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SPHERICAL SPACE FORMS IN K3: AN INSTANTON PARITY ARGUMENT

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Abstract. Let R be a connected subdiagram of the Dynkin diagram E_8, and let Y_R be the spherical three-manifold which bounds the corresponding plumbing of (-2)-spheres. We show that if a negative-definite four-manifold N with $H_1(N;\mathbb{Z}) = 0$ and oriented boundary Y_R is embedded in the K3 surface, then the intersection form of N is the negative-definite root lattice $-R$. As a consequence, the traces of -2 and -4 surgery on the left-handed trefoil do not embed in K3. Heegaard Floer correction terms and the theory of lattices reduce the theorem to the exclusion of one smaller lattice for each of $R = D_5$, D_6, D_7 and E_7. In these cases, stretching the K3 surface along Y_R exhibits a zero-dimensional moduli space of instantons as the zero set of a section of a real two-plane bundle over a compact surface. The quaternionic nature of the linearized anti-self-duality equations on K3 shows that the Euler number of this bundle is even, while the corresponding Donaldson invariant is odd.

Foreword

With the exception of this foreword, the present paper is entirely AI-generated, in response to prompts provided by the human authors. The mathematics was produced by GPT-6 Astra, and the paper was redrafted by Claude Fable 5.2. The main result is Theorem 1.1. A particular case states that the 4-manifold N obtained by adding a 2-handle to the 4-ball along a left-handed trefoil with framing -2 cannot be smoothly embedded in a $K3$ surface. (See Corollary 1.2.)

These results arose from prompts motivated by a question which the authors learned of from Anubhav Mukherjee. Let S be the standard 3-sphere in the neck of the connected sum $K3\#K3$. Does the connected sum contain any sphere different from S? Here one can interpret different as up to isotopy; or one can ask whether $\pm[S]$ is the only non-zero class in π_3 that is represented by an embedding. In the latter version, the question is closely related to the question of whether a diffeomorphism of $K3\#K3$ can "mix" the homology of the two summands. The clearest strategy to construct such a mixing diffeomorphism starts with a hypothetical embedded 2-sphere Σ in $K3\#K3$ with $\Sigma \cdot \Sigma = -2$ such that the class $[\Sigma] \in H_2(K3) \oplus H_2(K3)$ has non-zero projection onto both sides. Then reflection in $[\Sigma]$ arises from a diffeomorphism of $K3\#K3$ which does not preserve the direct sum decomposition. (Indeed, if $[\Sigma]$ is the sum of two such *primitive* classes, then it turns out that this would imply that the diffeomorphism group of $K3\#K3$ acts by the full isometry group of the homology latttice.) To try and construct such a 2-sphere, one might start with a cuspidal elliptic fiber on one side, and seek on the other side an embedded disk with relative self-intersection number -2 and boundary a trefoil in the 3-sphere S. Tubing the trefoil in S to the cusp gives the sought-after 2-sphere. But finding the disk in punctured $K3$ is equivalent to embedding the 4-dimensional handlebody N mentioned above, which

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Corollary 1.2 says cannot be done. Of course, only the simplest construction of such a sphere is ruled out by this argument, because the hypothetical sphere Σ might meet S in an arbitrary knot or link instead of the trefoil.

ChatGPT proved the corollary first, for the case of the -2 framed left-handed trefoil. It began by giving an argument conditional on the superconformal simple type conjecture [12, 7]. If N embeds in $K3$, then the manifold obtained by replacing this piece with an E_7 plumbing is a manifold whose Seiberg-Witten invariants can be computed sufficiently to obtain a contradiction to that conjecture. The AI then found an unconditional proof using Donaldson invariants, which is the one presented here.

In outline, here is the proof by contradiction as it applies to a hypothetical embedding of $N = X_{-2}(T_-)$ in $K3$. With a suitable SO(3) bundle on $K3$, one obtains a zero-dimensional ASD moduli space, and the Donaldson invariant is known to be ± 1. On the other hand, stretching the metric along ∂N leads to a gluing description which presents the moduli space as the zero set of a section of an obstruction bundle $L \to M$, where L has real rank 2 and M is a compact, oriented 2-manifold. This describes the Donaldson invariant as the Euler class $\langle e(L), [M]\rangle$. However, $[L] - [TM]$ has an interpretation as the element of $KO(M)$ arising from the index of the ASD deformation complex. The quaternionic structure of the tangent bundle of $K3$ implies that this element is even, and it follows that the Donaldson invariant of $K3$, computed as $\langle e(L), [M]\rangle$, is even, which contradicts the known result.

The justification of the required gluing result in section 5 of the paper is verbose. The AI initially found no suitable models for the required analytic steps in the literature. The human authors pointed to the cited papers of Taubes, which provided both the overall strategy for the gluing theorem and the analytic details. With this background, GPT-6 wrote a gluing argument which was later condensed by Fable.

After completing the proof of the corollary, the generalization given in Theorem 1.1 was suggested by the authors, as something that might follow from similar arguments. As noted in the paper, the instanton parity argument is needed only in the cases D_5, D_6, D_7 and E_7, to rule out the lattices $\langle -4\rangle$, $-(A_1 \oplus A_1)$, $-A_3$ and $-A_1$ respectively.

The above corollary and the parity argument in its proof are both new to the authors. Distributing the preprint in this form, the authors would be glad to hear from others if these results are known.

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1. Introduction

Let R be a Dynkin diagram of type A, D or E. We use the same letter for the corresponding root lattice, which we take to be positive definite. Let P_R be the four-manifold obtained by plumbing disk bundles of Euler number -2 over the two-sphere according to the diagram R, so that the intersection form of P_R is $-R$, and let $Y_R = \partial P_R$ be its oriented boundary. Thus Y_R is the quotient of S^3 by a finite subgroup of $SU(2)$. This paper is concerned with embeddings of the manifolds Y_R in the K3 surface. Our main result is the following.

Theorem 1.1. *Let R be a connected subdiagram of the Dynkin diagram E_8, and let N be a compact, connected, codimension-zero submanifold of the K3 surface whose oriented boundary is Y_R. Suppose that N is negative definite and that $H_1(N;\mathbb{Z}) = 0$. Then the intersection form of N is isomorphic to $-R$.*

The diagrams R to which the theorem applies are A_n for $1 \le n \le 7$, D_n for $4 \le n \le 7$, and E_6, E_7 and E_8. With the hypotheses of the theorem, the group $H_2(N;\mathbb{Z})$ is free and $H_3(N;\mathbb{Z})$ is zero (Proposition 7.1), so N has the same integral homology as the plumbing P_R.

In three cases, the conditions which the theorem imposes on N do not need to be assumed. If R is E_7, D_5 or D_7, and Y_R is embedded in the K3 surface in any manner, then it divides the K3 surface into two pieces, and one of the two is negative definite, has oriented boundary Y_R, and has vanishing first homology. Theorem 1.1 applies to that piece, whose intersection form is therefore $-R$. See Theorems 2.1 and 7.2.

For a knot $K \subset S^3$ and an integer m, let $X_m(K)$ denote the trace of m-surgery on K: the four-manifold obtained by attaching a two-handle to the four-ball along K with framing m.

Corollary 1.2. *Let K be the left-handed trefoil. Then the trace $X_{-2}(K)$ does not admit an orientation-preserving smooth embedding in the K3 surface. The same is true of $X_{-4}(K)$.*

Proof. As oriented manifolds, the boundary of $X_{-2}(K)$ is Y_{E_7} [1, Lemma 1.2], and the boundary of $X_{-4}(K)$ is Y_{D_5} [15]. The traces are simply connected, and their intersection forms are $\langle -2\rangle$ and $\langle -4\rangle$ respectively, rather than $-E_7$ and $-D_5$. □

Outline of the proof. The first part of the proof of Theorem 1.1 uses Heegaard Floer correction terms and the theory of lattices. The manifold N is spin, so the inequality of Ozsváth and Szabó [18] bounds $b_2(N)$ in terms of the correction terms of Y_R, and also determines $b_2(N)$ modulo 8. The classification of even definite lattices of small rank and determinant then shows that the intersection form of N is $-R$ when R is D_4, E_6 or E_8. The same holds for $R = A_n$, by an argument which also uses Donaldson's diagonalization theorem. In the four remaining cases, there is one possibility for the intersection form other than $-R$:

(1)

R	E_7	D_5	D_6	D_7
other possibility for Q_N	$\langle -2\rangle$	$\langle -4\rangle$	$-(A_1 \oplus A_1)$	$-A_3$

To exclude these four possibilities we use instantons. Consider the case $R = E_7$, and suppose that a K3 surface X is decomposed as $N \cup_Y C$, where $Y = Y_{E_7}$ and N has intersection form $\langle -2\rangle$. Let $P \to X$ be an $SO(3)$ bundle with $p_1(P) = -6$ whose second Stiefel–Whitney class is non-zero on the generator of $H_2(N)$. The moduli space of anti-self-dual connections in P has formal dimension zero, and the corresponding Donaldson invariant, which counts the points of this moduli space with signs, is ± 1. We examine the moduli space as the

metric on X is stretched along Y. On the manifold N with a cylindrical end attached, the bundle P carries a reducible anti-self-dual connection A_0 of action $1/8$, asymptotic to a reducible flat connection α on Y. A dimension count shows that, when the neck is long, every anti-self-dual connection in P is close to A_0 on N, and on C is close to a point of a moduli space M of instantons asymptotic to α, where M is a compact two-manifold. The connection A_0, however, belongs to a moduli space of formal dimension -3. Its stabilizer is a circle and, however the equations are perturbed, the cohomology group $H^2_{A_0}$ of its deformation complex is not zero: it is a copy of $\mathbb{C}$ on which the stabilizer acts with weight one. In this situation, gluing theory describes the moduli space on X not as a copy of M but as the zero set of a section of a real two-plane bundle $E \to M$, the bundle of obstruction spaces. The Donaldson invariant is therefore equal, modulo 2, to the Euler number of E.

