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PAIR CORRELATION OF L-ZEROS AND HOOLEY'S CONJECTURE

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ABSTRACT. In this article, we study the form factor for zeros of Dirichlet L-functions, pairing zeros of the same L-function and then averaging over primitive characters modulo a prime conductor. Unconditionally, we show for $|\alpha| < 1$ that the form factor agrees with the random matrix prediction from the Circular Unitary Ensemble (CUE). Under GRH and the Hardy–Littlewood twin primes conjecture, we extend this agreement to $|\alpha| < 2$. For $|\alpha| \geq 1$, we prove that the CUE prediction is equivalent to Hooley's conjecture for the variance of primes in arithmetic progressions with prime moduli. We also explore applications of these form-factor results to the simplicity of zeros.

1. INTRODUCTION

In 1973, assuming the Riemann hypothesis, Montgomery [25] studied the pair correlation of ζ-zeros. Writing the nontrivial zeros as $\frac{1}{2} + i\gamma, \frac{1}{2} + i\gamma'$ and $N(T)$ the number of zeros with $0 \leq \gamma \leq T$, he conjectured that, for every interval $[\alpha, \beta]$,

$$\lim_{T\to\infty} \frac{\#\left\{\gamma, \gamma' \in [0, T] : \frac{\gamma-\gamma'}{2\pi} \log T \in [\alpha, \beta]\right\}}{N(T)} = \mathbf{1}_{0\in[\alpha,\beta]} + \int_\alpha^\beta \left[1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right] \mathrm{d}u.$$

This is known as the *pair correlation conjecture*. The integrand in this conjecture coincides with the limiting pair-correlation density for the eigenvalues in the Circular Unitary Ensemble (CUE) model, establishing a connection between the random matrix theory and ζ-zeros.

This connection was later extended to a broad class of random matrix models for statistics of both zeros and values of L-functions: Rudnick–Sarnak (1996) [29] generalized the CUE prediction to high zeros of automorphic L-functions. Katz–Sarnak (1999) [18, 19] developed a philosophy that associates each reasonable family of L-functions with a random matrix group known as the *symmetry type* based on the statistics of low-lying zeros. Keating–Snaith (2000) [22, 21] formulated moment conjectures for $\zeta(\frac{1}{2} + it)$ and central values of L-functions based on the symmetry type. A detailed discussion of these topics can be found in Soundararajan's 2022 ICM plenary lecture [30].

Montgomery originally formulated his conjecture using the Fourier-side statistic

$$F(\alpha, T) = \frac{1}{N(T)} \sum_{0<\gamma,\gamma'\leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma'),$$

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where $w(u)$ is a suitable smooth weight with $w(0) = 1$. This statistic is known as the *form factor*. Montgomery's *strong pair correlation conjecture* states, in the sense of distributions, that

$$F(\alpha, T) \longrightarrow \delta(\alpha) + \min(|\alpha|, 1), \qquad (T \to \infty). \tag{1.1}$$

The continuous term has a *ramp* for $|\alpha| < 1$ and a *plateau* for $|\alpha| \geq 1$. The delta mass at the origin comes from the diagonal contribution $\gamma = \gamma'$. Montgomery himself verified this limit for $|\alpha| < 1$ under RH, allowing him to establish conditional results on the simplicity of and gaps between ζ-zeros.

Just as the abscissae of ζ-zeros govern the exponent in the error term of the prime number theorem, the pair correlation of the ordinates of these zeros has a prime-side interpretation. In 1987, Goldston–Montgomery [9] established under RH that the strong pair correlation conjecture[1] is equivalent to

$$J(X, H) := \int_1^X \left(\sum_{x < n \leq x+H} \Lambda(n) - H \right)^2 \mathrm{d}x \sim HX \log \frac{X}{H},$$

for $X^\varepsilon \leq H \leq X^{1-\varepsilon}$. In the same work, they showed that, under RH and the Hardy–Littlewood twin primes conjecture, the validity of (1.1) can be extended to $|\alpha| < 2$. Subsequently, Montgomery–Soundararajan (2004) [26] computed higher moment analogs under Hardy–Littlewood k-tuple conjectures, providing conditional support for $\sum_{x<n\leq x+H} \Lambda(n)$ to behave like a Gaussian with mean H and variance $H \log(X/H)$ as $X \to +\infty$ and $X^\varepsilon \leq H \leq X^{1-\varepsilon}$, whereas Cramér's model predicts a Gaussian behavior with variance $H \log X$ in the same regime.

A q-analog of $J(X, H)$ is the variance of primes in arithmetic progressions modulo q. Put

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a(q)}} \Lambda(n), \qquad G(x, q) = \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} \left| \psi(x; q, a) - \frac{x}{\varphi(q)} \right|^2.$$

Define the averaged variance by

$$H(x, Q) = \sum_{q \leq Q} G(x, q).$$

In 1966, Barban [1] and Davenport–Halberstam [5] studied this averaged variance. Refining their work, Montgomery (1970) [24] and Hooley (1975) [13] proved that, for every $A > 0$ and $\frac{x}{(\log x)^A} < Q \leq x$,

$$H(x, Q) = Qx \log Q - cQx + O_A\left(Q^{\frac{5}{4}} x^{\frac{3}{4}} + \frac{x^2}{(\log x)^A} \right),$$

where c is an explicit constant. Under GRH, Hooley (1975) [12] extended the validity of this asymptotic to $x^{\frac{1}{2}+\varepsilon} < Q \leq x$ with a slightly different remainder.

[1] More precisely, Goldston–Montgomery used a suitable uniform version of strong pair correlation; the distributional formulation above does not by itself assert this uniformity.

The individual-modulus problem $G(x,q)$ is substantially harder. In 1975, Hooley [11] conjectured, under unspecified conditions, that

$$G(x,q) \sim x \log q. \tag{1.2}$$

In 1996, Friedlander–Goldston [8] proved in the accessible range $q > x$ that

$$G(x,q) = x\log x - x - \frac{x^2}{\varphi(q)} + O_A\left(\frac{x}{(\log x)^A}\right) + O((\log q)^3), \tag{1.3}$$

and they conjectured that (1.2) holds throughout $x^{\frac{1}{2}+\varepsilon} < q < x$. In 2014, Keating–Rudnick [20] proved a function-field analog of Hooley's conjecture, their ranges suggesting that the asymptotic (1.2) may persist as far as $q \geq x^{\varepsilon}$.

In 2015, Fiorilli [6] gave heuristics predicting (1.2) for

$$(\log\log x)^{1+\delta} \leq q \leq x,$$

for every fixed $\delta > 0$. In 2023, Fiorilli–Martin [7] showed that this range is essentially optimal in the sense that the bound $G(x,q) \ll x\log q$ can fail when $q \asymp \log\log x$.

In 2026, Leung [23] computed higher moments and showed under the Hardy–Littlewood k-tuple conjectures that $\psi(x;q,a)$ behaves like a Gaussian with mean $\frac{x}{\varphi(q)}$ and variance $\frac{x\log q}{\varphi(q)}$ as the reduced residue class a varies, $x \to +\infty$, and $x^{1-\varepsilon} \leq q \leq x/\log x$.

To focus on the number-theoretic content and avoid technical complications arising from the sharp cutoff, we work with a smoothed version of $G(x,q)$. Given $W \in C_c^\infty(0,\infty)$, define

$$G_W(x,q) = \sum_{\substack{1\leq a\leq q\\(a,q)=1}} \left| \sum_{n\equiv a(q)} \Lambda(n) W\left(\frac{n}{x}\right) - \frac{1}{\varphi(q)} \sum_{(n,q)=1} \Lambda(n) W\left(\frac{n}{x}\right) \right|^2. \tag{1.4}$$

We formulate the corresponding smooth conjecture as follows:

Conjecture 1 (Smooth Hooley)**.** For every $\varepsilon > 0$ and every fixed $W \in C_c^\infty(0,\infty)$,

$$G_W(x,q) \sim \left(\int_0^\infty |W(u)|^2 \mathrm{d}u\right) x\log q,$$

as $x \to \infty$, uniformly for $x^{\varepsilon} \leq q \leq x$.

