Geometric Equivalence of the Riemann Hypothesis

Complex Extensions of SeqImpar and SeqPar,
Pole Correspondence, and Recovery of the Ordinates γ

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Abstract

We study the complex rational extensions of two sequences, called SeqImpar and SeqPar, and the way their geometric images relate to the nontrivial zeros of the Riemann zeta function. We prove that $I(z) = 5 - 4/(z-1)$ maps the critical line $\mathrm{Re}(z) = 1/2$ to the circle $|w - 9| = 4$, and that, for $w = I(\rho)$, rational inversion recovers exactly the coordinates $\mathrm{Re}(\rho)$ and $\gamma = \mathrm{Im}(\rho)$. We show that SeqPar induces a second rational curve and that its extension is injective in the half-plane $\mathrm{Re}(z) > 0$. To distinguish points corresponding to zeta zeros from the other points on these curves, we use a Dirichlet series weighted by the convergence errors of the sequences and prove the correspondence between its poles and the nontrivial zeros. This yields a geometric criterion rigorously equivalent to the Riemann Hypothesis: all relevant poles of the transformed function must lie on the indicated circle. We present formulas for extracting ordinates using residues and contour integrals, together with numerical checks involving the first five known zeros. The results constitute an exact reformulation of the hypothesis, not a proof of its truth or an independent algorithm for computing new zeros.

Keywords: SeqImpar; SeqPar; Riemann Hypothesis; zeta function; nontrivial zeros; complex geometry; Möbius transformations; meromorphic continuation.

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1 Introduction and logical scope

The Riemann Hypothesis (RH) states that every nontrivial zero $\rho = \beta + i\gamma$ of the zeta function $\zeta(s)$ satisfies $\beta = 1/2$. These zeros lie in the critical strip $0 < \mathrm{Re}(s) < 1$ and are symmetric with respect to the line $\mathrm{Re}(s) = 1/2$ [3]. This study develops the geometric interpretation of the sequences SeqImpar and SeqPar discussed in earlier manuscripts [1, 2].

The objective must be understood at three rigorously distinct levels:

- **Rational geometry:** map a point s to $I(s)$ or $P(s)$ and recover s by algebraic inversion;
- **Arithmetic selection:** characterize which of these points represent zeta zeros by means of a weighted series incorporating the von Mangoldt function;
- **Independent computation:** effectively locate previously unknown zeros with certified error using only a controlled evaluation of the transform—a problem not solved by the identities presented here.

The complex expressions below are natural extensions of the closed rational formulas. Values of a sequence at integer indices do not, without additional assumptions, determine a unique analytic extension; here the rational extension is fixed explicitly.

2 Complex extensions of the sequences

For $n \geq 3$, the closed formulas are

$$I_n = 5 - \frac{4}{n-1}, \qquad P_n = 5 - \frac{2}{n} - \frac{6}{n+1}. \tag{2.1}$$

We define the meromorphic extensions

$$I(s) = 5 - \frac{4}{s-1}, \qquad P(s) = 5 - \frac{2}{s} - \frac{6}{s+1}. \tag{2.2}$$

The first has a pole at $s = 1$; the second has poles at $s = 0$ and $s = -1$. In the critical strip $S = \{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1\}$, both are holomorphic. The ratio of their errors extends as

$$R(s) := \frac{5 - I(s)}{5 - P(s)} = \frac{2s(s+1)}{(s-1)(4s+1)}. \tag{2.3}$$

For $n \geq 3$, we have

$$R(n) = \frac{1}{2} + c(n), \qquad c(n) = \frac{7n+1}{2(n-1)(4n+1)} = O(n^{-1}). \tag{2.4}$$

The limit $R(n) \to 1/2$ is an algebraic fact about the sequence of weights; by itself, it does not imply the location of zeros on the critical line.

