sum is $\Delta(\mathcal{S}_1, \mathcal{T}_1, \ldots, \mathcal{S}_r, \mathcal{T}_r)$.

We remove the conditions one index at a time. For index i and a fixed choice at the other indices, with refinement $\mathcal{X}$ and conditions that depend only on σ, this adds the cells $\sigma \subseteq Z_i$ of $\mathcal{X} \vee \mathcal{S}_i \vee \mathcal{T}_i$ with sign $+1$ and those of $\mathcal{X} \vee \mathcal{T}_i$ with sign -1. Since Z_i is a union of faces of B and the relative interior of a cell lies in the relative interior of its carrier, the carrier of such a cell lies in Z_i, as in the proof of Lemma 9.3. There L_i is affine, so by Lemma 9.2 the two sets of cells coincide, and the added terms cancel. □

Lemma 9.5 (Collapse)**.** *Let $\mathcal{G}_1, \ldots, \mathcal{G}_m$ be as in the definition of Δ.*

(1) If $\mathcal{G}_i$ is trivial on a face E, then $\Delta_E(\mathcal{G}_1, \ldots, \mathcal{G}_m) = 0$. If $\mathcal{G}_i$ is trivial, then $\Delta(\mathcal{G}_1, \ldots, \mathcal{G}_m) = 0$.

(2) $\Delta(\mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_3, \ldots, \mathcal{G}_m) = -\Delta(\mathcal{G}_1, \mathcal{G}_3, \ldots, \mathcal{G}_m)$.

Proof. (1) By Lemma 9.2(2), $\mathcal{G}_J$ and $\mathcal{G}_J \vee \mathcal{G}_i$ have the same cells with carrier E, so the terms for J and $J \triangle \{i\}$ cancel. (2) Group the subsets $J \subseteq [m]$ by $J' = J \setminus \{1, 2\}$. The choice $J \cap \{1, 2\} = \varnothing$ gives $\mathcal{G}_{J'}$ with sign $(-1)^{m-|J'|}$. The other three choices give $\mathcal{G}_{J'} \vee \mathcal{G}_1$, with total sign $2(-1)^{m-|J'|-1} + (-1)^{m-|J'|} = (-1)^{m-1-|J'|}$. Both signs are the negatives of those in $\Delta(\mathcal{G}_1, \mathcal{G}_3, \ldots, \mathcal{G}_m)$. □

The lemma holds verbatim for subdivisions of any polyhedron. Part (2) uses only that the join is associative, commutative and idempotent, $\mathcal{G} \vee \mathcal{G} = \mathcal{G}$; we apply it below also to fans.

Corollary 9.6. *If one of the $2r$ envelopes is affine on B, then $\mathcal{M}_C(Y) = 0$. Otherwise let $\mathcal{G}_1, \ldots, \mathcal{G}_k$ be the distinct subdivisions among $\mathcal{S}_1, \mathcal{T}_1, \ldots, \mathcal{S}_r, \mathcal{T}_r$. Then*

$$\mathcal{M}_C(Y) = (-1)^k \Delta(\mathcal{G}_1, \ldots, \mathcal{G}_k).$$

Proof. An envelope is affine on B exactly when its subdivision is trivial. In the first case, Theorem 9.4 and Lemma 9.5(1) give $\mathcal{M}_C(Y) = 0$. Otherwise, apply Theorem 9.4 and then Lemma 9.5(2) a total of $2r - k$ times. This gives the sign $(-1)^{2r-k} = (-1)^k$. □

9.2 The sign rule

Following Proposition E.3, subdivisions $\mathcal{G}_1, \ldots, \mathcal{G}_k$ of B are *mutually transverse relative to every face* if for every face E, of dimension e, and every $x \in \operatorname{relint}(E)$, the minimal cells A_i of the restrictions of the $\mathcal{G}_i$ to E containing x satisfy $\dim \bigcap_i A_i = e - \sum_i (e - \dim A_i)$. Equivalently, the conormal spaces of the A_i in $\operatorname{dir}(E)^*$ form a direct sum. For $k = 2$ this is the relative transversality of Section 2.

Theorem 9.7 (Sign rule)**.** *Suppose that the k distinct subdivisions among $\mathcal{S}_1, \mathcal{T}_1, \ldots, \mathcal{S}_r, \mathcal{T}_r$ are mutually transverse relative to every face of B. Then*

$$(-1)^k \mathcal{M}_C(Y) \in (1+t)^k \mathbb{Z}_{\geq 0}[t].$$

In particular $\mathcal{M}_C(Y) = 0$ if $k > \dim B$.

Proof. If an envelope is affine on B, then $\mathcal{M}_C(Y) = 0$ by Corollary 9.6. Otherwise that corollary gives $(-1)^k \mathcal{M}_C(Y) = \Delta(\mathcal{G}_1, \ldots, \mathcal{G}_k)$. Apply Corollary G.3; the vanishing is part of Proposition E.3. □

Corollary 9.8 (Binomial bound)**.** *Under the hypotheses of Theorem 9.7, suppose $\mathcal{M}_C(Y) \neq 0$, and let ρ be the least degree in which it has a nonzero coefficient. Then $\rho \leq \dim B - k$ and*

$$(-1)^k \mathcal{M}_C(Y) \geq t^\rho (1+t)^{\dim B - \rho}.$$

Proof. Apply Theorem H.3, which assumes no regularity, to $\Delta(\mathcal{G}_1, \ldots, \mathcal{G}_k) = (-1)^k \mathcal{M}_C(Y)$. □

9.3 A poset criterion for transversality

Let $P^\circ = P \setminus (A \cup Y)$, the coordinate set of B, and let Γ be the graph on P° whose edges are the cover relations of P between elements of P°. Let $\kappa^-(q_i)$ and $\kappa^+(q_i)$ be the sets of connected components of Γ that contain a lower, respectively an upper, cover of q_i. For each nontrivial subdivision $\mathcal{G}$ among $\mathcal{S}_1, \mathcal{T}_1, \ldots, \mathcal{S}_r, \mathcal{T}_r$, let $\kappa(\mathcal{G})$ be the union of the sets $\kappa^-(q_i)$ with $\mathcal{S}_i = \mathcal{G}$ and $\kappa^+(q_i)$ with $\mathcal{T}_i = \mathcal{G}$. The pair $\mathcal{S}_i, \mathcal{T}_i$ is *isolated* if neither of them equals $\mathcal{S}_j$ or $\mathcal{T}_j$ for any $j \neq i$. Consider the condition

(PC) for distinct nontrivial subdivisions $\mathcal{G} \neq \mathcal{G}'$ among the $\mathcal{S}_i, \mathcal{T}_i$, the sets $\kappa(\mathcal{G})$ and $\kappa(\mathcal{G}')$ are disjoint, unless $\{\mathcal{G}, \mathcal{G}'\} = \{\mathcal{S}_i, \mathcal{T}_i\}$ for an isolated pair.

Proposition 9.9 (Product criterion)**.** *Suppose that no envelope of a switch in Y is affine on B, and that (PC) holds. Then the distinct subdivisions among $\mathcal{S}_1, \mathcal{T}_1, \ldots, \mathcal{S}_r, \mathcal{T}_r$ are mutually transverse relative to every face of B.*

Proof. Fix a face E of B, a point $x \in \operatorname{relint}(E)$, and put $V = \operatorname{dir}(E)$. For each subdivision $\mathcal{G}$ let $N(\mathcal{G}) \subseteq V^*$ be the conormal space of its minimal restricted cell at x. We show that these spaces form a direct sum.

Degenerate switches. Let $Y_d = \{q_j \in Y : L_j(x) = U_j(x)\}$ and $Y_n = Y \setminus Y_d$. For $q_j \in Y_d$, the set Z_j contains E, as in the proof of Lemma 9.3, so $L_j = U_j$ is affine on E. By Lemma 9.2(1), every subdivision equal to $\mathcal{S}_j$ or $\mathcal{T}_j$ is trivial on E, and its conormal space is zero. We discard these. Every remaining subdivision differs from $\mathcal{S}_j$ and $\mathcal{T}_j$ for all $q_j \in Y_d$.

A local product. For $q_i \in Y_n$ choose η_i with $L_i(x) < \eta_i < U_i(x)$. Let P_n be the poset P in which each $q_i \in Y_n$ is marked by η_i, with this extended marking λ_n, and each $q_j \in Y_d$ remains an order element. The marking stays order-preserving: for marked $a < q_i < b$, the branches with empty chains give $\lambda(a) \leq L_i(x) < \eta_i < U_i(x) \leq \lambda(b)$, and the elements of Y are pairwise incomparable. As in Lemma 9.1, the polytope $\Pi = Q_C(P_n, \lambda_n)$, with coordinates $P^\circ \cup Y_d$, is the set of (u, z) with $u \in B$, $L_j(u) \leq z_j \leq U_j(u)$ for $q_j \in Y_d$, and $L_i(u) \leq \eta_i \leq U_i(u)$ for $q_i \in Y_n$; here a chain meeting a marked q_i has it as an endpoint, and these inequalities say exactly $L_i(u) \leq \eta_i \leq U_i(u)$. By the saturated-chain form of (1) in Section 3.1, Π is defined by nonnegativity and by inequalities whose unmarked members are consecutive in a saturated chain. Each of these involves the coordinates of one connected component of the graph Γ_n on $P^\circ \cup Y_d$ whose edges are the cover relations of P.

Hence Π is the product, over the components R of Γ_n, of polytopes $\Pi_R \subseteq \mathbb{R}^R$. If c exposes E in B, then $c(u)$ exposes the face $F = \{(u,z) \in \Pi : u \in E\}$ of Π. Its projection is the set of $u \in E$ satisfying the inequalities $L_i(u) \le \eta_i \le U_i(u)$ ($q_i \in Y_n$), which are strict at x; so the projection agrees with E near x, and V is the projection of $\mathrm{dir}(F)$. A face of a product is a product of faces, so $V = \bigoplus_R V_R$, where $V_R \subseteq \mathbb{R}^{R \setminus Y_d}$ is the projection of $\mathrm{dir}(F_R)$. Consequently, restrictions to V of linear forms supported on disjoint sets of components of Γ_n span subspaces of V^* whose sum is direct.

Envelope conormals. Every lower branch $y_a + \sum_{c \in K} u_c$ of q_i is dominated on B by the branch of a saturated chain $a' \lessdot c_1 \lessdot \cdots \lessdot c_s \lessdot q_i$ with all $c_j \in C$. Indeed, refine the chain: an added chain element adds a nonnegative coordinate, and at an added marked or order element p the inequality of B for the part of the chain below p replaces that part by y_p. So L_i is the maximum of its saturated branches. The linear part of a saturated branch is zero, or supported on the unmarked elements among $a', c_1, \ldots, c_s$; these form a path in Γ ending at a lower cover of q_i. As in Section 3.3, $N(\mathcal{S}_i)$ is spanned by the restrictions to V of differences of linear parts of branches active at x. This holds for any representation of L_i as a maximum: for $v \in V$, all active linear parts agree at v exactly when the one-sided derivatives of L_i at x in the directions v and $-v$ add to zero, a condition on L_i alone. Thus $N(\mathcal{S}_i)$ consists of restrictions of forms supported on the components in $\kappa^-(q_i)$, and likewise $N(\mathcal{T}_i)$ for $\kappa^+(q_i)$.

Gluing. The graph Γ_n arises from Γ by adding the vertices $q_j \in Y_d$, each joined to its covers in P°. So a component of Γ lies in a larger component of Γ_n only if it belongs to $\kappa^-(q_j) \cup \kappa^+(q_j) \subseteq \kappa(\mathcal{S}_j) \cup \kappa(\mathcal{T}_j)$ for some $q_j \in Y_d$; here $\mathcal{S}_j, \mathcal{T}_j$ are nontrivial by hypothesis. Let $\mathcal{G}$ be a remaining subdivision and $q_j \in Y_d$. Since $\mathcal{G} \ne \mathcal{S}_j$, (PC) gives $\kappa(\mathcal{G}) \cap \kappa(\mathcal{S}_j) = \varnothing$ unless $\{\mathcal{G}, \mathcal{S}_j\}$ is an isolated pair $\{\mathcal{S}_i, \mathcal{T}_i\}$. Such a pair contains $\mathcal{S}_j$, so $i = j$ by isolation and $\mathcal{G} = \mathcal{T}_j$, which was discarded. The same holds for $\mathcal{T}_j$. Hence every component in $\kappa(\mathcal{G})$ is a component of Γ_n, and by (PC) two remaining subdivisions that do not form an isolated pair use disjoint sets of components of Γ_n.

Independence. Group the remaining subdivisions into isolated pairs and single subdivisions; an isolated pair is either discarded as a whole or remains as a whole. Conormal spaces of different groups lie in restrictions of forms supported on disjoint sets of components, so their sum is direct. For an isolated pair $\{\mathcal{S}_i, \mathcal{T}_i\}$ we have $q_i \in Y_n$, and $N(\mathcal{S}_i) \cap N(\mathcal{T}_i) = 0$ by Section 3.3, applied to q_i in $P \setminus (Y \setminus \{q_i\})$, whose deleted base is B and whose envelopes are L_i, U_i. Hence all the conormal spaces form a direct sum. □

The hypothesis that no envelope is affine is used only in the gluing step of the proof. Still, (PC) alone implies the conclusions of Theorem 9.7 and Corollary 9.8, because if an envelope is affine, then $\mathcal{M}_C(Y) = 0$ by Corollary 9.6.

9.4 Layered posets

Let $P = \{\hat{0}\} \sqcup P^- \sqcup Y \sqcup P^+ \sqcup \{\hat{1}\}$ with marks $\lambda(\hat{0}) = 0$ and $\lambda(\hat{1}) = 1$, chain part $C = P^- \cup P^+$ and $O = Y = \{q_1, \ldots, q_r\}$. Its order is generated by $\hat{0} < p < \hat{1}$ for every p, by $l < q_i$ for $l \in N_i^-$, and by $q_i < u$ for $u \in N_i^+$, where $N_i^- \subseteq P^-$ and $N_i^+ \subseteq P^+$ are nonempty. Thus P^-, Y and P^+ are antichains, and $l < u$ exactly when $l \in N_i^-$ and $u \in N_i^+$ for some i. The elements of P^- outside $\bigcup_i N_i^-$, and of P^+ outside $\bigcup_i N_i^+$, are *free*. We call such a marked poset *layered*, and N_i^- and N_i^+ the *lower* and *upper neighbourhoods* of q_i. Here

$$B = \{(x,y) \in [0,1]^{P^-} \times [0,1]^{P^+} : x_l + y_u \le 1 \text{ whenever } l < u\},$$

$L_i = \max(0, x_l : l \in N_i^-)$ and $U_i = 1 - \max(0, y_u : u \in N_i^+)$.

For a finite set W and subsets $D_j \subseteq W$, $j \in J$, let $\Phi_W(D_j : j \in J) = \sum_\sigma t^{\dim \sigma}$, the sum over the cones σ of the common refinement, in $\mathbb{R}^W$, of the fans of the functions $\max_{b \in D_j} x_b$. A set with at most one element gives the trivial fan $\{\mathbb{R}^W\}$, and $J = \varnothing$ gives $t^{|W|}$. For subsets $N_1, \dots, N_r$ of a finite set N, with union N', define the *lower polynomial*

$$\Lambda(N_1, \dots, N_r; N) = (2+t)^{|N \setminus N'|} \sum_{Z \subseteq N'} \sum_{J \subseteq [r]} (-1)^{r-|J|} \Phi_{N' \setminus Z}(N_j \setminus Z : j \in J).$$

Theorem 9.10 (Layered factorization)**.** *For a layered poset as above,*

$$\mathcal{M}_C(Y) = \Lambda(N_1^-, \dots, N_r^-; P^-) \cdot \Lambda(N_1^+, \dots, N_r^+; P^+).$$

Proof. By Theorem 9.4, $\mathcal{M}_C(Y) = \sum_E \Delta_E(\mathcal{S}_1, \mathcal{T}_1, \dots, \mathcal{S}_r, \mathcal{T}_r)$.

Faces that contribute. Suppose $x_l + y_u = 1$ on E, where $l \in N_i^-$ and $u \in N_i^+$. For $l' \in N_i^-$ and $u' \in N_i^+$ the inequalities $x_{l'} + y_u \le 1$ and $x_l + y_{u'} \le 1$ give $x_{l'} \le x_l$ and $y_{u'} \le y_u$ on E. So $L_i = x_l$, one of its pieces, on E; by Lemma 9.2(1), $\mathcal{S}_i$ is trivial on E, and $\Delta_E = 0$ by Lemma 9.5(1). If $x_l = 1$ on E for some $l \in N_i^-$, then $y_u = 0$ and so $x_l + y_u = 1$ on E for $u \in N_i^+$; the same holds if $y_u = 1$ on E for some $u \in N_i^+$. The remaining faces are the faces $E = E(Z, \varepsilon, Z', \varepsilon')$, where $Z \subseteq \bigcup_i N_i^-$ is the set of non-free l with $x_l = 0$ on E, ε assigns to each free l one of the states 0, 1 or open, and Z' and ε' are the same data for P^+. On such a face every other coordinate is nonconstant; we call these coordinates *moving*. No inequality $x_l + y_u \le 1$ is tight on E. Each such choice is realized by exactly one face: the point with value $1/4$ in every moving coordinate lies in its relative interior.

Cells. Fix such a face E, let $W^- = \bigcup_i N_i^- \setminus Z$ and $W^+ = \bigcup_i N_i^+ \setminus Z'$, and let o_E be the number of open free coordinates. On E all coordinates are nonnegative and $x_l = 0$ for $l \in Z$, so $L_i = \max(x_l : l \in N_i^- \setminus Z)$ there, and $L_i = 0$ if $N_i^- \subseteq Z$; likewise for U_i. Hence on E each $\mathcal{S}_i$ is the pullback of the fan of $\max_{l \in N_i^- \setminus Z} x_l$ on $\mathbb{R}^{W^-}$, and each $\mathcal{T}_i$ the pullback of a fan on $\mathbb{R}^{W^+}$. The cones of these fans and of their common refinements are invariant under adding multiples of $(1, \dots, 1)$ and under positive scaling, so every such cone meets $(0, 1/2)^{W^\pm}$ in its relative interior. A point of $\operatorname{aff}(E)$ whose moving coordinates lie in $(0, 1/2)$ lies in $\operatorname{relint}(E)$, since then every sum $x_l + y_u$ is less than 1. (The set $\operatorname{relint}(E)$ itself is not a product of an x-part and a y-part.) Choosing the two parts of a point in this way shows that, for $J, J' \subseteq [r]$, the cells of $\mathcal{S}_J \vee \mathcal{T}_{J'}$ with carrier E are the sets $(\sigma \times \tau \times \mathbb{R}^{o_E}) \cap E$, one for each cone σ of the refinement on $\mathbb{R}^{W^-}$ and each cone τ of the refinement on $\mathbb{R}^{W^+}$, of dimension $\dim \sigma + \dim \tau + o_E$. So their polynomial is $t^{o_E} \Phi_{W^-}(N_j^- \setminus Z : j \in J)\, \Phi_{W^+}(N_j^+ \setminus Z' : j \in J')$.

Factorization. The sign of the subfamily $\{\mathcal{S}_j : j \in J\} \cup \{\mathcal{T}_j : j \in J'\}$ in Δ is $(-1)^{r-|J|}(-1)^{r-|J'|}$. So Δ_E is t^{o_E} times the product of the alternating sums over J and over J'. The data (Z, ε) and (Z', ε') are chosen independently, and each free coordinate contributes $1 + 1 + t$ for its three states. Summing over the faces gives the product of the lower polynomials of the two families. □

Corollary 9.11 (Equal or disjoint neighbourhoods)**.** *In the layered setting, suppose that any two of the sets N_i^- are equal or disjoint, and likewise any two of the sets N_i^+. Let k be the number of distinct sets among all N_i^- and N_i^+, and f the number of free elements. Then*

$$\mathcal{M}_C(Y) = (-1)^k (2+t)^f \prod_D H_{|D|}(t), \qquad H_s(t) = t \sum_{j=2}^{s} \binom{s}{j} h_j(t),$$

the product over the k distinct sets, where $h_j(t) = (1+t)^j - t^j - t^{j-1} = (1+t)\big((1+t)^{j-1} - t^{j-1}\big)$ for $j \ge 2$. In closed form, $H_s(t) = t(2+t)^s - (1+t)^{s+1} + 1$. Every H_s with $s \ge 2$ is a nonzero element of $t(1+t)\mathbb{Z}_{\ge 0}[t]$, and $H_1 = 0$. For example $H_2 = t(1+t)$, $H_3 = 2t(1+t)(2+t)$ and $H_4 = t(1+t)(11 + 11t + 3t^2)$.

Proof. Consider the lower polynomial; the upper one is treated in the same way. For fixed Z, equal sets give equal fans, and Lemma 9.5(2), which uses only the equality of the complexes, reduces the alternating sum over J to the k^- distinct lower sets, with sign $(-1)^{r-k^-}$. Distinct sets are disjoint, so their fans live on disjoint blocks of coordinates, the refinement for a subfamily is a product, and the alternating sum is the product over the distinct sets D of $\Phi_{D\setminus Z}(D \setminus Z) - t^{|D\setminus Z|}$. The sum over Z factors over the blocks as well. The cones of the fan of $\max(x_1, \dots, x_j)$ correspond to the nonempty argmax sets I, of dimension $j-|I|+1$. So for $j \geq 2$ the factor is $t((1+t)^j - t^j) - t^j = t\, h_j$; for $j \leq 1$ the fan is trivial and the factor is 0. Summing over $Z \cap D$ gives $H_{|D|}$. The two signs multiply to $(-1)^{2r-k} = (-1)^k$. Positivity holds because $(1+t)^{j-1} - t^{j-1}$ is a nonzero element of $\mathbb{Z}_{\geq 0}[t]$ for $j \geq 2$. For the closed form, the binomial theorem gives $\sum_{j=0}^{s} \binom{s}{j}\big((1+t)^j - t^j - t^{j-1}\big) = (2+t)^s - (1+t)^s - (1+t)^s/t$, where the terms $j = 0$ and $j = 1$ are $-1/t$ and 0; removing the term $j = 0$ and multiplying by t gives $H_s(t) = t(2+t)^s - (1+t)^{s+1} + 1$. □

The hypothesis of Corollary 9.11 implies (PC): Γ has no edges, every element of P° is its own component, and distinct neighbourhoods are disjoint. So the corollary is a special case of Theorem 9.7, with the exact value added; a set with one element gives an affine envelope and $H_1 = 0$. Equality holds in Corollary 9.8 when $f = 0$ and all distinct sets have two elements, with $\rho = k = \dim B/2$.

A family of finite sets (an indexed family; repeated members are separate vertices) is *Berge-acyclic* if its incidence graph, the bipartite graph joining each element to the members containing it, is a forest. In a Berge-acyclic family, two distinct members share at most one element, and choosing one subset of each member gives again a Berge-acyclic family, since its incidence graph is a subgraph.

Lemma 9.12 (Cut spaces)**.** *Let $I_1, \dots, I_k$ be nonempty subsets of a finite set W, and let $K_j \subseteq (\mathbb{R}^W)^*$ be spanned by the forms $x_a - x_b$ with $a, b \in I_j$. Then the sum $K_1 + \cdots + K_k$ is direct if and only if the family $I_1, \dots, I_k$ is Berge-acyclic.*

Proof. Here $\dim K_j = |I_j| - 1$. Let $W_0 = \bigcup_j I_j$, and let ω be the number of components of the graph on W_0 joining a and b when they lie in a common I_j; it has the same components as the incidence graph. The sum $\sum_j K_j$ is spanned by the forms $x_a - x_b$ over the edges of this graph, so its dimension is $|W_0| - \omega$. The incidence graph has $|W_0| + k$ vertices, $\sum_j |I_j|$ edges and ω components, so it is a forest exactly when $\sum_j (|I_j| - 1) = |W_0| - \omega$, that is, when the sum is direct. □

In the language of hypergraphic polytopes, Lemma 9.12 says that the Minkowski sum of the simplices $\operatorname{conv}\{e_a : a \in I_j\}$ has dimension $\sum_j (|I_j| - 1)$ exactly when the family is Berge-acyclic; for such a family the sum is the product of the simplices [2, Section 2.5], using the edge ordering of [27, Lemma 3].

Theorem 9.13 (Hyperforest sign rule)**.** *In the layered setting, suppose that the distinct sets among $N_1^-, \dots, N_r^-$ form a Berge-acyclic family, and so do the distinct sets among $N_1^+, \dots, N_r^+$. Let k be the number of distinct sets among all of them, and f the number of free elements. Then*

$$(-1)^k \mathcal{M}_C(Y) \in (2+t)^f (1+t)^k \mathbb{Z}_{\geq 0}[t].$$

Proof. Consider the lower polynomial, with k^- distinct sets and f^- free elements of P^-; the upper one is treated in the same way. Fix Z and put $W = \bigcup_i N_i^- \setminus Z$. If some $N_i^- \setminus Z$ has at most one element, its fan is trivial and the inner sum vanishes by Lemma 9.5(1); this covers a distinct set with one element, for which $\mathcal{M}_C(Y) = 0$. Otherwise, if $N_i^- \setminus Z = N_j^- \setminus Z$, the two sets share at least two elements, so $N_i^- = N_j^-$. Fans of maxima over sets with at least two elements coincide

exactly when the sets do, so Lemma 9.5(2) applies exactly $r-k^-$ times and turns the inner sum into $(-1)^{r-k^-}$ times the value of Δ on the fans of $\max_{b\in D\setminus Z} x_b$ over the k^- distinct sets D. At a point $v\in\mathbb{R}^W$, the minimal cone of such a fan containing v is the cone of the argmax set $I=\operatorname{argmax}_{D\setminus Z} v$, which contains v in its relative interior; its conormal space is spanned by the forms x_a-x_b with $a,b\in I$. These sets I are subsets of the members of a Berge-acyclic family, so by Lemma 9.12 the conormal spaces form a direct sum. Thus the fans are mutually transverse relative to the only face of $\mathbb{R}^W$, namely $\mathbb{R}^W$ itself, where the empty refinement is the single cell $\mathbb{R}^W$. By Corollary G.3, which requires neither boundedness nor pointedness, this Δ lies in $(1+t)^{k^-}\mathbb{Z}_{\geq 0}[t]$. Summing over Z gives $(-1)^{r-k^-}\Lambda(N_1^-,\dots,N_r^-;P^-)\in(2+t)^{f^-}(1+t)^{k^-}\mathbb{Z}_{\geq 0}[t]$. By Theorem 9.10, $\mathcal{M}_C(Y)$ is the product of the two factors, and $(-1)^{r-k^-}(-1)^{r-k^+}=(-1)^k$. □

The proof of Theorem 9.13 can be made exact. For $s\geq 2$ let h_s be as in Corollary 9.11, and put $h_0=h_1=0$.

Proposition 9.14 (Exact value for Berge-acyclic families)**.** *Let $N_1,\dots,N_r$ be subsets of a finite set N with union N', and suppose that the distinct sets $D_1,\dots,D_k$ among them form a Berge-acyclic family. For $Z\subseteq N'$ let $\omega(Z)$ be the number of components of the graph on $N'\setminus Z$ joining a and b when they lie in a common set $D_j\setminus Z$. Then*

$$\Lambda(N_1,\dots,N_r;N)=(-1)^{r-k}(2+t)^{|N\setminus N'|}\sum_{Z\subseteq N'} t^{\omega(Z)}\prod_{j=1}^{k} h_{|D_j\setminus Z|}.$$

Proof. Fix Z and put $W=N'\setminus Z$ and $s_j=|D_j\setminus Z|$. If some $s_j\leq 1$, the inner sum of Λ and the term of Z both vanish, as in the proof of Theorem 9.13. Otherwise, as there, the inner sum is $(-1)^{r-k}\Delta$ of the fans of the maxima over the sets $D_j\setminus Z$, which are distinct and form a Berge-acyclic family. By Lemma 9.12 the map $\rho\colon \mathbb{R}^W\to\prod_j \mathbb{R}^{D_j\setminus Z}/\mathbb{R}(1,\dots,1)$, given by restriction, is onto, since its dual map is the sum map of the spaces K_j. For $\varnothing\neq I\subseteq D\subseteq W$, the cone of the fan of $\max_{b\in D} x_b$ with argmax set I has the relative interior $\{x:\operatorname{argmax}_D x=I\}$, whose image in $\mathbb{R}^D/\mathbb{R}(1,\dots,1)$ is a relatively open cone of dimension $|D|-|I|$. Hence, for $J\subseteq[k]$ and nonempty sets $I_j\subseteq D_j\setminus Z$ $(j\in J)$, the set of x with $\operatorname{argmax}_{D_j\setminus Z} x=I_j$ for all $j\in J$ is the preimage of a product: it is nonempty, relatively open, of dimension $|W|-\sum_{j\in J}(|I_j|-1)$, and its closure is a cone of the refinement; every cone arises once. So $\Phi_W(D_j\setminus Z: j\in J)=t^{|W|}\prod_{j\in J} a_j$ with $a_j=\sum_{\varnothing\neq I\subseteq D_j\setminus Z} t^{1-|I|}$, and

$$\Delta=t^{|W|}\prod_{j=1}^{k}(a_j-1),\qquad a_j-1=t^{1-s_j}\big((1+t)^{s_j}-t^{s_j}\big)-1=t^{1-s_j}h_{s_j}.$$

Finally $|W|-\sum_j(s_j-1)=\omega(Z)$, by the proof of Lemma 9.12. □

Each h_s with $s\geq 2$ is $1+t$ times a nonzero polynomial with nonnegative coefficients, so Proposition 9.14 gives Theorem 9.13 again, without Corollary G.3, and with Theorem 9.10 the exact value of $\mathcal{M}_C(Y)$. For disjoint sets, $\omega(Z)=k$ whenever the term is nonzero, the sum over Z factors, and Corollary 9.11 follows. For pairs forming a forest, only $Z=\varnothing$ contributes, which gives the forest case of Theorem 9.15. Since $h_s(0)=1$ for $s\geq 2$, and $\omega(Z)\geq\omega(\varnothing)$ for every nonzero term, the lowest term of $(-1)^k\mathcal{M}_C(Y)$ has degree equal to the total number of components of the two families, if every distinct set has at least two elements. Indeed, $\omega(Z)\geq\omega(\varnothing)$, since each component of the family keeps a reduced member, and a reduced member lies in one component. The power of $1+t$ can exceed the number of distinct sets. For the lower sets $\{c_1,c_2,c_3\}$ and $\{c_3,c_4,c_5\}$, the

set $Z = \varnothing$, the four sets $Z = \{c_i\}$ with $i \neq 3$, the four sets $Z = \{c_i, c_j\}$ with $i \leq 2 < j$ and $j \neq 3$, and the set $Z = \{c_3\}$ give $t\,h_3^2 + 4t\,h_2h_3 + 4t\,h_2^2 + t^2h_2^2 = t(1+t)^3(4t+9)$. With one common upper pair, $\mathcal{M}_C(Y) = -t^2(1+t)^4(4t+9)$, while $k = 3$.

Theorem 9.13 needs no transversality of the envelopes on the faces of B: by the proof of Theorem 9.10, on every face that contributes, the envelope subdivisions are pullbacks of the fans in the proof of Theorem 9.13. For example, the neighbourhoods $\{c_1, c_2\}$, $\{c_2, c_3\}$ below and $\{d_1, d_2\}$, $\{d_2, d_3\}$ above are Berge-acyclic, (PC) fails, and Theorem 9.10 gives $\mathcal{M}_C(Y) = t^2(1+t)^4$ with $k = 4$: on each side only $Z = \varnothing$ contributes, and the two walls give $(4t^3 + 4t^2 + t) - 2(2t^3 + t^2) + t^3 = t(1+t)^2$. When a family contains a Berge cycle, the conclusion can fail: see Example 9.20, whose lower sets share two elements. For three switches whose lower neighbourhoods are the edges of a triangle and whose upper neighbourhoods are one common pair, with $P^- = \{c_1, c_2, c_3\}$ and $P^+ = \{d_1, d_2\}$, Theorem 9.10 gives $\mathcal{M}_C(Y) = -t^2(1+t)^2(2+t)$, although $k = 4$: the upper factor is $t(1+t)$, and the lower factor uses only $Z = \varnothing$ and the braid fan on $\mathbb{R}^3$, which give $(6t^3 + 6t^2 + t) - 3(4t^3 + 4t^2 + t) + 3(2t^3 + t^2) - t^3 = -t(1+t)(2+t)$. The next theorem explains this value.

For a finite multigraph G (loops and parallel edges allowed), let $c(G)$ be its number of components and $\beta(G) = |E(G)| - |V(G)| + c(G)$ its cycle rank. For a finite graph G without loops or multiple edges, let T_G be its Tutte polynomial and χ_G its chromatic polynomial, and for $A \subseteq E(G)$ let $c(A)$ be the number of components of the spanning subgraph $(V(G), A)$. We use the subset expansion $T_G(x,y) = \sum_{A \subseteq E(G)} (x-1)^{c(A)-c(G)}(y-1)^{|A|-|V(G)|+c(A)}$ and the relation $\chi_G(q) = (-1)^{|V(G)|-c(G)} q^{c(G)} T_G(1-q, 0)$ [4]; thus $t^{c(G)} T_G(1+t, 0) = (-1)^{|V(G)|}\chi_G(-t)$.

Theorem 9.15 (Two-element neighbourhoods)**.** *In the layered setting, suppose that every N_i^- and every N_i^+ has two elements. Let G^- be the graph on $\bigcup_i N_i^-$ whose edges are the distinct sets N_i^-, define G^+ likewise, and let k be the total number of edges and f the number of free elements. Then*

$$\begin{aligned}\mathcal{M}_C(Y) &= (-1)^{k+\beta(G^-)+\beta(G^+)}(2+t)^f\, t^{c(G^-)+c(G^+)}\, T_{G^-}(1+t,0)\, T_{G^+}(1+t,0)\\ &= (-1)^{c(G^-)+c(G^+)}(2+t)^f \chi_{G^-}(-t)\,\chi_{G^+}(-t).\end{aligned}$$

In particular $\mathcal{M}_C(Y) \neq 0$, all its nonzero coefficients have the sign $(-1)^{k+\beta(G^-)+\beta(G^+)}$, and $(-1)^k\mathcal{M}_C(Y) \in (1+t)^k\mathbb{Z}_{\geq 0}[t]$ holds if and only if G^- and G^+ are forests.

Distinct pairs give distinct nontrivial subdivisions of B, with the walls $x_a = x_b$ and $y_a = y_b$, so k is also the number of distinct envelope subdivisions, as in Theorem 9.7.

Proof. Consider the lower polynomial. Write $G = G^-$, $V = V(G)$ and $k^- = |E(G)|$. Every element of V lies in a pair, so a nonempty Z meets a pair, whose fan becomes trivial; by Lemma 9.5(1) only $Z = \varnothing$ contributes, and Lemma 9.5(2) removes the repeated pairs with the sign $(-1)^{r-k^-}$. The fan of $\max(x_a, x_b)$ is the hyperplane $x_a = x_b$ with its two half-spaces, so for $J \subseteq E(G)$ the refinement is the graphic arrangement of J in $\mathbb{R}^V$, and the face of a point x is fixed by the signs of $x_a - x_b$, $ab \in J$. The components of the graph on V formed by the edges $ab \in J$ with $x_a = x_b$ are the parts of a partition π of V; the other edges of J join different parts and are oriented from the smaller to the larger value, acyclically. The face determines π and this orientation. Conversely, every partition π of V into parts connected by edges of J, with an acyclic orientation of the multigraph J/π on the parts formed by the edges of J between different parts, arises from exactly one open cone, of dimension $|\pi|$; two parts not joined by an edge of J may take equal values. (Multigraphs may have loops and parallel edges; a loop admits no acyclic orientation, and parallel edges must be oriented alike.) Splitting J into its edges inside and between the parts gives

$$\sum_{J \subseteq E(G)} (-1)^{|E(G)|-|J|}\Phi_V(J) = \sum_{\pi} t^{|\pi|} \prod_{D \in \pi} \mathrm{cs}(G[D]) \cdot \mathrm{ac}^{\pm}(G/\pi),$$

the sum over the partitions π of V into parts D with $G[D]$ connected, with G/π formed like J/π, where

$$\mathrm{cs}(H) = \sum_A (-1)^{|E(H)|-|A|}, \qquad \mathrm{ac}^{\pm}(H) = \sum_{A\subseteq E(H)} (-1)^{|E(H)|-|A|}\,\mathrm{ac}(V(H), A);$$

the first sum is over the edge sets A of the connected spanning subgraphs of H, and ac is the number of acyclic orientations. For a non-loop edge e, deletion and contraction give $\mathrm{ac}(H) = \mathrm{ac}(H-e) + \mathrm{ac}(H/e)$, where H/e keeps all edges other than e; hence $\mathrm{ac}^{\pm}(H) = \mathrm{ac}^{\pm}(H/e)$; a loop reverses the sign. Contracting a spanning forest therefore gives $\mathrm{ac}^{\pm}(H) = (-1)^{\beta(H)}$. (Equivalently, this inverts Stanley's formula $\mathrm{ac}(H) = (-1)^{|V(H)|}\chi_H(-1)$ [24], written with Whitney's expansion of χ_H [29]; for arrangements, compare Zaslavsky's count of regions [30].) Moreover $\mathrm{cs}(G[D]) = (-1)^{\beta(G[D])}\sum_A (-1)^{|A|-|D|+1}$, and $\sum_D \beta(G[D]) + \beta(G/\pi) = \beta(G)$. A partition π together with connected spanning edge sets of the $G[D]$ is the same as one set $A \subseteq E(G)$, with π the components of (V, A). So the sum equals

$$(-1)^{\beta(G)} \sum_{A\subseteq E(G)} t^{c(A)}(-1)^{|A|-|V|+c(A)} = (-1)^{\beta(G)} t^{c(G)} T_G(1+t, 0),$$

and the lower polynomial is $(-1)^{r-k^-+\beta(G)}(2+t)^{f^-}t^{c(G)}T_G(1+t,0)$, with f^- the number of free elements of P^-, and Theorem 9.10 gives the first formula. The second follows from $t^{c(G)}T_G(1+t,0) = (-1)^{|V(G)|}\chi_G(-t)$ and $|E(G)| + \beta(G) + |V(G)| \equiv c(G) \pmod 2$.

The polynomial $T_G(x, 0)$ has nonnegative coefficients and degree $|V(G)|-c(G) = |E(G)|-\beta(G)$, with leading coefficient 1 (the term $A = \varnothing$); this gives $\mathcal{M}_C(Y) \neq 0$ and the sign. For a forest, $T_G = x^{|E(G)|}$; so if both graphs are forests, then $(-1)^k\mathcal{M}_C(Y) = (2+t)^f t^{c(G^-)+c(G^+)}(1+t)^k$. If $\beta(G^-)+\beta(G^+) \geq 1$ and $(1+t)^k$ divided $\mathcal{M}_C(Y)$, then, as $1+t$ is prime to t and to $2+t$, the power $(1+t)^k$ would divide $T_{G^-}(1+t,0)\,T_{G^+}(1+t,0)$, a nonzero polynomial of degree $k-\beta(G^-)-\beta(G^+) < k$. □

The alternating sum over J in this proof is the graphic case of Remark G.6, which gives $(-1)^{m-\mathrm{rk}}t^{\,d-\mathrm{rk}}T(1+t,0)$ for m linear hyperplanes of $\mathbb{R}^d$, where rk is the rank of their normal vectors and T the Tutte polynomial of their matroid; here $m = |E(G)|$, $d = |V(G)|$ and $\mathrm{rk} = |V(G)|-c(G)$. The constant term $T_G(1,0)$ of $T_G(1+t,0)$ is the product, over the components of G, of the numbers of acyclic orientations with a unique source at a fixed vertex [17, Theorem 7.3]; so the lowest term of $\mathcal{M}_C(Y)$ is $\pm 2^f T_{G^-}(1,0)\,T_{G^+}(1,0)\,t^{c(G^-)+c(G^+)}$. For the triangle above, $T_{K_3}(x,0) = x^2+x$ gives the lower factor $-t(1+t)(2+t)$ again. A 4-cycle below and one common pair above, with no free elements, give $T_{C_4}(x,0) = x^3+x^2+x$ and $\mathcal{M}_C(Y) = t^2(1+t)^2(t^2+3t+3)$, whereas $k = 5$.

For a hypergraph $\mathcal{H}$ on a finite set V, that is, a family of subsets of V (its edges; repetitions allowed), let $\chi_{\mathcal{H}}(\nu)$ be the number of maps $V \to \{1, \ldots, \nu\}$ under which no edge is monochromatic. By inclusion–exclusion, $\chi_{\mathcal{H}}(\nu) = \sum_J (-1)^{|J|}\nu^{c(J)}$, the sum over the subfamilies J of edges, where $c(J)$ is the number of components of the graph on V that joins two elements lying in a common edge of J; so $\chi_{\mathcal{H}}$ is a polynomial. An edge with one element gives $\chi_{\mathcal{H}} = 0$, and an edge that contains another edge, or repeats it, can be omitted.

