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An unconditional estimate and the structure of the Bäcklund orbit

Anton Yu. Kalmykov *
Russia, Moscow
Alexander Timmermans
Holland, Utrecht

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Abstract

Let f be an arbitrary C^∞ solution of the sine-Gordon equation $f_{xy} = \sin f$ in the square $W = [-L, L]^2$, and let ω_n be obtained by n successive Bäcklund transformations with arbitrary real parameters $a_1, \dots, a_n > 0$ and arbitrary normalization constants c_j. For a solution ω let $r(\omega)$ be the associated smooth map of a surface of curvature $K = -1$ (Chebyshev net, common initial frame at the origin). We prove the unconditional estimate $\|r(\omega_n) - r((-1)^n f)\|_{L^\infty(W)} \le K_n \sum_{j=1}^n 1/a_j$, $K_n = 3L^2 + (1+L)\pi(2T_* + \pi)3^{n-1}$, $T_* = \sup_{|x|\le L} \int_{-L}^{L} |f_y|\, dy$, with no restrictions on the a_j, their ratios, the c_j, or L. The proof reduces the problem to a cascade of forced pendulum equations, controlled by an integral Lyapunov estimate and by cumulative estimates for the orthogonal Gauss-Weingarten propagator. For the Amsler seed $f = Y(xy)$ we show $T_* \le 2L^2$, which gives an explicit constant; the order $1/a$ is sharp. We also study the dependence on the data: smoothness along finite-dimensional families, continuity on the Fréchet data space, convergence of the immersions in C^0 and $W^{1,p}$ although the solutions ω_n themselves need not converge, partial limits along arbitrary index sets for $a_j \to \infty$ and for $a_j \to 0$, and a two-chamber structure of the orbit under Bäcklund chains.

Keywords: sine-Gordon equation; Bäcklund transformation; surfaces of constant negative curvature; Chebyshev nets; Amsler surfaces; Bianchi permutability theorem.
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*Independent researches, Corresponding author e-mail: anton@kalmykov.net

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1 Introduction

1.1 Historical context

The connection between surfaces of constant negative curvature in $\mathbb{R}^3$ and the sine-Gordon equation is one of the oldest topics in the geometric theory of integrable systems. If on a surface with $K = -1$ one chooses asymptotic coordinates in which both families of asymptotic lines are parametrized by arclength (a Chebyshev net), then the angle ω between them satisfies the equation $\omega_{xy} = \sin\omega$; conversely, every solution defines a smooth map whose image has $K = -1$ at the points where $\sin\omega \neq 0$ (see Remark 4.1). The classical corpus on pseudospherical coordinates and surfaces of constant curvature goes back to Darboux [3] and Eisenhart [4]; a modern exposition within soliton theory is in the monograph of Rogers and Schief [10].

The Bäcklund transformation—the machinery for generating new solutions of sine-Gordon (and, geometrically, new surfaces with $K = -1$) from a given one—was introduced by Bäcklund in his Lund paper of 1883 [1] as a geometric transformation of pseudospherical surfaces; the triply orthogonal variant was systematically studied by Bianchi [2]. The permutability property (the composition of two Bäcklund transformations with parameters a and b is independent of the order) is the Bianchi theorem; a modern exposition is in [10, Chs. 1–2]. The role of the parameter a as the "width of the transition layer" of the kink solution and the relation to multisoliton solutions are reflected in the classical works on sine-Gordon physics: the survey of Barone–Esposito–Magee–Scott [7], and Hirota's exact multisoliton solution [8]. The general program of "soliton surfaces"—immersions associated with spectral problems—was developed by Sym [9].

Separate mention should be made of surfaces of constant negative curvature with specific structure: the Kuen surface [5] and the family of Amsler surfaces [6]. The Amsler seed $f(x, y) = Y(xy)$, where Y solves $(uY')' = \sin Y$, occupies the position of a canonical example in the present work: its special structure (dependence only on xy) allows one to compute the constant T_* explicitly (Corollary 2 below).

1.2 Statement and result

Fix $n \geq 1$ real parameters $a_j > 0$ and constants $c_j \in \mathbb{R}$. Starting from the seed $\omega_0 = f$, we construct the chain $\omega_j = \mathrm{BT}_{a_j}(\omega_{j-1})$ (system (BT) below), normalized by the condition $\omega_j(0, 0) = c_j$. Each step changes the solution; the first question of the present text is: *how far can the immersion $r(\omega_n)$ move away from the immersion $r((-1)^n f)$ of the mirror seed if the parameters a_j are large?* The answer (Theorem 1) requires no assumptions except smoothness of the seed: the norm of the difference of immersions is majorized by a quantity linear in $\sum 1/a_j$, with an explicit constant independent of the normalization constants c_j and of the mutual ratios of the parameters, and with no restriction on the size of the window. It is essential that the constant is written in terms of the single integral characteristic of the seed $T_* = \sup_x \int |f_y|\, dy$—the total variation of its y-derivative along columns.

The proof splits into three independent blocks.

1. **Pendulum cascade.** Along each column $x = \mathrm{const}$, the difference of neighboring iterates is reduced to the chain of equations $\theta_j' = -a_j \sin\theta_j - F_j$ with known forcing F_j

(Lemmas 7, 9). No information about the location of solutions relative to the equilibria $\pi\mathbb{Z}$ is used: the only input is the integral Lyapunov estimate of Lemma 8, which asserts that for the equation $\theta' = -a\sin\theta - F$ the quantity $a \cdot \int |\sin\theta|$ is controlled by the constant π and the total integral of $|F|$.

2. **Orthogonal propagator.** Both immersions are constructed with the Gauss–Weingarten frame with a common initial frame; the difference of the frames satisfies a linear inhomogeneous equation with a skew-symmetric matrix, for which a cumulative (not Gronwall) norm estimate holds (Lemmas 4a–4b). Independence of the difference of immersions from the integration path (Lemma 4c) makes the choice of path a free parameter of the proof.

3. **Assembly.** The error on the axis (Lemmas 5–6, contribution $O(S \cdot L^2)$) and the error in the columns (Lemmas 10–11, contribution $\int d(\omega_n - \mathrm{ref}_n)\,dy$ with weight $(1 + L)$) add up to the final constant K_n.

The method is notable in that the estimate does not track the dynamics of the pendulums in phase (no count of transitions between the branches $2\pi\mathbb{Z}$): all phase data are reduced to the integral of $d(\omega_n - \mathrm{ref}_n)$, which is then majorized from above. This is precisely why the result is unconditional.

The second, complementary half of the paper (§11) studies the dependence of the immersions on the data, for which the Gronwall-type constants are allowed to degenerate at large a. *Smoothness.* The dependence on the data is C^∞: the Goursat solution map $\Psi\colon (\alpha, \beta) \mapsto f$ of the seed is a C^∞ bijection of the data space (Lemma 13), each Bäcklund step depends C^∞ on (a, c) and on the seed (Lemma 14), the immersion depends C^∞ on ω (Lemma 15), and their composition $(a, c, \alpha, \beta) \mapsto r(\omega_n)$ is C^∞ jointly in (x, y, p) into every C^k along every finite-dimensional smooth family of data (Theorem 17); along curves of data continuity follows from Corollary 18. Beyond the finite-dimensional statement, Φ_n is continuous—in fact locally Lipschitz in the C^k-seminorms, with a loss of $n+1$ derivatives—on the whole product $(0, \infty)^n \times \mathbb{R}^n \times \mathcal{D}$ with its Fréchet topology (Theorem 23). The total limit $a_j \to \infty$ of the immersions exists by Theorem 1—although the solutions themselves need not have a C^0 limit (Proposition 11.1)—with convergence in C^0 and every $W^{1,p}$ at sharp rates (Theorem 30), and, for the model kink, an explicit first-order limit off a boundary layer of width $O(1/a)$ (Proposition 11.3). In the coordinates $1/a_j$ the immersion extends to the closed orthant: partial limits $a_j \to \infty$ along arbitrary index sets, uniformly in the constants (Theorem 38), with a stratified Lipschitz structure at the faces (Corollary 41); the opposite limits $a_j \to 0$ are treated in the transposed chart (immersion r^σ), where no reflection occurs (Theorem 33, Corollary 40). The resulting global picture is a reflection chamber: the immersions of the chains of a fixed seed are covered by two alternating, mirror-symmetric, connected unions glued along the trailing faces $a_n = \infty$ (§11.9, Theorem 44; their disjointness is not decided), the "mirror" being a proper rotation (Corollary 28).

A companion interactive demonstration (numerical verification of the estimate for $n = 1, 2$ on the Amsler seed and on BT^2-chains) has been published in [11]; the numerical materials of the present article—in §12—are illustrations and are not used in the proof.

2 Statement and main result

Data. $L > 0$, $n \geq 1$; $f \in C^\infty(W)$ (here and below this means that all partial derivatives exist in the interior of W and extend continuously to the closed square; $C^k(W)$ is understood in the same way) is an *arbitrary* solution of $f_{xy} = \sin f$ in the window $W = [-L, L]^2$; the parameters $a_1, \dots, a_n > 0$ and the constants $c_1, \dots, c_n \in \mathbb{R}$ are *arbitrary*. Put

$$T_* := \sup_{|x| \leq L} \int_{-L}^{L} |f_y(x, y)|\, dy$$

(finite: f is smooth on the compact set W).

Chain. $\omega_0 := f$; ω_j is the solution of system (BT) with parameter a_j and seed ω_{j-1}, normalized by the condition $\omega_j(0, 0) = c_j$ (existence and uniqueness: Lemma 3):

$$(q + g)_x = -\frac{2}{a} \sin \frac{q - g}{2}, \qquad (q - g)_y = -2a \sin \frac{q + g}{2}. \tag{BT}$$

Reference. $\mathrm{ref}_n := (-1)^n f$. Denote $S := \sum_{j=1}^{n} 1/a_j$. The map $r(\omega)$ is from §4 (common initial frame at the origin).

Theorem 1.

$$\|r(\omega_n) - r(\mathrm{ref}_n)\|_{L^\infty(W)} \leq K_n \cdot S, \qquad K_n = 3L^2 + (1 + L) \cdot \pi(2T_* + \pi) \cdot 3^{n-1}.$$

Corollary 2 (Amsler seed)**.** *Let* $f(x, y) = Y(xy)$, $Y \in C^\infty(\mathbb{R})$, $(uY')' = \sin Y$, $Y(0) = \pi s$, $s \in (0, 1)$. *Then* $T_* \leq 2L^2$ *and*

$$\|r(\omega_n) - r(\mathrm{ref}_n)\|_{L^\infty(W)} \leq \left[3L^2 + (1 + L)\pi(4L^2 + \pi)3^{n-1}\right] \cdot S.$$

There are no conditions on the size of a_j, on their order and ratios, on the constants c_j, or on the size of the window. The constant depends only on (L, n, T_*); in Corollary 2 it depends only on (L, n).

Plan of the paper: §3—existence and uniqueness of the BT image; §4—construction of the immersion and its correctness; §5—estimate along the axis $y = 0$; §§6–7—pendulum cascade and integral estimates in the columns; §8—assembly; §9—completion of the proof of Theorem 1; §10—limitations of the result; §11—dependence on the seed and the parameters: smooth dependence, continuity on the data space, the total and partial limits, the infinitesimal regime; §11.9—the Bäcklund reflection chamber; §12—numerical illustrations (not part of the proof).

3 Preliminary facts

3.1 Elementary inequalities

Denote $d(t) := \mathrm{dist}(t, 2\pi\mathbb{Z})$, $d_\pi(t) := \mathrm{dist}(t, \pi\mathbb{Z})$.

$$d_\pi(t_1 + t_2) \leq d_\pi(t_1) + d_\pi(t_2). \tag{E1}$$

Proof. Choose $k_i \in \mathbb{Z}$ with $|t_i - k_i\pi| = d_\pi(t_i)$; then $(k_1+k_2)\pi \in \pi\mathbb{Z}$ and $|t_1+t_2-(k_1+k_2)\pi| \le d_\pi(t_1) + d_\pi(t_2)$. $\square$

$$d_\pi(t) \le (\pi/2) \cdot |\sin t|. \tag{E2}$$

Proof. Both sides are π-periodic and even, so it suffices to take $t \in [0, \pi/2]$, where $\sin t \ge 2t/\pi$ (concavity of sin). $\square$

$$|e^{i\alpha} - e^{i\beta}| \le d(\alpha - \beta). \tag{E3}$$

Proof. $|e^{i\alpha} - e^{i\beta}| = 2|\sin((\alpha-\beta)/2)|$ is even and 2π-periodic in $\alpha - \beta$, and for $\tau \in [0, \pi]$ we have $2\sin(\tau/2) \le \tau$. $\square$

$$d(2t) = 2 \cdot d_\pi(t). \tag{E4}$$

Proof. $d(2t) = \min_k |2t - 2\pi k| = 2 \cdot \min_k |t - \pi k|$. $\square$

3.2 Existence and uniqueness of the BT image

Lemma 3. *Let $g \in C^\infty(W)$ solve $g_{xy} = \sin g$, $a > 0$, $c \in \mathbb{R}$. Then on W there exists a unique $q \in C^\infty(W)$ satisfying (BT) and $q(0,0) = c$. It solves $q_{xy} = \sin q$.*

Proof. Put $u := (q+g)/2$, $w := (q-g)/2$. Then (BT) is equivalent to the system for a *single* function u:

$$u_x = A(x,y,u) := -\frac{1}{a}\sin(u-g), \qquad u_y = B(x,y,u) := g_y - a\sin u,$$

with $q = 2u - g$, $w = u - g$ (indeed, $w_y = u_y - g_y = -a\sin u \iff (q-g)_y = -2a\sin u$).

Compatibility. The partial derivatives for fixed u are:

$$A_y = \frac{1}{a}\cos(u-g)\, g_y, \quad A_u = -\frac{1}{a}\cos(u-g), \quad B_x = g_{xy} = \sin g, \quad B_u = -a\cos u.$$

Then

$$A_y + A_u B = \frac{1}{a}\cos(u-g) g_y - \frac{1}{a}\cos(u-g)(g_y - a\sin u) = \cos(u-g)\sin u,$$

$$B_x + B_u A = \sin g + \cos u \sin(u-g).$$

The difference equals

$$\cos(u-g)\sin u - \cos u \sin(u-g) - \sin g = \sin(u-(u-g)) - \sin g = 0.$$

Thus

$$A_y + A_u B = B_x + B_u A \tag{K}$$

identically.

Construction. A and B are smooth. On the compact set W we have the explicit bounds $|A| \le 1/a$ and $|A_u| \le 1/a$, so A is bounded and Lipschitz in u uniformly in (x,y); by the standard global existence-uniqueness theorem for ODEs, the equation $u_x = A(x,0,u)$ with

initial value $u(0,0) = (c + g(0,0))/2$ has a unique solution on the whole interval $x \in [-L, L]$. Then for each x solve $u_y = B(x, y, u)$ in $y \in [-L, L]$ with initial value $u(x, 0)$; here $|B_u| = a|\cos u| \le a$ and g_y is bounded on W, so B is globally Lipschitz in u and the solution exists on the whole interval. The solution depends smoothly on the parameter x, hence $u \in C^\infty(W)$.

Equality $u_x = A$ everywhere. Let $\Phi := u_x - A(x, y, u)$; $\Phi(x, 0) = 0$ by construction. Differentiating $u_y = B(x, y, u)$ with respect to x, we obtain $u_{xy} = B_x + B_u u_x$, while $\partial_y[A(x, y, u)] = A_y + A_u B$. Hence

$$\Phi_y = B_x + B_u u_x - A_y - A_u B = B_u \Phi + (B_x + B_u A - A_y - A_u B) = B_u \Phi$$

by (K). The linear homogeneous ODE with $\Phi(x, 0) = 0$ gives $\Phi \equiv 0$.

Uniqueness. Any solution of (BT) gives a solution of the system for u; by the uniqueness theorem for ODEs it coincides with the constructed one along the axis, and then along the columns.

q solves SG. From the system: $u_x = -(1/a)\sin w$, $w_y = -a \sin u$. Then

$$u_{xy} = -\frac{1}{a}\cos w\, w_y = \cos w \sin u, \qquad w_{yx} = -a \cos u\, u_x = \cos u \sin w,$$

whence $q_{xy} = u_{xy} + w_{xy} = \sin(u + w) = \sin q$. □

By induction, all $\omega_j \in C^\infty(W)$ are uniquely defined and solve SG.

4 The immersion map $r(\omega)$

For $\omega \in C^\infty(W)$, define the matrices (the rows of F are the vectors e_1, e_2, n):

$$P(p) = P_0 + pJ, \qquad P_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Q(\omega) = \begin{pmatrix} 0 & 0 & \sin\omega \\ 0 & 0 & -\cos\omega \\ -\sin\omega & \cos\omega & 0 \end{pmatrix}.$$

Definition of $r(\omega)$.

1. *Axis.* $F(\cdot, 0)$ solves $F_x = P(\omega_x(x, 0))F$, $F(0, 0) = I$; $r(x, 0) := \int_0^x e_1(\xi, 0)\, d\xi$.

2. *Columns.* For each x, the matrix $F(x, \cdot)$ solves $F_y = Q(\omega(x, y))F$ with initial value $F(x, 0)$ from step 1; $r(x, y) := r(x, 0) + \int_0^y (\cos\omega\, e_1 + \sin\omega\, e_2)\, dt$.

The frame at the origin is common for all ω: $F(0, 0) = I$, $r(0, 0) = 0$. **On the axis ω enters only through ω_x; in the columns, only through $\sin\omega$ and $\cos\omega$.**

Lemma 4a (norms)**.** *(a) $P(\cdot)$ and $Q(\cdot)$ are skew-symmetric, hence $F(x, y)$ is orthogonal and $\|F\|_{\mathrm{op}} = 1$. (b) The operator norm of a skew-symmetric 3×3 matrix equals the Euclidean length of its axis vector. Hence*

$$\|P(p)\| = \sqrt{1 + p^2}, \qquad \|Q(\omega)\| = 1,$$

$$\|P(p) - P(p')\| = |p - p'|, \qquad \|Q(\omega) - Q(\omega')\| = |e^{i\omega} - e^{i\omega'}| \le d(\omega - \omega')$$

by (E3).

Proof. (a) Since $P(p)$ and $Q(\omega)$ are skew-symmetric, the solution of $F' = MF$ with skew-symmetric M satisfies

$$(F^\top F)' = F^\top (M^\top + M)F = 0.$$

With $F(0) = I$ this gives $F^\top F = I$, so F is orthogonal and $\|F\|_{\mathrm{op}} = 1$.