3 The critical line is transformed into a circle

Theorem 3.1 (Exact image of the critical line under SeqImpar). If $s = 1/2 + it$, with $t \in \mathbb{R}$, and $w = I(s)$, then

$$\boxed{|w - 9| = 4}. \tag{3.1}$$

The image of the line is the circle with center 9 and radius 4, excluding $w = 5$, which is approached only as $|t| \to \infty$.

Proof. Writing $D = t^2 + 1/4$ and rationalizing the denominator,

$$I\left(\frac{1}{2} + it\right) = 5 - \frac{4}{-1/2 + it} = 5 + \frac{2}{D} + i\frac{4t}{D}. \tag{3.2}$$

If $w = u + iv$, then $u = 5 + 2/D$ and $v = 4t/D$. Therefore,

$$\begin{aligned}(u-9)^2 + v^2 &= \left(-4 + \frac{2}{D}\right)^2 + \frac{16t^2}{D^2} \\ &= 16 - \frac{16}{D} + \frac{4 + 16t^2}{D^2} = 16.\end{aligned}$$

The transformation $I(s)$ is a Möbius transformation, bijective on the Riemann sphere; hence the critical line completed with the point at infinity is mapped onto the full circle. Since $I(s) \neq 5$ for finite s, $w = 5$ is only a limit point of the image of the ordinary line.

The half-plane with $\gamma > 0$ corresponds to the upper arc of the circle; as $\gamma \to +\infty$, the image converges to 5 from above. □

4 Inversion and exact formulas for β and γ

Proposition 4.1 (Complete recovery of the coordinates). If $s = \beta + i\gamma \in S$ and $w = I(s) = u + iv$, then

$$s = 1 + \frac{4}{5 - w}, \tag{4.1}$$

and consequently,

$$\beta = 1 + \frac{4(5-u)}{(5-u)^2 + v^2}, \qquad \gamma = \frac{4v}{(5-u)^2 + v^2}. \tag{4.2}$$

If $\beta = 1/2$, then additionally

$$\gamma = \frac{v}{2(u-5)} \qquad (w \neq 5). \tag{4.3}$$

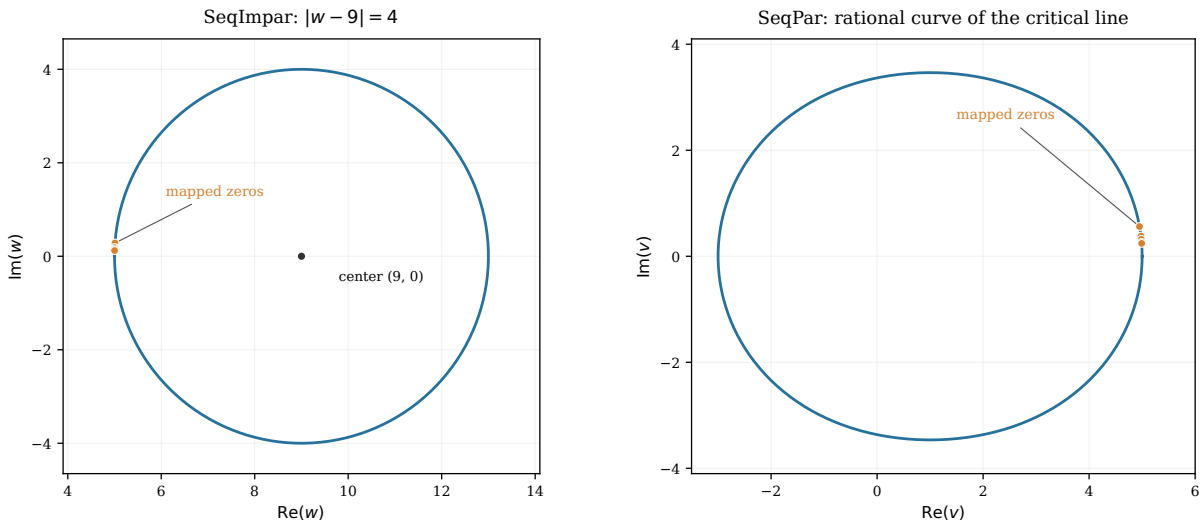


Figure 1: Images of the critical line: the circle under SeqImpar (left) and the rational curve under SeqPar (right). The five orange points use previously known zeros and do not represent independent discoveries.