The proof is completed by showing that this Euler number is even. The gluing construction provides a family of connections on X parametrized by M, and the index of the family of linearized operators $d^*_A \oplus d^+_A$ is the element $[TM] - [E]$ of $KO(M)$. For a hyperkähler metric on X, the operator $d^*_A \oplus d^+_A$ can be identified with a Dirac operator coupled to the real vector bundle $\operatorname{ad} P$, and so commutes with a quaternionic structure. The index of any family of such operators therefore lies in the image of $KSp \to KO$, and in particular its second Stiefel–Whitney class is zero. It follows that $w_2(E) = w_2(TM)$, so the Euler number of E has the same parity as the Euler number of the surface M, which is even.

The cases $R = D_5$, D_6 and D_7 are dealt with in the same way, and differ only in the numerical data.

The method of gluing in the presence of an obstruction goes back to Taubes [22] and Donaldson [3], and is a standard part of the theory [4, 5]. Obstructed gluing at a reducible connection, for cobordisms between rational homology spheres, is studied in [2, Section 5]. We have collected the analysis in Section 5, and the reader who is willing to take Proposition 4.1 on trust can omit that section.

Organization. Sections 2–4 deal with the case $R = E_7$. Section 2 contains the lattice-theoretic part of the argument, Section 3 describes the moduli spaces on the two pieces, and Section 4 contains the parity argument. The transversality and gluing results on which these depend are proved in Section 5. The cases $R = D_5$, D_6 and D_7 are treated in Section 6, and the remaining cases in Section 7, where the proof of Theorem 1.1 is completed.

2. The case $R = E_7$: constraints on the lattice

In this section and the next two, Y denotes the manifold Y_{E_7}. It is the quotient of S^3 by the binary octahedral group O^*, a group of order 48 whose abelianization is $\mathbb{Z}/2$. For this manifold we prove a statement which is a little stronger than Theorem 1.1, in that the hypotheses on N become conclusions.

Theorem 2.1. *Suppose that $Y = Y_{E_7}$ is smoothly embedded in a K3 surface X. Then Y separates X into two pieces,*

$$X = W \cup_Y C,$$

where W has oriented boundary Y and is negative definite. The groups $H_1(W;\mathbb{Z})$ and $H_1(C;\mathbb{Z})$ are zero, and the intersection form of W is isomorphic to $-E_7$. In particular, $b^+(C) = 3$ and $b^-(C) = 12$.

The proof occupies this section and the next three. Here we prove the following.

Proposition 2.2. *In the situation of Theorem 2.1, one of the two pieces, W, is negative definite and has oriented boundary Y. The groups $H_1(W;\mathbb{Z})$ and $H_1(C;\mathbb{Z})$ are zero, and the intersection form Q_W is isomorphic either to $-E_7$ or to $\langle -2\rangle$.*

Proof. Since $H_3(X;\mathbb{Z}) = 0$, the submanifold Y separates X into two pieces. Let V denote either of them. In the Mayer–Vietoris sequence, the groups $H_2(Y;\mathbb{Z})$ and $H_1(X;\mathbb{Z})$ are zero, so $H_2(V;\mathbb{Z})$ injects into $H_2(X;\mathbb{Z})$ and is therefore free, while $H_1(V;\mathbb{Z})$ is a quotient of $H_1(Y;\mathbb{Z})$ and is therefore finite. The exact sequence of the pair (V, Y) and Poincaré–Lefschetz duality then give

$$|H_1(Y;\mathbb{Z})| = |\det Q_V|\,|H_1(V;\mathbb{Z})|^2. \tag{2}$$

The left-hand side is 2, so $H_1(V;\mathbb{Z}) = 0$ and $|\det Q_V| = 2$. The form Q_V is even, because X is spin.

The manifold Y is an L-space and X is symplectic, so one of the two pieces has $b^+ = 0$ [16, Theorem 1.2]. (If b^+ were positive on both sides, then the Ozsváth–Szabó invariants of X would factor through $HF_{\mathrm{red}}(Y) = 0$, and they would vanish, which they do not for a symplectic manifold.) We call this piece W. It is negative definite, because Q_W is non-degenerate. To see which way the boundary of W is oriented, we use the correction terms of Y. There are two spin^c structures on Y, both of them spin, and their correction terms are

$$d(Y,\mathfrak{s}) = 7/4 \quad \text{and} \quad 1/4. \tag{3}$$

See [1, Lemma 1.2], where the values are given for $-Y$. If V is a negative-definite spin manifold whose boundary is a rational homology sphere, and $\mathfrak{s}$ is the spin structure which the boundary inherits, then

$$b_2(V) \le 4\, d(\partial V, \mathfrak{s})$$

by [18, Theorem 9.6]. Since both correction terms of $-Y$ are negative, the oriented boundary of W is Y and not $-Y$, and the rank $r = b_2(W)$ is at most 7. It is at least 1, because $|\det Q_W| = 2$.

Let L be the lattice $-Q_W$, which is even and positive definite, of determinant 2. Its discriminant group has one non-zero element, on which the discriminant quadratic form takes a value q which is either $1/2$ or $3/2$ modulo 2. The Milgram formula [14, Appendix 4] gives

$$e^{2\pi i r/8} = \frac{1 + e^{\pi i q}}{\sqrt{2}},$$

so $r \equiv 1 \pmod 8$ if $q = 1/2$, and $r \equiv 7 \pmod 8$ if $q = 3/2$. So either $r = 1$, in which case $L = \langle 2\rangle$, or $r = 7$ and $q = 3/2$. In the second case, the lattice $L \oplus A_1$ has an even unimodular overlattice of index 2. This overlattice is positive definite of rank 8, and is therefore isomorphic to E_8 [20, Chapter V]. In it, L is the orthogonal complement of a root, so L is isomorphic to E_7. □

To prove Theorem 2.1, it remains to show that Q_W cannot be $\langle -2\rangle$.

3. The moduli spaces when N has rank one

We suppose from now on that the second alternative in Proposition 2.2 occurs, and we write N for W. So X is a K3 surface, decomposed as

$$X = N \cup_Y C, \qquad Q_N \cong \langle -2\rangle,$$

and our aim is to derive a contradiction. By Proposition 2.2, the first homology groups of N and C are zero. The group $H_2(N;\mathbb{Z})$ is generated by a class x with $x \cdot x = -2$, and we have

$$\chi(N) + \sigma(N) = 1, \qquad b^+(C) = 3, \quad b^-(C) = 18, \qquad \chi(C) + \sigma(C) = 7. \tag{4}$$

The class x is primitive in $H_2(X;\mathbb{Z})$, because the intersection form of X is even.

3.1. **The bundle.** Since x is primitive and has square -2, there is an automorphism of the lattice $H_2(X;\mathbb{Z})$ which carries x to $e - f$, where e and f are the standard basis of a hyperbolic summand [8, Proposition 3.3(i)]. Let e' and f' be the standard basis of a second hyperbolic summand, orthogonal to the first, and let $w \in H^2(X;\mathbb{Z})$ be the class dual to $f + e' - 3f'$. Then

$$w[x] = 1, \qquad w^2 = -6. \tag{5}$$

Let $\mathcal{E} \to X$ be the $U(2)$ bundle with $c_1(\mathcal{E}) = w$ and $c_2(\mathcal{E}) = 0$, and let $P = \mathrm{ad}(\mathcal{E})$ be the associated $SO(3)$ bundle, so that

$$w_2(P) = w \bmod 2, \qquad p_1(P)[X] = -6.$$

We adopt the usual conventions for such bundles, as in [11]: we consider connections A in P, and the gauge group $\mathcal{G}$ consists of the automorphisms of $\mathcal{E}$ of determinant one, so that an irreducible connection has stabilizer $\{\pm 1\}$ in $\mathcal{G}$. We write κ for the action of a connection, normalized so that $\kappa = -p_1(P)/4$ on a closed manifold. For the bundle P we have $\kappa = 3/2$, and the moduli space $M(X, P)$ of anti-self-dual connections has formal dimension

$$8\kappa - 3(1 + b^+(X)) = 0.$$

The corresponding Donaldson invariant, which counts the points of a regular moduli space $M(X, P)$ with signs, is

$$D_X^w(1) = \pm 1. \tag{6}$$

This is the standard calculation for the K3 surface [11, Proposition 7.13]. We shall use only the fact that the number of points is odd.

3.2. **Flat connections on Y.** The restriction of w to N is the generator z of $H^2(N;\mathbb{Z}) \cong \mathbb{Z}$ with $z[x] = 1$, and its restriction to Y is the non-zero element of $H^2(Y;\mathbb{Z}) = \mathbb{Z}/2$. The flat connections in $P|_Y$, modulo gauge transformations of determinant one, therefore correspond to the conjugacy classes of homomorphisms $\pi_1(Y) = O^* \to U(2)$ whose determinant is the non-trivial character $\epsilon\colon O^* \to \{\pm 1\}$.

Lemma 3.1. (a) *There are two such conjugacy classes. They are represented by $\alpha = 1 \oplus \epsilon$ and by β, the pull-back of the irreducible two-dimensional representation of the symmetric group S_3 by the quotient maps $O^* \to S_4 \to S_3$.*

(b) *The rho-invariants of the corresponding flat connections in $P|_Y$ are*

$$\rho_Y(\alpha) = -4, \qquad \rho_Y(\beta) = -\frac{17}{3}.$$

(c) *Their Chern–Simons invariants satisfy*

$$\mathrm{cs}(\beta) - \mathrm{cs}(\alpha) = -\frac{1}{3} \pmod{\mathbb{Z}}.$$

Here $\rho_Y(\gamma)$ is the Atiyah–Patodi–Singer rho-invariant of the flat bundle $\operatorname{ad}\gamma$, and cs is the Chern–Simons invariant with values in $\mathbb{R}/\mathbb{Z}$. Our sign conventions for both are fixed by the way they enter the formulae (7) and (8) below. The proof of the lemma is a calculation with the character table of O^*, and is given in Section 3.4.

