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Closed-Form Adiabatic Asymptotics of the Solution of the Amsler Equation

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Abstract

We consider the regular solution $y(x;s)$ of the Amsler equation $(xy')' = \sin y$, $y(0) = \pi s$, $y'(0) = \sin \pi s$, with $s \in (0,1)$, and prove that as $x \to \infty$ it is approximated, with accuracy $O(\tau^{-5/2})$ for $\tau = 2\sqrt{x}$, by an explicit closed-form adiabatic ansatz $y_{\rm ad}$: a libration of the pendulum $w'' + \sin w = 0$ ($w = y - \pi$) whose parameter $m_0(\tau)$ is determined by the adiabatic invariant $\tau\,\ell(m_0) = \pi\nu$, where $\ell(m) = E(m) - (1-m)K(m)$ and $\nu = \pi^{-1} \ln \csc \frac{\pi s}{2}$, and whose phase is $\Phi = \tau f(m_0) + \Theta_0 - \frac{1}{8\tau}$ with $f(m) = E(1-m) - mK(1-m)$ and $\Theta_0 = C_2 + \nu(1 + \ln \frac{8}{\nu})$; explicitly, $y_{\rm ad}(x) = \pi + 2\arcsin\bigl(\sqrt{m_0}\,\mathrm{sn}\bigl(\frac{2K(m_0)}{\pi}\Phi \,\big|\, m_0\bigr)\bigr)$. The proof combines elementary adiabatic analysis of the damped pendulum (residual estimates, absence of low-order resonances, an approximate fundamental system) with two external inputs stated explicitly: Maevsky's asymptotics for y and the Its–Novokshenov/Kitaev connection formula for the sine-Gordon (D_8) reduction of the Painlevé III equation. The matching of the two determines the amplitude and phase constants in closed form, $C_1 = \sqrt{8\nu}$, $C_2 = \frac{5\pi}{4} + \arg\Gamma(1 + i\nu) - 3\nu \ln 2$; in particular, the result is a closed-form description of the large-x behaviour of every regular solution (up to the symmetries $y \mapsto -y$, $y \mapsto y + 2\pi$), conditional on the two external inputs and with no fitting parameters.

Keywords: Painlevé equations, Amsler equation, sine-Gordon equation, adiabatic invariants, connection formulas, elliptic integrals.

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1 Main result

Notation and conventions. Throughout, $s \in (0,1)$ is the parameter of the initial condition, $x \geq 0$ is the independent variable of the Amsler equation [1, 2], $\tau = 2\sqrt{x}$, and $y = y(x;s)$ denotes the solution of

$$(xy')' = \sin y, \qquad y(0) = \pi s, \quad y'(0) = \sin \pi s,$$

which is the regular solution at the singular point $x = 0$ (it exists, is unique and smooth on $[0,\infty)$, see Lemma 2.1); we also use $w = y - \pi$. The number $\nu = \nu(s) = \frac{1}{\pi} \ln \csc \frac{\pi s}{2} = -\frac{1}{\pi} \ln \sin \frac{\pi s}{2} > 0$ is the monodromy exponent (§ 7.3).

Elliptic integrals. For a parameter $m \in [0,1)$ (the parameter is $m = k^2$, the square of the modulus k) the complete elliptic integrals of the first and second kind are

$$K(m) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}}, \qquad E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \vartheta}\, d\vartheta.$$

$K(m)$ increases from $K(0) = \pi/2$ to $+\infty$ as $m \to 1$, and $E(m)$ decreases from $E(0) = \pi/2$ to $E(1) = 1$. For $m \in (0,1]$ the symbols $K(1-m)$, $E(1-m)$ denote the complementary integrals, i.e. K and E evaluated at the parameter $1 - m$. Expansions at $m \to 0$: $K(m) = \frac{\pi}{2}\big(1 + \frac{m}{4} + \frac{9m^2}{64} + \dots\big)$, $E(m) = \frac{\pi}{2}\big(1 - \frac{m}{4} - \frac{3m^2}{64} - \dots\big)$.

Jacobi sine. $\operatorname{sn}(u \,|\, m)$ is the Jacobi elliptic sine with parameter m: if $u = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}}$ then $\operatorname{sn}(u \,|\, m) = \sin \varphi$. It is odd, $4K(m)$-periodic, $|\operatorname{sn}| \leq 1$, and for $m \to 0$ it tends to $\sin u$. Consequently $\Phi \mapsto \operatorname{sn}\big(\frac{2K(m)}{\pi}\Phi \,\big|\, m\big)$ is 2π-periodic in Φ, and $\sqrt{m}\,|\operatorname{sn}| \leq \sqrt{m} < 1$, so the arcsin in y_{ad} is well defined with values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

The functions ℓ and f. The variable m in $\ell(m) = E(m) - (1-m)K(m)$ and $f(m) = E(1-m) - mK(1-m)$ is a dummy variable on $[0,1]$ (with $\ell(1) = E(1) = 1$); note $f(m) = \ell(1-m)$. One has $\ell(0) = 0$, $\ell(1) = 1$, $\ell'(m) = K(m)/2 > 0$ (§ 8), so ℓ is an increasing bijection $[0,1] \to [0,1]$. Hence for $\tau > \pi\nu$ the equation $\tau \ell(m_0) = \pi\nu$, i.e. $\ell(m_0) = \pi\nu/\tau \in (0,1)$, has a unique root $m_0 = m_0(\tau) \in (0,1)$; m_0 is smooth, decreasing in τ, $m_0 \to 1$ as $\tau \downarrow \pi\nu$, and $m_0(\tau) = \frac{4\nu}{\tau} - \frac{2\nu^2}{\tau^2} - \frac{\nu^3}{\tau^3} + O(\tau^{-4})$ as $\tau \to \infty$ (§ 14). This is why the statement is made for $\tau \geq \pi\nu + \delta$.

Gamma function. $\arg \Gamma(1 + i\nu) = \operatorname{Im} \ln \Gamma(1 + i\nu)$ is the continuous branch vanishing at $\nu = 0$ (§ 7.5). The phase constant C_2 matters only modulo 2π, because y_{ad} depends on Φ 2π-periodically.

Meaning of the quantities. The function $w_{\mathrm{ad}} = y_{\mathrm{ad}} - \pi = G(\Phi, m_0)$ is a libration of the pendulum $w'' + \sin w = 0$ with slowly varying parameter $m_0 = (H+1)/2$, where H is the pendulum energy (§ 2); its amplitude is $2 \arcsin \sqrt{m_0} \approx 4\sqrt{\nu}\,\tau^{-1/2}$, Φ is its phase, and $\Phi' = \omega(m_0) + \frac{1}{8\tau^2}$ with $\omega(m) = \frac{\pi}{2K(m)}$ (§ 8). The relation $\tau\ell(m_0) = \pi\nu$ is the adiabatic invariant of the model pendulum (the action times τ is constant for the undamped pendulum and is preserved asymptotically by the damped one); it defines the parameter of the ansatz and is not an exact invariant of the original solution. The constants C_1, C_2 are the amplitude

and the phase constant of the Maevsky asymptotics (M) below; C_δ is a constant depending on δ and s only.

Theorem 1.1. *Let $s \in (0,1)$, $\nu = \frac{1}{\pi}\ln\csc\frac{\pi s}{2}$, $\tau = 2\sqrt{x}$, and let y solve the Amsler equation*

$$(xy')' = \sin y, \qquad y(0) = \pi s, \quad y'(0) = \sin \pi s.$$

Define

$$m_0(\tau) \in (0,1): \quad \tau\ell(m_0) = \pi\nu, \qquad \ell(m) = E(m) - (1-m)K(m),$$

$$f(m) = E(1-m) - mK(1-m), \qquad \Phi = \tau f(m_0) + \Theta_0 - \frac{1}{8\tau},$$

$$\Theta_0 = C_2 + \nu\Big(1 + \ln\frac{8}{\nu}\Big), \qquad C_2 = \frac{5\pi}{4} + \arg\Gamma(1+i\nu) - 3\nu\ln 2,$$

$$y_{\mathrm{ad}}(x) = \pi + 2\arcsin\Big(\sqrt{m_0}\,\mathrm{sn}\big(\tfrac{2K(m_0)}{\pi}\Phi \,\big|\, m_0\big)\Big).$$

Then for every $\delta > 0$ there is $C_\delta > 0$ (depending on δ and s) such that

$$\boxed{|y(x) - y_{\mathrm{ad}}(x)| \le C_\delta\,\tau^{-5/2}, \qquad \tau \ge \pi\nu + \delta.}$$

Numerically, at $\nu = 1$ ($s \approx 0.02752$) the quantity $\tau^{5/2}|y - y_{\mathrm{ad}}|$ has maximum ≈ 4.04 on the computed interval $\tau \ge 4$, attained near $\tau \approx 5.8$; on the computed window $100 \le \tau \le \tau_{\max}$ the maximum is ≈ 1.83 (≈ 1.78 on $300 \le \tau \le \tau_{\max}$). These are finite-interval maxima; the theorem itself asserts boundedness of $\tau^{5/2}|y - y_{\mathrm{ad}}|$, and the numerics indicate that the effective constant for large τ is ≈ 1.8. The constant C_δ for small δ also accounts for the finite range $\pi\nu + \delta \le \tau \le \tau_*$ (§ 16).

External inputs. The proof uses two results that are not proved here.

(M) *Maevsky's asymptotics* (stated in full in § 15): there are constants $C_1 = C_1(s) > 0$ and $C_2 = C_2(s)$ such that, as $x \to \infty$,

$$y(x) = \pi + x^{-1/4}F_1(\theta) + x^{-3/4}F_3(\theta) + O(x^{-5/4}), \qquad \theta = 2\sqrt{x} - \tfrac{C_1^2}{16}\ln x + C_2,$$

with explicit trigonometric polynomials F_1, F_3 (§ 15); C_1 is the amplitude and C_2 the phase constant.

(IN) *Its–Novokshenov/Kitaev connection formulas* for the Painlevé III equation (D_8) introduced in § 3 (stated in § 6.2). Specialised to the Amsler family (§ 7) they determine the constants of (M) in closed form,

$$C_1 = \sqrt{8\nu}, \qquad C_2 = \frac{5\pi}{4} + \arg\Gamma(1+i\nu) - 3\nu\ln 2 \pmod{2\pi}.$$

Everything else is proved in this text: §§ 2 to 5 and §§ 8 to 14 unconditionally (using only classical facts on K, E, sn with references to the NIST DLMF [3]), § 7 is a derivation from (IN) (including the dictionary of § 6.2), and §§ 15 to 16 use (M) and (IN); the formulas for C_1, C_2 were also confirmed numerically (§ 7.5); a graphical comparison of the three curves is given in § A.

2 Setup

Lemma 2.1 (Existence, uniqueness, basic bounds)**.** *For every $y_0 \in (0,\pi)$ there is a unique function $y \in C^1[0,\infty)$ with $y(0) = y_0$ and $xy'(x) = \int_0^x \sin y\, d\xi$ (equivalently $(xy')' = \sin y$ with y regular at 0); $y \in C^\infty[0,\infty)$, $|y'| \le 1$, $y'(0) = \sin y_0$. With $w = y - \pi$, $\tau = 2\sqrt{x}$, $H = \frac{1}{2}w_\tau^2 - \cos w$ one has $H_\tau = -w_\tau^2/\tau \le 0$, $|w| < \pi$ for all τ, and $-1 < H < 1$ for $\tau > 0$; hence $m = (H+1)/2 \in (0,1)$.*

Proof. The equation is equivalent to $y(x) = y_0 + \int_0^x \frac{1}{\xi}\int_0^\xi \sin y(\eta)\, d\eta\, d\xi$. Let $y_{n+1} = y_0 + \int_0^x \frac{1}{\xi}\int_0^\xi \sin y_n$ with $y_0(x) \equiv y_0$, and $D_n(x) = \sup_{[0,x]} |y_{n+1} - y_n|$. Since $|\sin a - \sin b| \le |a-b|$, $D_n(x) \le \int_0^x D_{n-1}(\xi)\, d\xi$, and $D_0(x) \le x$; hence $D_n(x) \le x^{n+1}/(n+1)!$, the iteration converges locally uniformly on $[0,\infty)$, and the same estimate gives uniqueness. The limit satisfies $y'(x) = \frac{1}{x}\int_0^x \sin y = \int_0^1 \sin y(xt)\, dt$, so $|y'| \le 1$, $y'(0) = \sin y_0$, and $y \in C^k \Rightarrow y' \in C^k$, i.e. $y \in C^\infty$. For $\tau > 0$, $w'' + \frac{1}{\tau}w' + \sin w = 0$ gives $H_\tau = -w_\tau^2/\tau$. Now $H(0) = -\cos(\pi(s-1)) = \cos \pi s < 1$ (here $y_0 = \pi s$) and $H \ge 1$ whenever $w = \pm\pi$; as $w(0) \in (-\pi, 0)$ and H is non-increasing, w never reaches $\pm\pi$, so $|w| < \pi$ and $H < 1$. Finally $H \ge -\cos w \ge -1$, with $H = -1$ only if $w = 0$, $w_\tau = 0$ at that point, which by uniqueness for the regular ODE ($\tau > 0$) would force $w \equiv 0$, contradicting $w(0) \ne 0$; so $H > -1$. □

With $w = y - \pi$, $\tau = 2\sqrt{x}$:

$$w_{\tau\tau} + \tfrac{1}{\tau}w_\tau + \sin w = 0, \qquad H = \tfrac{1}{2}w'^2 - \cos w, \quad H' = -w'^2/\tau.$$

Libration parameter $m = (H+1)/2 \in (0,1)$. Initial data: $y'(0) = \sin \pi s$ in the x-variable; since $x = \tau^2/4$ and $w_\tau = y'\,\tau/2$, in the τ-variable this means

$$w_\tau(0) = 0, \qquad w_{\tau\tau}(0) = \tfrac{1}{2}\sin \pi s,$$

the local expansion being $w = \pi(s-1) + x\sin \pi s + O(x^2) = \pi(s-1) + \frac{\tau^2}{4}\sin \pi s + \ldots$. Notation:

$$L[w] = w'' + \tfrac{1}{\tau}w' + \sin w, \qquad L'[u] = u'' + \tfrac{1}{\tau}u' + \cos(w_{\mathrm{ad}})\, u,$$

$$G(\Phi, m) = 2\arcsin\Big(\sqrt{m}\,\mathrm{sn}\big(\tfrac{2K(m)}{\pi}\Phi \,\big|\, m\big)\Big), \qquad \omega(m) = \frac{\pi}{2K(m)}.$$

3 Reduction to Painlevé III (Case D_8)

Proposition 3.1. *If y solves the Amsler (radial sine-Gordon) equation $(xy')' = \sin y$ [1], then $u = e^{iy}$ satisfies*

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{u^2 - 1}{2x},$$

i.e. Painlevé III in the standard form

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + \gamma u^3 + \frac{\delta}{u}$$

with parameters

$$\alpha = \tfrac{1}{2}, \qquad \beta = -\tfrac{1}{2}, \qquad \gamma = 0, \qquad \delta = 0.$$

In Okamoto's classification, this is exactly the case D_8 (degenerate Painlevé III with $\gamma = \delta = 0$, $\alpha\beta \neq 0$); see Ohyama–Kawamuko–Sakai–Okamoto [4], where D_8 is defined by $\gamma = \delta = 0$, $\alpha\beta \neq 0$.

Proof. From $u = e^{iy}$: $u_x = iu\,y_x$ and $u_{xx} = iu\,y_{xx} - u\,y_x^2$. The Amsler equation $(xy')' = \sin y$, i.e. $xy_{xx} + y_x = \sin y$, gives $y_{xx} = (\sin y - y')/x$. Since $\sin y = (u - u^{-1})/(2i)$, one has $iu \sin y = (u^2 - 1)/2$. Hence

$$u_{xx} = iu\,\frac{\sin y - y_x}{x} - u\,y_x^2 = \frac{u^2-1}{2x} - \frac{iu\,y_x}{x} + \frac{(iu\,y_x)^2}{u} = \frac{u^2-1}{2x} - \frac{u_x}{x} + \frac{u_x^2}{u}.$$

□

4 Linearisation near $s = 1$

As $s \to 1$ ($y_0 \to \pi$), $w \to 0$, and $\sin w \approx w$:

$$(xw')' = -w \implies z^2 w_{zz} + z w_z + z^2 w = 0, \qquad z = 2\sqrt{x}.$$

This is the Bessel equation of order zero. The following proposition makes this precise.