(b) For a skew-symmetric 3×3 matrix $M = (m_{ij})$, define its axis vector $\alpha := (m_{32}, m_{13}, m_{21})$. A direct entry-by-entry verification gives $(Mv)_i = (\alpha \times v)_i$ for all $v \in \mathbb{R}^3$, hence $|Mv| \le |\alpha|\,|v|$ with equality when $v \perp \alpha$; therefore $\|M\|_{\mathrm{op}} = |\alpha|$. For $P_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ this gives $\alpha_{P_0} = (-1, 0, 0)$; for J it gives $\alpha_J = (0, 0, 1)$; for $Q(\omega)$ it gives $\alpha_{Q(\omega)} = (\cos\omega, \sin\omega, 0)$. Since $P(p) = P_0 + pJ$ has axis $\alpha_{P_0} + p\alpha_J = (-1, 0, p)$, we obtain

$$\|P(p)\| = \sqrt{1 + p^2}, \qquad \|Q(\omega)\| = 1.$$

The difference formula $\|P(p) - P(p')\| = |p - p'|$ follows from linearity in p, and $\|Q(\omega) - Q(\omega')\| = |e^{i\omega} - e^{i\omega'}| \le d(\omega - \omega')$ follows from (E3). □

Lemma 4b (orthogonal propagator)**.** *If $M(t)$ is skew-symmetric and $D' = M(t)D + R(t)$, then*

$$\|D(t)\| \le \|D(0)\| + \left| \int_0^t \|R(\tau)\|\, d\tau \right|.$$

Proof. For a fixed vector v, put $u(t) = D(t)v$. Then

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \langle u, Mu\rangle + \langle u, Rv\rangle = \langle u, Rv\rangle, \qquad \left|\frac{1}{2}\frac{d}{dt}\|u\|^2\right| \le \|u\|\,\|R\|\,\|v\|,$$

because $\langle u, Mu\rangle = 0$ (this gives a two-sided bound, valid for both signs of t). Hence, for $\epsilon > 0$ and the smooth positive function $\nu_\epsilon := (\|u\|^2 + \epsilon^2)^{1/2}$ (so that no differentiability issue arises at zeros of u),

$$|\nu_\epsilon'| = \left|\frac{\langle u, u'\rangle}{\nu_\epsilon}\right| \le \|R\|\,\|v\|.$$

Integrating gives $\nu_\epsilon(t) \le \nu_\epsilon(0) + \|v\|\left|\int_0^t \|R(\tau)\|\, d\tau\right|$, and letting $\epsilon \to 0$,

$$\|D(t)v\| \le \left(\|D(0)\| + \left|\int_0^t \|R(\tau)\|\, d\tau\right|\right)\|v\|.$$

Taking the supremum over $\|v\| = 1$ yields the claim. □

Lemma 4c (path independence and geometric meaning)**.** *Let $\omega \in C^\infty(W)$ solve $\omega_{xy} = \sin\omega$. Then throughout W we have $F_x = P(\omega_x)F$ and $r_x = e_1$. Consequently, the pair (F, r) coincides with the solution of the Gauss–Weingarten system of the Chebyshev net*

$$r_x = e_1, \qquad r_y = \cos\omega\, e_1 + \sin\omega\, e_2,$$

$$e_{1x} = -\omega_x e_2, \qquad e_{2x} = \omega_x e_1 + n, \qquad n_x = -e_2,$$

$$e_{1y} = \sin\omega\, n, \qquad e_{2y} = -\cos\omega\, n, \qquad n_y = -\sin\omega\, e_1 + \cos\omega\, e_2,$$

and ***does not depend on the integration path****.*

Proof. Zero-curvature condition. Since $P = P_0 + \omega_x J$, we have $P_y = \omega_{xy} J$. Since $Q = Q(\omega)$ depends on (x, y) only through ω, we have $Q_x = \omega_x\, Q'(\omega)$, where

$$Q'(\omega) := \partial_\omega Q(\omega) = \begin{pmatrix} 0 & 0 & \cos\omega \\ 0 & 0 & \sin\omega \\ -\cos\omega & -\sin\omega & 0 \end{pmatrix}.$$

A direct computation of commutators gives

$$[J, Q(\omega)] = Q'(\omega), \qquad [P_0, Q(\omega)] = \begin{pmatrix} 0 & \sin\omega & 0 \\ -\sin\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$[P, Q] = [P_0, Q] + \omega_x [J, Q] = [P_0, Q] + \omega_x Q'(\omega),$$

so the terms containing ω_x in $-Q_x + [P, Q]$ cancel:

$$-Q_x + [P, Q] = [P_0, Q].$$

Therefore

$$P_y - Q_x + [P, Q] = \omega_{xy} J + [P_0, Q] = (\sin\omega - \omega_{xy}) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$F_x = PF$. Let $\Psi := F_x - P(\omega_x)F$. Then $\Psi(x, 0) = 0$ by definition. Differentiating $F_y = QF$ with respect to x:

$$\Psi_y = Q\Psi + (Q_x + QP - P_y - PQ)F = Q\Psi$$

by the zero-curvature condition. The linear homogeneous ODE with $\Psi(x, 0) = 0$ gives $\Psi \equiv 0$.

$r_x = e_1$. Let $\Xi := r_x - e_1$, $\Xi(x, 0) = 0$. Then

$$\Xi_y = \partial_x(\cos\omega\, e_1 + \sin\omega\, e_2) - e_{1y}.$$

Substituting $e_{1x} = -\omega_x e_2$, $e_{2x} = \omega_x e_1 + n$, we obtain

$$\partial_x(\cos\omega\, e_1 + \sin\omega\, e_2) = \sin\omega\, n = e_{1y}.$$

Thus $\Xi_y = 0$ and $\Xi \equiv 0$.

Now F and r satisfy both systems $F_x = PF$, $F_y = QF$, $r_x = e_1$, $r_y = \cos\omega\, e_1 + \sin\omega\, e_2$ throughout W; the solution of such a compatible system with prescribed value at a point is unique, and one may integrate along any path. □

Lemma 4c applies both to ω_n and to ref_n, because both solve SG ($-f$ solves SG since sin is odd). Therefore the scheme "axis, then columns" is a device of the proof, not a choice of path in the definition of the surface.

Remark 4.1 (Regularity)**.** $r_x \times r_y = \sin\omega\, n$, therefore $r(\omega)$ is an immersion exactly where $\sin\omega \neq 0$. For arbitrary c_j this is not guaranteed: $r(\omega)$ is a smooth map, possibly with singular points. The estimates of the theorem do not depend on this.

5 The axis $y = 0$

Lemma 5 (derivative of the difference on the axis)**.** *Let $\delta_j := \omega_j - (-1)^j f$. Then **throughout the window** W and for all $j \geq 1$,*

$$|\partial_x \delta_j| \leq 2 \sum_{k \leq j} \frac{1}{a_k}.$$

In particular, $|\partial_x(\omega_n - \mathrm{ref}_n)| \leq 2S$ in W.

Proof. The x-row of (BT) is

$$\partial_x(\omega_j + \omega_{j-1}) = -\frac{2}{a_j} \sin \frac{\omega_j - \omega_{j-1}}{2},$$

that is,

$$\partial_x \omega_j = -\partial_x \omega_{j-1} - \frac{2}{a_j} \sin(\cdot).$$

Subtracting $(-1)^j \partial_x f$, we obtain

$$\partial_x \delta_j = -\partial_x \delta_{j-1} - \frac{2}{a_j} \sin(\cdot).$$

Hence $|\partial_x \delta_j| \leq |\partial_x \delta_{j-1}| + 2/a_j$, while $\delta_0 = \omega_0 - f = 0$. □

Lemma 6 (estimate of the frame and position on the axis)**.** *Let F_n, F_{ref} be the matrices F for ω_n and ref_n, and $D := F_n - F_{\mathrm{ref}}$. Then for all $x \in [-L, L]$, with $S := \sum 1/a_j$,*

$$\|D(x,0)\| \leq 2S|x|, \qquad |r(\omega_n)(x,0) - r(\mathrm{ref}_n)(x,0)| \leq Sx^2.$$

Proof. On the axis,

$$D_x = P_n D + (P_n - P_{\mathrm{ref}}) F_{\mathrm{ref}}, \qquad D(0,0) = 0,$$

where $P_n = P(\partial_x \omega_n(x,0))$ is skew-symmetric. By Lemma 4b, Lemma 4a(b), and Lemma 5,

$$\|D(x,0)\| \leq \int_0^{|x|} |\partial_x(\omega_n - \mathrm{ref}_n)(\xi,0)|\, d\xi \leq 2S|x|.$$

Furthermore, $\partial_x \Delta r$ is the first row of D, and its length is at most $\|D\|$; hence

$$|\Delta r(x,0)| \leq \int_0^{|x|} 2S\xi \, d\xi = Sx^2.$$

□

The discrepancy of the **values** $\omega_n(x,0) - \mathrm{ref}_n(x,0)$ themselves (for instance, nonzero modulo 2π for arbitrary c_j along the whole axis) does not appear here: on the axis the map sees only ω_x.

6 Column: pendulum cascade and Lyapunov function

Fix $x \in [-L, L]$; all functions are functions of $y \in [-L, L]$, and the prime denotes ∂_y. Let $\varphi(y) := f(x, y)$ and $h_j(y) := \omega_j(x, y)$, $h_0 = \varphi$. The y-row of (BT) is

$$h_j' = h_{j-1}' - 2a_j \sin \frac{h_j + h_{j-1}}{2}.$$

Introduce $\hat{h}_j := (-1)^j h_j$ ($\hat{h}_0 = \varphi$) and

$$z_j := \frac{\hat{h}_j - \varphi}{2}, \qquad z_0 = 0, \qquad \theta_j := z_j - z_{j-1}.$$

Then $z_n = \sum_{k \le n} \theta_k$, and, since $\mathrm{ref}_n = (-1)^n f$,

$$\omega_n - \mathrm{ref}_n = (-1)^n (\hat{h}_n - \varphi) = (-1)^n 2 z_n. \tag{6.1}$$

Lemma 7 (cascade)**.** *For $j = 1, \dots, n$,*

$$(z_j + z_{j-1})' = -\varphi' - a_j \sin \theta_j,$$

and therefore

$$\theta_j' = -a_j \sin \theta_j - F_j, \qquad F_j := \varphi' + 2 z_{j-1}'.$$

Proof. From $h_j = (-1)^j \hat{h}_j$ and $h_{j-1} = -(-1)^j \hat{h}_{j-1}$,

$$h_j + h_{j-1} = (-1)^j (\hat{h}_j - \hat{h}_{j-1}), \qquad h_j - h_{j-1} = (-1)^j (\hat{h}_j + \hat{h}_{j-1}).$$

Substituting into the y-row:

$$(-1)^j (\hat{h}_j + \hat{h}_{j-1})' = -2a_j \sin\left(\frac{(-1)^j (\hat{h}_j - \hat{h}_{j-1})}{2} \right) = -2a_j (-1)^j \sin\left(\frac{\hat{h}_j - \hat{h}_{j-1}}{2} \right),$$

whence

$$(\hat{h}_j + \hat{h}_{j-1})' = -2a_j \sin\left(\frac{\hat{h}_j - \hat{h}_{j-1}}{2} \right).$$

Since $\hat{h}_j = \varphi + 2z_j$, we obtain

$$2\varphi' + 2(z_j + z_{j-1})' = -2a_j \sin(z_j - z_{j-1}),$$

which is the first equality. The second follows from $\theta_j' = (z_j + z_{j-1})' - 2z_{j-1}'$. □

Lemma 8 (integral Lyapunov estimate)**.** *Let J be a compact interval, $\theta \in C^1(J)$, $F \in C^0(J)$, $a > 0$, and $\theta' = -a \sin \theta - F$ on J. Then*

$$a \cdot \int_J |\sin \theta| \, dy \le \pi + \int_J |F| \, dy.$$

No assumptions are made on the size of F relative to a, on the initial angle, or on the orientation of J.

Proof. Let $G(t) := d(t) = \operatorname{dist}(t, 2\pi\mathbb{Z})$: it is a 1-Lipschitz function with values in $[0, \pi]$; outside the points $\pi\mathbb{Z}$ it is differentiable. The function $\Psi := G \circ \theta$ is Lipschitz, hence absolutely continuous and differentiable almost everywhere. Consider the set $E := \{y \in J : \theta(y) \in \pi\mathbb{Z}\}$.

- If $y \notin E$, then

$$\Psi'(y) = \operatorname{sgn}(\sin\theta) \cdot \theta' = -a|\sin\theta| - \operatorname{sgn}(\sin\theta) \cdot F.$$

- If $y \in E$ is a limit point of E, then by continuity and discreteness of $\pi\mathbb{Z}$ there exist $y_k \to y$ with $\theta(y_k) = \theta(y)$; hence $\theta'(y) = 0$, and from the equation $F(y) = -\theta'(y) - a\sin\theta(y) = 0$. Then Ψ is differentiable at y with $\Psi'(y) = 0$, because

$$|\Psi(y+h) - \Psi(y)| \le |\theta(y+h) - \theta(y)| = o(h).$$

  The expression $-a|\sin\theta| - \operatorname{sgn}(\sin\theta)F$ there is also zero (with $\operatorname{sgn} 0 := 0$).

- Isolated points of E form a countable set of measure zero.

Thus almost everywhere

$$\Psi' = -a|\sin\theta| - \sigma F, \qquad \sigma := \operatorname{sgn}(\sin\theta) \in \{-1, 0, 1\}.$$

Integrating over $J = [\alpha, \beta]$ (absolute continuity):

$$\Psi(\beta) - \Psi(\alpha) = -a\int |\sin\theta| - \int \sigma F,$$

whence

$$a\int |\sin\theta| = \Psi(\alpha) - \Psi(\beta) - \int \sigma F \le \pi + \int |F|,$$

since $0 \le \Psi \le \pi$. □

Meaning. $d(\theta)$ is a Lyapunov function decreasing at rate $a|\sin\theta|$ in the absence of forcing; being bounded by the number π, it bounds the total "time away from equilibria" in both directions in y at once.

7 Cascade estimates

For fixed x (integrals over $y \in [-L, L]$):

$$T := \int |\varphi'|\,dy \le T_*, \qquad V_k := \int |z_k'|\,dy, \qquad I_k := \int |\sin\theta_k|\,dy, \qquad V_0 = 0.$$

Lemma 9 (cascade inequalities)**.** *For $k = 1, \dots, n$: (I) $a_k I_k \le \pi + T + 2V_{k-1}$; (II) $V_k \le (3^k - 1)(T + \pi/2)$; (III) $a_k I_k \le 3^{k-1}(2T + \pi)$.*

Proof. (I) Lemma 8 for θ_k, $a = a_k$, and $F = F_k$ from Lemma 7; here

$$\int |F_k| \le \int |\varphi'| + 2\int |z'_{k-1}| = T + 2V_{k-1}.$$

(II) From Lemma 7:

$$z'_k = -z'_{k-1} - \varphi' - a_k \sin\theta_k,$$

therefore

$$V_k \le V_{k-1} + T + a_k I_k.$$

By Part(I),

$$V_k \le 3V_{k-1} + 2T + \pi.$$

Induction: $V_0 = 0$; if $V_{k-1} \le (3^{k-1} - 1)(T + \pi/2)$, then

$$V_k \le 3(3^{k-1} - 1)(T + \pi/2) + 2(T + \pi/2) = (3^k - 1)(T + \pi/2).$$

(III) By Parts (I) and (II):

$$a_k I_k \le \pi + T + (3^{k-1} - 1)(2T + \pi) = 3^{k-1}(2T + \pi) - T \le 3^{k-1}(2T + \pi).$$

□

The estimate depends neither on a_j, nor on the initial values $z_k(0)$, nor on x.

Lemma 10 (integral estimate on a column)**.** *For each $x \in [-L, L]$,*

$$\Lambda(x) := \int_{-L}^{L} d(\omega_n - \mathrm{ref}_n)(x, y)\, dy \le \pi(2T + \pi) \sum_{k \le n} \frac{3^{k-1}}{a_k} \le \pi(2T_* + \pi)3^{n-1}S.$$

Proof. By (6.1), the evenness of d, and (E4),

$$d(\omega_n - \mathrm{ref}_n) = d(2z_n) = 2d_\pi(z_n).$$

Since $z_n = \sum \theta_k$, by (E1) and (E2),

$$d_\pi(z_n) \le \sum_k d_\pi(\theta_k) \le \frac{\pi}{2} \sum_k |\sin\theta_k|.$$

Therefore

$$\Lambda \le \pi \sum_k I_k \le \pi(2T + \pi) \sum_k \frac{3^{k-1}}{a_k}$$

by Lemma 9(III); it remains to use $T \le T_*$ and $3^{k-1} \le 3^{n-1}$. □

8 Assembly: error of the map

Lemma 11. *Let $\Lambda_y(x) := \int d(\omega_n - \mathrm{ref}_n)(x,t)\,dt$ over t between 0 and y (so that $\Lambda_y(x) \le \Lambda(x)$). Then for all $(x,y) \in W$,*

$$|r(\omega_n)(x,y) - r(\mathrm{ref}_n)(x,y)| \le Sx^2 + 2S|x||y| + (1+|y|)\Lambda_y(x).$$

Proof. Consider $y \ge 0$ (for $y < 0$ the argument is mirrored). In the column x, denote $D(y) := F_n(x,y) - F_{\mathrm{ref}}(x,y)$ and $d(y) := d(\omega_n - \mathrm{ref}_n)(x,y)$.

Frame. We have

$$D' = Q(\omega_n)D + (Q(\omega_n) - Q(\mathrm{ref}_n))F_{\mathrm{ref}},$$

the matrix $Q(\omega_n)$ is skew-symmetric, and by Lemma 4a(b) and the orthogonality of F_{ref},

$$\|(Q(\omega_n) - Q(\mathrm{ref}_n))F_{\mathrm{ref}}\| \le d(y).$$

Lemma 4b and Lemma 6 give

$$\|D(y)\| \le \|D(0)\| + \int_0^y d(t)\,dt \le 2S|x| + \Lambda_y(x).$$

Position. Write $\Delta r_y = r(\omega_n)_y - r(\mathrm{ref}_n)_y$ and split

$$\begin{aligned}\Delta r_y &= \big(\cos\omega_n\, e_{1,n} + \sin\omega_n\, e_{2,n}\big) - \big(\cos\omega_{\mathrm{ref}}\, e_{1,\mathrm{ref}} + \sin\omega_{\mathrm{ref}}\, e_{2,\mathrm{ref}}\big)\\ &= (\cos\omega_n - \cos\omega_{\mathrm{ref}})\, e_{1,n} + (\sin\omega_n - \sin\omega_{\mathrm{ref}})\, e_{2,n}\\ &\quad + \cos\omega_{\mathrm{ref}}\,(e_{1,n} - e_{1,\mathrm{ref}}) + \sin\omega_{\mathrm{ref}}\,(e_{2,n} - e_{2,\mathrm{ref}}),\end{aligned}$$

where $e_{i,n} := e_i(\omega_n)$, $e_{i,\mathrm{ref}} := e_i(\mathrm{ref}_n)$. The first two terms, expanded in the orthonormal pair $(e_{1,n}, e_{2,n})$, have norm

$$\big((\cos\omega_n - \cos\omega_{\mathrm{ref}})^2 + (\sin\omega_n - \sin\omega_{\mathrm{ref}})^2\big)^{1/2} = |e^{i\omega_n} - e^{i\omega_{\mathrm{ref}}}| \le d(y)$$

by (E3). The last two terms equal the row $(\cos\omega_{\mathrm{ref}}, \sin\omega_{\mathrm{ref}}, 0)$ acting on the matrix $D(y) = F_n(x,y) - F_{\mathrm{ref}}(x,y)$, whose norm is at most $|(\cos\omega_{\mathrm{ref}}, \sin\omega_{\mathrm{ref}}, 0)| \cdot \|D(y)\| = \|D(y)\|$. Hence $|\Delta r_y| \le d(y) + \|D(y)\|$.

Conclusion. Integrating over $t \in [0,y]$ and using Lemma 6 for $|\Delta r(x,0)|$:

$$|\Delta r(x,y)| \le |\Delta r(x,0)| + \int_0^y (d + \|D\|)\,dt \le Sx^2 + \Lambda_y + 2S|x|y + \int_0^y \Lambda_t\,dt \le Sx^2 + 2S|x||y| + (1+|y|)\Lambda_y(x),$$

since Λ_t is nondecreasing in t, and therefore $\int_0^y \Lambda_t\,dt \le y\Lambda_y$. □

9 Proofs of Theorem 1 and Corollary 2

Theorem 1. For $|x|, |y| \le L$, we have

$$Sx^2 + 2S|x||y| \le 3L^2 S,$$

and, by Lemma 10,

$$(1+|y|)\Lambda_y(x) \le (1+L)\Lambda(x) \le (1+L)\pi(2T_*+\pi)3^{n-1}S.$$

Substituting into Lemma 11 and taking the supremum over W:

$$\|r(\omega_n) - r(\mathrm{ref}_n)\|_{L^\infty(W)} \le [3L^2 + (1+L)\pi(2T_*+\pi)3^{n-1}]S = K_n S.$$

□

There is no case distinction according to the size of S: Lemmas 6, 10, and 11 hold for all data, and no Gronwall estimates are used in them.

Corollary 2 (Amsler). Let $Y \in C^\infty(\mathbb{R})$, $(uY')' = \sin Y$, $Y(0) = \pi s$.

- $|Y'| \le 1$. Since $Y \in C^\infty(\mathbb{R})$, Y' is bounded in a neighborhood of 0, so $uY'(u) \to 0$ as $u \to 0$. Integrating $(uY')' = \sin Y$ from 0 to u, we obtain

  $$uY'(u) = \int_0^u \sin Y,$$

  whence $|uY'(u)| \le |u|$, and $|Y'| \le 1$.

- $f(x,y) := Y(xy)$ *solves SG.* $\partial_x f = yY'(xy)$, and ∂_y of this expression equals

  $$Y'(u) + uY''(u) = (uY')' = \sin Y(u), \qquad u = xy.$$

- $T_* \le 2L^2$. $|\partial_y f| = |x| \cdot |Y'(xy)| \le L$, and the length of the integration interval is $2L$.

Applying Theorem 1 with this f:

$$\|\Delta r\| \le [3L^2 + (1+L)\pi(4L^2+\pi)3^{n-1}]S.$$

□

10 Remarks

Remark 10.1 (Sharpness in the sup-norm and integral form)**.** $\sup d(\omega_n - \mathrm{ref}_n)$ can reach π in thin layers if the initial data of the columns are far from the reference; the theorem needs only the integral $\Lambda(x) = \int d\,dy$, and it is small regardless of the number and location of such layers. The passage from the pointwise norm to the integral is the essence of the method.

Remark 10.2 (Axial constants)**.** The difference of the values $\omega_n(x,0) - \mathrm{ref}_n(x,0)$, generated by the constants c_j, is not seen by the map on the axis (Lemma 6: only ω_x occurs there) and enters the columns as the initial data of Lemma 8, whose contribution to the right-hand side does not exceed π, that is, $O(1/a)$ in the final normalization. Therefore the constants c_j are absent from the estimate.