The flat connection α is reducible: because the characters 1 and ϵ are distinct, its stabilizer in $\mathcal{G}/\{\pm 1\}$ is a circle. The flat connection β is irreducible. Both are non-degenerate, in that

$$H^1(Y;\operatorname{ad}\alpha) = H^1(Y;\operatorname{ad}\beta) = 0,$$

because $\pi_1(Y)$ is finite. We write h_γ for the dimension of the stabilizer of γ, so $h_\alpha = 1$ and $h_\beta = 0$.

Remark 3.2. The holonomy group of the $SO(3)$ connection $\operatorname{ad}\alpha$ has order two, so its centralizer in $SO(3)$ is a copy of $O(2)$. The elements in the non-identity component of $O(2)$ are automorphisms of $P|_Y$ which do not lift to automorphisms of $\mathcal{E}|_Y$ of determinant one, and they play no role in what follows.

3.3. **Moduli spaces on N and C.** Let N^+ and C^+ be the manifolds obtained from N and C by attaching cylindrical ends $[0,\infty)\times Y$ and $(-\infty,0]\times Y$, with metrics which are products on the ends. For $T > 0$, let X_T be the Riemannian manifold

$$X_T = N \cup \bigl([-T,T]\times Y\bigr) \cup C,$$

which is a copy of X containing a neck of length $2T$.

Let V be either N or C, and let γ be either α or β. We write $M_\kappa(V,\gamma)$ for the moduli space of anti-self-dual connections in P on V^+ which are asymptotic to γ on the end and have action κ. By the Atiyah–Patodi–Singer index theorem, the formal dimension of this moduli space is

$$d = 8\kappa - \frac{3}{2}\bigl(\chi(V)+\sigma(V)\bigr) - \frac{1}{2}h_\gamma + \frac{1}{2}\rho_{\partial V}(\gamma), \tag{7}$$

where $\rho_{\partial N} = \rho_Y$ and $\rho_{\partial C} = -\rho_Y$.

On $M_\kappa(V,\gamma)$, the action κ is determined modulo $\mathbb{Z}$ by γ. To find these residues, we consider reducible connections on N. A reduction of $P|_N$ to the circle has the form $P = \mathbb{R}\oplus L$, where L is a complex line bundle with $c_1(L) \equiv w \pmod 2$, so that $c_1(L) = kz$ with k odd. Because $b^+(N) = 0$ and $H_1(N;\mathbb{Z}) = 0$, each such line bundle carries an anti-self-dual connection on N^+ with curvature in L^2, unique up to gauge. Since $z^2 = -1/2$, the corresponding connection in P has action

$$\kappa = -\tfrac{1}{4}c_1(L)^2 = k^2/8.$$

The reductions for k and $-k$ give the same connection in P. So the reducible anti-self-dual connections on N^+ have action $1/8$, $9/8$, $25/8$ and so on, and there is one of each. We write A_0 for the reducible connection with action $1/8$. The limit of A_0 on the end is α, because α is the only reducible flat connection. So $\kappa \equiv 1/8$ on $M_\kappa(N,\alpha)$, and from Lemma 3.1(c) we obtain

$$\kappa \equiv \frac{1}{8} \text{ on } M_\kappa(N,\alpha), \qquad \kappa \equiv \frac{1}{8} - \frac{1}{3} = \frac{19}{24} \text{ on } M_\kappa(N,\beta) \pmod{\mathbb{Z}}. \tag{8}$$

The residues on C are obtained by subtracting these from $3/2$. In the following table we list the pairs of non-negative values κ_N and κ_C allowed by (8) with $\kappa_N + \kappa_C = 3/2$, together

with the formal dimensions d_N and d_C of $M_{\kappa_N}(N,\gamma)$ and $M_{\kappa_C}(C,\gamma)$ given by (7), (4) and Lemma 3.1(b).

(9)

γ	κ_N	d_N	κ_C	d_C
α	$1/8$	-3	$11/8$	2
α	$9/8$	5	$3/8$	-6
β	$19/24$	2	$17/24$	-2

In each row, $d_N + d_C + h_\gamma$ is zero, as it must be because $M(X,P)$ has formal dimension zero.

The connection A_0 belongs to a moduli space of formal dimension -3. Its deformation complex decomposes according to the decomposition $\operatorname{ad} P = \mathbb{R} \oplus L$. The summand with coefficients in $\mathbb{R}$ has $H^0 = \mathbb{R}$, the Lie algebra of the stabilizer of A_0, and its other cohomology groups are zero because $b_1(N)$ and $b^+(N)$ are zero. The summand with coefficients in L is a complex of complex vector spaces, with $H^0 = 0$, and its index is -1. So $H^2_{A_0}$ has complex dimension at least one, with equality if and only if $H^1_{A_0} = 0$.

In the next proposition, the moduli spaces are defined using perturbations of the anti-self-duality equations on N^+ and C^+, which are of the type described in Section 5.1 and are supported in compact subsets of the interiors of N and C. The same perturbations are used on X_T to define $M(X_T,P)$. Perturbing the equations does not affect the formal dimensions or the residues of κ. If the perturbations are small, then there is still exactly one reducible solution on N^+ with action $1/8$, and we continue to call it A_0.

Proposition 3.3. *The perturbations can be chosen so that the following hold.*

(a) *The moduli space $M_{1/8}(N,\alpha)$ consists of the single point $[A_0]$. The cohomology group $H^1_{A_0}$ is zero, and $H^2_{A_0}$ is isomorphic to $\mathbb{C}$, with the stabilizer of A_0 acting with weight one.*

(b) *The moduli spaces $M_{3/8}(C,\alpha)$ and $M_{17/24}(C,\beta)$ are empty. The moduli space*

$$M = M_{11/8}(C,\alpha)$$

is a compact smooth two-manifold. It consists of irreducible connections B with $H^2_B = 0$.

(c) *Let T_i be a sequence tending to infinity, and let $[A_i]$ belong to $M(X_{T_i},P)$. Then, after passing to a subsequence, there is a point $[B]$ in M such that the connections A_i converge to A_0 on compact subsets of N^+ and to B on compact subsets of C^+. No action is lost in the limit: $\kappa(A_0) + \kappa(B) = 3/2$.*

Proof. In Section 5.1 we show that the perturbations can be chosen so that the following hold. First, the connection A_0 has $H^1_{A_0} = 0$ and $H^2_{A_0} \cong \mathbb{C}$, and the irreducible parts of the moduli spaces on N^+ are regular (Lemma 5.1). Second, the moduli spaces on C^+ with $\kappa \le 3/2$ contain no reducibles and are regular (Lemma 5.2). Third, the action of every solution on N^+ or C^+ is greater than $-5/24$; since the residues of κ are $1/8$ and $19/24$ on N, and $3/8$ and $17/24$ on C, the action is then non-negative.

A regular moduli space of negative formal dimension is empty. So the irreducible part of $M_{1/8}(N,\alpha)$ is empty, which proves (a), and the two moduli spaces on C^+ in the second and third rows of (9) are empty also. The moduli space M is a smooth two-manifold. To see that it is compact, recall that a sequence in M has a subsequence which converges weakly to a point of some $M_{\kappa'}(C,\gamma)$ with $\kappa' \le 11/8$. If κ' is less than $11/8$, then the difference is accounted for by bubbles, each of which carries action at least 1, and by anti-self-dual

connections on the cylinder $\mathbb{R} \times Y$. The only possibilities allowed by the residues are $M_{3/8}(C, \alpha)$ and $M_{17/24}(C, \beta)$, and these are empty.

The proof of (c) is similar. A sequence $[A_i]$ as in (c) has a subsequence which converges weakly to a pair of solutions on N^+ and C^+, belonging to moduli spaces $M_{\kappa_N}(N, \gamma)$ and $M_{\kappa_C}(C, \gamma')$, where κ_N and κ_C are non-negative and $\kappa_N + \kappa_C \leq 3/2$. Equality holds only if there are no bubbles and no action is lost in the neck, in which case $\gamma = \gamma'$. Among the moduli spaces on C^+ with $\kappa_C \leq 3/2$, only M can be non-empty. So $\kappa_C = 11/8$ and $\kappa_N \leq 1/8$. Since κ_N is non-negative and congruent to $1/8$ or $19/24$ modulo $\mathbb{Z}$, it is equal to $1/8$, the limiting flat connection is α, and no action is lost. By (a), the limit on N^+ is A_0. □

3.4. **Proof of Lemma 3.1.** We refer to [9, Appendix A.4] for the character table of O^*. The group has three irreducible two-dimensional representations: the defining representation Q of $O^* \subset SU(2)$, the tensor product $Q \otimes \epsilon$, and β. The first two have determinant 1, and β has determinant ϵ. A reducible representation with determinant ϵ is the sum of two characters whose product is ϵ, and so is $1 \oplus \epsilon$. This proves (a).