Proof. The equation $w = 5 - 4/(s-1)$ is equivalent to $(s-1)(5-w) = 4$, which gives (4.1). Multiplying by $5 - \overline{w}$ and separating real and imaginary parts yields (4.2). For $\beta = 1/2$, the two components of (3.2) satisfy $v/(u-5) = 2\gamma$, proving (4.3). □

Here "exact" means that (4.2) is an algebraic identity for an already identified point w. By itself, it does not provide an independent closed-form expression for unknown values of γ.

Theorem 4.2 (Signed distance from the critical line). For $s \in S$ and $w = I(s)$, the identity

$$\boxed{\operatorname{Re}(s) - \frac{1}{2} = \frac{|w-9|^2 - 16}{2|w-5|^2}} \tag{4.4}$$

holds. Thus $\operatorname{Re}(s) = 1/2$ corresponds to $|w-9| = 4$; points with $\operatorname{Re}(s) > 1/2$ correspond to the exterior of the circle, and points with $\operatorname{Re}(s) < 1/2$ to its interior.

Proof. From the formula for β in (4.2) and $|w-5|^2 = (u-5)^2 + v^2$,

$$\begin{aligned} \beta - \frac{1}{2} &= \frac{|w-5|^2 + 8(5-u)}{2|w-5|^2} \\ &= \frac{(u-9)^2 + v^2 - 16}{2|w-5|^2}. \end{aligned}$$

The denominator is positive for $w = I(s)$, since $w \neq 5$. □

5 The complex curve produced by SeqPar

SeqPar also transforms the critical line into an explicit curve. For $s = 1/2 + it$,

$$\begin{aligned} P\left(\frac{1}{2} + it\right) = 5 &- \frac{1}{t^2 + \frac{1}{4}} - \frac{9}{t^2 + \frac{9}{4}} \\ &+ i\left(\frac{2t}{t^2 + \frac{1}{4}} + \frac{6t}{t^2 + \frac{9}{4}}\right). \end{aligned} \tag{5.1}$$

Call this curve $C_P = P(\{1/2 + it : t \in \mathbb{R}\})$.

Proposition 5.1 (Injectivity of SeqPar in the right half-plane). The function $P(s)$ is injective in $\operatorname{Re}(s) > 0$. Its inverse on this image is holomorphic and can be computed by selecting the root with positive real part of the quadratic equation

$$ds^2 + (d-8)s - 2 = 0, \qquad d = 5 - P(s). \tag{5.2}$$

In particular,

$$s = \frac{8 - d \pm \sqrt{d^2 - 8d + 64}}{2d}, \tag{5.3}$$

with the branch determined by $\operatorname{Re}(s) > 0$.

Proof. The identity $5 - P(s) = (8s+2)/(s(s+1))$ immediately yields (5.2). If s is one of its roots, the other is

$$s_* = -\frac{s+1}{4s+1}.$$

Writing $s = x + iy$ with $x > 0$,

$$\operatorname{Re}\left(\frac{s+1}{4s+1}\right) = \frac{(x+1)(4x+1) + 4y^2}{|4s+1|^2} > 0.$$

Thus $\operatorname{Re}(s_*) < 0$. Only one root can therefore lie in $\operatorname{Re}(s) > 0$, establishing injectivity.

Furthermore, $P'(s) = 2/s^2+6/(s+1)^2$ does not vanish in this region: its zeros are $(-1\pm i\sqrt{3})/4$. Therefore, the inverse is holomorphic on the image. □

Consequently, for an image $v = P(\rho)$, an exact expression for the ordinate is also obtained via SeqPar:

$$\gamma = \operatorname{Im}\left(\frac{8 - d + \epsilon\sqrt{d^2 - 8d + 64}}{2d}\right), \qquad d = 5 - v, \tag{5.4}$$

where $\epsilon \in \{-1, 1\}$ is chosen so that the root has positive real part. This condition selects a unique branch for ρ in the critical strip.