Remark 10.3 (Choice of the integration path as optimal control)**.** The difference Δr does not depend on the path (Lemma 4c)—the path is a free parameter *of the proof*: it determines only the tightness of the upper bound. For a monotone path γ, the bound functional decomposes as

$$F[\gamma] = Sx^2 \text{ (independent of the path)} + \int 2S\,x(t)\dot{y}\,dt + \int D_d(t)(|\dot{x}| + |\dot{y}|)\,dt + \int d\,\dot{y}\,dt,$$

where $D_d(t) = \int d\,ds$ is the accumulated d-contribution. For seeds for which $T(x) = \int |f_y(x,\cdot)|$ is minimal on the vertical (for example, for the Amsler seed $T(0) = 0$, since $f_y(0,y) = xY'(xy)|_{x=0} = 0$), the optimal path is $(0,0) \to (0,y) \to (x,y)$; it gives

$$|\Delta r(x,y)| \le Sx^2 + (1 + |x| + |y|)\Lambda(0)$$

and the estimate $K_n' = L^2 + (1+2L)\pi^2 3^{n-1}$ (for $\Lambda(0) \le \pi^2 3^{n-1} S$). *Proof.* By Lemma 4c, $F_x = P(\omega_x)F$ holds throughout W for both ω_n and ref_n. On $x = 0$ we have $D(0,0) = 0$, so Lemma 4b gives $\|D(0,t)\| \le \Lambda(0)$ and, as in the proof of Lemma 11, $|\Delta r(0,y)| \le \int_0^{|y|}(d + \|D\|)\,dt \le (1+|y|)\Lambda(0)$. Along the horizontal segment at height y, $D_x = P_n D + (P_n - P_{\mathrm{ref}})F_{\mathrm{ref}}$ and Lemma 5 (valid throughout W) give $\|D(x,y)\| \le \|D(0,y)\| + 2S|x| \le \Lambda(0) + 2S|x|$; since $\partial_x \Delta r$ is the first row of D, $|\Delta r(x,y)| \le |\Delta r(0,y)| + |x|\Lambda(0) + Sx^2$. Finally $\Lambda(0) \le \pi^2 3^{n-1} S$ by Lemma 10 with $T(0) = 0$. The base 3^{n-1} does not depend on the choice of path—this is a structural limitation of the cascade technique (Lemma 9(II)). For a general seed, minimization of $F[\gamma]$ is a substantial open problem (a possible formulation is in terms of Hamilton–Jacobi–Bellman for a functional with memory).

Remark 10.4 (Limitations of the result)**.** (a) The constant 3^{n-1} has not been optimized; uniformity in n is not claimed. However, the order $1/a$ is sharp for every fixed n: Corollary 26 exhibits a one-parameter family ($f \equiv 0$, $c_j = \pi$, $a_j = s^{-j}$) along which the immersion error is bounded below by $c_n \sum_j 1/a_j$ with $c_n = c_0/4^{n-1} > 0$. The constant c_n is exponentially small in n, so this does not give uniformity in n.
(b) The immersions are normalized by a common frame at the origin; the estimate compares the maps under *the same* frame at the origin.
(c) $r(\omega)$ is an immersion only when $\sin\omega \neq 0$ (see Remark 4.1); the estimate remains valid through singular points.
(d) The window W is compact; the dependence on L is polynomial.
(e) The statement is real: for complex parameters a_j the geometry has a different character (poles of complex solutions appear), and the theorem is not formulated outside the real case.

11 Dependence on the seed and on the Bäcklund parameters: smoothness, limits, and the infinitesimal regime

Throughout this section $n \ge 1$ is fixed, $W = [-L,L]^2$, and

$$\mathcal{S} := \{f \in C^\infty(W) : \ f_{xy} = \sin f \text{ in } W\}.$$

Theorem 1 describes the limit of $r(\omega_n)$ as *all* parameters a_j tend to infinity, uniformly in the normalization constants c_j. Here we study the dependence of the immersions on the data systematically: smooth dependence on the seed and on (a_j, c_j) at finite parameters (§11.2); the total limit $a_j \to \infty$ and its first-order (infinitesimal) structure (§11.4); one-step asymptotics outside the boundary layer (§11.6); the orientation-reversal symmetry and the optimal topology of the total limit (§11.5); the remaining limits—in the constants c_j (§11.7) and in $a_j \to 0$ (§11.7.2); partial limits along arbitrary index sets and the closed orthant (§11.8), including the partial limits $a_j \to 0$ (Corollary 40) and the stratified Lipschitz structure at the faces (Corollary 41); and unbounded seed data (§11.7). As everywhere in the paper, all statements are analytic; the numerical illustrations of §12 are not used.

Convention. Maps defined on the Fréchet spaces $C^\infty(W)$, $\mathcal{D}$, $\mathcal{S}$ are called C^∞ if their compositions with all C^∞ finite-dimensional families are C^∞ (jointly in (x, y, p)); continuity is understood in the Fréchet topology. Banach-space statements (implicit function theorem, Gronwall) are applied to the C^k-norms, $k < \infty$.

11.1 The Goursat parametrization of the seed

Every $f \in \mathcal{S}$ is determined by its *characteristic data*

$$\alpha(x) := f(x, 0), \qquad \beta(y) := f(0, y), \qquad \alpha(0) = \beta(0),$$

and conversely, arbitrary $\alpha, \beta \in C^\infty([-L, L])$ with $\alpha(0) = \beta(0)$ determine a unique element of $\mathcal{S}$ (this is the Goursat problem for the sine-Gordon equation; Lemma 13 below). We therefore introduce the data space

$$\mathcal{D} := \{(\alpha, \beta) \in C^\infty([-L, L])^2 : \ \alpha(0) = \beta(0)\}$$

(a closed affine subspace of the Fréchet space $C^\infty([-L, L])^2$) and the *solution map*

$$\Psi \colon \mathcal{D} \to \mathcal{S}, \qquad \Psi(\alpha, \beta) := f.$$

By "a C^∞ *family of seeds*" $p \mapsto g_p$ (p in an open set $P \subset \mathbb{R}^m$) we mean a map such that $g_p \in \mathcal{S}$ for all p and all parameter derivatives exist and are continuous as maps into $C^k(W)$ for every k; this is equivalent to joint smoothness of $(x, y, p) \mapsto g_p(x, y)$.

Lemma 12 (smooth dependence for first-order systems)**.** *Let $A, B \colon W \times \mathbb{R} \times P \to \mathbb{R}$ be C^∞ jointly ($P \subset \mathbb{R}^m$ open), let $u_0 \colon P \to \mathbb{R}$ be C^∞, and assume that for each p in a neighbourhood of p_0 the system*

$$u_x = A(x, y, u, p), \qquad u_y = B(x, y, u, p), \qquad u(0, 0) = u_0(p)$$

has a solution $u(\cdot, \cdot\,; p) \in C^\infty(W)$. (This holds, for instance, if A, B satisfy the compatibility identity $A_y + A_u B = B_x + B_u A$ and $|A|$, $|A_u|$, $|B_u|$ are bounded on $W \times \mathbb{R} \times \{|p - p_0| \le 1\}$: then the axis ODE in x and the column ODEs in y have global solutions, and the argument of Lemma 3 ($\Phi := u_x - A$ solves $\Phi_y = B_u \Phi$, $\Phi(x, 0) = 0$) gives $u_x = A$ on all of W. For the systems of Lemma 14 these conditions hold.) Then $(x, y, p) \mapsto u(x, y; p)$ is C^∞ jointly;

in particular $p \mapsto u(\cdot,\cdot\,;p)$ is a C^∞ family in the sense above. For any direction $\dot p$, the first variation $U := \partial_p u \cdot \dot p$ solves the linear system

$$U_x = A_u\, U + A_p \dot p, \qquad U_y = B_u\, U + B_p \dot p, \qquad U(0,0) = u_0'(p)\dot p,$$

and higher variations are obtained by differentiating this linear system.

Proof. Along the axis $y = 0$ the function $u(\cdot, 0; p)$ solves the ODE $u_x = A(x, 0, u, p)$ with initial value $u_0(p)$; the right-hand side is C^∞ in (x, u, p), so the classical theorem on smooth dependence of ODE solutions on parameters and data gives $u(x, 0; p)$ jointly C^∞ in (x, p). For fixed x, the column function $u(x, \cdot\,; p)$ solves $u_y = B(x, y, u, p)$ with initial value $u(x, 0; p)$, now with the pair (x, p) as parameter; the same classical theorem gives $u(x, y; p)$ jointly C^∞ in (x, y, p). Differentiating the two ODEs in p yields the stated linear system for U (on the axis first, then in the columns), and iteration yields all higher variations. □

Lemma 13 (smooth dependence of the seed on the characteristic data)**.** *The map $\Psi\colon \mathcal{D} \to \mathcal{S}$ is a well-defined bijection and is C^∞ in the C^∞-topologies. The first variation $h := \delta f$ in the direction $(\delta\alpha, \delta\beta)$ solves the linear Goursat problem*

$$h_{xy} = \cos f \cdot h, \qquad h(x, 0) = \delta\alpha(x), \qquad h(0, y) = \delta\beta(y), \tag{1}$$

and higher variations are obtained by differentiating (1).

Proof. Step 1: a Volterra estimate (no smallness of L is needed). For $\phi, h \in C^0(W)$ with $|\phi| \le 1$ put $(\mathcal{V}_\phi h)(x, y) := \int_0^x \int_0^y \phi\, h\, dt\, ds$ (oriented integrals). By induction on m, $|(\mathcal{V}_\phi^m h)(x, y)| \le \frac{|xy|^m}{(m!)^2} \|h\|_{C^0(W)}$; the induction step is $\int_0^{|x|} \int_0^{|y|} \frac{(st)^{m-1}}{((m-1)!)^2}\, dt\, ds = \frac{|xy|^m}{(m!)^2}$. Hence $\|\mathcal{V}_\phi^m\| \le L^{2m}/(m!)^2$, the Neumann series converges, and $\mathrm{Id} - \mathcal{V}_\phi$ is invertible on $C^0(W)$ with $\|(\mathrm{Id} - \mathcal{V}_\phi)^{-1}\| \le \sum_m L^{2m}/(m!)^2 \le e^{2L}$, for every $L > 0$.

Step 2: existence, uniqueness and regularity on all of W. Put $h_0(x, y) := \alpha(x) + \beta(y) - \alpha(0)$ and $T(f) := h_0 + \int_0^x \int_0^y \sin f\, dt\, ds$ on $C^0(W)$. Since $|\sin a - \sin b| \le |a - b|$, the same induction gives $|(T^m f_1 - T^m f_2)(x, y)| \le \frac{|xy|^m}{(m!)^2} \|f_1 - f_2\|_{C^0(W)}$, so T^m is a contraction for m large and T has exactly one fixed point $f \in C^0(W)$; it satisfies $f(x, 0) = \alpha(x)$, $f(0, y) = \beta(y)$. *Regularity.* The operator $\iint\colon C^k(W) \to C^{k+1}(W)$ is bounded: for $i, j \ge 1$, $\partial_x^i \partial_y^j \iint g = \partial_x^{i-1} \partial_y^{j-1} g$, and $\partial_x^i \iint g = \int_0^y \partial_x^{i-1} g\, dt$ (similarly in y) exist and are continuous for $i \le k+1$. (In general $\iint g$ is *not* C^{k+2}.) From $f = h_0 + \iint \sin f$: $f \in C^k \Rightarrow f \in C^{k+1}$, hence $f \in C^\infty(W)$ and $f_{xy} = \sin f$, i.e. $f \in \mathcal{S}$. Conversely, if $f \in \mathcal{S}$ has characteristic data (α, β), integrating $f_{xy} = \sin f$ over the rectangle with corners $(0, 0)$, (x, y) gives $f = h_0 + \iint \sin f$, so f is that fixed point. Thus $\Psi\colon \mathcal{D} \to \mathcal{S}$ is a well-defined bijection.

Step 3: smooth dependence in C^0. Let $\Phi\colon C^0(W) \times C^0(W) \to C^0(W)$, $\Phi(f, h_0) := f - h_0 - \iint \sin f$. It is C^∞ (the Nemytskii operator sin is C^∞ on the Banach space C^0, and $\iint$ is bounded linear), and $D_f \Phi(f, h_0) = \mathrm{Id} - \mathcal{V}_{\cos f}$ is invertible by Step 1. By the implicit function theorem in Banach spaces and the global uniqueness of Step 2, $h_0 \mapsto f$ is a C^∞ map $C^0(W) \to C^0(W)$. The map $(\alpha, \beta) \mapsto h_0$ is linear and continuous from $\mathcal{D}$ into $C^k(W)$ for every k. (Smoothness on the Fréchet space $\mathcal{D}$ is understood, as throughout this section, along C^∞ finite-dimensional families $p \mapsto (\alpha_p, \beta_p)$; all maps below are compositions of continuous linear maps with C^∞ maps between Banach spaces, so this is automatic.)

Step 4: bootstrap in C^k. We claim: for every k, the map $(\alpha,\beta)\mapsto f(\alpha,\beta)$ is C^∞ into $C^k(W)$. For $k=0$ this is Step 3. If it holds for k, then $(\alpha,\beta)\mapsto \sin f$ is C^∞ into $C^k(W)$ ($\sin\colon C^k\to C^k$ is C^∞, $C^k(W)$ being a Banach algebra), $\iint\colon C^k\to C^{k+1}$ is bounded linear and h_0 is linear and continuous into $C^{k+1}(W)$; the identity $f=h_0+\iint\sin f$ gives the claim for $k+1$.

Step 5: variational equation. Differentiating $f=h_0+\iint\sin f$ in the direction $(\delta\alpha,\delta\beta)$ gives

$$h=\delta\alpha(x)+\delta\beta(y)-\delta\alpha(0)+\int_0^x\int_0^y \cos f\cdot h,$$

whence $h_{xy}=\cos f\cdot h$, $h(x,0)=\delta\alpha(x)$, $h(0,y)=\delta\beta(y)$. The linear problem has a unique solution by Step 1 ($\mathrm{Id}-\mathcal{V}_{\cos f}$ is invertible), and higher variations follow by iteration. □

11.2 Smooth dependence at finite parameters

Lemma 14 (smooth dependence of one Bäcklund step)**.** *Let $P\subset\mathbb{R}^m$ be open, let $a\colon P\to(0,\infty)$ and $c\colon P\to\mathbb{R}$ be C^∞, and let $p\mapsto g_p$ be a C^∞ family of seeds in $\mathcal{S}$. Let $q_p\in C^\infty(W)$ solve* (BT) *with parameter $a(p)$ and seed g_p, normalized by $q_p(0,0)=c(p)$ (existence and uniqueness: Lemma 3; note that the seed must solve the sine-Gordon equation, otherwise the construction of Lemma 3 is not available). Then $p\mapsto q_p$ is a C^∞ family of seeds. Writing q, g, a, c for the values at a fixed p and considering a direction $\dot p$ with $\dot a$, $\dot c$, $h:=\delta g$ (so that $h_{xy}=\cos g\cdot h$ by Lemma 13), the first variation $V:=\delta q$ solves:*

- *for the variation in a (with $\delta g=0$, $\delta c=0$):*

$$V_x=\frac{2}{a^2}\sin\frac{q-g}{2}-\frac{1}{a}\cos\frac{q-g}{2}\,V,\qquad V_y=-2\sin\frac{q+g}{2}-a\cos\frac{q+g}{2}\,V,\qquad V(0,0)=0;$$

- *for the variation in c (with $\delta g=0$, $\delta a=0$):*

$$V_x=-\frac{1}{a}\cos\frac{q-g}{2}\,V,\qquad V_y=-a\cos\frac{q+g}{2}\,V,\qquad V(0,0)=\delta c;$$

- *for the variation in the seed:*

$$V_x=-h_x-\frac{1}{a}\cos\frac{q-g}{2}\,(V-h),\qquad V_y=h_y-a\cos\frac{q+g}{2}\,(V+h),\qquad V(0,0)=0.$$

Higher variations are obtained by differentiating these linear systems.

Proof. By Lemma 3, $q_p=2u_p-g_p$, where u_p solves

$$u_x=-\frac{1}{a}\sin(u-g),\qquad u_y=g_y-a\sin u,\qquad u(0,0)=\frac{c+g(0,0)}{2}.$$

The right-hand sides are C^∞ in (x,y,u,p) jointly: in u and p through a,c, and in (x,y,p) through g_p and $g_{p,y}$, which are jointly C^∞ by the definition of a smooth family (and, for the seed itself, by Lemma 13). Lemma 12 then gives that u_p, hence $q_p=2u_p-g_p$, is a C^∞ family.

For the variation systems, let $U := \delta u$; then $V = 2U - h$ and, from $u_x = -(1/a)\sin(u-g)$, $u_y = g_y - a\sin u$: *variation in a* ($h = 0$, $U = V/2$):

$$U_x = \frac{1}{a^2}\sin(u-g) - \frac{1}{a}\cos(u-g)\,U, \qquad U_y = -\sin u - a\cos u\,U, \qquad U(0,0) = 0,$$

and multiplying by 2 gives the first displayed system (use $u - g = (q-g)/2$, $u = (q+g)/2$). *variation in c*: $U(0,0) = \delta c/2$ and $U_x = -(1/a)\cos(u-g)U$, $U_y = -a\cos u\,U$; since $V = 2U$, the second displayed system follows. *variation in the seed*: from $\delta(u_x) = -(1/a)\cos(u-g)(U-h)$ and $\delta(u_y) = h_y - a\cos u\,U$ we get

$$U_x = -\frac{1}{a}\cos\frac{q-g}{2}\,(U-h), \qquad U_y = h_y - a\cos\frac{q+g}{2}\,U,$$

and substituting $U = (V+h)/2$ gives $V_x = -h_x - \frac{1}{a}\cos\frac{q-g}{2}(V-h)$ and $V_y = h_y - a\cos\frac{q+g}{2}(V+h)$. □

Lemma 15 (smooth dependence of the immersion)**.** *For $\omega \in C^\infty(W)$ let $(F(\omega), r(\omega))$ be constructed as in §4 (axis, then columns). Then $\omega \mapsto (F(\omega), r(\omega))$ is C^∞ from $C^\infty(W)$ to $C^\infty(W, \mathrm{SO}(3)) \times C^\infty(W, \mathbb{R}^3)$ in the sense that it is C^∞ into every C^k. For a C^∞ family $t \mapsto \omega(t)$ with $\eta = \partial_t\omega$, the variations $D = \partial_t F$, $V = \partial_t r$ satisfy, on the axis,*

$$D_x = P(\omega_x)D + \eta_x\, J\, F, \qquad V_x = \textit{first row of } D,$$

and, in the columns,

$$D_y = Q(\omega)D + Q'(\omega)\eta F, \qquad V_y = -\eta\sin\omega\, e_1 + \eta\cos\omega\, e_2 + \cos\omega\, D_{1\cdot} + \sin\omega\, D_{2\cdot},$$

where $D_{i\cdot}$ is the i-th row of D. If $\omega \in \mathcal{S}$, then by Lemma 4c also $F_x = P(\omega_x)F$ holds throughout W, and the full Gauss–Weingarten system is available.

Proof. On the axis, $F(\cdot, 0)$ solves $F_x = P(\omega_x(x,0))F$ with $F(0,0) = I$; the coefficient depends continuously (in every $C^k([-L, L])$) on $\omega_x(\cdot, 0)$, which depends continuously on $\omega \in C^{k+1}(W)$. Smooth dependence of ODE solutions on the coefficient function (a C^∞ map into the space of continuous matrix functions) gives that $\omega \mapsto F(\cdot, 0)$ is C^∞ into every C^k. In the columns, $F(x, \cdot)$ solves $F_y = Q(\omega(x,y))F$ with initial value $F(x, 0)$; treating x as a parameter and using the smooth dependence already established gives F jointly, and r follows by integrating the C^∞-in-(ω, F) integrands e_1 and $\cos\omega\, e_1 + \sin\omega\, e_2$. Differentiating in t yields the stated systems (P is affine in p, Q' is as in the proof of Lemma 4c, and $\partial_t r_y$ uses $\partial_t\cos\omega = -\eta\sin\omega$, $\partial_t\sin\omega = \eta\cos\omega$ together with $\partial_t e_i = D_{i\cdot}$). Iteration gives all higher variations. □

The next lemma quantifies the continuity of the immersion in ω by the same cumulative (Gronwall-free) technique as in Lemmas 6–11.