For (b), let $\pm\theta_g$ be the arguments of the eigenvalues of $g \in O^*$ in the representation Q. With Y oriented as the boundary of the negative-definite plumbing, the rho-invariant is given by

$$\rho_Y(\gamma) = \frac{1}{48} \sum_{g \neq 1} \cot^2\Big(\frac{\theta_g}{2}\Big) (\operatorname{tr} \operatorname{ad} \gamma(g) - 3).$$

The representations $\operatorname{ad} \alpha$ and $\operatorname{ad} \beta$ factor through the quotient S_4 of O^*, and each element of S_4 is the image of two elements $\pm g$ of O^*. In the following table, the third column is the sum of $\cot^2(\theta_g/2)$ over the two elements of O^* lying over an element of the given conjugacy class in S_4.

class in S_4	size	$\sum \cot^2(\theta_g/2)$	$\operatorname{tr} \operatorname{ad} \alpha - 3$	$\operatorname{tr} \operatorname{ad} \beta - 3$
(12)	6	2	−4	−4
(123)	8	10/3	0	−3
(1234)	6	6	−4	−4

The remaining conjugacy classes in S_4 make no contribution, because $\operatorname{ad} \alpha$ and $\operatorname{ad} \beta$ are trivial there. So

$$\rho_Y(\alpha) = \frac{-48 - 144}{48} = -4, \qquad \rho_Y(\beta) = \frac{-48 - 80 - 144}{48} = -\frac{17}{3}.$$

For (c), recall that $H^4(BO^*; \mathbb{Z})$ is cyclic of order 48, generated by $e = c_2(Q)$. We have $c_2(\alpha) = 0$. To find $c_2(\beta)$ we restrict to subgroups. On a Sylow 2-subgroup of O^*, which is a generalized quaternion group of order 16, the representation β is the sum of the trivial character and a non-trivial one, so $c_2(\beta)$ is zero modulo 16. On a cyclic subgroup of order 3, the representation β coincides with Q, so $c_2(\beta)$ is equal to e modulo 3. Hence $c_2(\beta) = 16e$. With the conventions of [9, Appendix B], the Chern–Simons invariant of Q is $-1/48$. Since α and β have the same determinant, it follows that

$$\operatorname{cs}(\beta) - \operatorname{cs}(\alpha) = -\frac{16}{48} = -\frac{1}{3} \pmod{\mathbb{Z}}.$$

In (8), we have used the fact that if two connections on N^+ are asymptotic to γ and γ', then the difference of their actions is congruent to $\operatorname{cs}(\gamma) - \operatorname{cs}(\gamma')$ modulo $\mathbb{Z}$. For this, the Chern–Simons invariant is defined using the form ζ with $d\zeta = (8\pi^2)^{-1} \operatorname{tr}(F \wedge F)$, which is

the normalization for which an anti-self-dual connection on a closed manifold has action $\kappa = -p_1/4$.

4. The parity argument

We continue with the notation of Section 3. The gluing theorem of Section 5 leads to the following description of the moduli space on X_T when T is large. We write $\mathcal{B}^*(X_T, P)$ for the space of irreducible connections in P modulo the gauge group $\mathcal{G}$, and τ for the perturbation of the equations on X_T, so that $M(X_T, P)$ is the set of solutions of $F_A^+ + \tau(A) = 0$ in $\mathcal{B}^*(X_T, P)$.

Proposition 4.1. *If T is sufficiently large then, after the addition of a further small perturbation to τ, the following hold.*

(a) *There are an open subset U of $\mathcal{B}^*(X_T, P)$, an oriented real two-plane bundle $E \to U$, and a smooth bundle map ι from E to the bundle over U whose fibre at $[A]$ is $\Omega^+(X_T; \operatorname{ad} P)$, such that the space*

$$\Sigma = \{\, ([A], v) : [A] \in U,\ v \in E_{[A]},\ F_A^+ + \tau(A) + \iota_A(v) = 0 \,\}$$

is cut out transversely and is a compact surface, diffeomorphic to M.

(b) *The moduli space $M(X_T, P)$ is regular and is contained in U. So $M(X_T, P)$ is the zero set of the section s of $E|_\Sigma$ given by $s([A], v) = v$, and s vanishes transversely. The number of its zeros is odd.*

The surface Σ is a thickened moduli space, of the kind which arises whenever a moduli space is described by a finite-dimensional obstruction bundle. To say that it is cut out transversely means the following. Write $\sigma(A) = F_A^+ + \tau(A)$. At a point $([A], v)$ of Σ, the linearization of the equation $\sigma(A) + \iota_A(v) = 0$, together with the Coulomb gauge condition, is the operator

$$\begin{aligned} \mathcal{D}_{(A,v)} \colon \Omega^1(X_T; \operatorname{ad} P) \oplus E_{[A]} &\longrightarrow \Omega^0(X_T; \operatorname{ad} P) \oplus \Omega^+(X_T; \operatorname{ad} P), \\ (a, u) &\longmapsto (d_A^* a,\ (d\sigma)_A(a) + (\partial_a \iota)(v) + \iota_A(u)), \end{aligned} \tag{10}$$

and we require this operator to be surjective. Its kernel is then the tangent space to Σ at $([A], v)$.

The last statement in part (b) holds because the number of points in the moduli space, counted with signs, is the Donaldson invariant (6). On the other hand, the number of zeros of a transverse section of $E|_\Sigma$ is given, modulo 2, by the second Stiefel–Whitney class. So Proposition 4.1 has the following consequence, in which $[\Sigma]$ is the fundamental class with coefficients $\mathbb{F}_2$:

$$\langle w_2(E), [\Sigma] \rangle = 1. \tag{11}$$

Because $\mathcal{G}/\{\pm 1\}$ acts freely on the irreducible connections, there is a universal $SO(3)$ bundle over $\mathcal{B}^*(X_T, P) \times X_T$. Its restriction to $\Sigma \times X_T$ carries a family of connections parametrized by Σ. So the operators

$$D_A = d_A^* \oplus d_A^+ \colon \Omega^1(X_T; \operatorname{ad} P) \longrightarrow \Omega^0(X_T; \operatorname{ad} P) \oplus \Omega^+(X_T; \operatorname{ad} P),$$

for $[A]$ in Σ, form a family of elliptic operators over Σ, which has an index $\operatorname{ind} D$ in $KO(\Sigma)$.

Lemma 4.2. *In $KO(\Sigma)$, we have*

$$[T\Sigma] - [E|_\Sigma] = \operatorname{ind} D. \tag{12}$$

Proof. The operators $\mathcal{D}_{(A,v)}$ in (10) form a family of surjective Fredholm operators over Σ, and the kernels form the bundle $T\Sigma$. So the index of this family is $[T\Sigma]$. On the other hand, $\mathcal{D}_{(A,v)}$ differs from the operator $(a,u) \mapsto D_A a$ by terms which are compact: the derivative of the perturbation τ is compact, and the terms involving ι have finite rank. The index of the family $(a,u) \mapsto D_A a$ is $\operatorname{ind} D + [E|_\Sigma]$. □

Proposition 4.3. *Let X be a K3 surface with any Riemannian metric, let B be a compact space, and let $\mathbf{P} \to B \times X$ be an $SO(3)$ bundle carrying a family of connections on X parametrized by B. Then the index of the family of operators $D_A = d_A^* \oplus d_A^+$ lies in the image of the map $KSp(B) \to KO(B)$. In particular, its first and second Stiefel–Whitney classes are zero.*

Proof. The index of the family does not depend on the Riemannian metric, so we may suppose that the metric is hyperkähler. The spin bundles S^+ and S^- of X are quaternionic line bundles: we regard their fibres as right $\mathbb{H}$-modules, and Clifford multiplication by one-forms and the Dirac operator are then $\mathbb{H}$-linear. For the hyperkähler metric, the bundle S^+ is flat and trivial. Let ϕ be a parallel section of S^+ of unit length. Clifford multiplication on ϕ gives isomorphisms of real vector bundles

$$\Lambda^1 \longrightarrow S^-, \qquad \Lambda^0 \oplus \Lambda^+ \longrightarrow S^+,$$

and because ϕ is parallel, these isomorphisms carry the operator $d^* \oplus d^+$ to the Dirac operator $\Gamma(S^-) \to \Gamma(S^+)$, up to constant factors. In the same way, the operator D_A becomes the Dirac operator coupled to the real vector bundle $\operatorname{ad} P$ with the connection A. This coupled Dirac operator is again $\mathbb{H}$-linear. The quaternionic structure is the same for every member of the family, so the index of the family is the class of a virtual quaternionic vector bundle on B. Finally, the real vector bundle which underlies a quaternionic vector bundle has $w_1 = 0$ and $w_2 = 0$. □

Proof of Theorem 2.1. By Proposition 2.2, we have only to exclude the case considered in Section 3, in which Q_N is $\langle -2 \rangle$. In that case, let Σ and E be as in Proposition 4.1. By Proposition 4.3, the class $\operatorname{ind} D$ in (12) has $w_1 = 0$ and $w_2 = 0$, and so the Stiefel–Whitney classes of $T\Sigma$ and $E|_\Sigma$ are equal. Because E is oriented, it follows that Σ is orientable and that

$$\langle w_2(E), [\Sigma] \rangle = \langle w_2(T\Sigma), [\Sigma] \rangle.$$

The right-hand side is the Euler number of Σ modulo 2, which is zero because Σ is a closed orientable surface. This contradicts (11). (If M is empty, then Σ is empty and (11) fails for that reason.) So Q_W is isomorphic to $-E_7$, and the values of $b^+(C)$ and $b^-(C)$ follow. □

5. Regularity and gluing

In this section we prove the transversality statements which were used in the proof of Proposition 3.3, and we prove Proposition 4.1. The methods are not new. The technique for gluing in the presence of an obstruction, in which the equations are first solved modulo a finite-dimensional space associated with the small eigenvalues of the linearized operator, goes back to Taubes [22]; see also [3] and [5, Section 7.2]. For manifolds with cylindrical ends, we follow [4, Chapter 4]. Some constants in our argument grow polynomially with the length of the neck, while the error terms decay exponentially, and to deal with this one needs a slice theorem in which the constants are controlled by the lowest eigenvalue of the Laplacian $d_A^* d_A$. Results of that kind are also due to Taubes: see [23, Section 6] and [21, Lemma A.1], and the account in [6, Theorems 1.1 and 1.2]. The small eigenvalues of

Dirac-type operators on manifolds with long necks, and the obstruction bundles to which they give rise in families, are discussed in [17]. The perturbations of the equations which we use are the holonomy perturbations of [10].

What is particular to the present situation is that the ends are rational homology spheres, on which the flat connections may be reducible. Moduli spaces of this sort are studied in detail in [13] and [2], and we refer to those papers at several points.

5.1. **Perturbations and regularity.** Let V be an oriented Riemannian four-manifold with a cylindrical end $[0,\infty)\times Y$, where Y is a rational homology sphere, and let $P\to V$ be an $SO(3)$ bundle, with the conventions of Section 3. For a non-degenerate flat connection γ on the end, the moduli spaces $M_\kappa(V,\gamma)$ are defined in the standard way [4, Chapter 4], using Sobolev spaces L^2_k with $k\geq 4$ and a small exponential weight on the end. For a solution A, we write H^0_A, H^1_A and H^2_A for the cohomology groups of its deformation complex, so that H^0_A is the Lie algebra of the stabilizer of A and H^2_A is the cokernel of the linearized equations. The solution is *regular* if $H^2_A=0$. The elements of H^2_A are represented by self-dual forms which decay exponentially on the end.