Theorem 9.16 (Chromatic form of the lower polynomial)**.** *Let $N_1, \ldots, N_r$ be nonempty subsets of a finite set N, with union N'. For $x \in \mathbb{R}^{N'}_{\geq 0}$ let $Z(x) = \{a : x_a = 0\}$ and $I_j(x) = \mathrm{argmax}_{b\in N_j} x_b$. The nonempty sets*

$$\sigma = \{x \in \mathbb{R}^{N'}_{\geq 0} : Z(x) = Z,\ I_j(x) = I_j \text{ for } j \in [r]\}$$

are relatively open convex cones. Call σ nonvanishing *if no* N_j *lies in* Z*, that is, if no* $\max_{b\in N_j} x_b$ *vanishes on* σ*, and let* $\mathcal{H}_\sigma$ *be the hypergraph on* $N'\setminus Z$ *with the edges* $I_1,\dots,I_r$*. Then*

$$\Lambda(N_1,\dots,N_r;N) = (-1)^r(2+t)^{|N\setminus N'|}\sum_\sigma (-1)^{\dim\sigma}\chi_{\mathcal{H}_\sigma}(-t),$$

the sum over the nonvanishing cones σ*.*

Proof. For $J\subseteq[r]$ let $\mathcal{O}_J$ be the partition of $\mathbb{R}^{N'}_{\ge 0}$ into the nonempty sets $\{x\in\mathbb{R}^{N'}_{\ge0} : Z(x) = Z,\ I_j(x) = I_j \text{ for } j\in J\}$, which are relatively open convex cones. On the relatively open face $\{x_a = 0 \text{ for } a\in Z,\ x_a>0 \text{ for } a\notin Z\}$ of $\mathbb{R}^{N'}_{\ge0}$ we have $\max_{b\in N_j} x_b = \max_{b\in N_j\setminus Z} x_b$ if $N_j\not\subseteq Z$, and $\max_{b\in N_j} x_b = 0$ otherwise. Every cone of the common refinement of the fans of $\max_{b\in N_j\setminus Z} x_b$, $j\in J$, on $\mathbb{R}^{N'\setminus Z}$ is invariant under adding multiples of $(1,\dots,1)$, so it meets the open positive orthant in its relative interior, with the same dimension. Hence the cells of $\mathcal{O}_J$ in this face correspond one to one, with the same dimensions, to these cones, and

$$F_J(t) := \sum_{\sigma\in\mathcal{O}_J} t^{\dim\sigma} = \sum_{Z\subseteq N'} \Phi_{N'\setminus Z}(N_j\setminus Z : j\in J),$$
$$\Lambda(N_1,\dots,N_r;N) = (2+t)^{|N\setminus N'|}\sum_{J\subseteq[r]}(-1)^{r-|J|}F_J(t).$$

The *braid cells* of $\mathbb{R}^{N'}_{\ge0}$, the sets of points with a given zero set and a given weak order of the remaining coordinates, partition $\mathbb{R}^{N'}_{\ge0}$. Each cell of $\mathcal{O}_J$ is a union of braid cells, since $Z(x)$ and the sets $I_j(x)$ are read off from the weak order. A relatively open polyhedral cone of dimension d is homeomorphic to $\mathbb{R}^d$, so its Euler characteristic with compact supports is $(-1)^d$, and this characteristic is additive over finite partitions into such cones (it is the o-minimal Euler characteristic, which is additive over finite partitions into definable sets [28]). So $\sum_{\beta\subseteq\sigma}(-1)^{\dim\beta} = (-1)^{\dim\sigma}$ for $\sigma\in\mathcal{O}_J$, the sum over the braid cells β in σ, and

$$F_J(t) = \sum_\beta (-1)^{\dim\beta}(-t)^{\dim\sigma_J(\beta)},$$

where $\sigma_J(\beta)$ is the cell of $\mathcal{O}_J$ that contains β. Let β have the zero set Z and the argmax sets I_j. The cell $\sigma_J(\beta)$ is cut out of the linear space $\{x_a = 0 \text{ for } a\in Z,\ x_a = x_b \text{ for } a,b\in I_j,\ j\in J\}$ by strict inequalities and contains β, so its dimension is the number $c_J(\beta)$ of components of the graph on $N'\setminus Z$ that joins a and b when $a,b\in I_j$ for some $j\in J$ with $N_j\not\subseteq Z$ (compare Lemma 9.12). If some N_j lies in Z, then $c_J(\beta)$ does not depend on whether $j\in J$, and $\sum_J(-1)^{r-|J|}(-t)^{c_J(\beta)} = 0$. Otherwise this sum is $(-1)^r\chi_{\mathcal{H}_\beta}(-t)$ by the inclusion–exclusion formula for χ, where $\mathcal{H}_\beta$ has the edges $I_1,\dots,I_r$. Finally, Z and the sets I_j, $j\in[r]$, are constant on each cell σ of $\mathcal{O}_{[r]}$; grouping the braid cells by these cells and using the Euler characteristic once more proves the theorem. □

For two-element sets, a nonvanishing cone with $\chi_{\mathcal{H}_\sigma}\neq 0$ has $x_a = x_b > 0$ for every pair $\{a,b\}$, so there is exactly one, the set of positive x that are constant on the components of G^-, and Theorem 9.16 gives the lower factor $(-1)^{r+c(G^-)}(2+t)^{f^-}\chi_{G^-}(-t)$ of Theorem 9.15, with f^- the number of free elements of P^-. For a Berge-acyclic family with k^- distinct sets, the distinct edges of each $\mathcal{H}_\sigma$ with $\chi_{\mathcal{H}_\sigma}\neq 0$ are k^- sets, one inside each distinct N_j, and they again form a Berge-acyclic family; repeated edges do not change χ. For such a hypergraph, $\chi_{\mathcal{H}}(\nu) = \nu^\omega\prod_e\big(\nu^{|e|-1}-1\big)$, where the product is over the distinct edges and ω is the number of components, isolated vertices included. This follows by removing, one at a time, an edge that meets the other edges in at most one vertex. Since $\nu-1$ divides $\nu^{s-1}-1$, every term of Theorem 9.16 is divisible by $(1+t)^{k^-}$: this is the divisibility in Theorem 9.13, term by term; compare Proposition 9.14.

Example 9.17 (Berge cycles that keep the sign rule)**.** Take three switches with one common upper pair $P^+ = \{d_1, d_2\}$, so that $\Lambda(N_1^+, N_2^+, N_3^+; P^+) = (-1)^{3-1}H_2 = t(1+t)$ (see the proof of Corollary 9.11), and write the lower elements as $1, 2, \ldots$; P^- is the union of the lower sets, so there are no free elements. In Theorem 9.16 only nonvanishing cones with $|I_j| \geq 2$ for all j contribute.

(a) Lower sets $\{1,2\}$, $\{1,2,3\}$, $\{3,4\}$, not Berge-acyclic. A contributing cone has $x_1 = x_2 = \alpha > 0$ and $x_3 = x_4 = \mu > 0$, and $\{1,2,3\}$ has at least two elements at its maximum, so $\alpha \geq \mu$. There are two such cones, $\alpha > \mu$ of dimension 2 and $\alpha = \mu$ of dimension 1. In both, the edge I_2 contains $\{1,2\}$, so $\chi_{\mathcal{H}_\sigma}(-t) = t^2(1+t)^2$, and the two terms cancel: $\Lambda(N_1^-, N_2^-, N_3^-; P^-) = 0$, hence $\mathcal{M}_C(Y) = 0$.

(b) Lower sets $\{1,2\}$, $\{1,2,3,4\}$, $\{3,5\}$, not Berge-acyclic. A contributing cone has $x_1 = x_2 = \alpha > 0$, $x_3 = x_5 = \mu > 0$ and $x_4 = \delta \geq 0$, and $\{1,2,3,4\}$ has at least two elements at its maximum: either $\alpha = \max(\alpha, \mu, \delta)$, or $\mu = \delta > \alpha$. In the first case I_2 contains $\{1,2\}$, and $\chi_{\mathcal{H}_\sigma}(-t)$ is $-t^3(1+t)^2$ if $\delta > 0$ and $t^2(1+t)^2$ if $\delta = 0$. There are six such cones: $I_2 = \{1,2\}$ with $\delta > 0$ (dimension 3) or $\delta = 0$ (dimension 2), $I_2 = \{1,2,3\}$ with $\delta > 0$ (dimension 2) or $\delta = 0$ (dimension 1), $I_2 = \{1,2,4\}$ (dimension 2) and $I_2 = \{1,2,3,4\}$ (dimension 1). Their terms $(-1)^{\dim\sigma}\chi_{\mathcal{H}_\sigma}(-t)$ add up to $t^3(1+t)^2(1-1-1+1) + t^2(1+t)^2(1-1) = 0$. In the second case there is one cone, $x_3 = x_4 = x_5 > x_1 = x_2 > 0$, of dimension 2, with the edges $\{1,2\}$, $\{3,4\}$, $\{3,5\}$ and $\chi_{\mathcal{H}_\sigma}(-t) = -t^2(1+t)^3$. So $\Lambda(N_1^-, N_2^-, N_3^-; P^-) = (-1)^3\big(-t^2(1+t)^3\big) = t^2(1+t)^3$ and $\mathcal{M}_C(Y) = t^3(1+t)^4$, with $k = 4$: the conclusion of Theorem 9.13 holds.

So Berge-acyclicity is not necessary in Theorem 9.13 once neighbourhoods may have three or more elements, even when $\mathcal{M}_C(Y) \neq 0$, as part (b) shows. ◇

Example 9.18 (Common lower and upper neighbourhoods)**.** Let $c_1, c_2 < q_1, q_2 < d_1, d_2$, with $C = \{c_1, c_2, d_1, d_2\}$ and $Y = \{q_1, q_2\}$. Here $\mathcal{S}_1 = \mathcal{S}_2$, $\mathcal{T}_1 = \mathcal{T}_2$, $k = 2$, and Corollary 9.11 gives $\mathcal{M}_C(Y) = H_2^2 = t^2(1+t)^2$. Thus the mixed differences of f_2, f_3, f_4 are 1, 2, 1. ◇

Example 9.19 (A negative mixed difference)**.** Let $c_1, c_2 < q_1$ and $c_3, c_4 < q_2$, and let $q_1, q_2 < d_1, d_2$, with all c_i, d_j in C. Here $\mathcal{T}_1 = \mathcal{T}_2$ and $k = 3$, so

$$\mathcal{M}_C(Y) = -H_2^3 = -t^3(1+t)^3 :$$

the mixed differences of $f_3, \ldots, f_6$ are $-1, -3, -3, -1$. This can also be read off from Proposition E.3 directly, applied to $\Delta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1) = -\mathcal{M}_C(Y)$; the three subdivisions are mutually transverse by Proposition 9.9, as noted after Corollary 9.11. Only cells that lie, within their carrier, in a wall of each of the three subdivisions contribute; in the notation of Proposition E.3, these are the cells with every $\alpha_i \geq 1$. The three walls $x_{c_1} = x_{c_2}$, $x_{c_3} = x_{c_4}$ and $y_{d_1} = y_{d_2}$ meet in a single cell with carrier B, of dimension 3, in the 6-dimensional base, and it contributes $t^3(1+t)^3$. Every facet of B makes one envelope affine: $x_{c_1} = 0$ gives $L_1 = x_{c_2}$, and similarly for the other x_{c_i}; $y_{d_j} = 0$ makes $U_1 = U_2$ affine; and the facets $x_{c_i} + y_{d_j} = 1$ are treated in the proof of Theorem 9.10. By Lemma 9.5(1), no proper face contributes. ◇

Example 9.20 (Mixed signs)**.** Let $c_1, c_2 < q_1$ and $c_1, c_2, c_3, c_4 < q_2$, and let $q_1 < d_1, d_2$ and $q_2 < d_3, d_4$, with all c_i, d_j in C. The lower sets $\{1,2\}$ and $\{1,2,3,4\}$ (indices of the c_i) are nested. The upper neighbourhoods are disjoint, and the proof of Corollary 9.11 gives $\Lambda(N_1^+, N_2^+; P^+) = H_2^2 = t^2(1+t)^2$. In $\Lambda(N_1^-, N_2^-; P^-)$, each Z meeting $\{1,2\}$ leaves at most one element of the first set and contributes 0. For $Z = \{3\}$, the fans of $\max(x_1, x_2)$ and $\max(x_1, x_2, x_4)$ on $\mathbb{R}^3$ give $(t + 4t^2 + 4t^3) - (t^2 + 2t^3) - (t + 3t^2 + 3t^3) + t^3 = 0$, and $Z = \{4\}$ is the same. For $Z = \{3,4\}$ the two fans coincide on $\mathbb{R}^2$, giving $(t + 2t^2) - 2(t + 2t^2) + t^2 = -t(1+t)$. For $Z = \varnothing$, the wall $x_1 = x_2$ cuts

exactly the cones of the fan of $\max(x_1, \dots, x_4)$ whose argmax set lies in $\{3, 4\}$, each into three cells; the refinement therefore has $2t^4+3t^3+t^2$ more cells than the second fan, and the alternating sum is $(2t^4+3t^3+t^2)-(2t^4+t^3)+t^4 = t^2(1+t)^2$. So $\Lambda(N_1^-, N_2^-; P^-) = t^2(1+t)^2-t(1+t) = t(1+t)(t^2+t-1)$, and

$$\mathcal{M}_C(Y) = t^3(1+t)^3(t^2+t-1) = -t^3 - 2t^4 + t^5 + 5t^6 + 4t^7 + t^8.$$

Both signs occur. By Theorem 9.7, the four envelope subdivisions of this example are not mutually transverse; the lower sets are not Berge-acyclic, and (PC) fails, since $\mathcal{S}_1$ and $\mathcal{S}_2$ share the components $\{c_1\}$ and $\{c_2\}$. ◇

Example 9.21 (A poset with crossed covers)**.** Let $c_1, c_2 < q_1 < d_3, d_4$ and $c_3, c_4 < q_2 < d_1, d_2$, add the covers $c_1 < d_1$ and $c_2 < d_2$, and take all c_i, d_j in C. This poset is not layered, and (PC) fails: the component $\{c_1, d_1\}$ of Γ is used by $\mathcal{S}_1$ and by $\mathcal{T}_2$. Example 9.24(a) below proves that its mixed difference is $t^3(1+t)^3(t^2+t-1)$, the value of Example 9.20. Hence, by Theorem 9.7, its four envelope subdivisions are not mutually transverse either. ◇

9.5 Crossed covers

We add relations between lower and upper elements of different switches to a layered poset and follow the faces of the base. For a poset X of this kind, with the same C, Y and marks, we write $B(X)$ for its base and $\mathcal{M}_C(X)$ for its mixed difference. Here X is a general poset of this kind, not the five-element poset of Section 1.

Lemma 9.22 (Cuts that do not meet)**.** *Let E be a face of B, and let $\mathcal{G}_1, \mathcal{G}_2$ be subdivisions of B, each trivial or a single cut by a hyperplane $\{w_k = 0\}$. If no point of* relint(E) *has $w_1 = w_2 = 0$, then $\Delta_E(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots, \mathcal{G}_m) = 0$ for all subdivisions $\mathcal{G}_3, \dots, \mathcal{G}_m$.*

Proof. Group the subsets $J \subseteq [m]$ by $J' = J \setminus \{1, 2\}$. The relatively open cells of a join are the nonempty intersections of relatively open cells of its factors. So the cells with carrier E of $\mathcal{G}_{J'} \vee \mathcal{G}_K$, $K \subseteq \{1, 2\}$, are those of $\tau \vee \mathcal{G}_K$, over the relatively open cells τ of $\mathcal{G}_{J'}$ with carrier E. Fix such a cell τ, of dimension v. If $\mathcal{G}_1$ is trivial or w_1 has constant sign on τ (possibly 0 on all of τ), joining with $\mathcal{G}_1$ changes no cell inside τ, and the four terms for τ cancel in pairs; likewise for w_2. Otherwise both walls cut τ and do not meet in it. The wall of w_2 in τ is convex and misses the wall of w_1, so it lies in one of the two halves of τ and cuts it, and w_2 has constant sign on the other half and on the wall of w_1 in τ. Then $\tau \vee \mathcal{G}_1 \vee \mathcal{G}_2$ has three cells of dimension v and two of dimension $v-1$, and $(3t^v + 2t^{v-1}) - 2(2t^v + t^{v-1}) + t^v = 0$, as in Section 6.1. □

Proposition 9.23 (Crossed covers)**.** *Let X_0 be a layered poset as in Section 9.4, and let X be obtained by adding relations $c_k < d_k$ $(k = 1, \dots, s)$, where $c_k \in \bigcup_i N_i^-$, $d_k \in \bigcup_j N_j^+$ and $c_k \not< d_k$ in X_0.*

(1) The faces of $B(X)$ on which no inequality $x_{c_k} + y_{d_k} \le 1$ is tight contribute $\mathcal{M}_C(X_0)$ to the face sum of Theorem 9.4.

(2) If $s = 2$, $N_a^- = \{c_1, c_2\}$ and $N_b^+ = \{d_1, d_2\}$ with $a \neq b$, the faces on which exactly one of the two inequalities is tight contribute 0, and

$$\mathcal{M}_C(X) = \mathcal{M}_C(X_0) - \sum_{E \subseteq B_{12}} \Delta_E(\mathcal{G} : \mathcal{G} \neq \mathcal{T}_b),$$

where B_{12} is the face of $B(X)$ on which both are tight, E runs over its faces, and $\mathcal{G}$ runs over the entries $\mathcal{S}_1, \mathcal{T}_1, \dots, \mathcal{S}_r, \mathcal{T}_r$ except the one entry $\mathcal{T}_b$ (an entry $\mathcal{T}_j$ with $j \neq b$ stays even if $\mathcal{T}_j = \mathcal{T}_b$).

Proof. No element of Y gains a lower or an upper cover, so the envelopes are those of X_0, and $B(X) = B(X_0) \cap \overline{R}$ with $R = \{x_{c_k} + y_{d_k} < 1 \text{ for all } k\}$.

(1) Let E be such a face and F the face of $B(X_0)$ with the same tight inequalities. Then $\operatorname{relint}(E) = \operatorname{relint}(F) \cap R$ and $\dim E = \dim F$, and the cells with carrier E of every join are the nonempty sets $\sigma \cap R$ over its cells σ with carrier F. If F is not one of the faces $E(Z, \varepsilon, Z', \varepsilon')$ of the proof of Theorem 9.10, some envelope is affine on F, hence on E, and $\Delta_E = \Delta_F = 0$ by Lemmas 9.2(1) and 9.5(1). If it is one of them, scaling the coordinates of $\bigcup_i N_i^- \cup \bigcup_i N_i^+$ of a point of σ by $\theta \in (0, 1]$ keeps it in $\operatorname{relint}(F)$ and in σ. Indeed, by the Cells step of the proof of Theorem 9.10, these cells are given by cones, which are invariant under positive scaling. For small θ the scaled point also lies in R. So every such σ meets R, and $\Delta_E = \Delta_F$. The same scaling of a point of $\operatorname{relint}(F)$ shows that each face $F = E(Z, \varepsilon, Z', \varepsilon')$ of $B(X_0)$ meets R, so it arises from exactly one such face E. Theorem 9.4 for X_0 gives the sum.

(2) $\mathcal{S}_a$ and $\mathcal{T}_b$ are single cuts by the walls of $w_S = x_{c_1} - x_{c_2}$ and $w_T = y_{d_1} - y_{d_2}$, and

$$w_S + w_T = (x_{c_1} + y_{d_1}) - (x_{c_2} + y_{d_2}).$$

If exactly one inequality is tight on E, the other is strict on $\operatorname{relint}(E)$, so the walls do not meet there, and $\Delta_E = 0$ by Lemma 9.22. On B_{12} we have $w_T = -w_S$, so $\mathcal{S}_a$ and $\mathcal{T}_b$ have the same cells contained in B_{12}. For a face E of B_{12}, a hyperplane supporting B along E supports every cell. So a cell of a join contained in E is an intersection of cells of the factors contained in E. Hence the joins of any $\mathcal{G}_{J'}$ with $\mathcal{S}_a$, with $\mathcal{T}_b$ and with both have the same cells contained in E, and the proof of Lemma 9.5(2) gives $\Delta_E(\dots, \mathcal{S}_a, \dots, \mathcal{T}_b, \dots) = -\Delta_E(\dots \text{ without } \mathcal{T}_b)$. Sum over the faces of B_{12} and use part (1). □

Example 9.24 (Three layers)**.** Write $H = t(1 + t)$; this is H_2 of Corollary 9.11.

(a) In Example 9.21, X_0 has $c_1, c_2 < q_1 < d_3, d_4$ and $c_3, c_4 < q_2 < d_1, d_2$, so $\mathcal{M}_C(X_0) = H^4$ by Corollary 9.11; Proposition 9.23(2) applies with $a = 1$, $b = 2$. On B_{12} put $y_{d_i} = 1 - x_{c_i}$ $(i = 1, 2)$; then B_{12} is the 6-dimensional polytope given by $0 \le x_{c_3}, x_{c_4} \le x_{c_1}, x_{c_2} \le 1$; $y_{d_3}, y_{d_4} \ge 0$; and $x_{c_i} + y_{d_j} \le 1$ $(i = 1, 2,\ j = 3, 4)$. Where one of these inequalities is tight, an envelope is affine: L_2 on each of $x_{c_3} = 0$, $x_{c_4} = 0$, $x_{c_3} = x_{c_i}$ and $x_{c_4} = x_{c_i}$; L_1 on each of $x_{c_i} = 0$, $x_{c_i} = 1$ and $x_{c_i} + y_{d_j} = 1$; and U_1 on $y_{d_j} = 0$ $(i = 1, 2,\ j = 3, 4)$. So, by Lemmas 9.2(1) and 9.5(1), only cells with carrier B_{12} count. There the walls $x_{c_1} = x_{c_2}$, $y_{d_3} = y_{d_4}$, $x_{c_3} = x_{c_4}$ are independent and meet at $x_{c_1} = x_{c_2} = \frac{1}{2}$, $x_{c_3} = x_{c_4} = y_{d_3} = y_{d_4} = \frac{1}{4}$; every sign region in $\operatorname{relint}(B_{12})$ is convex and nonempty, so $\Delta_{B_{12}}(\mathcal{S}_1, \mathcal{T}_1, \mathcal{S}_2) = t^6(1 + t^{-1})^3 = t^3(1 + t)^3 = H^3$. By Proposition 9.23, the faces with no, one, or two tight cross inequalities give H^4, 0 and $-H^3$, and $\mathcal{M}_C(X) = H^4 - H^3$.

(b) By the proof of Theorem 9.10, the faces of Example 9.20 with lower zero set Z give the Z-term of $\Lambda(N_1^-, N_2^-; P^-)$ times $\Lambda(N_1^+, N_2^+; P^+) = H^2$. Example 9.20 computes these terms: $Z = \varnothing$ gives H^2; Z meeting $\{1, 2\}$ gives 0; $Z = \{3\}$ or $Z = \{4\}$ gives 0; and $Z = \{3, 4\}$ gives $-H$, since on $x_{c_3} = x_{c_4} = 0$ we have $L_2 = L_1$ and Lemma 9.5(2) removes $\mathcal{S}_2$ with a minus sign. So both mixed differences split into the same three layers H^4, 0, $-H^3$; in particular they are equal.

(c) For the poset with $c_1, c_2 < q_1 < d_1, d_2$ and $c_3, c_4 < q_2 < d_3, d_4$ and the one added cover $c_3 < d_1$ (it returns in Remark 9.58), Proposition 9.23(1) gives H^4. On the facet $B' = \{x_{c_3} + y_{d_1} = 1\}$, of dimension 7, an envelope is affine wherever one of its inequalities is tight: L_1 for $x_{c_1} = 0$, $x_{c_2} = 0$, $x_{c_3} = 0$ and the rows of $c_1, c_2 < d_1, d_2$; L_2 for $x_{c_4} = 0$ and the rows of $c_4 < d_3, d_4$; U_1 for $y_{d_2} = 0$; U_2 for $y_{d_1} = 0$, $y_{d_3} = 0$, $y_{d_4} = 0$ and the rows of $c_3 < d_3, d_4$. The four walls

$x_{c_1} = x_{c_2}$, $x_{c_3} + y_{d_2} = 1$, $x_{c_3} = x_{c_4}$, $y_{d_3} = y_{d_4}$ are independent and meet in its relative interior, at $(x_{c_1}, \dots, x_{c_4}) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$ and $(y_{d_1}, \dots, y_{d_4}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. As in (a), only the cells with carrier B' count, and each sign region of the four walls in $\mathrm{relint}(B')$ is convex and nonempty. So $\mathcal{M}_C(X) = H^4 + t^{7-4}(1+t)^4 = t^3(1+t)^5$. ◇

9.6 Neighbourhoods with three or more elements

Fix one side of the layered setting: subsets $N_1, \dots, N_r$ of a finite set N with union N', the distinct sets $D_1, \dots, D_k$ among them, and $f = |N \setminus N'|$. Call $Z \subseteq N'$ *allowed* if $|D_j \setminus Z| \geq 2$ for all j. For an indexed family $F = (F_1, \dots, F_m)$ of subsets of a finite set W put

$$\Delta_W(F) = \sum_{J \subseteq [m]} (-1)^{m-|J|} \Phi_W(F_j : j \in J).$$

As in the proof of Theorem 9.13, Lemma 9.5 gives

$$\Lambda(N_1, \dots, N_r; N) = (-1)^{r-k}(2+t)^f \sum_{Z \text{ allowed}} \Delta_{N' \setminus Z}(D_1 \setminus Z, \dots, D_k \setminus Z), \tag{16}$$

where now the sets $D_j \setminus Z$ may coincide. For an indexed family of sets with at least two elements, call two members *linked* if they lie in a common 2-connected block of the incidence graph; the classes of the equivalence relation generated by linkage are the *groups*, and V_g is the union of the members of a group g. A member forms a group alone exactly when it lies on no cycle of the incidence graph, and the two members of a group with two members share at least two elements.

Lemma 9.25 (Blocks)**.** *Let F be an indexed family of subsets of a finite set W such that the union of F is W and each member of F has at least two elements, and let ω be the number of components of its incidence graph. Then the sets V_g form a Berge-acyclic family, $|W| = \omega + \sum_g (|V_g| - 1)$, and*

$$\Delta_W(F) = t^\omega \prod_g \delta_g, \qquad \delta_g = t^{-1} \Delta_{V_g}(\text{the members of } g).$$

Proof. In the block-cut forest of the incidence graph, contract each member that is a cut vertex together with the blocks containing it; a block containing no such member stays a single node. This gives a forest whose nodes are the groups and the elements that are cut vertices, and such an element e is joined exactly to the groups g with $e \in V_g$; every other element lies in exactly one V_g. So the incidence graph of the family $(V_g)_g$ is a forest. By Lemma 9.12 and its proof, the restriction map $\rho\colon \mathbb{R}^W \to \prod_g \mathbb{R}^{V_g}/\mathbb{R}(1, \dots, 1)$ is onto, and its kernel has dimension $|W| - \sum_g (|V_g| - 1) = \omega$. For a member D of g, the fan of $\max_{b \in D} x_b$ is the preimage under ρ of a fan on the factor of g, and every cone of a refinement of such fans on $\mathbb{R}^{V_g}$ contains $\mathbb{R}(1, \dots, 1)$. Hence, for every subfamily F_J, the cones of its refinement are the preimages of the products of the cones of the refinements of the subfamilies $F_{J \cap g}$, and $\Phi_W(F_J) = t^\omega \prod_g t^{-1} \Phi_{V_g}(F_{J \cap g})$. So the alternating sum over J factors over the groups. □

For $n \geq 1$ put $e_n = (1+t)^n - t^n$ and $\gamma_n = (1+t)^{n-1} - t^{n-1}$, so that $h_n = (1+t)\gamma_n$ and $\gamma_n \in \mathbb{Z}_{\geq 0}[t]$.

Lemma 9.26 (Two-member groups)**.** *Let $A \neq B$ be finite sets with $c = |A \cap B| \geq 2$, $p = |A \setminus B| \leq q = |B \setminus A|$ and $W = A \cup B$. Then*

$$t^{-1} \Delta_W(A, B) = \begin{cases} t\, h_c h_q, & p = 0, \\ (1+t)^2 \big(e_p e_q e_c - t^2 (1+t) \gamma_p \gamma_q \gamma_c \big), & p \geq 1, \end{cases}$$

which for $p \geq 1$ is a nonzero element of $(1+t)^2\mathbb{Z}_{\geq 0}[t]$, and for $p = 0$ lies in $t(1+t)^2\mathbb{Z}_{\geq 0}[t]$ and vanishes exactly when $q = 1$. Moreover $t^{-1}\Delta_A(A, A) = -h_{|A|}$.

Proof. For $n \geq 0$ let $a_n = \sum_S t^{1-|S|}$ over the nonempty subsets S of an n-element set; so $a_0 = 0$, and $a_n = t^{1-n}e_n$, $a_n - 1 = t^{1-n}h_n$ for $n \geq 1$. For $x \in \mathbb{R}^W$ let α, θ, β be the maxima of x over $A \setminus B$, $A \cap B$, $B \setminus A$ (with $\alpha = -\infty$ if $p = 0$). The pair $(\operatorname{argmax}_A x, \operatorname{argmax}_B x)$ determines, and is determined by, the position of θ relative to α and β and the argmax sets S of x in the parts that attain $\max_A x$ or $\max_B x$; its cone has dimension $|W|$ minus $\sum(|S| - 1)$ minus the number of equalities $\theta = \alpha$, $\theta = \beta$. With $X = a_p$, $U = a_c$, $Y = a_q$ and $w = t^{|W|}$, the nine positions give

$$\Phi_W(A, B) = w\big(U + (1 + t^{-1})(XU + UY) + (t^{-2} + 2t^{-1})XUY + XY\big),$$

and likewise $\Phi_W(A) = w(X + U + t^{-1}XU)$, $\Phi_W(B) = w(U + Y + t^{-1}UY)$. Hence

$$\Delta_W(A, B) = w\big(XYU(1 + t^{-1})^2 - (X - 1)(Y - 1)(U - 1)\big).$$

For $p = 0$ this is $t^2h_qh_c = t^2(1+t)^2\gamma_q\gamma_c$; for $p \geq 1$ it is $t(1+t)^2e_pe_qe_c - t^3h_ph_qh_c$, and $h_n = (1+t)\gamma_n$ gives the formula. Since $e_n = (1+t)\gamma_n + t^{n-1}$, the polynomial $e_pe_qe_c - (1+t)^3\gamma_p\gamma_q\gamma_c - t^{p+q+c-3}$ has nonnegative coefficients, and $(1 + t)^3 - t^2(1 + t) = (1 + t)(1 + 2t)$; so the bracket has nonnegative coefficients and contains $t^{p+q+c-3}$. The last claim follows from Lemma 9.5(2) and Proposition 9.14 for one set. □

Theorem 9.27 (Pendant pairs)**.** *Suppose that the distinct sets $D_1, \ldots, D_k$ of one side, each with at least two elements, satisfy:*

(i) every group $\{A, B\}$ with two members has a pendant pair*: a set $D_j = \{x, y\}$ with $x \in A \triangle B$ and $y \notin A \cup B$;*

(ii) every group has at most two members.

For allowed Z let $\omega(Z)$ be as in Proposition 9.14, and for a group g of the family $(D_j \setminus Z)_j$ let $\delta_g(Z) = h_s$ if g consists of one set with s elements, and $\delta_g(Z) = t^{-1}\Delta_{V_g}(A, B)$, as in Lemma 9.26, if $g = \{A, B\}$. Then

$$\Lambda(N_1, \ldots, N_r; N) = (-1)^{r-k}(2 + t)^f \sum_{Z \text{ allowed}} t^{\omega(Z)} \prod_g \delta_g(Z),$$

and $(-1)^{r-k}\Lambda(N_1, \ldots, N_r; N) \in (2 + t)^f(1 + t)^k\mathbb{Z}_{\geq 0}[t]$. If both sides satisfy (i) and (ii), then $(-1)^k\mathcal{M}_C(Y) \in (2 + t)^f(1 + t)^k\mathbb{Z}_{\geq 0}[t]$, where now k and f are counted over both sides, as in Theorem 9.13.

Proof. Fix an allowed Z. We first use only (ii). Call $(D_j \setminus Z)_j$ the *reduced family*. A 2-connected block of a subgraph lies in a 2-connected block of the graph, and two 2-connected blocks of the reduced family that share a member lie in 2-connected blocks of the family that share it. So the groups of the reduced family have at most two members. If $D_a \setminus Z = D_b \setminus Z$ with $a \neq b$, then $|D_a \cap D_b| \geq 2$, so $\{D_a, D_b\}$ is a group; thus two reduced sets coincide exactly when they come from a group $\{A, B\}$ and Z contains $A \triangle B$. By (i), Z then contains the element x of the pendant pair $\{x, y\}$ of $\{A, B\}$, so $|\{x, y\} \setminus Z| \leq 1$ and Z is not allowed. So the reduced sets are distinct, and by Lemma 9.25 the term of Z in (16) is $t^{\omega(Z)} \prod_g \delta_g(Z)$, where a one-member group gives h_s by Proposition 9.14. By that proposition and Lemma 9.26, each $\delta_g(Z)$ lies in $(1 + t)^{|g|}\mathbb{Z}_{\geq 0}[t]$. By Theorem 9.10, $\mathcal{M}_C(Y)$ is the product of the lower and the upper factor, and the claim follows. □

Under (ii), a set other than A and B meets $A \cup B$ in at most one element, so if it keeps at most one element when $A \triangle B$ is deleted, it is a pair as in (i). So (i) says exactly that the deletion $Z = A \triangle B$ is not allowed; this deletion makes A and B equal, and Lemma 9.26 gives $-h_c$, with a single factor $1+t$ and the wrong sign, that is, the sign opposite to that in Theorem 9.13. Theorem 9.27 explains Example 9.17: the pendant pairs are $\{3,4\}$ in (a), where every term vanishes, and $\{3,5\}$ in (b), where only $Z = \varnothing$ contributes, and its term is $t \cdot t\, h_2 h_2 \cdot h_2 = t^2(1+t)^3$.

Lemma 9.28 (Unions)**.** *If N_b is the union of the sets N_j with $N_j \subsetneq N_b$, then $\Lambda(N_1, \dots, N_r; N) = -\Lambda(N_j : j \neq b; N)$.*

Proof. Both families have the union N'. A cone as in Theorem 9.16 is nonvanishing for one family exactly when it is nonvanishing for the other, since $N_b \subseteq Z$ forces $N_j \subseteq Z$ for the sets N_j inside N_b. At a point x of a cone σ, the maximum over N_b is attained on some $N_a \subsetneq N_b$, so $I_a = I_b \cap N_a \subseteq I_b$. The edge I_b contains the edge I_a and can be omitted, so $\chi_{\mathcal{H}_\sigma} = \chi_{\mathcal{H}'_{\sigma'}}$, where $\sigma' \supseteq \sigma$ is the cone of the family without N_b and $\mathcal{H}'_{\sigma'}$ its hypergraph. Each σ' is the disjoint union of the cones σ in it, which are relatively open polyhedral cones, so $\sum_{\sigma \subseteq \sigma'} (-1)^{\dim \sigma} = (-1)^{\dim \sigma'}$, by the additivity of the Euler characteristic, as in the proof of Theorem 9.16. Hence the sums in Theorem 9.16 for the two families agree, and only the sign $(-1)^r$ changes. □

Lemma 9.29 (Private elements)**.** *Suppose that $N_b = N_a \cup P$ for some $a \neq b$, where P is nonempty and meets no N_j with $j \neq b$, and put $p = |P|$. Then*

$$(2+t)^p \Lambda(N_1, \dots, N_r; N) = \big(t(2+t)^p - (1+t)^{p+1}\big)\Lambda(N_j : j \neq b; N).$$

Proof. Let W be a finite set, P' any nonempty subset of W, $p' = |P'|$, $W' = W \setminus P'$, $\mathcal{G}$ an indexed family of subsets of W', and A' a member of $\mathcal{G}$. We first show

$$\Delta_W(\mathcal{G}, A' \cup P') = t\, h_{p'}\, \Delta_{W'}(\mathcal{G}). \tag{17}$$

If A' is empty, both sides vanish by Lemma 9.5(1). Otherwise let $m(y) = \max_{e \in A'} y_e$ for $y \in \mathbb{R}^{W'}$. For $J \subseteq \mathcal{G}$, the cones of the refinement of J in $\mathbb{R}^W = \mathbb{R}^{W'} \times \mathbb{R}^{P'}$ are the sets $\sigma \times \mathbb{R}^{P'}$, over the cones σ of its refinement in $\mathbb{R}^{W'}$, so $\Phi_W(J) = t^{p'} \Phi_{W'}(J)$. Adding the fan of $\max_{A' \cup P'} x$ replaces them by the following relatively open cones, where $(y, s) \in \mathbb{R}^{W'} \times \mathbb{R}^{P'}$.

(a) For each nonempty $I \subseteq P'$: the set where $y \in \sigma$, $\operatorname{argmax}_{e \in P'} s_e = I$ and $s_e > m(y)$ for $e \in I$, of dimension $\dim \sigma + p' - |I| + 1$.

(b) For each cone τ of the refinement of $J \cup \{A'\}$ in $\mathbb{R}^{W'}$, on which m is linear: for each nonempty $I \subseteq P'$, the set where $y \in \tau$, $\operatorname{argmax}_{e \in P'} s_e = I$ and $s_e = m(y)$ for $e \in I$, of dimension $\dim \tau + p' - |I|$; and the set where $y \in \tau$ and $s_e < m(y)$ for all $e \in P'$, of dimension $\dim \tau + p'$.

These sets are nonempty and partition each $\sigma \times \mathbb{R}^{P'}$; each is the intersection of $\sigma \times \mathbb{R}^{P'}$ with one cone of the fan of $\max_{A' \cup P'} x$ ($\operatorname{argmax} I$ in case (a), and the argmax over A' on τ together with I in case (b)), so it is one cone of the refinement. Over each σ the sets in (a) give $\sum_{\varnothing \neq I \subseteq P'} t^{\dim \sigma + p' - |I| + 1} = t^{\dim \sigma}\, t\big((1+t)^{p'} - t^{p'}\big)$, and over each τ the sets in (b) give $\sum_{I \subseteq P'} t^{\dim \tau + p' - |I|} = t^{\dim \tau}(1+t)^{p'}$ (the last set is the term $I = \varnothing$). Subtracting $\Phi_W(J) = t^{p'} \Phi_{W'}(J)$, we get

$$\Phi_W(J \cup \{A' \cup P'\}) - \Phi_W(J) = t\big((1+t)^{p'} - t^{p'} - t^{p'-1}\big)\Phi_{W'}(J) + (1+t)^{p'}\, \Phi_{W'}(J \cup \{A'\}),$$

and $(1+t)^{p'} - t^{p'} - t^{p'-1} = h_{p'}$, with $h_1 = 0$. Now

$$\Delta_W(\mathcal{G}, A' \cup P') = \sum_{J \subseteq \mathcal{G}} (-1)^{|\mathcal{G}| - |J|} \big(\Phi_W(J \cup \{A' \cup P'\}) - \Phi_W(J)\big),$$

and in this sum the second parts of the terms for J and $J\triangle\{A'\}$ cancel, since A' is a member of $\mathcal{G}$; this gives (17).

Now write each Z in the definition of Λ as $Z' \cup Q$ with $Z' = Z \setminus P$ and $Q = Z \cap P$, and put $W' = (N' \setminus P) \setminus Z'$, $\mathcal{G} = (N_j \setminus Z')_{j \neq b}$ and $A' = N_a \setminus Z'$. If $Q = P$, then $N_b \setminus Z = A'$, and Lemma 9.5(2) turns the inner sum into $-\Delta_{W'}(\mathcal{G})$. If $Q \neq P$, the inner sum is $\Delta_W(\mathcal{G}, A' \cup (P \setminus Q))$ with $W = W' \cup (P \setminus Q)$, which is $t\, h_{|P\setminus Q|}\Delta_{W'}(\mathcal{G})$ by (17) with $P' = P \setminus Q$. By Corollary 9.11, $\left(\sum_{Q \subsetneq P} t\, h_{|P\setminus Q|}\right) - 1 = H_p - 1 = t(2+t)^p - (1+t)^{p+1}$. The family without N_b has the union $N' \setminus P$, so on N it has p more free elements, which gives the factor $(2+t)^p$. □

Corollary 9.30 (Unions and private elements)**.** *Let U be the union of the sets $N_j \subseteq N_b$ with $j \neq b$, and suppose that $U \neq \varnothing$ and that $P = N_b \setminus U$ meets no N_j with $j \neq b$. Then the identity of Lemma 9.29 holds with $p = |P|$.*

Proof. If $P = \varnothing$, this is Lemma 9.28, or Lemma 9.5(2) if N_b equals another set. If U is one of the sets, it is Lemma 9.29. Otherwise add U as a new set. In the families with and without N_b, the set U is the union of the sets strictly inside it, so Lemma 9.28 changes the sign of both sides, and Lemma 9.29 applies with $N_a = U$. □

If $U = \varnothing$ and, as in the corollary, P meets no other N_j, then $N_b = P$ meets no other set. Since Λ is multiplicative over families on disjoint sets (for such families the refinements in (16) are products), the same identity holds with the factor $H_p = t(2+t)^p - (1+t)^{p+1} + 1$ instead. The factor $t(2+t)^p - (1+t)^{p+1}$ equals -1 for $p \leq 1$ and takes the value -1 at $t = -1$; for $p \geq 2$ it has constant term -1 and leading term $(p-1)t^p$, for instance $t^2 + t - 1$ for $p = 2$. In the setting of Corollary 9.30 (so $U \neq \varnothing$), suppose that N_b differs from the other sets, and let r', k', f' and Λ' be r, k, f and Λ for the smaller family, so that $r' - k' = r - k$, $k' = k - 1$ and $f' = f + p$. For $p \leq 1$, the larger family satisfies the conclusion of Theorem 9.13 for its side, $(-1)^{r-k}\Lambda \in (2+t)^f(1+t)^k\mathbb{Z}_{\geq 0}[t]$, if and only if $(-1)^{r'-k'}\Lambda' \in -(2+t)^{f'}(1+t)^{k'+1}\mathbb{Z}_{\geq 0}[t]$. For $p \geq 2$, if Λ' is nonzero with coefficients of one sign, then Λ has coefficients of both signs: its lowest coefficient has the sign opposite to that of Λ', and its highest coefficient has the same sign as Λ'. For instance, the lower sets $\{1,2\}$ and $\{1,2,3,4\}$ of Example 9.20, where 3 and 4 are free for $\{1,2\}$, give $(t^2+t-1)H_2 = t(1+t)(t^2+t-1)$. The lower sets $\{1,2,3\}$, $\{1,4,5\}$, $\{1,2,3,4,5\}$ give $-t(1+t)^3(4t+9)$, with the wrong sign, by Lemma 9.28 and Proposition 9.14, but two copies of this family on disjoint sets give $t^2(1+t)^6(4t+9)^2$ with $r = k = 6$. The two copies together contain a union and satisfy the conclusion of Theorem 9.13 for their side, although neither copy does on its own.

When every group has at most two members, the values of Lemma 9.26 determine the order of vanishing of Λ at $t = -1$, provided that a signed count E, defined below, is nonzero. A two-member group $\{A, B\}$ *collapses* under an allowed Z if $A \triangle B \subseteq Z$, that is, if $A \setminus Z = B \setminus Z$; let $\kappa(Z)$ be the number of collapsing groups. Put $T_Z = \Delta_{N'\setminus Z}(D_1 \setminus Z, \ldots, D_k \setminus Z)$, the term of Z in (16), let κ^* be the largest $\kappa(Z)$ over the allowed Z with $T_Z \neq 0$, and let $E = \sum(-1)^{|Z|}$, the sum over the allowed Z with $T_Z \neq 0$ and $\kappa(Z) = \kappa^*$. If every T_Z vanishes, put $\kappa^* = 0$ and $E = 0$.

