Maximal torus theory for crossed product quantum groups

TB 2.0 and GPT 6.2

Abstract

We give counterexamples to three proposed tests for characters, amenability and polynomial growth using the group-dual subgroups of a compact matrix quantum group. For an action on a compact connected Lie group, a single such subgroup detects all three properties exactly when the action preserves a maximal torus of the base. For a lattice bicrossed product whose compact factor has infinitely many components, failure of character detection is witnessed by a class function on that factor, and a torus of a quantum quotient detects exponential growth. These results suggest three conjectures. The character conjecture uses the quotient by the quantum center. The amenability and growth conjectures use tori of cocycle twists of quantum quotients.

hexagon:2611.00503v1

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Introduction

We can read every character of a compact connected Lie group on one maximal torus, since every conjugacy class meets it. For $G \subset U_N^+$ we use a family of tori. Throughout this paper $C(G)$ is the full C^*-algebra of G, and $C(G)_{central}$ is the norm-closed span of its irreducible characters. Making the fundamental matrix diagonal in the basis given by $Q \in U_N$ defines the torus $\widehat{\Gamma}_Q$ and the restriction $\pi_Q : C(G) \to C^*(\Gamma_Q)$. We begin with the three conjectures of [4].

Old conjecture 1 (Characters)**.** *If G is connected, then for each nonzero $x \in C(G)_{central}$ there is a $Q \in U_N$ such that $\pi_Q(x) \neq 0$.*

For the half-classical orthogonal group O_N^*, restriction to the diagonal torus distinguishes representations [7, Theorem 6.2]. For half-liberations see [4, Theorem 5.5] and [1, Chapter 14].

Old conjecture 2 (Amenability)**.** *The discrete dual $\widehat{G}$ is amenable exactly when all the groups Γ_Q, $Q \in U_N$, are amenable.*

For U_N^+ with $N \geq 2$ we need only the diagonal torus. Its group is F_N, so we can already read off nonamenability. For other free examples see [4, Theorem 4.4] and [1, Chapter 14].

Old conjecture 3 (Growth)**.** *Polynomial growth of Γ_Q for every $Q \in U_N$ is equivalent to polynomial growth of $\widehat{G}$.*

The dual of O_N^* has polynomial growth [7, Section 9]. For all three tests on the half-liberation $[H]$ of a connected conjugation-stable $H \subset U_N$, see [4, Theorem 5.5] and the crossed product model in [9].

We will prove in this paper that these 3 conjectures are all wrong. The first two counterexamples use an action of the tetrahedral group. For growth we use the quantum automorphism group of a five-dimensional algebra.

Proposition 4 (Characters)**.** *There is a connected crossed product $C(SO_5) \rtimes D$ of polynomial growth for which every torus kills the same nonzero virtual character, the discriminant of the defining representation of SO_5.*

We let $D = \mathbb{Z}^3 \rtimes \mathbb{Z}$ act through A_4. This action preserves no maximal torus of SO_5, so the discriminant vanishes on every torus of the crossed product (Theorem 5.7).

Proposition 5 (Amenability)**.** *For the tetrahedral rotation action of $D = \mathbb{Z}^3 \rtimes \mathbb{Z}$ on U_2^+, the crossed product is connected and has nonamenable dual. Every toral group is a quotient of the virtually abelian group $\mathbb{Z} \times D$.*

The rotations have no common axis, so the two diagonal weights coincide on every torus (Corollary 5.5). If we pass to the quotient U_2^+, we recover its free diagonal torus.

Proposition 6 (Growth)**.** *Let $G = G^+(\mathbb{C} \oplus M_2)$ be the quantum automorphism group for the canonical trace. Its dual grows exponentially, yet every toral group in every compact matrix quantum quotient of G has at most linear growth.*

Here every torus lies in SO_3, and there is no proper nontrivial quantum quotient (Theorem 6.3). But we can twist by a cocycle to get S_5^+, where a pair and a triple of points give $\mathbb{Z}_2 * \mathbb{Z}_3$.

For characters we keep the tori of G itself. If they miss an element of the character algebra, can we already see the failure on the compact factor? We say the tori *jointly detect* an algebra when their restrictions have trivial common kernel there.

Theorem 7 (Characters)**.** *In each of the following compact matrix cases, joint detection by the tori of G is equivalent on the three algebras $C(G)_{central}$, $C(K)_{central}$ and the pullback of $C(G/Z(G))_{central}$, where $Z(G)$ is the quantum center.*

1. *G is an ordinary crossed product of a compact group K by an amenable discrete group D.*
2. *G is a bicrossed product of $D = \mathbb{Z}^r$, $r \geq 1$, and a compact group K with infinitely many connected components.*

For an ordinary action we need only ask whether each conjugacy class of K meets an invariant abelian subgroup (Theorem 4.1). We treat the lattice case in Theorem 10.5 and pass to $G/Z(G)$ in Corollary 11.2.

Theorem 8 (Amenability)**.** *Let G be a compact matrix bicrossed product of a discrete group D and a compact group K. Its dual is amenable if and only if every toral group in every unitary dual 2-cocycle twist of a compact matrix quantum quotient of G is amenable. The torus $\widehat{D} \subset G$ alone detects amenability.*

We find this torus simply by evaluating at the identity of K. Coamenability of G is equivalent to amenability of D [14, Corollary 3.7]. Proposition 7.1 explains why all the other toral groups are amenable whenever D is.

Theorem 9 (Growth)**.** *Let $C(G) = C(L) \rtimes D$, with D finitely generated. Take for L a compact connected Lie group, U_2^+ with a rotation action, or $G^+(B)$ for the canonical trace on a finite-dimensional C^*-algebra B. In the first and third cases any action is allowed. Then $\widehat{G}$ has polynomial growth exactly when every toral group in every unitary dual 2-cocycle twist of a compact matrix quantum quotient of G has polynomial growth.*

For a compact connected Lie base, one torus suffices for all three tests precisely when D preserves a maximal torus of the base (Theorem 3.3). We prove both analytic tests for the three families and their cocycle twists in Theorem 7.4.

We can now state our three conjectures. Theorem 7 gives us a way to ask the first without choosing a crossed product presentation.

New conjecture 10 (Characters)**.** *Let G be a compact matrix quantum group of Kac type with quantum center $Z(G)$. If the tori of G jointly detect the norm-closed character algebra $C(G/Z(G))_{central}$, then they jointly detect $C(G)_{central}$. Here characters of $G/Z(G)$ are pulled back to G before restriction.*

We prove this for amenable chain groups in Section 11. In Section 12 we treat orthogonal easy quantum groups and free products of free unitary quantum groups. The latter can have a nonamenable chain group.

New conjecture 11 (Amenability)**.** *Let G be a compact matrix quantum group of Kac type. Its dual is amenable exactly when Λ is amenable for every diagram*

$$G \twoheadrightarrow H, \qquad H \sim K, \qquad \widehat{\Lambda} \subset K$$

where H is a compact matrix quantum quotient and K is a unitary dual 2-cocycle twist of H.

Theorem 8 gives us the bicrossed-product case. For the rotation actions on U_2^+, we can use the torus of the base quotient even when every toral group of G is amenable (Section 7).

New conjecture 12 (Growth)**.** *Let G be a compact matrix quantum group of Kac type. Its dual has polynomial growth exactly when Λ has polynomial growth for every diagram*

$$G \twoheadrightarrow H, \qquad H \sim K, \qquad \widehat{\Lambda} \subset K$$

where H is a compact matrix quantum quotient and K is a unitary dual 2-cocycle twist of H.

Theorem 9 covers our three families of crossed products. For lattice bicrossed products with infinitely many components in the compact factor, we can even exhibit a lamplighter group inside a toral group of an untwisted quotient (Theorem 10.3).

1 Tori and crossed products

Let v be a fundamental unitary corepresentation of a compact matrix quantum group G of Kac type [28]. For a unitary matrix Q of the same size as v, the quotient

$$C^*(\Gamma_Q) = C(G)/\langle (QvQ^*)_{ij} : i \neq j \rangle$$

is a full group C^*-algebra. Its diagonal coefficients are group-like unitaries and generate Γ_Q. We call any group-dual subgroup a *torus*. The groups $T_Q(G) = \widehat{\Gamma}_Q$ are the *standard tori*, and their indexed family is the *skeleton* of the given embedding [1, Chapter 13]. Every torus of G lies in one of its standard tori. On the discrete side this inclusion makes the toral group a quotient of Γ_Q. Indeed, a unitary corepresentation of a group dual splits into weight spaces indexed by group elements. An orthonormal basis of weight vectors therefore diagonalizes the restriction of

v [3, 4]. By Tannaka–Krein reconstruction [29], the intertwiners between tensor words in v and $\bar{v}$ determine the relations on the diagonal generators in each basis [4, Section 6].

Write $C(G)_{central}$ for the norm closure in $C(G)$ of the complex span of irreducible characters. Products correspond to tensor products and the involution corresponds to conjugation, so this span is a unital $*$-algebra. A virtual character is an integral linear combination of irreducible characters. For a family of tori with restriction maps π_i, joint detection of $C(G)_{central}$ means

$$C(G)_{central} \bigcap \left(\bigcap_i \ker \pi_i \right) = \{0\}$$

A single torus detects $C(G)_{central}$ if its restriction is injective, or equivalently isometric, there. Universal norms matter in this definition. For example the characters of $\widehat{D}$ are the elements of D, and $C(\widehat{D})_{central} = C^*(D)$ even when D is nonamenable.

For a discrete quantum group $\widehat{G}$, amenability means that G is coamenable. For growth choose a self-conjugate generating representation v containing 1 and use the volume

$$b_v(n) = \sum_{r \in v^{\otimes n}} (\dim r)^2$$

where we count each irreducible summand of $v^{\otimes n}$ once. Tensor powers of any other such generator give comparable lengths. We say that a torus detects an analytic property if its discrete group has that property exactly when $\widehat{G}$ does. Connectedness of G means that G has no nontrivial finite quantum quotient.

Let D act by automorphisms α_d on a compact group K. In the ordinary crossed product the copy of $C(K)$ has its original coproduct, and the canonical unitaries s_d, $d \in D$, satisfy

$$s_d s_e = s_{de}, \qquad s_d f s_d^* = f \circ \alpha_{d^{-1}}, \qquad \Delta(s_d) = s_d \otimes s_d$$

We always take the full crossed product, with its universal property for covariant representations. We assume compact matrix type. This holds for K a compact Lie group and D finitely generated, since the coefficients of a faithful representation of K and finitely many s_d generate. For a closed invariant abelian subgroup A put

$$\Gamma_A = \widehat{A} \rtimes D, \qquad (d \cdot \lambda)(a) = \lambda(\alpha_{d^{-1}}(a))$$

A character λ of A is a unitary function, so the covariance relation is precisely the group relation defining Γ_A.

Lemma 1.1. *Every torus of G is contained in a torus dual to $\widehat{A} \rtimes D$, where A is a closed D-invariant abelian subgroup of K. Every such A gives a torus of G.*

Proof. Restrict a Hopf quotient $C(G) \to C^*(\Lambda)$ to $C(K)$. Its image is $C(A)$ for a closed subgroup $A \subset K$, which is abelian because the image has cocommutative coproduct. Covariance makes the kernel of this restriction D-invariant. Thus the quotient factors through $C(A) \rtimes D = C^*(\widehat{A} \rtimes D)$. Conversely, equivariant restriction to A gives this Hopf quotient. □

For a compact group K, $C(K)_{central}$ consists of the continuous functions on K that are constant on conjugacy classes, by Peter–Weyl. The irreducible characters of the crossed product are $s_d \chi_u$ with $d \in D$ and $u \in Irr(K)$ [14, Theorem 6.1]. In particular

$$C(G)_{central} = C^*(C(K)_{central}, s_d \mid d \in D) \subset C(K) \rtimes D$$

For D trivial this gives the classical algebra of class functions. For abelian K the coefficient algebra is $C^*(\widehat{K} \rtimes D)$ and G is its own maximal torus. In between these cases the group unitaries

belong to $C(G)_{central}$ and can act nontrivially on the class functions. Faithfulness on $C(G)_{central}$ therefore involves the norm of sums $\sum_d f_d s_d$ with class-function coefficients.

We shall also use compact matrix quantum quotients $G \twoheadrightarrow H$. Write $\mathcal{R}_G$ for the algebra spanned by the matrix coefficients of finite-dimensional unitary representations of G. Pullback gives an inclusion $\mathcal{R}_H \subset \mathcal{R}_G$. Restriction to a quantum subgroup gives a surjection. Thus a torus of a quotient gives the pair of maps

$$\mathcal{R}_G \ \supset \ \mathcal{R}_H \ \longrightarrow \ \mathbb{C}[\Lambda]$$

We use these maps to compare the two discrete analytic properties. Character restrictions always use tori of G itself. We shall also use the following fact. The image of a polynomial Hopf $*$-homomorphism into a group algebra is the span of a subgroup. Indeed, if $\sum_g c_g g$ is in the image, slicing its coproduct at g puts each $c_g g$ in the image separately. The full completion of this subgroup algebra gives a torus of the source.