The equations are perturbed using holonomy perturbations, as in [10, Section 3]. In place of $F^+_A=0$ we consider

$$\sigma(A)=F^+_A+\tau(A)=0,$$

where τ is a sum of terms of the following kind. One takes a ball B in the interior of V, a map $q\colon S^1\times B\to V$ with $q(1,x)=x$, and a self-dual form ω supported in B. The corresponding term is obtained from the holonomy of A around the loops $q(\cdot,x)$, for x in B, by multiplying by ω and projecting to $\Omega^+(V;\operatorname{ad}P)$. The perturbations form a Banach space, they are supported in a compact subset of the interior of V, and they are gauge-equivariant: if A is compatible with a reduction $P=\mathbb{R}\oplus L$, then $\tau(A)$ takes values in the summand $\mathbb{R}$. Their use on manifolds with cylindrical ends, in the presence of reducible connections, is described in [13, Section 4] and [2, Section 3.2].

In [10], the map q is required to be a submersion. We require more, namely that each map $q(z,\cdot)$ is an embedding of B. The derivative of τ is then built from pull-backs by diffeomorphisms, so that it and its formal adjoint are bounded on Sobolev spaces of every order, negative orders included, which we need in the gluing argument. The usual transversality arguments apply without change to perturbations of this restricted kind.

A solution of the perturbed equations with action κ satisfies

$$\kappa=c_0\bigl(\|F^-_A\|^2_{L^2}-\|F^+_A\|^2_{L^2}\bigr)\geq -c_0\|\tau(A)\|^2_{L^2}, \tag{13}$$

where c_0 is a positive constant which depends only on the normalization of κ. The L^2 norm of $\tau(A)$ is bounded independently of A, by a constant which is small when τ is small. So for any $\varepsilon>0$, all solutions have $\kappa>-\varepsilon$ when τ is sufficiently small.

If a solution A is compatible with a reduction $P=\mathbb{R}\oplus L$, then the linearization of the equations at A in Coulomb gauge is the sum of a *tangential* operator, which acts on ordinary one-forms, and a *normal* operator D^ν_A, which acts on one-forms with values in L and is complex-linear. The circle in the stabilizer of A acts on L with weight one.

Lemma 5.1. *Suppose that $H_1(V;\mathbb{Z})=0$ and $b^+(V)=0$. Let α be a flat connection on the end with $H^1(Y;\operatorname{ad}\alpha)=0$ whose stabilizer is a circle. Let $P=\mathbb{R}\oplus L$ be a reduction of P which is compatible with α on the end, for which $c_1(L)$ is non-zero in $H^2(V;\mathbb{Q})$ and for which the normal operators have complex index -1. Then for all sufficiently small perturbations τ,*

there is exactly one solution A_0 compatible with this reduction, up to gauge. For a residual set of small perturbations, the normal operator $D^\nu_{A_0}$ is injective, so that

$$H^1_{A_0} = 0, \qquad H^2_{A_0} \cong \mathbb{C},$$

and the irreducible solutions in the moduli spaces $M_\kappa(V, \gamma)$ are regular, for all γ and for all κ below any given bound.

Proof. When $\tau = 0$, the solutions compatible with the reduction are the anti-self-dual connections in L with curvature in L^2, and there is exactly one of these up to gauge, because $b^+(V) = 0$ and $H_1(V; \mathbb{Z}) = 0$. The tangential operator at this solution has kernel $H^1(V; \mathbb{R}) = 0$, and its cokernel consists only of the constants. So the solution persists when τ is small, by the implicit function theorem. No others appear, because solutions in a line bundle cannot bubble and cannot lose action on the end; compare [13, Section 4.4].

The normal operators D^ν_τ at these solutions are complex Fredholm operators of index -1. Among such operators, those whose kernel has complex dimension j form a submanifold of real codimension $2j(j+1)$, whose normal space at an operator with kernel K and cokernel H is $\mathrm{Hom}_{\mathbb{C}}(K, H)$. Let K and H be the kernel and cokernel of D^ν_0. It is enough to show that the derivative of $\tau \mapsto D^\nu_\tau$ at $\tau = 0$ maps onto $\mathrm{Hom}_{\mathbb{C}}(K, H)$. For surjectivity onto a finite-dimensional space is an open condition, so the map $\tau \mapsto D^\nu_\tau$ is then transverse to all these submanifolds for τ near zero, and by the Sard–Smale theorem D^ν_τ is injective for a residual set of small τ. The surjectivity is proved by the argument which is used in [13, Section 4.6] and [2, Section 3.5] to obtain transversality in the directions normal to the reducible connections. A non-zero element of K is detected by the derivative of the holonomy around some loop in a compact subset of V, because the flat line bundle $L|_Y$ has no parallel sections. A non-zero element of H cannot vanish on an open set. And the only constraint on the derivative of a holonomy perturbation in the directions normal to the reducible connections is equivariance under the circle, which for a real-linear map from K to H is the same as complex-linearity. The perturbations can be taken to vanish to first order at A_0 along the reducible locus, so that A_0 does not move.

The regularity of the irreducible solutions for generic τ is proved as in [10, Sections 3 and 5]; see also [13, Section 4.6]. □

Lemma 5.2. *Suppose that $H_1(V; \mathbb{Z}) = 0$ and $b^+(V) > 0$, and that $w_2(P)$ is non-zero on the end. Then for a residual set of small perturbations τ, the moduli spaces $M_\kappa(V, \gamma)$ contain no reducible solutions and are regular, for all γ and for all κ below any given bound.*

Proof. Since $H^1(V; \mathbb{F}_2) = 0$, every reducible connection is compatible with a reduction $P = \mathbb{R} \oplus L$. The class $c_1(L)$ reduces modulo 2 to $w_2(P)$, so it is non-zero on the end; and it is non-zero in $H^2(V; \mathbb{Q})$, because $H^2(V; \mathbb{Z})$ has no torsion. The solutions compatible with such a reduction are the zeros of a Fredholm section of index $b_1(V) - b^+(V)$, which is negative, so for generic τ there are none; compare [13, Section 4.4]. The regularity of the irreducible solutions is proved as before. □

These lemmas supply what was needed in the proof of Proposition 3.3. Lemma 5.1 applies to the manifold N^+ of Section 3, with the reduction for which $c_1(L) = z$: the formal dimension -3 of $M_{1/8}(N, \alpha)$ is the sum of -1, from the stabilizer, and twice the complex index of the normal operator. Lemma 5.2 applies to C^+, where $b_1 - b^+ = -3$. The lower bound for the action comes from (13), with $\varepsilon = 5/24$.

5.2. **The gluing theorem.** Let X be a closed, oriented four-manifold, decomposed as $N \cup_Y C$ along a rational homology sphere Y, and let $P \to X$ be an $SO(3)$ bundle, with the same conventions as before. We form N^+, C^+ and X_T as in Section 3.3. We fix holonomy perturbations on N^+ and C^+, and we use their sum on X_T. Let α be a flat connection on $P|_Y$ with $H^1(Y; \operatorname{ad}\alpha) = 0$ whose stabilizer Γ_α is a circle. We make the following hypotheses.

(i) A_0 is a reducible solution on N^+, asymptotic to α, with $H^1_{A_0} = 0$. The space $H = H^2_{A_0}$ is isomorphic to $\mathbb{C}$, and the stabilizer Γ_{A_0}, which is a circle, acts on it with weight one.

(ii) M is a compact two-manifold which is a union of components of the moduli space of solutions on C^+ asymptotic to α. It consists of irreducible, regular solutions.

(iii) If T_i is a sequence tending to infinity, and $[A_i]$ belongs to $M(X_{T_i}, P)$, then after passing to a subsequence there is a point $[B]$ in M such that the connections A_i converge to A_0 on compact subsets of N^+ and to B on compact subsets of C^+, without loss of action.

Restriction to the end identifies Γ_{A_0} with Γ_α. Let $\widetilde{M} \to M$ be the moduli space of solutions framed at infinity, which is a principal Γ_α-bundle over M, and let

$$E_0 = \widetilde{M} \times_{\Gamma_\alpha} H$$

be the associated complex line bundle.

Theorem 5.3. *Suppose that hypotheses* (i)–(iii) *hold. Then there are constants T_0 and K such that the following is true for all $T \geq T_0$. Let ν be a holonomy perturbation on X_T, supported in the complement of the neck, whose norm is at most e^{-KT}. Then assertion* (a) *of Proposition 4.1 holds for the equations on X_T defined by $\tau + \nu$, with a diffeomorphism $M \to \Sigma$ which is covered by an isomorphism of oriented bundles $E_0 \to E|_\Sigma$; and the moduli space $M(X_T, P)$ is contained in U. For a residual set of such ν, the moduli space $M(X_T, P)$ is regular.*

Remark 5.4. The gluing parameter for the pair (A_0, B) is an element of Γ_α. Because Γ_{A_0} maps isomorphically to Γ_α, the gluing parameter is absorbed by the action of Γ_{A_0}, which is why Σ is a copy of M rather than of $\widetilde{M}$. In the situation of Remark 3.2, there is a second way to identify the two copies of $\operatorname{ad}\alpha$ over the neck, using the non-identity component of $O(2)$. The bundle which results is not P: its second Stiefel–Whitney class differs from $w_2(P)$ by the image of the non-zero element of $H^1(Y; \mathbb{F}_2)$ under the coboundary map to $H^2(X; \mathbb{F}_2)$, and this image is not zero because $H^1(N; \mathbb{F}_2)$ and $H^1(C; \mathbb{F}_2)$ are zero.

5.3. **Proof of Theorem 5.3.** We first take $\nu = 0$.