The two extensions are related without reference to zeta: if $d = 5 - I(s)$, then

$$P(s) = 5 - \frac{2d}{d+4} - \frac{3d}{d+2}. \tag{5.5}$$

Thus C_P is the rational image of the circle $|w - 9| = 4$ under the transformation (5.5). SeqPar provides an alternative geometry for the same critical line, although not generally a circle.

6 Selecting the zeta zeros among the points on the curve

The entire critical line, including points that are not zeros, satisfies (3.1). The geometric property therefore needs an analytic selector to distinguish points corresponding to zeros. For this purpose, we return to the construction of the weighted Dirichlet series [1].

Let $\Lambda(n)$ be the von Mangoldt function:

$$\Lambda(n) = \begin{cases} \log p, & n = p^k,\ p \text{ prime},\ k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

Initially, for $\operatorname{Re}(s) > 1$, define

$$F_\zeta(s) = \sum_{n=3}^{\infty} \frac{R(n)[\Lambda(n) - 1]}{n^s}, \qquad R(n) = \frac{5 - I_n}{5 - P_n}. \tag{6.2}$$

Theorem 6.1 (Meromorphic continuation and zero–pole correspondence). The function F_ζ admits a meromorphic continuation to $\operatorname{Re}(s) > 0$, given by

$$F_\zeta(s) = -\frac{1}{2}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{2}\zeta(s) + \frac{1}{2} + \frac{1 - \log 2}{2^{s+1}} + H_\zeta(s), \tag{6.3}$$

where

$$H_\zeta(s) = \sum_{n=3}^{\infty} \frac{c(n)[\Lambda(n) - 1]}{n^s} \tag{6.4}$$

is holomorphic for $\operatorname{Re}(s) > 0$. Its poles in the critical strip S are exactly the nontrivial zeros of $\zeta(s)$. If ρ has multiplicity m_ρ, then

$$\operatorname*{Res}_{s=\rho} F_\zeta(s) = -\frac{m_\rho}{2}. \tag{6.5}$$

Proof. Since $c(n) = O(1/n)$ and $|\Lambda(n) - 1| \leq \log n + 1$, the series (6.4) converges uniformly on every compact subset of $\operatorname{Re}(s) > 0$; hence H_ζ is holomorphic there. For $\operatorname{Re}(s) > 1$, the classical identities

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}, \qquad \zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

allow us to write

$$\sum_{n\geq 3}\frac{\Lambda(n)-1}{n^s}=-\frac{\zeta'(s)}{\zeta(s)}-\zeta(s)+1+\frac{1-\log 2}{2^s}.$$

Substitution of $R(n)=1/2+c(n)$ yields (6.3). At $s=1$, the residues of $-\zeta'/2\zeta$ and $-\zeta/2$ cancel. In the strip $0<\mathrm{Re}(s)<1$, zeta is holomorphic, and its zeros alone produce the poles of $-\zeta'/2\zeta$. If $\zeta(s)=(s-\rho)^{m_\rho}g(s)$ with $g(\rho)\neq 0$, then $\zeta'(s)/\zeta(s)=m_\rho/(s-\rho)+g'(s)/g(s)$, which gives (6.5). The residue is nonzero, so the correspondence holds in both directions. □

Remark 6.2. The arithmetic nature of the construction enters through $\Lambda(n)$. Values of the two rational sequences, without this information or an equivalent mechanism, do not select particular values of γ. The pole correspondence follows from a general mechanism: weights $a(n)=a+O(1/n)$ with $a\neq 0$ preserve the poles of $-a\zeta'/\zeta$ in the critical strip. The specific weight $R(n)$ is an exact realization of this principle, not an independent proof of RH.