In 2025, Hayani [10] established the analog of Fiorilli–Martin for $G_W(x,q)$, so we choose not to study the range $q < x^{\varepsilon}$ here.

Having formulated the prime-side problem, it is natural to ask whether there is an analog of the Goldston–Montgomery correspondence for $G_W(x,q)$. In this paper, we give an affirmative answer to this question when q is prime. To formulate the zero side of this correspondence, we begin with some notation.

Throughout the paper, q is an odd prime, χ is a Dirichlet character modulo q, χ_0 is the principal character, and the notation

$$\sum_{\chi}^{*}$$

denotes a sum over primitive characters modulo q; for prime q, these are precisely the $q-2$ nonprincipal characters. Write $e(z)=e^{2\pi iz}$ and use the Fourier-transform convention

$$\widehat{h}(\alpha)=\int_{\mathbb{R}} h(t)e(-\alpha t)\mathrm{d}t.$$

We call $h\in L^1(\mathbb{R})$ *band-limited* if $\operatorname{supp}\widehat{h}$ is compact and define its *bandwidth* by

$$B_h=\sup\{|\alpha|:\alpha\in\operatorname{supp}\widehat{h}\}.$$

Unless otherwise indicated, implied constants in O, $\ll$, and $\gg$ may depend on the fixed weights (f in sums over zeros and W in sums over integers).

For a nontrivial zero ρ_χ of $L(s,\chi)$, write

$$\rho_\chi=\frac{1}{2}+i\gamma_\chi,\qquad \gamma_\chi=-i\left(\rho_\chi-\frac{1}{2}\right). \tag{1.5}$$

This convention does not assume GRH: the ordinate γ_χ may be complex. We adopt the convention that $\sum_{\gamma_\chi}$ sums over zeros with the corresponding multiplicity. By the Paley–Wiener theorem [14, Theorem 7.3.1], a band-limited Schwartz function f extends to an entire function and decays rapidly as $|\Re z|\to\infty$ whenever $\Im z$ remains bounded. Because $|\Re\gamma_\chi|<\frac{1}{2}$, f may be evaluated at γ_χ and has rapid decay as $|\Re(\gamma_\chi)|\to\infty$. For an entire function f, put $f^{\#}(z)=\overline{f(\overline{z})}$.

Let f be a band-limited Schwartz function. Define the f-weighted form factor as

$$K(f,\alpha,q)=\frac{1}{q-2}\sideset{}{^*}\sum_{\chi}\left(\sum_{\gamma_\chi}f(\gamma_\chi)q^{i\alpha\gamma_\chi}\right)\left(\sum_{\gamma'_\chi}f^{\#}(\gamma'_\chi)q^{-i\alpha\gamma'_\chi}\right). \tag{1.6}$$

Let $\phi\in L^1(\mathbb{R})$ be band-limited. Define the f-weighted pair-correlation statistic by

$$R(f,\phi,q)=\frac{1}{q-2}\sideset{}{^*}\sum_{\chi}\sum_{\gamma_\chi,\gamma'_\chi}f(\gamma_\chi)f^{\#}(\gamma'_\chi)\phi\left(\frac{(\gamma_\chi-\gamma'_\chi)\log q}{2\pi}\right). \tag{1.7}$$

This quantity relates to the form factor via Fourier inversion:

$$R(f,\phi,q)=\int_{\mathbb{R}}\widehat{\phi}(\alpha)K(f,\alpha,q)\mathrm{d}\alpha. \tag{1.8}$$

Under GRH, $R(f,\phi,q)$ becomes the ordinary weighted pair-correlation statistic normalized by the local spacing $2\pi/\log q$.

As recalled above, Montgomery's pair-correlation density is the limiting CUE pair-correlation density; on the Fourier side, this corresponds to the ramp–plateau form factor $\min(|\alpha|,1)$. Rudnick–Sarnak (1996) [29] predicted CUE statistics for n-level correlations of high zeros of automorphic L-functions, which include $\zeta(s)$ and $L(s,\chi)$. Katz–Sarnak (1999) [18, 19] predicted unitary statistics for the low-lying zeros of families of Dirichlet L-functions. Hughes–Rudnick (2003) [15] subsequently verified the predicted one-level density for primitive characters of prime conductor and a restricted class of test functions, strengthening the evidence for unitary symmetry in this setting. Neither framework directly covers the fixed-height, q-aspect regime considered here, but it is natural to expect unitary statistics

in this regime as well. We therefore predict that our form factor $K(f,\alpha,q)$ should exhibit a plateau in the range $|\alpha|\geq 1$ as in Montgomery's strong pair correlation conjecture.

Conjecture 2. For every $B\geq 1$ and every band-limited Schwartz function f,

$$K(f,\alpha,q)\sim \frac{\log q}{2\pi}\int_{\mathbb{R}}|f(t)|^2\mathrm{d}t,$$

uniformly for $1\leq|\alpha|\leq B$ as $q\to\infty$ through primes.

Together with the expected ramp behavior for $|\alpha|<1$, Conjecture 2 leads to the following full pair-correlation conjecture:

Conjecture 3. For every band-limited $\phi\in L^1(\mathbb{R})$,

$$R(f,\phi,q)\sim \frac{\log q}{2\pi}\left(\int_{\mathbb{R}}|f(t)|^2\mathrm{d}t\right)\left(\widehat{\phi}(0)+\int_{\mathbb{R}}\min(|\alpha|,1)\widehat{\phi}(\alpha)\mathrm{d}\alpha\right), \tag{1.9}$$

as $q\to\infty$ through odd primes. Equivalently,

$$R(f,\phi,q)\sim \frac{\log q}{2\pi}\left(\int_{\mathbb{R}}|f(t)|^2\mathrm{d}t\right)\left[\phi(0)+\int_{\mathbb{R}}\left(1-\left(\frac{\sin\pi u}{\pi u}\right)^2\right)\phi(u)\mathrm{d}u\right]. \tag{1.10}$$

We will investigate these conjectures from the prime side. Our starting point is a relation between the form factor and Hooley's variance $G_W(x,q)$:

Theorem 1 (Form-factor–variance relation)**.** *Let $W\in C_c^\infty(0,\infty)$, let f be its partner under (4.2), and put $x=q^\alpha$. Uniformly for $x\geq 1$ and odd primes q,*

$$\begin{aligned}K(f,\alpha,q)=\frac{q-1}{q-2}\frac{G_W(x,q)}{x}+\frac{|W(1/x)|^2}{x}(\log q)^2\\ +O\left(\left(1+\frac{|W(1/x)|\log q}{\sqrt{x}}\right)\left(1+\left(\frac{G_W(x,q)}{x}\right)^{1/2}\right)\right).\end{aligned}$$

Under the convention $x=q^\alpha$, the range $|\alpha|<1$ implies $q\gg x$, where $G_W(x,q)$ is easy to evaluate, allowing us to deduce the following:

Theorem 2 (Ramp)**.** *For every band-limited Schwartz function f, as $q\to\infty$ through odd primes,*

$$\begin{aligned}K(f,\alpha,q)=\frac{|\alpha|\log q}{2\pi}\int_{\mathbb{R}}|f(t)|^2\mathrm{d}t+\frac{(\log q)^2}{4\pi^2}\left|\widehat{f}\left(-\frac{\alpha\log q}{2\pi}\right)\right|^2\\ +O\left(\left(1+\log q\left|\widehat{f}\left(-\frac{\alpha\log q}{2\pi}\right)\right|\right)\sqrt{\log(2q^{|\alpha|})}\right),\end{aligned}$$

uniformly for α in compact subsets of $(-1,1)$.

Remark 1. The compact support of $\widehat{f}$ implies that, whenever $|\alpha|\geq\sigma>0$,

$$\widehat{f}\left(-\frac{\alpha\log q}{2\pi}\right)=0,$$

for $q\gg_\sigma 1$. Thus, the second main term vanishes, and the error term in Theorem 2 simplifies to $O(\sqrt{\log q})$ in this range.