Definition 1.2. *The quotient test for amenability holds for G if amenability of $\widehat{G}$ is equivalent to amenability of every Λ in a diagram*

$$G \twoheadrightarrow H \supset \widehat{\Lambda}$$

with H a compact matrix quantum quotient and Λ finitely generated. The quotient test for growth uses polynomial growth on both sides of the equivalence.

2 Characters and invariant maximal tori

Throughout this section K is a compact connected Lie group. All its maximal tori are conjugate. A maximal torus T is invariant under D when every $d \in D$ sends the subgroup T onto itself. The action on T may be nontrivial. The forward construction uses the Weyl group to recover the character norm. For the converse we construct one virtual character that vanishes on all the tori at once.

Theorem 2.1. *For $C(G) = C(K) \rtimes D$ with K a compact connected Lie group, the tori of G detect its full character algebra if and only if D preserves a maximal torus T of K. In that case restriction to $C(T) \rtimes D$ is isometric on the character algebra. If there is no invariant maximal torus, a nonzero virtual character of K vanishes on every torus of G.*

Proof. Suppose first that D preserves T. Write $W = N_K(T)/T$. Restriction identifies $C(K)_{central}$ with $C(T)^W$ [11]. The identification is equivariant for D, which normalizes the action of W on T.

We check faithfulness in the universal norm. Start with a faithful representation of $C(G)$ on a Hilbert space H. Its class functions and group unitaries give a covariant pair (π, V) for $C(T)^W$ and D. The Weyl average $E : C(T) \to C(T)^W$ is a D-equivariant conditional expectation. Complete $C(T) \otimes H$, after dividing by null vectors, in the following inner product, taken linear in its first variable.

$$\langle f \otimes \xi, h \otimes \eta \rangle = \langle \pi(E(\bar{h} f))\xi, \eta \rangle$$

Left multiplication by $C(T)$ and the operators $f \otimes \xi \mapsto (d \cdot f) \otimes V_d \xi$ form a covariant pair on this completion. On $1 \otimes H$ the invariant functions and the group unitaries recover (π, V), since for $b \in C(T)^W$ one has $b \otimes \xi = 1 \otimes \pi(b)\xi$. This realizes the original norm on $C(G)_{central}$ after restriction to $C(T) \rtimes D$.

Conversely, fix a maximal torus T of K and a faithful unitary representation of K. Let F be the set of its distinct weights on T. The product

$$P_T = \prod_{\{\lambda,\mu\} \subset F, \ \lambda \neq \mu} (\lambda - \mu)^2$$

over unordered pairs belongs to $R(T)^W$, since W permutes the weights. It is nonzero because $R(T)$ is a Laurent polynomial ring. The restriction isomorphism $R(K) \simeq R(T)^W$ gives a nonzero virtual character P of K [11].

If $P(t) \neq 0$ at $t \in T$, the eigenspaces of t in the chosen representation are precisely the weight spaces of T. Any element of $C_K(t)$ preserves these spaces, so it commutes with T, which acts by scalars on each. Faithfulness and $C_K(T) = T$ give $C_K(t) = T$. By conjugacy, $C_K(x)$ is a maximal torus for every $x \in K$ with $P(x) \neq 0$.

If a D-invariant abelian subgroup A contains such an x, write $T_x = C_K(x)$. Then $A \subset T_x$ and $T_x \subset C_K(A) \subset C_K(x) = T_x$. Thus $C_K(A)$ is a D-invariant maximal torus. When no such torus exists, P vanishes on every invariant A. By Lemma 1.1 it vanishes on every torus of G. □

3 The connected center and the three detection properties

Suppose $C(G) = C(K) \rtimes D$ with K compact matrix and D finitely generated. If $\widehat{G}$ has polynomial growth, then so do $\widehat{K}$ and D. The converse depends on the action. We first give a condition under which one can read off the growth from these two factors.

If V is the coefficient space of a generating self-conjugate corepresentation containing the trivial representation, then $\dim V^n$ is the volume of the ball of radius n in $\widehat{K}$. Indeed, products of coefficients are coefficients of tensor products, and an irreducible of dimension m contributes a coefficient space of dimension m^2. We can equally use any finite-dimensional, self-adjoint algebraically generating space in $\mathcal{R}_K$ containing 1 to test polynomial growth. Two such filtrations are contained in each other after a linear change of radius [13].

Proposition 3.1. *Suppose some finite-dimensional, self-adjoint generating space $V \subset \mathcal{R}_K$ contains 1 and is invariant under D. Then $\widehat{G}$ has polynomial growth if and only if both $\widehat{K}$ and D have polynomial growth. Under this hypothesis the quotient test for growth in Definition 1.2 passes from K to G.*

Proof. Choose a finite symmetric generating set for D and let $B_D(n)$ be its word ball. Put $W = V + span\{s_d : d \in B_D(1)\}$ inside the algebraic crossed product. Moving group elements to the right in a word of length n gives the first inclusion below. Reading a word from each factor in succession gives the second inclusion, so that

$$W^n \subset \bigoplus_{d \in B_D(n)} V^n s_d \subset W^{2n}$$

The middle space has dimension $|B_D(n)| \dim V^n$. This proves the growth equivalence. If all the groups in the tests for G have polynomial growth, the quotient $G \twoheadrightarrow K$ supplies every test for K and the subgroup $\widehat{D} \subset G$ tests D itself. The last assertion follows. □

For compact connected Lie groups we can remove the invariance hypothesis on V. In this case one group-dual subgroup of G suffices for both analytic properties. The length-function approach to growth of more general bicrossed products is developed in [15].

Theorem 3.2. *Let K be a compact connected Lie group and let a finitely generated discrete group D act on K by automorphisms. Put $C(G) = C(K) \rtimes D$. Let $Z = Z(K)^0$ be the identity component of the center, and form the discrete group $L = \widehat{Z} \rtimes D$ using the induced action on its character group. Then $\widehat{L} \subset G$, and $\widehat{G}$ is amenable if and only if L is amenable. Likewise, $\widehat{G}$ has polynomial growth if and only if L has polynomial growth.*

Proof. Restriction to the characteristic subgroup Z is equivariant. It gives the Hopf quotient

$$C(K) \rtimes D \longrightarrow C(Z) \rtimes D = C^*(L)$$

By [14, Corollary 6.4], G is coamenable precisely when D is amenable. Since L is an extension of D by an abelian group, the same criterion holds for L. Polynomial growth of $\widehat{G}$ passes to L by [13, Proposition 4.1].

Conversely, suppose L has polynomial growth. Put $S = [K, K]$. This is a compact connected semisimple group, and multiplication $Z \times S \to K$ is a surjection with finite kernel [11]. It is D-equivariant. Pullback on functions and the identity on D give an injection of coefficient algebras

$$\mathcal{R}_K \rtimes_{alg} D \;\hookrightarrow\; (\mathcal{R}_Z \otimes \mathcal{R}_S) \rtimes_{alg} D \;\simeq\; \mathcal{R}_S \rtimes_{alg} L$$

Here $\rtimes_{alg}$ denotes the algebraic crossed product. Its elements are finite sums of coefficients times group unitaries. In the last algebra $\widehat{Z}$ acts trivially on S and D acts by its given automorphisms. The identification sends a product of a character of Z and s_d to the corresponding group element of L.

The outer automorphism group of S is finite, as follows from its semisimple root datum [11]. Inner automorphisms preserve the coefficient space of each irreducible. Thus the translates under D of the coefficient space of a faithful representation of S span a finite-dimensional space. Adding its adjoint and 1 gives an L-invariant generating space. The algebra $\mathcal{R}_S$ has polynomial growth [13]. Proposition 3.1, now with S and L as the two factors, gives polynomial growth of the algebra on the right. Its finitely generated subalgebra on the left has polynomial growth as well. □

Theorem 3.3. *Let K be a compact connected Lie group, let D be finitely generated and let $C(G) = C(K) \rtimes D$. The following are equivalent.*

1. *Some torus of G detects $C(G)_{central}$, amenability of $\widehat{G}$ and polynomial growth of $\widehat{G}$.*
2. *The family of all tori of G jointly detects $C(G)_{central}$.*
3. *The action of D preserves a maximal torus of K.*

For each such maximal torus T, the group $\Gamma_T = \widehat{T} \rtimes D$ gives a maximal group-dual subgroup of G satisfying the first condition.

Proof. A faithful restriction to one torus implies joint faithfulness. Theorem 2.1 gives the implication from the second condition to the third, including a single virtual character in the common kernel when the third condition fails.

Assume that D preserves T. Put $Z = Z(K)^0$. Since Z is central and connected, ZT is a compact connected abelian group, and maximality of T gives $Z \subset T$. The restriction $\widehat{T} \to \widehat{Z}$ is surjective by duality for compact abelian groups and is equivariant for D. At the level of full coefficient algebras the maps are

$$C(G) \;\longrightarrow\; C^*(\Gamma_T) \;\longrightarrow\; C^*(\Gamma_Z)$$

Here $\Gamma_Z = \widehat{Z} \rtimes D$ is the group L of Theorem 3.2. The first map is faithful on $C(G)_{central}$ by Theorem 2.1. Amenability and polynomial growth pass from $\widehat{G}$ to Γ_T under restriction to a quantum subgroup [4, 13]. Each property passes from Γ_T to its group quotient Γ_Z, and Theorem 3.2 transfers it back to $\widehat{G}$. This proves both equivalences for this same T.

Finally suppose a group-dual subgroup contains the torus dual to Γ_T. By Lemma 1.1 it lies in the torus dual to Γ_A for a closed D-invariant abelian $A \subset K$. Restriction of the resulting maps to $C(K)$ gives $T \subset A$. Since $C_K(T) = T$, we have $A = T$. This proves maximality. □

For $K = SU_2$ the criterion can be read directly from the trace. Identify the maximal tori with the unoriented axes in the Lie algebra $Lie(SU_2) \simeq \mathbb{R}^3$. The inner automorphism group SO_3 rotates these axes. Take $D = A_4$ acting by the rotations of a regular tetrahedron. An invariant axis would be fixed pointwise by each rotation of order three, yet two such rotations

have different axes. In this example $Z(K)^0 = 1$. The discrete group supplied by Theorem 3.2 is just A_4, so the dual is amenable and has polynomial growth.

A noncentral element of SU_2 has distinct eigenvalues and its centralizer is the unique maximal torus containing it. If an invariant abelian subgroup contained such an element, its centralizer would be an invariant maximal torus, as in Theorem 2.1. Every invariant abelian subgroup is therefore contained in $\{I, -I\}$. Writing χ for the defining character, we get the explicit virtual character

$$P = \chi^2 - 4, \qquad k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad P(k) = -4$$

Both central elements have $P = 0$, so every torus kills P.

For example take $K = \mathbb{T}^2$ and $D = \mathbb{Z}$, with the generator acting on $\widehat{K} = \mathbb{Z}^2$ by

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Both factors have polynomial growth. Here $Z = K$ and $L = \mathbb{Z}^2 \rtimes_M \mathbb{Z}$ has exponential growth. Indeed, with $e = (1, 0)$, the 2^n sums $\sum_{j=0}^{n-1} \epsilon_j M^j e$, $\epsilon_j \in \{0, 1\}$, are distinct. Projecting onto the expanding eigenline gives a nonzero multiple of a sum of powers of $(3 + \sqrt{5})/2 > 2$, in which the largest differing power decides equality. Each sum followed by the same final power t^n is a word of length at most $2n$ using the factors $e^{\epsilon_j} t$, in multiplicative notation. The torus in Theorem 3.2 thus sees growth created by the action.

4 Characters for amenable actions

We now allow an arbitrary compact group K, still assuming that $C(K) \rtimes D$ is of compact matrix type. The test can require more than one invariant abelian subgroup. Put

$$S = \{k \in K : [\alpha_d(k), \alpha_e(k)] = 1 \text{ for all } d, e \in D\}$$

For $k \in S$, the closed subgroup generated by its orbit is abelian and D-invariant. Conversely every point of an invariant abelian subgroup belongs to S. The geometric condition below is that the conjugates of S cover K.

Theorem 4.1. *Suppose D is amenable. The tori of G detect every nonzero element of $C(G)_{central}$ if and only if every conjugacy class of K meets a closed D-invariant abelian subgroup of K.*

Proof. The set S is closed, since for each pair d, e the equation $[\alpha_d(k), \alpha_e(k)] = 1$ is closed. Its conjugacy saturation $K \cdot S = \{hkh^{-1} : h \in K,\, k \in S\}$ is compact. If a class misses S, choose a continuous function on the compact Hausdorff class space $K/Ad(K)$ which equals one on this class and vanishes on the image of S. Its pullback belongs to $C(G)_{central}$ and vanishes on every torus by Lemma 1.1.