Lemma 16 (quantitative continuity of the immersion)**.** *Let $\omega \in \mathcal{S}$ and $\omega + \eta \in \mathcal{S}$, $\eta \in C^\infty(W)$. Put*

$$A(x) := \int_0^{|x|} |\eta_\xi(\xi, 0)|\, d\xi, \qquad B(x,y) := \int_{\min(0,y)}^{\max(0,y)} |\eta(x,t)|\, dt.$$

Then for all $(x, y) \in W$,

$$|r(\omega+\eta)(x,y) - r(\omega)(x,y)| \;\le\; \int_0^{|x|} A(\xi)\, d\xi \;+\; B(x,y) \;+\; |y|\,\big(A(x) + B(x,y)\big).$$

Proof. Let $F, \tilde{F}$ be the frames of ω and $\omega+\eta$ and $D := \tilde{F} - F$, $\Delta r := r(\omega+\eta) - r(\omega)$.

Axis. On $y = 0$, $D_x = P(\omega_x)D + \big(P(\tilde{\omega}_x) - P(\omega_x)\big)\tilde{F}$; the matrix $P(\omega_x)$ is skew-symmetric and, by Lemma 4a(b), $\|P(\tilde{\omega}_x) - P(\omega_x)\| = |\eta_x|$. Lemma 4b gives $\|D(x,0)\| \le A(x)$, and since $\partial_x \Delta r$ is the first row of D, $|\Delta r(x,0)| \le \int_0^{|x|} \|D(\xi,0)\|\, d\xi \le \int_0^{|x|} A(\xi) d\xi$.

Columns. For fixed x,

$$D_y = Q(\tilde{\omega})D + \big(Q(\tilde{\omega}) - Q(\omega)\big)F,$$

$Q(\tilde{\omega})$ is skew-symmetric, and by Lemma 4a(b) and (E3), $\|Q(\tilde{\omega}) - Q(\omega)\| \le d(\eta) \le |\eta|$. Lemma 4b gives $\|D(x,y)\| \le A(x) + B(x,y)$. For the position, decompose as in the proof of Lemma 11:

$$\Delta r_y = (\cos\tilde{\omega} - \cos\omega)\,\tilde{e}_1 + (\sin\tilde{\omega} - \sin\omega)\,\tilde{e}_2 + \cos\omega\, D_{1\cdot} + \sin\omega\, D_{2\cdot},$$

where the first two terms have norm $|e^{i\tilde{\omega}} - e^{i\omega}| \le |\eta|$ and the last two are bounded by $\|D\|$. Hence $|\Delta r_y| \le |\eta| + \|D\|$, and integrating over $t \in [0, y]$ (the case $y < 0$ is mirrored) gives

$$|\Delta r(x,y)| \le |\Delta r(x,0)| + \int_{\min(0,y)}^{\max(0,y)} \big(|\eta| + \|D\|\big) dt \le \int_0^{|x|} A\, d\xi + B(x,y) + |y|\big(A(x) + B(x,y)\big),$$

because $B(x,t)$ is nondecreasing as t runs from 0 to y. □

Theorem 17 (smooth dependence for finite parameters)**.** *Fix n. Let $P \subset \mathbb{R}^m$ be open, let $a\colon P \to (0,\infty)^n$ and $c\colon P \to \mathbb{R}^n$ be C^∞, and let α_p, β_p be such that $(x,p) \mapsto \alpha_p(x)$ and $(y,p) \mapsto \beta_p(y)$ are jointly C^∞ with $\alpha_p(0) = \beta_p(0)$ (equivalently, $p \mapsto g_p := \Psi(\alpha_p, \beta_p)$ is a C^∞ family of seeds). Then $\omega_j(p)$ $(j \le n)$, $F(\omega_n(p))$ and $r(\omega_n(p))$ are jointly C^∞ in (x,y,p); equivalently, $p \mapsto r(\omega_n(p))$ is C^∞ into every $C^k(W, \mathbb{R}^3)$ and $p \mapsto F(\omega_n(p))$ into every $C^k(W, \mathrm{SO}(3))$. In particular the maps $\widetilde{\Phi}_n(a,c,\alpha,\beta) := r(\omega_n)$ on $(0,\infty)^n \times \mathbb{R}^n \times \mathcal{D}$ and $\Phi_n(a,c,f) := r(\omega_n)$ on $(0,\infty)^n \times \mathbb{R}^n \times \mathcal{S}$ are C^∞ along every such finite-dimensional smooth family; this is the sense of smoothness used in this paper. Continuity on the whole product, including the Fréchet factor $\mathcal{D}$, is Theorem 23 below; Fréchet differentiability on $\mathcal{D}$ is not claimed.*

Proof. Induction over j. The seed $\omega_0 = g_p$ is a C^∞ family by Lemma 13, and Lemma 14 propagates the property along the chain (each $\omega_j(p)$ is again a seed by Lemma 3). Finally, Lemma 15 composes with $\omega \mapsto (F(\omega), r(\omega))$. If the seed is given as a smooth family $f_p \in \mathcal{S}$, then $\alpha_p = f_p(\cdot, 0)$, $\beta_p = f_p(0, \cdot)$ and $f_p = \Psi(\alpha_p, \beta_p)$, so Φ_n is covered as well. □

Corollary 18 (continuity along paths of data)**.** *In the setting of Theorem 17, let $t \mapsto p(t) \in P$ be a curve. If it is continuous, then $t \mapsto r(\omega_n(p(t)))$ is continuous in every $C^k(W, \mathbb{R}^3)$; if it is C^∞, then so is $t \mapsto r(\omega_n(p(t)))$ in every C^k. In particular this applies to deformations of the Bäcklund parameters and constants at a fixed seed ($P = (0,\infty)^n \times \mathbb{R}^n$) and to affine deformations $(\alpha_t, \beta_t) = (1-t)(\alpha_0, \beta_0) + t(\alpha_1, \beta_1)$ of the characteristic data. Moreover, for* every *continuous curve $t \mapsto (a(t), c(t), \alpha_t, \beta_t)$ in $(0,\infty)^n \times \mathbb{R}^n \times \mathcal{D}$ the curve $t \mapsto r(\omega_n(t))$ is continuous in every C^k (Theorem 23). A quantitative bound for the deformation is obtained by combining the variation systems of Lemmas 14–15 with Lemma 16 (applied with $\eta = \delta\omega_n$).*

Proof. Theorem 17 composed with the curve (C^∞ maps are continuous); the last assertion is Theorem 23. □

11.3 Continuity on the data space: Banach estimates

Theorem 17 is smoothness along finite-dimensional families. The following estimates give, by purely Banach-space arguments (no Fréchet calculus), continuity—in fact local Lipschitz continuity with respect to C^m-seminorms—of the whole chain on $(0,\infty)^n \times \mathbb{R}^n \times \mathcal{D}$. All constants are allowed to be large (they contain Gronwall factors, cf. Remark 11.2); only finiteness is used.

For $g \in C^k(W)$ put $[g]_k := \max_{i+j\le k} \|\partial_x^i \partial_y^j g\|_{L^\infty(W)}$ (for matrix- or vector-valued g: the maximum over entries; products of 3×3 matrices cost a harmless factor 3). Then $[g]_{k+1} = \max\big([g]_0, [g_x]_k, [g_y]_k\big)$.

Lemma 19 (Banach-algebra estimates). *Let $g, h \in C^k(W)$ and $\rho \ge 2^k \max([g]_k, [h]_k)$.*

(i) $[gh]_k \le 2^k [g]_k [h]_k$.

(ii) $[\sin g]_k \le \sinh\rho$, $[\cos g]_k \le \cosh\rho$, $[\sin g - \sin h]_k \le \cosh\rho\,[g-h]_k$, $[\cos g - \cos h]_k \le \sinh\rho\,[g-h]_k$.

(iii) $\big[\int_0^x\!\int_0^y g\big]_{k+1} \le c_L [g]_k$, $c_L := \max(1, L^2)$.

Proof. (i) is the Leibniz rule: $\partial_x^i\partial_y^j(gh)$ is a sum of products of derivatives with binomial weights summing to $2^{i+j} \le 2^k$. (ii) By (i), $N := 2^k[\cdot]_k$ is submultiplicative, so for $m \ge 1$ $N(g^m - h^m) \le m\rho^{m-1} N(g-h)$ (write $g^m - h^m = (g-h)\sum g^i h^{m-1-i}$) and $N(g^m) \le \rho^m$; hence $[g^m - h^m]_k \le m\rho^{m-1}[g-h]_k$ and $[g^m]_k \le \rho^m$. The series of sin and cos converge absolutely in $C^k(W)$ and their sums are $\sin\circ g$, $\cos\circ g$; summing, $\sum_m m|c_m|\rho^{m-1}$ equals $\cosh\rho$ for sin and $\sinh\rho$ for cos, and $\sum |c_m|\rho^m$ equals $\sinh\rho$, $\cosh\rho$ (for cos the term $m=0$ contributes $[1]_k = 1$). (iii) Let $G = \iint g$. Then $|G| \le L^2 [g]_0$; for $i,j \ge 1$, $\partial_x^i\partial_y^j G = \partial_x^{i-1}\partial_y^{j-1} g$; and for $1 \le i \le k+1$, $\partial_x^i G = \int_0^y \partial_x^{i-1} g\, dt$, similarly in y, bounded by $L[g]_k$. □

Lemma 20 (Lipschitz dependence of the seed on the data). *Let $(\alpha_i, \beta_i) \in \mathcal{D}$, $f_i = \Psi(\alpha_i, \beta_i)$ $(i = 1, 2)$, and $[\alpha_i]_m, [\beta_i]_m \le R$. Then $[f_i]_m \le C$ and*

$$[f_1 - f_2]_m \le C\big([\alpha_1 - \alpha_2]_m + [\beta_1 - \beta_2]_m\big),$$

with $C = C(m, L, R)$.

Proof. Write $f_i = h_0^i + \iint \sin f_i$ (Step 2 of Lemma 13), $\Delta f = f_1 - f_2$. $m = 0$: $|f_i| \le [h_0^i]_0 + L^2$, and $\Delta f = \Delta h_0 + \mathcal{V}_\varphi \Delta f$ with $\varphi := \int_0^1 \cos(f_2 + \tau\Delta f)\, d\tau$, $|\varphi| \le 1$, so Step 1 of that proof gives $[\Delta f]_0 \le e^{2L}[\Delta h_0]_0$. $m \to m+1$: by Lemma 19, $[f_i]_{m+1} \le [h_0^i]_{m+1} + c_L \sinh\rho_m$ and $[\Delta f]_{m+1} \le [\Delta h_0]_{m+1} + c_L \cosh\rho_m [\Delta f]_m$, with $\rho_m = 2^m \max [f_i]_m$. Finally $[h_0^1 - h_0^2]_k \le 2[\alpha_1 - \alpha_2]_k + [\beta_1 - \beta_2]_k$. □

Lemma 21 (Lipschitz dependence of one Bäcklund step). *Let $g_1, g_2 \in \mathcal{S}$, $a_i \in [R^{-1}, R]$, $|c_i| \le R$, $[g_i]_{k+1} \le R$, and let q_i be the solution of (BT) with seed g_i, parameter a_i, $q_i(0,0) = c_i$. Then $[q_i]_k \le C$ and*

$$[q_1 - q_2]_k \le C\big(|a_1 - a_2| + |c_1 - c_2| + [g_1 - g_2]_{k+1}\big), \qquad C = C(k, L, R).$$

Proof. Let $u_i = (q_i + g_i)/2$, so $u_{i,x} = -a_i^{-1}\sin(u_i - g_i)$, $u_{i,y} = g_{i,y} - a_i \sin u_i$, $q_i = 2u_i - g_i$ (Lemma 3); write $\Delta u = u_1 - u_2$, etc.

Bounds. $|u_i(0,0)| \le R$, $|u_i(x,0) - u_i(0,0)| \le LR$ (as $|u_x| \le 1/a$), and $|u_i(x,y) - u_i(x,0)| \le L(R+R)$; so $[u_i]_0 \le C$. If $[u_i]_j \le C$, then $[u_i]_{j+1} = \max([u_i]_0, [u_{i,x}]_j, [u_{i,y}]_j) \le C'$ by Lemma 19(ii), because $u_{i,x}$, $u_{i,y}$ are given by the formulas above; this gives $[u_i]_{k+1} \le C$, hence $[q_i]_k \le C$.

Level 0. On the axis let $v(x) = \Delta u(x,0)$. Then $|v'| \le |a_1^{-1} - a_2^{-1}| + a_2^{-1}(|v| + [\Delta g]_0)$, so by Gronwall $|v(x)| \le e^{LR}\big(|v(0)| + LR^2|\Delta a| + LR[\Delta g]_0\big)$, with $|v(0)| \le \frac{1}{2}(|\Delta c| + [\Delta g]_0)$. In a column, $\Delta u_y = \Delta g_y - \Delta a \sin u_1 - a_2(\sin u_1 - \sin u_2)$, so $|\Delta u_y| \le [\Delta g]_1 + |\Delta a| + R|\Delta u|$ and $|\Delta u(x,y)| \le e^{RL}\big(|v(x)| + L([\Delta g]_1 + |\Delta a|)\big)$. Thus $[\Delta u]_0 \le C(|\Delta a| + |\Delta c| + [\Delta g]_1)$.

Induction. Suppose $[\Delta u]_j \le C(|\Delta a| + |\Delta c| + [\Delta g]_{j+1})$. From $\Delta u_x = -(a_1^{-1} - a_2^{-1})\sin(u_1 - g_1) - a_2^{-1}\big(\sin(u_1 - g_1) - \sin(u_2 - g_2)\big)$ and the formula for Δu_y above, Lemma 19(ii) gives

$$[\Delta u_x]_j + [\Delta u_y]_j \le C\big(|\Delta a| + [\Delta u]_j + [\Delta g]_{j+1}\big),$$

hence $[\Delta u]_{j+1} \le C(|\Delta a| + |\Delta c| + [\Delta g]_{j+1})$, which is stronger than required. Finally $\Delta q = 2\Delta u - \Delta g$. □

Lemma 22 (Lipschitz dependence of the immersion on $\omega \in \mathcal{S}$)**.** *Let $\omega_1, \omega_2 \in \mathcal{S}$ with $[\omega_i]_{k+1} \le R$. Then $[F(\omega_i)]_k + [r(\omega_i)]_k \le C$ and*

$$[F(\omega_1) - F(\omega_2)]_k + [r(\omega_1) - r(\omega_2)]_k \le C\,[\omega_1 - \omega_2]_{k+1}, \qquad C = C(k, L, R).$$

Proof. Put $\eta = \omega_1 - \omega_2$, $D = F(\omega_1) - F(\omega_2)$, $\Delta r = r(\omega_1) - r(\omega_2)$. *Level* 0*:* by Lemma 16 (its proof gives $\|D\| \le A + B$), $[D]_0 \le L([\eta_x]_0 + [\eta]_0)$ and $[\Delta r]_0 \le (3L^2 + L)[\eta]_1$. *Higher levels:* since $\omega_i \in \mathcal{S}$, Lemma 4c gives $F_x = P(\omega_x)F$, $F_y = Q(\omega)F$, $r_x = e_1$, $r_y = \cos\omega\, e_1 + \sin\omega\, e_2$ throughout W. Hence

$$D_x = P(\omega_{1,x})D + \eta_x J F(\omega_2), \qquad D_y = Q(\omega_1)D + \big(Q(\omega_1) - Q(\omega_2)\big)F(\omega_2),$$

and $[F(\omega_i)]_{j+1} = \max([F]_0, [F_x]_j, [F_y]_j)$ is bounded inductively by Lemma 19. If $[D]_j \le C[\eta]_{j+1}$, then $[D_x]_j + [D_y]_j \le C([D]_j + [\eta]_{j+1})$ (Lemma 19(ii) for $Q(\omega_1) - Q(\omega_2)$), so $[D]_{j+1} \le C[\eta]_{j+2}$. For r: Δr_x is the first row of D, and $\Delta r_y = (\cos\omega_1 - \cos\omega_2)e_1(\omega_1) + (\sin\omega_1 - \sin\omega_2)e_2(\omega_1) + \cos\omega_2\,\Delta e_1 + \sin\omega_2\,\Delta e_2$, so $[\Delta r]_{j+1} = \max([\Delta r]_0, [\Delta r_x]_j, [\Delta r_y]_j) \le C[\eta]_{j+2}$. □

Theorem 23 (continuity on the data space)**.** *Fix n and $k \ge 0$, put $K := k + n + 1$, and let $B_R := \{(a, c, \alpha, \beta) :\ a_j \in [R^{-1}, R],\ |c_j| \le R,\ [\alpha]_K, [\beta]_K \le R\}$. There is $C = C(n, k, L, R)$ such that for $p_1, p_2 \in B_R$*

$$[F(\omega_n^{(1)}) - F(\omega_n^{(2)})]_k + [r(\omega_n^{(1)}) - r(\omega_n^{(2)})]_k \le C\Big(|a^{(1)} - a^{(2)}| + |c^{(1)} - c^{(2)}| + [\alpha^{(1)} - \alpha^{(2)}]_K + [\beta^{(1)} - \beta^{(2)}]_K\Big).$$

Consequently $\widetilde{\Phi}_n$ is continuous (locally Lipschitz in the seminorms $[\cdot]_K$) on $(0,\infty)^n \times \mathbb{R}^n \times \mathcal{D}$ into every $C^k(W, \mathbb{R}^3)$, and Φ_n is continuous on $(0,\infty)^n \times \mathbb{R}^n \times \mathcal{S}$ ($\mathcal{S} \subset C^\infty(W)$ with its C^∞-topology). In particular $t \mapsto r(\omega_n(t))$ is continuous for every continuous curve of data.

Proof. By Lemma 20 with $m = K$, the seeds satisfy $[\omega_0^{(i)}]_K \le C$ and $[\omega_0^{(1)} - \omega_0^{(2)}]_K \le C([\Delta\alpha]_K + [\Delta\beta]_K)$. Apply Lemma 21 successively with k replaced by $K - j$ ($j = 1, \dots, n$):

it gives $[\omega_j^{(i)}]_{K-j} \le C$ and $[\Delta\omega_j]_{K-j} \le C(|\Delta a_j| + |\Delta c_j| + [\Delta\omega_{j-1}]_{K-j+1})$; after n steps, $[\Delta\omega_n]_{k+1} \le C(|\Delta a| + |\Delta c| + [\Delta\omega_0]_K)$ and $[\omega_n^{(i)}]_{k+1} \le C$. Lemma 22 (both $\omega_n^{(i)} \in \mathcal{S}$) gives the estimate. Since $\mathcal{D}$ is metrizable and every convergent sequence in it is bounded in each seminorm $[\cdot]_K$, continuity follows; $\Phi_n = \widetilde{\Phi}_n \circ (f \mapsto (f(\cdot,0), f(0,\cdot)))$ with a continuous linear restriction map. □

Remark 11.1. Only continuity (with a loss of $n+1$ derivatives) is claimed; the constants contain $e^{O(RL)}$ factors and are useless as $a \to \infty$, where the cumulative estimates of Theorem 1 take over (Remark 11.2). Fréchet differentiability on $\mathcal{D}$ is not claimed.

Remark 11.2 (the continuity constants degenerate at large a)**.** The smoothness of Theorem 17 is a statement at *fixed* parameters; its implicit constants degenerate as the parameters grow. For example, the variation in c of one step (Lemma 14) satisfies, on the axis, $|V(x,0)| \le |\delta c|\, e^{L/a}$ (because $|V_x| \le |V|/a$), and along a column $|V(x,y)| \le |\delta c|\, e^{L/a} e^{aL}$ (because $|V_y| \le a|V|$). Thus the Gronwall bound for the Lipschitz constant of $c \mapsto r(\omega_1)$ grows like e^{aL}; we do not claim that this growth is attained (away from the layer the column flow contracts the variation); similar (in general worse, iterated) bounds hold for variations in a and along the chain. Uniform-in-a control is provided instead by the *cumulative* estimates of Theorem 1, which are independent of c and non-Gronwall. The two mechanisms complement each other: Theorem 17 for finite perturbations at fixed a, Theorem 1 for the regime $a \to \infty$.