Proposition 4.1. *Put $a := \pi(s-1) = w(0)$ and let J_0, Y_0 be the Bessel functions of order zero. For every $T > 0$ there are $a_T > 0$ and C_T such that for $|a| < a_T$ and $0 \le \tau \le T$,*

$$|w(\tau)| \le 2|a|, \qquad |w(\tau) - a\,J_0(\tau)| \le C_T |a|^3.$$

In particular $w \to 0$ and $w = aJ_0(\tau) + O(|a|^3)$ uniformly on $[0,T]$ as $s \to 1$.

Proof. (i) *Integral equation.* In the variable τ, $w'' + \frac{1}{\tau}w' + w = \rho(w)$ with $\rho(w) := w - \sin w$, $|\rho(w)| \le |w|^3/6$. By DLMF [3, §10.8.2], $Y_0(t) = \frac{2}{\pi}\ln\frac{t}{2}\,J_0(t) + h(t)$ with h entire (even) in t, hence $|h| \le H_T$ on $[0,T]$, and $|J_0| \le 1$. The Wronskian is $J_0 Y_0' - J_0' Y_0 = \frac{2}{\pi t}$, so variation of constants gives the solution of $u'' + \frac{1}{\tau}u' + u = f$ with $u(0) = u'(0) = 0$:

$$u(\tau) = \int_0^\tau g(\tau,\sigma) f(\sigma)\,d\sigma, \qquad g(\tau,\sigma) = \frac{\pi\sigma}{2}\big[J_0(\sigma)Y_0(\tau) - J_0(\tau)Y_0(\sigma)\big],$$

and $g(\tau,\sigma) = \sigma\,J_0(\sigma)J_0(\tau)\ln\frac{\tau}{\sigma} + \frac{\pi\sigma}{2}\big[J_0(\sigma)h(\tau) - J_0(\tau)h(\sigma)\big]$, so $|g(\tau,\sigma)| \le \sigma\ln\frac{\tau}{\sigma} + \pi H_T \sigma$ for $0 < \sigma \le \tau \le T$. The function $w - aJ_0 - \int_0^\tau g\,\rho(w)\,d\sigma$ solves the homogeneous Bessel equation, is bounded at 0 and vanishes at 0; since Y_0 is unbounded at 0 and $J_0(0) = 1$, it vanishes identically. Hence

$$w(\tau) = a\,J_0(\tau) + \int_0^\tau g(\tau,\sigma)\,\rho(w(\sigma))\,d\sigma, \qquad \int_0^\tau |g(\tau,\sigma)|\,d\sigma \le \frac{\tau^2}{4} + \frac{\pi H_T \tau^2}{2} \le \kappa_T := T^2\Big(\frac{1}{4} + \frac{\pi H_T}{2}\Big).$$

(ii) *Bootstrap.* Let $M(\tau) = \max_{[0,\tau]}|w|$, continuous with $M(0) = |a|$. By (i), $M(\tau) \le |a| + \frac{\kappa_T}{6}M(\tau)^3$. Choose a_T with $\frac{\kappa_T}{6}\,8a_T^2 < 1$. If M reached the value $2|a|$ at some first $\tau_* \le T$, then $2|a| \le |a| + \frac{\kappa_T}{6}\,8|a|^3 < 2|a|$, a contradiction. So $M \le 2|a|$ on $[0,T]$, and then $|w - aJ_0| \le \frac{\kappa_T}{6}(2|a|)^3 = \frac{4\kappa_T}{3}|a|^3$, i.e. the claim with $C_T = \frac{4\kappa_T}{3}$. □

Consistency with the nonlinear theory. $J_0(z) = \sqrt{2/\pi z}\,(\cos(z-\pi/4)+O(1/z))$, so the linearised solution oscillates with amplitude $\pi(1-s)\sqrt{2/\pi}\,z^{-1/2} = \sqrt{2\pi}(1-s)z^{-1/2}$; by §§ 7.3 to 7.5 the nonlinear oscillation amplitude is $4\sqrt{\nu}\,\tau^{-1/2}$ with $\nu = \pi^{-1}\ln\csc\frac{\pi s}{2} = \frac{\pi}{8}(1-s)^2+O((1-s)^4)$, i.e. $4\sqrt{\nu} = \sqrt{2\pi}(1-s)+O((1-s)^3)$ — the two amplitudes agree to leading order, and $\nu(s)\to 0$ as $s\to 1$, as claimed.

5 Lax pair and compatibility

Proposition 5.1 (Its–Prokhorov pair; [5])**.** *In this section u denotes the angle variable, $u = w = y-\pi$ (not the function e^{iy} of § 3). In the variable ξ (playing the role of τ),*

$$A(\lambda) = -\frac{i\xi^2\sigma_3}{16} - \frac{i\xi u_\xi\sigma_1}{4\lambda} + \frac{P_0(i\sigma_3)P_0^{-1}}{\lambda^2}, \quad U(\lambda) = -\frac{i\lambda\xi\sigma_3}{8} - \frac{iu_\xi\sigma_1}{2}, \quad P_0 = e^{-iu\sigma_1/2},$$

*satisfies the zero-curvature condition $A_\xi - U_\lambda + [A,U] = 0$ **iff** $u_{\xi\xi} + u_\xi/\xi + \sin u = 0$.*

Proof. Recall $[\sigma_1,\sigma_3] = -2i\sigma_2$, $[\sigma_2,\sigma_1] = -2i\sigma_3$, $[\sigma_2,\sigma_3] = 2i\sigma_1$. From $\frac{d}{d\theta}e^{-i\theta\sigma_1}\sigma_3 e^{i\theta\sigma_1} = -2\sigma_2$ at $\theta = 0$, solving the resulting rotation in the (σ_3,σ_2)-plane:

$$e^{-i\theta\sigma_1}\sigma_3 e^{i\theta\sigma_1} = \cos(2\theta)\,\sigma_3 - \sin(2\theta)\,\sigma_2,$$

so with $\theta = u/2$:

$$P_0(i\sigma_3)P_0^{-1} = i\cos u\,\sigma_3 - i\sin u\,\sigma_2.$$

Write $A = A_1+A_2+A_3+A_4$ and $U = U_1+U_2$ with

$$A_1 = -\tfrac{i\xi^2}{16}\sigma_3, \quad A_2 = -\tfrac{i\xi u_\xi}{4\lambda}\sigma_1, \quad A_3 = \tfrac{i\cos u}{\lambda^2}\sigma_3, \quad A_4 = -\tfrac{i\sin u}{\lambda^2}\sigma_2,$$

$$U_1 = -\tfrac{i\lambda\xi}{8}\sigma_3, \quad U_2 = -\tfrac{iu_\xi}{2}\sigma_1.$$

Commutators of equal Pauli matrices vanish; the nonzero ones are:

$$[A_1,U_2] = \left(\tfrac{-i\xi^2}{16}\right)\left(\tfrac{-iu_\xi}{2}\right)(2i\sigma_2) = -\frac{i\xi^2u_\xi}{16}\sigma_2, \qquad [A_2,U_1] = \left(\tfrac{-i\xi u_\xi}{4\lambda}\right)\left(\tfrac{-i\lambda\xi}{8}\right)(-2i\sigma_2) = +\frac{i\xi^2u_\xi}{16}\sigma_2,$$

$$[A_3,U_2] = \left(\tfrac{i\cos u}{\lambda^2}\right)\left(\tfrac{-iu_\xi}{2}\right)(2i\sigma_2) = \frac{i\cos u\,u_\xi}{\lambda^2}\sigma_2,$$

$$[A_4,U_2] = \left(\tfrac{-i\sin u}{\lambda^2}\right)\left(\tfrac{-iu_\xi}{2}\right)(-2i\sigma_3) = \frac{i\sin u\,u_\xi}{\lambda^2}\sigma_3,$$

$$[A_4,U_1] = \left(\tfrac{-i\sin u}{\lambda^2}\right)\left(\tfrac{-i\lambda\xi}{8}\right)(2i\sigma_1) = -\frac{i\xi\sin u}{4\lambda}\sigma_1.$$

Summing,

$$[A,U] = -\frac{i\xi\sin u}{4\lambda}\sigma_1 + \frac{i\cos u\,u_\xi}{\lambda^2}\sigma_2 + \frac{i\sin u\,u_\xi}{\lambda^2}\sigma_3.$$

Differentiating along ξ (through $u(\xi)$ and explicitly):

$$A_\xi = -\frac{i\xi}{8}\sigma_3 - \frac{i}{4\lambda}(u_\xi + \xi u_{\xi\xi})\sigma_1 - \frac{iu_\xi \cos u}{\lambda^2}\sigma_2 - \frac{iu_\xi \sin u}{\lambda^2}\sigma_3, \qquad U_\lambda = -\frac{i\xi}{8}\sigma_3.$$

Now collect $M := A_\xi - U_\lambda + [A, U]$ by Pauli components:

$$M\big|_{\sigma_3} = \Big(-\frac{i\xi}{8} - \frac{iu_\xi \sin u}{\lambda^2}\Big) + \frac{i\xi}{8} + \frac{i \sin u\, u_\xi}{\lambda^2} = 0,$$

$$M\big|_{\sigma_2} = -\frac{iu_\xi \cos u}{\lambda^2} + \frac{i \cos u\, u_\xi}{\lambda^2} = 0,$$

$$M\big|_{\sigma_1} = -\frac{i}{4\lambda}(u_\xi + \xi u_{\xi\xi}) - \frac{i\xi \sin u}{4\lambda} = -\frac{i\xi}{4\lambda}\Big(u_{\xi\xi} + \frac{u_\xi}{\xi} + \sin u\Big).$$

Hence $M \equiv 0$ iff $u_{\xi\xi} + u_\xi/\xi + \sin u = 0$. □

6 Monodromy and connection formulas (IN)

The monodromy data and the connection formulas of this section rest on the isomonodromy deformation method for the third Painlevé transcendent due to Novokshenov [6]; see also the monographs [7, 8].

6.1 Stokes multipliers and connection matrix

The monodromy data of a solution of the sine-Gordon Painlevé III equation are parametrised in [5, 9] by a pair (σ, η); the connection matrix is

$$E = \frac{1}{\sqrt{1+pq}}\begin{pmatrix} 1 & p \\ -q & 1 \end{pmatrix}, p = -i\,\frac{\sin 2\pi(\sigma+\eta)}{\sin 2\pi\eta}, \quad q = i\,\frac{\sin 2\pi(\sigma-\eta)}{\sin 2\pi\eta}, 1 + pq = \frac{\sin^2 2\pi\sigma}{\sin^2 2\pi\eta},$$

so that $E_{11} = e^{\pi\nu_{IP}} = \dfrac{\sin 2\pi\eta}{\sin 2\pi\sigma}$ ($\nu_{IP} = -\frac{1}{2\pi}\ln(1+pq)$). For the regular family under consideration $(\sigma, \eta) = (\frac{1}{4}, \frac{s}{4})$ (see § 7.3), hence

$$E = \begin{pmatrix} \sin\frac{\pi s}{2} & -i\cos\frac{\pi s}{2} \\ -i\cos\frac{\pi s}{2} & \sin\frac{\pi s}{2} \end{pmatrix} = \exp\Big(-\frac{i\pi(1-s)}{2}\sigma_1\Big),$$

$$E_{11} = e^{\pi\nu_{IP}} = \sin\frac{\pi s}{2}, \quad \nu_{IP} < 0, \quad |E_{12}| = \cos\frac{\pi s}{2}.$$

The Lax pair of § 5 is written for $u = w = y - \pi$, so $P_0(0) = e^{-i\pi(s-1)\sigma_1/2}$; in the derivation we use that, for the regular family, all Stokes multipliers at $\lambda = \infty$ are trivial, and this identification is part of the input (IN). The following theorem (Its–Novokshenov / Kitaev) is part of the input (IN) [10, 7, 11, 5].

6.2 Novokshenov's connection formula

Theorem 6.1. *[10, 11, 5, 9] For the Painlevé III family of § 3, let (α_N, β_N) be the Novokshenov parameters of the solution at the origin (distinct from the equation coefficients $(\alpha, \beta, \gamma, \delta)$ of § 3), and set*

$$\sigma = \frac{1}{4} + \frac{i\alpha_N}{8}, \qquad \eta = \frac{1}{4} + \frac{\beta_N + \alpha_N \ln 8}{4\pi} + \frac{i}{2\pi} \ln \frac{\Gamma(1/2 - i\alpha_N/4)}{\Gamma(1/2 + i\alpha_N/4)}.$$

Then $e^{\pi\nu_{IP}} = \frac{\sin 2\pi\eta}{\sin 2\pi\sigma}$, and for a solution u with leading asymptotic $b_+ e^{i\tau} \tau^{i\nu_{IP} - 1/2} + c.c.$,

$$b_\pm = -e^{\pi\nu_{IP}/2 \mp i\pi/4}\, 2^{1 \pm 2i\nu_{IP}}\, \frac{1}{\sqrt{2\pi}}\, \Gamma(1 \mp i\nu_{IP})\, \frac{\sin 2\pi(\sigma \mp \eta)}{\sin 2\pi\eta}.$$

Sign convention: b_+ carries the factor $e^{-i\pi/4}$.

Dictionary (part of (IN)). For a solution of $w_{\tau\tau} + \frac{1}{\tau} w_\tau + \sin w = 0$ the Novokshenov parameters are the coefficients of the behaviour at the origin, $w = \alpha_N \ln \tau + \beta_N + o(1)$ as $\tau \to 0$; for the regular solutions considered here (w analytic at $\tau = 0$) one has $\alpha_N = 0$, $\beta_N = w(0)$. The monodromy data of the regular solution with $y(0) = \pi s$ are as in § 6.1 (E as in § 6.1, determined by $(\sigma, \eta) = (\frac{1}{4}, \frac{s}{4})$; all Stokes multipliers trivial). The identifications of § 6.1 and of this paragraph are taken as part of the input (IN).

7 Specialisation to the Amsler family

7.1 Parameters

The regular solution has no logarithmic term at the origin, so $\alpha_N = 0$ in the (IN) notation. From $w(0) = \pi(s-1)$: $\beta_N = \pi(s-1)$. Then

$$\sigma = \frac{1}{4}, \qquad \eta = \frac{1}{4} + \frac{\beta_N}{4\pi} = \frac{s}{4},$$

the Γ-ratio being 1 since $\alpha_N = 0$.

7.2 Trigonometric evaluations

$$\sin 2\pi\sigma = 1, \qquad \sin 2\pi\eta = \sin \frac{\pi s}{2}, \qquad \sin 2\pi(\sigma \mp \eta) = \cos \frac{\pi s}{2}.$$

7.3 Monodromy exponent

$$e^{\pi\nu_{IP}} = \sin \frac{\pi s}{2} \implies \nu_{IP}(s) = \frac{1}{\pi} \ln \sin \frac{\pi s}{2} < 0.$$

With $\nu(s) = -\nu_{IP}(s) > 0$:

$$\boxed{\nu(s) = \frac{1}{\pi} \ln \frac{1}{\sin(\pi s/2)} = \frac{1}{\pi} \ln \csc \frac{\pi s}{2}.}$$

On $(0, 1)$: real, positive, analytic (indeed ν is analytic on $(0, 2)$); in the complex s-plane its only singularities are logarithmic branch points at the even integers $s = 2k$, $k \in \mathbb{Z}$ (the zeros of $\sin \frac{\pi s}{2}$).