Corollary 1 (Restricted-support pair correlation)**.** *For every band-limited $\phi \in L^1(\mathbb{R})$ with $B_\phi < 1$,*

$$R(f,\phi,q) \sim \frac{\log q}{2\pi}\left(\int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t\right)\left(\widehat{\phi}(0) + \int_{\mathbb{R}} |\alpha|\widehat{\phi}(\alpha)\mathrm{d}\alpha\right),$$

as $q \to \infty$ through odd primes.

We record a conditional application of Corollary 1 to the simplicity of L-zeros:

Corollary 2. *Assume GRH. For every nonnegative band-limited Schwartz function f,*

$$\sum_{\chi}^{*} \sum_{\substack{\rho_\chi \\ \gamma_\chi \text{ simple}}} |f(\gamma_\chi)|^2 \geq \left(\frac{2}{3} - o_f(1)\right) \sum_{\chi}^{*} \sum_{\gamma_\chi} |f(\gamma_\chi)|^2. \tag{1.11}$$

Remark 2. The estimate above is the smooth analog of the assertion that, for every nondegenerate fixed interval $[a,b]$,

$$\sum_{\chi}^{*} \sum_{\substack{\rho_\chi \\ \gamma_\chi \in [a,b] \\ \gamma_\chi \text{ simple}}} 1 \geq \left(\frac{2}{3} - o(1)\right) \sum_{\chi}^{*} \sum_{\gamma_\chi \in [a,b]} 1.$$

Formally, this is obtained from (1.11) by taking $|f|^2$ to be $\mathbf{1}_{[a,b]}$. We do not pursue here the approximation needed to pass from band-limited Schwartz weights to a sharp cutoff.

We can now state the affirmative answer announced above: for prime moduli, Hooley's conjecture for $G_W(x,q)$ is equivalent to Conjecture 2.

Theorem 3. *Fix $B \geq 1$. Restricted to prime moduli, the following are equivalent:*

(1) *For every $W \in C_c^\infty(0,\infty)$, as $x \to \infty$,*

$$G_W(x,q) \sim x \log q \int_0^\infty |W(u)|^2 \mathrm{d}u,$$

uniformly for primes q satisfying $x^{1/B} \leq q \leq x$.

(2) *For every band-limited Schwartz function f, as $q \to \infty$ through primes,*

$$K(f,\alpha,q) \sim \frac{\log q}{2\pi} \int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t,$$

uniformly for $1 \leq |\alpha| \leq B$.

It should be noted that the Goldston–Montgomery correspondence mentioned earlier was established under RH, but our Theorem 3 does not rely on any unproved hypotheses.

Theorem 4. *Assume GRH and the following smooth Hardy–Littlewood conjecture. For every $W \in C_c^\infty(0,\infty)$, extend W by zero to $\mathbb{R}$, put*

$$A_W(t) = \int_{\mathbb{R}} W(u)\overline{W(u+t)}\mathrm{d}u, \tag{1.12}$$

and, for $k \neq 0$, let

$$\mathfrak{S}(k) = \begin{cases} 2\displaystyle\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid k\\ p>2}}\frac{p-1}{p-2}, & k \text{ even},\\ 0, & k \text{ odd}. \end{cases}$$

The conjecture asserts that, for every $\eta > 0$, as $x \to \infty$ and uniformly for nonzero integers k with $|k| \ll x$,

$$\sum_n \Lambda(n)\Lambda(n+k)W\left(\frac{n}{x}\right)\overline{W\left(\frac{n+k}{x}\right)} = \mathfrak{S}(k)xA_W\left(\frac{k}{x}\right) + O\left(x^{\frac{1}{2}+\eta}\right). \tag{1.13}$$

Then, for every $\varepsilon > 0$, the asymptotic in Conjecture 1 holds uniformly for primes in the range $x^{\frac{1}{2}+\varepsilon} \le q \le x$. More precisely, uniformly for primes q satisfying $x^{\frac{1}{2}+\varepsilon} \le q \ll x$,

$$G_W(x,q) = x\log q\int_0^\infty |W(u)|^2\mathrm{d}u + O(x). \tag{1.14}$$

By Theorem 3, Theorem 4 has the following immediate consequence.

Corollary 3 (Hardy–Littlewood extension)**.** *Under the hypotheses of Theorem 4, Conjecture 2 holds in the range $1 \le |\alpha| < 2$, uniformly on compact subranges.*

Combining Theorem 2 with Corollary 3, we obtain the following conditional extensions of Corollary 1 and Corollary 2.

Corollary 4. *Under the hypotheses of Theorem 4, for every band-limited $\phi \in L^1(\mathbb{R})$ with $B_\phi < 2$,*

$$R(f,\phi,q) \sim \frac{\log q}{2\pi}\left(\int_{\mathbb{R}}|f(t)|^2\mathrm{d}t\right)\left(\widehat{\phi}(0) + \int_{\mathbb{R}}\min(|\alpha|,1)\widehat{\phi}(\alpha)\mathrm{d}\alpha\right),$$

as $q \to \infty$ through odd primes.

Corollary 5. *Under the hypotheses of Theorem 4, for every nonnegative band-limited Schwartz function f,*

$$\sum_\chi{}^{*}\sum_{\substack{\gamma_\chi\\ \rho_\chi \text{ simple}}}|f(\gamma_\chi)|^2 \ge \left(\frac{11}{12} - o_f(1)\right)\sum_\chi{}^{*}\sum_{\gamma_\chi}|f(\gamma_\chi)|^2. \tag{1.15}$$

The setup considered here is not the only q-analog of Montgomery's pair correlation for Dirichlet L-functions. Among previous works, Chandee–Lee–Liu–Radziwiłł [2] is closest to our zero-side family: they studied correlations for primitive Dirichlet L-functions with conductors $q \asymp Q$. By contrast, we fix a single prime conductor q and average only over the primitive characters modulo q.

Earlier, Özlük [27, 28] considered a related modulus-averaged statistic over all characters modulo $q \le Q$. In both Özlük's work and that of Chandee–Lee–Liu–Radziwiłł, the additional averaging over the modulus allows one to exploit Hooley's asymptotic evaluation of the Barban–Davenport–Halberstam variance $H(x,Q)$ in the range $x^{\frac{1}{2}+\varepsilon} < Q \le x$ under GRH, yielding a conditional form-factor result in the wider range $|\alpha| < 2$ and the lower bound $\frac{11}{12}$ for the proportion of simple zeros in the corresponding ranges. In our fixed-modulus setting,

GRH alone only gives the lower bound $\frac{2}{3}$, while the form factor result for $1 \le |\alpha| < 2$ and the ratio $\frac{11}{12}$ require the Hardy–Littlewood conjecture. This extra input is not needed in the modulus-averaged setting because the prime-correlation sum there can be rearranged into forms suitable for the circle method [24, p. 38] [28], the prime number theorem for arithmetic progressions [13, p. 210], or the asymptotic large sieve [3] [2, §3.4].

There are other versions of pair correlation conjectures formulated to understand other aspects of primes. Yıldırım (1991) [31] formulated a variant of pair correlation conjecture for Dirichlet L-functions related to primes in a fixed arithmetic progression in short intervals (a generalization of $J(X, H)$). Kandhil–Languasco–Moree (2026) [16] studied another variant related to the Elliott–Halberstam conjecture.

The paper is organized as follows. In section 4, we develop the technical results required for the proofs. In section 5, we prove Theorem 1. In section 6, we prove Theorem 2. In section 7, we establish the equivalence between Conjecture 2 and Hooley's conjecture in Theorem 3. In section 8, we prove Theorem 4, from which Corollary 3 follows. In section 9, we prove Corollary 1 and Corollary 4. In section 10, we prove Corollary 2 and Corollary 5. Finally, in section 11, we give a more technical comparison with related works and discuss the function field side of the subject.