Conversely take $x \in C(G)_{central}$. Since D is amenable the full and reduced crossed products agree. A coefficient of a norm limit is defined by $f_d = E_K(xs_d^*)$, where E_K is the conditional expectation onto $C(K)$. For a finite sum of characters each f_d is a class function, since automorphisms of K preserve class functions. Contractivity gives the same conclusion for x. The coefficients determine x, since their translates give all matrix entries in the regular representation. If $x \neq 0$, choose d and k with $f_d(k) \neq 0$. Some conjugate of k belongs to one of the subgroups A in the statement. Restriction to $C(A) \rtimes D$ has d-coefficient $f_d|_A$, by continuity from finite sums, and therefore detects x. □

Corollary 4.2. *Let F be a finite group and let $D = \mathbb{Z}^r$, $r \geq 1$, act by translations on $K = F^D$. The tori of the crossed product detect its full character algebra if and only if F is abelian.*

Proof. The functions at one site and the lattice generators give compact matrix generation. The shifts of $x \in K$ commute pairwise precisely when all its entries commute in F. For abelian F we can thus take $A = K$. If F is nonabelian, choose x whose entries include representatives of every conjugacy class of F. Conjugation in K acts site by site. If a conjugate of x had commuting entries, they would generate an abelian subgroup J meeting every class of F. A proper subgroup cannot do so, since its conjugates cover at most $1 + [F : J](|J| - 1) < |F|$ elements. Now apply Theorem 4.1. □

For $F = S_3$ two sites suffice. Write s for the sign character and c for the character of the two-dimensional irreducible representation. At the origin and a neighboring site e, take the virtual character

$$P(x) = (1 - s(x_0))(1 + s(x_e) - c(x_e))$$

Its value is 6 when x_0 is a transposition and x_e is a three-cycle, and is zero otherwise, by the character table of S_3. No transposition commutes with a three-cycle, even after independently conjugating these two entries. Thus every torus kills P. The same crossed product has exponential growth detected by a torus of a quotient (Corollary 10.4).

There is also a one-character obstruction on $S_3 \times (\mathbb{Z}_2)^{\mathbb{Z}^2}$. Let (m, n) act by conjugation by $(123)^m$ on S_3 and by the shift on the second factor. The projection of any invariant abelian subgroup to S_3 lies in A_3, since (123) cycles the three subgroups generated by transpositions. Consequently $1 - s$ on the first factor vanishes on every torus.

5 Rotations of the free unitary group

For a quantum base the weight spaces of a representation can play the role of the eigenlines in the SU_2 example. For the fundamental matrix of U_2^+ there are at most two such spaces on a group dual. This gives an axis criterion, and retaining two different weights now produces a nonabelian free group.

Let D be a finitely generated discrete group and let $\rho : D \to PU_2 = SO_3$ be a homomorphism. Write u for the fundamental matrix of U_2^+, so that $C(U_2^+)$ is generated by the entries of u subject to unitarity of u and $\bar{u}$. Choose a unitary lift U_d of each $\rho(d)$. The formula $u \mapsto U_d^* u U_d$ gives an action of D by quantum group automorphisms, since the scalar ambiguity in a lift cancels. We consider Wang's full crossed product [26]

$$C(G) = C(U_2^+) \rtimes D, \qquad s_d u s_d^* = U_d^* u U_d$$

Here s_d are the canonical group-like unitaries. If S is a finite generating set for D, a fundamental matrix for G is $v = u \oplus \bigoplus_{d \in S} s_d$. A point of $\mathbb{C}P^1 = S^2$ is a line in $\mathbb{C}^2$, and its antipode is the orthogonal line. An unoriented axis is thus an unordered pair of orthogonal lines.

Proposition 5.1. *In any Hopf quotient onto a group algebra, the image of u is either zI_2 or has two distinct weights on the lines of a $\rho(D)$-invariant axis. For each invariant axis the diagonal torus in an adapted basis is dual to $F_2 \rtimes D$, where D permutes the two free generators as it permutes the ends of the axis. If there is no invariant axis, every torus factors through the scalar quotient $C^*(\mathbb{Z} \times D)$, itself a toral quotient.*

Proof. Let the target be $C^*(\Lambda)$. With two distinct weights the image of u is $Pa + (I - P)b$, where P is a rank-one projection and $a, b \in \Lambda$. Conjugation by the image of s_d replaces a, b by their conjugates. The crossed-product relation also gives the same matrix with P replaced by $U_d^* P U_d$. Linear independence of the group elements shows that $U_d^* P U_d$ is either P or $I - P$. The two lines therefore form an invariant axis.

Conversely, choose an orthonormal basis on an invariant axis and leave the s_d blocks of v in place. Making u diagonal gives two free unitaries a, b. Each U_d is monomial in this basis, so

covariance requires exactly the indicated permutation of a, b by s_d. This is the presentation of $F_2 \rtimes D$. With no invariant axis, making u diagonal in any basis forces its two entries to agree. Its scalar value z then commutes with D, and an independent generator of $\mathbb{Z}$ realizes all these relations in $\mathbb{Z} \times D$. □

Set $\chi = Tr(u)$. The irreducibles of U_2^+ are indexed by words w in two letters corresponding to u and $\bar{u}$. Their characters χ_w are $*$-polynomials in χ, by the fusion rules [1, Theorem 6.20]. The irreducibles of G are indexed by (d, w), with character $s_d \chi_w$ [14, Theorem 6.1]. In particular

$$C(G)_{central} = C^*(\chi, s_d \mid d \in D) \ \subset C(G)$$

The covariance relation gives $s_d \chi s_d^* = \chi$.

Theorem 5.2. *The tori of G detect every nonzero element of $C(G)_{central}$ if and only if $\rho(D)$ preserves an unoriented axis. With an axis, restriction to its single torus is isometric on $C(G)_{central}$.*

Proof. If there is no axis, every torus sends χ to $2z$ by Proposition 5.1. Hence $\chi^*\chi - 4$ vanishes on all tori. It takes the value -4 in the copy of $C(U_2^+)$ inside the crossed product when we evaluate at

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in U_2$$

Fix an invariant axis and write $\sigma(d) = 0$ or 1 according as d fixes or exchanges its ends. Restriction to its torus sends $\chi/2$ to $(a + b)/2$. To check the norm, represent $C(G)$ faithfully on a Hilbert space K. Let T be the image of $\chi/2$ and V_d the image of s_d. Then T is a contraction, and V_d commutes with both T and T^*. On $K \oplus K$ put

$$A = \begin{pmatrix} T & \sqrt{I - TT^*} \\ \sqrt{I - T^*T} & -T^* \end{pmatrix}, \qquad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad B = JAJ$$

The identity $T\sqrt{I - T^*T} = \sqrt{I - TT^*}T$ gives unitarity of A. The assignments

$$a \mapsto A, \qquad b \mapsto B, \qquad d \mapsto (V_d \oplus V_d)J^{\sigma(d)}$$

represent $F_2 \rtimes D$. Indeed J exchanges A and B by conjugation, and $V_d \oplus V_d$ commutes with both. We have

$$(A + B)/2 = T \oplus (-T^*)$$

The first copy of K therefore reduces the images of $(a + b)/2$ and of all d. On this copy they act as T and V_d. Thus the original faithful representation of $C(G)_{central}$ is a subrepresentation of the toral one on these generators. This proves isometry after norm closure. □

The analytic conclusions use two facts about the ambient G which hold for every action in this family.

Proposition 5.3. *The quantum group G is noncoamenable and $\widehat{G}$ has exponential growth. If D is torsion-free, then G is connected.*

Proof. The diagonal torus of U_2^+ is dual to F_2, so U_2^+ is noncoamenable [4]. The crossed-product coamenability criterion [14, Corollary 6.4] gives noncoamenability of G.

Use word length in irreducibles with respect to $v \oplus \bar{v}$. The ball of radius n contains all 2^n distinct irreducibles $(1, w)$ with $|w| = n$. Indeed r_{xy} is a constituent of $r_x \otimes r_y$ for every pair of words x, y [1, Theorem 6.20]. Since the volume counts each irreducible with the square of its dimension, this gives exponential growth.

For connectedness, consider a nontrivial irreducible (d, w). If $d \neq 1$ and D is torsion-free, its tensor powers have pairwise distinct degrees d^n in the crossed-product fusion rules [14, Remark 6.3]. If $d = 1$, then w is nonempty and its n-th tensor power contains $(1, w^n)$. In either case the coefficients generate an infinite-dimensional algebra, which gives connectedness. □

Corollary 5.4. *Suppose D is torsion-free and has polynomial growth. If $\rho(D)$ preserves an axis, its associated torus detects $C(G)_{central}$, nonamenability and exponential growth. In the absence of an invariant axis every toral group has polynomial growth and every torus kills $\chi^*\chi - 4$.*

Proof. With an axis, one toral group contains F_2 as the subgroup generated by a, b, so it is nonamenable and has exponential growth. Theorem 5.2 gives character detection. Without an axis every toral group is a quotient of $\mathbb{Z} \times D$, hence amenable and of polynomial growth.

Together with Proposition 5.3 this proves the assertions. □

Corollary 5.5. *There is a connected $G \subset U_4^+$ with nonamenable dual of exponential growth for which every toral group has polynomial growth and every torus kills the same nonzero virtual character.*

Proof. Take $D = \mathbb{Z}^3 \rtimes \mathbb{Z}$, where the generator t cycles the standard basis of $\mathbb{Z}^3$. This group is torsion-free, and $\mathbb{Z}^3 \times \langle t^3 \rangle$ has index three. Map the basis to the three nonidentity elements of the Klein four group in A_4, in the order in which conjugation by (123) cycles them, and send t to (123). This gives a surjection onto the tetrahedral rotation group $A_4 \subset SO_3$. An invariant pair of antipodal points would be fixed pointwise, since A_4 is generated by elements of order three. Two tetrahedral rotations with distinct axes have no common fixed point. Corollary 5.4 now applies. The elements e_1, t generate D, so the fundamental matrix $u \oplus s_{e_1} \oplus s_t$ has size four. □

For compact Lie groups and for group duals, the skeleton determines the embedded group, and every proper subgroup in the same class has a strictly smaller standard torus for some Q [1, Theorem 13.12]. The tetrahedral action lets us compare two skeletons in the same U_4^+.

Theorem 5.6. *Let $G \subset U_4^+$ be the tetrahedral example of Corollary 5.5 and let H be its scalar torus, dual to $\mathbb{Z} \times D$. Then $H \subsetneq G$ and, as embedded subgroups of U_4^+,*

$$T_Q(H) = T_Q(G) \qquad (Q \in U_4)$$

The discrete group $\widehat{H}$ has polynomial growth, whereas $\widehat{G}$ is nonamenable and has exponential growth.

Proof. Equip H with the restriction of the fundamental matrix of G. For any inclusion $H \subset G$ with this convention, forming the diagonal quotient in the basis Q imposes the same off-diagonal relations on both sides. Thus $T_Q(H) = H \cap T_Q(G)$. In our example Proposition 5.1 gives $T_Q(G) \subset H$ for every Q, which proves the asserted equality. In the original block basis $T_1(G) = H$. The element $\chi^*\chi - 4$ from the proof of Theorem 5.2 is nonzero on G and vanishes on H, proving properness. Finally $\mathbb{Z} \times D$ is virtually abelian, and Proposition 5.3 gives the assertions about $\widehat{G}$. □

The same tetrahedral group gives a classical base of polynomial growth with an explicit character in the common kernel of all restrictions.

Theorem 5.7. *There is a connected compact matrix quantum group of Kac type, generated by two group-dual subgroups, for which all tori kill the same nonzero virtual character. Its discrete dual has polynomial growth.*

Proof. Let $K = SO_5$. Let $D = \mathbb{Z}^3 \rtimes \mathbb{Z}$ act through the surjection $D \to A_4$ of Corollary 5.5. Embed A_4 in SO_5 by even permutations of the first four coordinates and use conjugation. Form $C(G) = C(K) \rtimes D$. This G is connected. Indeed, an irreducible of nonidentity D-degree has tensor powers of infinitely many degrees, and in identity degree its coefficients belong to $C(K)$, where a nontrivial irreducible has infinite image by connectedness. The coefficients of the defining representation of SO_5, together with 1, span a D-invariant generating space V. Both the dual of SO_5 and the virtually abelian group D have polynomial growth [13]. Proposition 3.1 applied to V gives the growth assertion.

A maximal torus of SO_5 rotates two orthogonal real planes and fixes their orthogonal line. Its normalizer preserves this decomposition up to interchanging the two planes. If A_4 normalized such a torus, it would preserve each plane, since it has no quotient of order two. The same fact makes its determinant on each plane positive and its action on the line trivial. Its image would lie in $SO_2 \times SO_2$, contrary to the faithful nonabelian permutation action.

The defining representation of SO_5 has five distinct weights, each of multiplicity one. The virtual character in Theorem 2.1 is therefore the usual discriminant

$$P(k) = disc_X \det(XI_5 - k)$$

For instance it is nonzero at a rotation with eigenvalues $1, e^{\pm\pi i/3}, e^{\pm\pi i/2}$. Its expression as a polynomial in the characteristic coefficients writes it directly in terms of exterior power characters. By Theorem 2.1 it vanishes on every torus of G.