7.4 Amplitude

Derivation. Insert $\sigma = 1/4$, $\eta = s/4$ into Theorem 6.1. The ratio $\sin 2\pi(\sigma - \eta)/\sin 2\pi\eta = \cot \frac{\pi s}{2}$ is real positive, $|2^{1+2i\nu_{IP}}|^2 = 4$, and by Euler's identity $|\Gamma(1 - i\nu_{IP})|^2 = |\Gamma(1 + i\nu)|^2 = \pi\nu/\sinh(\pi\nu)$ with $\nu = -\nu_{IP}$. Hence

$$|b_+|^2 = e^{\pi\nu_{IP}} \cdot 4 \cdot \frac{\pi\nu}{\sinh \pi\nu} \cdot \frac{1}{2\pi} \cdot \cot^2 \frac{\pi s}{2} = \frac{2\nu}{\sinh \pi\nu} \cdot S \cdot \frac{\kappa^2}{S^2}, \qquad S = \sin \frac{\pi s}{2},\ \kappa = \cos \frac{\pi s}{2},$$

using $e^{\pi\nu_{IP}} = S$. Since $e^{\pi\nu} = 1/S$,

$$\sinh(\pi\nu) = \frac{e^{\pi\nu} - e^{-\pi\nu}}{2} = \frac{1/S - S}{2} = \frac{\kappa^2}{2S},$$

so

$$\boxed{|b_+|^2 = \frac{2\nu}{\kappa^2/(2S)} \cdot \frac{\kappa^2}{S} = 4\nu.}$$

7.5 Asymptotic constants in x-form

Translating via $x^{-1/4} = \sqrt{2}\,\tau^{-1/2}$ and $\ln x = 2\ln\tau - 2\ln 2$:

$$\boxed{C_1(s) = \sqrt{8\nu(s)} = \sqrt{-\frac{8}{\pi} \ln \sin \frac{\pi s}{2}},}$$

$$\boxed{C_2(s) = \frac{5\pi}{4} + \arg\Gamma(1 + i\nu) - 3\nu \ln 2 \pmod{2\pi},}$$

with $\arg\Gamma(1 + i\nu) = \operatorname{Im} \ln\Gamma(1 + i\nu)$ on the branch vanishing at $\nu = 0$. The -3 decomposes as -2 from $2^{1+2i\nu_{IP}}$ in b_+ and -1 from $\ln x = 2\ln\tau - 2\ln 2$. These constants admit an independent verification via the direct connection formula of Wong–Zhang [12] for the radial sine-Gordon equation $u_{xx} + \frac{1}{x}u_x + \sin u = 0$: with $r = 0$, $s_0 = w(0) = \pi(s - 1)$, their parameters give $\alpha^2 = -\frac{16}{\pi} \ln \sin \frac{\pi s}{2} = 16\nu$, i.e. $C_1 = \sqrt{8\nu}$, and their phase constant translates to $\frac{5\pi}{4} + \arg\Gamma(1 + i\nu) - 2\nu \ln 2$, which combined with $C = C_2 + \nu \ln 2$ of § 14 yields the same C_2 modulo 2π.

Numerical confirmation: end-to-end integration of the Amsler equation from $\tau = 10^{-4}$ to 630 at $\nu = 1$ gives $C_2(1) = 1.545908955$, matching the formula to $\sim 10^{-5}$. Moreover, local least-squares fits of the shifts $(\Delta\Theta, \Delta\nu)$ of the parameters (Θ_0, ν) in y_{ad} over windows of length 4π satisfy $\Delta\Theta, \Delta\nu = O(\tau^{-2})$. A rigorous bound from the main theorem alone is weaker: $|y - y_{\mathrm{ad}}| = O(\tau^{-5/2})$, the mode $P_1 = \partial_\Theta w_{\mathrm{ad}}$ is $O(\tau^{-1/2})$ (Lemma 12.1), but $P_2 = \partial_\nu w_{\mathrm{ad}} = \Pi - c(\tau)P_1$ with $c(\tau) = O(\ln\tau)$, so the two fitting columns are sheared and the smallest singular value of the window Jacobian is $O(\tau^{-1/2}/\ln\tau)$; hence $|\Delta\Theta| + |\Delta\nu| = O(\tau^{-2}\ln\tau)$ follows rigorously. The observed scaling is better, $O(\tau^{-2})$, and is reported here as a numerical observation. Numerically, at $\nu = 1$: $\tau^2\Delta\nu \approx -0.12$ for all $\tau = 50, \dots, 1200$, and $\tau^2\Delta\Theta$ moves slowly from -0.95 ($\tau = 50$) through -1.18 ($\tau = 200$) to -1.40 ($\tau = 1200$), i.e. $\Delta\Theta = O(\tau^{-2})$ up to a slowly varying factor.

8 Exact algebraic identities

Proposition 8.1. *For $m \in (0, 1)$:*

$$\ell'(m) = \frac{K(m)}{2}, \qquad f'(m) = -\frac{K(1-m)}{2}, \qquad K(m)f(m) + \ell(m)K(1-m) = \frac{\pi}{2}.$$

Proof. From $E' = \frac{E-K}{2m}$, $K' = \frac{E-(1-m)K}{2m(1-m)}$:

$$\ell' = \frac{E-K}{2m} + K - \frac{E-(1-m)K}{2m} = \frac{K}{2}.$$

$f(m) = \ell(1-m)$ gives $f' = -K(1-m)/2$. Legendre's relation $KE(1-m) + K(1-m)E - KK(1-m) = \frac{\pi}{2}$, after substituting $E(m) = \ell(m) + (1-m)K(m)$ and $E(1-m) = f(m) + mK(1-m)$ and collecting terms, gives the third identity. □

Proposition 8.2. *From $\tau\ell(m_0) = \pi\nu$:*

$$m_0' = -\frac{2\ell}{K\tau}, \qquad m_0'' + \frac{m_0'}{\tau} = \frac{2\ell(K^2 - 2\ell K')}{K^3\tau^2},$$

and ***exactly*** $\Phi' = \omega(m_0) + \frac{1}{8\tau^2}$.

Proof. Differentiating $\tau\ell(m_0) = \pi\nu$ at fixed ν: $\ell + \tau\ell' m_0' = 0$, so $m_0' = -\ell/(\tau K/2) = -2\ell/(K\tau)$. Differentiating $m_0' = -2\ell/(K\tau)$ and using $m_0' = -2\ell/(K\tau)$ again:

$$m_0'' = -2\Big[\frac{(\ell' K - \ell K')m_0'}{K^2\tau} - \frac{\ell}{K\tau^2}\Big],$$

$$m_0'' + \frac{m_0'}{\tau} = -\frac{2(\ell' K - \ell K')m_0'}{K^2\tau} = \frac{4\ell(\ell' K - \ell K')}{K^3\tau^2} = \frac{2\ell(K^2 - 2\ell K')}{K^3\tau^2},$$

using $\ell' = K/2$. For the last identity:

$$\Phi' = f + \tau f' m_0' + \frac{1}{8\tau^2} = f + \tau\Big(-\frac{K(1-m)}{2}\Big)\Big(-\frac{2\ell}{K\tau}\Big) + \frac{1}{8\tau^2} = f + \frac{\ell K(1-m)}{K} + \frac{1}{8\tau^2} = \frac{\pi}{2K} + \frac{1}{8\tau^2},$$

where the last step uses Proposition 8.1: $Kf + \ell K(1-m) = \pi/2$. □

Corollary 8.1. $1 - \Phi'^2/\omega^2 = -2 \cdot \frac{1}{8\tau^2} \cdot \frac{1}{\omega} - \frac{1}{64\tau^4\omega^2} = -\frac{1}{4\omega\tau^2} + O(\tau^{-4})$.

9 Fourier coefficients of G

Lemma 9.1. *For $m \le m_{\delta'} < 1$, G is odd and 2π-periodic in Φ, real-analytic jointly with m, and*

$$G(\Phi, m) = 2\sqrt{m}\sin\Phi + \frac{3}{8}m^{3/2}\sin\Phi + \frac{1}{24}m^{3/2}\sin 3\Phi + O(m^{5/2}),$$

i.e. $G = \sum_{k\ \text{odd}} g_k(m)\sin k\Phi$ with $g_k(m) = m^{k/2}h_k(m)$, h_k analytic on $[0, m_{\delta'}]$; even harmonics vanish. In particular

$$g_1(m) = 2m^{1/2} + \frac{3}{8}m^{3/2} + \frac{23}{128}m^{5/2} + O(m^{7/2}), \qquad g_3(m) = \frac{1}{24}m^{3/2} + \frac{1}{32}m^{5/2} + O(m^{7/2}).$$

Proof. **(a) Fourier series of** sn**.** The standard nome expansion (DLMF [3, §22.11]) gives, for $v = 2K(m)\Phi/\pi$ and nome $q(m)$,

$$\sqrt{m}\,\mathrm{sn}(v|m) = \frac{4}{\bar{K}}\sum_{n\ge 0}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin(2n+1)\Phi, \qquad \bar{K} = \frac{2K}{\pi},$$

absolutely and uniformly convergent for $m \le m_{\delta'}$, since $q(m_{\delta'}) < 1$. With the standard expansions (DLMF [3, §19.5] for K, [3, §22.2] for the nome; the coefficients were also confirmed numerically)

$$q(m) = \frac{m}{16} + \frac{m^2}{32} + \frac{21m^3}{1024} + O(m^4), \qquad \bar{K}(m) = 1 + \frac{m}{4} + \frac{9m^2}{64} + \frac{25m^3}{256} + O(m^4),$$

and $u := \sqrt{m}$, termwise series computation gives

$$\sqrt{q} = \frac{u}{4}\Big(1 + \frac{u^2}{4} + \frac{17u^4}{128}\Big) + O(u^7), \qquad \frac{1}{\bar{K}(1-q)} = 1 - \frac{3u^2}{16} - \frac{15u^4}{256} + O(u^6),$$

hence the first two harmonic amplitudes of $X := \sqrt{m}\,\mathrm{sn}(2K\Phi/\pi|m)$ are

$$a_0 = \frac{4\sqrt{q}}{\bar{K}(1-q)} = u + \frac{u^3}{16} + \frac{7u^5}{256} + O(u^7), \qquad a_1 = \frac{4q^{3/2}}{\bar{K}(1-q^3)} = \frac{u^3}{16} + \frac{u^5}{32} + O(u^7),$$

and $a_2 = O(u^5)$ (harmonic 5).

(b) Analytic structure. Each $a_j = m^{j+1/2}\times$(analytic in m), because $q = m/16\cdot(1+O(m))$ and $\bar{K}(1-q^{2j+1})$ are analytic and non-vanishing at $m = 0$. So the k-th harmonic of X ($k = 2j+1$) is $m^{k/2}\times$(analytic).

(c) Composition with arcsin**.** Since $|X| \le \sqrt{m} \le \sqrt{m_{\delta'}} < 1$,

$$G = 2\arcsin X = 2X + \frac{X^3}{3} + \frac{3X^5}{20} + O(X^7)$$

uniformly, and the series may be differentiated termwise in m on $m \ge m_1 > 0$. If X^{2n+1} contributes to harmonic k, the contributing products of harmonics $2j_i+1$ satisfy $\pm(2j_1+1)\pm\cdots\pm(2j_{2n+1}+1) = k$, hence $\sum_i(2j_i+1) \ge k$, so the m-power of such a product is $\sum_i(j_i+\frac{1}{2}) = \frac{1}{2}\sum_i(2j_i+1) \ge \frac{k}{2}$ (and is a half-integer, like $k/2$). Hence $g_k = m^{k/2}h_k(m)$

with h_k analytic on $[0, m_{\delta'}]$; oddness of G in Φ kills the even harmonics, and periodicity in Φ follows from $\mathrm{sn}(v+4K) = \mathrm{sn}(v)$, i.e. period 2π in Φ.

(d) Coefficients. Reduce powers by

$$\sin^3\Phi = \tfrac{3\sin\Phi - \sin 3\Phi}{4}, \qquad \sin^2\Phi\sin 3\Phi = \tfrac{1}{2}\sin 3\Phi - \tfrac{1}{4}\sin 5\Phi - \tfrac{1}{4}\sin\Phi,$$
$$\sin^5\Phi = \tfrac{10\sin\Phi - 5\sin 3\Phi + \sin 5\Phi}{16}.$$

All products involving a_2 or a_1^2 carry $\geq u^7$ in harmonics 1 and 3. Thus

$$g_1 = 2a_0 + \frac{a_0^3}{4} - \frac{a_0^2 a_1}{4} + \frac{3a_0^5}{32} + O(u^7):$$

$$2a_0 = 2u + \frac{u^3}{8} + \frac{7u^5}{128}, \quad \frac{a_0^3}{4} = \frac{u^3}{4} + \frac{3u^5}{64}, \quad -\frac{a_0^2 a_1}{4} = -\frac{u^5}{64}, \quad \frac{3a_0^5}{32} = \frac{3u^5}{32},$$

and the u^3 terms give $\frac{1}{8} + \frac{1}{4} = \frac{3}{8}$, while the u^5 terms give $\frac{7}{128} + \frac{6}{128} - \frac{2}{128} + \frac{12}{128} = \frac{23}{128}$. For the third harmonic:

$$g_3 = 2a_1 + \frac{1}{3}\Big(-\frac{a_0^3}{4} + \frac{3a_0^2 a_1}{2}\Big) + \frac{3}{20}\Big(-\frac{5a_0^5}{16}\Big) + O(u^7),$$

whose u^3 part is $\frac{2}{16} - \frac{1}{12} = \frac{1}{24}$ and whose u^5 part is $\frac{1}{16} - \frac{1}{64} + \frac{1}{32} - \frac{3}{64} = \frac{1}{32}$ (contributions of $2a_1$, $-\frac{a_0^3}{12}$, $\frac{a_0^2 a_1}{2}$, $-\frac{3a_0^5}{64}$). Substituting $u = \sqrt{m}$ gives the claim. □

Lemma 9.2 (Analytic structure and uniform expansions)**.** *Fix $m_* \in (0,1)$ and write $G(\Phi, m) = \sqrt{m}\,\hat{G}(\Phi, m)$. Then:*

(i) $\hat{G}$ is real-analytic on a neighbourhood of $[0, m_*] \times \mathbb{R}$, odd and 2π-periodic in Φ.

(ii) Near $m = 0$, $\hat{G}(\Phi, m) = \sum_{J \geq 0} m^J \hat{G}_J(\Phi)$, where $\hat{G}_J$ is an odd trigonometric polynomial of degree $\leq 2J+1$ containing only odd harmonics; $\hat{G}_0 = 2\sin\Phi$, $\hat{G}_1 = \frac{3}{8}\sin\Phi + \frac{1}{24}\sin 3\Phi$, $\hat{G}_2 = \frac{23}{128}\sin\Phi + \frac{1}{32}\sin 3\Phi + \frac{1}{640}\sin 5\Phi$ (the last coefficient is $\frac{1}{128} - \frac{1}{64} + \frac{3}{320} = \frac{1}{640}$: contributions of $2a_2$, $-\frac{a_0^2 a_1}{4}$, $\frac{3a_0^5}{320}$).

(iii) For every $N \geq 1$, $\hat{G} - \sum_{J<N} m^J \hat{G}_J = m^N \hat{R}_N$ with $\hat{R}_N$ real-analytic and bounded together with all its derivatives on $\mathbb{R} \times [0, m_*]$. Consequently $\partial_\Phi^i\big(\hat{G} - \sum_{J<N} m^J \hat{G}_J\big) = O(m^N)$ and $\partial_m^k\big(\hat{G} - \sum_{J<N} m^J \hat{G}_J\big) = O(m^{N-k})$ $(k \leq N)$, uniformly in Φ and $m \in [0, m_*]$.