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3. AI disclosure

Generative AI tools were used to assist with checking mathematical derivations and revising the exposition. The author takes full responsibility for the mathematical accuracy of the content.

4. Technical preliminaries

4.1. **The functional equation and zero symmetry.** For a primitive character χ modulo q, set

$$\kappa_\chi = \frac{1 - \chi(-1)}{2}$$

and define the completed L-function

$$\Lambda(s, \chi) = \left(\frac{q}{\pi}\right)^{\frac{s+\kappa_\chi}{2}} \Gamma\left(\frac{s + \kappa_\chi}{2}\right) L(s, \chi).$$

Its functional equation [4, §9] is

$$\Lambda(s, \chi) = \varepsilon_\chi \Lambda(1 - s, \overline{\chi}), \qquad |\varepsilon_\chi| = 1.$$

Therefore, with zeros counted with multiplicity, we have

$$\{\gamma_{\overline{\chi}}\} = \{-\gamma_\chi\}. \tag{4.1}$$

4.2. **The explicit formula.** Classically, one connects L-zeros to primes ad hoc by contour integration. The explicit formula packages this contour-integration step into a reusable identity. The following version is the appropriately renormalized specialization of Proposition 2.1 of Rudnick–Sarnak (1996) [29] to $L(s,\chi)$.

Lemma 1 (Explicit formula)**.** *Let h be a band-limited Schwartz function and let χ be primitive modulo q. Then*

$$\sum_{\gamma_\chi} h\left(\frac{\gamma_\chi}{2\pi}\right) = \widehat{h}(0)\log\frac{q}{\pi} - \sum_n \frac{\Lambda(n)}{\sqrt{n}}\left\{\chi(n)\widehat{h}(\log n) + \overline{\chi}(n)\widehat{h}(-\log n)\right\} + \int_{\mathbb{R}} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1+2\kappa_\chi}{4} + i\pi u\right)\mathrm{d}u.$$

4.3. **Fourier integrals.** For $W \in C_c^\infty(0,\infty)$, define its Mellin transform by

$$\widetilde{W}(s) = \int_0^\infty W(u)u^{s-1}\mathrm{d}u.$$

The formulas

$$f(t) = \widetilde{W}\left(\frac{1}{2} + it\right), \qquad W(u) = \frac{u^{-1/2}}{2\pi}\widehat{f}\left(\frac{\log u}{2\pi}\right) \tag{4.2}$$

give bijective correspondences between $W \in C_c^\infty(0,\infty)$ and band-limited Schwartz functions f. Thus either function uniquely determines the other; whenever f and W occur together below, they are related by (4.2) and are not independent choices. Plancherel and the second formula in (4.2), respectively, give

$$\int_0^\infty |W(u)|^2\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{R}} |f(t)|^2\mathrm{d}t, \qquad \frac{|W(1/x)|^2}{x} = \frac{1}{4\pi^2}\left|\widehat{f}\left(-\frac{\log x}{2\pi}\right)\right|^2. \tag{4.3}$$

5. Form-factor–variance relation

For $W \in C_c^\infty(0,\infty)$, put

$$\psi_W(x,\chi) = \sum_n \Lambda(n)\chi(n)W\left(\frac{n}{x}\right).$$

Character orthogonality gives

$$G_W(x,q) = \frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0} |\psi_W(x,\chi)|^2. \tag{5.1}$$

Moreover, when q is prime, $\sum_{\chi\neq\chi_0}$ sums over primitive characters modulo q.

Proof. For $x \geq 1$, specialize h in Lemma 1 by

$$h(u) = x^{2\pi i u}\widetilde{W}\left(\frac{1}{2} + 2\pi i u\right).$$

A direct calculation gives

$$\widehat{h}(\xi) = x^{-1/2} e^{\xi/2} W\left(\frac{e^{\xi}}{x}\right).$$

Consequently, Lemma 1 becomes

$$\begin{aligned}\sum_{\gamma_\chi} x^{i\gamma_\chi} \widetilde{W}\left(\frac{1}{2}+i\gamma_\chi\right) &= x^{-1/2} W\left(\frac{1}{x}\right) \log\frac{q}{\pi} \\ &\quad - x^{-1/2} \sum_n \Lambda(n)\chi(n) W\left(\frac{n}{x}\right) \\ &\quad - x^{-1/2} \sum_n \frac{\Lambda(n)\overline{\chi}(n)}{n} W\left(\frac{1}{nx}\right) \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}} x^{it} \widetilde{W}\left(\frac{1}{2}+it\right) \operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1+2\kappa_\chi}{4}+\frac{it}{2}\right) \mathrm{d}t.\end{aligned} \tag{5.2}$$

Since W is supported in a compact set away from zero, the third term on the right-hand side of (5.2) vanishes for $x \gg 1$ and is bounded for $1 \le x \ll 1$. The standard digamma estimates give

$$\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1+2\kappa}{4}+\frac{it}{2}\right) \ll \log(2+|t|),$$

uniformly for $\kappa \in \{0,1\}$. Since $\widetilde{W}(1/2+it)$ decays faster than any power and $|x^{it}|=1$, the integral in (5.2) is $O(1)$ uniformly for $x \ge 1$ and $\kappa_\chi \in \{0,1\}$. Therefore

$$\sum_{\gamma_\chi} \widetilde{W}\left(\frac{1}{2}+i\gamma_\chi\right) x^{i\gamma_\chi} = x^{-1/2} W\left(\frac{1}{x}\right) \log q - x^{-1/2}\psi_W(x,\chi) + O(1), \tag{5.3}$$

uniformly for $x \ge 1$ and primitive characters χ modulo q.

By (4.2), the left-hand side of (5.3) is the f-factor in (1.6). The same formula also evaluates the $f^{\#}$-factor after a change of variables using the zero symmetry (4.1):

$$\sum_{\gamma_\chi} f^{\#}(\gamma_\chi) x^{-i\gamma_\chi} = \sum_{\gamma_{\overline{\chi}}} g(\gamma_{\overline{\chi}}) x^{i\gamma_{\overline{\chi}}}, \qquad g(z) = f^{\#}(-z).$$

On the real axis, $g(t) = \overline{f(-t)}$; hence $\widehat{g}(u) = \overline{\widehat{f}(u)}$. The multiplicative weight corresponding to g under (4.2) is therefore $\overline{W}$. Applying (5.3) to g and $\overline{\chi}$, and using the reality of $\Lambda(n)$, gives

$$\begin{aligned}\sum_{\gamma_\chi} f^{\#}(\gamma_\chi) x^{-i\gamma_\chi} &= x^{-1/2}\overline{W(1/x)} \log q - x^{-1/2}\psi_{\overline{W}}(x,\overline{\chi}) + O(1) \\ &= x^{-1/2}\overline{W(1/x)} \log q - x^{-1/2}\overline{\psi_W(x,\chi)} + O(1).\end{aligned}$$

Multiplying the two factors gives

$$\begin{aligned}&\left(\sum_{\gamma_\chi} f(\gamma_\chi)x^{i\gamma_\chi}\right)\left(\sum_{\gamma'_\chi} f^{\#}(\gamma'_\chi)x^{-i\gamma'_\chi}\right)\\ &= \frac{|\psi_W(x,\chi)|^2}{x} + \frac{|W(1/x)|^2}{x}(\log q)^2 - \frac{2\log q}{x}\operatorname{Re}\left(W\left(\frac{1}{x}\right)\overline{\psi_W(x,\chi)}\right)\\ &\quad + O\left(1 + \frac{|W(1/x)|\log q}{\sqrt{x}} + \frac{|\psi_W(x,\chi)|}{\sqrt{x}}\right).\end{aligned} \tag{5.4}$$

It remains to average this expansion over χ. By Cauchy–Schwarz and (5.1), one has

$$\begin{aligned}\frac{1}{q-2}\sideset{}{^*}\sum_{\chi}\frac{|\psi_W(x,\chi)|}{\sqrt{x}} &\le \left(\frac{1}{q-2}\sideset{}{^*}\sum_{\chi}\frac{|\psi_W(x,\chi)|^2}{x}\right)^{1/2}\\ &= \left(\frac{q-1}{q-2}\frac{G_W(x,q)}{x}\right)^{1/2}.\end{aligned}$$