It remains to show generation. Let E be the diagonal sign subgroup of SO_5, and let C be the circle of rotations in the plane spanned by e_5 and $v = (e_1 + e_2 + e_3 + e_4)/2$. The group A_4 preserves E and fixes C pointwise. The conjugates of the infinitesimal rotation $e_5 \wedge v$ by E span all $e_5 \wedge e_i$, $1 \leq i \leq 4$. Indeed the first four signs are arbitrary and the fifth contributes an overall sign. Bracketing these rotations gives the remaining $e_i \wedge e_j$. It follows that finite products from E and C are dense in SO_5.

Take the tori with discrete groups $\widehat{E} \rtimes D$ and $\widehat{C} \rtimes D$ from Lemma 1.1. Suppose a Hopf $*$-ideal lies in the kernels of both restrictions and take an element $\sum_d f_d s_d$ of this ideal in $\mathcal{R}_G$. Its iterated coproduct vanishes after restriction to any ordered list of the two tori. The coefficient of $(d, \ldots, d)$ in that restriction is $(a_1, \ldots, a_m) \mapsto f_d(a_1 \cdots a_m)$, with a_i in the corresponding E or C. Density forces each f_d to vanish. Hence these two tori generate G, even though the discriminant vanishes on each of them separately. □

6 Tori of quotients and quantum automorphism groups

In the rotation examples of Section 5, pullback of the coefficients of U_2^+ gives a quantum quotient $G \twoheadrightarrow U_2^+$. Its diagonal torus is dual to F_2 for every action of D. The following observation explains when the quotient test passes from a base to its crossed product.

Proposition 6.1. *Suppose the quotient test for amenability in Definition 1.2 holds for a compact matrix quantum group K of Kac type. It also holds for G with $C(G) = C(K) \rtimes D$, for any action of a finitely generated discrete group D by quantum group automorphisms.*

Proof. Suppose every group tested for G is amenable. The inclusion $C(K) \subset C(G)$ gives a quantum quotient $G \twoheadrightarrow K$. Composing with this quotient shows that every group tested for K is also tested for G, so K is coamenable by hypothesis.

There is a second test inside G itself. The counit of K is invariant under the action of D, so evaluation at the counit on $C(K)$ and $s_d \mapsto d$ give a Hopf $*$-surjection $C(G) \to C^*(D)$. Thus $\widehat{D} \subset G$ and D is amenable. By the crossed-product criterion [14, Corollary 6.4], G is coamenable. In the other direction, coamenability passes to compact quantum quotients and then to their tori, as recalled in Proposition 7.1. □

For polynomial growth Proposition 3.1 gives an analogous conclusion when the action preserves a finite-dimensional generating space. We next examine how much these quotient tests can see for quantum automorphism groups. The exceptional algebra occurs already in dimension five. Consider the quantum automorphism group G of the algebra $B = \mathbb{C} \oplus M_2(\mathbb{C})$ with its canonical trace

$$\tau(z, X) = \frac{z + 2Tr(X)}{5}$$

where Tr is the ordinary matrix trace. This is the normalized trace of the left regular representation. The universal trace-preserving coaction on B gives a compact matrix quantum group of Kac type [2]. Conjugating the matrix summand gives a classical subgroup $PU_2 = SO_3 \subset G$.

Lemma 6.2. *Every group-dual subgroup of G is contained in this SO_3. The possible discrete groups are precisely $\mathbb{Z}$, the finite cyclic groups, and the Klein four group.*

Proof. A group-dual subgroup gives a grading $B = \bigoplus_g B_g$ and a coaction $\alpha(x) = x \otimes g$ for $x \in B_g$. Let $p : B \to \mathbb{C}$ be the scalar character. The algebra map

$$\varphi = (p \otimes id)\alpha : B \longrightarrow \mathbb{C}[\Gamma]$$

sends $x \in B_g$ to $p(x)g$. Its image is spanned by the group elements which occur. Since it is a finite-dimensional unital $*$-algebra, these elements form a finite subgroup $F \subset \Gamma$ and the image is $\mathbb{C}[F]$. If φ were injective, $|F| = 5$. Such a group is cyclic, whereas B is noncommutative. Hence $\ker \varphi \neq 0$. Composing φ with the counit gives p, so $\ker \varphi \subset \ker p = M_2(\mathbb{C})$. Simplicity of the matrix summand gives equality.

The map φ preserves degrees, so the matrix summand is a graded ideal. If e is its unit, comparison of homogeneous components in $ex = x$ shows that the neutral component e_1 acts as the identity on every homogeneous element of the ideal. Thus $e_1 e = e$, while $e_1 e = e_1$ since e_1 belongs to the ideal. Both e and $1 - e$ are therefore fixed. On $M_2(\mathbb{C})$ the coaction preserves the ordinary trace. The universal trace-preserving quantum automorphism group of $M_2(\mathbb{C})$ is $PU_2 = SO_3$. Indeed the quantum and classical groups here have the SO_3 fusion rules [2], and restriction sends the three-dimensional summand of M_2 to the usual rotation representation. Induction with the fusion rules identifies every irreducible on restriction, hence also its coefficient coalgebra. This proves the asserted factorization.

Since $C(SO_3)$ is commutative, every group-dual subgroup is classical abelian. A rotation through an angle other than 0 or π has a circle as its centralizer. A closed subgroup of this circle is finite cyclic or the full circle. If every nonidentity element is a half-turn, commuting half-turns about distinct axes have perpendicular axes, and two of them generate a Klein four group. This gives the list on taking duals, and each case occurs. □

Theorem 6.3. *The compact quantum group G has no proper nontrivial compact quantum quotient. Its discrete dual is nonamenable and has exponential growth. Nevertheless in every diagram $G \twoheadrightarrow H \supset \widehat{\Lambda}$ the group Λ has at most linear growth. In particular the two quotient tests in Definition 1.2 both fail, even for connected G.*

Proof. The irreducibles of G have the SO_3 fusion rules [2]. Writing them as $r_0 = 1, r_1, r_2, \ldots$, the coaction on B is $1 + r_1$ and

$$r_j \otimes r_k = \bigoplus_{s=|j-k|}^{j+k} r_s$$

Under a compact quantum quotient the polynomial Hopf algebra of H embeds in that of G. Its simple coefficient coalgebras give a subset of these irreducibles closed under tensor products and constituents. If it contains r_j with $j \geq 1$, then $r_j \otimes r_j$ contains r_1, and tensor powers of

r_1 contain every r_k. The inclusion is thus either trivial or onto, as in Wang's proof in [27]. In particular G has no nontrivial finite quantum quotient, so is connected.

For growth, put $d_k = \dim r_k$. Since $\dim B = 5$, the fusion rule gives

$$d_0 = 1, \qquad d_1 = 4, \qquad d_{k+1} = 3d_k - d_{k-1} \quad (k \geq 1)$$

In particular $d_{k+1} \geq 2d_k$ by induction. The ball of radius n for $1 + r_1$ has volume $\sum_{k=0}^{n} d_k^2$, which grows exponentially. Nonamenability in dimension five follows from the coamenability criterion of [2]. Finally H is trivial or isomorphic to G, so Lemma 6.2 accounts for every Λ in the test. □

This is a nonclassical quantum group all of whose group-dual subgroups are classical. Such a group is called *strange* in [3]. The same pair of matrix sizes also occurs in the computation of outer quantum symmetries in [16, Section 3.3, Example 2].

Corollary 6.4. *Let $G = G^+(\mathbb{C} \oplus M_2)$ with its canonical trace. Restriction to one circle is injective on the algebraic span of the irreducible characters, whereas every torus kills the same nonzero element of the norm-closed character algebra $C(G)_{central}$.*

Proof. Put $x = \chi_{r_1} = x^*$. The fusion rules in Theorem 6.3 express every irreducible character as a polynomial in x. On the subgroup SO_3 the action on $\mathbb{C} \oplus M_2$ splits as two trivial representations and the three-dimensional rotation representation. Hence $x|_{SO_3} = 1 + Tr(R)$. On rotations through angle θ about a fixed axis this is $2 + 2\cos\theta$, with range $[0, 4]$. Thus a polynomial in x vanishing on this circle is the zero polynomial.

Let h be the Haar state. Irreducibility and the fusion rule give $h(x) = 0$ and $h(x^2) = 1$. If x were positive, the inequality $x^2 \leq \|x\| x$ would contradict these two equalities. Therefore $x_- = \max(-x, 0)$ is a nonzero element of $C(G)_{central}$. Its restriction to SO_3 is zero, and all tori factor through that subgroup by Lemma 6.2. □

Is this failure specific to the algebra of dimension five? For the canonical trace, the possible block sizes give a complete answer.

Proposition 6.5. *Among the quantum groups $K = G^+(B)$ with the canonical trace, $B = \mathbb{C} \oplus M_2$ is the unique algebra, up to isomorphism, for which the tori of K fail to detect amenability or polynomial growth.*

Proof. Put $N = \dim_{\mathbb{C}} B$. For $N \leq 3$ we have the finite groups S_N. At $N = 4$ the irreducibles have the SO_3 fusion rules with $\dim r_1 = 3$ and hence $\dim r_j = 2j + 1$. The ball volumes $\sum_{j=0}^{n} (2j+1)^2$ are polynomial. Both cases are coamenable [2], and both properties pass to the tori [4, 13].

Let $N \geq 5$. We produce a trace-preserving group grading whose support generates a nonamenable group of exponential growth. Its coaction on B is faithful in the sense that its coefficients generate the group algebra, so its group dual embeds in K. By permanence this forces the ambient dual to be nonamenable and to have exponential growth. For a summand M_k one can give matrix units e_{ij} degree $g_i^{-1} g_j$ for any list of group elements g_i. On a scalar block $\mathbb{C}^m = C^*(\mathbb{Z}_m)$ one can use the Fourier grading by $\mathbb{Z}_m$. Both gradings preserve the canonical trace. Gradings on distinct direct summands combine as gradings by the free product of their groups, since products between summands vanish. All unused summands may be placed in the identity degree.

If B has a summand M_k with $k \geq 3$, use the list $1, a, b, 1, \ldots, 1$ in $F_2 = \langle a, b \rangle$. If it has two M_2 summands, grade each by $\mathbb{Z}$, giving F_2 once again. If there is just one M_2 and all other summands are scalar, two scalar summands together give $\mathbb{Z}_2$, and the matrix summand gives $\mathbb{Z}$. This gives $\mathbb{Z}_2 * \mathbb{Z}$ whenever there are at least two scalar summands. Finally, for $B = \mathbb{C}^N$, splitting off a pair and a triple gives $\mathbb{Z}_2 * \mathbb{Z}_3$. Each displayed group contains a free group on two generators. The sole unaccounted case is $\mathbb{C} \oplus M_2$, which fails both tests by Theorem 6.3. □

The exceptional algebra has nonamenable dual. Its failure of the quotient growth test leaves the following implication to be determined.

Question 6.6. *Suppose $\widehat{G}$ is amenable and every group Λ in a diagram $G \twoheadrightarrow H \supset \widehat{\Lambda}$ as in Definition 1.2 has polynomial growth. Must $\widehat{G}$ have polynomial growth?*

7 Two conjectures on amenability and growth

For $G = G^+(\mathbb{C} \oplus M_2)$ there is another way to recover a torus of exponential growth. A cocycle representing the nontrivial class on the Klein four group twists G into S_5^+ [22, Theorem 4.8]. The latter contains the dual of $\mathbb{Z}_2 * \mathbb{Z}_3$, acting on a pair and a triple of points [3]. More generally, for a finite-dimensional C^*-algebra B with its canonical trace, $G^+(B)$ is a unitary cocycle deformation of S_N^+, where $N = \dim_{\mathbb{C}} B$ [10, Theorem 3.1].

A unitary dual 2-cocycle twist gives a unitary monoidal equivalence which preserves the Hilbert space dimension of every representation. Conversely, a dimension-preserving unitary monoidal equivalence is realized, up to isomorphism, by such a twist [8]. Write $H \sim K$ for this equivalence. The identity quotient and the trivial twist are allowed throughout.

Proposition 7.1. *Let G be a compact matrix quantum group of Kac type. For every diagram*

$$G \twoheadrightarrow H, \qquad H \sim K, \qquad \widehat{\Lambda} \subset K$$

with H a compact matrix quantum quotient of G and K a unitary dual 2-cocycle twist of H, amenability of $\widehat{G}$ implies amenability of Λ, and polynomial growth of $\widehat{G}$ implies polynomial growth of Λ. A dimension-preserving unitary monoidal equivalence of G preserves the collection of groups Λ arising in these diagrams, up to isomorphism.

Proof. Growth passes through compact quantum quotients and subgroups [13]. Under $H \sim K$ a self-conjugate generating representation containing 1 and its image have exactly the same constituents in each tensor power, with the same dimensions. The two ball volumes agree at every radius.