Approximate solutions. Because α is non-degenerate, the connection A_0 approaches α exponentially on the end of N^+, and so do the solutions B representing the points m of M on the end of C^+, uniformly in m. We use cut-off functions to make A_0 and B equal to α at distance greater than T along the ends, and we join the results across the neck of X_T. This requires an identification of the two copies of α. The possible identifications form a copy of Γ_α, and because Γ_{A_0} maps onto Γ_α, they all give gauge-equivalent connections. So we obtain connections $A_T(m)$, whose gauge-equivalence classes depend only on m, with

$$\|\sigma(A_T(m))\|_{L^2_{k-1}} \leq Ce^{-\lambda T}, \tag{14}$$

and with similar bounds for the derivatives with respect to m. Here and below, C, λ and q denote positive constants which are independent of T and m. The manifolds X_T have

uniformly bounded geometry, so the constants in the Sobolev and elliptic estimates do not depend on T.

In the same way, the self-dual forms on N^+ which represent the elements of H can be cut off and transferred to X_T. The result depends on the identification, through the action of Γ_{A_0} on H, and we obtain linear maps

$$\iota_m \colon (E_0)_m \longrightarrow \Omega^+(X_T; \operatorname{ad} P).$$

The linearized equations. Let $\delta_m \colon \Omega^1(X_T; \operatorname{ad} P) \to \Omega^+(X_T; \operatorname{ad} P)$ be the derivative of σ at $A_T(m)$.

Lemma 5.5. *There is a constant $c > 0$ such that, for all m and all sufficiently large T, the operator $\delta_m \delta_m^*$ has exactly two eigenvalues in the interval $[0, Ce^{-\lambda T}]$, counted with multiplicity, and no other spectrum in $[0, c]$. If Π_m is the projection onto the span of the two eigenvectors, then $\Pi_m \circ \iota_m$ is an isomorphism from $(E_0)_m$ to the image of Π_m.*

Proof. The forms $\iota_m(v)$ are annihilated by δ_m^* up to an error of size $Ce^{-\lambda T}$, which accounts for two small eigenvalues and for the last statement. If the lemma were false, there would be self-dual forms u_i on X_{T_i}, with $T_i \to \infty$, which have L^2 norm 1, are orthogonal to the image of ι_{m_i}, and have $\delta_{m_i}^* u_i \to 0$ in L^2. On the neck, where $A_T(m)$ is the pull-back of α, we identify self-dual forms with time-dependent one-forms on Y by $\eta \mapsto dt \wedge \eta + *_Y \eta$, and $\delta\delta^*$ becomes the operator

$$\tfrac{1}{2}(-\partial_t^2 + \Delta_\alpha), \tag{15}$$

where Δ_α is the Hodge Laplacian on $\Omega^1(Y; \operatorname{ad} \alpha)$. This Laplacian is bounded below by a positive constant, because $H^1(Y; \operatorname{ad} \alpha) = 0$, so the L^2 norm of u_i cannot escape into the neck. A subsequence therefore converges on compact subsets of N^+ and C^+ to L^2 solutions of the adjoint equations, not both zero. But the limit on C^+ is zero, because the points of M are regular, and the limit on N^+ is an element of H which is orthogonal to H. □

The space $H^0(Y; \operatorname{ad} \alpha)$, which is not zero, does not enter in (15). It does affect the Laplacian $d_A^* d_A$ on zero-forms, and so the Coulomb gauge condition. For $A = A_T(m)$, and for connections A whose L^2_k distance from $A_T(m)$ is at most T^{-q}, we have

$$\|\xi\|_{L^2} \leq CT \, \|d_A \xi\|_{L^2}, \qquad \xi \in \Omega^0(X_T; \operatorname{ad} P). \tag{16}$$

This is obtained from a Poincaré inequality on a compact subset of C, where the connections representing the points of M are irreducible, by integrating along the neck. So $d_A^* d_A$ has an inverse G_A, whose norms as an operator between Sobolev spaces are bounded by powers of T, and $P_A = 1 - d_A G_A d_A^*$ is the projection onto the kernel of d_A^*. The powers of T do no harm, because they multiply terms which are exponentially small. For example, if $A = A_T(m)$ and R_m denotes the operator $\xi \mapsto [\sigma(A), \xi]$, then the gauge-invariance of σ gives $\delta_m d_A = R_m$, and so

$$\delta_m P_A \delta_m^* = \delta_m \delta_m^* - R_m G_A R_m^*, \tag{17}$$

where the last term has norm at most $CT^q e^{-2\lambda T}$, by (14). With Lemma 5.5, this shows that $(1 - \Pi_m)\delta_m$ has a right inverse S_m whose image lies in the kernel of d_A^* and whose norm, as an operator from L^2_{k-1} to L^2_k, is at most CT^q.

The thickened moduli space. For m in M, consider the equation

$$\sigma(A_T(m) + a) + \iota_m(v) = 0 \tag{18}$$

for a one-form a in the image of S_m and an element v of $(E_0)_m$. Its linearization at $(a, v) = (0, 0)$ is the map $(a, v) \mapsto \delta_m a + \iota_m(v)$, which is invertible, with an inverse of norm at most CT^q: this follows from the properties of S_m and from Lemma 5.5. The left-hand side of (18) is of size $Ce^{-\lambda T}$ at $(0, 0)$, and the non-linear terms are quadratic, with constants bounded by powers of T. So by the contraction mapping theorem, there is a Q such that, for large T, the equation has a unique solution (a_m, v_m) with

$$\|a\|_{L^2_k} + |v| \leq T^{-Q}. \tag{19}$$

The solution depends smoothly on m.

The map $m \mapsto [A_T(m)]$ embeds M in $\mathcal{B}^*(X_T, P)$, and up to an exponentially small error, the image of S_m is a complement in the Coulomb slice at $A_T(m)$ to the tangent space of the embedded surface. So the connections $A_T(m) + a$, with a in the image of S_m and $\|a\|_{L^2_k} < T^{-Q}$, represent the points of an open neighborhood U of the surface, each point once. This is an application of the slice theorem, in a form in which the size of the neighborhood is controlled by the lowest eigenvalue of $d^*_A d_A$, for which we have the bound (16); compare [23, Section 6] and [6, Theorem 1.2]. We define $E \to U$ and the map ι of Proposition 4.1 by pulling back E_0 and the maps ι_m by the projection $U \to M$. Then Σ is the set of points $([A_T(m) + a_m], v_m)$. It is a copy of M, it is cut out transversely, and the section s is $m \mapsto v_m$.

The moduli space is contained in U. We must show that, when T is large, every element of $M(X_T, P)$ is represented by a connection $A_T(m) + a$, with a in the image of S_m and satisfying (19). For then $(a, 0)$ is a solution of (18), and it coincides with (a_m, v_m) by uniqueness. Hypothesis (iii) does not give this directly, because the radius T^{-Q} in (19) shrinks with T.

Let $[A_i]$ be elements of $M(X_{T_i}, P)$, with $T_i \to \infty$, which converge to A_0 and B as in hypothesis (iii). Since no action is lost, A_i is close to α on the neck; and since α is non-degenerate, the difference $a = A_i - \alpha$, in a suitable gauge, decays exponentially from both ends of the neck [4, Chapter 4]:

$$\|a(t)\|_{L^2_k(Y)} \leq C(e^{-\lambda(t+T_i)} + e^{-\lambda(T_i - t)}), \qquad |t| \leq T_i. \tag{20}$$

Cutting A_i at $t = 0$ and extending the two halves by α, we obtain connections on N^+ and C^+ at which σ is of size $Ce^{-\lambda T_i}$ in the weighted norms. The linearized operator at A_0 is injective with closed range, because $H^1_{A_0} = 0$, and the points of the compact space M are regular. So these two connections lie within distance $Ce^{-\lambda T_i}$ of A_0 and of a solution representing a point m_i of M. It follows that, after a gauge transformation, A_i lies within distance $CT_i^q e^{-\lambda T_i}$ of $A_{T_i}(m_i)$, and this is much smaller than T_i^{-Q} when i is large.

The additional perturbation. If τ is replaced by $\tau + \nu$, where the norm of ν is at most e^{-KT} and K is large, then (14) still holds and the argument is unchanged. For a residual set of such ν, the moduli space $M(X_T, P)$ is regular: perturbations supported in the complement of the neck are sufficient for this, because an element of the cokernel which vanishes there is zero, by unique continuation. This completes the proof of Theorem 5.3.

5.4. **Proof of Proposition 4.1.** In the situation of Section 3, the stabilizer of α is a circle, and hypotheses (i), (ii) and (iii) are the three parts of Proposition 3.3. So Theorem 5.3 applies. It gives part (a) of Proposition 4.1 and the inclusion of $M(X_T, P)$ in U. For a generic choice of the additional perturbation, the moduli space $M(X_T, P)$ is regular, and its points are then the transverse zeros of the section s. For a generic choice it is also the case that there are no reducible solutions in P, and that the moduli spaces for the bundles with smaller action, whose formal dimensions are -8 or less, are empty. The moduli space $M(X_T, P)$ is then compact, and the number of its points, counted with signs, is the Donaldson invariant (6), which is odd.

6. The cases D_5, D_6 and D_7

Let n be 5, 6 or 7, and write Y_n for Y_{D_n}. This manifold is the quotient of S^3 by the binary dihedral group

$$\Gamma_m = \langle\, a, b \mid a^{2m} = 1,\ b^2 = a^m,\ bab^{-1} = a^{-1} \,\rangle, \qquad m = n - 2,$$

which has order $4m$. The group $H_1(Y_n; \mathbb{Z})$ is $\mathbb{Z}/4$ when n is odd and $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ when n is even. In this section we prove the following.

Proposition 6.1. *Let n be 5, 6 or 7, and let N be a negative-definite submanifold of a K3 surface X, with oriented boundary Y_n and with $H_1(N; \mathbb{Z}) = 0$. Then the intersection form of N is not isomorphic to $-L_n$, where*

$$L_5 = \langle 4 \rangle, \qquad L_6 = A_1 \oplus A_1, \qquad L_7 = A_3.$$

The proof follows Sections 3–5 closely, and we shall concentrate on the points at which it differs. Suppose then that $X = N \cup_{Y_n} C$, where Q_N is isomorphic to $-L_n$ and $H_1(N; \mathbb{Z}) = 0$.