Theorem 9.31 (The leading term at $t = -1$)**.** *Suppose that $D_1, \ldots, D_k$ each have at least two elements and that every group has at most two members (condition (ii) of Theorem 9.27). Then*

$$\Lambda(N_1, \ldots, N_r; N) = (-1)^{r-k}(2+t)^f(1+t)^{k-\kappa^*}R(t), \qquad R \in \mathbb{Z}[t], \quad R(-1) = (-1)^{|N'|+k}E.$$

If $\kappa^* = 0$, then $(-1)^{r-k}\Lambda(N_1, \ldots, N_r; N) \in (2+t)^f(1+t)^k\mathbb{Z}_{\geq 0}[t]$. If $E \neq 0$, the order of vanishing of Λ at $t = -1$ is exactly $k - \kappa^*$; in particular, if $\kappa^* \geq 1$ and $E \neq 0$, then $(1+t)^k$ does not divide Λ. If $E = 0$, the order is at least $k - \kappa^* + 1$.

Proof. As in the proof of Theorem 9.27, every group of a reduced family $(D_j \setminus Z)_j$ has at most two members, and two reduced members coincide only in a collapsing group. By Proposition 9.14 and Lemma 9.26, a nonzero δ_g equals $\varepsilon_g(1+t)^{v_g}\psi_g$ with $\psi_g \in \mathbb{Z}_{\geq 0}[t]$: for one member of size s, $\varepsilon_g = v_g = 1$ and $\psi_g = \gamma_s$; for two distinct members, $\varepsilon_g = 1$, $v_g = 2$, and $\psi_g = e_p e_q e_c - t^2(1+t)\gamma_p\gamma_q\gamma_c$ or $\psi_g = t\gamma_c\gamma_q$; for a collapsing group with $|A \setminus Z| = s$, $\varepsilon_g = -1$, $v_g = 1$ and $\psi_g = \gamma_s$. Since $\gamma_n(-1) = (-1)^n$ for $n \geq 2$ and $e_n(-1) = (-1)^{n+1}$ for $n \geq 1$, in every case $\psi_g(-1) = (-1)^{|V_g|+v_g+1}$. The numbers v_g add up to $k - \kappa(Z)$, and Lemma 9.25 gives $|N' \setminus Z| = \omega(Z) + \sum_g (|V_g| - 1)$. Hence every nonzero term is

$$T_Z = (-1)^{\kappa(Z)}(1+t)^{k-\kappa(Z)}P_Z, \qquad P_Z = t^{\omega(Z)}\prod_g \psi_g \in \mathbb{Z}_{\geq 0}[t], \qquad P_Z(-1) = (-1)^{|N'\setminus Z|+k-\kappa(Z)}.$$

If $\kappa^* = 0$, every term lies in $(1+t)^k\mathbb{Z}_{\geq 0}[t]$. In general the terms with $\kappa(Z) < \kappa^*$ are divisible by $(1+t)^{k-\kappa^*+1}$, and each of the others contributes $(-1)^{\kappa^*}P_Z(-1) = (-1)^{|N'|+k+|Z|}$ to $R(-1)$. The factor $(2+t)^f$ is 1 at $t = -1$. □

So, if no allowed Z with $T_Z \neq 0$ collapses a group (for instance, under condition (i) of Theorem 9.27, or for a Berge-acyclic family), the order of vanishing is k when $E \neq 0$ and larger otherwise. For the lower sets $\{c_1, c_2, c_3\}$ and $\{c_3, c_4, c_5\}$ of the computation after Proposition 9.14, the ten allowed Z give $E = 1 - 5 + 4 = 0$, consistent with the order of vanishing of $t(1+t)^3(4t+9)$ at $t = -1$, which is $3 > k = 2$.

Lemma 9.32 (Surplus)**.** *In the setting of Theorem 9.31, let z lie in exactly one D_j, which forms a group by itself, such that $|D_j \setminus (Z \cup \{z\})| \geq 2$ for every allowed Z. Then for every allowed $Z \not\ni z$ the set $Z \cup \{z\}$ is allowed, and $T_Z + T_{Z\cup\{z\}} \in (-1)^{\kappa(Z)}(1+t)^{k-\kappa(Z)+1}\mathbb{Z}_{\geq 0}[t]$. In particular $E = 0$.*

Proof. As z lies only in D_j, deleting it changes neither the groups, nor the components, nor κ (it removes only the bridge block $\{D_j \setminus Z, z\}$ of the incidence graph); only the factor $\delta_g = h_s$ of the group $\{D_j\}$ changes, to h_{s-1}, where $s = |D_j \setminus Z| \geq 3$. So T_Z and $T_{Z\cup\{z\}}$ vanish together, and the toggle $Z \mapsto Z\triangle\{z\}$ gives $E = 0$. Now $h_s + h_{s-1} = (1+t)^2\big((1+t)^{s-3} + (1+t)^{s-2} - t^{s-2}\big)$, and, as in the proof of Theorem 9.31, the other factors of T_Z multiply to $(-1)^{\kappa(Z)}(1+t)^{k-\kappa(Z)-1}$ times a polynomial with nonnegative coefficients. □

Example 9.33 (Conditions on the groups are not sharp)**.** Write the lower elements as $1, 2, \ldots$; there are no free elements and $r = k$.

(a) Take $\{1,2,3\}$, $\{1,2,4\}$, $\{1,5\}$, $\{5,6,7\}$, $\{6,8\}$. The groups have at most two members, and the group $\{\{1,2,3\},\{1,2,4\}\}$ has no pendant pair. The elements $1, 5, 6, 8$ lie in two-element sets. The allowed Z are the subsets of $\{2,3,4,7\}$ containing neither $\{2,3\}$ nor $\{2,4\}$. The reduced families are connected, and the choices on $\{2,3,4\}$ and on 7 are independent; so $\Lambda = t\,h_2^2\,S\,(h_3 + h_2)$, where the factor S is the sum of the contributions of the choices of $Z \cap \{2,3,4\}$. By Lemma 9.26, the choices $Z \cap \{2,3,4\} = \varnothing$, $\{2\}$ and $\{3,4\}$ give $(1+t)^2(1+2t)$, h_2^2 and $-h_2$, respectively, and the choices $\{3\}$ and $\{4\}$ leave a nested pair with $q = 1$; so $S = (1+t)(2t^2+4t+1)$. Since $h_3 + h_2 = 2(1+t)^2$ (Lemma 9.32 with $z = 7$), $\Lambda = 2t(1+t)^5(2t^2+4t+1)$ with $k = 5$: the conclusion of Theorem 9.13 holds for this side, although the pendant-pair condition fails. Here $\kappa^* = 1$ and $E = 1 - 1 = 0$, from $Z = \{3,4\}$ and $Z = \{3,4,7\}$.

(b) If $\{1,2\}$ and $\{1,2,3,4\}$ replace $\{1,2,3\}$ and $\{1,2,4\}$, then the elements $3, 4, 7$ can be deleted freely, and the same computation gives $S = t\,h_2^2 - h_2 = (1+t)(t^2+t-1)$ and $\Lambda = 2t(1+t)^5(t^2+t-1)$. The order of vanishing is again $k = 5$ and $E = 0$, but both signs occur. This is Lemma 9.29 with $P = \{3,4\}$: the family $\{1,2\}$, $\{1,5\}$, $\{5,6,7\}$, $\{6,8\}$ is Berge-acyclic, and Proposition 9.14 gives its sum $t\,h_2^3 h_3 + t\,h_2^4 = 2t(1+t)^5$.

(c) For $\{1,4\}$, $\{1,2,4\}$, $\{1,3,5\}$, $\{3,6\}$, Lemma 9.29 with $P = \{2\}$ and Proposition 9.14 for the Berge-acyclic family $\{1,4\}$, $\{1,3,5\}$, $\{3,6\}$, whose sum is $t\,h_2^2 h_3 + t\,h_2^3 = 2t(1+t)^4$, give $\Lambda = -2t(1+t)^4$ with $k = 4$: the order is k, and the sign is wrong. Only $Z = \{2\}$ and $Z = \{2,5\}$ give nonzero terms, so $\kappa^* = 1$ and $E = 0$.

In (a) and (b) the groups have the same unions V_g, namely $\{1,2,3,4\}$, $\{1,5\}$, $\{5,6,7\}$ and $\{6,8\}$, and in neither has the two-member group a pendant pair, but only (a) satisfies the conclusion of Theorem 9.13. ◇

When every group has at most two members and neither member of any two-member group lies inside the other, the coefficients have the sign of Theorem 9.13 by Corollary 9.43, although the factor $(1+t)^k$ may be lost; in (b) and (c) the member $\{1,2\}$, respectively $\{1,4\}$, lies inside the other member.

Groups with three members. We now allow groups with three members. In the rest of this subsection, A, B, C, Y, P and Q denote sets of elements, with two exceptions: Q also denotes the polynomial of (20), and C and Y keep their meaning in the mixed difference $\mathcal{M}_C(Y)$. For nonempty subsets $N_1, \dots, N_r$ of a finite set W, the proof of Theorem 9.16, restricted to the cells with $Z(x) = \varnothing$, gives

$$\Delta_W(N_1, \dots, N_r) = (-1)^r \sum_{\sigma} (-1)^{\dim \sigma} \chi_{\mathcal{H}_\sigma}(-t), \tag{18}$$

the sum over the cones $\sigma = \{x \in \mathbb{R}^W : \operatorname{argmax}_{b \in N_j} x_b = I_j \text{ for } j \in [r]\}$ of the common refinement, where $\mathcal{H}_\sigma$ is the hypergraph on W with the edges $I_1, \dots, I_r$. Indeed, each such cone is invariant under adding multiples of $(1, \dots, 1)$, so it meets the open positive orthant in its relative interior, with the same dimension; the braid cells and both uses of the Euler characteristic stay inside the open positive orthant. A cone with a one-element edge contributes 0, since such an edge does not change the number of components in the inclusion–exclusion formula for χ. The proof of Lemma 9.28 runs cone by cone in (18), so $\Delta_W(\mathcal{G}, U) = -\Delta_W(\mathcal{G})$ for an indexed family $\mathcal{G}$ of subsets of W and a union U of members of $\mathcal{G}$. From now on we put $e_0 = 0$ and $e_{-1} = \gamma_0 = -1/(t(1+t))$; then $e_n = (1+t)e_{n-1} + t^{n-1}$ for all $n \geq 0$. A condition in square brackets, such as $[y = 0]$, stands for 1 if it holds and 0 otherwise.

Lemma 9.34 (Blocks of equal membership)**.** *Let $\mathcal{F}$ be an indexed family of subsets of a finite set V, and let $X \subseteq V$ consist of $a \geq 1$ elements that lie in exactly the members of a nonempty subfamily $\mathcal{S}$, each member of $\mathcal{S}$ having an element outside X. For $0 \leq i \leq a$ let $\Delta(i)$ be the value of Δ for the family obtained by deleting $a - i$ elements of X from V and from the members. Then $\Delta(a) = e_a \Delta(1) + (t^a - t\,e_a)\Delta(0)$.*

Proof. Fix $J \subseteq \mathcal{F}$ and let $a \geq 1$. Write $x \in \mathbb{R}^V$ as (y, x_X) and put $\mu = \max_{i \in X} x_i$. The quotient family replaces X by one element ξ; the map $(y, x_X) \mapsto (y, \mu)$ preserves the argmax data, with ξ standing for $I = \operatorname{argmax}_{i \in X} x_i$. Call a cell σ of the quotient *on* if ξ lies in the argmax set of some member of J, and *off* otherwise; if $\mathcal{S} \cap J = \varnothing$, every cell is off. For $0 \leq i \leq a$, write $\Phi_i(J)$ for $\Phi(J)$ in the family of $\Delta(i)$. Write $\Phi_{\mathrm{on}}(J)$ and $\Phi_{\mathrm{off}}(J)$ for the sums of $t^{\dim \sigma}$ over the on cells and over the off cells σ of the quotient. Over an on cell σ and a nonempty $I \subseteq X$, the points with $(y, \mu) \in \sigma$, $x_i = \mu$ for $i \in I$ and $x_i < \mu$ otherwise form one cell, of dimension $\dim \sigma + a - |I|$; different I give different cells, and their sum is $t^{\dim \sigma} e_a$. An off cell is determined by a cell σ' of the family with X deleted (each member of $\mathcal{S}$ keeps an element outside X), on which the maxima m_j of the members of $\mathcal{S} \cap J$ are linear: it is $\{(y, \mu) : y \in \sigma',\ \mu < m_j(y) \text{ for all } j\}$, of dimension

$\dim\sigma'+1$, and its preimage $\{y\in\sigma',\ x_i<m_j(y)$ for all $i\in X$ and $j\}$ is one cell of dimension $\dim\sigma'+a$. So $\Phi_a(J)=e_a\Phi_{\rm on}(J)+t^{a-1}\Phi_{\rm off}(J)$. The off cells correspond one to one to the cells σ' (each set above is nonempty), with dimension $\dim\sigma'+1$, so $\Phi_{\rm off}(J)=t\,\Phi_0(J)$, and $a=1$ gives $\Phi_{\rm on}(J)=\Phi_1(J)-t\,\Phi_0(J)$; hence $\Phi_a(J)=e_a(\Phi_1(J)-t\,\Phi_0(J))+t^a\Phi_0(J)$ for all $a\ge0$. Take the alternating sum over J. □

We call a set such as X in Lemma 9.34, whose elements lie in exactly the same members, a *block of equal membership*, or simply a *block*. To avoid a clash, we call the blocks of an incidence graph, as in Lemma 9.25, 2-*connected blocks* from now on. As a function of a, every $Ke_a+K't^a$ with $K,K'\in\mathbb{Q}(t)$ satisfies this relation, and its values at $a=0,1$ determine it. For $p,q\ge1$ the formula of Lemma 9.26 holds also for $c\in\{0,1\}$ with these conventions:

$$t^{-1}\Delta_W(A,B)=(1+t)^2\big(e_pe_qe_c-t^2(1+t)\gamma_p\gamma_q\gamma_c\big),\tag{19}$$

since for $c=0$ both sides are $t\,h_ph_q$, and for $c=1$ Lemma 9.25 gives $h_{p+1}h_{q+1}=(1+t)^2e_pe_q$.

Let A,B,C be distinct sets, none inside another, such that every element of $V=A\cup B\cup C$ lies in at least two of them. Let $X_A=(B\cap C)\setminus A$, $X_B=(A\cap C)\setminus B$, $X_C=(A\cap B)\setminus C$ and $Y=A\cap B\cap C$, with sizes a,b,c,y and $n=a+b+c+y$. Then $A=X_B\cup X_C\cup Y$, and so on, and $a,b,c\ge1$, since $a=0$ would give $B\subseteq A$. The incidence graph has a cycle through one element of each of X_A,X_B,X_C, so $\{A,B,C\}$ is a group; we call it a *three-member core group* of shape (a,b,c,y). With $u=1+t$ and $\tau_k=t^{k-1}$ put

$$\begin{aligned}Q&=(2e_{y+2}-3t^{y+1})e_ae_be_c-t\,u^2e_y(e_ae_b\gamma_c+e_a\gamma_be_c+\gamma_ae_be_c)+t^3u^3e_{y-1}\gamma_a\gamma_b\gamma_c&(20)\\&=u^3(2+3t)u^{y-1}\gamma_a\gamma_b\gamma_c+u^2\big(2u\,e_y+(2+t)t^y\big)(\tau_a\gamma_b\gamma_c+\gamma_a\tau_b\gamma_c+\gamma_a\gamma_b\tau_c)\\&\quad+u(2+t)e_{y+1}(\gamma_a\tau_b\tau_c+\tau_a\gamma_b\tau_c+\tau_a\tau_b\gamma_c)+\big(2u^2e_y+(2+t)t^y\big)\tau_a\tau_b\tau_c.&(21)\end{aligned}$$

Theorem 9.35 (Three-member core groups)**.** *For a three-member core group of shape (a,b,c,y), $\Delta_V(A,B,C)=-t(1+t)Q$. The polynomial Q has degree $n-2$ and positive coefficients in all degrees $0,\dots,n-2$, and $Q(0)=2$ and $Q(-1)=(-1)^{n+1}$; so $t^{-1}\Delta_V(A,B,C)$ has the wrong sign in every coefficient and vanishes to order exactly 1 at $t=-1$.*

Proof. Let $F(a,b,c,y)$ be $-t(1+t)$ times (20), defined for $a,b,c,y\ge0$. In each variable it has the form $Ke_a+K't^a$, since $e_{n-1}=(e_n-t^{n-1})/(1+t)$ and $e_{y+2}=u^2e_y+u\,t^y+t^{y+1}$. By Lemma 9.34 applied to X_A (the members B and C have the elements of X_C and X_B outside it), it suffices to show $\Delta=F$ at $a=0$ and $a=1$. If $a=0$, then $A=B\cup C$, and $\Delta=-\Delta(B,C)$ by the remark after (18); by (19), with $|B\setminus C|=c$, $|C\setminus B|=b$ and $|B\cap C|=y$, this is $F(0,b,c,y)$, as $e_0=0$ and $\gamma_0=e_{-1}$. For $a=1$, apply the lemma to X_B (A and C have the elements of X_C and X_A outside it); if $b=0$, then again B is a union: $B=A\cup C$ with $A=X_C\cup Y$ and $C=X_A\cup Y$, and (19) applies to A,C. For $a=b=1$, apply the lemma to X_C in the same way. For $a=b=c=1$, apply it to Y (each member has two elements outside Y). The two remaining values follow from (18). For $y=0$, only the cone where the three coordinates are equal has all edges of size two, and $\Delta=\chi_{K_3}(-t)=-t(1+t)(2+t)=F(1,1,1,0)$. For $y=1$, with $V=\{1,2,3,4\}$ and $Y=\{4\}$, the cones with all edges of size at least two are: all coordinates equal (dimension 1, $\chi=\nu^4-3\nu^2+2\nu$); 4 and two of $1,2,3$ on top (three cones of dimension 2, $\chi=\nu^2(\nu-1)^2$); and $1,2,3$ equal and above 4 (dimension 2, $\chi=\nu^2(\nu-1)(\nu-2)$). So $\Delta=(t^4-3t^2-2t)-3t^2(1+t)^2-t^2(1+t)(2+t)=-t(1+t)(3t^2+6t+2)=F(1,1,1,1)$.

The equality of the two expressions (20) and (21) for Q follows from $e_k=u\,\gamma_k+\tau_k$, $2e_{y+2}-3t^{y+1}=2u^2e_y+(2+t)t^y$ and $e_y=u\,e_{y-1}+t^{y-1}$ (for $y=0$, $u^3u^{y-1}=u^2$). Each factor in (21) has positive coefficients in every degree from its lowest to its top, and $\gamma_k=0$ only for $k=1$. Consider

the term of (21) with γ at the blocks of size at least two and τ at the blocks of size one. If one of a, b, c equals 1, it has degree $n-2$ and positive coefficients in all degrees $0, \dots, n-2$. If $a, b, c \geq 2$, the first term has positive coefficients in degrees $0, \dots, n-3$, and $u^2(2u\,e_y + (2+t)t^y)\tau_a\gamma_b\gamma_c$ in degrees $a-1, \dots, n-2$; as $a-1 \leq n-3$, together they cover $0, \dots, n-2$. The other terms add nonnegative coefficients, and no term has degree above $n-2$. At $t=0$ only the first term of (20) survives and gives 2; at $t=-1$ only the first term survives, with $2e_{y+2}(-1) - 3(-1)^{y+1} = (-1)^y$ and $e_k(-1) = (-1)^{k+1}$ for $k \geq 1$. □

Theorem 9.36 (A side formed by a three-member core group)**.** *Suppose that the distinct sets of one side form a three-member core group of shape (a, b, c, y), so $k = 3$ and $N' = V$. Put $\hat{e}_x = (2+t)^x - u^x$, recall $H_x - 1 = t(2+t)^x - u^{x+1}$ (Corollary 9.11), and let $\widetilde{\Lambda} = (-1)^{r-3}(2+t)^{-f}\Lambda(N_1, \dots, N_r; N)$. Then*

$$\begin{aligned}\widetilde{\Lambda} = {} & -t\,u\big(2u^2\hat{e}_y + (2+t)u^y\big)\hat{e}_a\hat{e}_b\hat{e}_c + t\,u^2\hat{e}_y\big(\hat{e}_a\hat{e}_b(H_c-1) + \hat{e}_a(H_b-1)\hat{e}_c + (H_a-1)\hat{e}_b\hat{e}_c\big) \\ & - (H_y-1)(H_a-1)(H_b-1)(H_c-1) - (H_a+H_b+H_c-1),\end{aligned}$$

and $\widetilde{\Lambda} = -t(1+t)\Theta$ with $\Theta \in \mathbb{Z}[t]$ of degree $n-2$ and positive coefficients in all degrees $0, \dots, n-2$. So $(-1)^{r-3}\Lambda$ has negative coefficients in degrees $1, \dots, n+f$, and the conclusion of Theorem 9.13 fails for this side in every nonzero coefficient. Moreover $\Theta(-1) = 1-y$, so $\widetilde{\Lambda}$ vanishes to order exactly 1 at $t=-1$ if $y \neq 1$; if $y = 1$, the coefficient of $(1+t)^2$ in $\widetilde{\Lambda}$ is $4 - s_2$ with $s_2 = (a-1)(b-1) + (b-1)(c-1) + (c-1)(a-1)$.

The family is not Berge-acyclic, so this does not contradict Theorem 9.13. For the triangle (the shape $(1,1,1,0)$), $\widetilde{\Lambda} = -t(1+t)(2+t)$; for the shape $(1,1,1,1)$, $\widetilde{\Lambda} = -t(1+t)^2(3t+7)$.

Proof. For a Z that is not allowed, put $T_Z = 0$. By (16), $\widetilde{\Lambda} = \sum_Z T_Z$ over all $Z \subseteq V$, where T_Z depends only on the numbers (a', b', c', y') of elements of X_A, X_B, X_C, Y outside Z. Write $A' = A \setminus Z$, $B' = B \setminus Z$, $C' = C \setminus Z$ and $Y' = Y \setminus Z$, and let F be as in the proof of Theorem 9.35. Then $T_Z = F(a', b', c', y')$ for all shapes except those with $y' = 0$ and at least two of a', b', c' zero. At these shapes a set is empty and $T_Z = 0$, while $F = t\,h_x$ for the remaining one x of a', b', c' if $x \geq 1$, and $F = -1$ if $x = 0$. For the other shapes the equality holds as follows. For $a', b', c' \geq 1$, it is Theorem 9.35. For the shapes with one zero, one set is the union of the other two, as in the proof of Theorem 9.35. For $a' = b' = 0 < c'$, Lemma 9.5(2) and (17) give $T_Z = -t^2 h_{c'} h_{y'}$. Finally, $a' = b' = c' = 0$ gives $\Delta(Y', Y', Y') = t\,h_{y'}$. In each variable, $\sum_{x'} \binom{x}{x'} e_{x'} = \hat{e}_x$, $\sum_{x'} \binom{x}{x'} t^{x'} = u^x$ and $\sum_{x'} \binom{x}{x'} e_{x'-1} = (H_x - 1)/(t\,u)$, which turns the sum of $\binom{a}{a'}\binom{b}{b'}\binom{c}{c'}\binom{y}{y'}F(a', b', c', y')$ over all shapes into the first three terms. From this we subtract the values of F at the exceptional shapes: by Corollary 9.11 they add up to $H_a - 1$ on the shapes $(a', 0, 0, 0)$, and likewise for b and c; the shape $(0, 0, 0, 0)$, with value -1, lies on all three and is subtracted only once, so the total is $H_a + H_b + H_c - 1$.

Every T_Z is divisible by $t\,u$, and only the shapes with $a' = b' = c' = 0$, $y' \geq 2$ give positive terms $t\,h_{y'}$: the shapes with $a', b', c' \geq 1$ give $-t\,uQ$, those with one zero give $-\Delta(B', C') \in -t\,u^2\mathbb{Z}_{\geq 0}[t]$ by (19) and Lemma 9.26, and those with two zeros give $-t^2 h_{c'} h_{y'}$. Fix $\xi_A \in X_A$, $\xi_B \in X_B$, $\xi_C \in X_C$, and pair $Z = X_A \cup X_B \cup X_C \cup (Y \setminus Y')$ with $Z \setminus \{\xi_A, \xi_B, \xi_C\}$, of shape $(1, 1, 1, y')$. Since $\gamma_1 = 0$, the sum of the two terms is $-t\,u\big(2e_{y'+2} - 3t^{y'+1} - e_{y'-1}\big)$, and the coefficient of t^j in the bracket is $2\binom{y'+2}{j} - \binom{y'-1}{j} > 0$ for $j \leq y'-2$, $2\binom{y'+2}{j} > 0$ for $j = y'-1, y'$, and $2(y'+2) - 3 > 0$ for $j = y'+1$. So Θ has nonnegative coefficients. The contribution of $Z = \varnothing$ to Θ, or of the pair containing it, is positive in all degrees $0, \dots, n-2$, and only $Z = \varnothing$ reaches degree n.

At $t = -1$, the terms of the shapes with one or two zeros are divisible by u^2. The shapes with $a', b', c' \geq 1$ contribute $Q(-1) = (-1)^{n'+1}$ each to $\Theta(-1)$, where $n' = |V \setminus Z|$, in total $[y = 0]$; the shapes $(0, 0, 0, y')$ with $y' \geq 2$ contribute $(-1)^{y'+1}$ each, in total $1 - y - [y = 0]$. For $y = 1$, the

coefficient of u^2 in $\widetilde{\Lambda}$ is 1 from the shapes with $a', b', c' \geq 1$, and $1-(b-1)(c-1)$ from the shapes with $a' = 0$, and likewise for $b' = 0$ and $c' = 0$; in total $4 - s_2$. Here each shape with $a', b', c' \geq 1$ contributes $(dQ/dt)(-1) - Q(-1)$, and $(de_x/dt)(-1)/e_x(-1)$ is $-x$ for $x \geq 2$ and 0 for $x = 1$. $\square$

Proposition 9.37 (Hanging one-member groups)**.** *Suppose that the distinct sets $D_1, \dots, D_k$ of one side each have at least two elements, that three of them form a three-member core group $g = \{A, B, C\}$ of shape (a, b, c, y), that each of the others forms a group alone, and that the incidence graph is connected. Let $\widetilde{\Lambda} = (-1)^{r-k}(2+t)^{-f}\Lambda(N_1, \dots, N_r; N)$. Then $\widetilde{\Lambda} = -t(1+t)M$ with $M \in \mathbb{Z}[t]$ of degree $|N'|-2$ and positive coefficients in all degrees $0, \dots, |N'|-2$. So the conclusion of Theorem 9.13 fails for this side in every nonzero coefficient.*

Proof. Here $h_0 = h_1 = 0$, as before Proposition 9.14. Let $V = V_g$, and let $S_1, \dots, S_m$ be the other sets. Every element of V lies on a cycle through A, B and C: for $x \in A \cap B$ take $A, x, B, \xi_A, C, \xi_B, A$ with $\xi_A \in X_A$ and $\xi_B \in X_B$, and likewise for $x \in B \cap C$ and $x \in A \cap C$. As S_i lies on no cycle, the tree in the proof of Lemma 9.25 gives S_i one element r_i on the path from S_i to g. The other elements, $P_i = S_i \setminus \{r_i\}$, lie in no member of g; the sets P_i are pairwise disjoint and disjoint from V, and r_i lies in V or in one P_j.

Let T_Z be the term of Z in (16), with $T_Z = 0$ for a Z that is not allowed, and put $X = Z \cap V$; let a', b', c', y' be the numbers of elements of X_A, X_B, X_C, Y outside Z, as in the proof of Theorem 9.36. For an allowed Z the sets $A \setminus Z$, $B \setminus Z$, $C \setminus Z$ have a connected incidence graph: by the cycle if $a', b', c' \geq 1$, and otherwise one of them contains the other two. The set $S_i \setminus Z$ loses its link towards g exactly when $r_i \in Z$, so Lemma 9.25 gives

$$T_Z = T_g(X) \prod_{i=1}^{m} t^{[r_i \in Z]} h_{|S_i \setminus Z|}, \qquad T_g(X) = \Delta_{V \setminus X}(A \setminus X, B \setminus X, C \setminus X),$$

with $T_g(X) = 0$ if one of $A \setminus X$, $B \setminus X$, $C \setminus X$ has at most one element; both sides vanish for a Z that is not allowed. We sum over the elements outside V first, recursively from the leaves of the tree. For an element p let $W_p(0)$ and $W_p(1)$ be the products of α_{S_i}, respectively β_{S_i}, over the i with $r_i = p$ (empty products are 1), where for $S = \{r\} \cup P$

$$\alpha_S = \sum_{K \subseteq P} h_{1+|K|} W_K, \qquad \beta_S = \sum_{\varnothing \neq K \subseteq P} t\, h_{|K|} W_K, \qquad W_K = \prod_{p \in K} W_p(0) \prod_{p \in P \setminus K} W_p(1);$$

here K is the set of elements of P outside Z, and α_S, β_S collect the terms with $r \notin Z$, respectively $r \in Z$. Then $\widetilde{\Lambda} = \sum_{X \subseteq V} T_g(X) \prod_{v \in V} W_v([v \in X])$. By induction from the leaves, $W_p(0) \geq W_p(1) \geq 0$ coefficientwise: if this holds on P, then $\alpha_S, \beta_S \geq 0$, and as $h_{j+1} - t\, h_j = (1+t)^j$ for $j \geq 1$, $\alpha_S - \beta_S = \sum_{\varnothing \neq K \subseteq P} (1+t)^{|K|} W_K \geq 0$; products keep the order, as $w_1 w_2 - w_1' w_2' = w_1(w_2 - w_2') + (w_1 - w_1')w_2'$.

By the proof of Theorem 9.36, $T_g(X)$ lies in $-t(1+t)\mathbb{Z}_{\geq 0}[t]$, except for the full collapses $X = X_A \cup X_B \cup X_C \cup (Y \setminus Y')$ with $|Y'| = y' \geq 2$, where $T_g(X) = t\, h_{y'}$. Pair each full collapse X with $X \setminus \{\xi_A, \xi_B, \xi_C\}$, of shape $(1, 1, 1, y')$, for fixed ξ_A, ξ_B, ξ_C as there. The two terms share the factor $W^* = \prod_{v \in V \setminus \{\xi_A, \xi_B, \xi_C\}} W_v([v \in X]) \geq 0$, and their sum is

$$-t(1+t)W^* \Big(\big(2e_{y'+2} - 3t^{y'+1}\big) \prod_{\xi} W_\xi(0) - e_{y'-1} \prod_{\xi} W_\xi(1) \Big),$$

where the bracket is at least $\big(2e_{y'+2} - 3t^{y'+1} - e_{y'-1}\big) \prod_\xi W_\xi(0) \geq 0$. So $\widetilde{\Lambda} = -t(1+t)M$, where M is the sum of $-T/(t(1+t))$ over the unpaired terms T and over the sums T of the pairs; each of

these summands has nonnegative coefficients. The contribution of $X = \varnothing$ is $Q \prod_{v \in V} W_v(0)$, with Q as in Theorem 9.35; if $a = b = c = 1$ and $y \geq 2$, then $X = \varnothing$ is paired, and the pair contributes at least $\left(2e_{y+2} - 3t^{y+1} - e_{y-1}\right) \prod_{v \in V} W_v(0)$. As $\alpha_S \geq h_{|S|} \prod_{p \in P} W_p(0)$ (its term $K = P$), induction gives $\prod_{v \in V} W_v(0) \geq \prod_i h_{|S_i|}$. Here Q and $2e_{y+2} - 3t^{y+1} - e_{y-1}$ have positive coefficients in all degrees from 0 to $n-2$, and h_s in all degrees from 0 to $s-1$. So M has positive coefficients in all degrees $0, \ldots, n-2+\sum_i(|S_i|-1) = |N'|-2$, by Lemma 9.25. Finally $\deg T_Z \leq |N' \setminus Z|$, so $\deg M \leq |N'|-2$. □

When some groups are three-member core groups, Theorem 9.31 keeps its form if $\kappa(Z)$ is defined as k minus the order of T_Z at $t = -1$.

Theorem 9.38 (The leading term with three-member groups)**.** *Suppose that $D_1, \ldots, D_k$ each have at least two elements and that every group has at most two members or is a three-member core group (that is, every element of V_g lies in at least two members of g). For an allowed Z with $T_Z \neq 0$ let $\kappa(Z) = k - \operatorname{ord}_{t=-1} T_Z$, let κ^* be the largest $\kappa(Z)$, and let $E = \sum(-1)^{|Z|}$ over the Z with $T_Z \neq 0$ and $\kappa(Z) = \kappa^*$ ($\kappa^* = E = 0$ if every T_Z vanishes). Then:*

(1) $T_Z = (1+t)^{k-\kappa(Z)} P_Z$ with $P_Z(-1) = (-1)^{|N' \setminus Z| + k}$;

(2) $\kappa(Z)$ is the sum, over the groups, of the following numbers: 1 for a collapsing two-member group; 0 for any other group with at most two members, also when its two reduced sets meet in at most one element, so that the group splits into two one-member groups; and, for a three-member core group of shape (a', b', c', y') after deleting Z, 2 if $a', b', c' \geq 1$ or $a' = b' = c' = 0$, and 1 otherwise;

(3) $\Lambda(N_1, \ldots, N_r; N) = (-1)^{r-k}(2+t)^f (1+t)^{k-\kappa^*} R$ with $R \in \mathbb{Z}[t]$ and $R(-1) = (-1)^{|N'|+k} E$. If $E \neq 0$ the order of vanishing at $t = -1$ is exactly $k - \kappa^*$, and otherwise at least $k - \kappa^* + 1$.

If some group has three members and some $T_Z \neq 0$, then $\kappa^* \geq 1$, so if moreover $E \neq 0$, the conclusion of Theorem 9.13 fails for this side.

Proof. As in the proof of Theorem 9.27, the groups of $(D_j \setminus Z)_j$ refine those of the family. For a three-member core group with $T_Z \neq 0$, the reduced sets cover $V_g \setminus Z$ and form one reduced group: for $a', b', c' \geq 1$ by the cycle, and otherwise each case ($A' = B' \cup C'$, or $A' = B' \supseteq C' = Y'$, or $A' = B' = C'$) gives two pairs that share at least two elements. Let $m(g')$ be the number of indices j with $D_j \setminus Z$ in the reduced group g', so $\sum m(g') = k$. By Lemma 9.25, $T_Z = t^{\omega(Z)} \prod \delta_{g'}$. In every case $\delta_{g'} = \varepsilon(1+t)^{v(g')}\psi$ with $\varepsilon = \pm 1$ and $|\psi(-1)| = 1$; the sum of $m(g') - v(g')$ over the reduced groups g' of one group equals its number in (2); and $\varepsilon\psi(-1) = (-1)^{|V_{g'}|-1+m(g')}$. For one or two members, this follows from the proof of Theorem 9.31. For three members, it follows from the values in the proof of Theorem 9.36 ($-(1+t)Q$, the union case by (19), $-t\,h_x h_{y'}$ with $x \geq 2$, and $h_{y'}$), where $\varepsilon\psi(-1) = (-1)^{|V_{g'}|}$. With $|N' \setminus Z| = \omega(Z) + \sum(|V_{g'}| - 1)$ this gives (1) and (2), and (3) follows as in the proof of Theorem 9.31. □

For one three-member core group, $\kappa^* = 2$ and $E = (-1)^n(y-1)$, which gives $\Theta(-1) = 1 - y$ in Theorem 9.36. The core condition is needed in (1): for the sets $\{1,2,3\}$, $\{0,2\}$ and $\{0,1\}$, the term of $Z = \varnothing$ is $-t(1+t)^2$, with the sign opposite to (1). Three-member groups with an element in only one member are not covered.

Trees of groups. Theorems 9.27 and 9.36, and Proposition 9.37, concern special trees of groups. We now allow any side whose groups are single sets, two-member groups or three-member core groups. Let the distinct sets $D_1, \dots, D_k$ of one side each have at least two elements, and put $\widetilde{\Lambda} = (-1)^{r-k}(2+t)^{-f}\Lambda(N_1, \dots, N_r; N)$, so that $\widetilde{\Lambda} = \sum_Z T_Z$ over all $Z \subseteq N'$ by (16), with $T_Z = 0$ for a Z that is not allowed. As Λ is multiplicative over families on disjoint sets, we assume that the incidence graph is connected. By Lemma 9.25 and its proof, the graph whose nodes are the groups and the elements lying in at least two sets V_g is then a tree. Fix a group g_0. Every other group g has a *root* $r_g \in V_g$, the element of V_g on the path from g to g_0. For $X \subseteq V_g$ let $T_g(X) = \Delta_{V_g \setminus X}\big((D \setminus X)_{D \in g}\big)$, with $T_g(X) = 0$ if a member keeps at most one element; for a one-member group $\{S\}$, $T_g(X) = t\, h_{|S \setminus X|}$ by Proposition 9.14.

Lemma 9.39 (Tree sum)**.** *For a group $g \neq g_0$ and polynomials $W_v(0), W_v(1)$ at the elements $v \in V_g \setminus \{r_g\}$ put*

$$W_g(0) = \sum_X t^{-1} T_g(X) \prod_v W_v([v \in X]), \qquad W_g(1) = \sum_X T_g(X \cup \{r_g\}) \prod_v W_v([v \in X]),$$

summed over $X \subseteq V_g \setminus \{r_g\}$, with v in $V_g \setminus \{r_g\}$. For an element v let $W_v(0)$ and $W_v(1)$ be the products of the $W_g(0)$, respectively $W_g(1)$, over the groups g with $r_g = v$ (empty products are 1); this defines them recursively from the leaves of the tree. Then

$$\widetilde{\Lambda} = \sum_{X \subseteq V_{g_0}} T_{g_0}(X) \prod_{v \in V_{g_0}} W_v([v \in X]).$$

For a one-member group $g = \{S\}$ these are the α_S and β_S of the proof of Proposition 9.37; that proposition is the case of a three-member core group g_0 with one-member groups below it.

Proof. First, $T_Z = T_{g_0}(Z \cap V_{g_0}) \prod_{g \neq g_0} t^{-[r_g \notin Z]} T_g(Z \cap V_g)$. If Z is not allowed, some $D_j \setminus Z$ has at most one element, and D_j lies in one group, so both sides vanish. Otherwise, a 2-connected block of the incidence graph of $(D_j \setminus Z)_j$ is a 2-connected subgraph of the incidence graph of $(D_j)_j$, so it lies in one of its 2-connected blocks, and the groups of $(D_j \setminus Z)_j$ refine those of $(D_j)_j$. Lemma 9.25, applied to $(D_j \setminus Z)_j$ and to the members of each g minus Z, gives $T_Z = t^{\omega(Z) - \sum_g \omega_g(Z)} \prod_g T_g(Z \cap V_g)$, with $\omega_g(Z)$ the number of components for the members of g minus Z. Add the groups, starting with g_0, in an order in which each $g \neq g_0$ meets the earlier ones only in r_g: if $r_g \notin Z$, it joins a component of g to an earlier one, and otherwise nothing is joined; so $\omega(Z) = \sum_g \omega_g(Z) - |\{g \neq g_0 : r_g \notin Z\}|$. Second, every element lies in exactly one of the sets V_{g_0} and $V_g \setminus \{r_g\}$, $g \neq g_0$; split Z accordingly and sum from the leaves to g_0. □

Lemma 9.40 (A cone)**.** *Let $\mathcal{K}$ be the set of pairs $x_1(1,1) + x_2(1,0) + x_3(1+t,-t)$ with $x_1, x_2, x_3 \in \mathbb{Z}_{\geq 0}[t]$. Then $\mathcal{K}$ is closed under sums, under multiplication by $\mathbb{Z}_{\geq 0}[t]$ and under the componentwise product. A pair (w_0, w_1) of polynomials lies in $\mathcal{K}$ if and only if w_1 has a nonnegative constant term and $w_0 - w_1 - (1+2t)x \geq 0$, where $-t\,x$ is the sum of the terms of w_1 with negative coefficients.*

Proof. The pair $(1,1)$ is the unit, $(1,0)(w_0, w_1) = w_0(1,0)$, and $(1+t,-t)^2 = t^2(1,1) + (1+2t)(1,0)$; the rest is bilinearity. If $(w_0, w_1) = x_1(1,1) + x_2(1,0) + x_3(1+t,-t)$, then $x_1 = w_1 + t\,x_3$ and $x_2 = w_0 - w_1 - (1+2t)x_3$; the least $x_3 \geq 0$ with $w_1 + t\,x_3 \geq 0$ is x, and x_2 decreases in x_3. (The representation is not unique, as $(1+t,-t) + t(1,1) = (1+2t)(1,0)$.) □

Proposition 9.41 (Reduction)**.** *Put $s_g = 1$ if g is a three-member core group and $s_g = 0$ otherwise, and $s = \sum_g s_g$. Suppose that $(-1)^{s_g}(W_g(0), W_g(1)) \in \mathcal{K}$ for every group $g \neq g_0$ whenever the pairs*

$(W_v(0), W_v(1))$ *with* $v \in V_g \setminus \{r_g\}$ *all lie in* $\mathcal{K}$*, and that* $(-1)^{s_{g_0}} \sum_X T_{g_0}(X) \prod_v W_v([v \in X]) \geq 0$ *whenever the pairs* $(W_v(0), W_v(1))$ *with* $v \in V_{g_0}$ *all lie in* $\mathcal{K}$*. Then* $(-1)^s \widetilde{\Lambda} \geq 0$*. Each condition holds for all pairs in* $\mathcal{K}$ *as soon as it holds when every pair is* $(1,1)$*,* $(1,0)$ *or* $(1+t,-t)$*.*

Proof. For $g \neq g_0$ let $s(g)$ be the sum of $s_{g'}$ over g and the groups g' below it. By induction from the leaves, $(-1)^{s(g)}(W_g(0), W_g(1)) \in \mathcal{K}$: at an element with no group below it the pair is $(1,1)$; otherwise, by Lemma 9.40, $(-1)^{\sigma_v}(W_v(0), W_v(1)) \in \mathcal{K}$, where σ_v is the sum of $s(g)$ over the groups g with $r_g = v$, and since $W_g(0)$, $W_g(1)$ are linear in each pair, these signs factor out. At g_0, Lemma 9.39 gives $(-1)^s \widetilde{\Lambda} \geq 0$. The last claim holds since the conditions are linear in each pair and $\mathcal{K}$ is closed under sums and $\mathbb{Z}_{\geq 0}[t]$-multiples. □

Theorem 9.42 (Sign flip)**.** *Suppose that the distinct sets* $D_1, \ldots, D_k$ *of one side each have at least two elements, and that every group is a single set, a two-member group neither of whose members contains the other, or a three-member core group. Let* s *be the number of three-member core groups. Then*

$$(-1)^{r-k+s} \Lambda(N_1, \ldots, N_r; N) \in (2+t)^f \mathbb{Z}_{\geq 0}[t].$$

If both sides satisfy these hypotheses, then $(-1)^{k+s} \mathcal{M}_C(Y) \in (2+t)^f \mathbb{Z}_{\geq 0}[t]$*, where now* k *and* f *are counted over both sides, as in Theorem 9.13, and* s *is the total number of three-member core groups.*

For the three-member core groups, the proof uses an exact computer check of finitely many polynomial inequalities.[3]

Corollary 9.43 (Incomparable members)**.** *If the distinct sets of one side each have at least two elements and each of its groups is a single set or a two-member group neither of whose members contains the other, then* $(-1)^{r-k} \Lambda(N_1, \ldots, N_r; N) \in (2+t)^f \mathbb{Z}_{\geq 0}[t]$*. If both sides are of this kind, then* $(-1)^k \mathcal{M}_C(Y) \in (2+t)^f \mathbb{Z}_{\geq 0}[t]$*, where now* k *and* f *are counted over both sides, as in Theorem 9.13.*

Unlike Theorem 9.27, this gives no factor $(1+t)^k$: for the two-member group $\{\{1,2,3\},\{2,3,4\}\}$ ($r = k = 2$, $f = 0$) only $Z = \varnothing$, $\{2\}$, $\{3\}$ and $\{1,4\}$ contribute, and $\Lambda = t(1+t)^2(1+2t) + 2t(1+t)^2 - t(1+t) = t(1+t)(2+t)(1+2t)$, of order $1 < k$ at $t = -1$; here $Z = \{1,4\}$ makes the two sets equal. The proof of Corollary 9.43, Steps 1–5 below, uses no computer.