For a compact quantum quotient the counit restricts, and the inclusion of coefficient algebras is isometric for reduced norms because the Haar states agree and the reduced Haar state is faithful. Thus the quotient inherits a bounded reduced counit. For a compact quantum subgroup, take a self-conjugate generator v containing 1 and put $m = \dim v$. Restriction can only increase the multiplicity of 1 in $v^{\otimes 2n}$. This multiplicity is the $2n$-th Haar moment of χ_v, whose $2n$-th root tends to the reduced norm by faithfulness of the reduced Haar state. Hence if the ambient norm is m, the subgroup norm is also m. On both sides we can write $v = 1 \oplus w$, with $\chi_v = 1 + \chi_w$ and $\|\chi_w\| \leq m - 1$. Thus the spectrum is contained in $[2 - m, m]$. Norm m is therefore equivalent to the presence of the positive endpoint m, the quantum Kesten criterion for coamenability [4]. Under $H \sim K$ both m and the multiplicity of 1 in every $v^{\otimes 2n}$ are preserved. Thus the reduced character norm attains m on one side exactly when it does on the other. This also follows from the fusion-ring criterion [19, Theorem 4.5].

Finally a compact matrix quantum quotient is specified by a full rigid tensor subcategory of $Rep(G)$ closed under subobjects and finite sums and generated by one object and its conjugate. Its coefficient algebra is the span of the coefficient spaces of the simple objects in the subcategory. A monoidal equivalence carries each such subcategory to another and restricts to an equivalence there, preserving dimensions. Composing the inverse of the transported subcategory equivalence with $H \sim K$ gives the same tested group. The inverse equivalence proves equality of the collections. □

In particular, any finite succession of compact matrix quantum quotients and unitary dual cocycle twists before taking a torus reduces to a single quotient followed by a single twist. Indeed,

successive quotients compose, and a quotient taken after an equivalence can be transported back along that equivalence by the preceding proof.

Conjecture 7.2 (Amenability)**.** *Let G be a compact matrix quantum group of Kac type. Its discrete dual is amenable if and only if Λ is amenable in every diagram*

$$G \twoheadrightarrow H, \qquad H \sim K, \qquad \widehat{\Lambda} \subset K$$

where H is a compact matrix quantum quotient of G and K is a unitary dual 2-cocycle twist of H.

Conjecture 7.3 (Polynomial growth)**.** *Let G be a compact matrix quantum group of Kac type. Its discrete dual has polynomial growth if and only if Λ has polynomial growth in every diagram*

$$G \twoheadrightarrow H, \qquad H \sim K, \qquad \widehat{\Lambda} \subset K$$

where H is a compact matrix quantum quotient of G and K is a unitary dual 2-cocycle twist of H.

The open direction in each case is the existence of a witness. In the first case a nonamenable dual should yield a nonamenable Λ. In the second case the conclusion sought is a single Λ with growth that is not polynomial, even when $\widehat{G}$ is amenable. The group $\mathbb{Z}_p \wr \mathbb{Z}$ supplies such a witness in Corollary 10.4.

The permanence arguments of Propositions 6.1 and 3.1 also apply to Conjectures 7.2 and 7.3. Indeed, for $C(G) = C(L) \rtimes D$, every diagram for L is a diagram for G after composition with $G \twoheadrightarrow L$, and $\widehat{D} \subset G$ supplies the other factor. Thus the amenability implication passes from L to G for any action. The growth implication does so whenever D preserves a finite-dimensional generating space as in Proposition 3.1.

Theorem 7.4. *Conjectures 7.2 and 7.3 hold for $C(G) = C(L) \rtimes D$, where D is finitely generated and acts by quantum group automorphisms, for the following bases and actions.*

1. *L is a compact connected Lie group.*
2. *$L = U_2^+$ with the rotation action of Section 5.*
3. *$L = G^+(B)$ for a finite-dimensional C^*-algebra with its canonical trace.*

They also hold for every unitary dual cocycle twist of these crossed products.

Proof. In the first case Theorem 3.2 supplies a single group inside G testing both properties. In the second case the quotient $G \twoheadrightarrow U_2^+$ supplies F_2 without twisting.

In the third case $G \twoheadrightarrow L$. If $N = \dim B \geq 5$, Proposition 6.5 gives a group detecting both failures directly, except at $B = \mathbb{C} \oplus M_2$. In that case take the twist $K = S_5^+$ of L and its toral group $\Lambda = \mathbb{Z}_2 * \mathbb{Z}_3$. Equivalently, for every $N \geq 5$ one can use the same pair and triple in the model S_N^+ of [10, Theorem 3.1].

If $N \leq 4$, the base L is coamenable and has polynomial growth, as in Proposition 6.5. The counit supplies $\widehat{D} \subset G$, so the test includes D. When D is amenable, G is coamenable by [14, Corollary 6.4]. For growth, Proposition 3.1 applies to any action. For $N \leq 3$ use the whole finite-dimensional algebra, and for $N = 4$ use the coefficient space of $1 + r_1$. Indeed r_1 is the unique irreducible whose tensor square has three distinct constituents, so every automorphism preserves this space. Thus polynomial growth of D implies polynomial growth of $\widehat{G}$. The last assertion follows from Proposition 7.1 and invariance of the two properties. □

The third case also includes every compact quantum group L of Kac type having the SO_3 fusion rules. By [22] it is $G^+(B, \varphi)$ for a homogeneous faithful state, with $B = 1 + r_1$ as a representation. On this finite-dimensional algebra the invariant state is the canonical trace. To

see this, let U implement the coaction on $L^2(B,\varphi)$. Left multiplication $b \mapsto L_b$ intertwines the coaction with conjugation by U on $End(B)$. In Kac type $\bar{U}$ is unitary, so conjugation preserves the ordinary matrix trace. Thus $b \mapsto Tr(L_b)/\dim B$ is invariant. Uniqueness follows from the single copy of 1 in B, giving the assertion.

8 Bicrossed products with a compact Lie factor

Let D be discrete and K compact. The following formulas also apply to the infinite component groups of the next two sections. For a matched pair we use the factorization $dk = \alpha_d(k)\beta_k(d)$, with $\alpha_d(k) \in K$ and $\beta_k(d) \in D$. Factoring dhk and dsk in two ways gives

$$\alpha_d(hk) = \alpha_d(h)\alpha_{\beta_h(d)}(k), \qquad \beta_k(ds) = \beta_{\alpha_s(k)}(d)\beta_k(s)$$

Here α is a left action and β a right action. When β is trivial, α acts by group automorphisms and we recover the ordinary crossed product. Write G for the compact bicrossed product. Its coefficient algebra is the algebraic crossed product $\mathcal{R}_K \rtimes_{alg} D$, with $s_d f s_d^* = f \circ \alpha_{d^{-1}}$. Functions have their usual coproduct, and the canonical unitaries satisfy

$$\Delta(s_d) = \sum_{r \in \beta_K(d)} s_d 1_{\{k:\beta_k(d)=r\}} \otimes s_r$$

The orbits are finite by continuity and compactness. The Haar state integrates the identity coefficient against Haar measure on K. It is a trace since α preserves this measure. Evaluation at the identity of K gives $\widehat{D} \subset G$, and $\widehat{G}$ is amenable exactly when D is amenable [14, Theorem 3.4 and Corollary 3.7]. Thus for bicrossed products the remaining analytic question concerns growth when D itself has polynomial growth.

Theorem 8.1. *Let K be a compact Lie group, let D be finitely generated and put $K_0 = K^0$ and $Z = Z(K_0)^0$. In the bicrossed product the torus dual to $\widehat{Z} \rtimes D$ detects both amenability and polynomial growth. The action of D on the character group here comes from its action on K_0 by automorphisms.*

Proof. The continuous map $h \mapsto \beta_h(d)$ is constant on K_0, with value d. Each α_d fixes the identity and preserves K_0. By the matched-pair identities, its restriction θ_d to K_0 is an automorphism and

$$\alpha_d(hk) = \theta_d(h)\alpha_d(k) \qquad (h \in K_0)$$

Restriction of functions to K_0 and $s_d \mapsto s_d$ give the ordinary crossed product on K_0 as a quantum subgroup. Theorem 3.2 supplies the stated torus there, and the amenability assertion follows.

For growth, choose representatives $k_1 = 1, \ldots, k_b$ for the cosets $K_0 k_i$. Take a finite-dimensional self-adjoint generating space $V \subset \mathcal{R}_{K_0}$ containing 1 and stable under left and right translations. The coefficient spaces of a faithful representation, its conjugate and 1 give such a V. Use independent copies of V on the b cosets to obtain a space E on K. A coefficient on one coset extended by zero has finite-dimensional right translation span, since right translation permutes the cosets and acts on their coordinates by translations. Hence $E \subset \mathcal{R}_K$. Conversely, each coefficient on K restricts on each coset to a representative function on K_0. Since E includes the coset indicators, it generates $\mathcal{R}_K$.

Adjoin the unitaries for the same symmetric set of generators of D to E and to V, giving spaces W and W_0 in the two crossed products. For a word of degree d with coefficient f, record the b entries $f(hk_i)s_d$. Multiplying on the right by s_e replaces d by de. Multiplying on the right by $v \in E$ multiplies the ith entry on the right by the function $h \mapsto v(h\alpha_{d^{-1}}(k_i))$ in V. Indeed, commuting this function past s_d introduces $\theta_{d^{-1}}$ on h, giving precisely $v(\alpha_{d^{-1}}(hk_i))$. Thus the entries of a word of length n lie in W_0^n. Recording the entries degree by degree defines an injective

linear map into b copies of W_0^n, since the cosets cover K. Restriction to the first coset maps W^n onto W_0^n. We obtain

$$\dim W_0^n \;\le\; \dim W^n \;\le\; b \dim W_0^n$$

The polynomial growth assertion now follows from Theorem 3.2. □

9 The component group and the cyclic case

Let $D = \mathbb{Z}^r$, $r \ge 1$, and assume that K/K^0 is infinite. Every bicrossed product in this class is coamenable. It is also disconnected, since the Hopf inclusion of functions on K gives $G \twoheadrightarrow K$, and the profinite group K/K^0 has a nontrivial finite quotient. We first give the reduction used in both the cyclic and the higher-rank growth arguments. A group shift over a finite group F means a closed translation-invariant subgroup of F^D, with pointwise multiplication.

Lemma 9.1. *Suppose G is compact matrix and write $P = K/K^0$. The reverse action of K on D is by a finite group of lattice automorphisms. Its kernel H contains K^0. The matched pair descends to (D, P), and $Q = H/K^0$ is an infinite open subgroup of P on which D acts by automorphisms. As a D-system, Q is isomorphic to a group shift over a finite group.*

Proof. Let S be the union of the K-orbits of a basis of D and its negatives. This generating set is finite. The matched-pair identity applied to $(-s) + s$ gives $-\beta_k(s) = \beta_{\alpha_s(k)}(-s)$, so S is symmetric. For $s \in S$ one has

$$\beta_k(d + s) - \beta_k(d) = \beta_{\alpha_d(k)}(s) \in S$$

Thus β_k is an automorphism of the Cayley graph of (D, S) fixing zero. Here its linearity has an elementary proof. Each coordinate of the identity map is harmonic since $S = -S$, and so is each coordinate of β_k. Its difference along a basis vector is harmonic by translation invariance and takes only finitely many values by the displayed identity. At its maximum every neighbor has the same value. Connectedness makes the difference constant, which proves linearity. This is also a case of [21]. There are finitely many possible maps because the basis is sent into S. Substituting additivity into the identity for $\beta_k(d + s)$ gives $\beta_{\alpha_d(k)} = \beta_k$.

The kernel H of this right action is open, normal and D-invariant. On H the maps α_d are automorphisms, and $\alpha_d(hk) = \alpha_d(h)\alpha_d(k)$ for $h \in H$. Since $K^0 = H^0$, this identity gives descent to P. The finite index of H makes Q infinite and open in P.

Restriction to H gives the ordinary crossed product $C(H) \rtimes D$ as a quantum subgroup of G, hence it is compact matrix. Write finitely many algebra generators of its coefficient algebra as sums $\sum_j f_j s_j$. All the functions so obtained lie in the coefficient space of a finite-dimensional unitary representation ρ of H. Take the algebra generated by the D-translates of those coefficients and their adjoints. Crossing it with D contains the chosen generators, and taking the part of degree zero gives all of $\mathcal{R}_H$. Hence $h \mapsto (\rho(\alpha_d(h)))_{d \in D}$ embeds H as a closed subgroup of a compact Lie group shift.