Proof. (i) For $m \in [0,1)$ the function $\mathrm{sn}(v|m)$ is real-analytic in (v, m) (it extends holomorphically to a complex neighbourhood of the real set), and so is $K(m)$; thus $v = \frac{2K(m)}{\pi}\Phi$ and $\mathrm{sn}(v|m)$ are analytic in (Φ, m). Put $z = m\,\mathrm{sn}^2$; then $|z| \leq m_* < 1$ on the real set, hence in a complex neighbourhood, and $\frac{2\arcsin\sqrt{z}}{\sqrt{z}} = 2A(z)$, $A(z) = \sum_{n\geq 0} \frac{\binom{2n}{n}}{4^n(2n+1)} z^n$, is analytic for $|z| < 1$. Therefore $\hat{G} = 2\,\mathrm{sn}(v|m)\,A\big(m\,\mathrm{sn}^2(v|m)\big)$ is analytic. Oddness and 2π-periodicity were shown in Lemma 9.1.

(ii) With $u = \sqrt{m}$ and $X = \sqrt{m}\,\mathrm{sn}$, Lemma 9.1(a) gives $X/u = \sum_{j\geq 0} m^j \alpha_j(m) \sin(2j+1)\Phi$, $\alpha_j = (q/m)^{j+1/2}\, 4/(\bar{K}(1 - q^{2j+1}))$ analytic near $m = 0$ (since q/m is analytic and positive

there). Moreover $\hat G = 2\sum_{n\ge0} c_n\, m^n\,(X/u)^{2n+1}$ with $c_n = \binom{2n}{n}/(4^n(2n+1))$, and for $|m|$ small all series converge absolutely, so they may be multiplied out and regrouped. A product of $2n+1$ factors $m^{j_i}\alpha_{j_i}(m)\sin(2j_i+1)\Phi$, multiplied by m^n, has trigonometric degree $\le \sum_i(2j_i+1) = 2\big(\sum_i j_i + n\big)+1$ and is $O(m^{n+\sum j_i})$. Hence in the Taylor coefficient of m^J only products with $n+\sum j_i \le J$ occur, and these have degree $\le 2J+1$; only odd harmonics occur since every factor is a sine of an odd multiple of Φ, and oddness in Φ is preserved. The values of $\hat G_0, \hat G_1, \hat G_2$ are obtained from g_1, g_3 of Lemma 9.1 divided by $\sqrt{m}$, and, for the $\sin 5\Phi$ coefficient of $\hat G_2$, from $a_2 = u^5/256 + O(u^7)$ (so $2a_2/u = \frac{m^2}{128}$) together with the $\sin 5\Phi$-parts of $X^3/3$ and $3X^5/20$.

(iii) $\hat G$ is analytic on a complex neighbourhood of the compact set $[0, m_*]\times[0,2\pi]$ (and periodic), so Taylor's formula with integral remainder in m gives $\hat R_N$ with the stated properties; the derivative statements follow by differentiating. □

10 Exact residual

Proposition 10.1. *With $G = G(\Phi, m_0)$, $\Phi' = \omega + 1/(8\tau^2)$:*

$$L[w_{\mathrm{ad}}] = T_1 + T_2 + T_3 + T_4 + T_5,$$

$$T_1 = \sin G\Big(1 - \frac{\Phi'^2}{\omega^2}\Big), \quad T_2 = G_\Phi \Phi'', \quad T_3 = \frac{\Phi'}{\tau}\Big(G_\Phi - \frac{4\ell}{K} G_{\Phi m}\Big),$$

$$T_4 = G_{mm} m_0'^2, \quad T_5 = G_m\Big(m_0'' + \frac{m_0'}{\tau}\Big).$$

Proof. Chain rule $w' = G_\Phi\Phi' + G_m m_0'$, $w'' = G_{\Phi\Phi}\Phi'^2 + 2G_{\Phi m}\Phi' m_0' + G_\Phi\Phi'' + G_{mm}m_0'^2 + G_m m_0''$, with the exact identities $G_{\Phi\Phi} = -\sin G/\omega^2$ (differentiate G twice in Φ through $\psi = 2K\Phi/\pi$: $G_\psi = 2\sqrt{m}\,\mathrm{cn}$, $G_{\psi\psi} = -2\sqrt{m}\,\mathrm{sn}\,\mathrm{dn} = -\sin G$) and $m_0' = -2\ell/(K\tau)$. Collecting $G_{\Phi\Phi}\Phi'^2 + \sin G$ into T_1, the m_0'-terms into T_3, T_5 (using $m_0'/\tau = -2\ell/(K\tau^2)$), and the remaining terms into T_2, T_4. □

Lemma 10.1. *$B := G_\Phi - \frac{4\ell}{K} G_{\Phi m}$ satisfies*

$$B(\Phi, m) = -\frac{m^{3/2}}{2}\cos\Phi - \frac{m^{3/2}}{4}\cos 3\Phi + O(m^{5/2}),$$

uniformly in Φ, 2π-periodic; more precisely $B = m^{3/2}B_1(\Phi) + m^{5/2}B_2(\Phi) + O(m^{7/2})$ with trigonometric polynomials $B_1 = -\frac12\cos\Phi - \frac14\cos 3\Phi$ and B_2 of degree ≤ 5, whose $\cos\Phi$- and $\cos 3\Phi$-coefficients are $-\frac{29}{64}$ and $-\frac{21}{64}$.

Proof. By Lemma 9.2 write $G = \sqrt{m}\,\hat G$. Then $G_\Phi = \sqrt{m}\,\hat G_\Phi$ and $G_{\Phi m} = \frac{1}{2\sqrt{m}}\hat G_\Phi + \sqrt{m}\,\hat G_{\Phi m}$, so

$$B = \sqrt{m}\Big[\Big(1 - \frac{2\ell}{Km}\Big)\hat G_\Phi - \frac{4\ell}{K}\hat G_{\Phi m}\Big] = m^{3/2}\big[s(m)\hat G_\Phi - t(m)\hat G_{\Phi m}\big],$$

where $1 - \frac{2\ell}{Km} = m\,s(m)$ and $\frac{4\ell}{K} = m\,t(m)$ with s, t analytic on $[0, m_*]$ ($\ell = m\cdot$analytic, K analytic and positive), $t(0) = 2$, $s(0) = \frac18$ by the series for ℓ/K below. By Lemma 9.2(ii),

(iii) the bracket equals $B_1 + mB_2 + O(m^2)$ uniformly in Φ (also after differentiation in Φ) with trigonometric polynomials B_1, B_2 of degree $\le 3, \le 5$; and

$$B_1 = \frac{1}{8}\hat{G}_0' - 2\hat{G}_1' = \frac{1}{4}\cos\Phi - \frac{3}{4}\cos\Phi - \frac{1}{4}\cos 3\Phi = -\frac{1}{2}\cos\Phi - \frac{1}{4}\cos 3\Phi$$

(using $\hat{G}_0' = 2\cos\Phi$, $\hat{G}_1' = \frac{3}{8}\cos\Phi + \frac{1}{8}\cos 3\Phi$), which is the claim. The remaining statements about B_2 follow from the Fourier coefficients: since $G = \sum_{k\ \mathrm{odd}} g_k(m)\sin k\Phi$, the $\cos\Phi$-coefficient of B is $g_1 - \frac{4\ell}{K}g_1'$ and its $\cos 3\Phi$-coefficient is $3\big(g_3 - \frac{4\ell}{K}g_3'\big)$ (termwise differentiation is legitimate because the Fourier coefficients of the analytic function $\hat{G}$ decay geometrically, locally uniformly in m); harmonics $k \ge 5$ contribute only to $m^{5/2}B_2$ and beyond. By Lemma 9.1,

$$g_1 = 2m^{1/2} + \frac{3}{8}m^{3/2} + \frac{23}{128}m^{5/2} + O(m^{7/2}), \qquad g_1' = m^{-1/2} + \frac{9}{16}m^{1/2} + \frac{115}{256}m^{3/2} + O(m^{5/2}),$$

$$g_3 = \frac{1}{24}m^{3/2} + \frac{1}{32}m^{5/2} + O(m^{7/2}), \qquad g_3' = \frac{1}{16}m^{1/2} + \frac{5}{64}m^{3/2} + O(m^{5/2}),$$

and, from the standard series for K, E (as in §§ 8 to 9),

$$\ell = \frac{\pi}{2}\Big(\frac{m}{2} + \frac{m^2}{16} + \frac{3m^3}{128} + O(m^4)\Big), \qquad \frac{4\ell}{K} = 2m - \frac{m^2}{4} - \frac{m^3}{8} + O(m^4).$$

Then

$$\frac{4\ell}{K}g_1' = \Big(2m - \frac{m^2}{4} - \frac{m^3}{8}\Big)\Big(m^{-1/2} + \frac{9}{16}m^{1/2} + \frac{115}{256}m^{3/2}\Big) + O(m^{7/2})$$
$$= 2m^{1/2} + \frac{7}{8}m^{3/2} + \frac{81}{128}m^{5/2} + O(m^{7/2}),$$

so

$$g_1 - \frac{4\ell}{K}g_1' = \Big(\frac{3}{8} - \frac{7}{8}\Big)m^{3/2} + \Big(\frac{23}{128} - \frac{81}{128}\Big)m^{5/2} + O(m^{7/2}) = -\frac{1}{2}m^{3/2} - \frac{29}{64}m^{5/2} + O(m^{7/2}),$$

and

$$g_3 - \frac{4\ell}{K}g_3' = \frac{1}{24}m^{3/2} + \frac{1}{32}m^{5/2} - \Big(\frac{1}{8}m^{3/2} + \frac{9}{64}m^{5/2}\Big) + O(m^{7/2}) = -\frac{1}{12}m^{3/2} - \frac{7}{64}m^{5/2} + O(m^{7/2}),$$

whence $3\big(g_3 - \frac{4\ell}{K}g_3'\big) = -\frac{1}{4}m^{3/2} - \frac{21}{64}m^{5/2} + O(m^{7/2})$. □

Corollary 10.1 (Structure of the residual)**.** *Let $\Lambda \subset (0,\infty)$ be compact, $\delta > 0$, $\nu \in \Lambda$, $\tau \ge \pi\nu + \delta$, and $m_* := \max_{\nu\in\Lambda} \ell^{-1}\big(\frac{\pi\nu}{\pi\nu+\delta}\big) < 1$, so that $m_0 \le m_*$. Then*

$$\frac{2\pi\nu}{K(m_*)}\,\tau^{-1} \le m_0(\tau) \le 4\nu\,\tau^{-1}, \qquad R := L[w_{\mathrm{ad}}] = T_1 + \cdots + T_5,$$

$$T_1 = \tau^{-2}m_0^{1/2}S_1, \quad T_2 = \tau^{-1}m_0^{3/2}S_2' + \tau^{-3}m_0^{1/2}S_2'', \quad T_3 = \tau^{-1}m_0^{3/2}S_3, \quad T_4 = \tau^{-2}m_0^{1/2}S_4, \quad T_5 = \tau^{-2}m_0^{1/2}S_5,$$

where each $S = S(\Phi, m_0, 1/\tau)$ is real-analytic and 2π-periodic in Φ, and bounded together with all its derivatives on $\mathbb{R}\times[0,m_*]\times[0,1/\delta]$, uniformly for $\nu \in \Lambda$.

Proof. Since $\ell' = K/2 \geq \pi/4$ and $\ell' \leq K(m_*)/2$ on $[0, m_*]$, $\frac{\pi}{4}m \leq \ell(m) \leq \frac{K(m_*)}{2}m$, which gives the bounds on m_0 from $\ell(m_0) = \pi\nu/\tau$. Write $\ell = m\,\tilde{\ell}(m)$, $\frac{2\ell}{K} = m\lambda(m)$ with $\tilde{\ell}, \lambda$ analytic and positive, so $m_0' = -\frac{2\ell}{K\tau} = -\frac{m_0\lambda(m_0)}{\tau}$ and $m_0'' + \frac{m_0'}{\tau} = \frac{2\ell(K^2 - 2\ell K_m)}{K^3\tau^2} = \frac{m_0}{\tau^2}\,\varsigma(m_0)$ with ς analytic (Proposition 8.2, $K_m = dK/dm$ analytic). By Lemma 9.2, $G = \sqrt{m}\hat{G}$, $G_\Phi = \sqrt{m}\hat{G}_\Phi$, $G_m = m^{-1/2}(\frac{1}{2}\hat{G} + m\hat{G}_m)$, $G_{mm} = m^{-3/2}(-\frac{1}{4}\hat{G} + m\hat{G}_m + m^2\hat{G}_{mm})$, $\sin G = \sqrt{m}\,\hat{G}\,\mathrm{sinc}(\sqrt{m}\hat{G})$, and $B = m^{3/2}[s\hat{G}_\Phi - t\hat{G}_{\Phi m}]$ (Lemma 10.1). Moreover $1 - \Phi'^2/\omega^2 = -\frac{1}{4\omega\tau^2} - \frac{1}{64\omega^2\tau^4}$ and $\Phi'' = \omega'(m_0)m_0' - \frac{1}{4\tau^3} = -\frac{m_0}{\tau}\omega'\lambda - \frac{1}{4\tau^3}$. Substituting these into $T_1, \ldots, T_5$ of Proposition 10.1 gives the stated forms. □

Theorem 10.1. *For $\nu \in \Lambda$ and $\tau \geq \pi\nu + \delta$, uniformly in $\Theta \in \mathbb{R}$ (the constant Θ_0 in Φ replaced by an arbitrary Θ),*

$$|R| \leq C\tau^{-5/2}, \qquad |\partial_\Theta R| \leq C\tau^{-5/2}, \qquad |\partial_\nu R|_\Theta \leq C\tau^{-5/2}(1 + \ln\tau),$$

with $C = C(\Lambda, \delta)$.

Proof. By Corollary 10.1 and $m_0 \asymp \tau^{-1}$: $T_1 = O(\tau^{-2}\tau^{-1/2})$; $T_2 = O(\tau^{-1}\tau^{-3/2}) + O(\tau^{-3}\tau^{-1/2})$; $T_3 = O(\tau^{-1}\tau^{-3/2})$; $T_4, T_5 = O(\tau^{-2}\tau^{-1/2})$, i.e. $R = O(\tau^{-5/2})$ uniformly in Φ (hence in Θ) and $\nu \in \Lambda$. Since Θ enters only through Φ and the S_i are bounded with all Φ-derivatives, $\partial_\Theta R = \partial_\Phi R = O(\tau^{-5/2})$. For the ν-derivative at fixed Θ: $\partial_\nu m_0 = \frac{2\pi}{\tau K(m_0)} = O(\tau^{-1})$ and $\partial_\nu\Phi = \tau f'(m_0)\partial_\nu m_0 = -\frac{\pi K(1-m_0)}{K(m_0)} = O(1 + \ln\tau)$ (because $K(1-m) = \ln\frac{4}{\sqrt{m}} + O(m\ln\frac{1}{m})$, DLMF [3, §19.12.1], and $m_0 \geq c/\tau$). A term $\tau^{-a}m_0^b S(\Phi, m_0, 1/\tau)$ therefore changes by $\tau^{-a}m_0^b\big[\,b\,m_0^{-1}S + \partial_m S\,\big]\partial_\nu m_0 + \tau^{-a}m_0^b\partial_\Phi S\,\partial_\nu\Phi$, which is $O(\tau^{-a}m_0^b(1+\ln\tau))$ since $m_0^{-1}\partial_\nu m_0 = O(1)$. □

Term-by-term orders (elementary form of the bound on R). For $\tau \geq \pi\nu + \delta$: $m_0 \leq m_{\delta'} := \ell^{-1}\big(\frac{\pi\nu}{\pi\nu+\delta}\big) < 1$, and since $\ell(m) = \frac{\pi}{4}m\,(1 + O(m))$ uniformly on $[0, m_{\delta'}]$, $c_1/\tau \leq m_0(\tau) \leq c_2/\tau$ with $c_i = c_i(\nu, \delta)$; hence $m_0^p = \Theta(\tau^{-p})$ for every fixed p. Now (with $G = \sqrt{m}\hat{G}$ of Lemma 9.2, so that $G_\Phi = O(\sqrt{m})$, $G_m = O(m^{-1/2})$, $G_{mm} = O(m^{-3/2})$):

T_1: $|\sin G| \leq |G| \leq 2\arcsin\sqrt{m} = O(\tau^{-1/2})$ and $1 - \Phi'^2/\omega^2 = O(\tau^{-2})$;

T_2: $G_\Phi = O(\sqrt{m}) = O(\tau^{-1/2})$ and $\Phi'' = \omega'(m_0)m_0' - \frac{1}{4\tau^3} = O(\tau^{-2})$;

T_3: $\Phi'/\tau = O(\tau^{-1})$ and $B = O(m^{3/2}) = O(\tau^{-3/2})$ (Lemma 10.1);

T_4: $G_{mm} = O(m^{-3/2}) = O(\tau^{3/2})$ and $m_0'^2 = \big(\frac{2\ell}{K\tau}\big)^2 = O(m^2/\tau^2) = O(\tau^{-4})$;

T_5: $G_m = O(m^{-1/2}) = O(\tau^{1/2})$ and $m_0'' + \frac{m_0'}{\tau} = \frac{2\ell(K^2 - 2\ell K')}{K^3\tau^2} = O(m/\tau^2) = O(\tau^{-3})$.