Proof of Theorem 9.42. By Proposition 9.41 it suffices to check both conditions for one group, for every choice of its root, when every pair is $(1,1)$, $(1,0)$ or $(1+t,-t)$. We call

$$\sum_{X \subseteq V_g} T_g(X) \prod_{v \in V_g} W_v([v \in X])$$

the *sum* of g. For a set X of elements carrying such pairs and $K \subseteq X$ (the kept elements) put $W_K = \prod_{v \in K} W_v(0) \prod_{v \in X \setminus K} W_v(1)$ and

$$\bar{\mu}_X = \sum_K u^{|K|} W_K = \prod_{v \in X} \big(u W_v(0) + W_v(1)\big), \qquad \bar{\zeta}_X = W_\varnothing = \prod_{v \in X} W_v(1),$$

[3] It checks the fifteen identities of Step 6, the identity for Ξ, and the inequalities at all generator tuples, in exact integer polynomial arithmetic (Python 3 with SymPy; about one minute on one core). The ancillary files accompany this paper: the program `block_cone_certificate.py`, with SHA-256 `0bcaa32e4bb5bfc159461913004d4cc0ec0d43901e2dcf1cc856f8d658b79304`, and its output `block_cone_certificate_output.txt`, whose checksum is listed in the ancillary file `README.txt`.

$\bar e_X = \sum_K e_{|K|} W_K$ (with $e_0 = 0$), $\bar\eta_X = t\bar\mu_X - u(\bar\mu_X - \bar e_X)$ and $\bar\beta_X = \sum_{K\neq\varnothing} t\, h_{|K|} W_K = \bar\eta_X + \bar\zeta_X$. If i of the pairs on X are $(1,1)$, j are $(1,0)$ and l are $(1+t,-t)$, then

$$\bar\mu_X = (2+t)^i u^j (1+t+t^2)^l, \qquad \bar\mu_X - \bar e_X = \sum_K t^{|K|} W_K = u^i t^{j+2l}, \qquad \bar\zeta_X = [j=0](-t)^l.$$

Put $\iota_X = [j = l = 0]$ and $\vartheta_X = [j = 0 < l](-t)^l$, so $\bar\zeta_X = \iota_X + \vartheta_X$, and $\bar\eta^+_X = \bar\eta_X + \iota_X$, so $\bar\beta_X = \bar\eta^+_X + \vartheta_X$. Inequalities are coefficientwise. For a polynomial ϕ, we write $[t^j]\phi$ for the coefficient of t^j in ϕ; these square brackets are not a condition. We write $|\phi|$ for the polynomial whose coefficients are the absolute values of those of ϕ, so $|\phi+\phi'| \le |\phi| + |\phi'|$ and $|\phi\phi'| \le |\phi|\,|\phi'|$.

Step 1: one block. (a) $\bar e_X \ge 0$, and $\bar e_X \ge 1$ if $X \neq \varnothing$: termwise $(2+t)^i \ge u^i$, $u^j \ge t^j$, $(1+t+t^2)^l \ge t^{2l}$, and the constant term of $\bar e_X$ is $2^i - \iota_X$. (b) $0 \le \bar\eta^+_X \le t\bar e_X$: if $j \ge 1$ then $t\bar\mu_X \ge tu^i\, u\, t^{j-1}t^{2l} = u(\bar\mu_X - \bar e_X)$, and if $l \ge 1$ then $(1+t+t^2)^l \ge u\, t^{2l-1}$ gives the same; if $j = l = 0$ then $\bar\eta^+_X = t(2+t)^i - u^{i+1} + 1 = H_i \ge 0$ by Corollary 9.11; and $\bar\eta^+_X = t\bar e_X - (u^i t^{j+2l} - \iota_X)$. (c) $\bar\beta_X \ge 0$: this is (b) unless $j = 0 < l$, and then $[t^l]\bar\eta_X = [t^{l-1}]\bar\mu_X \ge 1$, since $u^i t^{2l}$ has order $2l > l$. (d) If $j = 0 < l$, every coefficient of $\bar e_X$ in degrees $0,\dots,2l-1$ is at least 1, and $\bar\mu_X \ge t^{l-1}(1+t+t^2)$: the coefficients of $(1+t+t^2)^l$ in degrees $0,\dots,2l$ are at least 1. (e) $|\bar\zeta_X| \le \bar\mu_X$; if $X \neq \varnothing$, then $|\bar\zeta_X| \le \bar e_X$, $|\vartheta_X| \le t\bar e_X$ and $\bar\beta_X \le u\bar e_X$, by (a), (d) and $\bar\beta_X = t\bar e_X - u^i t^{j+2l} + \bar\zeta_X$. (f) If $j = l = 0 < i$, then $H_i \le t(\bar e_X - 1)$, as $u^i - 1 \ge t$.

Step 2: single sets. For $g = \{S\}$ with root r and $P = S \setminus \{r\}$, the identity $h_{1+n} = u\, e_n$ gives $W_g(0) = u\bar e_P$ and $W_g(1) = \bar\beta_P$, so $W_g(1) \ge 0$ and $W_g(0) - W_g(1) = \bar\mu_P - \bar\zeta_P \ge 0$ by (c) and (e): the pair is $W_g(1)(1,1) + (W_g(0) - W_g(1))(1,0) \in \mathcal{K}$. For $g_0 = \{S\}$ the sum is $\bar\beta_S \ge 0$.

Step 3: the sum of a two-member group. Let $g = \{A, B\}$ with blocks $P = A \setminus B$, $Q = B \setminus A$, $C = A \cap B$, where $P, Q \neq \varnothing$. For kept parts of sizes p', q', c', the proof of Lemma 9.26 gives $T_g = t^{p'+q'+c'}\big(a_{p'}a_{q'}a_{c'}(1+t^{-1})^2 - (a_{p'}-1)(a_{q'}-1)(a_{c'}-1)\big)$, with a_n as in that proof, whenever both members keep an element: for $c' \ge 2$ by that lemma (this includes the cases $p' = 0$, $q' = 0$ and $A' = B'$, where A' and B' are the kept parts of A and B); for $c' = 1$ and for $c' = 0$ by Lemma 9.25, as $t\, h_{p'+1}h_{q'+1}$ and $t^2 h_{p'}h_{q'}$ (both 0, like T_g, when a member keeps one element). Since $t^n a_n = t\, e_n$ and $t^n(a_n - 1) = t\, h_n$ for $n \ge 1$, and $a_0 = 0$, summing over all kept parts gives $t\, u^2 \bar e_P \bar e_Q \bar e_C - \bar\eta_P\bar\eta_Q\bar\eta_C$; the choices of kept parts for which a member becomes empty, where the formula gives $-t\, h_{q'}$, $-t\, h_{p'}$ or 1 instead of 0, add up to $-\bar\zeta_C(\bar\zeta_P\bar\eta_Q + \bar\zeta_Q\bar\eta_P + \bar\zeta_P\bar\zeta_Q)$. So the sum $R(P,Q,C)$ of g is

$$R = t\, u^2 \bar e_P \bar e_Q \bar e_C - \bar\eta_P\bar\eta_Q\bar\eta_C + \bar\zeta_C(\bar\zeta_P\bar\eta_Q + \bar\zeta_Q\bar\eta_P + \bar\zeta_P\bar\zeta_Q). \tag{22}$$

Step 4: $R \ge 0$ for $P, Q \neq \varnothing$ and any C. Substituting $\bar\eta = \bar\eta^+ - \iota$ and $\bar\zeta = \iota + \vartheta$ gives

$$R = S_0 + \iota_P\bar\eta^+_Q\bar\eta^+_C + \iota_Q\bar\eta^+_P\bar\eta^+_C + \iota_C\bar\beta_P\bar\beta_Q - \iota_P\iota_Q\bar\beta_C + \vartheta_C\Psi,$$
$$S_0 = t\, u^2\bar e_P\bar e_Q\bar e_C - \bar\eta^+_P\bar\eta^+_Q\bar\eta^+_C, \qquad \Psi = \bar\eta^+_P\bar\zeta_Q + \bar\eta^+_Q\bar\zeta_P + \vartheta_P\vartheta_Q,$$

and by (b), $S_0 \ge (t+2t^2)\bar e_P\bar e_Q\bar e_C + t^2\bar e_P\bar e_Q(\bar\mu_C - \bar e_C - \iota_C) \ge 0$, with $\bar e_P, \bar e_Q \ge 1$. If $\vartheta_C = 0$, the only negative term is $-\bar\beta_C$, when $\iota_P = \iota_Q = 1$, and then $S_0 - \bar\beta_C \ge 2t^2\bar e_C + (1+t^2)(\bar\mu_C - \bar e_C - \iota_C) \ge 0$. Let $\vartheta_C = \pm t^l$, $l \ge 1$. If $\iota_P\iota_Q = 0$, then by (b) and (e) $|\Psi| \le (1+2t)\bar e_P\bar e_Q$, and $t^l \le t\bar e_C$ by (d), so $\vartheta_C\Psi \ge -(t+2t^2)\bar e_P\bar e_Q\bar e_C$. If $\iota_P = \iota_Q = 1$, where P and Q have p and q elements, then $\Psi = H_p + H_q \ge 0$ and $\bar\beta_C = t\bar e_C - u^i t^{2l} + \vartheta_C$; for l even, $R \ge 2t^2\bar e_C - t^l \ge 0$ by (d); for l odd, $\vartheta_C\Psi \ge -t\bar e_C \cdot t(\bar e_P + \bar e_Q)$ and $R \ge t\bar e_C\big((\bar e_P\bar e_Q - 1) + t\bar e_P(\bar e_Q - 1) + t\bar e_Q(\bar e_P - 1)\big) \ge 0$.

Step 5: the pairs $(W_g(0), W_g(1))$ of a two-member group. For $W_g(0)$, give the root the pair $(1,0)$, and then divide by t; for $W_g(1)$, give it the pair $(0,1)$. Write $\bar\mu_0, \bar e_0, \bar\zeta_0, \dots$ for the quantities of the block of the root without the root.

(i) $r \in C$. Then $W_g(1) = R(P, Q, C \setminus \{r\}) \geq 0$ by Step 4 and $W_g(0) - W_g(1) = \bar\mu_0 L + \bar\zeta_0 L'$ with $L = u^2 \bar e_P \bar e_Q - \bar\eta_P \bar\eta_Q \geq 0$ and $L + L' = u^2 \bar e_P \bar e_Q - \bar\beta_P \bar\beta_Q \geq 0$ by (a), (b), (c), (e). For $\bar\zeta_0 \geq 0$ this gives $W_g(0) - W_g(1) = (\bar\mu_0 - \bar\zeta_0)L + \bar\zeta_0(L + L') \geq 0$. For $\bar\zeta_0 = -t^l$ (l odd), $-L' = \Psi - \iota_P \iota_Q$; if $\iota_P \iota_Q = 0$, then $L \geq (1 + 2t)\bar e_P \bar e_Q$, $|\Psi| \leq (2t + t^2)\bar e_P \bar e_Q$ and $\bar\mu_0(1 + 2t) \geq t^{l-1}(1 + t + t^2)(1 + 2t) \geq t^l(2t + t^2)$ by (d), so $W_g(0) - W_g(1) = \bar\mu_0 L + t^l \Psi \geq 0$; if $\iota_P = \iota_Q = 1$, then $L \geq 2t$ and $W_g(0) - W_g(1) \geq 2t\bar\mu_0 - t^l \geq 0$.

(ii) $r \in P$ and $P \neq \{r\}$. Then $W_g(1) = R(P \setminus \{r\}, Q, C) \geq 0$ and $W_g(0) - W_g(1) = \bar\mu_0 L_1 + (\bar\mu_0 - \bar\zeta_0)\bar\zeta_C \bar\beta_Q$ with $L_1 = u^2 \bar e_Q \bar e_C - \bar\beta_C \bar\eta_Q \geq u \bar e_Q \bar e_C$, which is enough if $\bar\zeta_C \geq 0$, by (c) and (e); for $\bar\zeta_C = -t^l$ also $\bar\beta_C \leq t\bar e_C$, so $L_1 \geq (1 + 2t)\bar e_Q \bar e_C$, $\bar e_C \geq t^{l-1} u$, and $W_g(0) - W_g(1) \geq t^{l-1} u \bar e_Q \big(\bar\mu_0 u - t|\bar\zeta_0|\big) \geq 0$. In cases (i) and (ii) the pair is $W_g(1)(1, 1) + (W_g(0) - W_g(1))(1, 0)$.

(iii) $P = \{r\}$. Then $W_g(0) = u^2 \bar e_Q \bar e_C$ and $W_g(1) = \bar\beta_C \bar\eta_Q$. If $\iota_Q = 0$, then $W_g(1) \geq 0$ and $W_g(0) - W_g(1) \geq u \bar e_Q \bar e_C$. If $\iota_Q = 1$, where Q has q elements, take $x_3 = \bar\beta_C / t$: then $x_1 = \bar\beta_C H_q \geq 0$ and, with $\bar\beta_C = t\bar e_C - \rho_C$, $\rho_C = \bar\mu_C - \bar e_C - \bar\zeta_C$ (a multiple of t, and $\rho_C \geq -t^l$ with $l \geq 2$ if it has a negative term), $x_2 = u^2 \bar e_Q \bar e_C - \bar\beta_C H_q - u\bar\beta_C / t \geq u(t\bar e_C + \rho_C / t) \geq 0$ by (a), (e), (f) and (d). Steps 4 and 5 hold for Q in place of P.

Step 6: three-member core groups. Here the pairs must be multiplied by -1. In this step, for a block X, let $\bar\pi_X = \bar\mu_X - \bar e_X$. Let $g = \{A, B, C\}$ (now C is a member) have the blocks X_A, X_B, X_C, Y. Summing the formula of the proof of Theorem 9.36, $T_g = F(a', b', c', y')$ with the exceptions there, over all kept parts as in Step 3 (the factors e_n, $\gamma_n = e_{n-1}$, u^n and t^n give $\bar e$, $\bar\mu / u - \bar\pi / t$, $\bar\mu$ and $\bar\pi$, respectively), and removing the exceptional parts by inclusion and exclusion, gives minus the sum of g as

$$\begin{aligned}\Xi = {} & t(2 + 3t)\bar\mu_A \bar\mu_B \bar\mu_C \bar\mu_Y - t\, u(\bar\mu_Y + \bar\pi_Y) \textstyle\sum \bar\mu_A \bar\mu_B \bar\pi_C + t\, u\, \bar\pi_Y \sum \bar\mu_A \bar\pi_B \bar\pi_C + u\, \bar\pi_A \bar\pi_B \bar\pi_C \bar\pi_Y \\ & + \bar\zeta_Y \big(\textstyle\sum \bar\zeta_A \bar\zeta_B \bar\eta_C + 2 \bar\zeta_A \bar\zeta_B \bar\zeta_C \big),\end{aligned}$$

each sum over the three choices of the special block among X_A, X_B, X_C (written with the indices A, B, C). The polynomial Ξ is linear in each of the vectors $\mathbf{b}_X = (\bar\mu_X, \bar\pi_X, \bar\zeta_X)$. Adding to X an element whose pair is $(1, 1)$, $(1, 0)$ or $(1 + t, -t)$ multiplies $\mathbf{b}_X$ componentwise by $m_1 = (2 + t, u, 1)$, $m_2 = (u, t, 0)$ or $m_3 = (1 + t + t^2, t^2, -t)$, respectively, and $\mathbf{b}_\varnothing = \mathbf{d} = (1, 1, 1)$. Let $\mathcal{V}$ be the set of combinations, with coefficients in $\mathbb{Z}_{\geq 0}[t]$, of

$$\mathbf{d} = (1, 1, 1), \quad \mathbf{x} = (1, 0, 0), \quad \mathbf{y} = (t, t, 0), \quad \mathbf{z}_1 = (u, 0, -t), \quad \mathbf{z}_2 = \big(u(1 + t + t^2), 0, t^2\big).$$

Then $m_i \mathcal{V} \subseteq \mathcal{V}$, since

$$\begin{aligned}
& m_1 \mathbf{d} = \mathbf{d} + \mathbf{x} + \mathbf{y}, \quad m_1 \mathbf{x} = (2 + t)\mathbf{x}, \quad m_1 \mathbf{y} = u\mathbf{y} + t\mathbf{x}, \\
& m_1 \mathbf{z}_1 = \mathbf{z}_1 + u^2 \mathbf{x}, \quad m_1 \mathbf{z}_2 = \mathbf{z}_2 + u^2(1 + t + t^2)\mathbf{x}, \\
& m_2 \mathbf{d} = \mathbf{x} + \mathbf{y}, \quad m_2 \mathbf{x} = u\mathbf{x}, \quad m_2 \mathbf{y} = t\mathbf{y} + t\mathbf{x}, \\
& m_2 \mathbf{z}_1 = u^2 \mathbf{x}, \quad m_2 \mathbf{z}_2 = u^2(1 + t + t^2)\mathbf{x}, \\
& m_3 \mathbf{d} = \mathbf{z}_1 + t\mathbf{y}, \quad m_3 \mathbf{x} = (1 + t + t^2)\mathbf{x}, \quad m_3 \mathbf{y} = t^2 \mathbf{y} + (t + t^2)\mathbf{x}, \\
& m_3 \mathbf{z}_1 = \mathbf{z}_2, \quad m_3 \mathbf{z}_2 = t^2 \mathbf{z}_1 + u(1 + 2t + 2t^2 + 2t^3 + t^4)\mathbf{x}.
\end{aligned}$$

So every $\mathbf{b}_X$ lies in $\mathcal{V}$, and $\mathbf{b}_X$ lies in the cone $\mathcal{V}_1$ spanned by the fifteen vectors $m_i \mathbf{v}$, $\mathbf{v} \in \{\mathbf{d}, \mathbf{x}, \mathbf{y}, \mathbf{z}_1, \mathbf{z}_2\}$, if $X \neq \varnothing$. By linearity, $\Xi \geq 0$ follows from $\Xi \geq 0$ at the $680 \cdot 5$ tuples of generators: the vectors of X_A, X_B, X_C in $\mathcal{V}_1$, up to order by symmetry, and that of Y in $\mathcal{V}$. For a root r in X_A (in X_B, X_C by symmetry) or in Y, the pairs $-(W_g(0), W_g(1))$ are multilinear in the four block vectors; the block of r has the vector $m_2 \mathbf{v}$ for $W_g(0)$ (then divide by t) and $\mathbf{v}$ for $W_g(1)$, with $\mathbf{v} \in \mathcal{V}$ the vector of the block without r. Lemma 9.40 reduces their membership in $\mathcal{K}$ to $5 \cdot 120 \cdot 5$ tuples of generators for a root in X_A and to $680 \cdot 5$ tuples for a root in Y. The

program `block_cone_certificate.py` checks all these polynomials in exact arithmetic, and every such $W_g(0)$ is a polynomial. Two-member groups pass the same check, which gives a second proof of Steps 4 and 5. □

Groups with four members. Theorem 9.36 settles a side whose distinct sets form one three-member core group. For larger groups we express Λ in terms of blocks of equal membership. Fix one side as in (16), with nonempty members $N_1, \dots, N_r$, which may repeat. The *type* of an element of N' is the set of $j \in [r]$ with the element in N_j; we write 12 for the type $\{1,2\}$, 1234 for $\{1,2,3,4\}$, and so on. For a nonempty $T \subseteq [r]$ let X_T be the set of elements of type T and $a_T = |X_T|$, and let $\mathcal{E}$ be the set of types T with $X_T \neq \varnothing$; so N_j is the union of the X_T with $j \in T \in \mathcal{E}$. For $S \subseteq \mathcal{E}$ and $j \in [r]$ put $S_j = \{T \in S : j \in T\}$, and use the coordinates y_T, $T \in S$, on $\mathbb{R}^S$. Let $\hat{e}_a = (2+t)^a - (1+t)^a$, as in Theorem 9.36, and $\hat{\tau}_a = ((1+t)^a - 1)/t = \sum_{i=1}^{a} \binom{a}{i} \tau_i$, with $\tau_i = t^{i-1}$ as in (21).

Lemma 9.44 (Block formula)**.** *For $S \subseteq \mathcal{E}$, $J \subseteq [r]$ and a cone σ of the common refinement, in $\mathbb{R}^S$, of the fans of $\max_{T \in S_j} y_T$, $j \in J$ (trivial if $|S_j| \leq 1$), let $\mathrm{Top}(\sigma, J)$ be the set of $T \in S$ that lie in $\operatorname{argmax}_{T' \in S_j} y_{T'}$ on the relative interior of σ for some $j \in J$. Then*

$$(2+t)^{-f} \Lambda(N_1, \dots, N_r; N) = \sum_{S \subseteq \mathcal{E}} \sum_{J \subseteq [r]} (-1)^{r-|J|} \sum_{\sigma} t^{\dim \sigma} \prod_{T \in \mathrm{Top}(\sigma, J)} \hat{e}_{a_T} \prod_{T \in S \setminus \mathrm{Top}(\sigma, J)} \hat{\tau}_{a_T}. \tag{23}$$

Proof. By the definition of Λ, $(2+t)^{-f}\Lambda$ is the sum over $Z \subseteq N'$ and $J \subseteq [r]$ of $(-1)^{r-|J|} \Phi_{N' \setminus Z}(N_j \setminus Z : j \in J)$. Fix Z and J, put $W = N' \setminus Z$, $b_T = |X_T \setminus Z|$ and $S = \{T \in \mathcal{E} : b_T \geq 1\}$, and for $x \in \mathbb{R}^W$ let $y(x) \in \mathbb{R}^S$ be given by $y(x)_T = \max_{X_T \setminus Z} x$. Then $\max_{N_j \setminus Z} x = \max_{T \in S_j} y(x)_T$. A cell of the refinement in $\mathbb{R}^W$, with the argmax sets I_j of the nonempty sets $N_j \setminus Z$, $j \in J$, is mapped by $x \mapsto y(x)$ into the relative interior of the cone σ with the argmax sets $\{T : I_j \cap X_T \neq \varnothing\}$, and for $T \in \mathrm{Top}(\sigma, J)$ the set $I_T = I_j \cap X_T = \operatorname{argmax}_{X_T \setminus Z} x$ does not depend on j. Conversely, σ and nonempty sets $I_T \subseteq X_T \setminus Z$, $T \in \mathrm{Top}(\sigma, J)$, determine exactly one cell: the set of x with $y(x)$ in the relative interior of σ and $\operatorname{argmax}_{X_T \setminus Z} x = I_T$ for these T. On σ, a coordinate y_T with $T \notin \mathrm{Top}(\sigma, J)$ is only bounded from above, by the maxima over the S_j with $j \in J \cap T$, and these are attained in $\mathrm{Top}(\sigma, J)$. So, as in the proof of Lemma 9.34, every element of such a block $X_T \setminus Z$ varies in an open half-line or a line over the other coordinates, and the cell has dimension

$$\dim \sigma + \sum_{T \in \mathrm{Top}(\sigma, J)} (b_T - |I_T|) + \sum_{T \in S \setminus \mathrm{Top}(\sigma, J)} (b_T - 1).$$

As $\sum_I t^{b-|I|} = e_b$, the sum over the nonempty subsets I of a set with b elements, we get

$$\Phi_W(N_j \setminus Z : j \in J) = \sum_{\sigma} t^{\dim \sigma} \prod_{T \in \mathrm{Top}(\sigma, J)} e_{b_T} \prod_{T \in S \setminus \mathrm{Top}(\sigma, J)} t^{b_T - 1},$$

where the cones σ and the sets $\mathrm{Top}(\sigma, J)$ depend only on S and J. Exactly $\prod_T \binom{a_T}{b_T}$ sets Z give the numbers b_T, and $\sum_{b=1}^{a} \binom{a}{b} e_b = \hat{e}_a$ and $\sum_{b=1}^{a} \binom{a}{b} t^{b-1} = \hat{\tau}_a$. □

Lemma 9.45 (Top coefficient)**.** *Let $N_1, \dots, N_r$ be nonempty subsets of a finite set W with union W, and $n = |W|$. Call $F \subseteq W$* detachable *if every nonempty $G \subseteq F$ meets some N_j in exactly one element. Then the coefficient of t^n in $\Delta_W(N_1, \dots, N_r)$ is $(-1)^r \sum_F (-1)^{|F|}$, the sum*

over the detachable sets F. Consequently, the coefficient of $t^{|N'|}$ in $(2+t)^{-f}\Lambda(N_1,\dots,N_r;N)$ is $(-1)^r\sum_{S\in\mathcal{U}}\prod_{T\in S}(-a_T)$, where $\mathcal{U}$ is the set of $S\subseteq\mathcal{E}$ such that every nonempty $S'\subseteq S$ has an index j that lies in exactly one type of S'.

Proof. Let χ_c be the compactly supported Euler characteristic, as in the proof of Theorem 9.16. For $j\in[r]$ let $\mathcal{W}_j\subseteq\mathbb{R}^W$ be the set where $\max_{N_j}x$ is attained at least twice, and put $\mathcal{W}_K=\bigcap_{k\in K}\mathcal{W}_k$ for $K\subseteq[r]$, with $\mathcal{W}_\varnothing=\mathbb{R}^W$. For $J\subseteq[r]$, the cones of dimension n of the refinement of the fans of $\max_{N_j}x$, $j\in J$, are the closures of the sets of x with given one-element sets $\operatorname{argmax}_{N_j}x$, $j\in J$, which are open and convex and are the components of $\mathbb{R}^W\setminus\bigcup_{j\in J}\mathcal{W}_j$. So the coefficient of t^n in $\Phi_W(N_j:j\in J)$ is $(-1)^n\chi_c\big(\mathbb{R}^W\setminus\bigcup_{j\in J}\mathcal{W}_j\big)=(-1)^n\sum_{K\subseteq J}(-1)^{|K|}\chi_c(\mathcal{W}_K)$, and since $\sum_{J\supseteq K}(-1)^{r-|J|}=[K=[r]]$, the coefficient of t^n in $\Delta_W(N_1,\dots,N_r)$ is $(-1)^{n+r}\chi_c(\mathcal{W}_{[r]})$.

The braid cells of $\mathbb{R}^W$, the sets of points with a given weak order of the coordinates, partition $\mathbb{R}^W$, and a braid cell with l levels is homeomorphic to $\mathbb{R}^l$. For $x\in\mathbb{R}^W$ let $\mathcal{Q}(x)$ be the set of $e\in W$ that are the unique maximum of x on some N_j containing e. Then $\mathcal{W}_{[r]}=\{x:\mathcal{Q}(x)=\varnothing\}$, and $\mathcal{Q}$ is constant on each braid cell. Writing $[\mathcal{Q}(x)=\varnothing]=\sum_{F\subseteq\mathcal{Q}(x)}(-1)^{|F|}$ and summing over the braid cells gives

$$\chi_c(\mathcal{W}_{[r]})=\sum_{F\subseteq W}(-1)^{|F|}\chi_c(R_F),\qquad R_F=\{x:F\subseteq\mathcal{Q}(x)\}.$$

If $F\subseteq\mathcal{Q}(x)$ and $\varnothing\ne G\subseteq F$, let $e\in G$ have the least value x_e on G; some N_j has e as its unique maximum, so it contains no other element of G, and $N_j\cap G=\{e\}$. So $R_F=\varnothing$ unless F is detachable.

Let F be detachable. Then $R_F=\bigcap_{e\in F}A_e$, where A_e is the union of the open sets $\{x:x_e>x_{e'}$ for $e'\in N_j\setminus\{e\}\}$ over the j with $e\in N_j$, and A_e is preserved when a coordinate other than x_e decreases. So R_F is open, it is a union of braid cells, and it is preserved when x_v decreases, for every $v\notin F$. We use the following fact. Let V be a finite set, $v\in V$, and let $U\subseteq\mathbb{R}^V$ be open, a union of braid cells, and preserved when x_v decreases; let $U_v\subseteq\mathbb{R}^{V\setminus\{v\}}$ be the set of y with $(y,x_v)\in U$ for all small enough x_v. Then $\chi_c(U)=-\chi_c(U_v)$. Indeed, let β' be a braid cell of $\mathbb{R}^{V\setminus\{v\}}$ with l levels. The braid cells of $\mathbb{R}^V$ that restrict to β' put x_v below all levels, in the lowest level, between the two lowest levels, and so on, up to above all levels; these $2l+1$ cells have alternately $l+1$ and l levels. Those in U form an initial part of this list, since U is preserved when x_v decreases; this part is empty unless $\beta'\subseteq U_v$, and otherwise it ends with a cell with $l+1$ levels, since U is open. So these cells contribute $(-1)^{l+1}$ to $\chi_c(U)$ if $\beta'\subseteq U_v$, and 0 otherwise.

We apply this fact to the elements $v\notin F$, one at a time; an element below all others does not affect the conditions for the elements of F. This gives $\chi_c(R_F)=(-1)^{n-m}\chi_c(R'_F)$, where $m=|F|$ and $R'_F\subseteq\mathbb{R}^F$ is the set of y such that every $e\in F$ is the unique maximum of y on $N_j\cap F$ for some j with $e\in N_j$. As F is detachable, it has an order $w_1,\dots,w_m$ with $N_{j_i}\cap\{w_i,\dots,w_m\}=\{w_i\}$ for some j_i. If $m\ge1$, the condition for w_1 holds on all of $\mathbb{R}^F$, as $N_{j_1}\cap F=\{w_1\}$, and the other conditions are preserved when y_{w_1} decreases; so the fact gives $\chi_c(R'_F)=-\chi_c(R'_{F\setminus\{w_1\}})$, where $F\setminus\{w_1\}$ is detachable. By induction on m, with $R'_\varnothing=\mathbb{R}^\varnothing$ a point, $\chi_c(R'_F)=(-1)^m$. Hence $\chi_c(R_F)=(-1)^n$, and the first claim follows from the display above and the formula $(-1)^{n+r}\chi_c(\mathcal{W}_{[r]})$ for the coefficient of t^n.

For the consequence, only $Z=\varnothing$ reaches the degree $|N'|$ in the definition of Λ, so the coefficient is that of $t^{|N'|}$ in $\Delta_{N'}(N_1,\dots,N_r)$. Two elements of one block lie in the same members, so a detachable set meets each block in at most one element. If F meets the blocks X_T, $T\in S$, in one element each, a set $G\subseteq F$ meets N_j in exactly one element if and only if j lies in exactly one of the types of the blocks that G meets; so F is detachable if and only if $S\in\mathcal{U}$, and there are $\prod_{T\in S}a_T$ such sets F. □

Lemma 9.46 (The coefficient of $1+t$)**.** *Let $r \ge 1$, and for $J \subseteq [r]$ let ω_J be the number of components of the hypergraph on N' with the edges N_j, $j \in J$; an element in none of these edges is a component by itself. Then $(2+t)^{-f}\Lambda(N_1, \dots, N_r; N)$ vanishes at $t = -1$, and the coefficient of $1+t$ in its expansion in powers of $1+t$ is $\sum_{J\subseteq[r]}(-1)^{r-|J|}\omega_J$. This is $(-1)^r(a_{[r]}-1)$ if $[r] \in \mathcal{E}$, that is, if some element lies in all members, and it does not depend on the sizes a_T otherwise.*

Proof. By the proof of Theorem 9.16,

$$(2+t)^{-f}\Lambda = \sum_{\beta}(-1)^{\dim\beta}\sum_{J\subseteq[r]}(-1)^{r-|J|}(-t)^{\omega_J(\beta)},$$

the sum over the braid cells β of $\mathbb{R}^{N'}_{\ge 0}$, where, for the zero set Z and the argmax sets I_j of β, $\omega_J(\beta)$ (called $c_J(\beta)$ there) is the number of components of the graph on $N' \setminus Z$ joining two elements when both lie in I_j for some $j \in J$ with $N_j \not\subseteq Z$. Since $u = 1+t$, we have $(-t)^k \equiv 1 - ku$ modulo u^2. As $r \ge 1$, the inner sum vanishes at $u = 0$, and the coefficient of u is $-\sum_\beta(-1)^{\dim\beta}\varphi(\beta)$ with $\varphi(\beta) = \sum_J(-1)^{r-|J|}\omega_J(\beta)$.

If some j has $N_j \subseteq Z$ or $|I_j| = 1$, then $\omega_{J\cup\{j\}}(\beta) = \omega_J(\beta)$ for $j \notin J$, and $\varphi(\beta) = 0$; we drop these β. For the others let $H = \bigcup_j I_j$; an element of $N' \setminus (Z \cup H)$ is a component for every J and contributes $\sum_J(-1)^{r-|J|} = 0$, so $\varphi(\beta)$ depends only on the sets I_j. Sort these β into classes by H and their weak order on H, a braid cell β_H of $\mathbb{R}^H$ with positive values. The cells of a class fill the set of $x \in \mathbb{R}^{N'}_{\ge 0}$ with $x|_H \in \beta_H$ and $0 \le x_e < m_e(x|_H)$ for $e \notin H$, where m_e is the least of the maxima of the members containing e. This set is homeomorphic to $\beta_H \times [0,1)^{N'\setminus H}$, so the signs $(-1)^{\dim\beta}$ of the class add up to $\chi_c(\beta_H)\chi_c([0,1))^{|N'\setminus H|} = 0$, unless $H = N'$. If $H = N'$, then $Z = \varnothing$, and every level of β contains some I_j. If β has $l \ge 2$ levels, let M_i be the set of j with I_j in the i-th level; then $\omega_J(\beta)$ is the sum over i of the number of components of the i-th level for the edges I_j, $j \in J \cap M_i$. Each summand depends only on $J \cap M_i$, and $[r] \setminus M_i \ne \varnothing$, so its alternating sum over J is 0. So only the braid cell with one level remains, of dimension 1; there $I_j = N_j$ and $\omega_J(\beta) = \omega_J$, which gives the coefficient $\sum_J(-1)^{r-|J|}\omega_J$.

Two elements of one block lie in the same members, so a block that meets an edge N_j, $j \in J$, lies in one component, and $\omega_J = \bar\omega_J + \sum_{T\cap J=\varnothing} a_T$, where $\bar\omega_J$ is the number of components formed by the blocks X_T with $T \cap J \ne \varnothing$; it depends only on $\mathcal{E}$. As $\sum_{J\subseteq[r]\setminus T}(-1)^{r-|J|} = (-1)^r[T = [r]]$, the coefficient is $\sum_J(-1)^{r-|J|}\bar\omega_J + (-1)^r a_{[r]}[[r] \in \mathcal{E}]$. If $[r] \in \mathcal{E}$, the block $X_{[r]}$ lies in every member, so $\bar\omega_J = 1$ for $J \ne \varnothing$ and $\bar\omega_\varnothing = 0$, and the first sum is $-(-1)^r$. □

Since $(2+t)^f \equiv 1$ modulo $1+t$, Lemma 9.46 holds for Λ itself. For a three-member core group of shape (a,b,c,y) it gives $(-1)^{r-3}(1-y)$, which is $\Theta(-1) = 1-y$ in Theorem 9.36.

Theorem 9.47 (A side formed by a four-member group)**.** *Suppose that the distinct sets $D_1, \dots, D_4$ of one side each have at least two elements and form one group, that every element of N' lies in at least two of them, and that none of them is the union of the others inside it. Then exactly one of the following holds:*

- *(a) $\Lambda = 0$; this happens exactly when the sets are $\{p,q\}$, $\{p\} \cup X$, $\{q\} \cup X$ and X, with $|X| \ge 2$ and $p \ne q$ outside X;*
- *(b) the order of vanishing of Λ at $t = -1$ is at most 3;*
- *(c) the order is at least 4, and the coefficient of $t^{|N'|+f}$ in $(-1)^{r-4}\Lambda$ is negative.*

So the conclusion of Theorem 9.13 fails for this side unless $\Lambda = 0$.

The proof uses an exact computer enumeration of type sets and an exact check of finitely many polynomial identities and inequalities.[4]

Proof. By (16), $(-1)^{r-4}(2+t)^{-f}\Lambda$ depends only on $D_1,\dots,D_4$, and $(2+t)^f$ has the value 1 at $t=-1$ and the leading coefficient 1; so we may assume $r=4$, $N_j=D_j$ and $f=0$. Up to renaming the members, the side is then given by $\mathcal{E}$ and the sizes $a_T\ge 1$, $T\in\mathcal{E}$. Write $\Lambda=\sum_{m\ge1}c_m(1+t)^m$ (Lemma 9.46), and let $\Lambda_{\mathrm{top}}(a)=\sum_{S\in\mathcal{U}}(-1)^{|S|}\prod_{T\in S}a_T$ be the coefficient of $t^{|N'|}$ (Lemma 9.45); it is multilinear in the sizes.

(1) For 120 type sets $\mathcal{E}$ the hypotheses hold when every block has one element, and then for all sizes, since larger blocks keep the group and do not affect the other hypotheses. For 10 more sets $\mathcal{E}$ they hold only when certain blocks have at least two elements, and for no other $\mathcal{E}$, since all hypotheses depend only on which blocks have one element, and larger blocks never break them.

(2) For 99 of the 120, every coefficient of the polynomial $\Lambda_{\mathrm{top}}(1+x)$ in $x=(x_T)_{T\in\mathcal{E}}$ is at most 0 and its constant term is negative. So $\Lambda_{\mathrm{top}}(a)<0$ for all sizes, and (b) or (c) holds.

(3) For 4 of the remaining 21, $1234\notin\mathcal{E}$ and $c_1=-1$ by Lemma 9.46; so (b) holds, with order 1.

(4) The other 17 contain the type 1234, so $c_1=a_{1234}-1$ by Lemma 9.46, and (b) holds if $a_{1234}\ge2$. Let $a_{1234}=1$. For 8 of them, every coefficient of $\Lambda_{\mathrm{top}}(1+x)$ at $x_{1234}=0$ is at most 0 and the constant term is negative. For the last 9 we use c_2 and c_3. The coefficients of $(1+t)^i$ in $\hat{e}_a$ and $\hat{\tau}_a$ are $[a>i]\binom{a}{i}$ and $[a>i]$, respectively; for $i\le3$ and $a\ge3$ they equal $\binom{a}{i}$ and 1, except that both are 0 for $i=a=3$. By (23), for each type T with $a_T=3$, lowering both coefficients of $(1+t)^3$ from 1 to 0 alters only c_3, by $-\sum_{S\ni T}\sum_J(-1)^{r-|J|}\sum_\sigma(-1)^{\dim\sigma}$ (all other factors are 1 at $t=-1$), and this is 0, as $\sum_\sigma(-1)^{\dim\sigma}=(-1)^{|S|}$ for every J. So c_2 and c_3 are polynomials in the a_T on each region where every a_T is 1, 2 or at least 3; they are computed exactly from (23), with $a_T=3+z_T$, $z_T\ge0$, where $a_T\ge3$. A region is settled if c_2 or c_3 has a nonzero constant term and no coefficient of the opposite sign, or if Λ_{top} has a negative constant term and no positive coefficient. If c_2 is a nonzero constant times z_T plus a polynomial free of z_T, for some type T, then (b) holds where $c_2\ne0$. On the part where $c_2=0$, the solution z_T of $c_2=0$ is substituted into c_3 and Λ_{top}, and this part is tested and split in the same way; dropping the conditions on z_T only enlarges the part. If c_2 has no such form, a variable z_T is split into $z_T=0$ and $z_T\ge1$. All 1009 parts are settled, so (b) or (c) holds.

(5) The 10 sets of (1) are checked in the same way, with c_1, c_2, c_3 and Λ_{top} on the sizes allowed there; 780 parts are settled; the other 2 are those with $\mathcal{E}=\{12,13,234\}$ and $a_{12}=a_{13}=1$. These are the sides of (a), with $X=X_{234}$ and $|X|\ge2$. They satisfy the hypotheses: a cycle through p, q and an element of X links $\{p,q\}$, $\{p\}\cup X$ and $\{q\}\cup X$, and a cycle through two elements of X links X to them. Since $\hat{e}_1=\hat{\tau}_1=1$, (23) gives $\Lambda=\Lambda_1\hat{e}_{|X|}+\Lambda_2\hat{\tau}_{|X|}+\Lambda_3$, with $\Lambda_1,\Lambda_2,\Lambda_3\in\mathbb{Z}[t]$ independent of $|X|$. The program finds $\Lambda=0$ for $|X|=2,3,4$, and the determinant of the rows $(\hat{e}_a,\hat{\tau}_a,1)$, $a=2,3,4$, is $-(1+t)^3(2+t)^2\ne0$; so $\Lambda_1=\Lambda_2=\Lambda_3=0$, and $\Lambda=0$ for all $|X|$. In all other cases (b) or (c) holds, so $\Lambda\ne0$. □

Example 9.48 (Order four)**.** Let the blocks X_{12}, X_{123}, X_{1234}, X_{13}, X_{14} and X_{23} have the sizes 4, 2, 1, 4, 3 and 1, respectively, and let all other blocks be empty; so D_1 has 14 elements, D_2 and D_3 have 8, D_4 has 4, $|N'|=15$ and $f=0$. The hypotheses of Theorem 9.47 hold, and the ancillary

[4]The program `k4_certificate.py` checks the steps (1)–(5) of the proof (the type sets of step (1) come from the imported module `core_type_sets.py`) in exact integer arithmetic (Python 3, standard library only; about five minutes on one core). It and its output `k4_certificate_output.txt` are ancillary files that accompany this paper; its SHA-256 is `01b9d491afe750963e44b3a6da242592afa39dd2fa6c0d5e2c572ad5b8ed8ea7`, and the checksums of the programs it imports and of all outputs are listed in the ancillary file `README.txt`.

program `block_formula.py`, which evaluates the block formula (23) exactly, gives $\Lambda = -t(1+t)^4\Upsilon$ with

$$\Upsilon = 12551 + 98822t + 302117t^2 + 512660t^3 + 551841t^4 + 399748t^5 + 198994t^6 + 67548t^7 + 15009t^8 + 1975t^9 + 117t^{10}.$$

The output of the program, which contains this polynomial, is the file `block_formula_output.txt`; its checksum is listed in `README.txt`. So the order is exactly 4 and every nonzero coefficient has the wrong sign: case (c) occurs. Here $1234 \in \mathcal{E}$ and $a_{1234} = 1$, so the coefficient of $1 + t$ vanishes by Lemma 9.46. ◇

9.7 Comparable switches

Theorem 9.4 requires pairwise incomparable switches. Now let $q_1 < q_2$ be order elements, and define C_J for $J \subseteq \{1, 2\}$ and $\mathcal{M}_C(q_1, q_2)$ by the formulas of Section 9.1. We call $q_1 \lessdot q_2$ a *bridge* if q_2 is the only upper cover of q_1 and q_1 is the only lower cover of q_2. Equivalently, every element above q_1 is q_2 or lies above q_2, and every element below q_2 is q_1 or lies below q_1. For a bridge, let $\bar{P}$ be the poset obtained by identifying q_1 and q_2 to one order element q: on $P \setminus \{q_1, q_2\}$ it has the order of P, $x < q$ if and only if $x < q_1$, and $q < y$ if and only if $q_2 < y$. Since q_1 is not minimal and q_2 is not maximal (they are unmarked), $\bar{P}$ has the extrema and the marking of P, so it satisfies the bounded hypotheses of Section 1.