Remark 11.3 (what the flow deforms: first variation of the metric)**.** Corollary 18 asserts continuity of the flow; the computation behind it also identifies, to first order, *what* is deformed. Let $t \mapsto \omega(t)$ be a C^1 family of solutions with $\eta := \partial_t\omega|_{t=0}$ (so $\eta_{xy} = \cos\omega\,\eta$ by Lemma 13), and let $D = \partial_t F$, $\delta r = \partial_t r$ be the variations. Since $F(t)$ is orthogonal for all t (Lemma 4a(a)), the matrix $M := (\langle e_i, \delta e_j\rangle)_{ij} = FD^T$ is skew-symmetric. Writing $r_x = e_1$, $r_y = \cos\omega\, e_1 + \sin\omega\, e_2$ and $\delta r_y = -\eta\sin\omega\, e_1 + \eta\cos\omega\, e_2 + \cos\omega\,\delta e_1 + \sin\omega\,\delta e_2$, a direct computation gives

$$\langle r_x, \delta r_x\rangle = 0, \qquad \langle r_y, \delta r_y\rangle = 0, \qquad \langle r_x, \delta r_y\rangle + \langle r_y, \delta r_x\rangle = -\sin\omega\,\eta,$$

that is,

$$\operatorname{sym}\langle dr, d\,\delta r\rangle = \tfrac{1}{2}\,\delta I = -\sin\omega\,\eta\; dx \odot dy.$$

Thus the flow preserves the Chebyshev normalization ($E = G = 1$) and the curvature $K \equiv -1$ *exactly* along the whole family, and touches the metric only through the angle function; the linearized equation $\eta_{xy} = \cos\omega\,\eta$ is precisely the first-order condition of staying on the constant-curvature locus. Consequently the infinitesimal-bending identity $\operatorname{sym}\langle dr, d\,\delta r\rangle = 0$ (Darboux) holds along the flow if and only if $\sin\omega\cdot\eta \equiv 0$; if the nondegenerate locus $\{\sin\omega \neq 0\}$ is dense in W (e.g. for real-analytic solutions not identically $\equiv k\pi$), this forces $\eta \equiv 0$, and then the homogeneous linearized systems of Lemmas 15–14 with the normalization $F(0,0) = I$, $r(0,0) = 0$ give $D \equiv 0$ and $\delta r \equiv 0$. In other words: within the ASG normalization, the flow is metric-rigid—its only isometric directions are the infinitesimal rigid motions, which the normalization quotients out. Classical bendings of a $K = -1$ surface do exist, but they do not preserve the asymptotic gauge ($e = g = 0$ in II) and are therefore not directions of this flow. At a degenerate base the identity is vacuous: for the model kink family of Proposition 11.3 the base $\omega \equiv 0$ has $\sin\omega \equiv 0$, the metric changes only in the $O(s)$-wide

layer (at order $O(s^2)$ pointwise outside it), and the formal variation $q_s/s \to 2\pi\,\delta(y)$ is *not* a solution of the linearized equation—the family is a transition between branches ($k_- = k_+ \pm 1$, Proposition 11.5), not a C^1-path in the solution space, which is exactly why the infinitesimal limit Δ_* of Proposition 11.3 is a singular layer field.

11.4 The total limit $a_j \to \infty$

Theorem 24 (continuous extension to the Bäcklund chart at infinity)**.** *Fix n and $f \in \mathcal{S}$. As $\min_j a_j \to \infty$ (with arbitrary relative rates and arbitrary c_j),*

$$\|r(\omega_n) - r((-1)^n f)\|_{L^\infty(W)} \le K_n(f) \sum_{j=1}^{n} \frac{1}{a_j} \;\longrightarrow\; 0, \qquad K_n(f) = 3L^2 + (1+L)\pi(2T_*(f)+\pi)3^{n-1}.$$

Consequently, in the coordinates $s_j = 1/a_j \in (0,\infty)$, with $f \in \mathcal{S}$ carrying the $C^1(W)$-topology, the map $(s,c,f) \mapsto r(\omega_n)$ has at every boundary point $(0,c^0,f)$ the limit $r((-1)^n f)$, independent of c^0: $r(\omega_n(s,c,f')) \to r((-1)^n f)$ as $(s,c,f') \to (0,c^0,f)$.

Proof. For a fixed seed the estimate is Theorem 1 (unconditional in c_j and in the ratios of the a_j). Let now $f_m \to f$ in $C^1(W)$. Then $T_*(f_m) \le 2L\|\partial_y f_m\|_\infty$ is bounded, hence so is $K_n(f_m) \le K$. Put $\eta_m := (-1)^n(f_m - f)$; both $(-1)^n f$ and $(-1)^n f_m$ lie in $\mathcal{S}$, so Lemma 16 gives (with $A \le L\|\eta_x\|_\infty$, $B \le L\|\eta\|_\infty$)

$$\|r((-1)^n f_m) - r((-1)^n f)\|_\infty \le 2L^2\|\eta_{m,x}\|_\infty + (L+L^2)\|\eta_m\|_\infty \le (2L^2+L)\|f_m - f\|_{C^1},$$

with $\|g\|_{C^1} := \|g\|_\infty + \|g_x\|_\infty + \|g_y\|_\infty$. Therefore $\|r(\omega_n(s,c,f_m)) - r((-1)^n f)\|_\infty \le K\sum_j s_j + (2L^2+L)\|f_m - f\|_{C^1} \to 0$, uniformly in c. □

Theorem 24 concerns the *immersions*. The solutions ω_n themselves need not converge:

Proposition 11.1 (no C^0-limit for ω_n)**.** Let $n = 1$, $f \equiv 0$, $c = \pi$. Then

$$q_a(x,y) = 4\arctan\!\left(e^{-x/a - ay}\right)$$

solves (BT) with seed 0 and $q_a(0,0) = \pi$, but

$$q_a \to \begin{cases} 0, & y > 0 \\ \pi, & y = 0 \\ 2\pi, & y < 0 \end{cases} \qquad (a \to \infty)$$

pointwise on W, so the family $\{q_a\}$ has no limit in $C^0(W)$. Meanwhile $\|r(q_a) - r(0)\|_{L^\infty(W)} \le K_1/a \to 0$ by Theorem 1 (here $T_*(0) = 0$, $S = 1/a$): the immersion converges although the solution does not.

Proof. With $\theta = -x/a - ay$ and $q_a = 4\arctan e^\theta$ one has $\sin(q_a/2) = 2e^\theta/(1+e^{2\theta})$ and

$$\partial_x q_a = -\frac{4}{a}\frac{e^\theta}{1+e^{2\theta}} = -\frac{2}{a}\sin\frac{q_a}{2}, \qquad \partial_y q_a = -4a\frac{e^\theta}{1+e^{2\theta}} = -2a\sin\frac{q_a}{2},$$

which is (BT) with $g \equiv 0$; and $q_a(0,0) = 4\arctan 1 = \pi$. The pointwise limit is as stated (for $y = 0$, $e^{-x/a} \to 1$ uniformly on $[-L, L]$). If q_a converged in $C^0(W)$, the limit would be a uniform limit of continuous functions, hence continuous, and it would coincide with the displayed step function; but the step function is discontinuous at $y = 0$ while W contains points with $y = \pm\varepsilon$ on both sides. The last claim is Theorem 1. □

Proposition 11.2 (no C^1-extension of the immersion)**.** In the situation of Proposition 11.1, put $q_s := q_{1/s}$, $s > 0$. Then $\|r(q_s) - r(0)\|_{L^\infty(W)} = O(s)$, but the difference quotient $(r(q_s) - r(0))/s$ is unbounded in $C^1(W, \mathbb{R}^3)$; in particular $s \mapsto r(q_s)$ (extended by $r(0)$ at $s = 0$) is not a C^1-curve in $C^1(W, \mathbb{R}^3)$.

Proof. The bound $\|r(q_s) - r(0)\|_{L^\infty} \le K_1\, s$ is Theorem 1 with $n = 1$, $a = 1/s$, $T_*(0) = 0$. On the axis, $(q_s)_x = -2s\sin(q_s/2)$, so $|(q_s)_x| \le 2s$ and, by Lemma 4b with $\|P(p) - P(0)\| = |p|$ (Lemma 4a(b)),

$$\|F(q_s)(x,0) - F(0)(x,0)\| \le 2s|x|,$$

hence $e_i(q_s)(x,0) \to e_i(0)(x,0)$ uniformly on $[-L, L]$. Also $q_s(x,0) = 4\arctan e^{-sx} \to \pi$ uniformly. Therefore, at $y = 0$,

$$r_y(q_s)(x,0) = \cos(q_s)e_1(q_s) + \sin(q_s)e_2(q_s) \;\longrightarrow\; -e_1(0)(x,0),$$

while $r_y(0)(x,0) = e_1(0)(x,0)$. Hence $\sup_W |\partial_y(r(q_s) - r(0))| \ge 2 - o(1)$, and

$$\Big\|\frac{r(q_s) - r(0)}{s}\Big\|_{C^1(W)} \;\ge\; \frac{2 - o(1)}{s} \;\longrightarrow\; \infty.$$ □

Propositions 11.1 and 11.2 show that the convergence in Theorem 24 is genuinely C^0: the solution has a jump layer of size 2π and width $O(1/a)$ collapsing onto the axis $y = 0$. The natural question is whether a *first-order* (infinitesimal) limit exists. The answer is yes, in C^0 off the axis, and the limit is completely explicit; it is *not* obtained by differentiating at any finite parameter, and it retains a jump across the axis—the imprint of the boundary layer.

Proposition 11.3 (infinitesimal limit for the model kink)**.** In the situation of Proposition 11.1 ($n = 1$, $f \equiv 0$, $c = \pi$, $q_s = q_{1/s}$), the difference quotient $(r(q_s) - r(0))/s$ converges as $s \to 0^+$, uniformly on $\{|y| \ge \varepsilon'\}$ for every $\varepsilon' > 0$ and pointwise on all of W, to the map Δ_* defined by the following linear construction.

Let $F_0 := F(0)$ be the frame of the identically-zero solution with rows $e_1^{(0)}, e_2^{(0)}, n^{(0)}$, let $P_0, J, Q(\omega)$ be as in §4, and let $\Phi_0(y) := \exp(y\,Q(0))$. Define

(a) $\bar E$ as the solution of $\bar E_x = P_0\bar E - 2\,J\,F_0(x,0)$, $\bar E(0,0) = 0$;

(b) $\Delta_*(x,0) := \int_0^x \bar E_{1\cdot}(\xi,0)\,d\xi$;

(c) for $y > 0$:

$$\Delta_*(x,y) := \Delta_*(x,0) + \big(-2e_1^{(0)}(x,0) + 2e_2^{(0)}(x,0)\big) + \int_0^y E^*_{1\cdot}(x,t)\,dt,$$

where $E^*(x,y) := \Phi_0(y)\big(\bar E(x) + M_+F_0(x,0)\big)$;

(d) for $y < 0$:

$$\Delta_*(x,y) := \Delta_*(x,0) + \big(2e_1^{(0)}(x,0) + 2e_2^{(0)}(x,0)\big) + \int_0^y E^*_{1\cdot}(x,t)\,dt,$$

where $E^*(x,y) := \Phi_0(y)\big(\bar{E}(x) + M_- F_0(x,0)\big)$;

with the constant matrices

$$M_+ = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}, \qquad M_- = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}.$$

The limit Δ_* has a jump across the axis: $\Delta_*(0,0^+) - \Delta_*(0,0) = (-2,2,0) \neq 0$; in particular the convergence is not uniform near $y = 0$ (and not in C^1, Proposition 11.2). Moreover there is an absolute $c_0 > 0$ with

$$\|r(q_s) - r(0)\|_{L^\infty(W)} \;\geq\; c_0\, s \qquad \text{for all } s \leq s_0.$$

Proof. Preliminaries. Write $F_s := F(q_s)$, $F_0 := F(0)$, $D := F_s - F_0$, $E := D/s$, $\Delta r := r(q_s) - r(0)$, and

$$\hat{q}_s(z;x) := q_s(x,sz) = 4\arctan e^{-sx-z}, \qquad \hat{q}(z) := 4\arctan e^{-z}.$$

(The identity $q_s(x,sz) = 4\arctan(e^{-sx-z})$ is exact; throughout the proof we suppress the parameter x in $\hat{q}_s$, since in every estimate below the base point x is held fixed.) We use repeatedly: $0 < q_s < 2\pi$; $\arctan t \leq t$, hence $\hat{q}_s(z) \leq 4e^{sL}e^{-z}$ for $z \geq 0$ and, by $2\pi - 4\arctan u = 4\arctan(1/u)$, also $2\pi - \hat{q}_s(z) \leq 4e^{sL}e^{z}$ for $z \leq 0$; and the elementary layer integrals (substitute $t = e^{-z}$ and use $\sin(4\arctan t) = 4t(1-t^2)/(1+t^2)^2$, $1-\cos(4\arctan t) = 8t^2/(1+t^2)^2$):

$$\int_0^\infty \sin\hat{q}\,dz = 2, \quad \int_0^\infty (1-\cos\hat{q})\,dz = 2, \quad \int_{-\infty}^0 \sin\hat{q}\,dz = -2, \quad \int_{-\infty}^0 (1-\cos\hat{q})\,dz = 2. \tag{2}$$

Also $\|Q(\omega)\| = 1$ and $Q(\omega)$ skew-symmetric (Lemma 4a(a)), and $\|Q(\omega) - Q(\omega')\| = |e^{i\omega} - e^{i\omega'}| \leq d(\omega - \omega') \leq |\omega - \omega'|$ (Lemma 4a(b), (E3)).

1. Axis limit. On the axis $p_s(x) := (q_s)_x(x,0) = -2s\sin(q_s(x,0)/2) = -2s\sin(2\arctan e^{-sx})$, so $|p_s| \leq 2s$ and

$$p_s/s = -2\sin(2\arctan e^{-sx}) \to -2 \qquad \text{uniformly on } [-L,L].$$

From $F_{s,x} = P(p_s)F_s$, $F_{0,x} = P(0)F_0$ and the affinity of P,

$$D_x = P(p_s)D + p_s\, J\, F_0, \qquad \text{i.e.} \qquad E_x = P(p_s)E + (p_s/s)\, J\, F_0, \quad E(0,0) = 0.$$

Since $P(p_s) \to P_0 = P(0)$ and $(p_s/s) \to -2$ uniformly, continuous dependence for linear systems gives $E(\cdot,0) \to \bar{E}(\cdot)$ in $C^0([-L,L])$, where $\bar{E}$ is as in (a). Likewise $\Delta r(x,0)/s = \int_0^x E_{1\cdot}(\xi,0)\,d\xi \to \int_0^x \bar{E}_{1\cdot}(\xi,0)\,d\xi = \Delta_*(x,0)$, uniformly in x.

2. Cumulative frame estimate. In a column, $D_y = Q(q_s)D + \big(Q(q_s) - Q(0)\big)F_0$, and Lemma 4b together with $d(q_s, 2\pi\mathbb{Z}) \le 4e^{sL}e^{-|t|/s}$ gives, uniformly in (x,t),

$$\|D(x,t)\| \le \|D(x,0)\| + \int_0^{|t|} d(q_s(x,\tau), 2\pi\mathbb{Z})\, d\tau = O(s),$$

hence E is uniformly bounded and $e_i(q_s) \to e_i^{(0)}$ uniformly on W.

3. Frame limit. Fix x; let $\Phi_s(y)$ and $\Phi_0(y)$ be the fundamental matrices of $Y' = Q(q_s(x,y))Y$ and $Y' = Q(0)Y$ (both orthogonal, Lemma 4a(a)). Variation of constants and $t = sz$ give, for $y > 0$,

$$E(x,y) = \Phi_s(y)E(x,0) + \int_0^{y/s} \Phi_s(y)\Phi_s(sz)^{-1}\big(Q(\hat{q}_s(z)) - Q(0)\big)F_0(x,sz)\, dz.$$

Here $\|\Phi_s(y) - \Phi_0(y)\| \le \int_0^{|y|} \|Q(q_s) - Q(0)\|\, dt = O(s)$, and the integrand is dominated by $2 \cdot 4e^{sL}e^{-z}\|F_0\|_{L^\infty} \in L^1(0,\infty)$, uniformly in x. Splitting the integral at $z = Z$, letting $s \to 0$ (on $[0,Z]$ use $\hat{q}_s \to \hat{q}$ uniformly, $\Phi_s(y)\Phi_s(sz)^{-1} \to \Phi_0(y)$ uniformly, $F_0(x,sz) \to F_0(x,0)$, and $E(x,0) \to \bar{E}(x)$) and then $Z \to \infty$ (dominated convergence) yields

$$E(x,y) \;\longrightarrow\; \Phi_0(y)\Big(\bar{E}(x) + \Big[\int_0^\infty \big(Q(\hat{q}(z)) - Q(0)\big)dz\Big]F_0(x,0)\Big),$$

uniformly in (x,y) with $y \ge \varepsilon'$; by (2) the bracketed matrix is M_+ (entrywise: the $(1,3)$- and $(2,3)$-entries are $\int \sin\hat{q} = 2$ and $\int(1-\cos\hat{q}) = 2$, the others follow by skew-symmetry). For $y < 0$ the same formula holds with the *oriented* integral $\int_0^{y/s} = -\int_{y/s}^0$. With $z \in (y/s, 0)$ and the domination $2\pi - \hat{q}_s(z) \le 4e^{sL}e^z$ the limit is $\Phi_0(y)\big(\bar{E}(x) + M_-F_0(x,0)\big)$ with $M_- = -\int_{-\infty}^0 \big(Q(\hat{q}(z)) - Q(0)\big)dz$; by (2) its $(1,3)$-, $(2,3)$-entries are $-\int_{-\infty}^0 \sin\hat{q} = 2$ and $-\int_{-\infty}^0(1-\cos\hat{q}) = -2$, the others follow by skew-symmetry. This is E^* as in (d), uniformly on $\{|y| \ge \varepsilon'\}$.

4. Position limit. From $r_y(q_s) = \cos q_s\, e_1(q_s) + \sin q_s\, e_2(q_s)$, $r_y(0) = e_1^{(0)}$ and $e_1(q_s) = e_1^{(0)} + D_{1\cdot}$,

$$\Delta r(x,y) = \Delta r(x,0) + \int_0^y \Big[(\cos q_s - 1)\, e_1(q_s) + \sin q_s\, e_2(q_s) + D_{1\cdot}\Big](x,t)\, dt.$$

Divide by s, substitute $t = sz$, and use dominated convergence with the dominators $|\cos\hat{q}_s - 1| \le \hat{q}_s^2/2 \le 8e^{2sL}e^{-2z}$ and $|\sin\hat{q}_s| \le 4e^{sL}e^{-z}$ (resp. their mirror for $z < 0$):

$$\frac{1}{s}\int_0^y (\cos q_s - 1)e_1(q_s)\, dt \longrightarrow e_1^{(0)}(x,0)\int_0^\infty (\cos\hat{q} - 1)\, dz = -2e_1^{(0)}(x,0),$$

$$\frac{1}{s}\int_0^y \sin q_s\, e_2(q_s)\, dt \longrightarrow e_2^{(0)}(x,0)\int_0^\infty \sin\hat{q}\, dz = 2e_2^{(0)}(x,0),$$

and $\frac{1}{s}\int_0^y D_{1\cdot}\, dt = \int_0^y E_{1\cdot}(x,t)\, dt \to \int_0^y E_{1\cdot}^*(x,t)\, dt$ (uniformly on $y \ge \varepsilon'$). This is (c); the $y < 0$ computation is identical with the interval $(y/s, 0)$, the constants -2 and -2 of (2), and an overall change of sign from reversing the integration, giving (d).

5. Jump and sharpness. From (b),(c), $\Delta_*(x,0^+)-\Delta_*(x,0) = (-2e_1^{(0)}+2e_2^{(0)})(x,0)$; at $x=0$ this is $(-2,2,0)$. A uniform limit of continuous functions on W would be continuous, while Δ_* is discontinuous across $y=0$; hence the convergence cannot be uniform near the axis (it is uniform exactly on $\{|y|\geq\varepsilon'\}$). Finally, for fixed small $y_0>0$, $\Delta_*(0,y_0) = (-2,2,0)+O(y_0)$, so $|\Delta_*(0,y_0)|\geq\sqrt{2}$ for $y_0\leq y_0'$, and therefore $\|r(q_s)-r(0)\|_{L^\infty}\geq s\,|\Delta_*(0,y_0)|/2\geq c_0 s$ for all small s. □

Corollary 25 (sharpness of the order in Theorem 1)**.** *In the family of Proposition 11.1 ($n=1$, $f\equiv 0$, $c=\pi$, $a=1/s$) the immersion error is exactly of order $1/a$: the exponent 1 in $\sum 1/a_j$ cannot be improved.*

Proof. Immediate from Proposition 11.3: the first-order term Δ_* is nonzero, and the error is $s\,\Delta_* + o(s)$ off the axis. □

Corollary 26 (order-sharpness for every fixed n)**.** *Fix $n\geq 1$ and $L>0$. There exist constants $c_n = c_n(L)>0$ and $s_0 = s_0(n,L)>0$ such that for all $s\in(0,s_0)$, with*

$$f\equiv 0, \qquad c_j=\pi \quad (j=1,\dots,n), \qquad a_j := s^{-j},$$

one has

$$\|r(\omega_n)-r((-1)^n f)\|_{L^\infty(W)} = \|r(\omega_n)-r(0)\|_{L^\infty(W)} \;\geq\; c_n\sum_{j=1}^{n}\frac{1}{a_j}.$$

The constant may be taken $c_n = c_0/4^{n-1}$, where $c_0 = c_0(L)>0$ is the constant of Corollary 25.