Each product is $O(\tau^{-5/2})$, uniformly in Φ (all derivatives of $\hat{G}$ are bounded on $\mathbb{R} \times [0, m_{\delta'}]$, Lemma 9.2) and in ν on compacts.

11 Absence of resonance

Lemma 11.1. *At order $O(\tau^{-5/2})$:*

$$L[w_{\mathrm{ad}}] = -2\nu^{3/2}\tau^{-5/2}\cos 3\Phi + O(\tau^{-7/2}),$$

and the coefficients of $\sin\Phi$, $\cos\Phi$ vanish. More precisely, $R = -2\nu^{3/2}\tau^{-5/2}\cos 3\Phi + \tau^{-7/2}\rho_1(\Phi,\tau,\nu)$ with $|\rho_1| + |\partial_\Phi\rho_1| \le C$ for $\tau \ge \tau_0$, ν in a compact set.

Proof. **(a) Asymptotics of m_0.** From $\ell = \frac{\pi}{2}\big(\frac{m}{2} + \frac{m^2}{16} + \frac{3m^3}{128} + O(m^4)\big)$ and $\tau\ell(m_0) = \pi\nu$, reversion (§ 14, Step 2) gives

$$m_0 = \frac{4\nu}{\tau} - \frac{2\nu^2}{\tau^2} - \frac{\nu^3}{\tau^3} + O(\tau^{-4}),$$

hence, with all relative corrections $O(\tau^{-1})$:

$$m_0 = \frac{4\nu}{\tau}\big(1 + O(\tau^{-1})\big), \quad m_0' = -\frac{4\nu}{\tau^2}\big(1 + O(\tau^{-1})\big), \quad m_0'' + \frac{m_0'}{\tau} = \frac{4\nu}{\tau^3}\big(1 + O(\tau^{-1})\big).$$

Also, from $\omega = 1 - \frac{m}{4} + O(m^2)$ (so $\omega'(0) = -\frac{1}{4}$) and Proposition 8.2:

$$\Phi'' = \omega'(m_0)m_0' - \frac{1}{4\tau^3} = \frac{\nu}{\tau^2}\big(1 + O(\tau^{-1})\big).$$

(b) Leading harmonics of the terms. At leading order in m all m-powers are evaluated at $m = 4\nu/\tau$; every correction contributes only $O(\tau^{-7/2})$. From Lemma 9.1, $\sin G = G - \frac{G^3}{6} + O(m^{5/2}) = \big(2\sqrt{m} + \frac{3}{8}m^{3/2}\big)\sin\Phi + \frac{1}{24}m^{3/2}\sin 3\Phi - \frac{4}{3}m^{3/2}\sin^3\Phi + O(m^{5/2})$, i.e.

$$\sin G = \Big(2\sqrt{m} - \frac{5}{8}m^{3/2}\Big)\sin\Phi + \frac{3}{8}m^{3/2}\sin 3\Phi + O(m^{5/2}).$$

Since $G_\Phi, G_{\Phi m}, G_m, G_{mm}$ are odd-sine / even-cosine series respectively, only T_1, T_4, T_5 carry $\sin\Phi$, and only T_2, T_3 carry $\cos\Phi, \cos 3\Phi$, at this order. Using $1 - \Phi'^2/\omega^2 = -\frac{1}{4\omega\tau^2} + O(\tau^{-4})$, $\Phi'/\tau = \tau^{-1}(1 + O(\tau^{-1}))$, and Lemma 10.1:

$$T_1\big|_{\sin\Phi} = -\frac{2\sqrt{m}}{4\tau^2} = -\sqrt{\nu}\,\tau^{-5/2}\big(1 + O(\tau^{-1})\big),$$

$$T_2\big|_{\cos\Phi} = g_1\,\Phi'' = \Big(2\sqrt{m} + \frac{3}{8}m^{3/2}\Big)\frac{\nu}{\tau^2} = 2\sqrt{m}\,\frac{\nu}{\tau^2} + O(\tau^{-7/2}) = \frac{1}{2}m^{3/2}\tau^{-1} + O(\tau^{-7/2}),$$

$$T_3\big|_{\cos\Phi} = \frac{\Phi'}{\tau}\Big(-\frac{1}{2}m^{3/2}\Big) = -\frac{1}{2}m^{3/2}\tau^{-1} + O(\tau^{-7/2}), \qquad T_3\big|_{\cos 3\Phi} = -\frac{1}{4}m^{3/2}\tau^{-1} + O(\tau^{-7/2}),$$

$$T_4\big|_{\sin\Phi} = -\frac{1}{2}m^{-3/2}\cdot\frac{m^2}{\tau^2}\big(1 + O(\tau^{-1})\big) = -\frac{1}{2}m^{1/2}\tau^{-2}\big(1 + O(\tau^{-1})\big) = -\sqrt{\nu}\,\tau^{-5/2}\big(1 + O(\tau^{-1})\big),$$

$$T_5\big|_{\sin\Phi} = m^{-1/2}\cdot\frac{4\nu}{\tau^3}\big(1 + O(\tau^{-1})\big) = \frac{m^{1/2}}{\tau^2}\big(1 + O(\tau^{-1})\big) = 2\sqrt{\nu}\,\tau^{-5/2}\big(1 + O(\tau^{-1})\big),$$

where in the last two lines $m^{-1/2}m_0'^2 = -\frac{1}{2}m^{1/2}/\tau^2$ used $m_0'^2 = m^2\tau^{-2}(1+O(\tau^{-1}))$ and $m^{-1/2}\cdot\frac{4\nu}{\tau^3} = \frac{4\nu}{\tau^3}\sqrt{\frac{\tau}{4\nu}} = \frac{\sqrt{4\nu/\tau}}{\tau^2}$.

(c) Balance. In units of $2\sqrt{\nu}\,\tau^{-5/2}$:

$$\sin\Phi: \quad T_1 = -\tfrac{1}{2}, \quad T_4 = -\tfrac{1}{2}, \quad T_5 = +1 \quad \Rightarrow \quad \text{sum} = 0;$$

$$\cos\Phi: \quad T_2 = +\tfrac{1}{2}m^{3/2}\tau^{-1}, \quad T_3 = -\tfrac{1}{2}m^{3/2}\tau^{-1} \quad \Rightarrow \quad \text{sum} = 0;$$

$$\cos 3\Phi: \quad T_3 = -\tfrac{1}{4}m^{3/2}\tau^{-1} = -2\nu^{3/2}\tau^{-5/2}.$$

All remaining contributions to harmonics 1 and 3 are $O(\tau^{-7/2})$ (e.g. the $\sin 3\Phi$-part of T_1 is $-\frac{3m^{3/2}}{32\tau^2}$).

(d) Remainder. By Corollary 10.1, each $T_i = \tau^{-a_i}m_0^{b_i}S_i(\Phi, m_0, 1/\tau)$, and $m_0 = \frac{4\nu}{\tau}(1+O(\tau^{-1}))$, $S_i(\Phi, m_0, 1/\tau) = S_i(\Phi, 0, 0) + O(\tau^{-1})$ together with all Φ-derivatives. Hence $T_i = \tau^{-a_i-b_i}(4\nu)^{b_i}S_i(\Phi,0,0) + O(\tau^{-a_i-b_i-1})$; all $a_i + b_i = \frac{5}{2}$ except the second part of T_2 ($\frac{7}{2}$). The leading profiles $S_i(\Phi,0,0)$ are exactly the harmonics computed in (b), (c), so $R = -2\nu^{3/2}\tau^{-5/2}\cos 3\Phi + \tau^{-7/2}\rho_1$ with $\rho_1, \partial_\Phi\rho_1$ bounded. Uniformity in Θ is as in Theorem 10.1. □

12 Approximate fundamental system

Throughout §§ 12 to 13, ν ranges over a fixed compact $\Lambda \subset (0,\infty)$, $\tau \geq \tau_0$ with $\tau_0 = \tau_0(\Lambda)$ large, constants C depend only on Λ, and we use the one-parameter family $w_{\mathrm{ad}}(\tau;\Theta,\nu) = G(\Phi, m_0)$, $\Phi = \tau f(m_0) + \Theta - \frac{1}{8\tau}$, $\Theta = \Theta_0(\nu)$ at the end; $R = L[w_{\mathrm{ad}}]$ as in Theorem 10.1 (valid for every Θ).

Both the approximate fundamental system and the $O(\tau^{-5/2})$ bound below are obtained by elementary tools (variation of constants and oscillatory integral estimates); no WKB-type ansatz is used.

Lemma 12.1. *Let $P_1 = \partial_\Theta w_{\mathrm{ad}} = G_\Phi$, $\Pi := G_m\frac{2\pi}{\tau K}$, and*

$$P_2 = \partial_\nu\big[w_{\mathrm{ad}}(\tau;\Theta_0(\nu),\nu)\big] = \Pi - c(\tau)P_1, \qquad c(\tau) := \frac{\pi K(1-m_0)}{K(m_0)} - \Theta_0'(\nu),$$

$$\Theta_0'(\nu) = \frac{d\Theta_0}{d\nu} = -\ln\nu + \frac{d}{d\nu}\arg\Gamma(1+i\nu).$$

Put $E_i := L'[P_i]$. Then for $\tau \geq \tau_0$:

(i) $|P_1|, |P_1'|, |\Pi|, |\Pi'| \leq C\tau^{-1/2}$, $|P_2|, |P_2'| \leq C\tau^{-1/2}\ln\tau$;

(ii) $|E_1| \leq C\tau^{-5/2}$, $|E_2| \leq C\tau^{-5/2}\ln\tau$;

(iii) $c'(\tau) = \kappa(m_0)/\tau$ with $\kappa(m) = \frac{\pi^2\ell(m)}{2m(1-m)K(m)^3} = 1 + O(m)$; hence $|c'| \leq C/\tau$, $|c(\tau)| \leq C\ln\tau$, $|c(\sigma) - c(\tau)| \leq C\ln\frac{\sigma}{\tau}$ for $\sigma \geq \tau$;

(iv) $W := P_1P_2' - P_1'P_2$ satisfies $\tau W(\tau) = 8 + O(\tau^{-1})$; in particular $W \geq 4/\tau$. Moreover the exact identity $(\tau W)' = \tau(P_1E_2 - P_2E_1) = O(\tau^{-2}\ln\tau)$ holds, so also $\tau W = 8 + O(\tau^{-1}\ln\tau)$ by integration from ∞.

Proof. **(a) The identification of P_1, P_2.** $\partial_\Theta \Phi = 1$, so $P_1 = G_\Phi$. At fixed Θ, $\partial_\nu m_0 = \frac{2\pi}{\tau K}$ (differentiate $\tau\ell(m_0) = \pi\nu$ and use $\ell' = K/2$) and $\partial_\nu \Phi = \tau f'(m_0)\partial_\nu m_0 = -\frac{\pi K(1-m_0)}{K(m_0)}$ since $f'(m) = -K(1-m)/2$; the total derivative in ν with $\Theta = \Theta_0(\nu)$ adds $\Theta_0'(\nu)G_\Phi$. Hence $P_2 = \Pi + G_\Phi\big(\Theta_0' - \frac{\pi K(1-m_0)}{K}\big) = \Pi - cP_1$. The formula for Θ_0' follows from $\Theta_0 = C_2 + \nu(1+\ln\frac{8}{\nu})$ and $C_2 = \frac{5\pi}{4} + \arg\Gamma(1+i\nu) - 3\nu\ln 2$: $\Theta_0' = \frac{d}{d\nu}\arg\Gamma(1+i\nu) - 3\ln 2 + 1 + \ln\frac{8}{\nu} - 1 = -\ln\nu + \frac{d}{d\nu}\arg\Gamma(1+i\nu)$ because $\ln 8 = 3\ln 2$.

(b) Estimates (i). By Lemma 9.2, $P_1 = \sqrt{m_0}\,\hat G_\Phi = O(\tau^{-1/2})$, and

$$P_1' = \sqrt{m_0}\,\hat G_{\Phi\Phi}\Phi' + \Big(\tfrac{1}{2\sqrt{m_0}}\hat G_\Phi + \sqrt{m_0}\hat G_{\Phi m}\Big)m_0' = O(\tau^{-1/2}) + O(m_0^{-1/2}\tfrac{m_0}{\tau}) = O(\tau^{-1/2}).$$

Next $\Pi = \theta(\tau)\big(\sin\Phi + r_2(\Phi, m_0)\big)$ with $\theta = \frac{2\pi}{\tau K(m_0)\sqrt{m_0}} \asymp \tau^{-1/2}$, $|\theta'| \le C\theta/\tau$, $r_2 = \frac12\hat G + m_0\hat G_m - \sin\Phi = O(m_0)$ smooth (because $\frac12\hat G_0 = \sin\Phi$); so $|\Pi| \le C\tau^{-1/2}$, $\Pi' = \theta\Phi'\cos\Phi + O(\theta/\tau)$, $|\Pi'| \le C\tau^{-1/2}$. Finally $|c| \le C(1+\ln\tau)$ by (iii), and $P_2' = \Pi' - c'P_1 - cP_1'$ gives the bounds on P_2, P_2'.

(c) Estimates (ii). Differentiating $L[w_{\rm ad}(\Theta,\nu)] = R$ in the parameters gives $L'[\partial_\Theta w_{\rm ad}] = \partial_\Theta R$ and $L'[P_2] = \partial_\nu R|_\Theta + \Theta_0'\partial_\Theta R$ (here $L'[u] = u'' + \frac1\tau u' + \cos(w_{\rm ad})u$ and the operator L does not depend on the parameters). Theorem 10.1 gives $|E_1| \le C\tau^{-5/2}$, $|E_2| \le C\tau^{-5/2}(1+\ln\tau)$.

(d) Proof of (iii). From $c = \pi K(1-m_0)/K(m_0) - \Theta_0'$, $\frac{d}{dm}K(1-m) = -\frac{f(m)}{2m(1-m)}$ (since $\frac{dK}{d\mu} = \frac{E(\mu)-(1-\mu)K(\mu)}{2\mu(1-\mu)}$ at $\mu = 1-m$), $K_m = \frac{\ell}{2m(1-m)}$ and $m_0' = -\frac{2\ell}{K\tau}$:

$$\begin{aligned} c' &= \pi\Big[-\frac{f}{2m(1-m)K} - \frac{K(1-m)\,\ell}{2m(1-m)K^2}\Big]\Big(-\frac{2\ell}{K\tau}\Big) \\ &= \frac{\pi\ell}{m(1-m)K^3\tau}\big[Kf + \ell K(1-m)\big] = \frac{\pi^2\ell}{2m(1-m)K^3\tau}, \end{aligned}$$

by the Legendre relation $Kf + \ell K(1-m) = \pi/2$ (Proposition 8.1). As $m \to 0$, $\ell = \frac{\pi m}{4}(1+O(m))$, $K = \frac{\pi}{2}(1+O(m))$, so $\kappa(m) = 1 + O(m)$; $m_0 = O(\tau^{-1})$ gives $c' = \frac1\tau + O(\tau^{-2})$. Integrating $|c'| \le C/\sigma$ gives $|c(\sigma) - c(\tau)| \le C\ln\frac{\sigma}{\tau}$, and $|c(\tau)| \le C(1+\ln\tau)$ from $K(1-m_0) = \ln\frac{4}{\sqrt{m_0}} + O(m_0\ln\frac{1}{m_0})$ and $m_0 \asymp \tau^{-1}$.