Put $B = Q_C(P \setminus \{q_1, q_2\}, \lambda)$. Define L_1 on B by the formula for L in Section 3.1 with $q = q_1$, and U_2 by the formula for U with $q = q_2$, with chains in C and endpoints in $A \cup (O \setminus \{q_1, q_2\})$. These are the envelopes of q in $\bar{P}$, whose deleted base is B. Let $\mathcal{S}_1, \mathcal{T}_2$ be their effective affinity subdivisions of B, and let Δ be the alternating sum of Section 9.1 for subdivisions of this B.

Lemma 9.49 (Fibres over a bridge)**.** *Let $q_1 \lessdot q_2$ be a bridge, and write z_1, z_2 for the coordinates of q_1, q_2. After reordering coordinates, $Q_{C_J}(P, \lambda)$ is the set of $(u, z_1, z_2) \in B \times \mathbb{R}^2$ with*

$$\begin{aligned}
&L_1(u) \le z_1 \le z_2 \le U_2(u) && (J = \varnothing),\\
&0 \le z_1 \le z_2 - L_1(u), \quad z_2 \le U_2(u) && (J = \{1\}),\\
&L_1(u) \le z_1, \quad 0 \le z_2 \le U_2(u) - z_1 && (J = \{2\}),\\
&0 \le z_1, \quad 0 \le z_2, \quad z_1 + z_2 \le U_2(u) - L_1(u) && (J = \{1, 2\}),
\end{aligned}$$

and $L_1 \le U_2$ on B.

Proof. The inequalities (1) whose chains avoid q_1 and q_2 are those of B. If a chain has consecutive members $x < y$ with $x < q_i < y$, insert q_i. If q_i is a chain element, the new inequality has the additional term z_i on its left side; as $z_i \ge 0$, it implies the old one. If q_i is an order element, the new chain splits at q_i into two chains whose inequalities add up to the old one. So $Q_{C_J}(P, \lambda)$ is cut out by the inequalities of B, by $z_i \ge 0$ for $i \in J$, and by the inequalities of the chains that meet $\{q_1, q_2\}$ and admit no insertion. By the bridge property, such a chain is a run of consecutive members of K, q_1, q_2, H, for a chain K below q_1 and a chain H above q_2, possibly empty; its endpoints are marked or order elements, and its interior lies in C_J. For $J = \varnothing$ these chains give $L_1 \le z_1$, $z_1 \le z_2$ and $z_2 \le U_2$ (maxima over K and minima over H, as in Section 3.1); for $J = \{1\}$ they give $L_1 + z_1 \le z_2$ and $z_2 \le U_2$; for $J = \{2\}$ they give $L_1 \le z_1$ and $z_1 + z_2 \le U_2$; and for $J = \{1, 2\}$ they give $L_1 + z_1 + z_2 \le U_2$. Finally, a chain K below q_1 followed by a chain H above q_2 is a chain of $P \setminus \{q_1, q_2\}$, so $L_1 \le U_2$ on B. □

Theorem 9.50 (Bridges)**.** *Let $q_1 \lessdot q_2$ be a bridge of order elements. Then*

$$F_{Q_C(P,\lambda)} = F_{Q_{C\cup\{q_1\}}(P,\lambda)} = F_{Q_{C\cup\{q_2\}}(P,\lambda)}$$

and

$$\mathcal{M}_C(q_1,q_2) = (1+t)\big(F_{Q_{C\cup\{q\}}(\bar{P},\lambda)} - F_{Q_C(\bar{P},\lambda)}\big) = (1+t)\,\Delta(\mathcal{S}_1,\mathcal{T}_2).$$

In particular $\mathcal{M}_C(q_1,q_2) \in (1+t)^3\mathbb{Z}_{\geq 0}[t]$, and $\mathcal{M}_C(q_1,q_2) = 0$ if L_1 or U_2 is affine.

Proof. Write F_0, F_1, F_2, F_{12} for $F_{Q_{C_J}(P,\lambda)}$ with $J = \varnothing$, $\{1\}$, $\{2\}$ and $\{1,2\}$, respectively, and put $V = Q_C(\bar{P},\lambda)$ and $V' = Q_{C\cup\{q\}}(\bar{P},\lambda)$. By Section 3.1, V and V' are the sets of (u,z), $u \in B$, with $L_1(u) \leq z \leq U_2(u)$, respectively $0 \leq z \leq U_2(u) - L_1(u)$, and Lemma 2.1 on B gives $F_{V'} - F_V = \Delta(\mathcal{S}_1,\mathcal{T}_2)$.

By Lemma 9.49, $Q_C(P,\lambda)$ and $Q_{C\cup\{q_1\}}(P,\lambda)$ are the polytopes $\{L_1(u) \leq z_1 \leq z_2\}$ and $\{0 \leq z_1 \leq z_2 - L_1(u)\}$ over V, in the coordinates (u,z_2) on V. Apply Lemma 2.1 on V: the upper function z_2 is affine, so its effective affinity subdivision is the face complex of V, and the right side of (3) is 0. Thus $F_1 = F_0$. In the same way, $Q_C(P,\lambda)$ and $Q_{C\cup\{q_2\}}(P,\lambda)$ are $\{z_1 \leq z_2 \leq U_2(u)\}$ and $\{0 \leq z_2 \leq U_2(u) - z_1\}$ over V in the coordinates (u,z_1), with the affine lower function z_1, so $F_2 = F_0$.

Hence $\mathcal{M}_C(q_1,q_2) = F_{12} - F_2$. In the coordinates (u,z_2) on V', the polytopes $Q_{C\cup\{q_2\}}(P,\lambda)$ and $Q_{C\cup\{q_1,q_2\}}(P,\lambda)$ are $\{L_1(u) \leq z_1 \leq U_2(u) - z_2\}$ and $\{0 \leq z_1 \leq U_2(u) - z_2 - L_1(u)\}$ over V', and $L_1 \leq U_2 - z_2$ on V'. Put $\theta = U_2 - L_1$, a concave function with $\theta \geq 0$ on B. If $\theta = 0$ on B, then $L_1 = U_2$ is affine, $Q_{C\cup\{q_2\}}(P,\lambda)$ and $Q_{C\cup\{q_1,q_2\}}(P,\lambda)$ are both affinely isomorphic to B, and both sides vanish. Otherwise let $Z_q = \{u \in B : \theta(u) = 0\}$. If θ vanishes at a relative-interior point of a face of B, it vanishes on that face, since θ is concave and nonnegative. So Z_q is a union of proper faces of B, closed under taking faces; it need not be convex. Hence every maximal affinity domain of L_1, or of U_2, meets $B \setminus Z_q$, and its preimage in V' is full-dimensional. For a subdivision $\mathcal{G}$ of B that includes all faces, let $V' \vee \mathcal{G}$ be the face complex of V' refined by the preimage of $\mathcal{G}$. Then the effective affinity subdivisions of L_1 and of $U_2 - z_2$ on V' are $V' \vee \mathcal{S}_1$ and $V' \vee \mathcal{T}_2$, and Lemma 2.1 on V' gives

$$\mathcal{M}_C(q_1,q_2) = F_{V'} + F_{V'\vee\mathcal{S}_1\vee\mathcal{T}_2} - F_{V'\vee\mathcal{S}_1} - F_{V'\vee\mathcal{T}_2}.$$

Every relatively open cell c of such a $\mathcal{G}$ lies in Z_q or in $B \setminus Z_q$. If c lies in $B \setminus Z_q$, the cells of $V' \vee \mathcal{G}$ over c are $c \times \{0\}$, the set $\{(u,z) : u \in c,\ 0 < z < \theta(u)\}$ of dimension $\dim c + 1$, and the graphs of θ over the cells of $\mathcal{G} \vee \mathcal{S}_1 \vee \mathcal{T}_2$ inside c. On a face E of B contained in Z_q, $L_1 = U_2$ is both convex and concave, hence affine, and every affine piece of L_1 or of U_2 agrees with it on a face of E. So, for each $\mathcal{G} \in \{\mathcal{G}_\varnothing, \mathcal{S}_1, \mathcal{T}_2, \mathcal{S}_1 \vee \mathcal{T}_2\}$, where $\mathcal{G}_\varnothing$ is the face complex of B, the cells of $V' \vee \mathcal{G}$ over Z_q are the sets $E \times \{0\}$ for the faces E of B contained in Z_q, taken relatively open. Writing F_{Z_q} for the face polynomial of the subcomplex Z_q,

$$F_{V'\vee\mathcal{G}} = (1+t)(F_{\mathcal{G}} - F_{Z_q}) + F_{\mathcal{S}_1\vee\mathcal{T}_2},$$

and the alternating sum is $(1+t)\Delta(\mathcal{S}_1,\mathcal{T}_2)$.

Finally, Corollary 5.13 for $\bar{P}$ gives $F_{V'} - F_V \in (1+t)^2\mathbb{Z}_{\geq 0}[t]$. If L_1 or U_2 is affine, then $\mathcal{S}_1$ or $\mathcal{T}_2$ is trivial and $\Delta(\mathcal{S}_1,\mathcal{T}_2) = 0$. □

Example 9.51 (A bridge)**.** Let P have the cover relations $\hat{0} \lessdot a_i \lessdot q_1 \lessdot q_2 \lessdot b_j \lessdot \hat{1}$ $(i,j = 1,2)$, with marks 0 and 1 on $\hat{0}$ and $\hat{1}$, respectively, and let $C = \{a_1, a_2, b_1, b_2\}$. Then $\bar{P}$ is the layered poset with $Y = \{q\}$, $N^- = \{a_1, a_2\}$ and $N^+ = \{b_1, b_2\}$, so Corollary 9.11 gives $F_{Q_{C\cup\{q\}}(\bar{P},\lambda)} - F_{Q_C(\bar{P},\lambda)} = H_2^2 = t^2(1+t)^2$, and Theorem 9.50 gives $\mathcal{M}_C(q_1,q_2) = t^2(1+t)^3$. ◇

For comparable order elements $q_1 < q_2$ put $M = \{x \in P : q_1 < x < q_2\}$. We treat two further cases:

(T) $M \subseteq C$, every element above q_1 lies in M, is q_2 or lies above q_2, and every element below q_2 lies in M, is q_1 or lies below q_1;

(R) every maximal chain $q_1 \lessdot c_1 \lessdot \cdots \lessdot c_k \lessdot q_2$ of the interval $[q_1, q_2]$ has some $c_i \in O$.

For $M = \varnothing$, (T) says exactly that $q_1 \lessdot q_2$ is a bridge. Under (T), an upper cover y of an element $c \in M$ lies in $M \cup \{q_2\}$, since $y > q_1$, while $y > q_2$ would give $c < q_2 < y$; dually, a lower cover of c lies in $M \cup \{q_1\}$. Define B, L_1, U_2, $\mathcal{S}_1$, $\mathcal{T}_2$ and $\bar{P}$ by the same formulas as for a bridge. In $\bar{P}$ the elements of M are incomparable with q. Since q_1 and q_2 are unmarked, there are marked elements $a < q_1$ and $b > q_2$; every element above q_1 or q_2 lies above a, and every element below q_1 or q_2 lies below b. So $\bar{P}$ has the extrema and the marking of P, its deleted base at q is B, and the envelopes of q in $\bar{P}$ are L_1 and U_2. Put

$$D_M(u) = \max_{\gamma} \sum_{c \in \gamma} u_c \qquad (u \in B),$$

the maximum over the chains γ of M, the empty chain included, and let $\mathcal{D}_M$ be its effective affinity subdivision of B.

Lemma 9.52 (Fibres under (T))**.** *Assume (T), and write z_1, z_2 for the coordinates of q_1, q_2. After reordering coordinates, $Q_{C_J}(P, \lambda)$ is the set of $(u, z_1, z_2) \in B \times \mathbb{R}^2$ with*

$$\begin{aligned}
&L_1(u) \le z_1, \quad z_1 + D_M(u) \le z_2, \quad z_2 \le U_2(u) && (J = \varnothing),\\
&0 \le z_1, \quad L_1(u) + z_1 + D_M(u) \le z_2, \quad z_2 \le U_2(u) && (J = \{1\}),\\
&L_1(u) \le z_1, \quad 0 \le z_2, \quad z_1 + D_M(u) + z_2 \le U_2(u) && (J = \{2\}),\\
&0 \le z_1, \quad 0 \le z_2, \quad L_1(u) + z_1 + D_M(u) + z_2 \le U_2(u) && (J = \{1,2\}),
\end{aligned}$$

and $L_1 + D_M \le U_2$ on B.

Proof. As in the proof of Lemma 9.49, $Q_{C_J}(P, \lambda)$ is cut out by the inequalities of B, by $z_i \ge 0$ for $i \in J$, and by the inequalities of the chains that meet $\{q_1, q_2\} \cup M$ and admit no insertion of q_1 or q_2. Let such a chain have a member m in $\{q_1, q_2\} \cup M$. Every member is comparable with m, so by (T) it lies in exactly one of the classes $\{x : x < q_1\}$, $\{q_1\}$, M, $\{q_2\}$, $\{x : x > q_2\}$, and the classes occur in this order along the chain. If the chain has a member below q_1 and a later member in $M \cup \{q_2\}$ or above q_2, then q_1 is a member: otherwise the last member below q_1 and the next member would admit the insertion of q_1. The dual statement holds for q_2. The endpoints are not in $M \subseteq C$, so a chain with a member in M contains q_1 and q_2. Hence the chain is a run of consecutive members of K, q_1, γ, q_2, H, for a chain K below q_1, a chain γ of M and a chain H above q_2, possibly empty; its endpoints are marked or order elements, and its interior lies in C_J. Maxima over K and γ and minima over H, as in Section 3.1, give the four systems. Finally, K, γ and H together form a chain of $P \setminus \{q_1, q_2\}$ with interior in C, so $L_1 + D_M \le U_2$ on B. □

The four polytopes are thus triangle fibrations over B with fixed edge normals, that is, polytopes Ω as in Lemma 9.53 below. That lemma counts their faces as in Lemma 9.3.

Lemma 9.53 (Faces of triangle fibrations)**.** *Let $n_1, n_2, n_3 \in \mathbb{R}^2$ be pairwise linearly independent with $n_1 + n_2 + n_3 = 0$, let h_1, h_2, h_3 be finite minima of affine functions on B with effective affinity subdivisions $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and suppose that $g = h_1 + h_2 + h_3 \ge 0$ on B. Put*

$$\Omega = \{(u, \zeta) \in B \times \mathbb{R}^2 : \langle n_e, \zeta \rangle \le h_e(u) \ (e = 1, 2, 3)\}, \qquad Z = \{u \in B : g(u) = 0\}.$$

For $I \subseteq \{1,2,3\}$ with $|I| \le 2$ let $\mathcal{R}_I = \bigvee_{e\in I} \mathcal{H}_e$, the face complex of B for $I = \varnothing$. Then

$$F_\Omega = F_Z + \sum_{|I|\le 2} t^{2-|I|}\big(F_{\mathcal{R}_I} - F_Z\big),$$

where F_Z is the face polynomial of the complex of faces of B contained in Z.

Proof. Since g is concave and nonnegative, Z is a union of faces of B, closed under taking faces, as in the proof of Theorem 9.50. On a face of B contained in Z the concave functions h_e have sum 0, so each is affine there, by the Jensen argument in the proof of Lemma B.2. A cell of $\mathcal{R}_I$ contained in Z has its carrier in Z, since a relative-interior point of the cell lies in the relative interior of its carrier. By Lemma 9.2, the cells of $\mathcal{R}_I$ contained in Z are therefore exactly the faces of B contained in Z. For $u \in B \setminus Z$ the fibre of Ω is a triangle with outer normals n_1, n_2, n_3 and nonempty interior; for $u \in Z$ it is a point, at which all three inequalities are tight, because they add up to $0 \le g(u) = 0$.

Every nonempty face of Ω is exposed by a functional $\psi(u,\zeta) = c(u) + \langle \beta, \zeta\rangle$. The normal fan of the triangle is complete, so $\beta = \sum_{e\in I} \alpha_e n_e$ with $\alpha_e > 0$ for a unique I with $|I| \le 2$. For $u \in B \setminus Z$, the maximum of ψ on the fibre is attained exactly on the face of the triangle on which the edges in I are tight, and for every $u \in B$ it equals

$$\varphi(u) = c(u) + \sum_{e\in I} \alpha_e h_e(u).$$

By the Jensen argument, the affinity subdivision of φ is $\mathcal{R}_I$. As in the proof of Lemma 9.3, the face exposed by ψ is

$$G_I(\sigma) = \{(u,\zeta) \in \Omega : u \in \sigma,\ \langle n_e, \zeta\rangle = h_e(u)\ (e \in I)\}$$

with $\sigma = \operatorname{argmax}_B \varphi \in \mathcal{R}_I$. Conversely, for $|I| \le 2$ and $\sigma \in \mathcal{R}_I$ choose an affine ℓ with $\sigma = \operatorname{argmax}_B(\ell + \sum_{e\in I} h_e)$; then $\ell(u) + \langle \sum_{e\in I} n_e, \zeta\rangle$ exposes $G_I(\sigma)$.

If $\sigma \not\subseteq Z$, then $\operatorname{relint}(\sigma)$ misses Z, the h_e with $e \in I$ are affine on σ, and over $\operatorname{relint}(\sigma)$ the fibre of $G_I(\sigma)$ is the face of a nondegenerate triangle on which exactly the edges in I are tight. So $\dim G_I(\sigma) = \dim \sigma + 2 - |I|$; moreover σ is the projection of $G_I(\sigma)$, and I is the set of e with $\langle n_e, \zeta\rangle = h_e(u)$ on all of $G_I(\sigma)$. If $\sigma \subseteq Z$, then $G_I(\sigma)$ is the graph of the point fibre over σ, a face of B contained in Z, the same for every I. Hence the nonempty faces of Ω correspond bijectively to the pairs (I,σ) with $\sigma \in \mathcal{R}_I$ and $\sigma \not\subseteq Z$, together with the faces of B contained in Z, with the dimensions just given. Counting these faces by dimension gives the formula. □

Theorem 9.54 (Chain elements between comparable switches)**.** *Assume (T). Then*

$$\mathcal{M}_C(q_1,q_2) = (1+t)\,\Delta(\mathcal{S}_1,\mathcal{T}_2) + t\,\Delta(\mathcal{D}_M,\mathcal{S}_1,\mathcal{T}_2),$$

and $\Delta(\mathcal{S}_1,\mathcal{T}_2) = F_{Q_{C\cup\{q\}}(\bar P,\lambda)} - F_{Q_C(\bar P,\lambda)}$, and $\mathcal{M}_C(q_1,q_2) \in (1+t)^3\mathbb{Z}_{\ge 0}[t]$. If D_M is affine on B, for instance if M is empty or a chain, then $\mathcal{M}_C(q_1,q_2) = (1+t)\Delta(\mathcal{S}_1,\mathcal{T}_2)$; for $M = \varnothing$ this is Theorem 9.50. If L_1 or U_2 is affine, then $\mathcal{M}_C(q_1,q_2) = 0$.

Proof. Write F_0, F_1, F_2, F_{12} for $F_{Q_{C_J}(P,\lambda)}$ with $J = \varnothing$, $\{1\}$, $\{2\}$ and $\{1,2\}$, respectively, and put $G = U_2 - L_1 - D_M \ge 0$. By Lemma 9.52, each $Q_{C_J}(P,\lambda)$ is a polytope Ω as in Lemma 9.53 in the coordinates $\zeta = (z_1, z_2)$, with $g = G$ and hence with the same Z for all four J. The normals

n_1, n_2, n_3, the functions h_1, h_2, h_3 and their subdivisions are as follows, by the Jensen argument for the sums:

J	normals n_1, n_2, n_3	functions h_1, h_2, h_3	subdivisions
$\varnothing$	$(-1,0),\ (0,1),\ (1,-1)$	$-L_1,\ U_2,\ -D_M$	$\mathcal{S}_1,\ \mathcal{T}_2,\ \mathcal{D}_M$
$\{1\}$	$(-1,0),\ (0,1),\ (1,-1)$	$0,\ U_2,\ -L_1 - D_M$	$\mathcal{G}_\varnothing,\ \mathcal{T}_2,\ \mathcal{S}_1 \vee \mathcal{D}_M$
$\{2\}$	$(-1,0),\ (0,-1),\ (1,1)$	$-L_1,\ 0,\ U_2 - D_M$	$\mathcal{S}_1,\ \mathcal{G}_\varnothing,\ \mathcal{D}_M \vee \mathcal{T}_2$
$\{1,2\}$	$(-1,0),\ (0,-1),\ (1,1)$	$0,\ 0,\ U_2 - L_1 - D_M$	$\mathcal{G}_\varnothing,\ \mathcal{G}_\varnothing,\ \mathcal{S}_1 \vee \mathcal{D}_M \vee \mathcal{T}_2$

In $\mathcal{M}_C(q_1, q_2) = F_{12} - F_1 - F_2 + F_0$ the terms F_Z cancel. For $I = \{2\}$ and $I = \{1,3\}$ the complex $\mathcal{R}_I$ is the same for J and $J \cup \{1\}$, and for $I = \{1\}$ and $I = \{2,3\}$ it is the same for J and $J \cup \{2\}$; for $I = \varnothing$ it is always the face complex. So these terms cancel. The case $I = \{1,2\}$ contributes $F_{\mathcal{G}_\varnothing} - F_{\mathcal{T}_2} - F_{\mathcal{S}_1} + F_{\mathcal{S}_1 \vee \mathcal{T}_2} = \Delta(\mathcal{S}_1, \mathcal{T}_2)$, and the case $I = \{3\}$ contributes t times

$$F_{\mathcal{D}_M} - F_{\mathcal{S}_1 \vee \mathcal{D}_M} - F_{\mathcal{D}_M \vee \mathcal{T}_2} + F_{\mathcal{S}_1 \vee \mathcal{D}_M \vee \mathcal{T}_2} = \Delta(\mathcal{S}_1, \mathcal{T}_2) + \Delta(\mathcal{D}_M, \mathcal{S}_1, \mathcal{T}_2).$$

This proves the identity. By Section 3.1 for q in $\bar{P}$ and Lemma 2.1 on B, as in the proof of Theorem 9.50, $F_{Q_{C\cup\{q\}}(\bar{P},\lambda)} - F_{Q_C(\bar{P},\lambda)} = \Delta(\mathcal{S}_1, \mathcal{T}_2)$, and Corollary 5.13 for $\bar{P}$ shows that it lies in $(1+t)^2\mathbb{Z}_{\geq 0}[t]$. If D_M, L_1 or U_2 is affine, its subdivision is trivial, and the corresponding terms vanish by Lemma 9.5(1).

It remains to show that $\mathcal{D}_M, \mathcal{S}_1, \mathcal{T}_2$ are mutually transverse relative to every face of B; then Corollary G.3 puts $\Delta(\mathcal{D}_M, \mathcal{S}_1, \mathcal{T}_2)$ in $(1+t)^3\mathbb{Z}_{\geq 0}[t]$. We follow the proof of Proposition 9.9. Fix a face E of B, a point $x \in \mathrm{relint}(E)$, and put $V = \mathrm{dir}(E)$. If $G(x) = 0$, then $G = 0$ on E, so the concave functions $-L_1$, $-D_M$ and U_2 have sum 0 on E; they are affine there, and all three conormal spaces are zero by Lemma 9.2(1). Otherwise choose η_1, η_2 with $L_1(x) < \eta_1$ and $\eta_1 + D_M(x) < \eta_2 < U_2(x)$, and mark q_1, q_2 by η_1, η_2. By (T) and $M \subseteq C$, a marked element comparable with q_1 or q_2 lies below q_1 or above q_2; so the empty branches and $D_M(x) \geq 0$ show that the marking stays order-preserving. As in Lemma 9.52, where a marked q_i is an endpoint, the polytope Π of this marking is the set of $u \in B$ with $L_1(u) \leq \eta_1$, $\eta_1 + D_M(u) \leq \eta_2$ and $\eta_2 \leq U_2(u)$, and these inequalities are strict at x. As in the proof of Proposition 9.9, with $Y = \{q_1, q_2\}$ and P°, Γ as in Section 9.3, restrictions to V of linear forms supported on disjoint sets of components of Γ span subspaces of V^* whose sum is direct. The branches of L_1 have all their members below q_1, so their saturated refinements in that proof avoid q_1, q_2 and M; likewise for U_2 above q_2. So, as there, $N(\mathcal{S}_1)$ and $N(\mathcal{T}_2)$ consist of restrictions of forms supported on the components in $\kappa^-(q_1)$ and $\kappa^+(q_2)$. If $M \neq \varnothing$, the empty chain is dominated in D_M, since $u_c \geq 0$ for $c \in M$, and every chain of M refines, by adding elements of $M \subseteq C$, to a maximal chain of $[q_1, q_2]$. So D_M is the maximum of the linear forms $\sum_j u_{c_j}$ over the saturated chains $q_1 \lessdot c_1 \lessdot \cdots \lessdot c_k \lessdot q_2$, and the conormal space $N(\mathcal{D}_M)$ consists of restrictions of forms supported on the interval M. By the description of the covers of the elements of M given after (T), the set M is a union of components of Γ, and these are not in $\kappa^-(q_1) \cup \kappa^+(q_2)$. Hence the sum of $N(\mathcal{D}_M)$ and $N(\mathcal{S}_1) + N(\mathcal{T}_2)$ is direct. Finally $N(\mathcal{S}_1) \cap N(\mathcal{T}_2) = 0$ by Section 3.3, applied to q in $\bar{P}$, whose deleted base is B and whose envelopes L_1, U_2 satisfy $L_1(x) < U_2(x)$. So the three conormal spaces form a direct sum. □

Example 9.55 (Chain elements between switches)**.** Let P have the cover relations $\hat{0} \lessdot a_i \lessdot q_1 \lessdot c_j \lessdot q_2 \lessdot b_k \lessdot \hat{1}$ $(i,j,k = 1,2)$, with marks 0 and 1 on $\hat{0}$ and $\hat{1}$, respectively, and let $C = \{a_1, a_2, c_1, c_2, b_1, b_2\}$. Then (T) holds with $M = \{c_1, c_2\}$, and B is the 6-dimensional polytope given by $x \geq 0$ and $x_{a_i} + x_{c_j} + x_{b_k} \leq 1$ (the other inequalities are implied). Here $L_1 = \max(x_{a_1}, x_{a_2})$, $U_2 = 1 - \max(x_{b_1}, x_{b_2})$ and $D_M = \max(x_{c_1}, x_{c_2})$, so $\mathcal{S}_1$, $\mathcal{T}_2$ and $\mathcal{D}_M$ are the cuts by the walls

$x_{a_1} = x_{a_2}$, $x_{b_1} = x_{b_2}$ and $x_{c_1} = x_{c_2}$. On a face where some x_{a_i} or x_{b_k} vanishes, L_1 or U_2 is affine; on a face where $x_{a_i} + x_{c_j} + x_{b_k} = 1$, the other inequalities give $x_{a_{i'}} \le x_{a_i}$, so $L_1 = x_{a_i}$ is affine. By Lemmas 9.2(1) and 9.5(1), only the carriers B, $\{x_{c_1} = 0\}$, $\{x_{c_2} = 0\}$ and $\{x_{c_1} = x_{c_2} = 0\}$ count, and D_M is affine on the last three. The walls are independent and meet the relative interiors of these faces, for instance at the points with all nonzero coordinates equal to $\frac{1}{4}$, and every sign region there is convex and nonempty. So $\Delta(\mathcal{D}_M, \mathcal{S}_1, \mathcal{T}_2) = t^6(1+t^{-1})^3 = t^3(1+t)^3$ and $\Delta(\mathcal{S}_1, \mathcal{T}_2) = (t^6+2t^5+t^4)(1+t^{-1})^2 = t^2(1+t)^4$, and Theorem 9.54 gives $\mathcal{M}_C(q_1, q_2) = t^2(1+t)^3(2t^2+2t+1)$. $\diamond$

Proposition 9.56 (Order elements between comparable switches)**.** *Assume (R). Put $B = Q_C(P \setminus \{q_1, q_2\}, \lambda)$, and for $\{i, j\} = \{1, 2\}$ let L_i, U_i be the envelopes of q_i in $P \setminus \{q_j\}$, whose deleted base is B, with effective affinity subdivisions $\mathcal{S}_i, \mathcal{T}_i$. Then Lemma 9.1, Theorem 9.4, Corollary 9.6, Theorem 9.7 and Proposition 9.9 hold for $Y = \{q_1, q_2\}$ with these envelopes; in particular*

$$\mathcal{M}_C(q_1, q_2) = \Delta(\mathcal{S}_1, \mathcal{T}_1, \mathcal{S}_2, \mathcal{T}_2).$$

Proof. We use the insertion argument from the proof of Lemma 9.49. A chain whose inequality involves both q_1 and q_2 contains both, and its members strictly between them lie in C. Extend these members, with q_1 and q_2, to a maximal chain of $[q_1, q_2]$. By (R) it has an order element o, which is not a member and is comparable with every member. Inserting o splits the chain into one through q_1 but not q_2 and one through q_2 but not q_1, whose inequalities add up to the old one. A chain through q_i but not q_j is a chain of $P \setminus \{q_j\}$, so Section 3.1 for q_i in $P \setminus \{q_j\}$ gives the conditions of Lemma 9.1 and $L_i \le U_i$ on B. The proofs of Theorem 9.4, Corollary 9.6 and Theorem 9.7 use only Lemmas 9.1, 9.2, 9.3 and 9.5, the Jensen argument and Appendices B–H, so they apply.

In the proof of Proposition 9.9, three steps use that Y is an antichain; the others apply unchanged, with Section 3.3 applied to q_i in $P \setminus \{q_j\}$.

(a) The marking. If q_1 and q_2 are both marked, then for an order element o with $q_1 < o < q_2$, given by (R), the empty chains of $q_1 < o$ and $o < q_2$ give $U_1 \le y_o \le L_2$ on B; so $\eta_1 < U_1(x) \le L_2(x) < \eta_2$.

(b) The polytope Π. A marked q_i and an order element $q_j \in Y_d$ are never interior members of a chain. So a chain through q_1 and q_2 is $q_1 < K < q_2$ with $K \subseteq C$; inserting o as above splits it into $q_1 < K' < o$ and $o < K'' < q_2$, whose inequalities follow from $w_1 \le U_1(u)$ and $L_2(u) \le w_2$, where w_i is η_i or z_i. So Π has the description given there.

(c) The envelope conormals. Call a lower chain $a < K < q_2$ of q_2 in $P \setminus \{q_1\}$ *straddling* if it has consecutive members $x < y$ with $x < q_1 < y$. The refinement in that proof inserts elements strictly between consecutive members, so it keeps a chain non-straddling and never inserts q_1 into it. A straddling branch is dominated on B by a non-straddling one: the members of K after x lie in $(q_1, q_2) \cap C$; extend them, with q_1 and q_2, to a maximal chain of $[q_1, q_2]$ and insert its order element o. The part $a < K' < o$ is a chain of $P \setminus \{q_1, q_2\}$, so $y_a + \sum_{c \in K'} u_c \le y_o$ on B, and the branch is at most the branch of $o < K'' < q_2$, a chain above q_1. Hence L_2 is the maximum of the branches of saturated chains of P that avoid q_1; their unmarked members form paths in Γ ending at a lower cover of q_2. Dually, U_1 is the minimum of the branches of saturated chains that avoid q_2. The branches of L_1 lie below q_1 and those of U_2 above q_2, so their refinements avoid q_2, respectively q_1. $\square$

If no marked element lies in $[q_1, q_2]$, then under (R) the elements strictly between q_1 and q_2 on every maximal chain of $[q_1, q_2]$ form a path in Γ from an upper cover of q_1 to a lower cover of q_2, so $\kappa^+(q_1)$ and $\kappa^-(q_2)$ meet. If $\mathcal{T}_1$ and $\mathcal{S}_2$ are distinct and both nontrivial, then $\{\mathcal{T}_1, \mathcal{S}_2\}$ is not an isolated pair: it equals $\{\mathcal{S}_1, \mathcal{T}_1\}$ only if $\mathcal{S}_2 = \mathcal{S}_1$, it equals $\{\mathcal{S}_2, \mathcal{T}_2\}$ only if $\mathcal{T}_1 = \mathcal{T}_2$, and in either case that pair is not isolated. Hence (PC) holds only if $\mathcal{T}_1 = \mathcal{S}_2$, as in Example 9.57, or if U_1 or L_2 is affine, and in the latter case $\mathcal{M}_C(q_1, q_2) = 0$ by Corollary 9.6.

Example 9.57 (Order elements between switches)**.** Let P have the cover relations $\hat{0} \lessdot a_i \lessdot q_1 \lessdot o_j \lessdot q_2 \lessdot b_k \lessdot \hat{1}$ $(i, j, k = 1, 2)$, with marks 0 and 1 on $\hat{0}$ and $\hat{1}$, respectively, and let $C = \{a_1, a_2, b_1, b_2\}$, so that $o_1, o_2 \in O$ and (R) holds. Then $L_1 = \max(x_{a_1}, x_{a_2})$, $U_1 = \min(y_{o_1}, y_{o_2})$, $L_2 = \max(y_{o_1}, y_{o_2})$ and $U_2 = 1 - \max(x_{b_1}, x_{b_2})$. So $\mathcal{T}_1 = \mathcal{S}_2$ is the cut by $y_{o_1} = y_{o_2}$, and there are $k = 3$ distinct subdivisions; Corollary 9.6 gives $\mathcal{M}_C(q_1, q_2) = -\Delta(\mathcal{S}_1, \mathcal{T}_1, \mathcal{T}_2)$. No envelope is affine, and the graph Γ has no edges, so (PC) holds, and Proposition 9.56 with Theorem 9.7 gives $-\mathcal{M}_C(q_1, q_2) \in (1+t)^3 \mathbb{Z}_{\geq 0}[t]$. Explicitly, the 6-dimensional polytope B is cut out by $x_{a_i} \geq 0$, $x_{a_i} \leq y_{o_j}$, $y_{o_j} + x_{b_k} \leq 1$ and $x_{b_k} \geq 0$ (the other inequalities are implied), and where one of them is tight, L_1 or U_2 is affine; for instance $x_{a_i} = y_{o_j}$ gives $x_{a_{i'}} \leq x_{a_i}$. So only the carrier B counts; there the three walls are independent and meet at $x_{a_i} = x_{b_k} = \frac{1}{4}$, $y_{o_j} = \frac{1}{2}$. Hence $\mathcal{M}_C(q_1, q_2) = -t^6(1 + t^{-1})^3 = -t^3(1+t)^3$: the coincidence $\mathcal{T}_1 = \mathcal{S}_2$ reverses the sign. ◇

Remark 9.58 (Open points)**.** The conclusion of Theorem 9.7 holds in cases where (PC) fails; Theorem 9.13 gives layered ones. In that theorem, k counts distinct sets; this is the number of distinct subdivisions whenever $\mathcal{M}_C(Y) \neq 0$, since a set with at least two elements determines its subdivision and a set with one element forces $\mathcal{M}_C(Y) = 0$. For $c_1, c_2 < q_1 < d_1, d_2$ and $c_3, c_4 < q_2 < d_3, d_4$ with the additional cover relation $c_3 \lessdot d_1$, Example 9.24(c) gives $t^3(1+t)^5$, consistent with Theorem 9.7 for $k = 4$; whether transversality holds there is open. By Theorem 9.15 and its proof, Berge-acyclicity is necessary in Theorem 9.13 when all neighbourhoods have two elements, also for the lower and the upper factor separately. For larger sets it is not (Example 9.17). Three questions remain open.

(1) A necessary and sufficient poset condition for mutual transversality.

(2) A sharp criterion for the conclusion of Theorem 9.13 when some neighbourhood has at least three elements. Known results:

- when every group has at most two members, Section 9.6 gives a sufficient condition and the leading term at $t = -1$, and Example 9.33 shows that the condition is not sharp;
- for one three-member core group the conclusion fails in every nonzero coefficient by Theorem 9.36, also with one-member groups attached (Proposition 9.37);
- when every group is a single set, a two-member group neither of whose members contains the other, or a three-member core group, $(-1)^{r-k+s}\Lambda$ has nonnegative coefficients, with s the number of three-member core groups (Theorem 9.42);
- the conclusion fails whenever the distinct sets of a side form one group with three or four members, with every element in at least two of them and with no member equal to the union of the other members inside it, unless $\Lambda = 0$ (Theorems 9.36 and 9.47; for four members $\Lambda = 0$ only for $\{p, q\}$, $\{p\} \cup X$, $\{q\} \cup X$ and X, with $|X| \geq 2$).

(3) A formula for comparable switches outside the cases (T) and (R) of Section 9.7.

A Introduction and conventions

Appendices B–H prove the results on transverse polyhedral subdivisions that are used in the main text. Section 2 restates those for one and two subdivisions. Section 9 cites the results for m subdivisions (Proposition E.3, Corollary G.3, Remark G.6, Theorem H.3) directly. This appendix fixes their conventions. The appendices do not use the main text.

The common refinement of two subdivisions can acquire cells that are absent from either subdivision separately. A useful comparison retains the dimension of each cell and the smallest face of the original base containing it. For a nonempty polyhedron or finite polyhedral complex $\mathcal{K}$, write

$$F_{\mathcal{K}}(t) = \sum_{\substack{A \in \mathcal{K} \\ A \neq \varnothing}} t^{\dim A}.$$

For a polyhedron this means its face complex. We count each nonempty geometric cell once, include whole polyhedra, and add no faces at infinity. Dimensions, affine hulls and relative interiors are intrinsic. Polynomial inequalities are coefficientwise. The lowest degree of a nonzero polynomial with nonnegative coefficients is the least degree with a positive coefficient; it may be zero. For a polynomial f, $[t^k]f$ denotes its coefficient of t^k.

A finite polyhedral subdivision of a nonempty polyhedron B is a finite face-to-face complex with support B, including every nonempty face of each of its cells. Its cells lie in B. Restricting to any face of B gives a subdivision of that face. The *carrier* of a cell is the smallest face of B containing it. For subdivisions $\mathcal{S}, \mathcal{T}$, their common refinement $\mathcal{R} = \mathcal{S} \vee \mathcal{T}$ consists of their nonempty cell intersections and faces, counted without duplication. For a face E of B and $x \in \operatorname{relint}(E)$, let S and T be the minimal cells of the restrictions of $\mathcal{S}$ and $\mathcal{T}$ to E that contain x in their relative interiors. The pair is *transverse relative to every face* if

$$\dim(S \cap T) = \dim S + \dim T - \dim E$$

for every such E, x. Here E runs over the faces of B itself, not over the faces that a truncation adds.

For $e = \dim E$, let $s_E(k)$, $t_E(k)$ and $r_E(k)$ count the k-cells with carrier E of $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{R}$, respectively, and set

$$\delta_{E,k} = r_E(k) - s_E(k) - t_E(k) + \mathbf{1}_{k=e}, \qquad \delta_E(t) = \sum_k \delta_{E,k} t^k.$$

The total defect is

$$\Delta(t) = \sum_E \delta_E(t) = F_B + F_{\mathcal{R}} - F_{\mathcal{S}} - F_{\mathcal{T}}.$$

The effective affinity subdivision of a finite maximum L of affine functions on B is the complex formed by the maximal convex subsets of B of dimension $\dim B$ on which L is affine, together with all their nonempty faces; a finite minimum is treated after negation. The affine functions of a finite maximum or minimum are its *branches*; a branch is *effective* if it agrees with the maximum, or minimum, on a set of dimension $\dim B$. A subdivision is called *regular* in the support results when it is the effective affinity subdivision of a maximum of finitely many affine functions, with all nonempty faces included. No rationality assumption is made.

The hypotheses differ across the results:

- For arbitrary finite transverse pairs, each δ_E is in $(1+t)^2\mathbb{Z}_{\geq 0}[t]$ (Corollary F.4).
- For regular transverse pairs on a d-dimensional base, a nonzero Δ has its positive coefficients exactly in the degrees $r, \ldots, d$, with $0 \leq r \leq d-2$, and $\Delta(t) \geq t^r(1+t)^{d-r}$ (Theorems C.4 and D.1). By Corollary D.2, bounded examples attain this bound.
- A single finite subdivision has $F_{\mathcal{S}} - F_B \in (1+t)\mathbb{Z}_{\geq 0}[t]$ without transversality; terminal support is established for affinity subdivisions (Proposition F.5 and Theorem C.6).

- For m subdivisions the higher-order Euler statement requires mutual transversality and gives divisibility by $(1+t)^m$ (Proposition E.3). For every m, the quotient has nonnegative integer coefficients without regularity or boundedness assumptions (Corollary G.3).

- For any $m \geq 1$ finite subdivisions, mutually transverse relative to every face and not necessarily regular, a nonzero Δ has its positive coefficients exactly in the degrees $r, \dots, d$, with $0 \leq r \leq d-m$, and $\Delta(t) \geq t^r(1+t)^{d-r}$; bounded examples attain the bound (Theorem H.3 and Corollaries H.4 and H.6). For $m = 2$ and $m = 1$ this removes the regularity hypotheses of the second and third items (Remark H.5).