Proof. Induction on n. *Base $n=1$:* $a_1 = 1/s$ (chain constant π), $f\equiv 0$, so $\omega_1 = q_s = 4\arctan(e^{-sx-y/s})$, and Corollary 25 gives $\|r(q_s)-r(0)\|_\infty\geq c_0 s = c_0/a_1$ for all $s\leq s_0(1)$, where $s_0(1)>0$ and $c_0>0$ are the constants of that corollary; here $\sum_{j=1}^{1}1/a_j = 1/a_1$. This proves the claim for $n=1$ with $c_1 := c_0$ and $s_0(1)$ as above.

Step. Assume the claim for $n-1$ with constant c_{n-1}, and append $a_n = s^{-n}$ (with chain constant π, not to be confused with the estimate constant c_n). Write $g := \omega_{n-1}$. By Theorem 1 applied with $n=1$ to the seed g (which solves SG by Lemma 3),

$$\|r(\omega_n)-r(-g)\|_{L^\infty}\leq\frac{K_1(g)}{a_n} = K_1(g)\,s^n, \qquad K_1(g) = 3L^2+(1+L)\pi\big(2T_*(g)+\pi\big).$$

By Lemma 27, $r(-g) = r(g)R$ with $R=\mathrm{diag}(1,-1,-1)$ orthogonal, and $r(0)R = r(0)$ (the immersion of the identically zero solution, $r(0)(x,y) = (x+y)\varepsilon_1$, is invariant under R); hence $\|r(-g)-r(0)\|_\infty = \|r(g)-r(0)\|_\infty$. The triangle inequality gives

$$\|r(\omega_n)-r(0)\|_\infty\geq\|r(g)-r(0)\|_\infty - K_1(g)\,s^n\geq c_{n-1}\sum_{j=1}^{n-1}\frac{1}{a_j} - K_1(g)\,s^n.$$

By Lemma 35 with $f\equiv 0$ (so $T_*(f)=0$), $T_*(g)\leq\pi(3^{n-1}-1)$ uniformly in s, hence

$$K_1(g)\leq C_n := 3L^2+(1+L)\pi\big(2\pi(3^{n-1}-1)+\pi\big)$$

independently of s. Also $\sum_{j=1}^{n-1} 1/a_j = s + O(s^2) \geq s$ for $s \leq \frac{1}{2}$. Choose

$$s_0(n) := \min\Big(s_0(n-1),\ \tfrac{1}{2},\ \big(c_{n-1}/(2C_n)\big)^{1/(n-1)}\Big) > 0;$$

then for $s \in (0, s_0(n))$ the induction hypothesis applies (as $s < s_0(n-1)$), $C_n s^n \leq c_{n-1}s/2$, and therefore

$$\|r(\omega_n) - r(0)\|_\infty \ \geq\ \tfrac{c_{n-1}}{2}\, s.$$

Since $\sum_{j=1}^{n} 1/a_j = s + O(s^2) \leq 2s$ for $s \leq \frac{1}{2}$, we conclude

$$\|r(\omega_n) - r(0)\|_\infty \ \geq\ \tfrac{c_{n-1}}{4} \sum_{j=1}^{n} \frac{1}{a_j},$$

which is the claim with $c_n := c_{n-1}/4 = c_0/4^{n-1}$ and $s_0(n)$ as defined above. □

Remark 11.4 (scope of Corollary 26)**.** Corollary 26 shows that for every *fixed* n the exponent 1 in the right-hand side of Theorem 1 cannot be improved, and it does so constructively (the single one-parameter family $f \equiv 0$, $c_j = \pi$, $a_j = s^{-j}$, $s \to 0$). Two limitations are inherited from the construction and are not removed by it:
(a) the constant $c_n = c_0/4^{n-1}$ is exponentially small in n, so uniformity in n is *not* obtained (this is consistent with Remark 10.4(a));
(b) the constant K_n of Theorem 1 is not touched: the corollary gives order-sharpness, not an optimal universal constant.

11.5 Orientation reversal, mirrors, and the optimal topology

For two solutions $\omega, \tilde{\omega} \in \mathcal{S}$ define

$$\delta(\omega, \tilde{\omega}) := \sup_{|x| \leq L} \big|\partial_x(\omega - \tilde{\omega})(x, 0)\big|, \qquad \Lambda(\omega, \tilde{\omega}) := \sup_{|x| \leq L} \int_{-L}^{L} d(\omega - \tilde{\omega})(x, y)\, dy.$$

Lemma 27 (orientation reversal)**.** *Let $R = \Sigma := \mathrm{diag}(1, -1, -1)$. For every $\omega \in C^\infty(W)$,*

$$F(-\omega) = \Sigma\, F(\omega)\, R, \qquad r(-\omega) = r(\omega)\, R.$$

Proof. Rows of F are e_1, e_2, n; r is a row vector, R acts from the right. Put $F' := \Sigma F(\omega) R$, $r' := r(\omega)R$. One checks directly $\Sigma P(p)\Sigma = P(-p)$ and $\Sigma Q(\omega)\Sigma = Q(-\omega)$ (conjugation multiplies the entry (i, j) by $\sigma_i\sigma_j$, $\sigma = (1, -1, -1)$). Hence on the axis $F'_x = \Sigma P(\omega_x)F(\omega)R = P(-\omega_x)F'$ and in the columns $F'_y = Q(-\omega)F'$; $F'(0, 0) = \Sigma R = I$. By uniqueness for these ODEs, $F' = F(-\omega)$. Further $r'_x = e_1(\omega)R = e'_1$, and $r'_y = (\cos\omega\, e_1 + \sin\omega\, e_2)R = \cos(-\omega)e'_1 + \sin(-\omega)e'_2$ (because $e'_2 = -e_2 R$); with $r'(0, 0) = 0$ this identifies $r' = r(-\omega)$. □

Corollary 28 (the mirror as a proper rotation)**.** $r(\mathrm{ref}_n) = r\big((-1)^n f\big) = r(f)R^n$. *Thus Theorem 1 reads $\|r(\omega_n) - r(f)R^n\|_{L^\infty(W)} \leq K_n S$: for even n the limit surface is $r(f)$ itself, for odd n it is $r(f)$ rotated by π about the axis $\mathbb{R}e_1(0, 0)$. The "mirror image" $r(-f)$ is therefore* congruent by a proper rotation *($\det R = +1$) to $r(f)$: the map $\omega \mapsto -\omega$ reverses the orientation (normal and e_2) but does not change the chirality of the surface.*

Proof. By Lemma 27 applied to $\omega = (-1)^{j-1}f$, one has $r\big((-1)^j f\big) = r\big((-1)^{j-1}f\big)R$ for each $j \geq 1$; induction gives $r\big((-1)^n f\big) = r(f)R^n$. Since $R = \operatorname{diag}(1,-1,-1)$ satisfies $R^2 = I$, for even n the limit surface is $r(f)$ and for odd n it is $r(f)R$, the rotation by π about the axis $\mathbb{R}e_1(0,0)$. Finally $\det R = +1$, so $r(-f) = r(f)R$ is a congruence by a *proper* rotation of $\mathbb{R}^3$: proper rotations preserve chirality, while the orientation of the frame is reversed (Lemma 27: e_2 and n change sign). □

Lemma 29 (comparison lemma)**.** *Let $\omega, \tilde{\omega} \in \mathcal{S}$, let $F, \tilde{F}$ and $r, \tilde{r}$ be the corresponding frames and immersions, $D = F - \tilde{F}$, $\delta = \delta(\omega, \tilde{\omega})$, and $\Lambda_y(x) = \left|\int_0^y d(\omega - \tilde{\omega})(x,t)\,dt\right|$. Then for all $(x,y) \in W$*

$$\|D(x,y)\| \leq \delta|x| + \Lambda_y(x), \qquad |r - \tilde{r}\,|(x,y) \leq \tfrac{1}{2}\delta x^2 + \delta|x||y| + (1+|y|)\Lambda_y(x),$$

$$|\partial_x(r - \tilde{r}\,)| \leq \|D\|, \qquad |\partial_y(r - \tilde{r}\,)| \leq d(\omega - \tilde{\omega}) + \|D\|.$$

In particular $\|r - \tilde{r}\,\|_\infty \leq \frac{3}{2}L^2\delta + (1+L)\Lambda(\omega, \tilde{\omega})$.

Proof. This is the proof of Lemmas 6 and 11 with $2S$ replaced by δ, written for an arbitrary pair. On the axis, $D_x = P(\omega_x)D + (P(\omega_x) - P(\tilde{\omega}_x))\tilde{F}$, $D(0,0) = 0$; by Lemmas 4a(b), 4b and $\|\tilde{F}\| = 1$, $\|D(x,0)\| \leq \delta|x|$; $\partial_x(r - \tilde{r})$ is the first row of D, so $|(r - \tilde{r})(x,0)| \leq \frac{1}{2}\delta x^2$. In the column, $D_y = Q(\omega)D + (Q(\omega) - Q(\tilde{\omega}))\tilde{F}$ and $\|(Q(\omega) - Q(\tilde{\omega}))\tilde{F}\| \leq d(\omega - \tilde{\omega})$ (Lemma 4a), so $\|D(x,y)\| \leq \|D(x,0)\| + \Lambda_y(x)$. Finally $\partial_y(r - \tilde{r}) = (\cos\omega - \cos\tilde{\omega})e_1 + (\sin\omega - \sin\tilde{\omega})e_2 + (\cos\tilde{\omega}, \sin\tilde{\omega}, 0)D$ has length at most $|e^{i\omega} - e^{i\tilde{\omega}}| + \|D\| \leq d(\omega - \tilde{\omega}) + \|D\|$ (by (E3)); integrate in y and use that $t \mapsto \Lambda_t(x)$ is monotone. □

Theorem 30 (convergence as $\min a_j \to \infty$ in C^0, $W^{1,p}$, and sharpness)**.** *Let K_n be as in Theorem 1 and $K_n' := 2L + \pi(2T_* + \pi)3^{n-1}$. For all $a_j > 0$, $c_j \in \mathbb{R}$:*

(i) $\|r(\omega_n) - r(\mathrm{ref}_n)\|_{L^\infty(W)} \leq K_n S$;

(ii) $\|\partial_x(r(\omega_n) - r(\mathrm{ref}_n))\|_{L^\infty(W)} \leq K_n' S$ *and* $\|F_n - F_{\mathrm{ref}}\|_{L^\infty(W)} \leq K_n' S$;

(iii) *for every $1 \leq p < \infty$,*

$$\|\partial_y(r(\omega_n) - r(\mathrm{ref}_n))\|_{L^p(W)} \leq \pi^{1-\frac{1}{p}}\big(2L\pi(2T_* + \pi)3^{n-1}S\big)^{\frac{1}{p}} + (4L^2)^{\frac{1}{p}}K_n' S;$$

(iv) *hence $r(\omega_n) \to r(\mathrm{ref}_n)$ in $W^{1,p}(W)$ for all $p < \infty$, with rate $S^{1/p}$ for the derivative $\partial_y r$, and the family $\{r(\omega_n)\}$ is uniformly Lipschitz ($|r_x| = |r_y| = 1$).*

The rates in (i) *and* (iii) *cannot be improved, and the convergence does not hold in $W^{1,\infty}$ (Proposition 11.4).*

Proof. (i) is Theorem 1. Apply Lemma 29 to ω_n, ref_n: here $\delta \leq 2S$ (Lemma 5) and $\Lambda_y(x) \leq \Lambda(x) \leq \pi(2T_* + \pi)3^{n-1}S$ (Lemma 10), which gives $\|D\| \leq (2L + \pi(2T_* + \pi)3^{n-1})S = K_n' S$ and (ii). For (iii): $|\partial_y \Delta r| \leq d_n + K_n' S$, $d_n := d(\omega_n - \mathrm{ref}_n) \leq \pi$, so $\iint d_n^p \leq \pi^{p-1}\iint d_n \leq \pi^{p-1} \cdot 2L \cdot \pi(2T_* + \pi)3^{n-1}S$, and Minkowski gives (iii). (iv) follows from (i)–(iii). □

Proposition 11.4 (sharpness; no C^1-convergence and no differentiability at infinity)**.** Let $n = 1$, $f \equiv 0$, $c_1 = \pi$, $a = 1/s$. Then $\omega_1 = q_s := 4\arctan e^{-sx - y/s}$, $\mathrm{ref}_1 \equiv 0$, $r(0)(x,y) = (x+y)\varepsilon_1$ with ε_i the standard basis, and for $0 < s \leq s_0$ ($s_0 = s_0(L)$ small):

(a) $\|\partial_y(r(q_s)-r(0))\|_{L^\infty(W)} \geq 2$: no convergence in C^1 (the deficit is attained already at the origin, refining Proposition 11.2);

(b) $\|\partial_y(r(q_s)-r(0))\|_{L^p(W)} \geq \frac{1}{4}(4Ls)^{1/p}$, $1 \leq p < \infty$: the rate of Theorem 30(iii) is sharp;

(c) $\|r(q_s)-r(0)\|_{L^\infty(W)} \geq s$ for $s \leq \frac{1}{4}$: the rate of Theorem 30(i) is sharp (cf. Corollary 25);

(d) the difference quotients $\Delta_s := (r(q_s)-r(0))/s$ are bounded but not equicontinuous: $|\Delta_s(0,s) - \Delta_s(0,-s) + 4\tanh(1)\,\varepsilon_1| \leq 4s$ for all $0 < s \leq L$. In particular $s \mapsto r(\omega_1) \in C^0(W)$ is not differentiable at $s = 0$ (even one-sidedly), although it is Lipschitz there by Theorem 30(i).

Moreover the ε_2-component of $\Delta_s(0,s)$ equals $\int_0^1 \sin\hat\Theta\, d\tau + O(s)$, where $\hat\Theta(\tau) = 4\arctan e^{-\tau}$ and $\sin\hat\Theta > 0$ on $(0,\infty)$; it is therefore positive for small s.

Proof. That q_s solves (BT) with $g \equiv 0$ and $q_s(0,0) = \pi$ is a direct check ($\theta = -sx - y/s$, $(q_s)_x = -2s\sin\frac{q_s}{2}$, $(q_s)_y = -\frac{2}{s}\sin\frac{q_s}{2}$; this is the family of Proposition 11.1 with $a = 1/s$). For $\omega \equiv 0$: $P(0) = P_0$ and $Q(0)$ leave e_1 unchanged, so $e_1 \equiv \varepsilon_1$ and $r(0) = (x+y)\varepsilon_1$. (a) At the origin the frame is I, so $r_y(q_s)(0,0) = \cos\pi\,\varepsilon_1 = -\varepsilon_1$, whereas $r_y(0)(0,0) = \varepsilon_1$. (b) $\partial_y\Delta r = (\cos q_s, \sin q_s, 0)F(q_s) - (1,0,0)F(0) = (\cos q_s - 1, \sin q_s, 0)F(0) + (\cos q_s, \sin q_s, 0)(F(q_s) - F(0))$, so $|\partial_y\Delta r| \geq |e^{iq_s} - 1| - \|F(q_s) - F(0)\|$. Here $|e^{iq_s} - 1| = 2|\sin(q_s/2)| = 2\,\mathrm{sech}(sx + y/s) \geq 2\,\mathrm{sech}\,2 > \frac{1}{2}$ on $\{|y| \leq s\}$ if $sL \leq 1$, and by Lemma 29, $\|F(q_s) - F(0)\| \leq \delta L + \Lambda \leq (2L + \pi^2)s$ (using $\delta \leq 2s$, $\Lambda \leq \pi^2 s$ from Lemmas 5, 10, with $T_*(0) = 0$). For s small this is $\leq \frac{1}{4}$ on a set of measure $4Ls$. (c),(d) Compute at $x = 0$. There $q_s(0,y) = \hat\Theta(y/s)$ and $\|F(q_s)(0,y) - I\| \leq |y|$ (Lemma 4b, $\|Q\| = 1$), so $|e_i(0,t) - \varepsilon_i| \leq |t|$. Hence, for $T \in \mathbb{R}$, with $g(\tau) = (\cos\hat\Theta(\tau) - 1)\varepsilon_1 + \sin\hat\Theta(\tau)\,\varepsilon_2$,

$$\Delta_s(0,sT) = \frac{1}{s}\int_0^{sT} \big[(\cos q_s - 1)e_1 + \sin q_s\, e_2 + (e_1 - \varepsilon_1)\big]\,dt = \int_0^T g(\tau)\,d\tau + E, \qquad |E| \leq 2sT^2.$$

(The integrand of the error is bounded by $(2+1+1)|t|$.) Now $\hat\Theta(-\tau) = 2\pi - \hat\Theta(\tau)$, so $\sin\hat\Theta$ is odd and $\cos\hat\Theta$ is even, and $\sin(\hat\Theta/2) = \mathrm{sech}\,\tau$; hence $\int_{-1}^1 g = -2\int_{-1}^1 \mathrm{sech}^2\tau\,d\tau\,\varepsilon_1 = -4\tanh 1\,\varepsilon_1$, which gives (d). From (d), $\max(|\Delta_s(0,s)|, |\Delta_s(0,-s)|) \geq 2\tanh 1 - 2s \geq 1$ for $s \leq \frac{1}{4}$, which is (c). The ε_2-component is $\int_0^1 \sin\hat\Theta\,d\tau + O(s)$, and $\sin\hat\Theta > 0$ on $(0,\infty)$. A uniformly convergent sequence of continuous functions on a compact set is equicontinuous, whence the non-differentiability. □

11.6 One step outside the boundary layer

Propositions 11.1–11.3 exhibit a boundary layer of width $O(1/a)$ at the axis, across which ω jumps by 2π. Outside the layer the solution is close to an explicit quasi-equilibrium:

Proposition 11.5 (one-step asymptotics off the boundary layer)**.** Let $g \in \mathcal{S}$, $a > 0$, $c \in \mathbb{R}$, and let q_a be the solution of (BT) with seed g and $q_a(0,0) = c$ (Lemma 3); put $u := (q_a + g)/2$ and $u_0 := (c + g(0,0))/2 = u(0,0)$. Assume

$$d_0 := \mathrm{dist}(u_0, \pi\mathbb{Z}) > 0.$$

Let k_+ be the even and k_- the odd integer nearest to u_0/π (both well defined because $u_0/\pi \notin \mathbb{Z}$, i.e. precisely because $d_0 > 0$: the nearest *even* integer is ambiguous only when u_0/π is an odd integer, the nearest *odd* one only when u_0/π is an even integer). Then for every $\varepsilon > 0$ there are $a_0 = a_0(d_0, \varepsilon, L, \|g\|_{C^2(W)})$ and $C = C(d_0, \varepsilon, L, \|g\|_{C^2(W)})$ such that for all $a \geq a_0$, uniformly in $x \in [-L, L]$,

$$\sup_{y\in[\varepsilon,L]} \Big| q_a(x,y) + g(x,y) - 2\pi k_+ - \frac{2}{a} g_y(x,y) \Big| \leq \frac{C}{a^2},$$
$$\sup_{y\in[-L,-\varepsilon]} \Big| q_a(x,y) + g(x,y) - 2\pi k_- + \frac{2}{a} g_y(x,y) \Big| \leq \frac{C}{a^2}.$$

Proof. Write $G := \|g_y\|_{C^0}$, $G_2 := \|g_{yy}\|_{C^0}$ and let a be large (several lower bounds below). The potential u solves (Lemma 3)

$$u_x = -\frac{1}{a}\sin(u - g), \qquad u_y = g_y - a\sin u. \tag{3}$$

Step 0 (axis drift). From (3), $|u_x| \leq 1/a$, so $|u(x,0) - u_0| \leq L/a \leq d_0/2$ for $a \geq 2L/d_0$. Hence $\operatorname{dist}(u(x,0), \pi\mathbb{Z}) \geq d_0/2$ and, since $|u_0/\pi - k_+| \leq 1 - \delta'$ with $\delta' := d_0/\pi$ and $L/(a\pi) \leq \delta'/2$,

$$u(x,0) \in \big((k_+ - 1)\pi + \tfrac{d_0}{2},\ (k_+ + 1)\pi - \tfrac{d_0}{2}\big) \quad \text{for all } x.$$

Step 1 (basin, forward). Fix x and integrate the second equation of (3) in y from the initial value $u(x,0)$. Set $\rho_a := K/\sqrt{a}$ with $K := \pi(G+1)$, and assume $a \geq a_1$ so that $a\sin(d_0/4) \geq 2G$ and $(2K/\pi)\sqrt{a} \geq 2G$. *Invariance:* the interval $I_a := [k_+\pi - \rho_a,\ k_+\pi + \rho_a]$ is invariant: at $u = k_+\pi + \rho_a$ (recall k_+ is even, so $\sin u = \sin \rho_a > 0$)

$$u_y \leq G - a\sin\rho_a \leq G - a\cdot\tfrac{2}{\pi}\rho_a = G - \tfrac{2K}{\pi}\sqrt{a} < 0,$$

and symmetrically at $u = k_+\pi - \rho_a$. Moreover u cannot leave $((k_+ - 1)\pi,\ (k_+ + 1)\pi)$: at distance ρ_a from an endpoint the same estimate shows u_y points inwards. *Attraction:* while $\rho_a \leq |u - k_+\pi| \leq \pi - d_0/4 < \pi$, the sign of $\sin(u - k_+\pi)$ is that of $u - k_+\pi$, so $u_y = g_y - a\sin(u - k_+\pi)$ points towards $k_+\pi$ whenever $a|\sin(u - k_+\pi)| > G$. We bound the speed in two sub-regions. For $\rho_a \leq |u - k_+\pi| \leq \pi/2$ we have $|\sin(u - k_+\pi)| \geq \frac{2}{\pi}|u - k_+\pi| \geq \frac{2}{\pi}\rho_a$, so

$$|u_y| \geq a\cdot\tfrac{2}{\pi}\rho_a - G = \tfrac{2K}{\pi}\sqrt{a} - G > 0.$$

For $\pi/2 < |u - k_+\pi| \leq \pi - d_0/4$ we have $|\sin(u - k_+\pi)| = \sin\big(\pi - |u - k_+\pi|\big) \geq \sin(d_0/4)$, so

$$|u_y| \geq a\sin(d_0/4) - G \ \geq\ \tfrac{a}{2}\sin(d_0/4) > 0$$

for $a \geq a_1$ (by the definition of a_1). In both sub-regions the velocity is directed towards $k_+\pi$ and bounded below by $\min\big(\frac{2K}{\pi}\sqrt{a} - G,\ \frac{a}{2}\sin(d_0/4)\big)$, so u enters I_a within time

$$\tau_1 \leq \frac{\pi/2}{\frac{2K}{\pi}\sqrt{a} - G} + \frac{\pi}{\frac{a}{2}\sin(d_0/4)} \leq \frac{C_1}{\sqrt{a}},$$

uniformly in x (enlarging C_1).