(e) Proof of (iv). Since $P_2 = \Pi - cP_1$: $W = P_1\Pi' - P_1'\Pi - c'P_1^2$ (the terms cP_1P_1' cancel). By (b), $P_1 = 2\sqrt{m_0}\cos\Phi + O(\tau^{-3/2})$, $P_1' = -2\sqrt{m_0}\Phi'\sin\Phi + O(\tau^{-3/2})$, $\Pi = \theta(\sin\Phi + O(\tau^{-1}))$, $\Pi' = \theta\Phi'\cos\Phi + O(\theta\tau^{-1})$. Thus

$$P_1\Pi' - P_1'\Pi = 2\sqrt{m_0}\,\theta\,\Phi'(\cos^2\Phi + \sin^2\Phi) + O(\tau^{-2}) = \frac{4\pi\Phi'}{\tau K(m_0)} + O(\tau^{-2}),$$

using $\sqrt{m_0}\,\theta = \frac{2\pi}{\tau K}$, and $c'P_1^2 = O(\tau^{-1}\cdot\tau^{-1}) = O(\tau^{-2})$. With $\Phi' = \omega(m_0) + \frac{1}{8\tau^2} = \frac{\pi}{2K} + O(\tau^{-2})$ we get $\tau W = \frac{2\pi^2}{K(m_0)^2} + O(\tau^{-1}) = 8 + O(\tau^{-1})$, because $K(m_0) = \frac{\pi}{2}(1+O(m_0))$. The Abel identity: $W' = P_1P_2'' - P_1''P_2$, and $P_i'' = E_i - \frac1\tau P_i' - \cos(w_{\rm ad})P_i$, so $W' = P_1E_2 - P_2E_1 - \frac1\tau W$, i.e. $(\tau W)' = \tau(P_1E_2 - P_2E_1) = \tau\big(O(\tau^{-1/2})O(\tau^{-5/2}\ln\tau) + O(\tau^{-1/2}\ln\tau)O(\tau^{-5/2})\big) = O(\tau^{-2}\ln\tau)$; integrating from τ to ∞ (where $\tau W \to 8$) gives $\tau W(\tau) - 8 = O(\tau^{-1}\ln\tau)$. □

13 Cauchy formula and $O(\tau^{-5/2})$ bound

Lemma 13.1 (Oscillatory integrals)**.** *Let $\tau \ge 1$ and $\Phi \in C^2[\tau,\infty)$ with $\Phi' \in [\frac12,\frac32]$, $|\Phi''| \le A\sigma^{-2}$; let $a \in C^1$ with $|a| \le B\sigma^{-q}$, $|a'| \le B\sigma^{-q-1}$ for some $q>0$, and $k \ge 1$ an integer. Then*

(a) $\left|\int_\tau^\infty a\, e^{ik\Phi} d\sigma\right| \le \dfrac{C_{q,A}\, B}{k}\, \tau^{-q}$;

(b) if moreover $\lambda \in C^1[\tau,\infty)$, $\lambda(\tau)=0$, $|\lambda'| \le \Lambda_0\sigma^{-1}$, then $\left|\int_\tau^\infty \lambda\, a\, e^{ik\Phi} d\sigma\right| \le \dfrac{C_{q,A}\, B\Lambda_0}{k}\, \tau^{-q}$.

Proof. (a) Integrate by parts with $e^{ik\Phi} = \frac{(e^{ik\Phi})'}{ik\Phi'}$: the boundary term at ∞ vanishes ($a \to 0$), the one at τ is at most $\frac{2B\tau^{-q}}{k}$, and the remaining integral is at most $\frac1k\int_\tau^\infty \big(2|a'| + 4|a||\Phi''|\big)d\sigma \le \frac{B}{k}\big(\frac2q + \frac{4A}{q+1}\big)\tau^{-q}$. (b) Apply the same argument to λa: $|\lambda(\sigma)| \le \Lambda_0 \ln\frac{\sigma}{\tau}$, so $|\lambda a| \le \Lambda_0 B\sigma^{-q}\ln\frac{\sigma}{\tau} \to 0$; the boundary term at τ vanishes since $\lambda(\tau)=0$; and $|(\lambda a)'| \le \Lambda_0 B\sigma^{-q-1}(1+\ln\frac{\sigma}{\tau})$, $\int_\tau^\infty \sigma^{-q-1}(1+\ln\frac{\sigma}{\tau})d\sigma = \tau^{-q}(\frac1q + \frac{1}{q^2})$, while $\int_\tau^\infty |\lambda a||\Phi''| \le \Lambda_0 BA\int \sigma^{-q-2}\ln\frac{\sigma}{\tau}\, d\sigma \le C\tau^{-q}$. □

Lemma 13.2. *Let $\varepsilon \in C^2[\tau_0,\infty)$ solve*

$$L'[\varepsilon] = -R - N(\varepsilon), \qquad N(\varepsilon) := \sin(w_{\mathrm{ad}}+\varepsilon) - \sin w_{\mathrm{ad}} - \cos(w_{\mathrm{ad}})\,\varepsilon, \quad |N| \le \tfrac12\varepsilon^2,$$

(this is the equation satisfied by $\varepsilon = w - w_{\mathrm{ad}}$ for any solution w of $L[w]=0$), and assume the a priori decay $|\varepsilon(\tau)| \le C_0\tau^{-3/2}$ for $\tau \ge \tau_0$. Then $|\varepsilon(\tau)| \le C_\delta\tau^{-5/2}$ for $\tau \ge \tau_0$ (with τ_0 possibly increased), C_δ depending on C_0, ν, δ. The decay assumption excludes the homogeneous solutions P_1, P_2, whose exact order is $\tau^{-1/2}$, resp. $\tau^{-1/2}\ln\tau$; for $\varepsilon = y - y_{\mathrm{ad}}$ it is supplied by (M) and Lemma 15.1.

Proof. **Step 1 (variation of constants with the approximate system).** By Lemma 12.1(iv), $W \ne 0$, so any pair $(\varepsilon,\varepsilon')$ can be written $\varepsilon = \xi_1P_1 + \xi_2P_2$, $\varepsilon' = \xi_1P_1' + \xi_2P_2'$, i.e. $\xi_1'P_1 + \xi_2'P_2 = 0$. Then $L'[\varepsilon] = \xi_1E_1 + \xi_2E_2 + \xi_1'P_1' + \xi_2'P_2'$, so $\xi_1'P_1' + \xi_2'P_2' = -R - N - \xi\cdot E$ and

$$\xi_1' = \frac{P_2}{W}\,\zeta, \qquad \xi_2' = -\frac{P_1}{W}\,\zeta, \qquad \zeta := R + N + \xi_1E_1 + \xi_2E_2 = R + \zeta_2.$$

(Note that, since $E_i \ne 0$, P_1, P_2 are not exact solutions, which is why the terms $\xi\cdot E$ are kept.)

Step 2 (ξ has a limit). By Lemma 12.1, $1/W \le C\sigma$, $|P_2| \le C\sigma^{-1/2}\ln\sigma$, $|P_1| \le C\sigma^{-1/2}$, $|R| \le C\sigma^{-5/2}$, $|N| \le \frac12 C_0^2\sigma^{-3}$, hence

$$|\xi'| \le C\sigma^{1/2}\ln\sigma\big(\sigma^{-5/2} + \sigma^{-3} + |\xi|\,\sigma^{-5/2}\ln\sigma\big) \le C\big(\sigma^{-2}\ln\sigma + |\xi|\,\sigma^{-2}\ln^2\sigma\big).$$

Since the coefficients are integrable on $[\tau_0,\infty)$, Gronwall's inequality shows that ξ is bounded, and then ξ' is integrable, so $\xi(\tau) \to \xi_\infty = (a,b)$ with $|\xi(\tau) - \xi_\infty| \le C\tau^{-1}\ln^2\tau$.

Step 3 ($\xi_\infty = 0$). Write $\varepsilon = \alpha P_1 + \beta\Pi$ with $\alpha = \xi_1 - c(\tau)\xi_2$, $\beta = \xi_2$ (because $P_2 = \Pi - cP_1$). By Step 2, $\beta \to b$ and $\alpha = a - c(\tau)b + o(1)$ (using $|c| \le C\ln\tau$ and $|\xi_2 - b| \le C\tau^{-1}\ln^2\tau$).

If $(a,b)\neq 0$, then $\max(|\alpha|,|\beta|)\geq\kappa_0>0$ for large τ (if $b\neq 0$ then $|\beta|\to|b|$; if $b=0$ then $\alpha\to a\neq 0$). By Lemma 12.1(b), $P_1=4\sqrt{\nu}\,\sigma^{-1/2}\cos\Phi+O(\sigma^{-3/2})$ and $\Pi=2(\nu\sigma)^{-1/2}\sin\Phi+O(\sigma^{-3/2})$ (use $m_0=\frac{4\nu}{\sigma}(1+O(\sigma^{-1}))$), and on any interval $[\sigma,\sigma+13]$ the phase Φ runs through more than a full period ($\Phi'\geq\frac12$) while α,β,c change by $O(\sigma^{-1})$; therefore $\sup_{[\sigma,\sigma+13]}|\varepsilon|\geq\kappa_1\sigma^{-1/2}\max(|\alpha|,|\beta|)-o(\sigma^{-1/2})\geq\frac{\kappa_1\kappa_0}{2}\sigma^{-1/2}$, contradicting $|\varepsilon|\leq C_0\sigma^{-3/2}$. Hence $\xi_\infty=0$ and

$$\xi_1(\tau)=-\int_\tau^\infty\frac{P_2\zeta}{W}d\sigma,\qquad \xi_2(\tau)=\int_\tau^\infty\frac{P_1\zeta}{W}d\sigma.$$

Substituting $P_2=\Pi-c(\sigma)P_1$ and regrouping with $\varepsilon=\big(\xi_1-c(\tau)\xi_2\big)P_1+\xi_2\Pi$:

$$\varepsilon=P_1\Big[-\int_\tau^\infty\frac{\Pi\zeta}{W}d\sigma+\int_\tau^\infty\big(c(\sigma)-c(\tau)\big)\frac{P_1\zeta}{W}d\sigma\Big]+\Pi\int_\tau^\infty\frac{P_1\zeta}{W}d\sigma.\tag{$\star$}$$

This regrouping is exactly what removes the apparent logarithm $c(\tau)\sim\ln\tau$ of P_2.

Step 4 (the resonance structure of R). Put $H_1:=P_1R/W$, $H_2:=\Pi R/W$. By § 14 (Step 2), $m_0=\sigma^{-1}\mu(1/\sigma)$ with μ analytic, $\mu(0)=4\nu$. By Lemma 9.2, Corollary 10.1 and Lemma 12.1, P_1, Π, R, W are of the form $\sigma^{-p}\times$(a function $\hat h(\Phi,1/\sigma)$, real-analytic in a neighbourhood of $\mathbb{R}\times[0,1/\tau_0]$, 2π-periodic and bounded in a complex strip $|\mathrm{Im}\,\Phi|<\eta$), with $p=\frac12,\frac12,\frac52,1$ respectively, and $\sigma W\geq 4$ so that $1/(\sigma W)$ is again of this form ($\sigma W=8$ at $1/\sigma=0$). Hence

$$H_i(\sigma)=\sigma^{-2}\hat h_i\big(\Phi(\sigma),1/\sigma\big),\qquad \hat h_i(\Phi,0)=h_i^0(\Phi),$$

$$h_1^0=\tfrac18\cdot4\sqrt{\nu}\cos\Phi\cdot(-2\nu^{3/2}\cos3\Phi)=-\nu^2\cos\Phi\cos3\Phi,$$
$$h_2^0=\tfrac18\cdot2\nu^{-1/2}\sin\Phi\cdot(-2\nu^{3/2}\cos3\Phi)=-\tfrac{\nu}{2}\sin\Phi\cos3\Phi,$$

(by Lemma 11.1 and the leading forms of P_1,Π,W), and both h_i^0 have mean zero in Φ (they contain only the harmonics $2\Phi,4\Phi$). Expand in Fourier series in Φ: $H_i=\sum_{n\in\mathbb{Z}}\beta_n^{(i)}(\sigma)e^{in\Phi(\sigma)}$ with $\beta_n^{(i)}(\sigma)=\sigma^{-2}\hat\beta_n^{(i)}(1/\sigma)$, $|\hat\beta_n^{(i)}|\leq Ce^{-\eta|n|}$ (analyticity in the strip), $|\partial_\sigma\beta_n|\leq C\sigma^{-3}e^{-\eta|n|}$, and, because the mean of h_i^0 vanishes, $\beta_0^{(i)}=O(\sigma^{-3})$, $\beta_0^{(i)\prime}=O(\sigma^{-4})$. By Lemma 13.1 with $q=2$ (note $\Phi'=\omega+O(\sigma^{-2})\in[\frac12,\frac32]$, $\Phi''=O(\sigma^{-2})$) and $\sum_{n\neq0}e^{-\eta|n|}/|n|<\infty$:

$$I_i(\tau):=\int_\tau^\infty H_i\,d\sigma=O(\tau^{-2})\ (i=1,2),\qquad I_3(\tau):=\int_\tau^\infty\big(c(\sigma)-c(\tau)\big)H_1\,d\sigma=O(\tau^{-2}).$$

For I_1,I_2: the $n\neq0$ terms by Lemma 13.1(a), the $n=0$ term is bounded by $\int_\tau^\infty C\sigma^{-3}=O(\tau^{-2})$. For I_3: the $n\neq0$ terms by Lemma 13.1(b) with $\lambda=c(\sigma)-c(\tau)$, $\Lambda_0=C$ (Lemma 12.1(iii)); the $n=0$ term is bounded by $C\int_\tau^\infty\ln\frac{\sigma}{\tau}\,\sigma^{-3}d\sigma=\frac{C}{4\tau^2}$ ($\int_1^\infty\ln u\,u^{-3}du=\frac14$). (A purely absolute bound would give $\int_\tau^\infty\ln\frac{\sigma}{\tau}\,\sigma^{-2}d\sigma=\tau^{-1}$, which is insufficient; the oscillation is essential.)

Inserting $\zeta=R+\zeta_2$ into ($\star$), the R-part is

$$\varepsilon^{(R)}=P_1\big[-I_2+I_3\big]+\Pi\,I_1=O(\tau^{-1/2}\tau^{-2})=O(\tau^{-5/2}),$$

and likewise $\xi_2^{(R)}=I_1=O(\tau^{-2})$, $\xi_1^{(R)}=-\int P_2R/W=-I_2+c(\tau)I_1+I_3=O(\tau^{-2}\ln\tau)$.

Step 5 (the remaining terms; two passes). By ($\star$) and $|\Pi/W|, |P_1/W| \le C\sigma^{1/2}$, the ζ_2-part $\varepsilon^{(2)}$ satisfies

$$|\varepsilon^{(2)}(\tau)| \le C\tau^{-1/2}\int_\tau^\infty \sigma^{1/2}\Big(1+\ln\frac{\sigma}{\tau}\Big)|\zeta_2(\sigma)|\,d\sigma, \qquad |\xi(\tau)| \le C\tau^{-2}\ln\tau + C\int_\tau^\infty \sigma^{1/2}\ln\sigma\,|\zeta_2|\,d\sigma,$$

where $\zeta_2 = N + \xi_1 E_1 + \xi_2 E_2$, $|\zeta_2| \le \frac{1}{2}C_0^2\sigma^{-3} + C|\xi|\sigma^{-5/2}\ln\sigma$.