The fibre comparison identities of Appendix B extend Fourier's proofs of [10, Lemmas 4.1 and 4.3], stated there for chain-order fibrations, to general polyhedral data. Weighted alternating face sums for Minkowski sums were studied by Fukuda and Weibel [15]. The nonnegative quotient argument uses classical double-complex algebra [26] and its m-direction analogue (Lemma G.1), together with a relative cellular construction for the subdivisions. No identification with a local or mixed h-polynomial is needed.

Appendix B establishes the comparison identities. Appendix C develops crossing and support arguments, followed by the sharp quantitative bound in Appendix D. Appendix E treats Euler identities, and Appendix F proves pair-quotient positivity, first for bounded relative carriers and then by truncation. Appendix G establishes positivity for any number of mutually transverse subdivisions, by an induction on successive relative-chain quotients and an m-direction rank identity. Appendix H localizes these quotients at a point and proves the binomial bound and terminal support for any number of mutually transverse subdivisions, without regularity.

B Comparison identities

B.1 Bounded fibre comparison

Let B be a polytope and let $L \leq U$ be a convex and a concave function, respectively, each given by finitely many affine pieces. Let $\mathcal{S}, \mathcal{T}$ be their affinity subdivisions, including all nonempty faces, and put $\mathcal{R} = \mathcal{S} \vee \mathcal{T}$. Carriers and transversality relative to every face are as in Appendix A.

Lemma B.1 (Fourier's comparison tool)**.** *Put* $Q = \{(u,z) : u \in B, L(u) \leq z \leq U(u)\}$ *and* $Q' = \{(u,w) : u \in B, 0 \leq w \leq U(u) - L(u)\}$. *Then*

$$F_{Q'} - F_Q = F_B + F_{\mathcal{R}} - F_{\mathcal{S}} - F_{\mathcal{T}}. \tag{24}$$

Under relative transversality, for every carrier E of dimension e, the carrier defects of Appendix A satisfy

$$\delta_{E,k} \geq 0. \tag{25}$$

These statements follow from the proofs of Lemmas 4.1 and 4.3 in [10], whose stated setting is the chain-order fibration. The arguments work for the displayed general polytopal data relative to $\mathrm{aff}(B)$, including degenerate bases. Lemma B.2 below records them for polyhedra that may be unbounded; its proof is Fourier's proof of [10, Lemmas 4.1 and 4.3], extended to that setting. We will also use the counting mechanism in the carrierwise proof: for $k < e$, the refinement families of original k-cells with carrier E are nonempty and mutually disjoint; for $k = e$, full cells form a connected bipartite intersection graph whose edges are refinement cells. These facts identify the surplus cells used below.

B.2 Finite polyhedral subdivisions

Let B be a nonempty polyhedron, and let $L \le U$ be finite-valued functions on B, with L a maximum and U a minimum of finitely many affine functions. Let $\mathcal{S}, \mathcal{T}$ be their effective affinity subdivisions, including all nonempty faces, and let $\mathcal{R} = \mathcal{S} \vee \mathcal{T}$. Carriers and relative transversality are as in Appendix A.

Lemma B.2 (Polyhedral comparison tool)**.** *Put*

$$Q = \{(u, z) : u \in B,\ L(u) \le z \le U(u)\}, \qquad Q' = \{(u, w) : u \in B,\ 0 \le w \le U(u) - L(u)\}.$$

Then

$$F_{Q'} - F_Q = F_B + F_{\mathcal{R}} - F_{\mathcal{S}} - F_{\mathcal{T}}.$$

If $\mathcal{S}, \mathcal{T}$ are transverse relative to every face of B, the right-hand side is coefficientwise nonnegative. More precisely, for each carrier E of dimension e and each k,

$$r_E(k) - s_E(k) - t_E(k) + \mathbf{1}_{k=e} \ge 0,$$

where the three cell counts refer to $\mathcal{R}, \mathcal{S}, \mathcal{T}$ respectively. No boundedness assumption on B is required.

Proof. Set $H = U - L \ge 0$ and $Z = \{u \in B : H(u) = 0\}$. If $x \in \operatorname{relint}(E)$ and $H(x) = 0$, then for any $y \in E$ one can extend the segment from y to x a short distance past x while remaining in E. Concavity and nonnegativity force $H(y) = 0$. Hence Z is a union of faces of B. On each such face $L = U$ is both convex and concave, and therefore affine. Both restricted subdivisions are trivial there. Write F_Z for the polynomial of this subcomplex, with $F_Z = 0$ if it is empty.

For an effective lower branch ℓ_i, the valid inequality $z \ge \ell_i(u)$ has tight face

$$\{(u, L(u)) : u \in B,\ L(u) = \ell_i(u)\}.$$

Projection identifies this face with its affinity domain. Taking faces gives all lower graph faces: if a lower graph face has a relative-interior point in one of these tight faces, that tight face contains the whole graph face. The upper graph is treated identically. These arguments use supporting inequalities and the transitivity of the face relation, so unbounded graph faces cause no change. The two collections overlap exactly in the lifted base faces belonging to Z.

For completeness, let a nonempty face of Q be exposed by a functional $c(u) + \alpha z$ that attains a finite maximum on Q. If $\alpha < 0$, the face lies on the lower graph; if $\alpha > 0$, it lies on the upper graph. If $\alpha = 0$, surjectivity of the projection onto B shows that the face is the full interval lift of a face E of B. For $E \subseteq Z$ this was already counted. Otherwise $H > 0$ on $\operatorname{relint}(E)$ and this lift has dimension $\dim E + 1$. Thus

$$F_Q = F_{\mathcal{S}} + F_{\mathcal{T}} - F_Z + t(F_B - F_Z).$$

This count includes Q itself.

The corresponding formula for Q' is

$$F_{Q'} = F_B + F_{\mathcal{H}} - F_Z + t(F_B - F_Z),$$

where $\mathcal{H}$ is the affinity subdivision of H. For a concave function g, points u_i and weights $\lambda_i \ge 0$ with $\sum_i \lambda_i = 1$, the number $g(\sum_i \lambda_i u_i) - \sum_i \lambda_i g(u_i)$ is nonnegative; we call it the *Jensen deficit* of g. The Jensen deficits of the concave functions U and $-L$ add up to that of H. Consequently

H is affine on a convex set exactly when both L and U are affine there. Both complexes consist of all faces of their full cells, and by this Jensen argument their full cells coincide. Hence $\mathcal{H} = \mathcal{R}$, proving the identity. In particular, if $Z = B$, the formulas remain valid without adding a spurious dimension to the collapsed fibres.

We prove the inequality separately for each carrier E and dimension k, putting $e = \dim E$. If $e < k$ there are no such cells. For $e > k$, each original k-cell S has a nonempty family of refinement k-cells contained in it. Indeed, a finite collection of lower-dimensional cells cannot cover $\operatorname{relint}(S)$. Each chosen refinement cell has the same carrier: its relative interior lies in $\operatorname{relint}(S)$ and hence in $\operatorname{relint}(E)$. These families are disjoint for distinct original cells of one subdivision. They are disjoint across the two subdivisions as well: a shared k-cell would give, at a relative-interior point, an intersection of dimension at least k, whereas transversality gives $2k - e < k$. Thus $r_E(k) \geq s_E(k) + t_E(k)$.

For $e = k$, form the bipartite graph whose vertices are the full cells of $\mathcal{S}|_E$ and $\mathcal{T}|_E$, and whose edges are their full-dimensional intersections. Its edges correspond exactly to the full cells of $\mathcal{R}|_E$. The dual graph of $\mathcal{R}|_E$ is connected. To see this without boundedness, choose interior points of any two full cells off the finitely many wall hyperplanes, and join them by a generic polygonal path inside the convex set $\operatorname{relint}(E)$. The path can avoid the finite codimension-at-least-two skeleton and cross only facet interiors. Its segments meet finitely many cells because the subdivision is finite. For $e = 1$ an interval suffices, and for $e = 0$ the claim is immediate.

At a facet crossed by this path, the two original cell labels cannot both change. Otherwise an $\mathcal{S}$-wall and a $\mathcal{T}$-wall of codimension one would share a codimension-one refinement face in $\operatorname{relint}(E)$, contrary to the transverse intersection dimension $e-2$. Thus successive refinement cells correspond to edges of the bipartite graph sharing a vertex. Every original full cell is incident with an edge, since a finite union of lower-dimensional cells cannot cover its relative interior. The bipartite graph is therefore connected. Its number of edges is at least its number of vertices minus one, giving $r_E(e) \geq s_E(e) + t_E(e) - 1$. Summing over carriers proves the coefficientwise inequality. □

B.3 An epigraph identity

Lemma B.3 (Epigraph and ray-product faces)**.** *Let B be a nonempty polyhedron, let L be a finite maximum of affine functions on B, and let $\mathcal{S}$ be its effective affinity subdivision. Then*

$$F_{\operatorname{epi}_B(L)} = F_{\mathcal{S}} + tF_B, \qquad F_{B\times[0,\infty)} = (1+t)F_B, \tag{26}$$

and $F_{\mathcal{S}} \geq F_B$ coefficientwise.

Proof. The graph faces of $\operatorname{epi}_B(L)$ are indexed by $\mathcal{S}$, preserving dimensions. Every other face is the full epigraph lift over a face E of B, of dimension $\dim E + 1$. Indeed, in an exposing functional the vertical coefficient is negative for a graph face or zero for a full lift; a positive coefficient is unbounded on a vertical ray. Graph faces and full lifts are disjoint, and hence (26) holds. Each k-face E of B contains a k-cell of the restricted finite subdivision with carrier E. Such cells for different carriers are distinct, so $F_{\mathcal{S}} \geq F_B$ coefficientwise. □

C Crossing and terminal support

This appendix first counts faces and cofaces and finds the surplus from an ordinary crossing. It then proves terminal support for two transverse subdivisions, of a polytope and of a polyhedron, and for one affinity subdivision.

C.1 Face and coface counts

Lemma C.1 (Face and coface lower bounds)**.** *A full-dimensional s-polytope has at least $\binom{s+1}{j+1}$ faces of dimension j, for $-1 \le j \le s$; for $j = -1$, and only in this lemma, the empty face is counted. If E is a nonempty e-face of a nonempty d-polyhedron B, then for every $0 \le j \le d-e$ there are at least $\binom{d-e}{j}$ cofaces of E of dimension $e+j$. No pointedness or boundedness of B is required.*

Proof. Choose a facet F and a vertex v outside it. The j-faces lying in F and those containing v are disjoint families. Induction applied to F and the vertex figure at v gives respectively $\binom{s}{j+1}$ and $\binom{s}{j}$ faces, whose sum is the asserted bound. For vertices the latter family contributes v itself, corresponding to the empty face of the vertex figure. For $j = s$ the only face is the polytope itself, and the case $s = 0$ is immediate.

Put $q = d - e$. The normal cone of E in $\operatorname{dir}(B)^*$ is pointed, of dimension q, and its faces correspond, in reverse order, to the cofaces of E. A cone face of dimension ℓ corresponds to a coface of dimension $d - \ell$. For $q > 0$, a compact transverse section is a $(q-1)$-polytope. The face lower bound above gives at least $\binom{q}{j}$ cofaces of dimension $e + j$ for $0 \le j \le q$: the endpoints are E and B, and the intervening counts come from the section. The case $q = 0$ has only $E = B$. □

C.2 The surplus from a crossing

Lemma C.2. *Suppose that two subdivisions satisfying the relative transversality of Appendix A have an ordinary crossing in* $\operatorname{relint}(E)$*, where* $\dim E = e$*: locally there is one wall from each subdivision, their normals are independent, and all four full-dimensional sign regions occur. Then $\delta_E(t) \ge t^{e-2}(1+t)^2$.*

Proof. At the crossing point the minimal original cells are two walls of dimension $e-1$, while their refinement intersection has dimension $e-2$. This refinement cell lies in neither family assigned to an original $(e-2)$-cell: its relative-interior point lies in the relative interior of a strictly larger original cell on each side. It is a surplus cell in degree $e-2$, since distinct cells of a subdivision have disjoint relative interiors.

Each wall is divided by the other into at least two refinement $(e-1)$-cells. The two refinement families are disjoint by relative transversality, so there are at least two surplus cells in degree $e-1$. Finally the two full cells on each side meet in all four combinations. They give a four-cycle in the connected bipartite intersection graph of the carrier; by convexity, two full cells have a connected intersection. Thus the number of edges minus the number of vertices plus one is at least one, which gives the surplus in degree e. Other coefficients are nonnegative by the carrier inequality of Lemma B.2 (its carrier part uses neither $L \le U$ nor regularity, and B may be unbounded; see its proof). All these cells have carrier E, since each chosen relative-interior point lies in $\operatorname{relint}(E)$. □

C.3 Terminal support for bounded transverse subdivisions

Regular subdivisions are as in Appendix A. We work in the affine hull of the underlying polytope.

Theorem C.3. *Let B be a nonempty d-dimensional polytope, and let $\mathcal{S}, \mathcal{T}$ be finite regular affinity subdivisions transverse relative to every nonempty face of B. Put $\mathcal{R} = \mathcal{S} \vee \mathcal{T}$ and*

$$\Delta(t) = F_B(t) + F_{\mathcal{R}}(t) - F_{\mathcal{S}}(t) - F_{\mathcal{T}}(t),$$

counting each nonempty geometric cell once. Then either $\Delta = 0$, or its positive coefficients occur exactly in degrees $r, r+1, \dots, d$ for some $0 \le r \le d-2$. In particular $\Delta = 0$ when $d < 2$.

Proof. We first fix realizations of the two subdivisions. The proof then has three steps. We find a carrier E that contributes to the lowest degree r of Δ and contains a mixed cell (defined below). The local product of fans at that cell, with Lemma C.2, gives positive coefficients in every degree from r to $e = \dim E$. Finally, an ordinary crossing in $\operatorname{relint}(E)$ lifts to every face containing E, and this gives the remaining degrees up to d.

Since $\mathcal{S}$ is regular, it is the affinity subdivision of a maximum L of finitely many affine functions. Likewise $\mathcal{T}$ is the affinity subdivision of a minimum U of finitely many affine functions; to obtain U, negate a convex realization of $\mathcal{T}$ if necessary. Since B is compact, adding a large enough constant to U gives $L \le U$, and this changes neither affinity subdivision. So the carrier counts and their nonnegativity in (25) apply. Both these carrier counts and the associated fibre identity are recorded in Lemma B.1 and proved for general polyhedral data in Lemma B.2 (compare [10, Lemmas 4.1 and 4.3]). The proof of Lemma B.2 also shows that, for every face E of B, the intersection graph of the full cells of $\mathcal{S}|_E$ and $\mathcal{T}|_E$ is connected.

Descent from the lowest degree. Let E be a face of dimension e, and let $s_E(k)$, $t_E(k)$, $r_E(k)$ and $\delta_{E,k}$ be as in Appendix A. Call a refinement cell *mixed* if its dimension is strictly less than the dimensions of both original cells containing its relative interior. For $k < e$, assign to each original k-cell with carrier E all refinement k-cells contained in it. Each assigned family is nonempty. The families are pairwise disjoint, both within and across the two subdivisions, by the carrierwise argument in the proof of Lemma B.2. A refinement k-cell outside all these families has both original-cell dimensions greater than k, so it is mixed. Conversely, a mixed cell belongs to none of these families. Consequently

$$\begin{aligned}\delta_{E,k} = {} & \#\{\text{mixed refinement } k\text{-cells with carrier } E\} \\ & + \sum_{A\in\mathcal{S},\,\dim A=k,\,\operatorname{car}(A)=E} (\nu(A)-1) + \sum_{A\in\mathcal{T},\,\dim A=k,\,\operatorname{car}(A)=E} (\nu(A)-1), \end{aligned} \tag{27}$$

where $\nu(A)$ is the size of the family assigned to A, and $\operatorname{car}(A)$ denotes the carrier of A. We call the family of A *excessive* if $\nu(A) > 1$.

Now suppose that the family of an original k-cell A of $\mathcal{S}$ is excessive. The refinement subdivides A into $\nu(A) \ge 2$ cells of dimension k, so this subdivision of A has an internal $(k-1)$-face G separating two of them, with $\operatorname{relint}(G) \subset \operatorname{relint}(A) \subset \operatorname{relint}(E)$. At a point of $\operatorname{relint}(G)$, the minimal $\mathcal{S}$-cell is A. If the minimal $\mathcal{T}$-cell is T, transversality gives

$$k - 1 = k + \dim T - e, \qquad \text{so } \dim T = e - 1.$$

Since $k < e$, the cell G is mixed, and $\delta_{E,k-1} > 0$. The argument for $\mathcal{T}$ is the same. When $k = 0$, a point has no nontrivial subdivision, so no family is excessive.

For each carrier E we also have the Euler identity

$$\sum_{k=0}^{e} (-1)^k \delta_{E,k} = 0. \tag{28}$$

Indeed, for any subdivision, the cells with carrier E have alternating sum $(-1)^e$, the relative Euler characteristic of $(E, \partial E)$. This value does not depend on the subdivision. Applying it to $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ and to the face complex of B, whose only cell with carrier E is E itself, shows that the four terms of δ_E cancel.

Now suppose that $\Delta \ne 0$. Let r be its lowest degree, and choose E with $\delta_{E,r} > 0$. Since $\Delta = \sum_E \delta_E$ and every carrier coefficient is nonnegative, all carrier coefficients in degrees below r

vanish. A cell with carrier E has dimension at most e, so $r \le e$; and $e = r$ would contradict (28), since $\delta_{E,r}$ would then be its only nonzero term. Hence $e > r$. An excessive family in degree r would give a positive coefficient in degree $r-1$, which is impossible. So, by (27), there is a mixed refinement r-cell with carrier E. If its original-cell dimensions are a and b, then

$$r = a + b - e, \qquad a > r, \quad b > r.$$

It follows that $a, b < e$ and $e \ge r + 2$.

A local product of fans. Choose p in the relative interior of this mixed cell; then $p \in \operatorname{relint}(E)$. Set $\sigma = e - a$ and $\tau = e - b$; then $\sigma, \tau \ge 1$ and $r = e - \sigma - \tau$. A branch of L is *active* at p if it agrees with L at p, and likewise for U. The differences of the branches of L active at p span the conormal space N_S of the minimal $\mathcal{S}$-cell at p in $\operatorname{dir}(E)^*$, and the differences of the active branches of U span the conormal space N_T of the minimal $\mathcal{T}$-cell. Indeed, the minimal face of the epigraph of L at $(p, L(p))$ is defined by the active branches, taken as equalities. Eliminating the height coordinate turns these equalities into exactly the branch differences, as equations for the affine hull of the cell, and all other inequalities are strict near p. For U, the same argument uses the hypograph. Relative transversality gives $N_S \cap N_T = \{0\}$, and N_S and N_T have dimensions σ and τ.

Choose a splitting

$$\operatorname{dir}(E) = K \oplus X \oplus Y, \qquad \dim K = r, \quad \dim X = \sigma, \quad \dim Y = \tau,$$

where K is the common kernel of $N_S + N_T$, X is a complement of K in the annihilator of N_T, and Y is a complement of K in the annihilator of N_S. The sum is direct. Indeed, the forms in N_S vanish on K and on Y. Applied to a vanishing sum, they show that its X-term is annihilated by N_S; it is also annihilated by N_T, so it lies in $K \cap X = \{0\}$. The forms in N_T do the same for the Y-term. So the differences of the branches of L depend only on the X-component, and those of U only on the Y-component.

Choose a neighbourhood of p so small that the branches inactive at p remain inactive on it. Subtract one active branch of L from all active branches of L, and take the convex hull in X^* of the resulting gradients, including the zero gradient of the subtracted branch itself. This gradient polytope is full-dimensional, because the differences span X^*. Its normal fan is complete and pointed, and it is exactly the local affinity fan of L in X. Do the same in Y^* for the negatives of the active branches of U. Near p, the refinement is then the product of K with these two fans.

A pointed complete fan in dimension σ has cones of every dimension $0, \dots, \sigma$: take a flag of faces of one full-dimensional cone. Choose cones of dimensions $i < \sigma$ and $j < \tau$ in the two fans. Their product with K is the germ at p of a refinement cell of dimension $r + i + j$. The two original cells that contain its relative interior have dimensions $r + i + \tau$ and $r + \sigma + j$, both strictly larger. These germs lie in the relative interiors of cells of the subdivisions and in $\operatorname{relint}(E)$, so they are mixed, with carrier E. As i and j vary, all dimensions $r, \dots, e-2$ occur, and by (27) each corresponding coefficient $\delta_{E,k}$ is positive.

For dimension $e-2$, take $i = \sigma - 1$ and $j = \tau - 1$. The two original cells then have dimension $e-1$, and a relative-interior point of the germ is an ordinary crossing of two effective walls, with four full regions. Lemma C.2 gives positive carrier coefficients in degrees $e-2$, $e-1$ and e. So $\delta_{E,k} > 0$ for every $k = r, \dots, e$.

Lifting an ordinary crossing to a coface. We now prove that an ordinary crossing in $\operatorname{relint}(E)$ persists: the relative interior of every face E' containing E also contains an ordinary crossing. Take such a crossing p, and choose vectors $v_x, v_y \in \operatorname{dir}(E)$ dual to its two independent wall normals. Use

coordinates x, y in these directions, centred at p, and complete them by residual coordinates z on $\mathrm{aff}(E')$. Write z_0 for the residual coordinate of p.

First we describe E' near p. Every defining inequality of E' that is tight at p is tight on all of E, since $p \in \mathrm{relint}(E)$, so its linear part vanishes on $\mathrm{dir}(E)$. In a small enough product neighbourhood of p, the inequalities that are strict at p stay strict, and the tight ones involve only z. Thus, near p, the set E' is the product of the (x, y)-plane with a residual feasible set Q. The set Q is full-dimensional in the residual space, since E' is full-dimensional in $\mathrm{aff}(E')$ and x, y are free. Near z_0, the set Q is the feasible polyhedral cone at z_0 intersected with a neighbourhood, so z_0 lies in the closure of the interior of Q. Hence a point $z \in \mathrm{int}(Q)$ near z_0, together with small enough x and y, gives a point of $\mathrm{relint}(E')$.

Next we describe L and U near p. We keep every branch active at p, including the branches that are redundant on E. An active branch ℓ of L agrees with L on the whole minimal wall cell of $\mathcal{S}$ through p. Indeed, on that cell the difference $L - \ell$ is affine and nonnegative, and it vanishes at a relative-interior point; so it vanishes on the affine hull of the cell. Therefore the linear parts of the differences of the active branches of L vanish on v_y, and, for the same reason, those for U vanish on v_x. After we subtract one affine function from L and another from U, the local envelopes have the forms

$$\max_i \big(\lambda_i x + A_i(z)\big), \qquad \min_j \big(\mu_j y + B_j(z)\big), \tag{29}$$

where $A_i(z_0) = B_j(z_0) = 0$. At least two distinct λ_i and two distinct μ_j occur, since both original walls are effective. Branches inactive at p remain inactive once the neighbourhood is small enough.

The choices are made in the following order. First, fix the product neighbourhood, small enough that the branches inactive at p stay inactive. Then choose $\varepsilon > 0$ so small that x and y may vary in $[-\varepsilon, \varepsilon]$. Next, take z so close to z_0 that, in the maximum, the form with the smallest slope exceeds every form with a larger slope at $x = -\varepsilon$, and the form with the largest slope exceeds every form with a smaller slope at $x = \varepsilon$. In the minimum, the analogous comparisons hold with the order of the slopes reversed. At z_0 these are strict comparisons between forms of distinct slopes, so they persist for z near z_0. The generic feasible z is chosen last, as follows.

First we remove redundant forms. We regard two affine forms as equal if they coincide on $\mathrm{aff}(E')$. Two forms with the same slope have the same intercept only for z in an affine hyperplane of the residual space, unless the forms are equal. For three forms with distinct slopes, the condition that their three lines pass through one point is also an affine condition on z, after clearing the fixed slope differences. If that condition holds for every z, the form with the intermediate slope is redundant. Explicitly, for $\lambda_1 < \lambda_2 < \lambda_3$ put

$$\theta = \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}.$$

Concurrence for every z means $A_2 = \theta A_1 + (1 - \theta) A_3$ on $\mathrm{aff}(Q)$, and hence

$$\lambda_2 x + A_2(z) = \theta\big(\lambda_1 x + A_1(z)\big) + (1 - \theta)\big(\lambda_3 x + A_3(z)\big), \qquad 0 < \theta < 1.$$

So removing this form changes neither the maximum nor the minimum. We repeatedly remove such forms and any repeated copies of a form. The finite list strictly shrinks, and its extreme slopes remain. We do this on both sides of (29).

For the remaining forms, the exceptional sets of z, where two parallel lines coincide or three lines pass through one point, are finitely many proper hyperplanes. They cannot cover the open set $\mathrm{relint}(Q)$ near z_0, so we choose z there outside all of them. The residual space is zero-dimensional only when $E' = E$ and E is two-dimensional; then every possible concurrence is an identity, and these forms were removed above.

At this z, the maximum is attained by the form of smallest slope at $x = -\varepsilon$ and by the form of largest slope at $x = \varepsilon$. So, as a function of x, it has a breakpoint x_* in $(-\varepsilon, \varepsilon)$. For the same reason, the minimum has a breakpoint y_*. At each breakpoint exactly two forms are active, and their slopes are distinct, because triple concurrences and coincident parallel lines were excluded. The inactive forms are strictly away from the envelope at the breakpoints, so the same two forms stay active nearby, and the place where they are equal is an affine function of z. The point (x_*, y_*, z) lies in $\operatorname{relint}(E')$, by the description of E' near p given above. One of the two wall normals has a nonzero x-coefficient and a zero y-coefficient, and the other has the reverse. So the normals are independent, and all four sign regions occur near this point.

Finally, we restore the removed forms; this creates no new effective wall. Indeed, if a positive convex combination of forms is active in a maximum or minimum, then every form of the combination is active as well, and the gradient differences of the combination lie in the span of theirs. So (x_*, y_*, z) is an ordinary crossing for the original subdivisions on E'. Forms with intermediate slopes may become active after the lift: for example, $\max(-x, x, z)$ has one wall for $z \le 0$ and two walls for $z > 0$.

Conclusion. Return to the carrier E found in the descent from the lowest degree. It has an ordinary crossing, and $\delta_{E,k} > 0$ for every $k = r, \dots, e$. Choose a flag of cofaces from E to B with one face in every dimension $e, \dots, d$; such a flag exists because the face lattice of a polytope is graded. By the lifting argument, the relative interior of each face of this flag contains an ordinary crossing. By Lemma C.2, a flag face of dimension h contributes positively in degrees $h-2, h-1, h$. Together these cover every degree $r, \dots, d$. Since every carrier coefficient is nonnegative, nothing cancels, and the minimality of r gives zero coefficients below r. Since $e \ge r+2$, we have $r \le d-2$. □

C.4 Terminal support for finite polyhedral subdivisions

We now extend Theorem C.3 to polyhedra. Regular subdivisions are as in Appendix A. No relation between the two realizing envelopes is assumed.

Theorem C.4. *Let B be a nonempty d-dimensional polyhedron, possibly unbounded or with lineality. Let $\mathcal{S}, \mathcal{T}$ be finite regular affinity subdivisions transverse relative to every nonempty face of B, and put $\mathcal{R} = \mathcal{S} \vee \mathcal{T}$. Counting all nonempty cells, including unbounded cells, set*

$$\Delta(t) = F_B(t) + F_{\mathcal{R}}(t) - F_{\mathcal{S}}(t) - F_{\mathcal{T}}(t).$$

Then either $\Delta = 0$, or its positive coefficients occur exactly in degrees $r, \dots, d$ for some $0 \le r \le d-2$. In particular $\Delta = 0$ if $d < 2$.

Proof. We work in the affine hull of B. Since $\mathcal{S}$ and $\mathcal{T}$ are regular, $\mathcal{S}$ is the affinity subdivision of a maximum L of finitely many affine functions, and $\mathcal{T}$ is that of a minimum U of finitely many affine functions. We do not require $L \le U$: on an unbounded base, a constant shift need not give this inequality. The proof has two parts. First we prove the carrier inequalities and the carrier Euler identity without using $L \le U$. Then we reuse the local arguments of the proof of Theorem C.3 on the original polyhedron.

Carrier counts without ordered envelopes. Let E be a face of dimension e, and let $\delta_{E,k}$ be as in Appendix A. The carrier-counting part of the proof of Lemma B.2 uses neither $L \le U$ nor boundedness. So $\delta_{E,k} \ge 0$ for all k, and Lemma C.2 holds here, with this nonnegativity in place of (25).

The Euler identity on each original carrier. We prove the carrier Euler identity without assuming transversality on any artificial boundary. There are finitely many nonempty faces of B and finitely many cells of $\mathcal{S}, \mathcal{T}, \mathcal{R}$. Choose a relative-interior point in each of them. In $\operatorname{aff}(B) \cong \mathbb{R}^d$, take a closed full-dimensional box W with all these points in its interior, and put $B_W = B \cap W$. For each original face E, the set $E_W = E \cap W$ is a face of B_W of the same dimension and

$$\operatorname{relint}(E_W) = \operatorname{relint}(E) \cap \operatorname{int}(W).$$

An exposing functional for E still exposes E_W; the dimension and relative-interior identity follow because the selected relative-interior point lies strictly inside W.

Restrict each subdivision to B_W, including every face of every restricted cell. Every old cell A becomes $A \cap W$ of the same dimension, with its chosen witness in its relative interior. This map on old cells is injective. If two truncated cells agreed, their common set would contain the chosen relative-interior points of both old cells. Their old intersection is a face of each; containing both witnesses forces both old cells to equal that face. Any new face created by truncation lies on the box boundary. It cannot have carrier E_W for an old E, whose relative interior lies in $\operatorname{int}(W)$. Moreover an old cell has carrier E exactly when its truncation has carrier E_W, as witnessed at the chosen interior point. These assertions hold for the common refinement as well as for both original subdivisions, since all its cells were included when choosing the box.

Consequently every cell count with an original carrier is preserved, in every dimension. Apply the ordinary bounded relative Euler identity to the subdivisions of the polytope E_W. The alternating sum over cells with carrier E_W is $(-1)^e$ for each subdivision, where $e = \dim E$. Transferring the counts back to E gives

$$\sum_{k=0}^{e} (-1)^k \delta_{E,k} = 0. \tag{30}$$

This is the analogue of (28). The box is used only for this Euler calculation. The bounded terminal-support theorem is not applied to B_W: its artificial faces need not satisfy transversality.

The local proof on the original polyhedron. We now work again on B, with no box faces involved. Identity (27) depends only on the disjoint refinement families just obtained, so it still holds. If an original k-cell has more than one full refinement cell, that is, if its family is excessive, then a compact generic path inside its relative interior crosses an internal refinement facet. Transversality there gives an original cell of the other subdivision, of dimension $e - 1$, exactly as in the descent from the lowest degree in the proof of Theorem C.3. Thus an excessive family gives a mixed cell one degree lower, and a point has no excessive family. Together with (30) and nonnegativity, the same descent gives a mixed r-cell in a carrier E with $e = \dim E \geq r + 2$, where r is the lowest degree of Δ.

At a relative-interior point of that mixed cell, the conormal splitting and the complete pointed normal fans of the active gradient polytopes are exactly those in the proof of Theorem C.3. They use finitely many affine branches and a neighbourhood inside $\operatorname{relint}(E)$, not the boundedness of E. The zero gradient is still included after one active branch is subtracted. The product fans give mixed cells in every degree $r, \ldots, e - 2$, and an ordinary crossing. The crossing count above gives the remaining degrees $e - 1$ and e.

The lifting argument of that proof is local as well. Let $E' \supseteq E$ be a coface. The linear parts of the defining inequalities of E' that are tight at the crossing vanish on $\operatorname{dir}(E)$, and all other inequalities stay strict in a small enough neighbourhood. So, as in that proof, near the crossing the set E' is the product of the plane of the two free wall coordinates with a full-dimensional residual

feasible set in $\mathrm{aff}(E')$. All active branches, including those with intermediate slopes, depend on the two free coordinates in the same separated way. The choices are made in the same order: the neighbourhood, the radius ε of the free coordinates, the neighbourhood of z_0, and finally the generic residual point z. There are finitely many exceptional hyperplanes, and the concurrences that hold for every z are removed by the same convex-combination identities. The two breakpoints give an ordinary crossing in $\mathrm{relint}(E')$, and restoring the removed forms adds no effective wall.

Coface flags without vertices. The normal cone of E in $\mathrm{dir}(B)^*$ is pointed and has dimension $d-e$. Indeed, if both a covector and its negative were normal at E, that covector would be constant on B, hence zero in this relative dual space. The normals of the facets containing E generate this cone and span the annihilator of $\mathrm{dir}(E)$, giving the dimension assertion. Polarity of the tangent cone at a point of $\mathrm{relint}(E)$ matches the faces of this normal cone, in reverse order, with the cofaces of E in B: a j-dimensional face of the cone corresponds to a coface of dimension $d-j$. So a graded flag of faces of the pointed cone gives a coface of every dimension $e, e+1, \dots, d$. If $E = B$, the normal cone is zero and the flag is trivial.

Lift the ordinary crossing to each face of the flag. By the crossing count, a carrier of dimension h contributes positively in degrees $h-2$, $h-1$ and h. Together with the degrees $r, \dots, e$ already obtained, these cover $r, \dots, d$. Since every carrier coefficient is nonnegative, nothing cancels, and the minimality of r gives zero coefficients below r. Finally $r \le e-2 \le d-2$, which proves the theorem. □

Remark C.5. Theorem C.4 gives no lower bound on r in terms of the lineality of B. For $B = \mathbb{R}^2$ and the two affinity subdivisions defined by $\max(0, x)$ and $\max(0, y)$, Δ is

$$t^2 + (4t^2 + 4t + 1) - 2(2t^2 + t) = (1 + t)^2.$$

The base has two-dimensional lineality, but the refinement has a mixed vertex. A subdivision cell need not inherit the lineality of its base.

C.5 Support for one affinity subdivision

Theorem C.6 (Support of one affinity subdivision)**.** *Let B be a nonempty polyhedron of dimension d, let L be the maximum of finitely many affine functions on B, and let $\mathcal{S}$ be its effective affinity subdivision, including all nonempty faces, with each geometric cell counted once. Then $F_{\mathcal{S}} - F_B = 0$ if and only if L is affine on B. Otherwise its positive coefficients occur exactly in degrees $r, r+1, \dots, d$, for some $0 \le r \le d-1$. The polyhedron and its cells may be unbounded and may have lineality.*

Proof. For a face E of B of dimension e, let $s_E(k)$ count the k-cells of $\mathcal{S}$ whose carrier is E, and put

$$\varepsilon_E(k) = s_E(k) - \mathbf{1}_{k=e}.$$

The restricted finite subdivision covers E, so it has at least one e-cell with carrier E. Consequently every $\varepsilon_E(k)$ is nonnegative, and

$$F_{\mathcal{S}}(t) - F_B(t) = \sum_{E \neq \varnothing} \sum_{k} \varepsilon_E(k) t^k.$$

If the difference is nonzero, let r be its lowest degree, and choose E with $\varepsilon_E(r) > 0$. Necessarily $e \ge r$. If $e = r$, there are at least two full-dimensional affinity cells on E. For $e > 0$, a compact

polygonal path between interior points of two such cells, chosen in relint(E) and generic with respect to the finite codimension-two skeleton, crosses an internal facet. This gives $\varepsilon_E(r-1) > 0$, a contradiction. For $e = 0$ there is only one geometric cell. Thus $e > r$, and there is an r-cell with carrier E.

Choose p in the relative interior of that cell, hence in relint(E), and let K be its r-dimensional tangent space inside dir(E). In a neighbourhood of p only the affine forms active at p matter. Their gradient differences span the conormal of K: the active equalities define the minimal epigraph face at $(p, L(p))$, and eliminating height gives the equations for the cell's affine hull. Every other inequality is strict at p. Subtract one active form and pass to dir(E)/K. The convex hull of the remaining gradients, together with the zero gradient of the subtracted form, is therefore a full-dimensional polytope of dimension $e - r$. Its normal fan is complete and pointed and is exactly the local affinity fan in this quotient. A face flag in a full cone gives cell germs of every dimension from r to $e - 1$. They lie in relint(E) and extend to distinct cells of the same dimensions and carrier. There are also at least two full cones, since a positive-dimensional polytope has at least two vertices. Hence

$$\varepsilon_E(k) > 0 \qquad (r \le k \le e).$$

In particular there is an ordinary wall of the subdivision in relint(E), that is, an $(e-1)$-cell with two adjacent full-dimensional effective affinity regions.

We next show that an ordinary wall persists in every coface $E' \supseteq E$. Choose a point p_0 in the relative interior of such a wall. Inside dir(E) choose a direction v transverse to the wall and use coordinates $p_0 + xv + z$ in aff(E'), where z belongs to a complementary space. The linear part of every defining inequality of E' tight at p_0 vanishes on dir(E), and in particular on v. All other inequalities are strict in a sufficiently small neighbourhood. Thus, near p_0, the set E' is the product of the x-axis with a residual polyhedral cone Q in the z-space. This cone is full-dimensional there: the tangent cone of E' at p_0 is full-dimensional in dir(E') and has the free direction v. Points with small $z \in$ relint(Q) and small x lie in relint(E').

We keep all branches active at p_0, including branches redundant on E. After subtracting a common affine form, they have the form

$$\lambda_i x + A_i(z), \qquad A_i(0) = 0.$$

Their restrictions along the wall agree: L is affine on the wall cell, and each active branch minus L is affine, nonpositive there, and zero at its relative-interior point p_0. It is therefore identically zero on that cell. Since this is an effective wall on E, the minimum and maximum slopes λ_i are distinct. First fix a small neighbourhood where inactive branches stay inactive, and choose $\eta > 0$ so that the points with $x = \pm\eta$ and $z = 0$ lie in it. Next restrict z to a sufficiently small neighbourhood of zero. At $x = -\eta$ every maximizing branch then has the minimum slope, and at $x = \eta$ every maximizing branch has the maximum slope. This follows from the positive gaps between distinct slopes and the finiteness of the branch family.

We now choose z generically. Delete duplicate affine forms. For three distinct slopes, concurrence of their three affine functions of x is an affine equation in z. If it holds identically, the middle-slope form is an affine convex combination of the two extreme-slope forms on the full local affine span and can be removed without changing the maximum. Repeating this deletion leaves only proper concurrence hyperplanes. Equal-slope forms either were duplicates or have equal intercepts only on a proper hyperplane. Choose a small $z \in$ relint(Q) outside these finitely many proper hyperplanes. In zero residual dimension there are no proper exceptional hyperplanes to avoid, and the same conclusion holds. The maximum as a function of x changes maximizing slope between $-\eta$ and η. At each breakpoint exactly two remaining forms are active, with different slopes. Their

equality defines an ordinary wall in $\operatorname{relint}(E')$. Restoring a deleted convex-combination form adds neither an effective region nor a gradient direction at a tie: if it is active, both its constituents are active, and its gradient difference lies in their span. Thus the wall is an ordinary wall of the original effective subdivision as well. If $h = \dim E'$, this wall and its two adjacent full cells give

$$\varepsilon_{E'}(h-1) > 0, \qquad \varepsilon_{E'}(h) > 0.$$

By the paragraph "Coface flags without vertices" in the proof of Theorem C.4, which holds also for unbounded polyhedra, E lies in a flag of faces of every dimension $e, \dots, d$. Lifting the wall through this flag fills every degree from e to d. Together with the previously obtained degrees $r, \dots, e$, and nonnegative carrier contributions, this proves the asserted interval of positive coefficients.

Finally, if L is affine its effective complex is just the face complex of B. Conversely, if there were only one full affinity region, its closed domain would cover $\operatorname{relint}(B)$ and, by continuity, all of B, so L would be affine. A nonaffine L consequently has at least two full regions, and the coefficient in degree d is positive. The argument from the lowest degree above also gives $r < e \le d$, completing the proof. □

D A quantitative bound from the lowest degree

In this section the polynomial variable is z, since s and t denote conormal ranks.

Theorem D.1. *Let B be a nonempty d-dimensional polyhedron, and let $\mathcal{S}, \mathcal{T}$ be finite regular affinity subdivisions transverse relative to every nonempty face of B. Set $\mathcal{R} = \mathcal{S} \vee \mathcal{T}$ and*

$$\Delta(z) = F_B(z) + F_{\mathcal{R}}(z) - F_{\mathcal{S}}(z) - F_{\mathcal{T}}(z).$$

If $\Delta \neq 0$ and r is its lowest degree, then

$$\Delta(z) \ge z^r (1+z)^{d-r} \tag{31}$$

coefficientwise. Moreover $0 \le r \le d-2$ by Theorem C.4.

Proof. We use the carrierwise nonnegativity, the identity (27) and the local product of normal fans from the proofs of Theorems C.3 and C.4. All dimensions and conormal spaces below are relative to the affine hull of the indicated carrier.

A local deficit is bounded by its carrier deficit. First consider any mixed k-cell with carrier E, where $e = \dim E$, and take p in its relative interior. Let $s, t > 0$ be the conormal ranks of its two original cells, so $k = e - s - t$. The local product model at p is a k-dimensional affine factor times two complete pointed normal fans of dimensions s and t. Let $\mathcal{A}_s(z)$ and $\mathcal{A}_t(z)$ count all cones of those fans, including their zero cones, by dimension. The polynomials F_B, $F_{\mathcal{R}}$, $F_{\mathcal{S}}$, $F_{\mathcal{T}}$ of this local model are, respectively,

$$z^{k+s+t}, \qquad z^k \mathcal{A}_s(z)\mathcal{A}_t(z), \qquad z^{k+t}\mathcal{A}_s(z), \qquad z^{k+s}\mathcal{A}_t(z).$$

Their alternating sum is

$$z^k \big(\mathcal{A}_s(z) - z^s\big)\big(\mathcal{A}_t(z) - z^t\big). \tag{32}$$

This local deficit is a coefficientwise lower bound for $\sum_i \delta_{E,i} z^i$. To justify the comparison despite its negative terms, we use the exact mixed-cell and excessive-family decomposition (27), which also

holds for the transverse complete fans of the local model. Every local cell germ comes from a unique global cell meeting relint(E) and has that cell's dimension. Distinct cones of a gradient polytope's normal fan have distinct active sets: each exposed face is the convex hull of the gradients active there. These active sets, or their pairs for the refinement, distinguish the local germs. A global cell has a constant active set on its relative interior, so distinct germs cannot come from the same cell. This applies to both original subdivisions and their refinement.