Step 2 (tracking, forward). For $y \geq \tau_1$ write $u = k_+\pi + e_a + v$ with $e_a(y) := g_y(x,y)/a$ (the exact equilibrium branch would be e_a), so

$$v' = u_y - e_a' = g_y - a\sin(e_a + v) - \frac{g_{yy}}{a} = -a\,v - \frac{g_{yy}}{a} + a\,\theta, \qquad |\theta| \leq \frac{(|e_a| + |v|)^3}{6},$$

using $\sin(e_a+v) = e_a+v+O\big((|e_a|+|v|)^3\big)$ and $g_y = ae_a$. At $y = \tau_1$, $|v| \leq \rho_a + G/a \leq K'/\sqrt{a}$. Choose $a \geq a_2$ so that $5\ln a/a \leq \varepsilon/3$ and $C_1/\sqrt{a} \leq \varepsilon/3$. *Round 1:* while $|e_a| + |v| \leq K''/\sqrt{a}$ the forcing is bounded by $G_2/a + C''/\sqrt{a} \leq C'''/\sqrt{a}$, so

$$|v(y)| \leq e^{-a(y-\tau_1)}\frac{K'}{\sqrt{a}} + \int_{\tau_1}^{y} e^{-a(y-t)}\frac{C'''}{\sqrt{a}}\,dt \leq e^{-a(y-\tau_1)}\frac{K'}{\sqrt{a}} + \frac{C'''}{a\sqrt{a}};$$

at $y \geq \tau_1 + 5\ln a/a$ the exponential term is $O(a^{-11/2})$, so $|v| \leq C_4/a^{3/2}$, and $|e_a|+|v| \leq C_5/a$. *Round 2:* now the forcing is bounded by $G_2/a + C_6/a^2 \leq C_7/a$, and repeating the estimate gives, for $y \geq \tau_1 + 10\ln a/a$,

$$|v(y)| \leq e^{-a(y-\tau_1-5\ln a/a)}\frac{C_4}{a^{3/2}} + \frac{C_7}{a^2} \leq \frac{C_8}{a^2}.$$

Since $\tau_1 + 10\ln a/a \leq \varepsilon$ for $a \geq a_0$, this holds on $[\varepsilon, L]$, uniformly in x. Finally

$$q_a = 2u - g = 2\pi k_+ - g + \tfrac{2}{a}g_y + 2v \qquad \Rightarrow \qquad \Big|q_a + g - 2\pi k_+ - \tfrac{2}{a}g_y\Big| \leq \frac{2C_8}{a^2},$$

which is the first estimate (k_+ even, so $(-1)^{k_+} = +1$).

Step 3 (backward). For $y < 0$ substitute $\tau = -y$ and $\tilde{u}(\tau) := u(x,-\tau)$, so

$$\tilde{u}_\tau = -g_y(x,-\tau) + a\sin\tilde{u}.$$

Now the stable equilibria (in the τ-direction) are the odd multiples: near $\tilde{u} = k_-\pi + w$, $\tilde{u}_\tau = -g_y - a\sin w$ with equilibrium $w = -g_y(x,-\tau)/a$; the linearized rate is $+a > 0$ towards it. The initial value $\tilde{u}(0) = u(x,0)$ lies in $((k_- - 1)\pi + d_0/2,\ (k_- + 1)\pi - d_0/2)$ by Step 0 (with $k_- = k_+ \pm 1$), and the same basin argument (Step 1, with I_a centred at $k_-\pi$ and the inward checks using $\cos(k_-\pi) = -1$, i.e. the sign of $\sin\tilde{u}$ reversed) gives $|\tilde{u} - k_-\pi| \leq \rho_a$ for $\tau \geq \tau_1$, and the same two-round tracking with $e_a^- := -g_y(x,-\tau)/a$ gives $|\tilde{u} - k_-\pi - e_a^-| \leq C_8/a^2$ for $\tau \in [\varepsilon, L]$. Returning to $y = -\tau$:

$$u(x,y) = k_-\pi - \frac{g_y(x,y)}{a} + O(a^{-2}), \qquad q_a = 2u - g = -g + 2\pi k_- - \frac{2}{a}g_y + O(a^{-2}),$$

uniformly on $y \in [-L, -\varepsilon]$. This is the second estimate (k_- odd, so $(-1)^{k_-} = -1$). □

Remark 11.5 (structure of the layer)**.** *(a)* In general $k_- = k_+ \pm 1$: the two half-windows are governed by different constants, and q_a jumps by 2π across the layer of width $O(\ln a/a)$ around $y = 0$. This is consistent with Proposition 11.1 (there $u_0 = \pi/2$, $k_+ = 0$, $k_- = 1$, and the half-window limits are 0 and 2π); in particular a *single* window-wide constant k (as one might naively guess from $k_+ = \mathrm{round}\big((c + g(0,0))/(2\pi)\big)$ at the origin) is correct only on the half-window $\{y > 0\}$, and only when that round happens to be even. *(b)* The

hypothesis $d_0 > 0$ excludes the "stall" values $u_0 \in \pi\mathbb{Z}$—and this is all that is needed: under $d_0 > 0$ the nearest even and the nearest odd integers $k_\pm$ are automatically unique (they are ambiguous only when u_0/π is an integer, which $d_0 > 0$ excludes). By Step 0 of the proof the hypothesis can be checked directly on the data: $d_0 > 0$ is equivalent to $(c+g(0,0))/(2\pi) \notin \mathbb{Z}$. *(c)* Inside the layer no uniform $O(1/a^2)$ expansion of q_a exists; the contribution of the layer to the immersion is $O(1/a)$ (Theorem 1), and Proposition 11.3 shows this contribution is genuinely of first order—the layer, not the exterior, carries the sharp constant. *(d)* For $n > 1$ the exterior expansion would have to be propagated through subsequent Bäcklund steps, and the layer of each step interacts with the next one; we do not pursue this here (the resulting partial limits are nevertheless controlled by Theorem 38 below).

11.7 Other limits and cases

11.7.1 The normalization constants c_j

Lemma 31 (4π-periodicity in the constants)**.** *Fix a seed $g \in \mathcal{S}$ and $a > 0$. If q solves (BT) with seed g, then $q + 4\pi$ solves (BT) with the same seed g and the same a. Consequently $q_{c+4\pi} = q_c + 4\pi$ (Lemma 3), the whole chain satisfies $\omega_n(c_j + 4\pi) = \omega_n(c_j) + 4\pi$ in the shifted component, and the immersion does not change: $r(\omega + 4\pi k) = r(\omega)$ for every $k \in \mathbb{Z}$, because on the axis the map sees only ω_x and in the columns only $\sin\omega, \cos\omega$ (§4).*

Proof. Both equations of (BT) are invariant under $q \mapsto q + 4\pi$: $\sin\big(\frac{(q+4\pi)\pm g}{2}\big) = \sin\big(\frac{q\pm g}{2} + 2\pi\big) = \sin\frac{q\pm g}{2}$, while the left-hand sides are unchanged. Uniqueness (Lemma 3) identifies the shifted solution with the BT image of the shifted constant; induction along the chain gives the statement for ω_n, and the last claim follows from the definition of r. □

The period is 4π, *not* 2π:

Example 1 (different branches for c and $c + 2\pi$)**.** Let $n = 1$, $g \equiv 0$. Then

$$q_\pi = 4\arctan\big(e^{-x/a-ay}\big), \qquad q_{3\pi} = 4\pi - 4\arctan\big(e^{-x/a-ay}\big)$$

are the solutions of (BT) with $q(0,0) = \pi$ and $q(0,0) = 3\pi$ respectively (direct verification as in Proposition 11.1; note $q_{3\pi} \neq q_\pi + 2\pi$, so $c \mapsto q$ is not 2π-periodic). The immersions differ: on the axis $\partial_x e_1(0,0)$ is the first row of $P(\omega_x(0,0))$, namely $(0, -\omega_x(0,0), 0)$, and

$$\omega_x(0,0) = \mp\frac{2}{a} \quad \text{for } q_\pi,\ q_{3\pi},$$

so $\partial_x e_1(0,0) = (0, \pm 2/a, 0)$; equal immersions would have equal frames along the axis ($e_1 = \partial_x r(\cdot, 0)$), a contradiction for $a > 0$.

Remark 11.6 (the limit $c_j \to \infty$)**.** By Lemma 31 the maps $(a_1, \dots, a_n, c_1, \dots, c_n, f) \mapsto r(\omega_n)$ factor through $(\mathbb{R}/4\pi\mathbb{Z})^n$. Hence: (i) if $c_j \to \infty$ and each c_j converges in $\mathbb{R}/4\pi\mathbb{Z}$, then $r(\omega_n)$ converges (Theorem 17); (ii) convergence may fail without such a limit: by Example 1, along a sequence with c alternating between π and 3π the immersions have two distinct limit points; (iii) the estimate of Theorem 1 is uniform in the c_j regardless—the non-convergence reflects the normalization, not the geometry.

11.7.2 The limit $a_j \to 0$

Let $(\sigma u)(x,y) := u(y,x)$. Then $\sigma\mathcal{S} = \mathcal{S}$ and σ preserves W: if u solves $u_{xy} = \sin u$, then $\partial_{xy}(\sigma u) = \partial_{yx} u \circ \sigma = \sin u \circ \sigma = \sin(\sigma u)$.

Lemma 32 (exact symmetries of the Bäcklund system)**.** *For $g \in \mathcal{S}$, $a > 0$, $c \in \mathbb{R}$:*

$$\mathrm{BT}_a(g;c) = \sigma\ \mathrm{BT}_{1/a}(-\sigma g;c), \qquad \mathrm{BT}_a(-g;-c) = -\,\mathrm{BT}_a(g;c).$$

Proof. Let $q = \mathrm{BT}_a(g;c)$, $Q = \sigma q$, $G = -\sigma g$. Then $(Q+G)_x = \sigma\big((q-g)_y\big) = -2a\sin\frac{\sigma q+\sigma g}{2} = -\frac{2}{A}\sin\frac{Q-G}{2}$ with $A = 1/a$, and $(Q-G)_y = \sigma\big((q+g)_x\big) = -\frac{2}{a}\sin\frac{\sigma q-\sigma g}{2} = -2A\sin\frac{Q+G}{2}$; $Q(0,0) = q(0,0) = c$. By uniqueness (Lemma 3), $Q = \mathrm{BT}_A(G;c)$, which is the first identity. The second: $(q,g) \mapsto (-q,-g)$ leaves (BT) unchanged and maps $c \mapsto -c$. □

Theorem 33 (the limit $a_j \to 0$)**.** *Define $r^\sigma(\omega)(x,y) := r(\sigma\omega)(y,x)$ (immersion of the same Chebyshev net, normalized at the origin by the frame I adapted to the y-lines: $r^\sigma_y(0,0) = \varepsilon_1$) and $T^\sigma_*(f) := \sup_{|y|\le L}\int_{-L}^{L}|f_x(x,y)|\,dx$. Then for all $a_j > 0$, $c_j \in \mathbb{R}$*

$$\|r^\sigma(\omega_n) - r^\sigma(f)\|_{L^\infty(W)} \le \big[3L^2 + (1+L)\pi(2T^\sigma_*(f)+\pi)3^{n-1}\big]\sum_{j=1}^{n} a_j.$$

For the Amsler seed $T^\sigma_* \le 2L^2$. Thus, in contrast to $a_j \to \infty$ (the reflection $\omega \mapsto -\omega$ at each step), $a_j \to 0$ acts as the identity: no reflection occurs.

Proof. Put $\zeta_j := \sigma\omega_j$. By Lemma 32 (first identity applied to the pair (ω_{j-1}, c_j), then the second identity), $\zeta_j = \mathrm{BT}_{A_j}(-\zeta_{j-1};c_j) = -\,\mathrm{BT}_{A_j}(\zeta_{j-1};-c_j)$ with $A_j = 1/a_j$. Let $\eta_j := (-1)^j\zeta_j$. Using that identity repeatedly, $\eta_j = \mathrm{BT}_{A_j}(\eta_{j-1};(-1)^j c_j)$ with $\eta_0 = \sigma f \in \mathcal{S}$ (induction: if $\zeta_{j-1} = (-1)^{j-1}\eta_{j-1}$, then $\zeta_j = -\,\mathrm{BT}_{A_j}((-1)^{j-1}\eta_{j-1};-c_j) = (-1)^j\,\mathrm{BT}_{A_j}(\eta_{j-1};(-1)^j c_j)$). So η is a standard chain with parameters A_j and seed σf. Theorem 1 gives $\|r(\eta_n) - r((-1)^n\sigma f)\| \le K_n(\sigma f)\sum 1/A_j = K_n(\sigma f)\sum a_j$, where $T_*(\sigma f) = T^\sigma_*(f)$ (relabeling $x \leftrightarrow y$). Now $\sigma\omega_n = (-1)^n\eta_n$ and $\sigma f = (-1)^n\big((-1)^n\sigma f\big)$; by Lemma 27, $r(\sigma\omega_n) - r(\sigma f) = (r(\eta_n) - r((-1)^n\sigma f))R^n$ and R is orthogonal. Finally $r^\sigma(\omega) = r(\sigma\omega)\circ\sigma$ is a reparametrization. For $f = Y(xy)$, $|f_x| = |y|\,|Y'(xy)| \le L$. □

Remark 11.7 (open)**.** Combined regimes (some $a_j \to 0$, others $\to \infty$) follow from Lemma 32 only for a chain in which *all* the parameters tend to 0 or all to ∞; the partial limits for $a_j \to 0$ along an arbitrary index set are Corollary 40 below. A genuinely mixed theorem (some $a_j \to 0$ and some $a_k \to \infty$ simultaneously) requires a comparison lemma for the two normalizations simultaneously and is not proved here.

11.7.3 Unbounded seed data

Proposition 11.6 (growth of T_* on a sequence of seeds)**.** Let $\beta_\lambda(y) = \lambda\phi(y)$ with $\phi \in C^\infty([-L,L])$, $\phi(0) = 0$, $\phi' \not\equiv 0$, and $\alpha_\lambda \equiv 0$. Then $f_\lambda = \Psi(\alpha_\lambda, \beta_\lambda)$ satisfies

$$T_*(f_\lambda) \;\ge\; \lambda\int_{-L}^{L}|\phi'(y)|\,dy.$$

Hence $T_*(f_\lambda) \to \infty$ linearly in λ, and the constant $K_n(f_\lambda)$ of Theorem 1 grows at least linearly.

Proof. On the characteristic line $x = 0$, $f_\lambda(0, y) = \beta_\lambda(y) = \lambda\phi(y)$, so $\partial_y f_\lambda(0, y) = \lambda\phi'(y)$ and

$$T_*(f_\lambda) = \sup_{|x|\le L} \int_{-L}^{L} |\partial_y f_\lambda(x, y)|\, dy \ge \int_{-L}^{L} \lambda|\phi'(y)|\, dy. \qquad \square$$

Remark 11.8 (no compactification of the seed data)**.** Smoothness in Theorem 17 is local in the data. If $(\alpha_\lambda, \beta_\lambda)$ leaves every bounded subset of $\mathcal{D}$, each $f_\lambda = \Psi(\alpha_\lambda, \beta_\lambda)$ is still a perfectly regular seed, and $\lambda \mapsto r(\omega_n(a, c, f_\lambda))$ is C^∞ on compact λ-intervals (Theorem 17); but by Proposition 11.6 the constant $K_n(f_\lambda)$ may grow without bound, so the rate in Theorem 24 is guaranteed uniformly in λ only while $T_*(f_\lambda)$ stays bounded. Whether the actual error grows with λ is not investigated (Proposition 11.6 concerns the constant of the *upper bound* only). No continuous extension of the flow to a "boundary of the data space" is claimed.

Remark 11.9 (periodicity of the seed itself)**.** For a *single* solution, $r(f + 2\pi k) = r(f)$: the axis sees only f_x and the columns only $\sin f, \cos f$ (§4). This does not descend to the chain: the Bäcklund system (BT) depends on the seed itself through $(q \pm g)/2$, and $g \mapsto g + 2\pi$ flips the sign of $\sin\big((q - g)/2\big)$ (it corresponds to $a \mapsto -a$), so $\mathrm{BT}_a(g + 2\pi)$ is not $\mathrm{BT}_a(g)$ shifted. The correct period of the whole construction is 4π (Lemma 31, Example 1).

11.8 Partial limits

Let $J \subset \{1, \dots, n\}$ be the set of indices whose parameters a_j will tend to infinity, $K = \{1, \dots, n\} \setminus J$. Put $c_0 := f(0, 0)$; then $w_j(0, 0) = (c_j - c_{j-1})/2$ is the value at the origin of $w = (q - g)/2$ in the j-th step. For $g \in \mathcal{S}$ and $w_0 \in \mathbb{R}$ denote by $\mathrm{BT}_a^\flat(g; w_0)$ the Bäcklund image q with $(q - g)(0, 0)/2 = w_0$, i.e. $\mathrm{BT}_a(g;\, g(0, 0) + 2w_0)$.

Definition 34 (reduced chain)**.** $\hat g_0 := f$; $\hat g_j := -\hat g_{j-1}$ for $j \in J$; $\hat g_j := \mathrm{BT}_{a_j}^\flat\big(\hat g_{j-1};\, \frac12(c_j - c_{j-1})\big)$ for $j \in K$.

For $J = \{1, \dots, n\}$, $\hat g_n = (-1)^n f = \mathrm{ref}_n$. For $J \ne \{1, \dots, n\}$ the reduced chain depends on a_k, $k \in K$, and on c. Note that $\hat g_j$ need not satisfy $\hat g_j(0, 0) = c_j$ across the steps $j \in J$; the definition fixes the normalization w_0 of each Bäcklund step, which is all the comparison below uses.

Lemma 35 (bound on the variation in y)**.** *For all $a_1, \dots, a_k > 0$ and all c_j:* $\sup_{|x|\le L} \int_{-L}^{L} |\partial_y \omega_k|\, dy \le \mathcal{T}_k := T_* + 2(3^k - 1)(T_* + \pi/2)$.