Thus, averaging (5.4) over primitive characters yields

$$\begin{aligned}K(f,\alpha,q) &= \frac{q-1}{q-2}\frac{G_W(x,q)}{x} + \frac{|W(1/x)|^2}{x}(\log q)^2\\ &\quad - \frac{2\log q}{x}\operatorname{Re}\left(W\left(\frac{1}{x}\right)\frac{1}{q-2}\sideset{}{^*}\sum_{\chi}\overline{\psi_W(x,\chi)}\right)\\ &\quad + O\left(1 + \frac{|W(1/x)|\log q}{\sqrt{x}} + \left(\frac{G_W(x,q)}{x}\right)^{1/2}\right).\end{aligned} \tag{5.5}$$

Here the factor $(q-1)/(q-2)$ records the different normalizations in (5.1) and (1.6). The Cauchy–Schwarz argument above shows that the third term on the right hand side of (5.5) contributes

$$\ll \frac{|W(1/x)|\log q}{\sqrt{x}}\left(\frac{G_W(x,q)}{x}\right)^{1/2}.$$

Combining this with the error already present in (5.5), the total contribution of the errors is

$$\ll \left(1 + \frac{|W(1/x)|\log q}{\sqrt{x}}\right)\left(1 + \left(\frac{G_W(x,q)}{x}\right)^{1/2}\right),$$

which is the error stated in the theorem. □

6. The ramp theorem

We attack the range $|\alpha| < 1$ from the prime side by first establishing a smoothed (1.3).

Lemma 2 (Smooth Friedlander–Goldston)**.** *For every $W \in C_c^\infty(0,\infty)$,*

$$G_W(x,q) = \left(\int_0^\infty |W(u)|^2 \mathrm{d}u\right) x\log x + O(x),$$

uniformly for $x \ge 1$ and primes $q \gg x$.

Proof. Expanding (1.4) gives

$$G_W(x,q) = \sum_{\substack{m\equiv n(q)\\(mn,q)=1}} \Lambda(m)\Lambda(n) W\left(\frac{m}{x}\right)\overline{W\left(\frac{n}{x}\right)} - \frac{1}{\varphi(q)}\left|\sum_{(n,q)=1}\Lambda(n)W\left(\frac{n}{x}\right)\right|^2.$$

If $q \gg x$, the support of W is contained in $(0, q/x)$, so $m \equiv n \pmod q$ forces $m = n$. Since q is prime, the coprimality conditions may then be removed, and

$$G_W(x,q) = \sum_n \Lambda(n)^2\left|W\left(\frac{n}{x}\right)\right|^2 - \frac{1}{q-1}\left|\sum_n \Lambda(n)W\left(\frac{n}{x}\right)\right|^2. \tag{6.1}$$

The second term is $O(x)$ because $q \gg x$ and $\sum_{n\ll x}\Lambda(n) \ll x$.

To evaluate the first term in (6.1), put $V = |W|^2$. Since $\Lambda(n)^2$ and $\Lambda(n)\log n$ agree at primes, we have

$$\left|\sum_n \left(\Lambda(n)^2 - \Lambda(n)\log n\right) V\left(\frac{n}{x}\right)\right| = \left|\sum_{\substack{p^k\ll x\\k\geq 2}} (k-1)(\log p)^2 V\left(\frac{p^k}{x}\right)\right|$$
$$\ll \sum_{\substack{p^k\ll x\\k\geq 2}} (k-1)(\log p)^2 \ll \log(2x)\sum_{\substack{p^k\ll x\\k\geq 2}}\log p \ll x^{1/2}\log(2x).$$

By Mellin inversion, one has

$$\sum_n \Lambda(n)\log n\, V\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{(2)}\left(\frac{\zeta'}{\zeta}\right)'(s)\widetilde{V}(s)x^s \mathrm{d}s. \tag{6.2}$$

Using the classical de la Vallée Poussin zero-free region, we can shift the contour slightly past the double pole at $s = 1$, eventually arriving at

$$\sum_n \Lambda(n)\log n\, V\left(\frac{n}{x}\right) = x\widetilde{V}(1)\log x + x\widetilde{V}'(1) + O\left(xe^{-c_0\sqrt{\log(2x)}}\right),$$

where $c_0 > 0$ is an absolute constant.

Since $\widetilde{V}(1) = \int_0^\infty |W(u)|^2\mathrm{d}u$ and $\widetilde{V}'(1) = O(1)$, combining this asymptotic with the $O(x^{1/2}\log(2x))$ error from replacing $\Lambda(n)^2$ by $\Lambda(n)\log n$ evaluates the first term in (6.1) up to $O(x)$. Together with the $O(x)$ bound for the second term, this proves the lemma. □

Remark 3. The key point is the implication $m \equiv n \pmod q \Rightarrow m = n$ for $m, n < q$. With $x = q^\alpha$, this implication holds in compact subranges of $0 \leq \alpha < 1$ and fails beyond $\alpha = 1$, identifying the arithmetic origin of the ramp–plateau transition.

Proof of Theorem 2. Fix $\eta > 0$, suppose $0 \le \alpha \le 1-\eta$, and put $x = q^{\alpha}$. Therefore, for $q \gg_{\eta} 1$, one has $q \gg x$. Substitution of Lemma 2 into Theorem 1 gives

$$K(f,\alpha,q) = \left(\int_0^{\infty} |W(u)|^2 \mathrm{d}u\right) \log x + \frac{|W(1/x)|^2}{x}(\log q)^2 + O\left(\left(1 + \frac{|W(1/x)|\log q}{\sqrt{x}}\right)\sqrt{\log(2x)}\right). \tag{6.3}$$

Using $\log x = \alpha \log q$ and the two identities in (4.3) proves the asserted formula.

For $-1+\eta \le \alpha < 0$, define

$$W_-(u) = u^{-1}W(1/u), \qquad \widetilde{W}_-(s) = \widetilde{W}(1-s), \qquad f_-(t) = f(-t).$$

By (4.1), replacing each zero ordinate of $L(s,\chi)$ by the negative of the corresponding ordinate of $L(s,\overline{\chi})$ in (1.6) gives

$$\begin{aligned} K(f,\alpha,q) &= \frac{1}{q-2}\sideset{}{^*}\sum_{\chi}\left(\sum_{\gamma_{\overline{\chi}}} f(-\gamma_{\overline{\chi}})q^{-i\alpha\gamma_{\overline{\chi}}}\right)\left(\sum_{\gamma'_{\overline{\chi}}} f^{\#}(-\gamma'_{\overline{\chi}})q^{i\alpha\gamma'_{\overline{\chi}}}\right) \\ &= \frac{1}{q-2}\sideset{}{^*}\sum_{\chi}\left(\sum_{\gamma_{\chi}} f_-(\gamma_{\chi})q^{-i\alpha\gamma_{\chi}}\right)\left(\sum_{\gamma'_{\chi}} (f_-)^{\#}(\gamma'_{\chi})q^{i\alpha\gamma'_{\chi}}\right) \\ &= K(f_-,-\alpha,q). \end{aligned} \tag{6.4}$$

Moreover, a change of variables gives

$$\int_{\mathbb{R}} |f_-(t)|^2 \mathrm{d}t = \int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t, \qquad \widehat{f_-}(\xi) = \widehat{f}(-\xi).$$

Applying the positive-α result to $K(f_-,-\alpha,q)$ completes the proof. □

7. The plateau and Hooley's conjecture

Proof of Theorem 3. Fix $B \ge 1$. Suppose first that the prime-side asymptotic holds. Fix a band-limited Schwartz function f, let W be its unique partner, and put $x = q^{\alpha}$ with $1 \le \alpha \le B$. Then $x^{1/B} \le q \le x$, so Theorem 1 gives

$$K(f,\alpha,q) = \left(\int_0^{\infty} |W(u)|^2 \mathrm{d}u\right)\log q + \frac{q-1}{q-2}o_W(\log q) + \frac{|W(1/x)|^2}{x}(\log q)^2 + O\left(\left(1 + \frac{|W(1/x)|\log q}{\sqrt{x}}\right)\sqrt{\log q}\right), \tag{7.1}$$

uniformly in this range. By Remark 1 and (4.3), the second main term vanishes and the error simplifies to $O(\sqrt{\log q})$. The first identity in (4.3) therefore turns (7.1) into the positive-α plateau. Finally, (6.4) extends this conclusion to $1 \le |\alpha| \le B$.