By the descending chain theorem for compact Lie group shifts [18, Theorem 3.2], the closed D-invariant subgroups of H satisfy the descending chain condition. It passes to Q by preimages. For U open normal in the profinite group Q set $N_U = \bigcap_{d \in D} \alpha_d(U)$. The family of these invariant subgroups is closed under finite intersections and its total intersection is trivial. A minimal member is therefore trivial. For the corresponding U, reading the quotient $Q \to Q/U$ at all translates embeds Q as a closed group shift on the finite alphabet Q/U. □

We will use the following observation when locating a torus of a quotient. For $E = nD$ and a closed E-invariant abelian subgroup $A \subset Q$ we have

$$\mathcal{R}_G \;\supset\; \mathcal{R}_P \rtimes_{alg} E \;\longrightarrow\; \mathbb{C}[\widehat{A} \rtimes E]$$

Indeed E is characteristic in D and the orbit indicators in $\Delta(s_d)$ descend to P. Thus the middle algebra is the coefficient Hopf $*$-algebra of (E, P). The last arrow restricts functions to A, is onto by Peter–Weyl, and respects the coproduct because β is trivial on A. Whenever $\widehat{A} \rtimes E$ is finitely generated, the middle algebra can be replaced by a compact matrix coefficient algebra. Lift group generators to it and take the finitely many coefficient spaces of irreducibles needed to contain the lifts. Those spaces and their adjoints generate a coefficient Hopf $*$-algebra of compact matrix type still surjecting onto the group algebra. For $P = K$ and $E = D$ the arrows already give a torus in G itself.

We now specialize to $D = \mathbb{Z}$. Write $\mathbb{Z}_p$ for the cyclic group of prime order p.

Lemma 9.2. *Let X be an infinite closed shift-invariant subgroup of $F^{\mathbb{Z}}$, where F is a finite group and T is the bilateral shift. For some prime p, X contains a closed invariant subgroup isomorphic, with its shift, to $(\mathbb{Z}_p)^{\mathbb{Z}}$.*

Proof. Let P_l be the group of words of length l appearing in X. The subgroups $\{a \in F : 1^l a \in P_{l+1}\}$ decrease with l and stabilize, say at $r \geq 1$. The possible letters following a word of length $l \geq r$ form a coset of this stabilized group. They are contained in the coset for the suffix of length r, so the two cosets coincide. Induction and closedness now show that a configuration belongs to X whenever all its blocks of length $r + 1$ occur in X.

As X is infinite, $|P_l|$ is unbounded. For $l > 2r$ with $|P_l| > |F|^{2r}$, restriction to the first and last r letters has nontrivial kernel. Pad a nonidentity word in this kernel with identities on both sides. Every block of length $r+1$ is admissible, so we have a nonidentity element of finite support in X.

Among such elements choose one with the smallest support cardinality. A suitable power y has prime order p and still has minimal support. For $j \neq 0$ the commutator $[y, T^j y]$ is supported in $supp(y) \cap (supp(y) + j)$, strictly smaller than $supp(y)$. It is therefore trivial. Thus all translates of y commute, even at sites where their supports overlap.

Arbitrary products of powers of these translates are defined coordinatewise, as only finitely many factors contribute at each site. They give a continuous shift-equivariant homomorphism $(\mathbb{Z}_p)^{\mathbb{Z}} \to X$ with nontrivial compact image A. On taking Pontryagin duals, $\widehat{A}$ embeds as a nonzero submodule of the regular module $R = \mathbb{F}_p[z, z^{-1}]$, where z acts by translation. Every nonzero ideal of this Laurent polynomial ring is principal and is isomorphic to R as an R-module. Dualizing gives the required isomorphism with its shift. □

Theorem 9.3. *Let G be a compact matrix bicrossed product of $\mathbb{Z}$ and a compact group K with infinitely many connected components. For some prime p there is a compact matrix quantum quotient L of G such that*

$$G \twoheadrightarrow L \supset \widehat{\Lambda}, \qquad \Lambda = \mathbb{Z}_p \wr \mathbb{Z}$$

When K is profinite one can take $L = G$. In all cases $\widehat{G}$ is amenable and has exponential growth.

Proof. Use P, Q from Lemma 9.1. Lemma 9.2 gives a closed shift-invariant subgroup $A \subset Q$ isomorphic with its shift to $(\mathbb{Z}_p)^{\mathbb{Z}}$. Its character group consists of the finitely supported functions from $\mathbb{Z}$ to $\mathbb{Z}_p$, with the dual shift. Taking $E = D$ and restricting functions on P to A gives the toral group

$$\widehat{A} \rtimes D = \left(\bigoplus_{n \in \mathbb{Z}} \mathbb{Z}_p \right) \rtimes \mathbb{Z} = \mathbb{Z}_p \wr \mathbb{Z}$$

One lamp and the shift generate this group, so the compact matrix quotient obtained by lifting these generators suffices. For profinite K we have $K = P$, and the torus already lies in G.

Finally the shift can visit n consecutive sites, at each site choosing whether to switch the lamp on. This gives 2^n distinct group elements of length at most $2n$. Lifts of the two generators

have bounded length in $\mathcal{R}_G$, so the same exponential lower bound holds there after a linear change of radius. Amenability follows from that of $\mathbb{Z}$ by [14, Corollary 3.7]. □

For the full shift $K = (\mathbb{Z}_p)^{\mathbb{Z}}$ with trivial reverse action, G itself is the dual of $\mathbb{Z}_p \wr \mathbb{Z}$. Theorem 9.3 shows that every compact matrix bicrossed product in the stated class contains this example at the level of tori of quotients.

10 Lattice actions, growth and characters

In two dimensions a group shift can be infinite while its only finitely supported configuration is the identity. For instance, Ledrappier's three-dot system [20] consists of the arrays over $\mathbb{F}_2$ satisfying

$$x_{i,j+1} = x_{i,j} + x_{i+1,j}$$

For a finitely supported configuration, take a zero row below its support. The recurrence makes every row above it zero as well. Yet projection onto one horizontal row gives the full binary shift. Indeed, a predecessor of any two-sided row a is obtained by fixing b_0 and solving $b_i + b_{i+1} = a_i$ in both directions. We first explain what replaces the finite-support argument for an arbitrary finite group alphabet.

Lemma 10.1. *Let $D = \mathbb{Z}^r$ with $r \geq 1$, let F be a finite group and let X be an infinite closed shift-invariant subgroup of F^D. Some sublattice nD, $n \geq 1$, preserves an infinite closed abelian subgroup of X.*

Proof. We induct on $|F|$. By passing to the image at one coordinate we can suppose that all coordinate maps are onto F. If F is abelian, take X itself. If $1 < N < F$ is normal, put $Y = X \cap N^D$. When Y is infinite apply induction there. When Y is finite the image of X in $(F/N)^D$ is infinite. By induction that image contains an infinite abelian subgroup invariant under some nD and closed in the image. Let B be its preimage in X. All commutators in B lie in Y. Choose an open normal subgroup V of the profinite group B with $V \cap Y = 1$. Then $[V, B] \subset V \cap Y = 1$, so $Z(B)$ is open in B. This center is infinite, closed, abelian and invariant under nD.

It remains to treat a nonabelian simple F. Put two sites in the same class if their coordinate maps on X have the same kernel. Within one class the coordinates determine one another by automorphisms of F. On distinct classes the values are independent. Indeed, on finitely many sites Scott's lemma [23, p. 328] writes the subdirect product as a product of full diagonal subgroups. The diagonal blocks are exactly the sites with equal kernels. Compactness then permits arbitrary choices on one representative of each class. In particular a configuration can be set equal to the identity off any union of classes while keeping its values on that union.

Let $L \leq D$ stabilize the class of 0. The classes are indexed by D/L, an infinite set since X is infinite. Configurations supported on the class of 0 form a copy of F. The action of L on this copy has finite image in $Aut(F)$. Denote its kernel by L_0. Choose $n \geq 1$ annihilating the torsion of the finitely generated abelian group D/L_0. Then $n(D/L_0)$ is torsion-free, and L/L_0 is finite. Hence $nD \cap L \subset L_0$.

Take a nonidentity element in this supported copy of F. Two of its nD-translates either have disjoint supports or are equal, since a translation in nD stabilizing the class acts trivially there. There are infinitely many supports because the image of nD has finite index in D/L. The closure of the group generated by these translates is infinite, abelian and nD-invariant. □

Lemma 10.2. *Let E be a free abelian group of finite positive rank. Suppose M is an infinite discrete abelian group of bounded exponent, finitely generated as a $\mathbb{Z}[E]$-module. Then $M \rtimes E$ contains $\mathbb{Z}_p \wr \mathbb{Z}$ for some prime p.*

Proof. If each basis element of E had finite orbits on each of a finite set of module generators, their full E-orbits would be finite by commutativity. These finitely many elements would generate M as a group, making it finite. Thus some basis element t has an infinite orbit. Among elements with infinite t-orbit take a of least order. If its order is composite, take a prime p dividing it. The orbit of pa is finite by minimality, so $b = (t^k - 1)a$ is killed by p for some $k > 0$. The orbit of b is also finite by minimality. Its translates generate a finite t-invariant subgroup, modulo which t^k fixes a. This would give a a finite orbit. Hence a has prime order p.

A nonzero Laurent polynomial relation over $\mathbb{F}_p$ between the translates of a would make their span a quotient of $\mathbb{F}_p[z, z^{-1}]/(f)$ with $f \neq 0$, a finite group. Thus the translates are independent and $\langle a, t \rangle = \mathbb{Z}_p \wr \mathbb{Z}$. □

Theorem 10.3. *Let G be a compact matrix bicrossed product of $D = \mathbb{Z}^r$, $r \geq 1$, and a compact group K with infinitely many connected components. There exist a compact matrix quantum quotient L of G, a finitely generated metabelian group Λ and a prime p such that*

$$G \twoheadrightarrow L \supset \widehat{\Lambda}, \qquad \mathbb{Z}_p \wr \mathbb{Z} \ \subset \ \Lambda$$

In particular $\widehat{G}$ is amenable and the tori of its untwisted quotients detect its exponential growth.

Proof. Let $P = K/K^0$ and take $Q \subset P$ from Lemma 9.1. Identify Q with a group shift in F^D for a finite group F. Lemma 10.1 gives a closed infinite abelian $A \subset Q$ invariant under $E = nD$. Put $M = \widehat{A}$ and $\Lambda = M \rtimes E$. Since E is characteristic in D, restriction to A gives the coefficient maps of Section 9,

$$\mathcal{R}_G \ \supset \ \mathcal{R}_P \rtimes_{alg} E \ \longrightarrow \ \mathbb{C}[\Lambda]$$

The character group $M = \widehat{A}$ is infinite of exponent dividing the exponent of F. Take one site in each coset of E in D. At each of these sites the image of A is finite abelian. The characters of these finitely many images and all their E-translates separate the points of A, so they generate M by Pontryagin duality. Thus M is finitely generated over $\mathbb{Z}[E]$, and $\Lambda = M \rtimes E$ is finitely generated as a group. Lift a finite set of group generators to the middle algebra. The finitely many coefficient spaces containing the lifts generate the required compact matrix quotient L on adjoining their adjoints.

Lemma 10.2 gives the indicated subgroup of Λ. Words in its lamp and shift already give exponential growth in Λ and, by lifting generators, in $\mathcal{R}_G$. Coamenability follows from amenability of D by [14, Corollary 3.7]. □

For an ordinary crossed product there is also a shorter route through a single lattice direction. It gives the lamplighter group itself as the discrete group tested by the quotient.

Corollary 10.4. *Under the assumptions of Theorem 10.3, suppose D acts on K by automorphisms and the reverse action is trivial. There is a compact matrix quantum quotient of G containing the dual of $\mathbb{Z}_p \wr \mathbb{Z}$ for some prime p.*

Proof. Here $Q = P = K/K^0$. Embed P in F^D by Lemma 9.1 and let Y_i be its image on the coordinate line $\mathbb{Z}e_i$. At least one Y_i is infinite. Otherwise the shift on each finite Y_i has a common period m_i for its points. Translating the line shows that every configuration in P is periodic in each direction with period m_i, leaving only finitely many choices in all.

For this infinite Y_i, restriction to the line is an equivariant group surjection $P \to Y_i$ for $\mathbb{Z}e_i$. Pullback and the unitary s_{e_i} give the Hopf $*$-subalgebra $\mathcal{R}_{Y_i} \rtimes_{alg} \mathbb{Z}e_i \subset \mathcal{R}_G$. Its compact quantum group is compact matrix, since the functions at site zero and the shift generate the algebra. Lemma 9.2 supplies an invariant copy of $(\mathbb{Z}_p)^{\mathbb{Z}}$ in Y_i. Restricting to it gives the required torus. □

Theorem 10.3 establishes Conjectures 7.2 and 7.3 for the stated lattice bicrossed products. Amenability is automatic by [14, Corollary 3.7], and the exponential-growth group already occurs with the trivial twist. The inclusion $\mathbb{Z}_p \wr \mathbb{Z} \subset \Lambda$ here is an inclusion of discrete groups. Corollary 10.4 obtains the lamplighter dual itself as a torus in the ordinary crossed-product case.

We turn to characters. Pullback from the compact factor puts $C(K)_{central}$ inside $C(G)_{central}$. If all tori kill a nonzero function in this subalgebra, then joint character detection fails. For lattice bicrossed products the converse also holds, even when the reverse action is nontrivial.