Pass A. Let $\mu(\tau) := \sup_{\sigma\ge\tau}|\xi(\sigma)|$ (finite and $\to 0$, Steps 2–3). Then $\mu(\tau) \le C\tau^{-2}\ln\tau + C\tau^{-3/2} + C\mu(\tau)\tau^{-1}\ln^2\tau$ (the N-part gives $\int_\tau^\infty \sigma^{1/2}(1+\ln\frac{\sigma}{\tau})\,\sigma^{-3}d\sigma = O(\tau^{-3/2})$ with no logarithm), so for $\tau \ge \tau_1$ (with $C\tau_1^{-1}\ln^2\tau_1 \le \frac{1}{2}$) $\mu(\tau) \le C\tau^{-3/2}$, and hence $|\varepsilon| = |\xi_1 P_1 + \xi_2 P_2| \le C\tau^{-2}\ln\tau$.

Pass B. By Pass A, $|\xi| \le C\sigma^{-3/2}$, so $|N| \le \frac{1}{2}|\varepsilon|^2 \le C\sigma^{-4}\ln^2\sigma$ and $|\xi\cdot E| \le C\sigma^{-3/2}\cdot\sigma^{-5/2}\ln\sigma = C\sigma^{-4}\ln\sigma$, hence $|\zeta_2| \le C\sigma^{-4}\ln^2\sigma$ and $|\varepsilon^{(2)}| \le C\tau^{-1/2}\int_\tau^\infty \sigma^{-7/2}\big(1+\ln\frac{\sigma}{\tau}\big)\ln^2\sigma\,d\sigma \le C\tau^{-3}\ln^2\tau$. Together with Step 4,

$$|\varepsilon| \le |\varepsilon^{(R)}| + |\varepsilon^{(2)}| \le C\tau^{-5/2} + C\tau^{-3}\ln^2\tau \le C_\delta\tau^{-5/2}.$$

□

Remark. Lemma 13.2 is an independent strengthening: it applies to *any* solution w of $L[w] = 0$ with the a priori decay $\varepsilon = O(\tau^{-3/2})$. The proof of Theorem 1.1 itself does not invoke it: in § 16 the bound $O(\tau^{-5/2})$ for large τ follows directly from Theorem 15.1 and Lemma 15.1.

14 Phase structure

Lemma 14.1. *With m_0 from $\tau\ell(m_0) = \pi\nu$:*

$$\Phi(\tau) = \tau - \nu\ln\tau + C + \frac{6\nu^2-1}{8\tau} + \frac{5\nu^3}{8\tau^2} + O(\tau^{-3}), \qquad C = \nu\ln 2 + C_2.$$

The only logarithm is $-\nu\ln\tau$; there is no $\ln\tau/\tau^k$ for $k \ge 1$.

Proof. **Step 1 (series for ℓ).** From the hypergeometric series

$$K = \frac{\pi}{2}\sum_{n\ge0}\Big(\frac{\binom{2n}{n}}{4^n}\Big)^2 m^n, \qquad E = \frac{\pi}{2}\sum_{n\ge0}\Big(\frac{\binom{2n}{n}}{4^n}\Big)^2\frac{m^n}{1-2n},$$

one computes $\ell = E - (1-m)K$:

$$\ell = \frac{\pi}{2}\Big(\frac{m}{2} + \frac{m^2}{16} + \frac{3m^3}{128} + O(m^4)\Big).$$

Step 2 (reversion). $\ell'(0) = K(0)/2 = \pi/4 \neq 0$, so by the inverse function theorem $m_0(\tau) = \ell^{-1}(\pi\nu/\tau)$ is real-analytic in $1/\tau$ for $\tau > \pi\nu$. Substituting $m_0 = d_1\tau^{-1} + d_2\tau^{-2} + d_3\tau^{-3} + O(\tau^{-4})$ into $\frac{2}{\pi}\ell(m_0) = \frac{2\nu}{\tau}$ and matching powers:

$$\frac{d_1}{2} = \frac{2\nu}{\tau}\Big|_{\tau^{-1}} \Rightarrow\ d_1 = 4\nu; \qquad \frac{d_2}{2} + \frac{d_1^2}{16} = 0 \Rightarrow\ d_2 = -2\nu^2; \qquad \frac{d_3}{2} + \frac{2d_1d_2}{16} + \frac{3d_1^3}{128} = 0 \Rightarrow\ d_3 = -\nu^3.$$

Step 3 (ω). From the series of K,

$$\omega(m) = \frac{1}{1 + \frac{m}{4} + \frac{9m^2}{64} + \frac{25m^3}{256} + O(m^4)} = 1 - \frac{m}{4} - \frac{5m^2}{64} - \frac{11m^3}{256} + O(m^4).$$

Composing with Step 2:

$$\omega(m_0) = 1 - \frac{\nu}{\tau} - \frac{3\nu^2}{4\tau^2} - \frac{5\nu^3}{4\tau^3} + O(\tau^{-4})$$

(the τ^{-2} coefficient is $\frac{\nu^2}{2} - \frac{5\nu^2}{4} = -\frac{3\nu^2}{4}$; the τ^{-3} coefficient is $\frac{\nu^3}{4} + \frac{5\nu^3}{4} - \frac{11\nu^3}{4} = -\frac{5\nu^3}{4}$). **Step 4 (integration).** By Proposition 8.2, exactly,

$$\Phi' = \omega(m_0) + \frac{1}{8\tau^2} = 1 - \frac{\nu}{\tau} + \Big(\frac{1}{8} - \frac{3\nu^2}{4}\Big)\frac{1}{\tau^2} - \frac{5\nu^3}{4\tau^3} + O(\tau^{-4}),$$

and integrating in τ:

$$\Phi = \tau - \nu\ln\tau + C + \frac{3\nu^2/4 - 1/8}{\tau} + \frac{5\nu^3}{8\tau^2} + O(\tau^{-3}) = \tau - \nu\ln\tau + C + \frac{6\nu^2 - 1}{8\tau} + \frac{5\nu^3}{8\tau^2} + O(\tau^{-3}).$$

Step 5 (the constant). The standard complementary-modulus expansions (DLMF [3, §19.12.1]) give, as $m \to 0$, with $L := \ln\frac{4}{\sqrt{m}}$:

$$K(1-m) = L + \frac{m}{4}(L-1) + O(m^2\ln m), \qquad E(1-m) = 1 + \frac{m}{2}\Big(L - \frac{1}{2}\Big) + O(m^2\ln m),$$

hence

$$f(m) = E(1-m) - mK(1-m) = 1 - \frac{m}{2}L - \frac{m}{4} + O(m^2L).$$

With $m_0 = \frac{4\nu}{\tau}\big(1 + O(\tau^{-1})\big)$: $L(m_0) = \ln 2 + \frac{1}{2}\ln\frac{\tau}{\nu} + O(\tau^{-1})$, so

$$\tau f(m_0) = \tau - 2\nu L(m_0) - \nu + O\Big(\frac{\ln\tau}{\tau}\Big) = \tau - \nu\ln\tau + \nu\ln\nu - \nu - 2\nu\ln 2 + O\Big(\frac{\ln\tau}{\tau}\Big).$$

Adding $\Theta_0 = C_2 + \nu(1 + \ln\frac{8}{\nu})$ and $-\frac{1}{8\tau}$, the constants combine:

$$\nu\ln\nu - \nu - 2\nu\ln 2 + C_2 + \nu + \nu\ln 8 - \nu\ln\nu = C_2 + \nu\ln 2,$$

since $\ln 8 - 2\ln 2 = \ln 2$. Hence $C = \nu\ln 2 + C_2$. Finally, ℓ^{-1} is analytic at 0 (Step 2), so Φ' is a convergent power series in $1/\tau$ for $\tau \geq \tau_*(\nu)$ (and real-analytic in $1/\tau$ for all $\tau > \pi\nu$); hence its integral contains the single logarithm $-\nu\ln\tau$ and no terms $\ln\tau/\tau^k$. □

15 Matching with Maevsky

Theorem 15.1. *[13] (Theorem 15; the statement is reproduced in the published abstract (avtoreferat) of the dissertation, pp. 6–7, which is publicly available) Let y be the regular solution of the Amsler problem with $y(0)=\pi s$, $s\in(0,1)$. There are constants $C_1=C_1(s)>0$, $C_2=C_2(s)$ such that, as $x\to\infty$,*

$$y(x)=\pi+x^{-1/4}F_1(\theta)+x^{-3/4}F_3(\theta)+O(x^{-5/4}),\qquad \theta=2\sqrt{x}-\tfrac{C_1^2}{16}\ln x+C_2,$$

$$F_1=C_1\sin\theta,\qquad F_3=\frac{C_1^3}{32}\sin\theta+\frac{C_1^3}{192}\sin 3\theta+\Big(\frac{3C_1^5}{512}-\frac{C_1}{16}\Big)\cos\theta.$$

Remark 15.1. *The two-term structure of Theorem 15.1 agrees with the published asymptotics of Maevsky [14], Theorem 1, via the dictionary $\alpha^2=\nu$, $f_3=f'''(0)=-1$; its leading coefficients can also be re-derived from the connection formulas (IN) of § 6.2, and the constants C_1,C_2 are confirmed independently by the radial sine-Gordon connection formula of Wong–Zhang [12] (§ 7.5). The sharp remainder estimate $O(x^{-5/4})$ is Theorem 15 of [13], whose statement is reproduced in the publicly available abstract of the dissertation.*

Lemma 15.1. *Let $M:=\pi+x^{-1/4}F_1(\theta)+x^{-3/4}F_3(\theta)$ denote the Maevsky approximant. Under (M) and § 7: $|y_{\mathrm{ad}}-M|=O(\tau^{-5/2})$.*

Proof. By § 7.5 (which derives C_1,C_2 from Theorem 6.1), $C_1=\sqrt{8\nu}$; moreover

$$\theta=\tau-\frac{C_1^2/16}{1}\big(2\ln\tau-2\ln 2\big)+C_2=\tau-\nu\ln\tau+\nu\ln 2+C_2,$$

which by Lemma 14.1 is exactly the smooth part $\tilde\Phi:=\tau-\nu\ln\tau+C$ of Φ; in particular the constants in (M) and in Lemma 14.1 agree. Write both expansions in the phase $\tilde\Phi$. **Adiabatic side.** By Lemma 9.1 and Lemma 14.1 ($\Phi=\tilde\Phi+\frac{a_1}{\tau}+\frac{a_2}{\tau^2}+O(\tau^{-3})$, $a_1=\frac{6\nu^2-1}{8}$, $a_2=\frac{5\nu^3}{8}$):

$$y_{\mathrm{ad}}-\pi=2\sqrt{m_0}\sin\Phi+\frac{3}{8}m_0^{3/2}\sin\Phi+\frac{1}{24}m_0^{3/2}\sin 3\Phi+O(m_0^{5/2}).$$

Here $2\sqrt{m_0}=4\sqrt{\nu}\,\tau^{-1/2}\big(1-\frac{\nu}{4\tau}+O(\tau^{-2})\big)=4\sqrt{\nu}\,\tau^{-1/2}-\nu^{3/2}\tau^{-3/2}+O(\tau^{-5/2})$, and $\sin\Phi=\sin\tilde\Phi+\frac{a_1}{\tau}\cos\tilde\Phi+O(\tau^{-2})$. Collecting all terms through order $\tau^{-3/2}$:

$$\begin{aligned}y_{\mathrm{ad}}-\pi=\;&4\sqrt{\nu}\,\tau^{-1/2}\sin\tilde\Phi\\&+\Big(\underbrace{-\nu^{3/2}+3\nu^{3/2}}_{=2\nu^{3/2}}\Big)\tau^{-3/2}\sin\tilde\Phi\\&+\frac{\nu^{3/2}}{3}\tau^{-3/2}\sin 3\tilde\Phi\\&+4\sqrt{\nu}\,a_1\,\tau^{-3/2}\cos\tilde\Phi\\&+O(\tau^{-5/2}),\end{aligned}$$

using $\frac{3}{8}m_0^{3/2} = 3\nu^{3/2}\tau^{-3/2} + O(\tau^{-5/2})$, $\frac{1}{24}m_0^{3/2} = \frac{\nu^{3/2}}{3}\tau^{-3/2} + O(\tau^{-5/2})$, and $4\sqrt{\nu}\,a_1 = 3\nu^{5/2} - \frac{1}{2}\nu^{1/2}$.

Maevsky side. With $x^{-1/4} = \sqrt{2}\,\tau^{-1/2}$, $x^{-3/4} = 2\sqrt{2}\,\tau^{-3/2}$, $C_1^3 = (8\nu)^{3/2} = 16\sqrt{2}\,\nu^{3/2}$, $C_1^5 = (8\nu)^{5/2} = 128\sqrt{2}\,\nu^{5/2}$, and $\theta = \tilde{\Phi}$:

$$M - \pi = 4\sqrt{\nu}\,\tau^{-1/2}\sin\tilde{\Phi} + \tau^{-3/2}\Big[2\nu^{3/2}\sin\tilde{\Phi} + \frac{\nu^{3/2}}{3}\sin 3\tilde{\Phi} + \Big(3\nu^{5/2} - \frac{1}{2}\nu^{1/2}\Big)\cos\tilde{\Phi}\Big] + O(\tau^{-5/2}),$$

where the $\sin\tilde{\Phi}$-coefficient of F_3 is $\sqrt{2}\tau^{-3/2}\cdot\frac{C_1^3}{32} = 2\nu^{3/2}\tau^{-3/2}$, the $\sin 3\tilde{\Phi}$-coefficient is $2\sqrt{2}\cdot\frac{16\sqrt{2}}{192}\nu^{3/2} = \frac{\nu^{3/2}}{3}$, and the $\cos\tilde{\Phi}$-coefficient is $2\sqrt{2}\big(\frac{3\cdot 128\sqrt{2}}{512}\nu^{5/2} - \frac{2\sqrt{2}}{16}\nu^{1/2}\big) = 3\nu^{5/2} - \frac{1}{2}\nu^{1/2}$.

Comparison. The two expansions coincide term by term through order $\tau^{-3/2}$; the remainders are $O(m^{5/2}) = O(\tau^{-5/2})$ (Lemma 9.1) and $O(x^{-5/4}) = O(\tau^{-5/2})$ (M). □

16 Proof of the main theorem

Step 1 (large τ). By Theorem 15.1, $|y - M| \le Cx^{-5/4} = O(\tau^{-5/2})$ for $\tau \ge \tau_*(s)$, and by Lemma 15.1, $|M - y_{\mathrm{ad}}| = O(\tau^{-5/2})$. Hence

$$|y - y_{\mathrm{ad}}| \le |y - M| + |M - y_{\mathrm{ad}}| = O(\tau^{-5/2}) + O(\tau^{-5/2}) = O(\tau^{-5/2}), \qquad \tau \ge \tau_*.$$

Step 2 (finite range $\pi\nu + \delta \le \tau \le \tau_*$). By Lemma 2.1 ($H' = -w'^2/\tau \le 0$, $H(0) = \cos\pi s < 1$, $H \ge 1$ at $w = \pm\pi$, $w(0) \in (-\pi, 0)$) we have $|w| < \pi$, i.e. $y \in (0, 2\pi)$. Also $y_{\mathrm{ad}} - \pi = G \in (-\pi, \pi)$ because $|X| \le \sqrt{m_0} < 1$. Since $\tau \le \tau_*$, we have $2\pi \le 2\pi\,\tau_*^{5/2}\,\tau^{-5/2}$, hence $|y - y_{\mathrm{ad}}| < 2\pi \le 2\pi\,\tau_*^{5/2}\,\tau^{-5/2}$ on this range. Taking

$$C_\delta = \max\Big\{C,\ 2\pi\,\tau_*^{5/2}\Big\}$$

proves the bound for all $\tau \ge \pi\nu + \delta$. ■

17 Status summary

In the table, "Proved under" lists only the external inputs; all internal lemmas used in the proofs are themselves proved in the text.