Conversely, assume the form-factor asymptotic. Fix $W \in C_c^{\infty}(0,\infty)$, and let f be its unique partner. Given $x^{1/B} \le q \le x$, put $\alpha = \log x/\log q$, so that $1 \le \alpha \le B$. By Remark 1 and

(4.3), Theorem 1 reduces to

$$K(f,\alpha,q)=\frac{q-1}{q-2}\frac{G_W(x,q)}{x}+O\left(1+\left(\frac{G_W(x,q)}{x}\right)^{1/2}\right). \tag{7.2}$$

Put $Y=G_W(x,q)x^{-1}$. Then (7.2) and the assumed form-factor asymptotic imply

$$Y\ll \log q+Y^{\frac12},$$

which implies $Y\ll\log q$, so the error in (7.2) is $O(\sqrt{\log q})$, which is negligible compared with the main term. The first identity in (4.3) now recovers the prime-side asymptotic in Theorem 3, uniformly for $x^{1/B}\le q\le x$. □

8. The Hardy–Littlewood extension

We now use the smooth Hardy–Littlewood conjecture stated in Theorem 4 to establish an asymptotic for $G_W(x,q)$ in the range $x^{\frac12+\varepsilon}\le q\ll x$. The extension to $1\le|\alpha|<2$ then follows from Theorem 3. The derivation in this section is a smooth adaptation of arguments in §2 and §4 of Friedlander–Goldston [8].

Lemma 3. *For $W\in C_c^\infty(0,\infty)$, put*

$$A_W(t)=\int_{\mathbb{R}}W(u)\overline{W(u+t)}\mathrm{d}u.$$

Uniformly for odd primes q with $q\ll x$,

$$\sum_{j\neq0}\mathfrak{S}(jq)A_W\left(\frac{jq}{x}\right)=\frac{x}{q-1}|\widetilde{W}(1)|^2-\left(\int_0^\infty|W(u)|^2\mathrm{d}u\right)\log\frac{x}{q}+O(1). \tag{8.1}$$

Proof. Put

$$h_W(t)=A_W(t)+A_W(-t)\qquad(t\ge0).$$

Since $\mathfrak{S}(k)=\mathfrak{S}(-k)$, Mellin inversion gives

$$\sum_{j\neq0}\mathfrak{S}(jq)A_W\left(\frac{jq}{x}\right)=\frac{1}{2\pi i}\int_{(2)}\left(\frac{x}{q}\right)^s D_q(s)\widetilde{h}_W(s)\mathrm{d}s, \tag{8.2}$$

where

$$D_q(s)=\sum_{j\ge1}\frac{\mathfrak{S}(jq)}{j^s}=\mathfrak{S}(2q)\zeta(s)\zeta(s+1)F_q(s),$$

$$F_q(s)=2^{-s}\left(1-2^{-s-1}\right)\left(1-q^{-s-1}\right)\prod_{\substack{p>2\\p\neq q}}\left(1+\frac{1}{(p-2)p^s}\right)\left(1-p^{-s-1}\right). \tag{8.3}$$

The product defining $F_q(s)$ converges absolutely for $\operatorname{Re}s>-\frac12$ and satisfies $F_q(s)\ll1$ uniformly over odd primes q and s with $-\frac14\le\operatorname{Re}s\le2$.

Shifting the line of integration in (8.2) to $\operatorname{Re} s = -\frac{1}{4}$, we have

$$\sum_{j\neq 0} \mathfrak{S}(jq) A_W\left(\frac{jq}{x}\right) = \operatorname*{Res}_{s=1}\left(\left(\frac{x}{q}\right)^s D_q(s)\widetilde{h}_W(s)\right) + \operatorname*{Res}_{s=0}\left(\left(\frac{x}{q}\right)^s D_q(s)\widetilde{h}_W(s)\right) + \mathcal{E}_W(x,q), \tag{8.4}$$

where

$$\mathcal{E}_W(x,q) = \frac{1}{2\pi i}\int_{(-\frac{1}{4})}\left(\frac{x}{q}\right)^s D_q(s)\widetilde{h}_W(s)\mathrm{d}s. \tag{8.5}$$

As $s\to 1$, one has $\zeta(s)\sim (s-1)^{-1}$ and hence $(s-1)D_q(s)\to q/(q-1)$. Since

$$\widetilde{h}_W(1) = \int_{\mathbb{R}} A_W(t)\mathrm{d}t = |\widetilde{W}(1)|^2,$$

the residue at $s=1$ is

$$\operatorname*{Res}_{s=1}\left(\left(\frac{x}{q}\right)^s D_q(s)\widetilde{h}_W(s)\right) = \frac{x}{q-1}|\widetilde{W}(1)|^2. \tag{8.6}$$

For the residue at zero, logarithmic differentiation of the $F_q(s)$ in (8.3) gives

$$\begin{aligned}\frac{F_q'(s)}{F_q(s)} &= -\log 2 + \frac{2^{-s-1}\log 2}{1-2^{-s-1}} + \frac{q^{-s-1}\log q}{1-q^{-s-1}} \\ &\quad + \sum_{\substack{p>2\\ p\neq q}}\left(-\frac{((p-2)p^s)^{-1}\log p}{1+((p-2)p^s)^{-1}} + \frac{p^{-s-1}\log p}{1-p^{-s-1}}\right), \\ \frac{F_q'(0)}{F_q(0)} &= -\log 2 + \log 2 + \frac{\log q}{q-1} + \sum_{\substack{p>2\\ p\neq q}}\left(-\frac{\log p}{p-1} + \frac{\log p}{p-1}\right) = \frac{\log q}{q-1}.\end{aligned}$$

Together with $\mathfrak{S}(2q)F_q(0)=1$, this yields

$$\mathfrak{S}(2q)F_q(s) = 1 + \frac{\log q}{q-1}s + O_q(s^2).$$

Since

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}(\log 2\pi)s + O(s^2), \qquad \zeta(s+1) = \frac{1}{s} + \gamma + O(s),$$

we obtain the explicit Laurent expansion

$$D_q(s) = -\frac{1}{2s} - \frac{1}{2}\left(\gamma + \log 2\pi + \frac{\log q}{q-1}\right) + O_q(s). \tag{8.7}$$

Since $h_W(0) = 2\int_0^\infty |W(u)|^2\mathrm{d}u$, the meromorphic continuation of its Mellin transform satisfies

$$\widetilde{h}_W(s) = \frac{2}{s}\int_0^\infty |W(u)|^2\mathrm{d}u + \frac{\mathrm{d}}{\mathrm{d}s}\left(s\widetilde{h}_W(s)\right)\Big|_{s=0} + O(s). \tag{8.8}$$

Combining (8.7) and (8.8) with $(\frac{x}{q})^s = 1 + s\log(\frac{x}{q}) + O(s^2)$ shows that the residue at $s=0$ is

$$\begin{aligned} &-\left(\int_0^\infty |W(u)|^2\mathrm{d}u\right)\log\frac{x}{q} - \left(\int_0^\infty |W(u)|^2\mathrm{d}u\right)\left(\gamma + \log 2\pi + \frac{\log q}{q-1}\right) \\ &\quad - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(s\widetilde{h}_W(s)\right)\Big|_{s=0}. \end{aligned} \tag{8.9}$$