7 Geometric equivalence of the Riemann Hypothesis

Define the open images of the critical strip:

$$\Omega_I=I(S),\qquad \Omega_P=P(S),\qquad S=\{s:0<\mathrm{Re}(s)<1\}. \tag{7.1}$$

On Ω_I, the inverse transformation $\Phi_I(w)=1+4/(5-w)$ is well-defined. For $v\in\Omega_P$, denote by $\Phi_P(v)$ the holomorphic branch of (5.3) whose real part is positive. Construct

$$G_I(w)=F_\zeta(\Phi_I(w)),\qquad G_P(v)=F_\zeta(\Phi_P(v)). \tag{7.2}$$

Theorem 7.1 (Geometric equivalence via SeqImpar). The following equivalence is rigorous:

$$\boxed{\text{RH}\quad\Longleftrightarrow\quad\begin{array}{c}\text{every pole of } G_I \text{ in } \Omega_I\\ \text{lies on the circle } |w-9|=4.\end{array}} \tag{7.3}$$

More precisely, poles of G_I inside the circle correspond to zeta zeros with $\beta<1/2$, whereas exterior poles correspond to zeros with $\beta>1/2$.

Proof. By Theorem 6.1, the poles of F_ζ in S are precisely the nontrivial zeros of zeta. Since I and Φ_I are biholomorphisms between S and Ω_I, the composition $G_I=F_\zeta\circ\Phi_I$ has poles exactly at $w_\rho=I(\rho)$. By Theorem 4.2, $|w_\rho-9|=4$ is equivalent to $\mathrm{Re}(\rho)=1/2$. Hence all zeros lie on the critical line if and only if all relevant poles lie on the circle. □

Corollary 7.2 (Geometric equivalence via SeqPar). With the curve C_P defined in (5.1),

$$\boxed{\text{RH}\quad\Longleftrightarrow\quad\text{every pole of } G_P \text{ in } \Omega_P \text{ lies on } C_P.} \tag{7.4}$$

Proof. The injectivity and the existence of a holomorphic inverse of P for $\mathrm{Re}(s)>0$ were proved in Proposition 5.1. The poles of G_P are the images $P(\rho)$ of the nontrivial zeros, and $P(\rho)\in C_P$ is equivalent to $\mathrm{Re}(\rho)=1/2$ by injectivity. □

The corresponding residue in the w coordinate varies with the point:

$$\operatorname*{Res}_{w=w_\rho}G_I(w)=-\frac{m_\rho(5-w_\rho)^2}{8}. \tag{7.5}$$

Indeed, $\Phi_I'(w)=4/(5-w)^2$, and the change-of-variable rule for residues gives (7.5). This observation matters for contour integrals taken directly in the w-plane.

8 Recovery of γ using poles and contour integrals

Proposition 8.1 (Recovery formula from an isolated pole). If $\Gamma_w \subset \Omega_I$ is a positively oriented closed contour enclosing exactly one pole w_ρ of G_I, with no poles on the curve, then

$$w_\rho = \frac{\oint_{\Gamma_w} wG_I(w)\,dw}{\oint_{\Gamma_w} G_I(w)\,dw}. \tag{8.1}$$

Consequently, its ordinate is

$$\gamma_\rho = \frac{4\,\mathrm{Im}(w_\rho)}{|5 - w_\rho|^2}. \tag{8.2}$$

Proof. By Theorem 7.1, the pole of G_I corresponds to a nontrivial zero ρ. The residues of wG_I and G_I at this point are $w_\rho a_\rho$ and $a_\rho \neq 0$, respectively. The residue theorem gives (8.1); (8.2) follows from (4.2). □

In the original s-plane, a contour Γ_s inside the critical strip, with no poles on the boundary, counts the zeros with multiplicity:

$$N_{\Gamma_s} = -\frac{1}{\pi i}\oint_{\Gamma_s} F_\zeta(s)\,ds. \tag{8.3}$$

If it encloses a single distinct zero, its position can be recovered directly from

$$\rho = \frac{\oint_{\Gamma_s} sF_\zeta(s)\,ds}{\oint_{\Gamma_s} F_\zeta(s)\,ds}. \tag{8.4}$$

These are exact characterization formulas. Their effective evaluation presupposes that the meromorphic continuation can be computed and controlled on the contour without circularly using the location of the very zero being sought.