Statement	Status
Lemma 2.1 (existence, bounds)	Proved
Proposition 3.1 (Painlevé III reduction, D_8)	Proved
Proposition 4.1 (Bessel linearisation, with remainder $O(\|a\|^3)$ on $[0,T]$)	Proved
Proposition 5.1 (Lax pair + compatibility)	Proved
§ 6 (Monodromy and (IN))	Theorem (external input)
§ 7 (Specialisation)	Derived
§ 8 (Exact identities)	Proved
Lemma 9.1, Lemma 9.2 (Fourier coefficients and analytic structure)	Proved
Proposition 10.1, Theorem 10.1, Corollary 10.1, Lemma 10.1	Proved
Lemma 11.1	Proved
Lemma 12.1 (basis P_1, P_2)	Proved (direct Wronskian computation and Abel's identity)
Lemma 13.1, Lemma 13.2	Proved (hypothesis $\varepsilon = O(\tau^{-3/2})$, supplied by Theorem 15.1, Lemma 15.1; conclusion $O(\tau^{-5/2})$)
Lemma 14.1	Proved
Lemma 15.1	Proved under Theorem 15.1, Theorem 6.1
§ 16 (Main theorem)	Proved under Theorem 15.1, Theorem 6.1 (Step 1: large τ; Step 2: finite range by boundedness)

■

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A Comparison between adiabatic and Maevsky asymptotic solutions

Figure 1 compares the numerical solution of the Amsler equation with the adiabatic ansatz and with the Maevsky approximant.

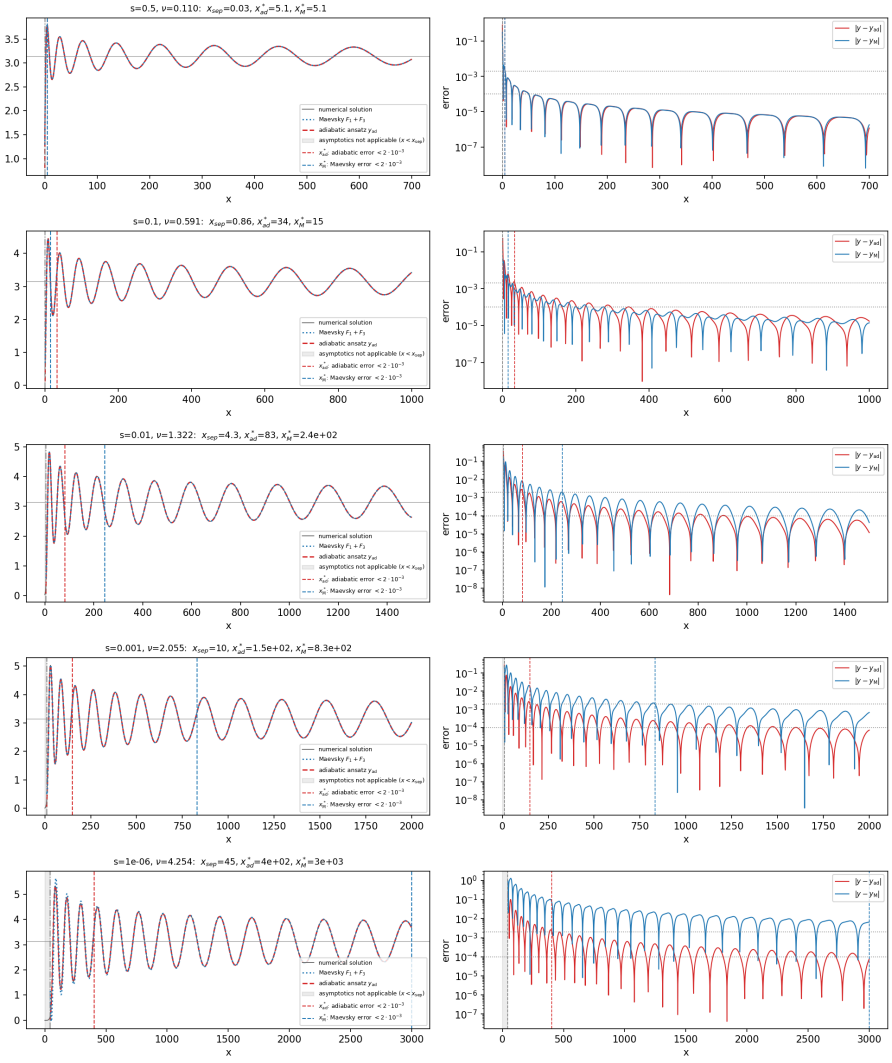


Figure 1: Comparison of the numerical solution of the Amsler equation with the adiabatic ansatz y_{ad} and the Maevsky approximant M for $s = 0.5,\ 0.1,\ 0.01,\ 10^{-3},\ 10^{-6}$ (rows). Left: solutions; right: absolute errors. The shaded region $x < x_{\mathrm{sep}} = (\pi\nu/2)^2$ ($\tau < \pi\nu$) lies outside the domain of definition of the ansatz, and the dotted horizontal line is the error threshold $2 \cdot 10^{-3}$ used to define the points x^*_{ad}, x^*_{M} beyond which the error stays below it. The code is given in § A.1.

A.1 Python code for the plot above

```
"""
Comparison of the numerical solution of the Amsler equation (x y')' = sin y,
y(0) = pi*s, y'(0) = sin(pi*s), with
  (a) the adiabatic ansatz y_ad (libration of the damped pendulum with
      adiabatic invariant tau*ell(m0) = pi*nu), and
  (b) the Maevsky approximant M = pi + x^{-1/4} F1(theta) + x^{-3/4} F3(theta),
      theta = 2*sqrt(x) - (C1^2/16) ln x + C2,
      F1 = C1 sin(theta),
      F3 = (C1^3/32) sin(theta) + (C1^3/192) sin(3*theta)
           + (3*C1^5/512 - C1/16) cos(theta),
with C1 = sqrt(8*nu), C2 = 5*pi/4 + arg Gamma(1+i*nu) - 3*nu*ln 2,
nu = (1/pi) ln csc(pi*s/2), tau = 2*sqrt(x).

Both approximants are evaluated ONLY on their domain of validity x >= x_sep,
where x_sep = (pi*nu/2)^2 (tau >= pi*nu). The region x < x_sep is shaded.

Performance: the elliptic-integral root m0(tau) is found by a fully
vectorized bisection (60 iterations on the whole array), so the script
runs in seconds even on a 4000-point uniform tau grid.

Plotting: the numerical solution is thin and behind (zorder=1); the dashed
adiabatic curve and the dotted Maevsky curve are on top (zorder=2,3).
Legend: left panels -- lower right; right panels -- upper right.
X-axis: linear (USE_LOG_X = True switches to log).

Run:  python amsler_compare.py   ->  saves amsler_compare_s.png
Requires: numpy, scipy, matplotlib.
"""

import numpy as np
from scipy.integrate import solve_ivp
from scipy.special import gamma, ellipk, ellipe, ellipj
import matplotlib.pyplot as plt

TOL = 2e-3          # threshold defining x_ad* and x_M*
S_VALUES = [0.5, 0.1, 0.01, 0.001, 1e-6]
X_MAXS   = [700, 1000, 1500, 2000, 3000]
USE_LOG_X = False   # x-axis scale: linear (original behavior)
N_PTS    = 4000     # uniform grid size in tau (~90 pts per period even at
                    # the largest tau_max of the table)

def monodromy_exponent(s):
    """nu(s) = (1/pi) ln csc(pi*s/2) > 0."""
    return -np.log(np.sin(np.pi * s / 2)) / np.pi

def ell_of_m(m):
    """ell(m) = E(m) - (1-m) K(m),  m in [0,1] (parameter m = k^2).
       INCREASING bijection [0,1] -> [0,1]: ell(0)=0, ell(1)=1, ell'=K/2>0."""
    return ellipe(m) - (1 - m) * ellipk(m)

def f_of_m(m):
    """f(m) = E(1-m) - m K(1-m) = ell(1-m)."""
```

```
    return ellipe(1 - m) - m * ellipk(1 - m)

def phase_constant(nu):
    """C2 = 5*pi/4 + arg Gamma(1+i*nu) - 3*nu*ln 2 (branch vanishing at nu=0)."""
    return 5 * np.pi / 4 + float(np.imag(np.log(gamma(1 + 1j * nu)))) \
        - 3 * nu * np.log(2)

def solve_m0_vec(tau, nu):
    """Vectorized: unique root m0 in (0,1) of ell(m0) = pi*nu/tau for all tau
       where the root exists (pi*nu/tau in (0,1), i.e. tau > pi*nu); NaN
       otherwise. Bisection on the whole array at once: ell is increasing,
       ell(0)=0, ell(1)=1. 60 iterations -> ~1e-15 precision."""
    tau = np.atleast_1d(np.asarray(tau, dtype=float))
    target = np.pi * nu / tau
    ok = (target > 0) & (target < 1)
    m0 = np.full(tau.shape, np.nan)
    if np.any(ok):
        lo = np.full(tau.shape, 1e-15)
        hi = np.full(tau.shape, 1 - 1e-13)
        for _ in range(60):
            mid = 0.5 * (lo + hi)
            ell = ell_of_m(mid)
            lo = np.where(ok & (ell < target), mid, lo)   # ell too small -> right
            hi = np.where(ok & (ell >= target), mid, hi)
        m0[ok] = 0.5 * (lo + hi)[ok]
    return m0

def adiabatic(x, s):
    """Adiabatic ansatz y_ad(x; s). NaN where no root m0 exists."""
    nu = monodromy_exponent(s)
    tau = 2 * np.sqrt(x)
    m0 = solve_m0_vec(tau, nu)
    Theta0 = phase_constant(nu) + nu * (1 + np.log(8 / nu))
    Phi = tau * f_of_m(m0) + Theta0 - 1 / (8 * tau)
    # guard against NaN in ellipj
    sn = np.zeros_like(tau)
    ok = np.isfinite(m0)
    if np.any(ok):
        u = 2 * ellipk(m0[ok]) * Phi[ok] / np.pi
        sn[ok] = ellipj(u, m0[ok])[0]
    y = np.pi + 2 * np.arcsin(np.clip(np.sqrt(np.where(ok, m0, 0.0)) * sn,
                                      -1.0, 1.0))
    y[~ok] = np.nan
    return y

def maevsky(x, s):
    """Maevsky approximant M(x; s). Asymptotics as x -> +infinity:
       meaningful only for x >= x_sep (after the first libration)."""
    nu = monodromy_exponent(s)
    C1 = np.sqrt(8 * nu)
    theta = 2 * np.sqrt(x) - C1**2 / 16 * np.log(x) + phase_constant(nu)
    F1 = C1 * np.sin(theta)
    F3 = (C1**3 / 32) * np.sin(theta) + (C1**3 / 192) * np.sin(3 * theta) \
         + (3 * C1**5 / 512 - C1 / 16) * np.cos(theta)
```

```
    return np.pi + x**(-0.25) * F1 + x**(-0.75) * F3

def numerical(s, x_max, n=N_PTS):
    """Regular solution via the tau-variable form w'' + w'/tau + sin w = 0,
       w(0) = pi(s-1), w'(0) = 0;  returns (x, y).
       Grid: UNIFORM in tau (not geomspace) so that the number of points per
       oscillation period (period ~ 2*pi in tau) is constant over the range:
       2*pi*n/tau_max points per period, >= ~90 even for the largest tau_max."""
    a = np.pi * (s - 1)

    def rhs(t, u):
        w, wp = u
        return [wp, -wp / t - np.sin(w)]

    # start slightly off the singular point using the local expansion
    t0 = 1e-6
    w0 = a + t0**2 / 4 * np.sin(np.pi * s)
    wp0 = t0 / 2 * np.sin(np.pi * s)
    tau_max = 2 * np.sqrt(x_max)
    ts = np.linspace(t0, tau_max, n)
    sol = solve_ivp(rhs, (t0, tau_max), [w0, wp0], t_eval=ts,
                    method="DOP853", rtol=1e-11, atol=1e-13)
    tau = sol.t
    x = (tau / 2)**2
    y = np.pi + sol.y[0]
    mask = x > 1e-4
    return x[mask], y[mask]

def threshold(x, err):
    """Smallest x0 within the validity domain such that err < TOL for all
       x >= x0 (NaNs ignored)."""
    below = (err < TOL) | np.isnan(err)
    for i in range(len(x) - 1, -1, -1):
        if not below[i]:
            return x[i + 1] if i + 1 < len(x) else x[-1]
    return x[0]

def main():
    fig, axes = plt.subplots(len(S_VALUES), 2, figsize=(13, 3.1 * len(S_VALUES)))
    for row, (s, xmax) in enumerate(zip(S_VALUES, X_MAXS)):
        x, y = numerical(s, xmax)
        nu = monodromy_exponent(s)
        x_sep = (np.pi * nu / 2) ** 2

        # ---- validity domain: both approximants only for x >= x_sep ----
        valid = x >= x_sep
        y_ad = adiabatic(x, s)
        y_M = maevsky(x, s)
        y_ad = np.where(valid, y_ad, np.nan)
        y_M = np.where(valid, y_M, np.nan)

        err_ad = np.abs(y - y_ad)          # NaN outside validity domain
        err_M = np.abs(y - y_M)
        x_ads, x_Ms = threshold(x, err_ad), threshold(x, err_M)
```

```
        # ---------- left panel: solutions ----------
        ax = axes[row, 0]
        # numerical solution: thin, BEHIND (zorder=1)
        ax.plot(x, y, color="0.35", lw=0.8, zorder=1,
                label="numerical solution")
        # Maevsky: dotted blue, ON TOP (zorder=3)
        ax.plot(x, y_M, color="tab:blue", lw=1.4, ls=":", zorder=3,
                label="Maevsky $F_1+F_3$")
        # adiabatic: dashed red, ON TOP (zorder=2)
        ax.plot(x, y_ad, color="tab:red", lw=1.4, ls="--", zorder=2,
                label="adiabatic ansatz $y_{\\rm ad}$")
        ax.axhline(np.pi, color="gray", lw=0.5)
        ax.axvspan(x[0], x_sep, color="gray", alpha=0.15, zorder=0,
                   label="asymptotics not applicable ($x<x_{\\rm sep}$)")
        ax.axvline(x_sep, color="gray", ls="-.", lw=1)
        ax.axvline(x_ads, color="tab:red", ls="--", lw=1,
                   label="$x^*_{\\rm ad}$: adiabatic error $<2\\cdot10^{-3}$")
        ax.axvline(x_Ms, color="tab:blue", ls="--", lw=1,
                   label="$x^*_{\\rm M}$: Maevsky error $<2\\cdot10^{-3}$")
        if USE_LOG_X:
            ax.set_xscale("log")
        ax.set_title(f"s={s:g}, $\\nu$={nu:.3f}:  $x_{{sep}}$={x_sep:.2g}, "
                     f"$x^*_{{ad}}$={x_ads:.2g}, $x^*_{{M}}$={x_Ms:.2g}",
                     fontsize=9)
        ax.set_xlabel("x")
        ax.legend(fontsize=6, loc="lower right")

        # ---------- right panel: errors ----------
        ax = axes[row, 1]
        ax.semilogy(x, err_ad, color="tab:red", lw=1,
                    label="$|y - y_{\\rm ad}|$")
        ax.semilogy(x, err_M, color="tab:blue", lw=1,
                    label="$|y - y_{\\rm M}|$")
        ax.axhline(TOL, color="gray", ls=":", lw=0.8)
        ax.axhline(1e-4, color="gray", ls=":", lw=0.8)
        ax.axvspan(x[0], x_sep, color="gray", alpha=0.15, zorder=0)
        for xv, c in [(x_sep, "gray"), (x_ads, "tab:red"), (x_Ms, "tab:blue")]:
            ax.axvline(xv, color=c, ls="--", lw=0.8)
        if USE_LOG_X:
            ax.set_xscale("log")
        ax.set_xlabel("x")
        ax.set_ylabel("error")
        ax.legend(fontsize=7, loc="upper right")

    fig.tight_layout()
    fig.savefig("amsler_compare_s.png", dpi=300)
    print("saved amsler_compare_s.png")

if __name__ == "__main__":
    main()
```