The last two terms in (8.9) are $O(1)$ uniformly in q. Finally, the standard polynomial bounds for $\zeta(s)$, the uniform bound for $F_q(s)$ above, and the rapid vertical decay of $\widetilde{h}_W(s)$ give

$$\mathcal{E}_W(x,q) \ll \left(\frac{x}{q}\right)^{-\frac{1}{4}} = \left(\frac{q}{x}\right)^{\frac{1}{4}} \ll 1.$$

Inserting this estimate and the two residue evaluations into (8.4) proves (8.1). □

Proof of Theorem 4. We may suppose that x is sufficiently large, so that q is odd. Since the inner n-sum is nonzero only when $|j| \ll x/q$, summing (1.13) in these ranges and applying Lemma 3 gives

$$\begin{aligned} &\sum_{j\neq 0}\sum_n \Lambda(n)\Lambda(n+jq)W\left(\frac{n}{x}\right)\overline{W\left(\frac{n+jq}{x}\right)} \\ &\quad = x\sum_{j\neq 0}\mathfrak{S}(jq)A_W\left(\frac{jq}{x}\right) + O\left(x^{\frac{3}{2}+\eta}q^{-1}\right) \\ &\quad = \frac{x^2}{q-1}|\widetilde{W}(1)|^2 - x\left(\int_0^\infty |W(u)|^2\mathrm{d}u\right)\log\frac{x}{q} + O\left(x + x^{\frac{3}{2}+\eta}q^{-1}\right). \end{aligned} \tag{8.10}$$

Opening (1.4) and using orthogonality of residue classes gives

$$\begin{aligned} G_W(x,q) = &\sum_{(n,q)=1}\Lambda(n)^2\left|W\left(\frac{n}{x}\right)\right|^2 \\ &+ \sum_{j\neq 0}\sum_n \Lambda(n)\Lambda(n+jq)W\left(\frac{n}{x}\right)\overline{W\left(\frac{n+jq}{x}\right)} \\ &- \frac{1}{q-1}\left|\sum_{(n,q)=1}\Lambda(n)W\left(\frac{n}{x}\right)\right|^2 + O\left((\log(2x))^2\right), \end{aligned} \tag{8.11}$$

where the error comes from dropping the coprimality condition in the sum over $j \neq 0$.

The diagonal calculation in the proof of Lemma 2 gives

$$\sum_{(n,q)=1}\Lambda(n)^2\left|W\left(\frac{n}{x}\right)\right|^2 = x\log x\int_0^\infty |W(u)|^2\mathrm{d}u + O(x). \tag{8.12}$$

Contour integration and GRH also give

$$\sum_{(n,q)=1}\Lambda(n)W\left(\frac{n}{x}\right) = \sum_n \Lambda(n)W\left(\frac{n}{x}\right) + O(\log(2x)) = x\widetilde{W}(1) + O\left(x^{\frac{1}{2}+\eta}\right).$$

After squaring and dividing by $q-1$, this becomes

$$\frac{1}{q-1}\left|\sum_{(n,q)=1}\Lambda(n)W\left(\frac{n}{x}\right)\right|^2=\frac{x^2}{q-1}|\widetilde{W}(1)|^2+O\left(x^{\frac{3}{2}+\eta}q^{-1}\right). \tag{8.13}$$

Substituting (8.10), (8.12), and (8.13) into (8.11), we obtain

$$\begin{aligned}G_W(x,q)&=x\left(\int_0^\infty|W(u)|^2\mathrm{d}u\right)\left(\log x-\log\frac{x}{q}\right)+O\left(x+x^{\frac{3}{2}+\eta}q^{-1}\right)\\&=x\log q\int_0^\infty|W(u)|^2\mathrm{d}u+O\left(x+x^{\frac{3}{2}+\eta}q^{-1}\right).\end{aligned}$$

Taking $\eta=\varepsilon$ and using $q\geq x^{\frac{1}{2}+\varepsilon}$ proves (1.14). □

Remark 4. Combining Theorem 4 with Lemma 2 gives, for every $\varepsilon>0$,

$$G_W(x,q)=x\min(\log x,\log q)\int_0^\infty|W(u)|^2\mathrm{d}u+O(x), \tag{8.14}$$

as $x\to\infty$, uniformly for primes q satisfying $q\geq x^{\frac{1}{2}+\varepsilon}$.

9. Pair correlation

Proof of Corollary 1. Start with the unconditional identity (1.8). Choose σ with $B_\phi<\sigma<1$; then Theorem 2 is uniform on the support of $\widehat{\phi}$. The first ramp term contributes

$$\frac{\log q}{2\pi}\left(\int_{\mathbb{R}}|f(t)|^2\mathrm{d}t\right)\int_{\mathbb{R}}|\alpha|\widehat{\phi}(\alpha)\mathrm{d}\alpha.$$

Changing variables by $u=-\alpha\log q/(2\pi)$ and applying dominated convergence and Plancherel, we find that the second term of Theorem 2 contributes

$$\begin{aligned}&\frac{(\log q)^2}{4\pi^2}\int_{\mathbb{R}}\widehat{\phi}(\alpha)\left|\widehat{f}\left(-\frac{\alpha\log q}{2\pi}\right)\right|^2\mathrm{d}\alpha\\&\quad=\frac{\log q}{2\pi}\int_{\mathbb{R}}\widehat{\phi}\left(-\frac{2\pi u}{\log q}\right)|\widehat{f}(u)|^2\mathrm{d}u\\&\quad\sim\frac{\log q}{2\pi}\widehat{\phi}(0)\int_{\mathbb{R}}|f(t)|^2\mathrm{d}t.\end{aligned}$$

The contributions coming from the O-terms are

$$\int_{\mathbb{R}}|\widehat{\phi}(\alpha)|\sqrt{\log(2q^{|\alpha|})}\mathrm{d}\alpha\ll_\phi\sqrt{\log q},$$

$$\begin{aligned}&\log q\int_{\mathbb{R}}|\widehat{\phi}(\alpha)|\left|\widehat{f}\left(-\frac{\alpha\log q}{2\pi}\right)\right|\sqrt{\log(2q^{|\alpha|})}\mathrm{d}\alpha\\&\quad=2\pi\int_{\mathbb{R}}\left|\widehat{\phi}\left(-\frac{2\pi u}{\log q}\right)\right||\widehat{f}(u)|\sqrt{\log 2+2\pi|u|}\mathrm{d}u\ll_\phi 1.\end{aligned}$$

Both of these are $o(\log q)$, proving the corollary. □

Proof of Corollary 4. Combining (8.14) with Theorem 1, (6.4), and (4.3), and using the observation in Remark 1 when α is away from zero, we obtain, uniformly on compact subsets of $0 < |\alpha| < 2$,

$$K(f, \alpha, q) \sim \frac{\log q}{2\pi} \min(|\alpha|, 1) \int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t.$$

Now choose B with $B_\phi < B < 2$. By a smooth partition of unity on the Fourier side, write $\phi = \phi_0 + \phi_1$, where $B_{\phi_0} < 1$ and, for some $\sigma > 0$,

$$\operatorname{supp} \widehat{\phi_1} \subset [-B, -\sigma] \cup [\sigma, B].$$

The contribution of ϕ_0 is given by Corollary 1, while substitution of the uniform asymptotic into (1.8) gives the asserted contribution of ϕ_1. Recombining the two contributions proves the asserted asymptotic. The identity

$$\widehat{\phi}(0) + \int_{\mathbb{R}} \min(|\alpha|, 1) \widehat{\phi}(\alpha) \mathrm{d}\alpha = \phi(0) + \int_{\mathbb{R}} \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) \phi(u) \mathrm{d}u$$

shows that it is equivalent to (1.10). □

10. Simplicity of L-zeros

Proofs of Corollary 2 and Corollary 5. Assume GRH and fix a nonnegative band-limited Schwartz function f. If ρ_χ is a zero, let m_{ρ_χ} denote its multiplicity, and put

$$M_f(q) = \frac{1}{q-2} \sideset{}{^*}\sum_{\chi} \sum_{\gamma_\chi} |f(\gamma_\chi)|^2,$$

$$S_f(q) = \frac{1}{q-2} \sideset{}{^*}\sum_{\chi} \sum_{\substack{\rho_\chi \\ \gamma_\chi \text{ simple}}} |f(\gamma_\chi)|^2.$$