6.1. **The two pieces.** The relation (2) holds for N and for C, with $|H_1(Y_n; \mathbb{Z})| = 4$ on the left. If $H_1(C; \mathbb{Z})$ were non-zero, it would have order 2 and Q_C would be unimodular. But Q_C is even, and its signature is $n - 20$, which is not divisible by 8. So $H_1(C; \mathbb{Z}) = 0$. As in Section 3, we then have

$$\chi(N) + \sigma(N) = 1, \qquad b^+(C) = 3, \qquad \chi(C) + \sigma(C) = 7. \tag{21}$$

Because the first homology groups of N and C are zero, the lattices $H_2(N; \mathbb{Z})$ and $H_2(C; \mathbb{Z})$ are primitive sublattices of $H_2(X; \mathbb{Z})$, each the orthogonal complement of the other, and the restriction map $H^2(X; \mathbb{Z}) \to H^2(N; \mathbb{Z})$ is surjective.

6.2. **The bundle.** We identify $H^2(N; \mathbb{Z})$ with the dual lattice L_n^*, so that the square of a class c in $H^2(N; \mathbb{Z})$ is $-|c|^2$, where $|c|^2$ denotes the norm in L_n^*. We choose a class w_N in $H^2(N; \mathbb{Z})$ as follows. When $n = 5$, it is a generator of $\langle 4 \rangle^*$. When $n = 6$, it is z_1, where z_1 and z_2 are the basis of L_6^* dual to the basis of roots. When $n = 7$, we use the standard model of A_3 as the lattice of vectors in $\mathbb{Z}^4$ with coordinate sum zero, and we take $w_N = \omega_1$, where

$$\omega_i = e_i - \tfrac{1}{4}(e_1 + e_2 + e_3 + e_4).$$

In each case $|w_N|^2 = (n - 4)/4$, and

$$\text{the only elements of } w_N + 2L_n^* \text{ of norm } (n-4)/4 \text{ are } \pm w_N. \tag{22}$$

This is clear when n is 5 or 6. When $n = 7$, the vectors of norm $3/4$ in A_3^* are the eight vectors $\pm\omega_i$, and for $i \neq j$ the vectors $(\omega_i - \omega_j)/2$ and $(\omega_i + \omega_j)/2$ have norms $1/2$ and $1/4$, so they do not belong to A_3^*.

Lemma 6.2. *There is a class w in $H^2(X;\mathbb{Z})$ with $w|_N = w_N$ and $w^2 = -6$.*

Proof. Let $\Lambda = H_2(C;\mathbb{Z})$. It is an even, indefinite lattice of rank $26-n$, and its discriminant form is the negative of that of $H_2(N;\mathbb{Z})$. When n is odd, the discriminant group is cyclic of order 4, and the discriminant form takes the value $1/4$ or $3/4$ on a generator. When $n = 6$, it takes the values $1/2$, $1/2$ and 1 on the three non-zero elements. In neither case is there a non-zero isotropic element. By Meyer's theorem, Λ contains a primitive vector u with $u^2 = 0$. Let d be the positive generator of the ideal $u \cdot \Lambda$ in $\mathbb{Z}$. Then u/d represents an isotropic element of the discriminant group, which is non-zero if $d > 1$. So $d = 1$, and there is a vector v in Λ with $u \cdot v = 1$. After adding a multiple of u to v, we may suppose that $v^2 = 0$, so that u and v span a hyperbolic plane in Λ.

Now let w_0 be any class in $H^2(X;\mathbb{Z})$ which restricts to w_N. If we identify $H^2(X;\mathbb{Z})$ with $H_2(X;\mathbb{Z})$, then the classes which restrict to zero on N are the elements of Λ. So we may subtract a combination of u and v from w_0 to make it orthogonal to both, without changing its restriction to N. The square of w_0 is even, and the class

$$w = w_0 + u - \tfrac{1}{2}(6 + w_0^2)\, v$$

has the required properties. □

Let P be the adjoint bundle of the $U(2)$ bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = w$ and $c_2(\mathcal{E}) = 0$. As in Section 3, we have $\kappa = 3/2$, the moduli space $M(X,P)$ has formal dimension zero, and the Donaldson invariant is ± 1.

6.3. **Flat connections.** The group $H^2(Y_n;\mathbb{Z})$ is the group of characters of Γ_m, which has four elements. Let δ be the character with

$$\delta(a) = 1, \qquad \delta(b) = -1,$$

and let χ be the character which corresponds to the restriction of w to Y_n. When n is odd, the class w_N generates $L_n^*/L_n \cong \mathbb{Z}/4$, so χ has order four and $\chi^2 = \delta$. When $n = 6$, the character χ is not trivial and is not δ, because its self-linking number is non-zero, whereas δ corresponds to the non-zero element of D_6^*/D_6 which has self-linking number zero. In all three cases, therefore,

$$\chi(a) = -1,$$

and the four characters of Γ_m are 1, δ, χ and $\delta\chi$.

Lemma 6.3. (a) *Up to conjugacy, there are two homomorphisms $\Gamma_m \to U(2)$ with determinant χ. They are*

$$\alpha_0 = 1 \oplus \chi, \qquad \alpha_1 = \delta \oplus \delta\chi.$$

(b) *The rho-invariants of the corresponding flat connections are*

$$\rho_{Y_n}(\alpha_0) = \rho_{Y_n}(\alpha_1) = -m.$$

Proof. The irreducible two-dimensional representations of Γ_m are the representations ρ_j, for $1 \le j \le m-1$, given by

$$\rho_j(a) = \begin{pmatrix} e^{\pi i j/m} & 0 \\ 0 & e^{-\pi i j/m} \end{pmatrix}, \qquad \rho_j(b) = \begin{pmatrix} 0 & (-1)^j \\ 1 & 0 \end{pmatrix}.$$

See [9, Appendix A.2]. The determinant of ρ_j is 1 or δ, according as j is odd or even, and is never χ. A reducible representation with determinant χ is the sum of two characters whose product is χ, and there are two such pairs. This proves (a).

For (b), we use the formula for the rho-invariant from Section 3.4, with $4m$ in place of 48. For both α_0 and α_1, the character of the adjoint representation is $1+\chi+\overline{\chi}$. So the contribution of a^j is zero when j is even, and when j is odd it is $-4\cot^2(\pi j/2m)$. Each of the $2m$ elements ba^j has $\theta=\pi/2$, and the sum of $\operatorname{tr}\operatorname{ad}\alpha_i - 3$ over these elements is $-4m$. Since

$$\sum_{\substack{0<j<2m\\ j \text{ odd}}} \cot^2\Big(\frac{\pi j}{2m}\Big) = m^2 - m,$$

we obtain $\rho_{Y_n}(\alpha_i) = \big(-4(m^2-m)-4m\big)/4m = -m$. □

The flat connections α_0 and α_1 have the same adjoint representation, but they are not equivalent under the gauge group $\mathcal{G}$ of determinant-one automorphisms, and we shall see that they are distinguished by the action. Both are reducible and non-degenerate. Their stabilizers in $\mathcal{G}/\{\pm1\}$ are circles, because the characters 1 and χ are distinct. (When $n=6$, the holonomy group of $\operatorname{ad}\alpha_0$ has order two, and Remarks 3.2 and 5.4 apply. When n is odd, the holonomy group contains a rotation of order four, and its centralizer in $SO(3)$ is already a circle.)

6.4. **The moduli spaces.** Since α_0 and α_1 are both reducible, the residues of the action can be found from reducible connections on N. A decomposition of $\mathcal{E}|_N$ as $L_1\oplus L_2$ is determined by the class $c = c_1(L_1) - c_1(L_2)$, which lies in $w_N + 2H^2(N;\mathbb{Z})$. For each such c there is a reducible anti-self-dual connection on N^+, unique up to gauge; the classes c and $-c$ give the same connection. Its action is $|c|^2/4$, and its limit on the end is the sum of the two characters which are the restrictions to Y_n of $c_1(L_1) = (w_N+c)/2$ and $c_1(L_2) = (w_N-c)/2$. Let

$$\kappa_0 = \frac{n-4}{16}.$$

For $c=w_N$, the action is κ_0 and the limit is $1\oplus\chi = \alpha_0$. By (22), this is the only reducible connection with action κ_0, and we call it A_0. Reducible connections with limit α_1 are obtained by taking c as in the following table, in which $\omega_2 = e_1+e_2-\frac12(e_1+e_2+e_3+e_4)$.

n	5	6	7
c	$3w_N$	z_1+2z_2	$\omega_1-2\omega_2$
$\lvert c\rvert^2/4$	9/16	5/8	11/16

In each case, the classes $(w_N \pm c)/2$ restrict to δ and $\delta\chi$ on Y_n, and the action is $\kappa_0+1/2$. So

$$\kappa \equiv \kappa_0 \ \text{ on } M_\kappa(N,\alpha_0), \qquad \kappa \equiv \kappa_0 + \frac12 \ \text{ on } M_\kappa(N,\alpha_1) \pmod{\mathbb{Z}}. \tag{23}$$

From (7), (21) and Lemma 6.3, we obtain the following table, which takes the place of (9).

γ	κ_N	d_N	κ_C	d_C
α_0	κ_0	-3	$3/2-\kappa_0$	2
α_1	$\kappa_0+1/2$	1	$1-\kappa_0$	-2
α_0	κ_0+1	5	$1/2-\kappa_0$	-6

(24)

Lemma 5.1 applies to N^+, with the reduction for which $c=w_N$, and Lemma 5.2 applies to C^+. We choose the perturbations so small that all solutions have $\kappa > -\kappa_0$, which by (23) implies that $\kappa \ge 0$. The three parts of Proposition 3.3 then hold, with the following changes.