For degree $i < e$, local mixed cells therefore inject into global mixed cells with carrier E. For each local original i-cell, its refinement i-cell family injects into the global family of the same original cell. Thus $\nu_{\text{local}} - 1 \le \nu_{\text{global}} - 1$ for each such family; both are nonempty. The other global families have nonnegative excess. The exact decomposition now proves the desired comparison in every degree below e.

For degree e, let m_S, m_T be the numbers of full cones in the two local fans. Their full-cell intersection graph is the complete bipartite graph with these vertex classes. Its vertices and edges inject into the connected global intersection graph of the carrier E. In particular, a pair of global full cells cannot produce two distinct refinement edges, since their intersection is convex. The local top coefficient is

$$m_S m_T - m_S - m_T + 1 = (m_S - 1)(m_T - 1),$$

the cycle rank of this connected subgraph. Cycle rank cannot decrease under inclusion into a connected graph: its independent cycles remain independent in the larger graph. The global carrier coefficient in degree e is the number of edges minus the number of vertices plus one of the global intersection graph, that is, its cycle rank, so it dominates the local coefficient as well. This proves the comparison with (32) in all degrees.

Lower bounds for the two fan polynomials. By Lemma C.1, a full-dimensional s-polytope has at least $\binom{s+1}{j+1}$ faces of dimension j.

An i-cone of the normal fan corresponds to an $(s-i)$-face of its full-dimensional gradient polytope. Therefore

$$\mathcal{A}_s(z) \ge \sum_{i=0}^{s} \binom{s+1}{i} z^i.$$

Since $s \ge 1$, subtracting z^s leaves coefficients at least $\binom{s+1}{i} \ge \binom{s}{i}$ for $i < s$, and a top coefficient at least $s \ge 1$. Hence $\mathcal{A}_s(z) - z^s \ge (1+z)^s$, including $s = 1$. The same argument applies to t. Both factors have nonnegative coefficients, so (32) yields

$$\sum_i \delta_{E,i} z^i \ge z^k (1+z)^{s+t} = z^k (1+z)^{e-k} \tag{33}$$

whenever there is a mixed k-cell with carrier E.

A lift preserving both conormal ranks. The descent from the lowest degree in the proof of Theorem C.4 gives a mixed r-cell in a carrier E of dimension e, with positive ranks s, t and $s + t = e - r$. We show that every coface $E' \supseteq E$ of dimension h has a mixed cell of dimension $h - s - t = r + h - e$, with the same two ranks.

At a relative-interior point p of the original mixed cell, use the splitting $\operatorname{dir}(E) = K \oplus X \oplus Y$ constructed in the local-product argument of Theorem C.3 (where the ranks s, t are called σ, τ). Choose free coordinates $x \in \mathbb{R}^s$ on X and $y \in \mathbb{R}^t$ on Y, dual to the two conormal spaces, and residual coordinates u in $\operatorname{aff}(E')$, centred so that $p = (0, 0, 0)$. The linear part of every feasibility row tight at p vanishes on $\operatorname{dir}(E)$, while the other rows stay strict locally. Thus x, y are locally

free over a residual feasible set Q full-dimensional in its residual space, with 0 in the closure of $\mathrm{int}(Q)$. All envelope branches active at p agree along their corresponding original minimal cell. On that cell the envelope is affine, and an active branch minus the envelope is affine, has one sign, and vanishes at the relative-interior point p; it therefore vanishes throughout the cell. Their lower differences have no y component and their upper differences have no x component. After subtracting a common affine term on each side, the local envelopes are

$$\max_i f_i(x,u), \quad f_i(x,u) = a_i^\mathsf{T} x + A_i(u), \qquad \min_j g_j(y,u), \quad g_j(y,u) = b_j^\mathsf{T} y + B_j(u), \tag{34}$$

where $A_i(0) = B_j(0) = 0$. The convex hulls of the x-gradients and of the y-gradients are full-dimensional, of dimensions s and t. All branches inactive at p remain inactive throughout a sufficiently small fixed neighbourhood.

For every residual u, consider the auxiliary lower maximum over only the branches active at p, on all of $\mathbb{R}^s$. Its epigraph is a nonempty pointed polyhedron. Indeed, a line direction (v, c) would require $a_i^\mathsf{T} v = c$ for every i. Since the gradients a_i affinely span $\mathbb{R}^s$, this forces $v = 0$ and then $c = 0$. A nonempty pointed polyhedron has a vertex. At such a vertex, $s + 1$ active inequalities have independent normals, so their slopes are affinely independent. The upper minimum has the same property after negation.

Call a set $I = \{i_0, i_1, \ldots, i_s\}$ of lower branches active at p a *lower basis* if their slopes are affinely independent; upper bases are defined in the same way, after negation. Enumerate the finitely many lower bases. Their equality systems have the form

$$G_I(x,u) = M_I x + c_I(u) = 0,$$

where M_I is the invertible matrix of slope differences and $c_I(0) = 0$. The corresponding candidate vertex is

$$x_I(u) = -M_I^{-1} c_I(u).$$

It is affine in u and vanishes at zero. Every vertex of the auxiliary envelope is one of these candidates, and at least one candidate is feasible for every u, in the sense made precise by the residuals below. Because there are finitely many fixed invertible matrices, all candidates obey a common bound $\|x_I(u)\| \le C\|u\|$ near zero. The upper candidates $y_J(u)$ satisfy the analogous bound. Consequently all feasible candidates are inside the original free-coordinate neighbourhood when u is sufficiently small; the omitted branches stay inactive there.

For each basis and every lower branch, define its affine feasibility residual

$$\rho_{I,i}(u) = f_i(x_I(u), u) - f_{i_0}(x_I(u), u).$$

Feasibility requires $\rho_{I,i}(u) \le 0$ for all i. Define the corresponding upper residuals, with the reverse inequalities. Choose $u \in \mathrm{relint}(Q)$ close to zero and outside the zero hyperplanes of the residuals that are not identically zero, for all lower and upper bases; and then choose any feasible lower basis I and upper basis J. Such u exists because Q is full-dimensional; feasible bases exist at every u by the preceding epigraph argument. In zero residual dimension, all residuals are identities and there are no proper exceptional hyperplanes.

Suppose an additional lower branch is active at $x_I(u)$. Its residual vanishes at the generic u, and hence is identically zero in the residual space. Write its difference from the base branch as

$$f_i - f_{i_0} = \alpha^\mathsf{T} x + c_i(u), \qquad \gamma^\mathsf{T} = \alpha^\mathsf{T} M_I^{-1}.$$

Subtract the basis combination $\gamma^\mathsf{T} G_I(x,u)$ from this difference. The x part cancels, leaving $c_i(u) - \gamma^\mathsf{T} c_I(u)$. This is exactly the residual obtained on substituting $x_I(u)$, so it vanishes identically. Thus

the full affine difference is a linear combination of the s basis differences on $\operatorname{aff}(E')$, including its residual dependence. Every additional active branch has this property. The full lower conormal rank is therefore exactly s, since the basis projects independently onto the x coordinates. The upper rank is exactly t by the same argument.

The lower conormal span has zero y projection and injective x projection, while the upper span has zero x projection and injective y projection. Their intersection is zero. At the feasible point $(x_I(u), y_J(u), u) \in \operatorname{relint}(E')$, the common cell consequently has dimension $h - s - t = r + h - e$; its two original cells have dimensions $h - s$ and $h - t$, both larger. This is the required mixed cell with preserved ranks. The envelope vertices need not be simple: genericity controls the extra active residuals.

Summing over all cofaces. Put $q = d - e$. Lemma C.1 gives at least $\binom{q}{j}$ cofaces of dimension $e + j$, for $0 \le j \le q$, including the endpoints E and B.

On each coface of dimension $e+j$, the lifted mixed cell has dimension $r+j$ and conormal ranks s, t. Applying (33) there gives a carrier contribution at least $z^{r+j}(1+z)^{s+t}$. Different cofaces are different carriers, so these bounds add without double counting. All other carrier contributions are nonnegative. Therefore

$$\Delta(z) \ge z^r(1+z)^{s+t} \sum_{j=0}^{d-e} \binom{d-e}{j} z^j = z^r(1+z)^{d-r},$$

as required. The bound $r \le d-2$ comes from Theorem C.4, not from this inequality alone. □

For the attainment construction we return to t as the polynomial variable.

Corollary D.2 (Bounded attainment in every allowed dimension)**.** *For every pair of integers $0 \le r \le d-2$, there is a bounded d-dimensional polytope B with two finite regular affinity subdivisions $\mathcal{S}, \mathcal{T}$, transverse relative to every nonempty face of B, such that*

$$F_B(t) + F_{\mathcal{S}\vee\mathcal{T}}(t) - F_{\mathcal{S}}(t) - F_{\mathcal{T}}(t) = t^r(1+t)^{d-r}.$$

In particular, the bound in Theorem D.1 is attained by bounded examples for every allowed (r, d).

Proof. We first construct an example in dimension $r+2$, the *seed*. For $r \ge 1$, let $W \subset \mathbb{R}^r$ be any full-dimensional bounded polytope with facet description $\ell_i(w) \ge 0$, where the ℓ_i are affine. Set

$$B_0 = \{(x, y, w) : |x| \le \ell_i(w), \quad |y| \le \ell_i(w) \text{ for every } i\}.$$

These finitely many affine inequalities imply $w \in W$. The fibre over w is the square of half-width $\min_i \ell_i(w)$, which is positive in $\operatorname{relint}(W)$ and zero on its boundary. Compactness of W and boundedness of all the ℓ_i on W make B_0 bounded. The positive-width fibres over $\operatorname{relint}(W)$ show that its dimension is $r+2$. For $r = 0$ use $B_0 = [-1, 1]^2$ instead. In both cases take the affinity subdivisions of the maxima of finitely many affine functions

$$L = \max(0, x), \qquad M = \max(0, y).$$

Thus the subdivisions are regular; their walls are the two coordinate hyperplanes.

We check all proper faces before counting the full carrier. Suppose $r \ge 1$ and a proper face E has a relative-interior point $(0, 0, w)$. The point w cannot belong to $\operatorname{relint}(W)$, since then every defining inequality of B_0 would be strict. Choose i with $\ell_i(w) = 0$. All four inequalities

$$\ell_i(w) - x \ge 0, \quad \ell_i(w) + x \ge 0, \quad \ell_i(w) - y \ge 0, \quad \ell_i(w) + y \ge 0$$

are tight at a relative-interior point of E, so each is tight throughout E. Hence $x = y = \ell_i(w) = 0$ on E. Both envelope restrictions are affine there and both restricted subdivisions are trivial. They are transverse on E and their carrier deficit is zero.

For any other proper face, the two walls do not meet simultaneously in its relative interior. This includes faces meeting only one wall or neither wall. At each relative-interior point at most one restricted subdivision has a nontrivial wall; the other minimal cell is full-dimensional in E. The transverse dimension formula therefore holds. If one envelope is affine on all of E, its restricted subdivision is the face complex of E, and the restricted defect cancels identically.

When both restrictions are nonaffine, we count the proper-carrier deficit directly. Each effective wall cuts $\operatorname{relint}(E)$ in a nonempty convex set, and these two sets are disjoint. Each wall section therefore lies entirely on one side of the other wall and is not subdivided by it. The only cells below top dimension with carrier E are the two unchanged wall cells of dimension $\dim E - 1$. They appear once each in the refinement and once each among the original subdivisions, so their contributions cancel. There are three full-dimensional refinement regions: each of the two disjoint hyperplane cuts splits just one existing convex region. Each original subdivision has two full regions. The top carrier deficit is consequently $1 + 3 - 2 - 2 = 0$. There is no lower-dimensional cell with carrier E, since that would require the two walls to meet in $\operatorname{relint}(E)$. This also covers one-dimensional faces with two distinct interior cuts; a point has only the trivial subdivisions. For the square used when $r = 0$, the walls have no common boundary point, and the same proper-face reasoning applies. Thus every proper carrier contributes zero, and transversality holds on every proper face.

In the relative interior of B_0 the two walls are independent, so transversality holds there as well. Their intersection is one r-cell, namely $\{(0, 0, w) : w \in W\}$ when $r \geq 1$, or the origin when $r = 0$. It has carrier B_0; every proper face of it lies on a proper face of B_0. Each original subdivision has one full-carrier wall of dimension $r + 1$ and two full regions of dimension $r + 2$. Their common refinement has the joint r-cell, four half-wall cells of dimension $r + 1$, and four full regions. All four sign regions are nonempty, because the square fibres have positive width in the interior. The contribution of the full carrier is therefore

$$t^{r+2} + \big(t^r + 4t^{r+1} + 4t^{r+2}\big) - 2\big(t^{r+1} + 2t^{r+2}\big) = t^r(1 + t)^2.$$

Together with the vanishing of all proper-carrier deficits, this gives the complete defect polynomial of the seed.

It remains to increase dimension without changing the lowest degree. Embed a bounded polytope B at height one and take its pyramid with apex at the origin at height zero. Cone every subdivision cell to the apex, retaining its base cell. If the original realizing envelope has affine forms $a_i \cdot z + b_i$, use the homogeneous forms $a_i \cdot Z + b_i h$ on the pyramid. At $(Z, h) = (hz, h)$, with $h > 0$, their maximum is h times the original envelope value at z. Its affinity domains are exactly the cones over the old affinity domains. In particular no two distinct full affinity regions merge: restricting to $h = 1$ would then merge the corresponding old regions. At the apex all forms agree. This realizes the coned subdivisions as finite regular affinity subdivisions.

The nonempty faces of the pyramid are the base faces $E \times \{1\}$, their cones, and the apex, with E ranging over all nonempty faces of B. Moreover

$$\operatorname{relint}(\operatorname{cone}(E)) = \{(hz, h) : z \in \operatorname{relint}(E),\ 0 < h < 1\},$$

where $\operatorname{cone}(E)$ denotes the bounded cone from the apex to $E \times \{1\}$. At a relative-interior point of such a face, minimal cells are cones over the old minimal cells. Their intersection is the cone over the old intersection, and the cell, intersection and carrier dimensions each increase by one. Thus

$$\dim(S \cap T) = \dim S + \dim T - \dim E$$

is preserved after coning. Base-face restrictions are unchanged and the apex restrictions are trivial. This proves transversality relative to every face of the pyramid, including its boundary faces.

If F is the face polynomial of the face complex of B, of either subdivision, or of their common refinement, then the coned complex has face polynomial $(1+t)F+1$: its cells are the old cells, their cones, and the single apex. Intersections of nonempty cones away from the apex correspond to the old intersections; if the old cells are disjoint, their cones meet only at that already counted apex. Thus the refinement is also exactly the coned old refinement. The four added apex terms cancel in the defect, and pyramiding multiplies it by $1+t$. Applying this operation $d-r-2$ times to the seed gives a bounded d-polytope with defect $t^r(1+t)^{d-r}$, as claimed. □

Remark D.3. These examples are general polyhedral constructions, not built from posets. Section 4 realizes the case $r=2$ by a switch of a poset, and Theorem 6.1 classifies the posets with equality in Theorem 1.2.

E Weighted carrier Euler identities

Identities for weighted alternating face sums of Minkowski sums in relative general position were proved by Fukuda and Weibel [15, Theorems 2 and 4]; their proof uses polynomial differentiation for exact face decompositions and Euler counts in normal-fan subdivisions [15, Lemmas 7 and 10].

The carrier identity below uses the same weighted Euler principle, but its defect includes the base-face term and counts all nonempty cells. We give a direct proof for finite polyhedral subdivisions, including unbounded and nonregular ones.

Lemma E.1 (Weighted carrier Euler identity)**.** *Let B be a nonempty polyhedron and let $\mathcal{S},\mathcal{T}$ be finite polyhedral subdivisions, including all nonempty faces, transverse relative to every nonempty face of B. Put $\mathcal{R}=\mathcal{S}\vee\mathcal{T}$. For a face E of dimension e, let $s_E(k),t_E(k),r_E(k)$ count the k-cells of the three complexes with carrier E, and set*

$$\delta_E(t)=\sum_k\bigl(r_E(k)-s_E(k)-t_E(k)+\mathbf{1}_{k=e}\bigr)t^k.$$

Then

$$\delta_E(-1)=0,\qquad \sum_k k(-1)^k[t^k]\delta_E(t)=0.$$

Consequently $(1+t)^2$ divides every $\delta_E(t)$ and the total defect $F_B+F_{\mathcal{R}}-F_{\mathcal{S}}-F_{\mathcal{T}}$. For any nonzero such polynomial, -1 is a root of multiplicity at least two. Regularity of the subdivisions is not required.

Proof. Fix a carrier E. For every refinement cell C with carrier E, let $A_S(C)$ and $A_T(C)$ be the unique minimal cells of $\mathcal{S}$ and $\mathcal{T}$ whose relative interiors contain $\operatorname{relint}(C)$. Set

$$c=\dim C,\qquad a(C)=\dim A_S(C),\qquad b(C)=\dim A_T(C).$$

At a relative-interior point of C the transversality hypothesis gives

$$c=a(C)+b(C)-e. \tag{35}$$

Both original cells have carrier E, since their relative interiors meet $\operatorname{relint}(E)$. Thus the map $C\mapsto A_S(C)$ is defined on *all* refinement cells with carrier E and lands in original $\mathcal{S}$-cells with that carrier. Conversely, for every such original cell A, the relative interiors of the refinement cells

with $A_S(C) = A$ partition relint(A). Indeed each point has a unique minimal refinement cell, and being in relint(A) makes A its unique minimal original $\mathcal{S}$-cell. Since relint$(A) \subseteq$ relint(E), all these refinement cells have carrier E as well. The same statements hold for $C \mapsto A_T(C)$.

We justify the relative Euler count for this partition, also when A is unbounded. Choose one closed box W in aff(B) containing, in its interior, a relative-interior witness of every nonempty face of B and every cell of $\mathcal{S}, \mathcal{T}, \mathcal{R}$. Each original cell intersected with W retains its dimension. For the induced subdivision of A, the old cells are the refinement cells contained in A, together with their faces. The old cells with carrier A correspond exactly to truncated cells with carrier $A \cap W$. Indeed, the old relative-interior witnesses lie inside W, whereas each new face produced by truncation lies on its boundary. Such a new face cannot have carrier $A \cap W$, whose relative interior is relint$(A) \cap$ int(W). Distinct old cells stay distinct: if their truncations coincided, their common old face would contain a relative-interior witness of each, forcing the old cells to coincide. Carrier preservation follows from the same witnesses. Thus the bounded relative Euler identity on $A \cap W$ gives

$$\sum_{C : A_S(C) = A} (-1)^{\dim C} = (-1)^{\dim A}. \tag{36}$$

The same argument works for the $\mathcal{T}$-cells, and for each subdivision restricted to E. In particular the total signed sum of cells with carrier E is $(-1)^e$ in each of the three subdivisions. Subtracting these sums proves $\delta_E(-1) = 0$.

For a complex $\mathcal{K}$, write

$$\Omega_{\mathcal{K}}(E) = \sum_{C \in \mathcal{K},\ \mathrm{car}(C) = E} \dim C \, (-1)^{\dim C}.$$

Using (35), and then regrouping by the unique minimal cells, yields

$$\begin{aligned}
\Omega_{\mathcal{R}}(E) &= \sum_{C \in \mathcal{R},\ \mathrm{car}(C) = E} (-1)^c \big(a(C) + b(C) - e\big) \\
&= \sum_{A \in \mathcal{S},\ \mathrm{car}(A) = E} \dim A \sum_{C : A_S(C) = A} (-1)^c + \sum_{D \in \mathcal{T},\ \mathrm{car}(D) = E} \dim D \sum_{C : A_T(C) = D} (-1)^c - e(-1)^e \\
&= \Omega_{\mathcal{S}}(E) + \Omega_{\mathcal{T}}(E) - e(-1)^e.
\end{aligned}$$

The middle line counts each refinement cell once in each of the two indicated sums. This proves the weighted identity. After moving terms it reads $\sum_k k(-1)^k [t^k] \delta_E(t) = 0$, that is, $\delta'_E(-1) = 0$; so both the value and the first derivative of δ_E vanish at -1. Polynomial division gives the asserted divisibility in $\mathbb{Z}[t]$, since $(1+t)^2$ is monic, and summing over carriers gives it for the total defect. □

Remark E.2. The weighted identity alone gives divisibility. Nonnegativity of the pair quotient follows from Corollary F.4.

E.1 Higher-order identities

The same minimal-cell Euler grouping also gives a higher-order form of the identity. Here mutual transversality, rather than pairwise transversality, is essential.

Proposition E.3 (Higher-order carrier Euler identity)**.** *Let $\mathcal{S}_1, \ldots, \mathcal{S}_m$, with $m \geq 1$, be finite polyhedral subdivisions of a nonempty d-polyhedron B, including all nonempty faces. Assume that*

the subdivisions are mutually transverse relative to every face: if $p \in \operatorname{relint}(E)$ and A_i is the minimal restricted $\mathcal{S}_i$-cell containing p, then, with $e = \dim E$,

$$\dim \bigcap_{i=1}^{m} A_i = e - \sum_{i=1}^{m} (e - \dim A_i).$$

For $J \subseteq [m]$, write $\mathcal{S}_J = \bigvee_{i \in J} \mathcal{S}_i$, with $\mathcal{S}_\varnothing$ the face complex of B, and put $\mathcal{R} = \mathcal{S}_{[m]}$. Define

$$P_{J,E}(t) = \sum_{\substack{D \in \mathcal{S}_J \\ \operatorname{car}(D) = E}} t^{\dim D}, \qquad \Delta_E(t) = \sum_{J \subseteq [m]} (-1)^{m-|J|} P_{J,E}(t).$$

For $C \in \mathcal{R}$ with carrier E, let $A_i(C)$ be its unique minimal original $\mathcal{S}_i$-cell, and set $\alpha_i(C) = e - \dim A_i(C)$. Then

$$\Delta_E(t) = \sum_{\substack{C \in \mathcal{R} \\ \operatorname{car}(C) = E}} t^{\dim C} \prod_{i=1}^{m} \bigl(1 - (-t)^{\alpha_i(C)}\bigr). \tag{37}$$

In particular, $(1+t)^m$ divides $\Delta_E(t)$ in $\mathbb{Z}[t]$, and $\Delta_E = 0$ if $m > e$. Thus it also divides

$$\Delta(t) = \sum_{J \subseteq [m]} (-1)^{m-|J|} F_{\mathcal{S}_J}(t),$$

which is zero when $m > d$. Neither boundedness nor regularity of the subdivisions is required.

Proof. At a common relative-interior point, mutual transversality says that the conormal spaces of the original cells form a direct sum in $\operatorname{dir}(E)^*$. The same is true for any subfamily. Consequently the unique minimal $\mathcal{S}_J$-cell associated to C is $A_J(C) = \bigcap_{i \in J} A_i(C)$, has carrier E, and satisfies

$$\dim A_J(C) = e - \sum_{i \in J} \alpha_i(C) = c + \sum_{i \notin J} \alpha_i(C), \qquad c = \dim C.$$

For $J = \varnothing$ this means $A_J(C) = E$. For each $D \in \mathcal{S}_J$ with carrier E, the relative interiors of the cells C satisfying $A_J(C) = D$ partition $\operatorname{relint}(D)$. The relative Euler count, also for unbounded D, gives

$$\sum_{C : A_J(C) = D} (-1)^{\dim C} = (-1)^{\dim D}.$$

This is the grouping proved in (36): apply its witness-box argument to the finitely many subfamily refinements. Equivalently, it is the additivity of the compactly supported Euler characteristic, since a relatively open convex polyhedron of dimension k has compactly supported Euler characteristic $(-1)^k$. Regrouping therefore gives the polynomial identity

$$P_{J,E}(t) = \sum_{\substack{C \in \mathcal{R} \\ \operatorname{car}(C) = E}} (-1)^c (-t)^{\dim A_J(C)}.$$

Indeed, the cells C with $A_J(C) = D$ contribute $(-1)^{\dim D}(-t)^{\dim D} = t^{\dim D}$. In the alternating sum over J, replace J by its complement. For each fixed C the inner sum becomes

$$(-1)^c (-t)^c \sum_{K \subseteq [m]} (-1)^{|K|} \prod_{i \in K} (-t)^{\alpha_i(C)} = t^c \prod_{i=1}^{m} \bigl(1 - (-t)^{\alpha_i(C)}\bigr),$$

proving (37). If some $\alpha_i(C) = 0$, this summand is zero. Otherwise every factor is divisible by $1 + t$ in $\mathbb{Z}[t]$. If $m > e$, the equality $\sum_i \alpha_i(C) = e - c \le e$ forces a zero factor in every summand. Summing over carriers proves the total assertions. □

Remark E.4. The exponent m cannot be increased. On $[-1,1]^m$ take the m coordinate cuts $x_i = 0$, realized by $|x_i|$. They are mutually transverse on every face. Only the origin has all $\alpha_i > 0$, and there $c = 0$ and $\alpha_i = 1$; on a proper face a fixed coordinate makes one subdivision trivial. Formula (37) gives $\Delta = (1+t)^m$. Taking the product with an unsubdivided $(d-m)$-cube gives $(1+t)^m(2+t)^{d-m}$ for every $d \geq m$, so the root at -1 has multiplicity exactly m.

Pairwise transversality alone does not suffice. In $\mathbb{R}^2$, use the three line subdivisions $x = 0$, $y = 0$, $x+y = 0$. Each single complex has polynomial $t+2t^2$, each pair refinement has $1+4t+4t^2$, and the full refinement has $1+6t+6t^2$. Together with $F_{\mathbb{R}^2} = t^2$, their alternating defect is $-2-3t-t^2 = -(1+t)(2+t)$. The three conormals at the origin are not a direct sum. The proposition establishes divisibility for mutually transverse families. Nonnegativity of the quotient, for every m, is proved separately in Corollary G.3.

F Nonnegative quotients

We use a standard consequence of the square decomposition of bounded double complexes over a field; see [26, Theorem 3 and Proposition 6]. We include the elementary rank argument to isolate the algebra from the geometric construction that follows.

Lemma F.1 (Ranks in an exact double complex)**.** *Let $D^{p,q}$ be a finite-dimensional double complex supported in finitely many bidegrees, with differentials h, v of bidegrees $(1,0),(0,1)$ satisfying $h^2 = v^2 = hv + vh = 0$. If every row and every column is exact, and*

$$\rho_{p,q} = \operatorname{rank}(vh : D^{p,q} \longrightarrow D^{p+1,q+1}),$$

then

$$\dim D^{p,q} = \rho_{p,q} + \rho_{p-1,q} + \rho_{p,q-1} + \rho_{p-1,q-1}.$$

Proof. Set $I^{p,q} = \operatorname{im}(h : D^{p-1,q} \to D^{p,q})$. Anticommutation makes each $I^{p,*}$ a column subcomplex. Replace the vertical differential on column p by $(-1)^p v$, so that the horizontal maps commute with these column differentials. Row exactness gives short exact sequences of column complexes

$$0 \longrightarrow I^{p,*} \longrightarrow D^{p,*} \xrightarrow{h} I^{p+1,*} \longrightarrow 0.$$

Starting to the left of the finite support, where $I^{p,*} = 0$, the long exact cohomology sequence and column exactness show inductively that every $I^{p,*}$ is exact. The incoming and outgoing vertical ranks at $I^{p+1,q}$ are $\rho_{p,q-1}$ and $\rho_{p,q}$, respectively; signs do not affect ranks. Therefore $\dim I^{p+1,q} = \rho_{p,q-1} + \rho_{p,q}$. Finally row exactness gives $\dim D^{p,q} = \dim I^{p,q} + \dim I^{p+1,q}$, proving the formula. □

Theorem F.2 (Nonnegative bounded carrier quotients)**.** *Let $\mathcal{S}, \mathcal{T}$ be finite polyhedral subdivisions of a nonempty polytope B, including all nonempty faces, and suppose they are transverse relative to every face of B. Put $\mathcal{R} = \mathcal{S} \vee \mathcal{T}$. For each face E of dimension e, the carrier defect satisfies*

$$\delta_E(t) = \sum_k \big(r_E(k) - s_E(k) - t_E(k) + \mathbf{1}_{k=e}\big)t^k = (1+t)^2 Q_E(t), \qquad Q_E(t) \in \mathbb{Z}_{\geq 0}[t].$$

Consequently $(F_B + F_{\mathcal{R}} - F_{\mathcal{S}} - F_{\mathcal{T}})/(1+t)^2$ has nonnegative integer coefficients. Regularity of the subdivisions is not required.

Proof. Fix E and use cellular chains with real coefficients. For $\mathcal{U} = \mathcal{S}, \mathcal{T}, \mathcal{R}$, let $K_{\mathcal{U}}$ be the cellular chain complex of the induced subdivision of the pair $(E, \partial E)$. Its basis consists of the cells with

carrier E. These are the oriented polytopal cells themselves, not simplices of an auxiliary triangulation. The finite polytopal complexes are regular CW complexes, and ∂E is a subcomplex, so their oriented cellular chains compute the relative homology. Since a polytope is a closed ball, each complex has one-dimensional homology in degree e, generated by its relative fundamental chain, and zero homology in every other degree. For $e = 0$ this is the pair consisting of one point and the empty set.

Subdivision maps and the double grading. Subdivision sends each oriented cell A to the signed sum of the refined cells of dimension $\dim A$ in A. This is a chain map: each interior codimension-one cell belongs to exactly two top-dimensional cells, with opposite induced orientations, whereas each boundary one belongs to exactly one. The interior contributions cancel, while boundary facets give precisely the subdivided oriented boundary of A. Cells in ∂E are discarded consistently on both sides. The map is injective, since distinct original cells of the same dimension have disjoint relative-interior supports in the refinement. Identify $K_{\mathcal{S}}$ and $K_{\mathcal{T}}$ with their images in $K_{\mathcal{R}}$. Their fundamental chains of E agree; write $K_0 = \mathbb{R}[E]$ for the subcomplex spanned by this chain in degree e.

For a refinement cell C of carrier E, let $A_S(C), A_T(C)$ be its minimal original cells. Their relative interiors meet at a point of $\operatorname{relint}(C)$, so that point lies in the relative interior of their intersection. The intersection is a refinement cell, and uniqueness of the minimal cell gives $C = A_S(C) \cap A_T(C)$. Give C bidegree

$$p = e - \dim A_S(C), \qquad q = e - \dim A_T(C).$$

Transversality gives $\dim C = e - p - q$. If a facet C' of C still has carrier E, each minimal original cell of C' is a face of the corresponding original cell of C. Thus neither p nor q decreases, and their sum increases by one. Exactly one increases by one. The relative cellular boundary consequently splits as $h + v$ of bidegrees $(1, 0)$ and $(0, 1)$. Sorting the identity $\partial^2 = 0$ by bidegree gives $h^2 = v^2 = hv + vh = 0$.

The subdivision image of an original $\mathcal{S}$-cell of dimension k is supported in bidegree $(e-k, 0)$: a refined k-cell in its relative interior must have a full-dimensional minimal $\mathcal{T}$-cell by transversality. Because subdivision is a chain map, its boundary is again in this image on $q = 0$; hence v vanishes on the image, while h is its original boundary. Likewise $K_{\mathcal{T}}$ lies on $p = 0$ and h vanishes on it. Both images are therefore sub-double-complexes.

Below degree e, the two images have distinct bidegrees and hence zero intersection. If an e-chain lies in both images, its boundary lies in their lower-degree intersection and is zero. The relative top cycles of $K_{\mathcal{R}}$ are exactly $\mathbb{R}[E]$, since there are no $(e+1)$-chains and top homology is one-dimensional. Thus

$$K_{\mathcal{S}} \cap K_{\mathcal{T}} = K_0.$$

It follows that the quotient double complex

$$D = K_{\mathcal{R}}/(K_{\mathcal{S}} + K_{\mathcal{T}})$$

has ordinary-degree dimensions

$$\sum_{p+q=e-k} \dim D^{p,q} = r_E(k) - s_E(k) - t_E(k) + \mathbf{1}_{k=e} = \delta_{E,k}. \tag{38}$$

Exact rows and columns. Fix q. Before taking the quotient, the h-row is a direct sum over the $\mathcal{T}$-cells T_0 of dimension $e - q$ and carrier E. The block for T_0 consists of refinement cells whose

minimal $\mathcal{T}$-cell is T_0. Indeed, along an h-boundary the minimal $\mathcal{T}$-cell is a face of T_0 of the same dimension, so it is T_0 itself. This block is precisely the relative cellular complex of the induced subdivision of $(T_0, \partial T_0)$, read in degree $p = e - q - \dim C$. Facets in ∂T_0 leave the block or lie in ∂E and are discarded. Even if T_0 touches ∂E, its relative interior lies in $\mathrm{relint}(E)$, so no interior cell of the block is missing. Its only homology is at $p = 0$, represented by the subdivision fundamental chain $[T_0]$.

First quotient by $K_{\mathcal{T}}$. In each row its generators are exactly these fundamental cycles at $p = 0$, one for each T_0. The quotient rows are therefore exact. The remaining image of $K_{\mathcal{S}}$ lies entirely in row $q = 0$ and is $K_{\mathcal{S}}/K_0$. This is an exact h-complex, because $K_0 \to K_{\mathcal{S}}$ induces an isomorphism on the only nonzero relative homology group. Quotienting the already exact row $q = 0$ by this exact subcomplex preserves exactness; all other rows are unchanged. Thus every row of D is exact. Interchanging $\mathcal{S}$ and $\mathcal{T}$ proves that every column is exact as well.

Nonnegative ranks. Apply Lemma F.1 to D. Combining its rank formula with (38) and grouping the four occurrences of each rank gives

$$\delta_E(t) = (1+t)^2 \sum_{\rho_{p,q}>0} \rho_{p,q}\, t^{e-p-q-2}.$$

If $\rho_{p,q} > 0$, its target $D^{p+1,q+1}$ is nonzero, so $e - p - q - 2 \geq 0$. The displayed quotient is therefore a polynomial with nonnegative integer coefficients. Summing over carriers proves the total assertion. □

Lemma F.3 (Relative-interior transversality suffices)**.** *Let E be a nonempty polytope with two finite polyhedral subdivisions of E, including all nonempty faces, transverse at every point of* $\mathrm{relint}(E)$*. Carriers are taken relative to E. No transversality condition is imposed on its proper faces. Counting only cells with carrier E, one still has $\delta_E(t) \in (1+t)^2\mathbb{Z}_{\geq 0}[t]$.*

Proof. Use the fixed-carrier construction in the proof of Theorem F.2. The relative chain complexes, subdivision injections and fundamental classes need no transversality assumption. Every cell retained after quotienting out ∂E, and every retained facet used to split its boundary, has its relative interior in $\mathrm{relint}(E)$. Thus the codimension grading, the axis supports and the intersection of the two images require only the stated interior hypothesis.

For the row block associated to T_0, every retained interior facet has the same minimal $\mathcal{T}$-cell T_0 and is an h-term. A facet on ∂T_0 but inside $\mathrm{relint}(E)$ has larger $\mathcal{T}$-codimension and leaves the block through v; a facet on ∂E was already discarded. Hence the block is exactly the relative complex of $(T_0, \partial T_0)$ without any boundary transversality assumption. The two successive quotients make the rows exact as before; the column argument is symmetric. The rank formula and its nonnegative coefficients are unchanged. □

Corollary F.4 (Nonnegative polyhedral carrier quotients)**.** *Let B be a nonempty polyhedron and let $\mathcal{S}, \mathcal{T}$ be finite polyhedral subdivisions transverse relative to every nonempty face of B. Then each carrier defect and their total satisfy*

$$\delta_E(t) \in (1+t)^2\mathbb{Z}_{\geq 0}[t], \qquad F_B + F_{\mathcal{S}\vee\mathcal{T}} - F_{\mathcal{S}} - F_{\mathcal{T}} \in (1+t)^2\mathbb{Z}_{\geq 0}[t].$$

No boundedness, pointedness or regularity assumption is required.

Proof. Choose a closed box W in $\operatorname{aff}(B)$ whose interior contains a relative-interior witness for every nonempty face of B and every cell of $\mathcal{S}, \mathcal{T}, \mathcal{R} = \mathcal{S} \vee \mathcal{T}$. For an original carrier E, the polytope $E_W = E \cap W$ has dimension $\dim E$ and

$$\operatorname{relint}(E_W) = \operatorname{relint}(E) \cap \operatorname{int}(W).$$

Each original cell A with carrier E gives a cell $A \cap W$ with carrier E_W, the same dimension, and nonempty relative interior $\operatorname{relint}(A) \cap \operatorname{int}(W)$. Distinct original cells remain distinct, since their relative interiors are disjoint and every one retains such an interior piece. Old cells on a proper face of E remain on its proper truncated face. A new cell on ∂W cannot meet $\operatorname{relint}(E_W)$ and is not counted. Thus the three original carrier polynomials are preserved exactly. The truncated refinement is the common refinement of the truncated subdivisions, since the cell intersections commute with truncation.

At a point of $\operatorname{relint}(E_W)$ no box constraint is active. The minimal cells of the two truncated subdivisions, and their intersection, therefore have the same local dimensions as the original minimal cells at that point. Transversality holds throughout $\operatorname{relint}(E_W)$. Apply Lemma F.3 to E_W. It yields the asserted nonnegative quotient for the original δ_E. Sum over the original carriers E. □

Proposition F.5 (A single-subdivision quotient)**.** *For any finite polyhedral subdivision $\mathcal{S}$ of a nonempty polyhedron B, including all nonempty faces, the carrier excess and its total satisfy*

$$\sum_k \big(s_E(k) - \mathbf{1}_{k=\dim E}\big)t^k \in (1+t)\mathbb{Z}_{\geq 0}[t], \qquad F_{\mathcal{S}} - F_B \in (1+t)\mathbb{Z}_{\geq 0}[t].$$

No regularity or transversality hypothesis is needed.

Proof. First let the carrier E be bounded. Its relative subdivision complex $K_{\mathcal{S}}$ has only the fundamental homology class $[E]$. Thus $Q = K_{\mathcal{S}}/\mathbb{R}[E]$ is acyclic, with $\dim Q_k = s_E(k) - \mathbf{1}_{k=\dim E}$. Put $\beta_k = \operatorname{rank}(\partial_k : Q_k \to Q_{k-1})$, with $\beta_0 = 0$. Ordinary rank-nullity and exactness give $\dim Q_k = \beta_k + \beta_{k+1}$, hence

$$\sum_k \dim Q_k \, t^k = (1+t)\sum_{k\geq 1} \beta_k t^{k-1}.$$

This also covers a zero-dimensional carrier, for which $Q = 0$. For an unbounded carrier, the witness-box construction of Corollary F.4 preserves its single-subdivision carrier counts exactly. Apply the bounded argument to that relative carrier; no transversality condition is involved. Finally sum over the original carriers. □

Remark F.6. The unbounded reduction uses bounded relative cellular pairs; it does not apply an all-faces transversality theorem to the artificial box faces. The double-complex argument is stronger than total acyclicity and a double root alone: two contractible two-term complexes in degrees $(0,1)$ and $(3,4)$ have dimension polynomial $1 + t + t^3 + t^4 = (1+t)^2(1 - t + t^2)$. Exactness in both directions is the additional ingredient.

G Any number of mutually transverse subdivisions

This appendix extends the nonnegative pair quotient of Appendix F to any number m of mutually transverse subdivisions. Lemma G.1 extends the rank identity of Lemma F.1 to m directions; Theorem G.2 and Corollary G.3 then give the nonnegative quotient, by an induction on successive relative-chain quotients.

Lemma G.1 (Rank identity for m directions)**.** *Let $m \geq 1$, and let $D = \bigoplus_{p \in \mathbb{Z}^m} D^p$ be a finite-dimensional complex supported in finitely many multidegrees. Suppose its differentials $d_1, \dots, d_m$ have the unit coordinate degrees, square to zero and anticommute pairwise. Write $\mathbf{1} = (1, \dots, 1)$. Assume that the complex is exact in each direction, and put*

$$\rho_p = \operatorname{rank}(d_m \cdots d_1 : D^p \longrightarrow D^{p+\mathbf{1}}),$$

with $\rho_p = 0$ outside the support. Then

$$\dim D^p = \sum_{\varepsilon \in \{0,1\}^m} \rho_{p-\varepsilon}.$$

Proof. We induct on m. For $m = 1$, exactness and rank-nullity give $\dim D^p = \rho_p + \rho_{p-1}$. For $m = 2$ this is Lemma F.1; the following step from $m-1$ to $m \geq 2$ is the same argument. Write $p = (p_1, q)$ with $q \in \mathbb{Z}^{m-1}$, and put $I^{p_1,q} = \operatorname{im}(d_1 : D^{p_1-1,q} \to D^{p_1,q})$. By anticommutation, each slice $I^{p_1,*}$ is closed under $d_2, \dots, d_m$. On slice p_1 replace each of $d_2, \dots, d_m$ by $(-1)^{p_1}$ times itself. The adjusted differentials still square to zero and anticommute pairwise, and d_1 commutes with them. Exactness in the first direction gives short exact sequences

$$0 \longrightarrow I^{p_1,*} \longrightarrow D^{p_1,*} \xrightarrow{d_1} I^{p_1+1,*} \longrightarrow 0$$

of complexes for each adjusted differential. Each slice $D^{p_1,*}$ is exact in the directions $2, \dots, m$, because it contains every line in those directions through its points. Starting to the left of the finite support, the long exact cohomology sequences in these directions show inductively that every $I^{p_1,*}$ is exact in all of them. The rank of the adjusted composite $d_m \cdots d_2$ on $I^{p_1+1,q}$ equals $\rho_{p_1,q}$, because d_1 maps $D^{p_1,q}$ onto $I^{p_1+1,q}$ and sign adjustments do not affect ranks. The identity for $m-1$ directions, applied to $I^{p_1+1,*}$, therefore gives

$$\dim I^{p_1+1,q} = \sum_{\varepsilon' \in \{0,1\}^{m-1}} \rho_{p_1,q-\varepsilon'}.$$

Adding this formula to the one for $I^{p_1,q}$, and using $\dim D^{p_1,q} = \dim I^{p_1,q} + \dim I^{p_1+1,q}$, proves the assertion. □

Theorem G.2 (Local quotients for m subdivisions)**.** *Let E be a nonempty e-polytope and $m \geq 1$. Let $\mathcal{S}_1, \dots, \mathcal{S}_m$ be finite polyhedral subdivisions of E, including all nonempty faces. Suppose they are mutually transverse at every point of* $\operatorname{relint}(E)$*: if A_i is the minimal original $\mathcal{S}_i$-cell at such a point, then*

$$\dim(A_1 \cap \cdots \cap A_m) = e - \sum_{i=1}^{m} (e - \dim A_i).$$

No transversality on the proper faces of E is required. For $J \subseteq [m] = \{1, \dots, m\}$ let $\mathcal{S}_J = \bigvee_{j \in J} \mathcal{S}_j$, with $\mathcal{S}_\varnothing$ the face complex of E, and define

$$P_{J,E}(t) = \sum_{\substack{A \in \mathcal{S}_J \\ \operatorname{car}(A) = E}} t^{\dim A}.$$

Then

$$\Delta_E(t) = \sum_{J \subseteq [m]} (-1)^{m-|J|} P_{J,E}(t) \in (1+t)^m \mathbb{Z}_{\geq 0}[t].$$

No regularity assumption is needed.