For $0 < \beta < 2$, let

$$\Phi_\beta(u) = \left(\frac{\sin \pi\beta u}{\pi\beta u}\right)^2, \qquad \widehat{\Phi_\beta}(\alpha) = \max\left(0, \frac{\beta - |\alpha|}{\beta^2}\right). \tag{10.1}$$

Applying Lemma 1 to $|f|^2$ gives

$$M_f(q) = \frac{\log q}{2\pi} \int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t + O(1). \tag{10.2}$$

Since Φ_β is nonnegative and $\Phi_\beta(0) = 1$, positivity under GRH implies

$$\frac{1}{q-2} \sideset{}{^*}\sum_{\chi} \sum_{\gamma_\chi} |f(\gamma_\chi)|^2 m_{\rho_\chi} \le R(f, \Phi_\beta, q).$$

The inequality $\mathbf{1}_{m=1} \ge 2 - m$ for every positive integer m therefore gives

$$S_f(q) \ge 2M_f(q) - R(f, \Phi_\beta, q). \tag{10.3}$$

For $0 < \beta < 1$, Corollary 1 and (10.2) give

$$R(f, \Phi_\beta, q) \sim \left(\frac{1}{\beta} + \frac{\beta}{3}\right) M_f(q),$$

because

$$\widehat{\Phi_\beta}(0) + \int_{\mathbb{R}} |\alpha| \widehat{\Phi_\beta}(\alpha) \mathrm{d}\alpha = \frac{1}{\beta} + \frac{\beta}{3}.$$

It follows from (10.3) that $S_f(q) \geq (2 - \frac{1}{\beta} - \frac{\beta}{3} - o_{f,\beta}(1))M_f(q)$. Letting $\beta \to 1^-$ proves Corollary 2.

Under the additional smooth Hardy–Littlewood conjecture in Theorem 4, Corollary 4 applies to Φ_β when $B_{\Phi_\beta} = \beta \in (1, 2)$. In this case,

$$\widehat{\Phi_\beta}(0) + \int_{\mathbb{R}} \min(|\alpha|, 1) \widehat{\Phi_\beta}(\alpha) \mathrm{d}\alpha = 1 + \frac{1}{3\beta^2},$$

so (10.3) gives $S_f(q) \geq (1 - \frac{1}{3\beta^2} - o_{f,\beta}(1))M_f(q)$. Letting $\beta \to 2^-$ proves Corollary 5. □

11. Comparison with related work

Our study may be seen as a q-analog of Montgomery's [25] 1973 investigation of the pair correlation of ζ-zeros. He considered the pair correlation of a single $\zeta(s)$. We compute the pair correlation of zeros of a single $L(s, \chi)$ and then average over primitive χ modulo q. His results concerned zeros in $[0, T]$ as $T \to \infty$, while we keep the height level fixed and let $q \to \infty$ through primes. Montgomery's pair correlation statistics is connected to primes in short intervals via the Goldston–Montgomery correspondence [9], while ours is connected to primes in arithmetic progressions via Theorem 1.

Montgomery, Yıldırım [31], and Kandhil–Languasco–Moree [16] studied form factors for zeros whose ordinates lie in an enlarging height window ($[0, T]$ or $[-T, T]$ with $T \to \infty$). Montgomery considered pairs of zeros of a single $\zeta(s)$, while Yıldırım and Kandhil–Languasco–Moree paired a zero of $L(s, \chi_1)$ with a zero of $L(s, \chi_2)$ and then summed over all $\chi_1, \chi_2 \bmod q$, so the two zeros in a pair need not belong to the same L-function.

Özlük [27], Chandee–Lee–Liu–Radziwiłł [2], and our work studied form factors of zeros of Dirichlet L-functions with ordinates in a fixed height window. These works constructed form factors by first pairing zeros of the same $L(s, \chi)$ and then average over χ, but they averaged χ differently. Özlük averaged over all χ modulo $q \leq Q$, so the same primitive L-function could reappear through induced characters, whereas Chandee–Lee–Liu–Radziwiłł averaged directly over primitive characters with conductors $q \asymp Q$. We only average over primitive characters with a fixed conductor q, requiring q to be prime.

The family we consider has also been studied through other statistics: Hughes–Rudnick [15] studied its one-level density, with a scaling that restricted attention to low-lying zeros whose ordinates are $\ll 1/\log q$. We study pair correlation over this same family in a fixed height window as the prime conductor tends to infinity.

In 2014, Keating–Rudnick [20] studied the function-field analog of the variance of primes in arithmetic progressions. In their model, the variance is related to traces of powers of Frobenius matrices, the function field analog of the form factor. To describe their setting, let $\mathbb{F}_{\mathfrak{q}}$ be a finite field, let $Q \in \mathbb{F}_{\mathfrak{q}}[T]$ be squarefree of degree $d \geq 2$, and let $\mathcal{M}_n$ denote the set of

monic polynomials of degree n. Set

$$G(n;Q)=\sum_{\substack{A \bmod Q\\(A,Q)=1}}\left|\sum_{\substack{N\in\mathcal{M}_n\\N\equiv A \pmod Q}}\Lambda_{\mathfrak{q}}(N)-\frac{\mathfrak{q}^n}{\Phi(Q)}\right|^2,$$

where $\Phi(Q)$ is the number of invertible residue classes modulo Q and $\Lambda_{\mathfrak{q}}$ is the analog of the von Mangoldt function.

Keating–Rudnick first treated the range $1\le n<d$ by observing that congruence modulo Q forces equality between monic polynomials of degree n. They obtained

$$G(n;Q)\sim n\mathfrak{q}^n, \tag{11.1}$$

as $\mathfrak{q}\to\infty$ with n and d fixed. This is parallel to our observation in section 6.

For the range $n\ge d$, let $\Theta_\chi\in U(d-1)$ denote the unitarized Frobenius conjugacy class attached to a primitive odd character χ modulo Q. Keating–Rudnick showed that

$$G(n;Q)\sim \mathfrak{q}^n\left\langle\left|\operatorname{tr}\Theta_\chi^n\right|^2\right\rangle,$$

as $\mathfrak{q}\to\infty$ with n and d fixed, where the brackets denote the average over primitive odd characters modulo Q. Katz's 2013 equidistribution theorem for squarefree conductors [17] implies, for fixed n and d as $\mathfrak{q}\to\infty$, that

$$\lim_{\mathfrak{q}\to\infty}\frac{G(n;Q)}{\mathfrak{q}^n}=\int_{U(d-1)}|\operatorname{tr}U^n|^2\,\mathrm{d}U=\min(n,d-1).$$

Together with (11.1) for $n<d$, this parallels the expected asymptotic in Hooley's conjecture

$$G_W(x,q)\sim\left(\int_0^\infty|W(u)|^2\mathrm{d}u\right)x\min(\log q,\log x),\qquad q\ge x^\varepsilon,$$

as $x\to\infty$. The eigenangles of Θ_χ correspond to the ordinates of zeros of $L(s,\chi)$, and $|\operatorname{tr}\Theta_\chi^n|^2$ corresponds to the form factor. Under these identifications, Katz's theorem in the evaluation of $G(n;Q)$ played the role of our form-factor conjecture, Conjecture 2, in the study of $G_W(x,q)$.

A further distinction concerns the modulus. Keating–Rudnick allowed Q to be squarefree while we required q to be prime. This is because in the function field setting, when Q is squarefree, most characters modulo Q are primitive and odd, allowing Katz's theorem to be applied. In our work, relaxing the requirement of q to be prime will introduce nontrivial contributions from characters of different conductors and terms with $(n,q)>1$ in relevant calculations. We therefore restrict q to prime moduli.

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