Theorem 10.5. *Let G be a compact matrix bicrossed product of $D = \mathbb{Z}^r$, $r \geq 1$, and a compact group K with infinitely many connected components. The tori of G are jointly faithful on $C(G)_{central}$ if and only if they are jointly faithful on $C(K)_{central}$.*

Proof. Suppose the tori are jointly faithful on $C(K)_{central}$, and fix a nonzero $x \in C(G)_{central}$. We use right coefficients, writing an element of the algebraic span of the characters as

$$y = \sum_{n \in D} s_n f_n, \qquad f_n \in \mathcal{R}_K$$

For any character χ, the sum of the diagonal coefficients gives $\Delta(\chi) = \Delta^{op}(\chi)$, so the same equality holds for y. Comparing the right coefficients of bidegree (n, s) in the coproduct formula of Section 8, we obtain, for $k, l \in K$,

$$1_{\{\beta_k(n)=s\}} f_n(kl) = 1_{\{\beta_l(s)=n\}} f_s(lk)$$

Take $l = 1$ and $s = \beta_k(n)$. Since $\beta_1(s) = s$, a nonzero value $f_n(k)$ forces $\beta_k(n) = n$. Taking $n = s = 0$ gives $f_0(kl) = f_0(lk)$, so f_0 is a class function. Both facts can also be read from the character formula in [15, Theorem 3.1].

Let $E : C(K) \rtimes D \to C(K)$ take the coefficient at 0. Since D is amenable, the full and reduced crossed products agree and E is faithful. On the algebraic span just considered, E takes values in the class functions. By continuity the same holds on $C(G)_{central}$, which is a C^*-algebra. In particular $E(x^*x)$ is a nonzero positive class function. Choose a torus restriction $\pi : C(G) \twoheadrightarrow C^*(\Gamma)$ on which it survives. The image of $C(K)$ is $C(A)$ for a closed abelian subgroup $A \subset K$, since restriction is a Hopf map into a cocommutative algebra. Covariance makes A invariant under each α_n. We may therefore choose $a \in A$ with $E(x^*x)(a) > 0$.

By Lemma 9.1, $\sigma = \beta_a$ is a lattice automorphism of finite order m, the image of β is finite, and $\beta_{\alpha_n(b)} = \beta_b$. Put

$$A_0 = \{b \in A : \beta_b \in \langle \sigma \rangle\}, \qquad N = 1 + \sigma + \cdots + \sigma^{m-1} \ \in End(D)$$

Since the image of β is finite, A_0 is open in A. It contains a and is invariant under the lattice action. We have $N\beta_b = N$ for every $b \in A_0$, and $Nn = mn$ on the subgroup fixed by σ. In particular N is injective on the lattice degrees that can have a nonzero coefficient at a.

We first restrict the chosen torus over A_0. The projection $p = 1_{A_0}$ commutes with $C(A)$ and with every $\pi(s_n)$ by invariance. These generate $C^*(\Gamma)$, so p is central. The Hopf inclusion $C(A) = C^*(\widehat{A}) \subset C^*(\Gamma)$ identifies $\widehat{A}$ with a subgroup of Γ. Writing $F = A_0^{\perp}$ for the finite subgroup of characters of A trivial on A_0, we have

$$p = \frac{1}{|F|} \sum_{h \in F} h$$

Centrality and linear independence of group elements give $gFg^{-1} = F$ for every $g \in \Gamma$. The corner $pC^*(\Gamma)$ is therefore the full group algebra of Γ/F. Indeed the unitary representations on which p acts as the identity are precisely those trivial on F. Composing π with the group

quotient gives a torus restriction π_0 with $\pi_0(C(K)) = C(A_0)$, since compression on $C(A)$ is restriction to A_0.

Write $v_n = \pi_0(s_n)$ and let u_j denote the canonical unitaries of $C^*(D)$. The following assignments give a map ρ into $C^*((\Gamma/F) \times D)$ by covariance.

$$f \longmapsto \pi_0(f) \otimes 1, \qquad s_n \longmapsto v_n \otimes u_{Nn}$$

It intertwines coproducts on $C(K)$. In the coproduct of s_n, the summand with second degree s has first-leg indicator $1_{\{b:\beta_b(n)=s\}}$. After applying π_0 this indicator can survive only if $s = \beta_b(n)$ for some $b \in A_0$. We then have $Ns = Nn$, so the two added group factors in this summand are both u_{Nn}. This proves compatibility with the coproduct on s_n as well. The image of the polynomial Hopf algebra is a Hopf $*$-subalgebra of $\mathbb{C}[(\Gamma/F) \times D]$, hence is the group algebra of a subgroup. Passing to its full completion gives a torus of G through which ρ factors.

Finally extend evaluation at a on $C(A_0)$ to a state of $C^*(\Gamma/F)$. In its cyclic representation let ξ be the unit vector inducing this state. For $f \in C(K)$, $\pi_0(f)\xi = f(a)\xi$, by applying the state to $|f - f(a)|^2$. Use the regular representation on $\ell^2(D)$ for the other group factor. Although v_n need not be group-like, it is unitary. With the right coefficients of y we obtain

$$\rho(y)(\xi \otimes \delta_0) = \sum_n f_n(a)\, v_n\xi \otimes \delta_{Nn}$$

The nonzero summands are orthogonal, since their degrees are fixed by σ and N is injective there. We obtain

$$\|\rho(y)(\xi \otimes \delta_0)\|^2 = \sum_n |f_n(a)|^2 = E(y^*y)(a)$$

The last equality follows by extracting degree zero in the product using the right coefficients. Approximate x in norm by such y. Both sides are continuous, so the squared norm at x equals $E(x^*x)(a) > 0$. The torus just constructed detects x. The reverse implication follows from $C(K)_{central} \subset C(G)_{central}$. □

11 Character detection and the quantum center

The compact factor in Theorem 10.5 belongs to the given bicrossed-product presentation. To obtain a character test defined for G alone, consider its quantum center $Z(G)$. We recall the description by the chain group [12, 17]. Let Γ be the group with one generator g_u for each $u \in Irr(G)$ and relations

$$g_w = g_u g_v \qquad \text{whenever } w \in u \otimes v$$

In particular $g_1 = e$ and $g_{\bar{u}} = g_u^{-1}$. The group Γ grades the polynomial Hopf algebra by assigning degree g_u to every matrix coefficient u_{ij}. Restriction to the quantum center is given on matrix coefficients by $u_{ij} \mapsto \delta_{ij} g_u$, so $Z(G) = \widehat{\Gamma}$ [17, Remark 2.7]. An irreducible representation descends to $G/Z(G)$ precisely when its degree is e. Taking the norm closure of the pulled-back characters in $C(G)$ gives

$$C(G/Z(G))_{central} = \overline{span}\{\chi_u : u \in Irr(G),\ g_u = e\}$$

Throughout this section restrictions on this subalgebra are taken to the tori of G.

Write I for the common kernel of the torus restrictions on $C(G)_{central}$. Define P_e on the span of the characters by keeping only the terms of degree e. To see its extension to the norm closure, let U_g be the canonical unitary in $C^*(\Gamma)$ and define the grading map by

$$\delta(u_{ij}) = u_{ij} \otimes U_{g_u}, \qquad P_e = (id \otimes \tau)\delta$$

Here $\tau(U_g)$ is one for $g = e$ and zero otherwise. The map δ is the coproduct followed by restriction of the second factor to the center, with values in the spatial tensor product $C(G) \otimes C^*(\Gamma)$. Slicing by the trivial representation $U_g \mapsto 1$ gives the identity, so δ is injective. The slice defining P_e is positive and contractive. On $C(G)_{central}$ it is a conditional expectation onto $C(G/Z(G))_{central}$.

Proposition 11.1. *For any compact matrix quantum group G of Kac type, $P_e(I)$ is contained in I. If the chain group is amenable, every nonzero $x \in I$ gives a nonzero element $P_e(x^*x)$ in $I \cap C(G/Z(G))_{central}$. The same conclusion holds for any nonzero finite linear combination of irreducible characters in I, for an arbitrary chain group.*

Proof. Fix a torus restriction $\pi : C(G) \twoheadrightarrow C^*(\Lambda)$. For a homogeneous of degree g, the assignment

$$a \longmapsto \pi(a) \otimes U_g \ \in \ \mathbb{C}[\Lambda \times \Gamma]$$

on the polynomial algebra preserves multiplication and the involution. It also preserves coproducts, since both legs of the coproduct of a matrix coefficient of an irreducible of degree g have degree g. The image is a Hopf $*$-subalgebra of a group algebra, hence the group algebra of a subgroup. The resulting map to its full completion is a torus restriction of G. Its composition with the natural map to $C^*(\Lambda) \otimes C^*(\Gamma)$ agrees with $(\pi \otimes id)\delta$ on the polynomial algebra and therefore on $C(G)$. For $x \in I$ the enlarged torus kills x. Slicing gives $\pi(P_e x) = 0$. This holds for every π.

When Γ is amenable, the regular trace τ is faithful on $C^*(\Gamma)$, so $id \otimes \tau$ is faithful on positive elements of the spatial tensor product. Injectivity of δ now gives $P_e(x^*x) \neq 0$ for $x \neq 0$. For a finite sum $x = \sum_g x_g$ of homogeneous characters or their linear combinations we have directly

$$P_e(x^*x) = \sum_g x_g^* x_g$$

This is nonzero as soon as some x_g is nonzero. In both cases x^*x belongs to I, and the first assertion finishes the proof. □

Corollary 11.2. *Let $G \twoheadrightarrow K$ be a quantum quotient with K a classical compact group. Whenever the tori of G kill a nonzero $f \in C(K)_{central}$, they also kill a nonzero element of $C(G/Z(G))_{central}$. In particular New conjecture 10 holds for both crossed-product classes in Theorem 7.*

Proof. Regard representations of K as representations of G by pullback. For $u, v \in Irr(K)$ the tensor products $u \otimes v$ and $v \otimes u$ are isomorphic. Any common irreducible constituent therefore gives $g_u g_v = g_v g_u$ in the chain group of G. The subgroup A generated by the degrees coming from K is abelian. Consequently

$$\delta(C(K)_{central}) \ \subset \ C(K)_{central} \otimes C^*(A)$$

Here $C^*(A)$ embeds in $C^*(\Gamma)$, and the restriction of τ to it is the faithful regular trace of the abelian group A. Thus $P_e(f^*f)$ is nonzero. Proposition 11.1 shows that it belongs to $I \cap C(G/Z(G))_{central}$.

For an ordinary crossed product with amenable discrete factor, Theorem 4.1 and its proof give a nonzero invisible class function on K whenever joint detection on $C(G)_{central}$ fails. For a lattice bicrossed product with infinitely many compact-factor components, Theorem 10.5 gives precisely that implication. Applying the first assertion proves the character conjecture in both cases. □

A counterexample to New conjecture 10 would therefore have a nonamenable chain group. In such an example every nonzero character polynomial would survive on some torus, but a nonzero norm limit of character polynomials would vanish on all of them. When $Z(G)$ is trivial, $G/Z(G) = G$ and the conjecture holds immediately.

12 Examples and partition calculations

An old torus is still available for the two analytic tests. For characters the verification concerns the norm closure in $C(G)_{central}$.

Proposition 12.1. *Within the compact matrix Kac setting, each verification of Old conjecture 2 or 3 gives a verification of New conjecture 11 or 12, respectively. Joint detection of $C(G)_{central}$ implies New conjecture 10. This last conjecture also holds whenever $\widehat{G}$ is amenable or G has an irreducible fundamental representation.*

Proof. Every old torus occurs in the new family by taking the identity quotient and the trivial twist. Thus a witness to failure of either analytic property still works, while Proposition 7.1 proves the forward implications. Joint detection of $C(G)_{central}$ is the conclusion of New conjecture 10, with or without connectedness.

For the two other character assertions, let Γ be the chain group. Its dual is the central torus $Z(G) \subset G$, so amenability of $\widehat{G}$ gives amenability of Γ by Proposition 7.1. If a fundamental representation u is irreducible, every irreducible occurs in a tensor word in $u, \bar{u}$. Its degree is thus a power of g_u, and Γ is cyclic. In both cases apply Proposition 11.1. □

In particular, all three new conjectures hold for the classical compact matrix groups and group duals of [4, Proposition 4.1]. On a classical group each element belongs to the closure of the cyclic subgroup it generates, so restrictions even separate all continuous functions. A group dual is its own torus.

All three also hold for every compact quantum subgroup of O_N^*. Indeed, polynomial growth of the dual of O_N^* [7, Section 9] passes to the dual of each such subgroup [13, Proposition 4.1] and implies amenability [13, Theorem 4.6]. Propositions 7.1 and 12.1 now apply. This covers the half-classical examples of [4, Theorem 5.5], including $[H]$ from the crossed-product construction of [9], and works for the new tests without a connectedness assumption.

Orthogonal easy quantum groups. For New conjecture 10, even the partitions describing the higher tensor powers are unnecessary in the following case.

Theorem 12.2. *If $S_N \subset G \subset O_N^+$ in the permutation representation, then New conjecture 10 holds. In particular it holds for every orthogonal easy quantum group.*

Proof. For $N \geq 2$, the commutant of S_N on $\mathbb{C}^N$ is spanned by the identity and the projection onto the constant vectors. Indeed a commuting matrix has one common diagonal entry and one common off-diagonal entry. The commutant of the fundamental u of G is a unital $*$-subalgebra of this two-dimensional algebra. If it consists of scalars, u is irreducible and Proposition 12.1 applies.