9 Numerical experiment with the first five zeros

We take the first five known nontrivial zeros in the upper half-plane, $\rho_k = 1/2 + i\gamma_k$, with ordinates supplied by known mathematical tables [4, 3]. We evaluate the rational expressions (2.2) at each ρ_k. Table 1 records the rounded results.

Table 1: Known ordinates and complex images under the two sequences.

k	γ_k	$\mathrm{Re}\,I(\rho_k)$	$\mathrm{Im}\,I(\rho_k)$	$\mathrm{Re}\,P(\rho_k)$	$\mathrm{Im}\,P(\rho_k)$
1	14.134725141735	5.009997978	0.282637333	4.950455477	0.561077922
2	21.022039638772	5.004523084	0.190168914	4.977476226	0.379053413
3	25.010857580146	5.003195945	0.159866651	4.984066092	0.318969355
4	30.424876125860	5.002160007	0.131435868	4.989220917	0.262446800
5	32.935061587739	5.001843372	0.121423131	4.990798404	0.242511139

For the first zero,

$$\rho_1 \approx 0.5 + 14.134725141734694\,i,$$

we obtain

$$w_1 = I(\rho_1) \approx 5.009997977667446 + 0.282637332605110\,i.$$

The inverse formula on the critical line gives

$$\gamma_1 = \frac{\mathrm{Im}(w_1)}{2[\mathrm{Re}(w_1) - 5]} \approx 14.134725141734694. \tag{9.1}$$

Evaluating $P(\rho_1)$ gives $4.950455476919354 + 0.561077921550546\,i$. The two images agree with identity (5.5).

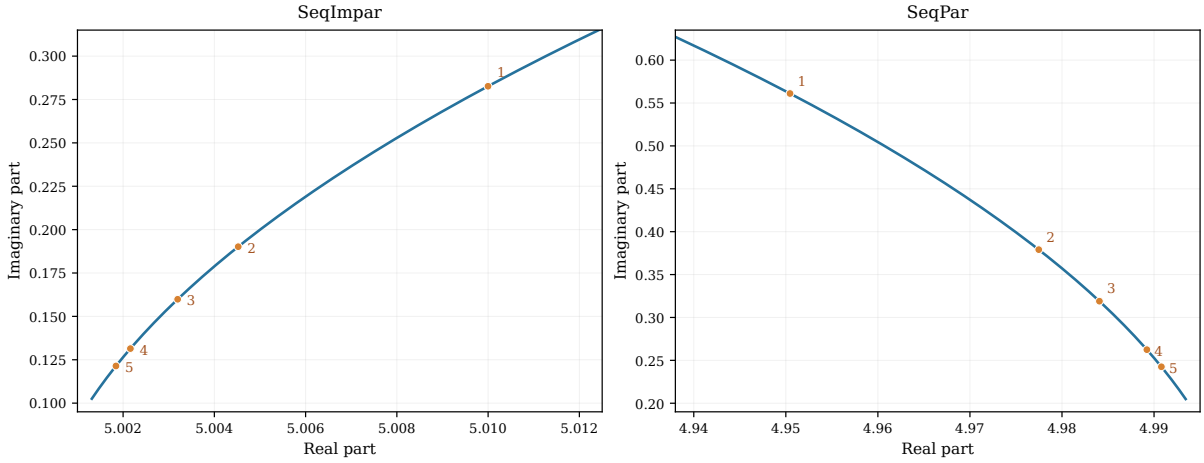


Figure 2: Magnified views of the regions near $5 + 0i$, where the images of zeros with increasing ordinates accumulate. The labels indicate $k = 1, \ldots, 5$.