Proof. We use real cellular chains relative to $(E, \partial E)$, with the oriented polytopal cells themselves as bases. Let K_J be the relative complex for $\mathcal{S}_J$ and $K = K_{[m]}$. For $J = \varnothing$, this is the single relative fundamental e-cell. Subdivision maps send each oriented cell to its fundamental sum of refined cells. They are injective chain maps, by disjoint relative-interior supports and cancellation of internal oriented facets. These maps are coherent under composition: subdividing a fundamental chain twice gives the same fundamental chain in the final refinement. We regard every K_J as its image in K.

Multigrading. For a cell C of the common refinement $\mathcal{S}_{[m]}$ with carrier E, let $A_i(C)$ be its minimal original $\mathcal{S}_i$-cell and put $p_i = e - \dim A_i(C)$. The minimal cell for any subfamily is the intersection of the corresponding $A_i(C)$. These cells have a common relative-interior point, and the relative interior of their intersection is the intersection of their relative interiors: locally all inequalities other than the affine-hull equalities are strict. In particular $C = \bigcap_{i=1}^{m} A_i(C)$. Mutual transversality makes the m conormal spaces a direct sum in $\operatorname{dir}(E)^*$. A subfamily of a direct sum is direct, so every subfamily of $\mathcal{S}_1, \dots, \mathcal{S}_m$ is again mutually transverse on $\operatorname{relint}(E)$, and the minimal cell of C for the subfamily indexed by J has dimension $e - \sum_{i \in J} p_i$. In particular, with $|p| = p_1 + \cdots + p_m$,

$$\dim C = e - |p|.$$

Along a retained facet, each minimal original cell moves to one of its faces; no p_i decreases and their sum increases by one. The relative boundary consequently splits into m differentials d_i of the unit coordinate degrees. Sorting the identity $\partial^2 = 0$ by multidegree shows that they square to zero and anticommute pairwise. This is a finite multicomplex supported in the first orthant $\mathbb{Z}^m_{\geq 0}$. Facets on ∂E have already been discarded. The cohomology of d_j is taken along the lines on which all coordinates other than p_j are fixed, and long exact sequences are taken line by line.

For $I \subseteq [m]$, write $U_I = K_{[m]\setminus I}$ for its embedded image in K; use subscripts such as U_{12} for $U_{\{1,2\}}$, and write $I + j$ for $I \cup \{j\}$. Let A be a cell of $\mathcal{S}_J$ with carrier E, where $J = [m] \setminus I$. Since $\mathcal{S}_J$ refines $\mathcal{S}_j$ for $j \in J$, the relative interior of A lies in the relative interior of a single $\mathcal{S}_j$-cell. A refined cell of the same dimension in the subdivision image of A has its relative interior in $\operatorname{relint}(A)$, hence the same minimal original cells for all $j \in J$. The dimension formula for the subfamily indexed by J then forces $p_i = 0$ for $i \in I$. Thus the image of A lies in a single multidegree, with $|p| = e - \dim A$. Hence U_I is multigraded and supported on $p_i = 0$ for $i \in I$. It is closed under the total boundary and hence, by its multigrading, under each component. Its differentials d_i with $i \in I$ vanish, since they would land at $p_i = 1$. By coherence, $U_{I'} \subseteq U_I$ when $I \subseteq I'$, and

$$U_\varnothing = K, \qquad U_{[m]} = \mathbb{R}[E].$$

Since subdivision maps are injective and the image of a c-cell lies in ordinary degree $e - |p| = c$,

$$\sum_p \dim U_I^p \, t^{e-|p|} = P_{[m]\setminus I, E}(t). \tag{39}$$

Directional homology of the subspaces. For $I \subseteq [m]$ and $j \notin I$, the following identification is natural under all the embedded inclusions:

$$H^a_{d_j}(U_I) = \begin{cases} U_{I+j}, & a = 0, \\ 0, & a \neq 0. \end{cases} \tag{40}$$

Here $a = p_j$ and the other gradings are retained. We prove the assertion using explicit fundamental-cycle representatives, rather than an abstract homology isomorphism. Put $J = [m] \setminus I$. The d_j-complex of $U_I = K_J$ splits into blocks indexed by carrier-E cells A of $\mathcal{S}_{J\setminus\{j\}}$. The minimal subfamily

cells are the intersections just described. Along a d_j-facet the coordinates p_i with $i \in J \setminus \{j\}$ do not change, so the corresponding minimal cells move to faces of the same dimensions, hence are the same cells. Thus their intersection A remains fixed.

The block over A is precisely the relative cellular complex of the $\mathcal{S}_j$-induced subdivision of $(A, \partial A)$, read in degree $p_j = \dim A - \dim C$. Cells in $\operatorname{relint}(A)$ have carrier E, even if A touches ∂E. A facet on ∂A changes the minimal $\mathcal{S}_i$-cell for some $i \in J \setminus \{j\}$, so it is a term of that d_i, or it lies in ∂E and was already discarded. Since A is bounded and convex, the only homology is its relative fundamental cycle at $p_j = 0$. These cycles are exactly the embedded images in U_{I+j} of the cells A. Coherence of subdivision maps proves naturality. For $J \setminus \{j\} = \varnothing$ the block is the subdivision of $(E, \partial E)$ itself, with the same conclusion.

There are no incoming chains at $p_j = -1$. Consequently these zero-degree homology groups are kernels: U_{I+j} is the kernel of d_j on U_I in the hyperplane $p_j = 0$. More generally, in every multigraded subquotient of K the d_l-cohomology at $p_l = 0$ is the kernel of d_l there.

Subfamilies. Fix $j \in [m]$, and let $\mathcal{S}'$ be the family obtained by omitting $\mathcal{S}_j$, indexed by $[m] \setminus \{j\}$ in the inherited order. It is a family of $m-1$ subdivisions of E, mutually transverse on $\operatorname{relint}(E)$; for $m = 1$ it is empty. Its relative complex K' is $K_{[m]\setminus\{j\}}$, embedded in K as U_j. By the homogeneity shown above, the multigrading of K' is that of K restricted to the hyperplane $p_j = 0$; the embedding is a multigraded chain map, so it identifies each differential of K' with the corresponding d_i, $i \neq j$. By coherence, the subspace U'_I of K' attached to $I \subseteq [m] \setminus \{j\}$ is U_{I+j}.

Successive quotients. For $0 \leq k \leq m$ put

$$Q_k = K/(U_1 + \cdots + U_k), \qquad Q_0 = K.$$

These are multicomplexes supported in the first orthant. Call a family *eligible* if it consists of $m \geq 0$ finite polyhedral subdivisions of a nonempty polytope E, including all nonempty faces, mutually transverse at every point of $\operatorname{relint}(E)$; for $m = 0$ the family is empty and $K = \mathbb{R}[E]$. We prove the following statement $\mathrm{P}(k)$, for every eligible family and every $0 \leq k \leq m$, by induction on m and, for a fixed family, on k.

(a) In every multidegree, $\dim Q_k = \sum_{I \subseteq [k]} (-1)^{|I|} \dim U_I$.

(b) Q_k is d_i-exact for every $i \leq k$.

(c) For every $l > k$:

(c1) $H^a_{d_l}(Q_k) = 0$ for $a \neq 0$;

(c2) the kernel of d_l on Q_k in the hyperplane $p_l = 0$ is the image of U_l;

(c3) $U_l \cap (U_1 + \cdots + U_k) = U_{1l} + \cdots + U_{kl}$.

For $m = 0$ there is nothing to prove: $K = \mathbb{R}[E]$, and (b) and (c) are empty. For $k = 0$, statement (a) is trivial, (b) is empty, and (c1), (c2) are (40) with $I = \varnothing$; (c3) reads $U_l \cap 0 = 0$.

Now let $m \geq 1$. Assume P for every eligible family of $m-1$ subdivisions and every $k \leq m-1$, and assume $\mathrm{P}(k)$ for $\mathcal{S}_1, \ldots, \mathcal{S}_m$ with some $k < m$. Put $j = k+1$, and let $\mathcal{S}'$ be the family obtained by omitting $\mathcal{S}_j$, as in the previous paragraph. Since $j = k+1$, its first k members are $\mathcal{S}_1, \ldots, \mathcal{S}_k$, and its quotients are $Q'_k = K'/(U'_1 + \cdots + U'_k)$.

The piece. Let V be the image of U_j in Q_k. By (c3) with $l = j$,

$$V \cong U_j/\big(U_j \cap (U_1 + \cdots + U_k)\big) = U_j/(U_{1j} + \cdots + U_{kj}) = Q'_k,$$

an identification of multicomplexes in the directions other than j. Moreover V lies in the hyperplane $p_j = 0$, and d_j vanishes on it. Since $k \le m-1$, statement P(k) holds for $\mathcal{S}'$ by induction on m, and it gives the following: V is d_i-exact for $i \le k$; for $l > j$, its d_l-cohomology vanishes off $p_l = 0$, and its d_l-kernel at $p_l = 0$ is the image of $U'_l = U_{jl}$; and in every multidegree

$$\dim V = \sum_{I \subseteq [k]} (-1)^{|I|} \dim U_{I+j}.$$

We use the short exact sequence of multicomplexes

$$0 \longrightarrow V \longrightarrow Q_k \longrightarrow Q_{k+1} = Q_k/V \longrightarrow 0$$

and its long exact sequences in each direction.

Directions $i \le k$. Both V and Q_k are d_i-exact, hence so is Q_{k+1}.

Direction j. By (c1) and (c2) with $l = j$, the d_j-cohomology of Q_k is concentrated at $p_j = 0$, where it is the kernel, the image of U_j, which is V. The d_j-cohomology of V is V itself, at $p_j = 0$, and the map between them is the identity of V. The long exact sequence shows that Q_{k+1} is d_j-exact.

Directions $l > j$. The d_l-cohomology of V and of Q_k vanishes off $p_l = 0$. Hence that of Q_{k+1} vanishes in degrees $a \ge 1$, and in negative degrees there is nothing. This is (c1) for $k+1$. Since $H^1_{d_l}(V) = 0$, the map from $H^0_{d_l}(Q_k)$ to $H^0_{d_l}(Q_{k+1})$ is onto. Both are kernels at $p_l = 0$, so the kernel of d_l on Q_{k+1} at $p_l = 0$ is the image of U_l. This is (c2) for $k+1$. For (c3), let $x \in U_l \cap (U_1 + \cdots + U_{k+1})$. Its image in Q_k lies in V and, since U_l lies at $p_l = 0$ with $d_l = 0$, in the d_l-kernel of V at $p_l = 0$. That kernel is the image of U_{jl}. Choose $y \in U_{jl}$ with $x - y \in U_1 + \cdots + U_k$. As $U_{jl} \subseteq U_l$, statement (c3) for k gives

$$x - y \in U_l \cap (U_1 + \cdots + U_k) = U_{1l} + \cdots + U_{kl}.$$

Thus $x \in U_{1l} + \cdots + U_{k+1,l}$. The reverse inclusion holds because $U_{il} \subseteq U_i \cap U_l$. This is (c3) for $k+1$.

Dimensions. In every multidegree,

$$\dim Q_{k+1} = \dim Q_k - \dim V = \sum_{I \subseteq [k]} (-1)^{|I|} \dim U_I - \sum_{I \subseteq [k]} (-1)^{|I|} \dim U_{I+j} = \sum_{I \subseteq [k+1]} (-1)^{|I|} \dim U_I.$$

This is (a) for $k+1$, and completes the induction.

The dimension and the nonnegative quotient. By P(m) the quotient

$$D = Q_m = K/(U_1 + \cdots + U_m)$$

is exact in all m directions. By (a) and (39), with $J = [m] \setminus I$,

$$\sum_p \dim D^p \, t^{e-|p|} = \sum_{I \subseteq [m]} (-1)^{|I|} P_{[m]\setminus I, E}(t) = \Delta_E(t).$$

Apply Lemma G.1 to D. Each composite rank ρ_q occurs at the 2^m multidegrees $q + \varepsilon$, $\varepsilon \in \{0,1\}^m$, of ordinary degrees $e - |q| - |\varepsilon|$. Hence

$$\Delta_E(t) = (1+t)^m \sum_{\rho_q > 0} \rho_q \, t^{e-|q|-m}.$$

A nonzero composite has nonzero target $D^{q+\mathbf{1}}$, whose ordinary degree is $e - |q| - m \ge 0$. The displayed quotient therefore has nonnegative integer coefficients. In particular $\Delta_E = 0$ when $e < m$, as in Proposition E.3. □

Corollary G.3 (Polyhedral quotients for m subdivisions)**.** *Let $\mathcal{S}_1, \dots, \mathcal{S}_m$, with $m \geq 1$, be finite polyhedral subdivisions of a nonempty polyhedron B, mutually transverse relative to every nonempty face of B, as in Proposition E.3. Then each carrier defect satisfies $\Delta_E(t) \in (1+t)^m \mathbb{Z}_{\geq 0}[t]$, and*

$$\sum_{J \subseteq [m]} (-1)^{m-|J|} F_{\mathcal{S}_J}(t) \in (1+t)^m \mathbb{Z}_{\geq 0}[t],$$

where $\mathcal{S}_\varnothing$ is the face complex of B. Boundedness, pointedness and regularity are not required.

Proof. For a nonempty face E of B, restricting each $\mathcal{S}_i$ to E gives a subdivision of E, and the cells of each $\mathcal{S}_J$ with carrier E are cells of the restricted refinement. Choose one closed box in $\operatorname{aff}(B)$ whose interior contains relative-interior witnesses of every base face and of every cell of all 2^m subfamily refinements. The old-carrier preservation argument of Corollary F.4 applies simultaneously to these finitely many complexes. For each original carrier E, all 2^m carrier polynomials are unchanged by truncation. At points of the relative interior of the truncated carrier no box constraint is active. Minimal original cells and their intersections retain their local dimensions, so mutual transversality holds there. Apply Theorem G.2 to this bounded relative carrier. Transfer its nonnegative quotient to the original carrier and sum over carriers. □

Remark G.4. For $m = 1$ the induction shows that $K_1/\mathbb{R}[E]$ is exact, the relative statement in the proof of Proposition F.5. For $m = 2$ it recovers the row and column exactness used for Theorem F.2 and Lemma F.3. For $m = 3$ the successive quotients are K/U_1, $K/(U_1 + U_2)$ and D. The pieces added after U_1 are U_2/U_{12} and $U_3/(U_{13} + U_{23}) = K_{12}/(K_1 + K_2)$, the two-subdivision quotient of $\mathcal{S}_1, \mathcal{S}_2$. Statement (c3) at $k = 2$ is the relation $U_3 \cap (U_1 + U_2) = U_{13} + U_{23}$, which is derived from exactness, not assumed.

Remark G.5. By Remark E.4, the power m is sharp also for the nonnegative quotient of Corollary G.3; Section 9 interprets these sums for simultaneous switches.

Remark G.6 (Hyperplane arrangements)**.** Without mutual transversality the power of $1 + t$ can drop; for central hyperplane arrangements we compute the exact power. Let $H_1, \dots, H_m$ be linear hyperplanes of $\mathbb{R}^d$ (repetitions are allowed), let $\mathcal{S}_i$ be the subdivision of $\mathbb{R}^d$ into H_i and its two closed half-spaces, and let $\mathcal{M}$ be the matroid of the normal vectors on $[m]$, with Tutte polynomial $T_{\mathcal{M}}$. Take $B = \mathbb{R}^d$, which has one face, and $\mathcal{S}_\varnothing = \{\mathbb{R}^d\}$, so that $F_{\mathcal{S}_\varnothing} = t^d$, as in Proposition E.3. Then

$$\Delta(t) = \sum_{J \subseteq [m]} (-1)^{m-|J|} F_{\mathcal{S}_J}(t) = (-1)^{m - \operatorname{rk} \mathcal{M}}\, t^{\,d - \operatorname{rk} \mathcal{M}}\, T_{\mathcal{M}}(1+t, 0).$$

Indeed, the cells of $\mathcal{S}_J$ (each counted once, as a set) are exactly the closed faces of the arrangement $\{H_j\}_{j \in J}$. The indices $j \in J$ such that H_j contains a given face form a set G that is closed in $\mathcal{M}|_J$: if $j \in J$ lies in the closure of G, then H_j contains $X_G = \bigcap_{i \in G} H_i$, which contains the face. The face has dimension $d - \operatorname{rk} G$, and the relative interiors of the faces with this G are exactly the regions of the restriction of the arrangement to X_G, whose matroid is $(\mathcal{M}/G)|_{J \setminus G}$ (parallel elements may occur). By Zaslavsky's theorem [30], a central arrangement has $(-1)^{\operatorname{rk} \mathcal{N}} \chi_{\mathcal{N}}(-1)$ regions, where $\chi_{\mathcal{N}}$ is the characteristic polynomial of its matroid $\mathcal{N}$. For an arrangement in a space V, the characteristic polynomial of the arrangement is $q^{\dim V - \operatorname{rk} \mathcal{N}} \chi_{\mathcal{N}}(q)$. Moreover, $\chi_{\mathcal{N}}$ depends only on the lattice of flats, so parallel elements do no harm. With Whitney's expansion [29] of $\chi_{\mathcal{N}}$, in its matroid form [4], a central arrangement with matroid $\mathcal{N}$ on a set R has $\sum_{P \subseteq R} (-1)^{|P| - \operatorname{rk}_{\mathcal{N}} P}$ regions. Since $\operatorname{cl}_{\mathcal{M}|_J} G = J \cap \operatorname{cl}_{\mathcal{M}} G$, the pairs (J, G) with G a flat of $\mathcal{M}|_J$ correspond one to one, via $J = G \sqcup R$, to the pairs (G, R) with $G \subseteq [m]$ arbitrary and $R \subseteq [m] \setminus \operatorname{cl}_{\mathcal{M}} G$. Sum over R first:

only $P = [m] \setminus \operatorname{cl}_{\mathcal{M}} G$ survives, the signs combine to $(-1)^{(m-\operatorname{rk}\mathcal{M})-(|G|-\operatorname{rk} G)}$, and the remaining sum, over all subsets $G \subseteq [m]$, is the subset expansion of $T_{\mathcal{M}}(1+t,0)$.

The matroid $\mathcal{M}$ has no loops, since every H_j is a proper subspace. The polynomial $T_{\mathcal{M}}$ has nonnegative coefficients and is multiplicative over the connected components of $\mathcal{M}$; for a connected loopless $\mathcal{N}$ the polynomial $T_{\mathcal{N}}(x,0)$ has constant term 0 and a positive coefficient of x (Crapo's invariant [5, 4]); and $T_{\mathcal{M}}(x,0)$ has leading term $x^{\operatorname{rk}\mathcal{M}}$. Hence $\Delta \neq 0$, its nonzero coefficients are exactly those in degrees $d - \operatorname{rk}\mathcal{M}, \dots, d$, they all have the sign $(-1)^{m-\operatorname{rk}\mathcal{M}}$, and $1+t$ divides Δ exactly $\kappa(\mathcal{M})$ times, where $\kappa(\mathcal{M})$ is the number of connected components of $\mathcal{M}$. A loopless one-element component is a coloop, so $\kappa(\mathcal{M}) = m$ exactly when every element is a coloop, that is, when the normals are linearly independent. At a point v the minimal cell of $\mathcal{S}_i$ containing v is H_i if $v \in H_i$, and a closed half-space otherwise; so independent normals give mutual transversality at every point, and at the origin mutual transversality means independent normals. So $(1+t)^m$ divides Δ if and only if the $\mathcal{S}_i$ are mutually transverse, and then $\Delta = t^{d-m}(1+t)^m$. When $m - \operatorname{rk}\mathcal{M}$ is odd, every nonzero coefficient of Δ is negative, so the transversality hypothesis of Corollary G.3 cannot be dropped; and four distinct lines through the origin in $\mathbb{R}^2$ give $\Delta = (1+t)(3+t)$, positive but not divisible by $(1+t)^4$. For three planes through a common line in $\mathbb{R}^3$, $T_{\mathcal{M}} = x^2 + x + y$ and $\Delta = -t(1+t)(2+t)$.

H A binomial bound for any number of subdivisions

Throughout this section $\mathcal{S}_1, \dots, \mathcal{S}_m$, with $m \geq 1$, are finite polyhedral subdivisions of a nonempty d-polyhedron B, including all nonempty faces, mutually transverse relative to every nonempty face of B as in Proposition E.3. We write

$$\Delta(t) = \sum_{J \subseteq [m]} (-1)^{m-|J|} F_{\mathcal{S}_J}(t) = \sum_{E} \Delta_E(t),$$

the sum running over the nonempty faces E of B. For a cell C of the full refinement with carrier E, where $e = \dim E$, put $\alpha_i(C) = e - \dim A_i(C)$, where $A_i(C)$ is its minimal original $\mathcal{S}_i$-cell. We call C *fully mixed* if $\alpha_i(C) \geq 1$ for every i. No regularity is assumed in this section.

H.1 Localization

Lemma H.1 (Localization)**.** *Let E be a nonempty e-polytope with finite polyhedral subdivisions $\mathcal{S}_1, \dots, \mathcal{S}_m$, including all nonempty faces, mutually transverse at every point of* $\operatorname{relint}(E)$*, and let $p \in \operatorname{relint}(E)$. Every cell of every $\mathcal{S}_J$ containing p has carrier E. Put*

$$L_p(t) = \sum_{J \subseteq [m]} (-1)^{m-|J|} \sum_{\substack{A \in \mathcal{S}_J \\ p \in A}} t^{\dim A}.$$

Write $\Delta_E = (1+t)^m R_E$ and $L_p = (1+t)^m R_{(p)}$ with the quotients of Theorem G.2, the second for the subdivisions induced on a small box around p. Then

$$R_E \geq R_{(p)} \qquad \text{and} \qquad \Delta_E \geq L_p$$

coefficientwise. If C_0 is the cell of $\mathcal{S}_{[m]}$ with $p \in \operatorname{relint}(C_0)$, $k = \dim C_0$ and $\alpha_i = \alpha_i(C_0)$, then

$$L_p(t) = t^k \prod_{i=1}^{m} \bigl(a_i(t) - t^{\alpha_i}\bigr), \tag{41}$$

where $a_i(t)$ counts, by dimension, all cones of the local fan of $\mathcal{S}_i$ at p modulo its lineality space, including the zero cone.

Proof. Let K, U_I and $D = K/(U_1 + \cdots + U_m)$ be as in the proof of Theorem G.2. A cell contains p if and only if it contains C_0, the least cell containing p; such a cell meets $\mathrm{relint}(E)$ at p, so its carrier is E. Let K_{out} be the span of the cells with carrier E that do not contain C_0. The faces of such a cell do not contain C_0, so K_{out} is a multigraded subcomplex, and K/K_{out} has as basis the cells containing C_0.

Choose a closed box W in $\mathrm{aff}(E)$ around p, so small that $W \subseteq \mathrm{relint}(E)$ and W meets only the cells containing p. The induced subdivisions of the polytope W include all nonempty faces and are mutually transverse on $\mathrm{int}(W)$, so they satisfy the hypotheses of Theorem G.2. The cells of their common refinement with carrier W are the sets $C \cap W$ with $C \supseteq C_0$; they have the dimensions of the corresponding C, and distinct C give distinct sets. A facet of $C \cap W$ is either $C' \cap W$ for a facet C' of C containing C_0, or C intersected with a facet of W; the latter lies in ∂W and is discarded. Hence the relative cellular complex of $(W, \partial W)$ is K/K_{out}, with the same incidence numbers and, since the local minimal cells have the same codimensions, the same multigrading. Projecting the subdivision chain of a cell A of $\mathcal{S}_{[m]\setminus\{i\}}$ keeps its top pieces containing C_0: nothing if $p \notin A$, and otherwise the subdivision chain of $A \cap W$ in the local refinement, with the same orientations. Every local cell arises in this way. Consequently

$$D_{(p)} = K/(K_{\mathrm{out}} + U_1 + \cdots + U_m)$$

is the multicomplex of Theorem G.2 for the local model, and the projection $f : D \to D_{(p)}$ is onto in every multidegree. By statement (a) of P(m) in that proof, the dimension polynomial of $D_{(p)}$ is L_p. The composite $d_m \cdots d_1$ commutes with f, so f maps its image in D^q onto its image in $(D_{(p)})^q$. Therefore every composite rank of D is at least the corresponding rank of $D_{(p)}$, and Lemma G.1, applied to both complexes, gives $R_E \geq R_{(p)}$; multiplying by $(1+t)^m$ gives $\Delta_E \geq L_p$.

For (41), the conormal spaces N_i of the minimal cells at p form a direct sum in $\mathrm{dir}(E)^*$. Choose subspaces $X_i \subseteq \mathrm{dir}(E)$ with $N_j|_{X_i} = 0$ for $j \neq i$ and $N_i|_{X_i}$ a perfect pairing, and let X_0 be the common kernel of the N_i. Then $\mathrm{dir}(E) = X_0 \oplus X_1 \oplus \cdots \oplus X_m$, $\dim X_0 = k$, $\dim X_i = \alpha_i$, and $\mathrm{dir}(A_i(C_0)) = X_0 \oplus \bigoplus_{l \neq i} X_l$. Near p, each cell of $\mathcal{S}_i$ through p coincides with p plus its tangent cone, whose lineality space contains $\mathrm{dir}(A_i(C_0))$. Thus $\mathcal{S}_i$ is locally $\mathrm{dir}(A_i(C_0)) \times \Sigma_i$, where Σ_i is a complete fan in X_i with least cone $\{0\}$, and cells through p correspond to cones of the same dimension. The local refinement $\mathcal{S}_J$ is $X_0 \times \prod_{i \in J} \Sigma_i \times \prod_{i \notin J} X_i$, with polynomial $t^k \prod_{i \in J} a_i \prod_{i \notin J} t^{\alpha_i}$, and the alternating sum factors as stated. □

Lemma H.2 (Cones of a complete pointed fan)**.** *Let Σ be a complete fan in $\mathbb{R}^b$, $b \geq 1$, whose cones are pointed, and let $a(t)$ count its cones by dimension. Then $a(t) - t^b \geq (1+t)^b$.*

Proof. Let σ be a maximal cone; it is pointed of dimension b. For $1 \leq j \leq b$, its j-dimensional faces correspond to the $(j-1)$-faces of a compact cross-section, a $(b-1)$-polytope, so by Lemma C.1 there are at least $\binom{b}{j}$ of them, and they are cones of Σ. The zero cone gives the constant term. Since $\sigma \neq \mathbb{R}^b$, the fan has at least two maximal cones, so the coefficient of t^b in $a - t^b$ is at least one. □

For a regular subdivision the local fan is the normal fan of a full-dimensional gradient polytope, and the count in the proof of Theorem D.1 gives the same bound; the lemma does not need regularity.

H.2 The bound

Theorem H.3. *If $\Delta \neq 0$ and r is its lowest degree, then $0 \leq r \leq d-m$ and*

$$\Delta(t) \geq t^r(1+t)^{d-r}$$

coefficientwise.

Proof. The lowest degree. By Proposition E.3,

$$\Delta(t) = \sum_E \sum_{\operatorname{car}(C)=E} t^{\dim C} \prod_{i=1}^{m} \bigl(1-(-t)^{\alpha_i(C)}\bigr).$$

Only fully mixed cells contribute, and each contributes a polynomial with lowest term $t^{\dim C}$. Hence the lowest nonzero coefficient of Δ is in the degree equal to the least dimension of a fully mixed cell and equals the number of such cells; in particular $\Delta \neq 0$ if and only if a fully mixed cell exists, and r is that least dimension. Fix a fully mixed r-cell C_0 with carrier E of dimension e, ranks $\alpha_i = \alpha_i(C_0) \geq 1$ with $\sum_i \alpha_i = e-r$, and a point $p \in \operatorname{relint}(C_0)$. Then $r \leq e-m \leq d-m$.

A local bound. Let C be a fully mixed c-cell with carrier E', $h = \dim E'$, and minimal-cell codimensions $\beta_i \geq 1$. Apply Lemma H.1 at a point of $\operatorname{relint}(C)$; for an unbounded carrier, first truncate as in the proof of Corollary G.3, with a box whose interior also contains that point. By Lemma H.2,

$$\Delta_{E'}(t) \geq t^c \prod_{i=1}^{m} (1+t)^{\beta_i} = t^c(1+t)^{h-c}. \tag{42}$$

A lift to every coface. Let $E' \supseteq E$ be a face of dimension $h = e+j$. We show that E' contains a fully mixed cell of dimension at most $r+j$. Use the splitting $\operatorname{dir}(E) = X_0 \oplus X_1 \oplus \cdots \oplus X_m$ of the proof of Lemma H.1 at p, and a complement V of $\operatorname{dir}(E)$ in $\operatorname{dir}(E')$. Rows tight on E' are equalities of $\operatorname{aff}(E')$; the other rows tight at p cut out the tangent cone $\operatorname{dir}(E)+Q$ of E' at p, where $Q \subseteq V$ is a full-dimensional polyhedral cone. Its lineality space is $\operatorname{dir}(E)$, because $p \in \operatorname{relint}(E)$ and E is a face, so Q is pointed. The cell $A_i = A_i(C_0)$ is also the minimal $\mathcal{S}_i$-cell at p in E'. Near p, inside E', the subdivision $\mathcal{S}_i$ is $\operatorname{dir}(A_i) \times \Phi_i$, where Φ_i is a finite fan covering $X_i \times Q$. Since Q is pointed, $X_i \times \{0\}$ is a face of $X_i \times Q$; so for every cone σ of Φ_i the intersection $\sigma \cap (X_i \times \{0\})$ is a face of σ and a cone of Σ_i.

Fix $v \in \operatorname{int}(Q)$. The sets $\sigma \cap (X_i \times \{v\})$, $\sigma \in \Phi_i$, form a finite complete polyhedral subdivision of $X_i \times \{v\}$, and this subdivision has a vertex. Otherwise all its cells have one common lineality space $L \neq 0$: a polyhedron and its faces have the same lineality space, and the full cells are connected through common facets, since the complement of the codimension-two skeleton is connected. Take a full cone σ whose section is a full cell, and a point (x,v) of that cell. Then $(x+l,v) \in \sigma$ for all $l \in L$; since σ is a cone, $(sx+l,sv) \in \sigma$ for all $s>0$ and $l \in L$, and letting $s \to 0$ gives $L \times \{0\} \subseteq \sigma$. Then the cone $\sigma \cap (X_i \times \{0\})$ of Σ_i contains a line, which is impossible.

Choose a vertex x_i of the section of Φ_i at v for each i. For $s>0$, sx_i is a vertex of the section at sv. The point $q_s = p + (0, sx_1, \ldots, sx_m, sv)$, in the coordinates of $X_0 \oplus X_1 \oplus \cdots \oplus X_m \oplus V$, lies in $\operatorname{relint}(E')$ and, for small s, near p; there $\mathcal{S}_i$ depends only on the X_i and V coordinates. Let M_i be the cone of Φ_i with (sx_i, sv) in its relative interior. Because the affine subspace $H = X_i \times \{sv\}$ passes through a relative-interior point of M_i, we have $\operatorname{relint}(M_i \cap H) = \operatorname{relint}(M_i) \cap H$, so $M_i \cap H$ is the vertex, and $0 = \dim(M_i \cap H) \geq \dim M_i - \dim V$. Thus $\dim M_i \leq j$, and the minimal $\mathcal{S}_i$-cell at q_s has dimension at most $j + \dim A_i = h - \alpha_i$, that is, codimension $\beta_i \geq \alpha_i \geq 1$ in E'. By mutual transversality at q_s, the cell of the common refinement there has dimension $c' = h - \sum_i \beta_i \leq h - (e-r) = r+j$, and it is fully mixed.

Summing over cofaces. On such a coface, $c' \geq r$ by the first step, and (42) gives

$$\Delta_{E'}(t) \geq t^{c'}(1+t)^{r+j-c'}(1+t)^{e-r} \geq t^{r+j}(1+t)^{e-r}.$$

By Lemma C.1, E has at least $\binom{d-e}{j}$ cofaces of dimension $e+j$, for $0 \leq j \leq d-e$. Distinct cofaces are distinct carriers, and every other carrier contribution is nonnegative by Corollary G.3. Therefore

$$\Delta(t) \geq t^r(1+t)^{e-r} \sum_{j=0}^{d-e} \binom{d-e}{j} t^j = t^r(1+t)^{d-r}.$$

□

Corollary H.4 (Terminal support)**.** *Either $\Delta = 0$, or the positive coefficients of Δ occur exactly in the degrees $r, r+1, \ldots, d$ for some $0 \leq r \leq d-m$. For $m = 1$, where mutual transversality holds automatically, this holds for $F_{\mathcal{S}} - F_B$, for every finite polyhedral subdivision $\mathcal{S}$ of B, and $F_{\mathcal{S}} - F_B$ vanishes exactly when $\mathcal{S}$ is the face complex of B; otherwise $F_{\mathcal{S}} - F_B \geq t^r(1+t)^{d-r}$ with $r \leq d-1$.*

Proof. Below r every coefficient vanishes by the definition of r. From r to d, Theorem H.3 gives, in each degree k, a coefficient at least $\binom{d-r}{k-r} \geq 1$. No coefficient is negative, by Corollary G.3. For $m = 1$, a subdivision other than the face complex has a cell of smaller dimension than its carrier, which is fully mixed: some face E of B, of dimension e, is not a cell, so at least two e-cells have carrier E, and two of them share a facet that meets $\operatorname{relint}(E)$ (connect them through facets meeting $\operatorname{relint}(E)$, as in the proof of Lemma B.2). □

Remark H.5. For $m = 2$, Theorem H.3 and Corollary H.4 give the conclusions of Theorems C.3, C.4 and D.1 without the regularity hypothesis. For $m = 1$, Corollary H.4 contains the conclusion of Theorem C.6 for an arbitrary finite polyhedral subdivision. We keep the earlier proofs: they are local and constructive, and for two regular subdivisions the lift in the proof of Theorem D.1 gives more, a fully mixed cell with the same ranks on every coface.

H.3 Attainment

Corollary H.6 (Bounded attainment for m subdivisions)**.** *For all integers $m \geq 1$ and $0 \leq r \leq d-m$ there are a bounded d-polytope B and m regular affinity subdivisions of it, mutually transverse relative to every nonempty face, with $\Delta(t) = t^r(1+t)^{d-r}$.*

Proof. First let the dimension be $r+m$. For $r \geq 1$, let $W \subset \mathbb{R}^r$ be a full-dimensional bounded polytope with facet description $\ell_k(w) \geq 0$, and put

$$B_0 = \{(x_1, \ldots, x_m, w) : |x_i| \leq \ell_k(w) \text{ for all } i, k\};$$

for $r = 0$ put $B_0 = [-1, 1]^m$. As in the proof of Corollary D.2, B_0 is bounded of dimension $r+m$. Let $\mathcal{S}_i$ be the affinity subdivision of $\max(0, x_i)$; its wall is $x_i = 0$.

Let y be a relative-interior point of a face E. If $\ell_k(w) = 0$ at y for some k, all the inequalities $|x_i| \leq \ell_k(w)$ are tight at y, hence on E, so $x = 0$ on E and every $\mathcal{S}_i$ is trivial on E. Otherwise, for $T = \{i : x_i(y) = 0\}$, no inequality involving x_i with $i \in T$ is tight at y; so $\operatorname{dir}(E)$ contains the coordinate directions of the x_i with $i \in T$, the forms x_i with $i \in T$ are independent on $\operatorname{dir}(E)$, and the other walls miss y. Thus the family is mutually transverse relative to every face. A fully mixed cell on a proper face would need $x = 0$ at a relative-interior point with every $\mathcal{S}_i$ nontrivial; by the first case, every $\ell_k(w)$ is then positive, so no inequality is tight there, since each involves some x_i, and that point would be interior to B_0. Hence proper carriers contribute nothing. The only fully

mixed cell with carrier B_0 is $\{0\} \times W$ (the origin when $r = 0$), of dimension r with all $\alpha_i = 1$. By Proposition E.3, $\Delta = t^r(1+t)^m$.

Now take pyramids as in the proof of Corollary D.2. The homogenized maxima keep the subdivisions regular, the coned refinements are the refinements of the cones, and every $F_{\mathcal{S}_J}$ becomes $(1+t)F_{\mathcal{S}_J} + 1$. The 2^m added constants cancel in the alternating sum, so Δ is multiplied by $1+t$. Coning raises the dimensions of every cell, every intersection and every carrier by one, so mutual transversality is kept on every face. After $d - r - m$ pyramids we obtain a bounded d-polytope with $\Delta = t^r(1+t)^{d-r}$. □

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References

[1] I. Ahmad, G. Fourier and M. Joswig, *Order and chain polytopes of maximal ranked posets*, Discrete Appl. Math. 390 (2026), 28–39, doi:10.1016/j.dam.2026.03.036 and arXiv:2309.01626v2 (theorem numbers as in this version).

[2] C. Benedetti, N. Bergeron and J. Machacek, *Hypergraphic polytopes: combinatorial properties and antipode*, J. Comb. 10(3) (2019), 515–544; arXiv:1712.08848v2 (section numbers as in this version).

[3] B. Bhandari, D. Cunningham, G. Morrell, S. Oh and P. Smith, *Gap between the number of facets of the two poset polytopes*, Ann. Comb. (2025), published online, doi:10.1007/s00026-025-00776-9 and arXiv:2405.17646v1 (2024; theorem numbers as in this version).

[4] T. Brylawski and J. Oxley, *The Tutte polynomial and its applications*, in: N. White (ed.), Matroid Applications, Encyclopedia Math. Appl. 40, Cambridge Univ. Press, 1992, 123–225.

[5] H. H. Crapo, *A higher invariant for matroids*, J. Combin. Theory 2 (1967), 406–417.

[6] X. Fang and G. Fourier, *Marked chain-order polytopes*, European J. Combin. 58 (2016), 267–282; arXiv:1508.02232v2

[7] X. Fang, G. Fourier, J.-P. Litza and C. Pegel, *A continuous family of marked poset polytopes*, SIAM J. Discrete Math. 34(1) (2020), 611–639, doi:10.1137/18M1228529 and arXiv:1712.01037v1 (2017; theorem numbers as in this version).

[8] X. Fang, G. Fourier and C. Pegel, *The Minkowski property and reflexivity of marked poset polytopes*, Electron. J. Combin. 27(1) (2020), P1.27, doi:10.37236/8144 (theorem numbers as in this version); arXiv:1807.03970v2

[9] G. Fourier, *Marked poset polytopes: Minkowski sums, indecomposables, and unimodular equivalence*, J. Pure Appl. Algebra 220(2) (2016), 606–620, doi:10.1016/j.jpaa.2015.07.007 and arXiv:1410.8744v2

[10] G. Fourier, *The Hibi–Li face-number conjecture*, arXiv:2609.23629v1 (September 20, 2026).

[11] G. Fourier, *The Hibi–Li face-number conjecture*, second version, arXiv:2609.23629v2 (October 29, 2026).

[12] G. Fourier, *On facet gaps of order and chain polytopes*, arXiv:2608.19989v1 (2026).

[13] R. Freij-Hollanti and T. Lundström, *f-vector inequalities for order and chain polytopes*, Math. Scand. 130(3) (2024), 467–486, doi:10.7146/math.scand.a-143995 and arXiv:2312.13890v3 (theorem numbers as in this version).

[14] R. Freij-Hollanti, T. Lundström and A. Mori, *Two-dimensional faces of order and chain polytopes*, Electron. J. Combin. 33(3) (2026), P3.67, doi:10.37236/14909 and arXiv:2509.17541v2

[15] K. Fukuda and C. Weibel, *A linear equation for Minkowski sums of polytopes relatively in general position*, European J. Combin. 31(2) (2010), 565–573, doi:10.1016/j.ejc.2009.03.023 and arXiv:0712.0027v2 (2008; theorem and lemma numbers as in this version).

[16] D. Goel, *The Hibi–Li (2016) conjecture is fully resolved*, Zenodo (2026), doi:10.5281/zenodo.23704734 and arXiv:2610.40703

[17] C. Greene and T. Zaslavsky, *On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs*, Trans. Amer. Math. Soc. 280(1) (1983), 97–126.

[18] T. Hibi and N. Li, *Unimodular equivalence of order and chain polytopes*, Math. Scand. 118 (2016), 5–12; arXiv:1208.4029v1 (conjecture number as in this version).

[19] T. Hibi, N. Li, Y. Sahara and A. Shikama, *The numbers of edges of the order polytope and the chain polytope of a finite partially ordered set*, Discrete Math. 340(5) (2017), 991–994; arXiv:1508.00187v3

[20] J. Lee, *A refinement of the edge theorem for order and chain polytopes*, arXiv:2610.07413v2 (2026).

[21] T. Lundström, *Combinatorics of poset polytopes*, doctoral thesis, Aalto University, Doctoral Theses 104/2026 (2026), aaltodoc.aalto.fi/handle/123456789/145155

[22] A. Mori, *Triangular faces of the order and chain polytope of a maximal ranked poset*, Discrete Math. 348(8) (2025), 114480, doi:10.1016/j.disc.2025.114480 and arXiv:2404.00263v3 (theorem numbers as in this version).

[23] C. Pegel, *The face structure and geometry of marked order polyhedra*, arXiv:1610.01393v2 (2017).

[24] R. P. Stanley, *Acyclic orientations of graphs*, Discrete Math. 5 (1973), 171–178.

[25] R. P. Stanley, *Two poset polytopes*, Discrete Comput. Geom. 1 (1986), 9–23; author's copy

[26] J. Stelzig, *On the structure of double complexes*, J. London Math. Soc. 104(2) (2021), 956–988, doi:10.1112/jlms.12453 and arXiv:1812.00865v3

[27] J. Taylor, *Chromatic symmetric functions of hypertrees*, Electron. J. Combin. 24(2) (2017), P2.2.

[28] L. van den Dries, *Tame Topology and O-minimal Structures*, London Math. Soc. Lecture Note Ser. 248, Cambridge Univ. Press, 1998.

[29] H. Whitney, *A logical expansion in mathematics*, Bull. Amer. Math. Soc. 38 (1932), 572–579.

[30] T. Zaslavsky, *Facing up to arrangements: face-count formulas for partitions of space by hyperplanes*, Mem. Amer. Math. Soc. 1(154) (1975).