In (a), the moduli space is $M_{\kappa_0}(N, \alpha_0)$. In (b), the moduli spaces on C^+ in the second and third rows of (24) are empty, and the compact two-manifold is

$$M = M_{3/2-\kappa_0}(C, \alpha_0).$$

The proofs are the same. For the compactness of M and for part (c), we note that an anti-self-dual connection on $\mathbb{R} \times Y_n$ which runs between α_0 and α_1 has action at least $1/2$, while a bubble, or a non-trivial anti-self-dual connection on the cylinder with the same limit at both ends, has action at least 1. So the only moduli spaces on C^+ which could arise from weak limits are the ones in the table.

The hypotheses of Theorem 5.3 are therefore satisfied, and Proposition 4.1 holds in the present situation. The argument of Section 4 then gives a contradiction, as before, and Proposition 6.1 is proved.

7. The remaining cases

In this section we complete the proof of Theorem 1.1. Throughout, R is a connected subdiagram of the Dynkin diagram E_8, and N is a submanifold of a K3 surface X which satisfies the hypotheses of the theorem: it is negative definite, its oriented boundary is Y_R, and $H_1(N; \mathbb{Z}) = 0$. We write C for the closure of the complement of N in X.

Proposition 7.1. *The group $H_2(N; \mathbb{Z})$ is free, and $H_3(N; \mathbb{Z})$ is zero. The intersection form Q_N either is isomorphic to $-R$, or is one of the four forms listed in (1). The group $H_1(C; \mathbb{Z})$ is zero, except possibly when $R = A_7$, in which case it is either zero or $\mathbb{Z}/2$.*

Proof. The complement C is connected, and both N and C are spin. As in the proof of Proposition 2.2, for $V = N$ or C the group $H_2(V; \mathbb{Z})$ is free, the group $H_1(V; \mathbb{Z})$ is finite, and

$$|H_1(Y_R; \mathbb{Z})| = |\det Q_V| \, |H_1(V; \mathbb{Z})|^2. \tag{25}$$

The boundary map $H_4(X; \mathbb{Z}) \to H_3(Y_R; \mathbb{Z})$ in the Mayer–Vietoris sequence is an isomorphism, so $H_3(N; \mathbb{Z})$ and $H_3(C; \mathbb{Z})$ are zero. From (25) we see that $L = -Q_N$ is an even, positive-definite lattice whose determinant is $|H_1(Y_R; \mathbb{Z})| = \det R$. Let r be its rank.

Correction terms. The spin^c structures on Y_R are indexed by the elements λ of R^*/R, in such a way that the spin structures correspond to the elements with $2\lambda = 0$, and the formula of [19, Corollary 1.5] for the boundary of a negative-definite plumbing gives

$$d(Y_R, \mathfrak{s}_\lambda) = \frac{\operatorname{rk} R}{4} - \min_{v \in R} |\lambda + v|^2. \tag{26}$$

From the minimal norms of the cosets of the root lattices in their duals, we obtain the correction terms of the spin structures:

(27)

R	$d(Y_R, \mathfrak{s})$, $\mathfrak{s}$ spin
A_n, n even	$n/4$
A_n, n odd	$n/4$, $-1/4$
D_n, n odd	$n/4$, $n/4 - 1$
D_n, n even	$n/4$, $n/4 - 1$, 0, 0
E_6	$3/2$
E_7	$7/4$, $1/4$
E_8	2

(For D_n the four cosets have minimal norms 0, 1, $n/4$ and $n/4$, and the last two have order two only when n is even. For A_n the coset corresponding to $j \in \mathbb{Z}/(n+1)$ has minimal norm $j(n+1-j)/(n+1)$, and it has order two when $j = (n+1)/2$.)

Let $\mathfrak{s}$ be the spin structure which Y_R inherits from N. Then

$$r \leq 4\, d(Y_R, \mathfrak{s}), \qquad r \equiv 4\, d(Y_R, \mathfrak{s}) \pmod 8. \tag{28}$$

The inequality is [18, Theorem 9.6], and the congruence follows from the grading shift of the map which the punctured manifold N induces on HF^∞. If $r = 0$, then N is a rational homology ball and $d(Y_R, \mathfrak{s}) = 0$.

The diagrams E_6, E_7 and E_8. From (27) and (28), the rank r is 6 for E_6, it is 1 or 7 for E_7, and it is 0 or 8 for E_8. In the last case, $r = 0$ is excluded because the correction term is not zero, and L is an even unimodular lattice of rank 8, so it is isomorphic to E_8. For E_6, the discriminant form of L is the negative of the discriminant form of A_2. So $L \oplus A_2$ has an even unimodular overlattice, which is isomorphic to E_8, and L is the orthogonal complement of a sublattice A_2 in E_8, which is E_6. The case of E_7 was treated in Proposition 2.2.

The diagrams D_n. The determinant of L is 4, by (25) and the hypothesis that $H_1(N;\mathbb{Z})$ is zero, so r is not zero. (For D_4 and D_6, the correction terms alone do not exclude $r = 0$.) From (27) and (28), we then have $r = 4$ for D_4, and

$$r \in \{1, 5\}, \qquad r \in \{2, 6\}, \qquad r \in \{3, 7\}$$

for D_5, D_6 and D_7 respectively. An even positive-definite lattice of determinant 4 is $\langle 4\rangle$ if its rank is 1, and is $A_1 \oplus A_1$ if its rank is 2. When $r \geq 3$, the discriminant form of L is determined by the linking form of Y_R and the value of r modulo 8, through Milgram's formula, and it is the negative of the discriminant form of the lattice L' in the following table.

r	3	4	5	6	7
L'	D_5	D_4	A_3	$A_1 \oplus A_1$	$\langle 4\rangle$
L	A_3	D_4	D_5	D_6	D_7

So $L \oplus L'$ has an even unimodular overlattice, isomorphic to E_8, in which L is the orthogonal complement of L'. The Weyl group of E_8 acts transitively on the sublattices of E_8 of each of the types A_3, D_4 and D_5, on the pairs of orthogonal roots, and on the vectors of norm 4. So the complement can be computed in the standard model, in which E_8 consists of the vectors x in $\mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8$ with $\sum x_i$ even. A sublattice D_k in the first k coordinates, where $D_3 = A_3$, has complement D_{8-k} in the remaining coordinates. The orthogonal roots $e_1 - e_2$ and $e_1 + e_2$ have complement D_6, and the vector $2e_1$ has complement D_7. This gives the last row of the table.

The diagrams A_n. When n is odd, the spin structure with correction term $-1/4$ cannot be the one inherited from N. So $d(Y_R, \mathfrak{s}) = n/4$, and from (28) and the fact that $n \leq 7$ we obtain $r = n$. To identify the lattice, let V be the disk bundle over the two-sphere with Euler number $-(n+1)$. Its boundary is $-Y_{A_n}$, so we can form the closed manifold $Z = N \cup V$. It is negative definite and has $H_1(Z;\mathbb{Z}) = 0$, so by Donaldson's theorem its intersection form is diagonalizable. Changing the sign, we identify $H_2(Z;\mathbb{Z})$ with the standard lattice $\mathbb{Z}^{n+1}$. Let $v \in \mathbb{Z}^{n+1}$ be the class of the zero section of V, so that $|v|^2 = n+1$. The vector v is primitive, because the quotient of $H_2(Z;\mathbb{Z})$ by $H_2(V;\mathbb{Z})$ is a subgroup of $H_2(N, Y_{A_n};\mathbb{Z})$, which is free. The lattice L is contained in the orthogonal complement of v, and it is equal to it because the two have the same rank and determinant.

Because v is primitive, the reduction of $v^\perp$ modulo 2 is the kernel of the linear form $u \mapsto v \cdot u$ on $(\mathbb{Z}/2)^{n+1}$. Because $v^\perp$ is even, the linear form $u \mapsto |u|^2$, which is not zero, vanishes on this kernel. So the two linear forms are equal, which means that every coordinate of v is odd. Since $|v|^2 = n+1$, every coordinate is ± 1, and $v^\perp$ is isomorphic to A_n.

The complement. By (25), the group $H_1(C;\mathbb{Z})$ is zero when $|H_1(Y_R;\mathbb{Z})|$ is square-free. When $|H_1(Y_R;\mathbb{Z})| = 4$, which is the case for A_3 and for D_n, the group $H_1(C;\mathbb{Z})$ could only have order 2, and Q_C would then be even and unimodular. But the signature of Q_C is $r - 16$, with $1 \le r \le 7$, and this is not divisible by 8. In the remaining case, $R = A_7$, the relation (25) allows $H_1(C;\mathbb{Z})$ to have order 1 or 2. □

Proof of Theorem 1.1. By Proposition 7.1, either Q_N is isomorphic to $-R$ or we are in one of the four cases of (1). The first of these is excluded by Theorem 2.1, in which N is the piece called W, and the other three are excluded by Proposition 6.1. □

For $R = D_5$ and $R = D_7$, as for $R = E_7$, the hypotheses of Theorem 1.1 can be removed.

Theorem 7.2. *Let n be 5 or 7, and suppose that $Y_n = Y_{D_n}$ is smoothly embedded in a K3 surface X. Then Y_n separates X into two pieces, $X = W \cup_{Y_n} C$, where W has oriented boundary Y_n and is negative definite. The groups $H_1(W;\mathbb{Z})$ and $H_1(C;\mathbb{Z})$ are zero, and the intersection form of W is isomorphic to $-D_n$.*

Proof. As in the proof of Proposition 2.2, the manifold Y_n is an L-space, and one of the two pieces, W, is negative definite. By (27), the correction terms of the two spin structures on Y_n are $n/4$ and $n/4 - 1$. Both are positive, so the inequality of [18, Theorem 9.6] shows that the oriented boundary of W is Y_n and that $b_2(W) \le n$. By (25), the group $H_1(W;\mathbb{Z})$ has order 1 or 2. If the order were 2, then Q_W would be even and unimodular, and its rank would be divisible by 8. The rank is at most 7, so W would be a rational homology ball. The correction term of the spin structure inherited from W would then be zero, which it is not. So $H_1(W;\mathbb{Z}) = 0$, and Theorem 1.1 shows that Q_W is isomorphic to $-D_n$. The group $H_1(C;\mathbb{Z})$ is zero by Proposition 7.1. □

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