Otherwise u has two irreducible summands on the constants and their orthogonal complement. Both summands are self-conjugate, since their projections are real and u is orthogonal. Their degrees generate the chain group and square to the identity. This group is therefore a quotient of the amenable group $\mathbb{Z}_2 * \mathbb{Z}_2$. Apply Proposition 11.1. The case $N = 1$ is immediate. For an orthogonal easy quantum group the inclusion of S_N follows because every partition map intertwines the permutation action [5]. □

We can now account for the four free families of [4, Theorem 4.4]. The preceding theorem gives New conjecture 10 for O_N^+, S_N^+ and H_N^+, and the fundamental representation of U_N^+ is irreducible already on U_N. For $N \geq 3$ the standard diagonal groups of O_N^+, H_N^+ are $\mathbb{Z}_2^{*N}$, while that of U_N^+ is F_N. For S_N^+ with $N \geq 5$, a pair and a triple of points give $\mathbb{Z}_2 * \mathbb{Z}_3$. These groups witness nonamenability and exponential growth in the indicated ranks, as does F_2 for U_2^+. All remaining ranks have polynomial growth and amenable dual [4, Theorem 4.4]. Thus the two analytic verifications transfer through Proposition 12.1. For the character verification above, no passage from polynomial injectivity to norm injectivity is needed. The same transfer applies wherever the diagonal groups of the other orthogonal easy families in [4, Section 6] are known to be nonamenable. Theorem 12.2 supplies the character assertion for these families as well.

Torus relations from partitions. For an orthogonal easy quantum group and a real orthogonal basis change Q, take a partition map T_p between tensor powers of lengths k, l. Set

$$T_p^Q = Q^{\otimes l} T_p (Q^t)^{\otimes k}$$

On the diagonal torus the intertwiner equation reads

$$(T_p^Q)_{j,i}\, g_{i_1} \cdots g_{i_k} = g_{j_1} \cdots g_{j_l}\, (T_p^Q)_{j,i}$$

Whenever the scalar coefficient is nonzero, we can cancel it and identify the two words. It suffices to impose these relations on any family generating the tensor category [4, Section 6]. For unitary categories one also uses the conjugate basis on conjugate tensor factors and inverse letters there. For New conjectures 11 and 12, a compact matrix quantum quotient selects a full tensor subcategory closed under subobjects, conjugates and finite sums. A dimension-preserving unitary fiber functor realizes the allowed twist [8]. One may then use the same coefficient rule on the resulting concrete intertwiners. A single presentation exhibiting a group without the required property suffices.

Here is a small example where the basis matters. Write $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ and take

$$Q = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad a^2 + b^2 = 1$$

Proposition 12.3. *For H_2^+ the three possibilities for Γ_Q are*

$$\Gamma_Q \simeq \begin{cases} D_\infty & ab = 0 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & |a| = |b| \\ \mathbb{Z}_2 & ab \neq 0,\ |a| \neq |b| \end{cases}$$

The same list covers all real orthogonal bases up to row signs.

Proof. The pair partition makes the two generators g, h involutions. The four-block fixes $e_1^{\otimes 4} + e_2^{\otimes 4}$. These two tensors present the orthogonal cubic relations [5, 6]. Bending two legs of the four-block gives the projection onto the span of $e_i \otimes e_i$. Intertwining this projection with $u^{\otimes 2}$ says exactly that distinct entries in any row or column have product zero.

If $ab = 0$ the only supported four-letter words are g^4, h^4. Otherwise the coefficient of $ghgh$ is $2a^2b^2$, so the two involutions commute. A word with three copies of one letter and one of the other has coefficient $\pm ab(a^2 - b^2)$. When this is zero the remaining relations have even multiplicity of each letter, and commuting involutions satisfy them all. When it is nonzero they force $g = h$, and one involution satisfies every relation. Row signs do not change the support of either tensor. □

The original basis gives D_∞. The normalized Fourier matrix gives

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \Gamma_Q \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

In the Fourier basis we can check commutativity directly. Put $w = gh$. Since g and h are involutions, we have

$$u_{11} = \frac{g + h}{2}, \qquad u_{11}^2 = \frac{2 + w + w^{-1}}{4}$$

The cubic relations make u_{11}^2 a projection. Squaring the second expression and equating it to itself gives $w^2 + w^{-2} = 2$. Linear independence in the group algebra forces $w^2 = 1$, so g and h commute. This corrects the Fourier-basis calculation in the arXiv version of [4, proof of Theorem 4.4].

Computing the two tensor supports over the integers, `tori_partitions.py` finds 4, 4 and 632 primitive signed pairs in the respective cases for $\max(|a|, |b|) \leq 16$ before normalization.

Proposition 12.4. *New conjectures 10–12 hold for the free complex reflection groups H_N^{s+}, for $N \geq 2$ and $s \geq 3$. Their standard diagonal group is the free product of N copies of $\mathbb{Z}_s$, and their chain group is $\mathbb{Z}_s$.*

Proof. In the defining presentation of [6], the entries of u are normal partial isometries satisfying $u_{ij}^s = u_{ij}u_{ij}^*$. Setting all off-diagonal entries to zero leaves arbitrary unitaries g_i with $g_i^s = 1$. Conversely any such family gives a diagonal solution of all the relations. In the partition description a block of s equal colors imposes $g_i^s = 1$ in this basis. The resulting group contains $\mathbb{Z}_s * \mathbb{Z}_s$. The kernel of the map from their free product onto $\mathbb{Z}_s^2$ acts freely on the free product tree. The quotient graph is the complete bipartite graph on $s + s$ vertices. Counting its edges and vertices gives the free rank of the kernel

$$s^2 - 2s + 1 = (s-1)^2 \geq 2$$

This one torus therefore verifies both analytic tests.

The classical diagonal phase matrices and the permutation matrices act irreducibly together, so u is irreducible. Its degree generates the chain group. The tensor $\sum_i e_i^{\otimes s}$ is fixed by the block of size s, so $g_u^s = 1$. Scalar matrices of order s give a central subgroup on which u has the defining degree of order s. This proves the claimed equality, and Proposition 12.1 proves the character assertion. □

The script counts words by syllable length, using the generators and their inverses. For $\mathbb{Z}_3 * \mathbb{Z}_3$ it finds the following ball volumes at radii $0, \ldots, 6$

$$1,\ 5,\ 13,\ 29,\ 61,\ 125,\ 253$$

At the same radii in D_∞ the volumes are $1, 3, 5, 7, 9, 11, 13$.

A test with nonamenable chain group. The following dilation argument also gives the full norm in the single-factor calculation of [4, Theorem 4.4].

Theorem 12.5. *Let $C(G)$ be the unital full free product of the coefficient algebras of $U_{N_1}^+, \ldots, U_{N_r}^+$ with their usual coproducts, where $r \geq 1$ and $N_i \geq 2$. The diagonal torus of the block fundamental representation is isometric on $C(G)_{central}$. All three new conjectures hold, and the chain group is F_r.*

Proof. Write χ_i for the fundamental character of the i-th factor. The fusion rules for a single factor [1, Theorem 6.20] and the alternating-word description of the free product irreducibles [25, Theorem 3.10] give $C(G)_{central} = C^*(\chi_1, \ldots, \chi_r)$. Let g_{ij} be the jth diagonal generator of the ith factor. The restriction of its character is

$$\pi(\chi_i) = g_{i1} + \cdots + g_{iN_i}$$

All these generators are free, both within each factor and across different factors.

Fix a representation of $C(G)$ on a Hilbert space E. The operators $T_i = \chi_i/N_i$ in this representation are contractions. For each i form the unitary on $E \oplus E$

$$V_i = \begin{pmatrix} T_i & \sqrt{1 - T_i T_i^*} \\ \sqrt{1 - T_i^* T_i} & -T_i^* \end{pmatrix}$$

Summing the conjugates over the N_i-th roots of unity cancels the off-diagonal entries, giving

$$D_z = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}, \qquad \sum_{z^{N_i}=1} D_z V_i D_z^* = N_i \begin{pmatrix} T_i & 0 \\ 0 & -T_i^* \end{pmatrix}$$

Assign these unitaries $D_z V_i D_z^*$ to the generators of the i-th block. Freeness gives a representation of the full toral group algebra on $E \oplus E$. The first summand reduces all the character images and recovers the original χ_i simultaneously. Taking the supremum over representations proves equality of norms for every $*$-polynomial in the characters and hence on $C(G)_{central}$.

The toral group $F_{\sum_i N_i}$ witnesses nonamenability and exponential growth. The r fundamental degrees generate the chain group and map to the free generators z_i of F_r under $u_i \mapsto z_i I_{N_i}$. This quotient is central, as one checks on the coefficients of u_i. Thus the chain group is F_r. □

Twisting the free product. The preceding isometry uses a particular torus. To follow that torus under a twist we keep track of the bases of the fundamental blocks.

Corollary 12.6. *For the free product G of Theorem 12.5, every unitary dual 2-cocycle is a coboundary. In every twist, the diagonal torus of the corresponding fundamental blocks has group $F_{N_1+\cdots+N_r}$, and restriction to this torus is isometric on the full character algebra.*

Proof. The cocycle defines a dimension-preserving unitary fiber functor on $Rep(G)$. On $U_{N_i}^+$ such a functor is classified by an invertible matrix M of size N_i, up to left and right unitaries [8, Theorem 6.2 and Remark 6.5]. For $P = M^*M$ the two trace conditions give

$$Tr(P) = Tr(P^{-1}) = N_i, \qquad Tr(P + P^{-1} - 2I) = 0$$

Each eigenvalue $\lambda > 0$ of P contributes $\lambda + \lambda^{-1} - 2 = (\lambda - 1)^2/\lambda$ to the last trace. All contributions are nonnegative, so every eigenvalue equals one and $P = I$. It follows that M is equivalent to the identity and the restricted functor is unitarily monoidally isomorphic to the forgetful functor.

The representation category of the free product has the universal property of the free product of the factor categories [24, Proposition 5.1]. Hence the factor isomorphisms extend to a unitary monoidal isomorphism on all of $Rep(G)$. On a tensor word its value is the tensor product of the factor isomorphisms, composed with the tensor structure maps of the functors. By [8, Proposition 4.5] this says precisely that the cocycle is a coboundary.

The induced Hopf $*$-isomorphism matches each fundamental block up to unitary change of basis. Any such changes can be absorbed independently by automorphisms of the original free product, since $u_i \mapsto W_i u_i W_i^*$ preserves biunitarity and has an inverse. The isomorphism then identifies the corresponding diagonal quotients and preserves character norms by Theorem 12.5. □

For an explicit twist take two U_2^+ factors, with fundamental matrices u, v and toral letters g_1, g_2, h_1, h_2, respectively. Define $p : F_4 \to A = \mathbb{Z}_2^2$ on the generators by

$$p(g_1) = (1,0), \qquad p(h_1) = (0,1), \qquad p(g_2) = p(h_2) = 0$$

On A take the bicharacter $\sigma(x,y) = (-1)^{x_1 y_2}$. Its values on the two basis elements depend on their order. Since every coboundary on A is symmetric, σ represents a nontrivial class. Composing with p gives a cocycle on F_4. The diagonal restriction $\mathcal{R}_G \to \mathbb{C}[F_4]$ then gives the cocycle on the dual of G.

For coefficients a, b of respective left and right A-weights (x, x') and (y, y'), the twisted product is

$$a \star b = \sigma(x,y)\overline{\sigma(x',y')}\, ab$$

For example, the order of the two coefficients affects the sign

$$u_{12} \star v_{11} = -u_{12} v_{11}, \qquad v_{11} \star u_{12} = v_{11} u_{12}$$

For a word in the free generators and their inverses put

$$b(\ell_1 \cdots \ell_k) = (-1)^{\sum_{i<j} p(\ell_i)_1 p(\ell_j)_2}$$

An inverse pair contributes zero internally, since a letter has weight on at most one coordinate. Its terms with every other letter cancel. Thus b is defined on F_4, and splitting the sum between words gives

$$b(st) = b(s)b(t)\sigma(p(s), p(t))$$

For a coefficient of left and right F_4-degrees (s, t), multiplication by $b(s)\overline{b(t)}$ untwists the product. This follows by applying the preceding identity to the left and right degrees separately. It fixes the entries of u and v, since b is one on single letters. In $\mathbb{C}[F_4]$, a group element has equal left and right degrees. Hence diagonal restriction kills coefficients with $s \neq t$. For $s = t$ the untwisting sign is one. Thus in this example the original restriction map itself survives on the common polynomial coalgebra, with

$$\chi_u \longmapsto g_1 + g_2, \qquad \chi_v \longmapsto h_1 + h_2$$

The script `tori_twists.py` checks this formula on all 69,905 formal coefficient words through length four, including adjoints and the empty word. All 341 character $*$-words through that length give identical toral expansions. Corollary 12.6 gives the isometry on the norm closure.

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