Experimental interpretation. The numbers in Table 1 validate exact identities at previously known zeros. This test does not independently extract the ordinates. For large $|\gamma|$, $I(1/2+i\gamma)-5 \sim 4i/\gamma$, and numerical recovery requires additional precision because the points accumulate near $w = 5$.

10 Limiting detector of zeros on the critical line

The polar behavior of F_ζ suggests an analytic test. For $\varepsilon > 0$ and $t \in \mathbb{R}$, define

$$D_\varepsilon(t) = \varepsilon F_\zeta\left(\frac{1}{2} + \varepsilon + it\right). \tag{10.1}$$

If $\rho = 1/2 + i\gamma$ is a zero of multiplicity m, the local expansion

$$F_\zeta(s) = -\frac{m}{2(s-\rho)} + O(1)$$

implies, for fixed t,

$$\lim_{\varepsilon \to 0^+} D_\varepsilon(t) = \begin{cases} -m/2, & \zeta(1/2+it) = 0 \text{ of order } m, \\ 0, & \zeta(1/2+it) \neq 0. \end{cases} \tag{10.2}$$

Near $t = \gamma$, the singular contribution has magnitude

$$\left|F_\zeta\left(\frac{1}{2} + \varepsilon + it\right)\right| \approx \frac{m}{2\sqrt{\varepsilon^2 + (t-\gamma)^2}}, \tag{10.3}$$

which motivates detecting peaks as ε decreases. Approximation (10.3) refers to the *local singular part*; an additional holomorphic contribution remains present. The procedure detects zeros on the critical line but does not certify the absence of zeros off it. To work numerically without circularity, it requires an independent evaluation of the meromorphic continuation with explicitly controlled error.

11 Limitations, meaning, and research frontier

The results may be classified as follows:

1. **Proved:** the rational complex extensions (2.2) and their inverses on the indicated regions.
2. **Proved:** the circular image $|w - 9| = 4$ of the critical line under SeqImpar and the corresponding rational curve under SeqPar.
3. **Proved:** formulas (4.2), (4.4), and (8.2) for the coordinates and ordinates of identified poles/zeros.
4. **Proved:** the geometric equivalences (7.3) and (7.4), conditional only on the definitions and classical analytic identities used in the proof.
5. **Not proved:** the absence of poles of G_I outside the circle, or the location of all zeros by autonomous computation; this would be equivalent to RH itself.

The circular geometry results from the action of a Möbius transformation on a line; the mere existence of a circle does not constrain the zeros. The restriction arises when the special set of poles of the transform is carried through this change of coordinates. Likewise, decomposition using $\Lambda(n)$ recodes the logarithmic derivative of zeta and does not eliminate the original difficulty of meromorphic continuation and pole localization.

As a research direction, one may seek an evaluation of G_I or G_P using the specific coefficients of the sequences and prime data, with effective estimates for truncation and continuation. Substantive mathematical progress would require a new bound, proved without assuming RH, that restricts the location of the poles or yields certified approximations to their coordinates.

12 Conclusion

The sequences SeqImpar and SeqPar admit complex rational extensions whose structure allows the geometry of the critical line to be carried to explicit curves. SeqImpar maps this line exactly to the circle $|w - 9| = 4$ and supplies the inversion

$$\rho = 1 + \frac{4}{5 - w_\rho}, \qquad \gamma_\rho = \frac{4\,\mathrm{Im}(w_\rho)}{|5 - w_\rho|^2}.$$

Selecting the points w_ρ through the poles of the weighted series F_ζ yields the rigorous reformulation

$$\boxed{\text{RH} \iff \begin{array}{c}\text{all poles of } G_I \text{ associated with nontrivial zeros} \\ \text{lie on } |w-9|=4\end{array}}.$$

A parallel construction holds for SeqPar's curve C_P. These characterizations do not prove RH; they make the geometric equivalence explicit and locate the remaining difficulty in independently determining the poles of the transform.

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