Proof. In the notation of §6, $\omega_k = h_k = \varphi + 2z_k$, so $\int |h_k'|\, dy \le T + 2V_k$, and $V_k \le (3^k - 1)(T + \pi/2)$ by Lemma 9(II), which holds for all $a_j > 0$; take the supremum in x and use $T \le T_*$. $\square$

Lemma 36 (a large step)**.** *Let $g \in \mathcal{S}$, $q = \mathrm{BT}_a(g; c)$. Then $\delta(q, -g) \le 2/a$ and $\Lambda(q, -g) \le \pi(\pi + T(g))/a$, where $T(g) := \sup_{|x|\le L} \int_{-L}^{L} |g_y|\, dy$.*

Proof. $\partial_x(q + g) = -\frac{2}{a} \sin\frac{q-g}{2}$ throughout W (the first equation of (BT)), which gives the first claim. For the second apply the case $n = 1$ of Lemmas 8–10 with seed g: $d(q + g) = 2d_\pi\big((q + g)/2\big) \le \pi\big|\sin\frac{q+g}{2}\big|$ by (E4) and (E2), and $aI_1 \le \pi + T(g)$ by Lemma 9(I). $\square$

Lemma 37 (a finite step is stable). *Let $g, \hat{g} \in \mathcal{S}$, $a > 0$, $q = \mathrm{BT}^{\flat}_a(g; w_0)$, $\hat{q} = \mathrm{BT}^{\flat}_a(\hat{g}; w_0)$ (the same w_0). Then, with $\delta = \delta(g, \hat{g})$ and $\Lambda = \Lambda(g, \hat{g})$,*

$$\delta(q, \hat{q}) \le \kappa\, \delta, \qquad \Lambda(q, \hat{q}) \le \mu\, \Lambda + \nu\, \delta,$$

$$\kappa = 1 + \tfrac{2L}{a} e^{L/a}, \qquad \mu = 1 + 4aLe^{aL}, \qquad \nu = 4L^2 e^{aL + L/a}.$$

Proof. Let $w = (q - g)/2$, $\hat{w} = (\hat{q} - \hat{g})/2$. On the axis, $w_x = u_x - g_x = -\frac{1}{a}\sin w - g_x$ (with $u = w + g$), so $|\partial_x(w - \hat{w})| \le \delta + \frac{1}{a}|w - \hat{w}|$ and $w(0,0) = \hat{w}(0,0) = w_0$; Gronwall gives $|w - \hat{w}\,|(x,0) \le a\delta(e^{|x|/a} - 1) \le \delta|x|e^{L/a}$. Since $q_x = -g_x - \frac{2}{a}\sin w$, $|\partial_x(q - \hat{q})(x,0)| \le \delta + \frac{2}{a}|w - \hat{w}| \le \delta\kappa$. In the columns $w_y = u_y - g_y = -a\sin(w + g)$, and $|\sin\alpha - \sin\beta| \le |e^{i\alpha} - e^{i\beta}| \le d(\alpha - \beta) \le d(\alpha_1 - \beta_1) + d(\alpha_2 - \beta_2)$ for $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2$; hence $|(w - \hat{w})_y| \le a|w - \hat{w}| + a\, d(g - \hat{g})$, and Gronwall gives $|w - \hat{w}\,|(x,y) \le e^{a|y|}\big(|w - \hat{w}\,|(x,0) + a\Lambda\big)$. Finally $q - \hat{q} = 2(w - \hat{w}) + (g - \hat{g})$, so $d(q - \hat{q}) \le 2|w - \hat{w}| + d(g - \hat{g})$, and, integrating over an interval of length $2L$, $\Lambda(q, \hat{q}) \le 4L \sup|w - \hat{w}| + \Lambda \le 4Le^{aL}(L\delta e^{L/a} + a\Lambda) + \Lambda$. □

Theorem 38 (partial limits). *For all $a_j > 0$, $c_j \in \mathbb{R}$ and all $J \subset \{1, \dots, n\}$,*

$$\|r(\omega_n) - r(\hat{g}_n)\|_{L^\infty(W)} \le \Big(\tfrac{3}{2}L^2 + 1 + L\Big) \sum_{j \in J} \frac{\pi(\pi + \mathcal{T}_{j-1})}{a_j} \prod_{\substack{k \in K \\ k > j}} \rho_k, \qquad \rho_k := \kappa_k + \mu_k + \nu_k,$$

where κ_k, μ_k, ν_k are the constants of Lemma 37 for $a = a_k$, and $\hat{g}_n$ is the reduced chain of Definition 34. In particular, if $a_j \to \infty$ for $j \in J$ while a_k, $k \in K$, stay in a compact subset of $(0, \infty)$, then $r(\omega_n) \to r(\hat{g}_n)$ uniformly with rate $O(\sum_{j \in J} 1/a_j)$, uniformly in the constants c_j. The steps $j \in J$ may occur anywhere in the chain, including non-trailing positions.

Proof. Let $E_j := \max\{\delta(\omega_j, \hat{g}_j),\, \Lambda(\omega_j, \hat{g}_j)\}$, $E_0 = 0$. If $j \in J$: $\omega_j - \hat{g}_j = (\omega_j + \omega_{j-1}) - (\omega_{j-1} - \hat{g}_{j-1})$, so by Lemma 36 (with seed ω_{j-1} and $T(\omega_{j-1}) \le \mathcal{T}_{j-1}$ by Lemma 35), the subadditivity of d, and the same splitting for δ,

$$\delta(\omega_j, \hat{g}_j) \le \delta(\omega_{j-1}, \hat{g}_{j-1}) + \frac{2}{a_j}, \qquad \Lambda(\omega_j, \hat{g}_j) \le \Lambda(\omega_{j-1}, \hat{g}_{j-1}) + \frac{\pi(\pi + \mathcal{T}_{j-1})}{a_j};$$

since $\pi(\pi + \mathcal{T}_{j-1}) \ge 2$, $E_j \le E_{j-1} + \sigma_j/a_j$ with $\sigma_j = \pi(\pi + \mathcal{T}_{j-1})$. If $j \in K$: both ω_j and $\hat{g}_j$ are $\mathrm{BT}^{\flat}_{a_j}$-images with the same $w_0 = \frac{1}{2}(c_j - c_{j-1})$ (for ω_j this holds because $\omega_{j-1}(0,0) = c_{j-1}$; for $\hat{g}_j$ it is the definition), so Lemma 37 gives $E_j \le \rho_j E_{j-1}$. By induction, $E_n \le \sum_{j \in J} \frac{\sigma_j}{a_j} \prod_{k \in K,\, k > j} \rho_k$. Lemma 29 gives $\|r(\omega_n) - r(\hat{g}_n)\| \le \frac{3}{2}L^2\delta_n + (1 + L)\Lambda_n \le (\frac{3}{2}L^2 + 1 + L)E_n$. □

Corollary 39 (uniformity in the other parameters; the closed orthant). (i) *For $J = \{n\}$: $\|r(\omega_n) - r(-\omega_{n-1})\|_\infty \le (\frac{3}{2}L^2 + 1 + L)\,\pi(\pi + \mathcal{T}_{n-1})/a_n$, uniformly in $a_1, \dots, a_{n-1} > 0$ and in all c_j; this strengthens Proposition 11.7 below.* (ii) *Put $\varepsilon_j := 1/a_j$. The map $(\varepsilon, c) \mapsto r(\omega_n) \in C^0(W)$, defined for $\varepsilon \in (0, \infty)^n$, extends continuously to $[0, \infty)^n \times \mathbb{R}^n$: for every $(\varepsilon^0, c^0) \in [0, \infty)^n \times \mathbb{R}^n$ the limit of $r(\omega_n(\varepsilon, c))$ as $(\varepsilon, c) \to (\varepsilon^0, c^0)$ through the interior exists and equals $r(\hat{g}_n)$, computed with $J = J(\varepsilon^0) := \{j : \varepsilon^0_j = 0\}$ and the parameters $(1/\varepsilon^0_k)_{k \in K}$, c^0. The convergence is Lipschitz in ε_J, uniformly in c, and the boundary value is a smooth function of (ε_K, c).*

Proof. (i) is Theorem 38 with $J = \{n\}$: here $\hat{g}_j = \omega_j$ for $j < n$ (all previous steps are K-steps with the same seed and the same w_0, so $\hat{g}_j(0,0) = c_j$ inductively and uniqueness, Lemma 3, identifies $\hat{g}_j = \omega_j$), and $\hat{g}_n = -\omega_{n-1}$; the constant $\mathcal{T}_{n-1}$ depends only on (L, n, T_*), which gives the uniformity. (ii) Near ε^0 the parameters $a_k = 1/\varepsilon_k$, $k \in K$, stay in a compact subset of $(0,\infty)$, so Theorem 38 bounds $\|r(\omega_n) - r(\hat{g}_n)\|$ by $C\sum_{j\in J}\varepsilon_j$, uniformly in c; and $\hat{g}_n$ is a smooth function of (ε_K, c) (Theorem 17: the reduced chain is a chain of Bäcklund steps with parameters $a_k = 1/\varepsilon_k$ and smoothly varying seeds, the steps $j \in J$ being the exact sign flip $g \mapsto -g$). The corner case $J = \{1,\dots,n\}$ is Theorem 1. □

Corollary 40 (partial limits $a_j \to 0$ along an index set)**.** *Let $J \subset \{1,\dots,n\}$, K its complement, and let $a_j \to 0$ for $j \in J$ while a_k, $k \in K$, stay in a compact subset of $(0,\infty)$. Let $\eta = \eta(\sigma f; A, \tilde{c})$ be the chain of the proof of Theorem 33 (seed σf, parameters $A_j := 1/a_j$, constants $\tilde{c}_j := (-1)^j c_j$), and let $\hat{g}^\eta_n$ be its reduced chain (Definition 34) for the index set J. Then, uniformly in $c_1,\dots,c_n$,*

$$\|r^\sigma(\omega_n) - r\big((-1)^n\hat{g}^\eta_n\big)\circ\sigma\|_{L^\infty(W)} \le \Big(\tfrac{3}{2}L^2 + 1 + L\Big)\sum_{j\in J} a_j\,\pi\big(\pi + \mathcal{T}^\eta_{j-1}\big)\prod_{\substack{k\in K\\ k>j}}\rho_k,$$

where $\mathcal{T}^\eta_{j-1} := T^\sigma_*(f) + 2(3^{j-1}-1)\big(T^\sigma_*(f) + \pi/2\big)$ and ρ_k are the constants of Lemma 37 for the parameters $A_k = 1/a_k$. In particular, $r^\sigma(\omega_n) \to r((-1)^n\hat{g}^\eta_n)\circ\sigma$ uniformly with rate $O(\sum_{j\in J} a_j)$. For $J = \{1,\dots,n\}$ this recovers Theorem 33 (there with the constant of Theorem 1, which differs from the one above).

Proof. Apply Theorem 38 to the η-chain: its large parameters are $A_j \to \infty$ exactly for $j \in J$, its K-step parameters $A_k = 1/a_k$ stay in a compact subset of $(0,\infty)$, and its constants $\tilde{c}_j = (-1)^j c_j$ range over all of $\mathbb{R}^n$ as c does, so the uniformity in $\tilde{c}$ is the uniformity in c. This gives

$$\|r(\eta_n) - r(\hat{g}^\eta_n)\| \le \Big(\tfrac{3}{2}L^2+1+L\Big)\sum_{j\in J}\frac{\pi(\pi+\mathcal{T}^\eta_{j-1})}{A_j}\prod_{\substack{k\in K\\ k>j}}\rho_k = \Big(\tfrac{3}{2}L^2+1+L\Big)\sum_{j\in J} a_j\,\pi\big(\pi+\mathcal{T}^\eta_{j-1}\big)\prod_{\substack{k\in K\\ k>j}}\rho_k,$$

where $\mathcal{T}^\eta_{j-1}$ is the quantity of Lemma 35 for the η-chain, whose seed σf has $T_*(\sigma f) = T^\sigma_*(f)$ (relabeling $x \leftrightarrow y$). On the other hand, by the construction of the proof of Theorem 33, $\sigma\omega_n = (-1)^n\eta_n$ and $r^\sigma(\omega_n) = r(\sigma\omega_n)\circ\sigma$; hence, by Lemma 27 ($r((-1)^n u) = r(u)R^n$ with orthogonal R) and the compatibility of σ with the norms,

$$\|r^\sigma(\omega_n) - r\big((-1)^n\hat{g}^\eta_n\big)\circ\sigma\|_{L^\infty(W)} = \|r\big((-1)^n\eta_n\big) - r\big((-1)^n\hat{g}^\eta_n\big)\|_{L^\infty(W)} = \|r(\eta_n) - r(\hat{g}^\eta_n)\|_{L^\infty(W)},$$

which is the claim. For $J = \{1,\dots,n\}$ the reduced chain is $\hat{g}^\eta_n = (-1)^n\sigma f$, so the left-hand side is $\|r^\sigma(\omega_n) - r^\sigma(f)\|$ and the statement reduces to Theorem 33 (whose constant, coming from Theorem 1, differs from the one above). □

Corollary 41 (stratified Lipschitz at the faces of the orthant)**.** *Fix $(\varepsilon^0, c^0) \in [0,\infty)^n \times \mathbb{R}^n$, put $J := \{j : \varepsilon^0_j = 0\}$ and K its complement, and fix $\delta > 0$ with $\varepsilon^0_k \ge 2\delta$ for $k \in K$. Then for all $\varepsilon, \bar{\varepsilon}$ in the stratum*

$$S_\delta := \big\{\varepsilon :\ |\varepsilon_k - \varepsilon^0_k| \le \delta\ (k\in K),\ 0 < \varepsilon_j \le \delta\ (j \in J)\big\}$$

and all $|c-c^0|\le\delta$, $|\bar c-c^0|\le\delta$,

$$\|r(\omega_n(\varepsilon,c))-r(\omega_n(\bar\varepsilon,\bar c))\|_{L^\infty(W)}\le C_*\Big(\sum_{k\in K}|\varepsilon_k-\bar\varepsilon_k|+\sum_{j\in J}(\varepsilon_j+\bar\varepsilon_j)+|c-\bar c|\Big),$$

with $C_*=C_*(n,L,T_*,\delta,\varepsilon^0)$*; in particular* $C_1=\big(\frac32L^2+1+L\big)\max_{j\in J}\pi(\pi+\mathcal{T}_{j-1})\prod_{k\in K}\bar\rho_k$*, where* $\bar\rho_k:=\max\{\kappa+\mu+\nu$ *(Lemma 37 with parameter* a*)* $:\ a\in[1/(\varepsilon_k^0+\delta),\,1/(\varepsilon_k^0-\delta)]\}$.

Proof. By Theorem 38 (as in the proof of Corollary 39(ii)), uniformly in c,

$$\|r(\omega_n(\varepsilon,c))-r(\hat g_n(\varepsilon_K,c))\|\le\Big(\tfrac32L^2+1+L\Big)\sum_{j\in J}\varepsilon_j\prod_{\substack{k\in K\\k>j}}\rho_k(\varepsilon_k)\le C_1\sum_{j\in J}\varepsilon_j,$$

since on the stratum $a_k=1/\varepsilon_k$ stays in the compact interval $[1/(\varepsilon_k^0+\delta),1/(\varepsilon_k^0-\delta)]\subset(0,\infty)$, so that $\rho_k\le\bar\rho_k$ (Lemma 37), and $\rho_k\ge1$. The same holds for $(\bar\varepsilon,\bar c)$. Finally, $(\varepsilon_K,c)\mapsto\hat g_n(\varepsilon_K,c)$ is C^∞ on the compact stratum (Corollary 39(ii) and Theorem 17), and $g\mapsto r(g)$ is locally Lipschitz from C^1 to C^0 (Lemma 22), so

$$\|r(\hat g_n(\varepsilon_K,c))-r(\hat g_n(\bar\varepsilon_K,\bar c))\|\le C_2\Big(\sum_{k\in K}|\varepsilon_k-\bar\varepsilon_k|+|c-\bar c|\Big).$$

The triangle inequality gives the claim with $C_*=\max(C_1,C_2)$. □

Remark 11.10 (what is not claimed)**.** The constant of Theorem 38 contains e^{a_kL} for the finite steps $k\in K$ *after* a large step; the estimate is therefore useful only for fixed (or slowly growing) finite parameters, and it is not asserted that this exponential dependence is necessary. Corollary 41 gives Lipschitz control only in the *tangential* variables (ε_K,c); in the normal directions ε_j, $j\in J$, the bound has the two-sided form $\varepsilon_j+\bar\varepsilon_j$ (through the base point), and a bound of the form $|\varepsilon_j-\bar\varepsilon_j|$ is not claimed. Proposition 11.4(d) shows, for the model kink ($n=1$), that $\varepsilon\mapsto r(\omega_1)$ is not differentiable at $\varepsilon=0$ in C^0 (although it is Lipschitz at that point, Theorem 30(i)); it does not by itself exclude a Lipschitz bound between interior points. Finally, an asymptotic expansion of $r(\omega_n)$ in powers of $1/a_j$ is in general not available in C^0 (for the model kink, Proposition 11.4); whether one exists in L^p, $p<\infty$, is not investigated here.

11.9 The Bäcklund reflection chamber

Theorem 24 describes the *total* limit (all $a_j\to\infty$ simultaneously). In this subsection we show that the *one-step* (trailing) limit is already available from it, that the Bäcklund step commutes with the sign change $g\mapsto-g$ of the seed, and that these two facts combine into a precise version of the "graph of levels connected through infinity" picture: the orbit of a fixed seed splits into two alternating, mirror-symmetric, connected unions, glued along the faces $a_n=\infty$. Throughout, we fix $f\in\mathcal{S}$ and write, for $\varepsilon\in\{+1,-1\}$ and $n\ge0$,

$$M_n^\varepsilon:=\Big\{r\big(\omega_n(a,c,\varepsilon f)\big)\ :\ a\in(0,\infty)^n,\ c\in\mathbb{R}^n\Big\}\ \subset\ C^0(W,\mathbb{R}^3),\qquad M_0^\varepsilon=\{r(\varepsilon f)\},$$

for the *levels* of the orbit (no manifold structure is claimed or needed).

Lemma 42 (levels are path-connected)**.** *Each M_n^ε is path-connected in $C^0(W,\mathbb{R}^3)$.*

Proof. The parameter set $(0,\infty)^n \times (\mathbb{R}/4\pi\mathbb{Z})^n$ is path-connected (Lemma 31 allows the reduction modulo 4π), and Φ_n is continuous into every $C^k(W,\mathbb{R}^3)$ by Theorem 17. The image of a parameter path under Φ_n is a path in M_n^ε joining the two given points (Corollary 18). □

Lemma 43 (sign equivariance of one Bäcklund step)**.** *If q solves (BT) with seed $g \in \mathcal{S}$, parameter a and normalization $q(0,0) = c$, then $-q$ solves (BT) with seed $-g$, the same a, and normalization $-q(0,0) = -c$; by uniqueness (Lemma 3) it is* the *normalized transform of $-g$. Consequently, along chains,*

$$-\omega_n(a,c,f) \;\equiv\; \omega_n(a,-c,-f) \quad \text{on } W, \qquad \text{hence} \qquad M_n^{-\varepsilon} = \big\{r(-\omega) :\; r(\omega) \in M_n^\varepsilon\big\}. \tag{4}$$

Proof. Write $q = 2u - g$ with u solving $u_x = -(1/a)\sin(u-g)$, $u_y = g_y - a\sin u$, $u(0,0) = (c + g(0,0))/2$ (Lemma 3), and put $g' := -g$, $c' := -c$, $u' := -u$, $q' := 2u' - g'$. Then $u'_x = -u_x = (1/a)\sin(u-g) = -(1/a)\sin(u'-g')$, $u'_y = -u_y = -g_y + a\sin u = g'_y - a\sin u'$, and $u'(0,0) = -u(0,0) = (c' + g'(0,0))/2$, so q' solves (BT) with seed g' and $q'(0,0) = c'$; uniqueness gives $q' = -q$. Iterating along the chain (each ω_j is again a seed, Lemma 3) gives (4); the displayed equality of level sets follows by re-parametrizing $c \mapsto -c$. □

Proposition 11.7 (one-step mirror at the trailing end)**.** Fix $n \geq 1$, the previous parameters $(a_1,\dots,a_{n-1})$, $(c_1,\dots,c_{n-1})$, and the seed f. Then, as $a_n \to \infty$, uniformly in c_n,

$$\big\|r(\omega_n) - r(-\omega_{n-1})\big\|_{L^\infty(W)} \;\leq\; \frac{K_1(\omega_{n-1})}{a_n} \;\longrightarrow\; 0, \qquad K_1(g) = 3L^2 + (1+L)\pi\big(2T_*(g) + \pi\big) < \infty.$$

Proof. $\omega_{n-1} \in \mathcal{S}$ by Lemma 3, and (ω_{n-1}, ω_n) *is* the one-step chain from the seed $g := \omega_{n-1}$ with parameter a_n and constant c_n. Apply Theorem 24 with $n = 1$ and seed g: it gives $\|r(\omega_1^{(g)}) - r(-g)\|_{L^\infty} \leq K_1(g)/a_1$, unconditionally in c_1. □

Remark 11.11 (the branch ambiguity is immaterial for immersions)**.** Proposition 11.7 lands at $r(-\omega_{n-1})$ *without* any 2π-ambiguity, because Theorem 1 is unconditional in c_n. The branch shifts $2\pi k$ that a naive mirroring prescription would suggest are in any case invisible at the level of immersions, since $r(\omega + 2\pi k) = r(\omega)$ for a single solution (Remark 11.9).

Theorem 44 (the two mirror unions are connected)**.** *Fix $f \in \mathcal{S}$ and define the two alternating unions*

$$\mathcal{C}_+ := \bigcup_{n\geq 0} M_n^{(-1)^n} = M_0^+ \cup M_1^- \cup M_2^+ \cup M_3^- \cup \cdots, \qquad \mathcal{C}_- := \bigcup_{n\geq 0} M_n^{(-1)^{n+1}}.$$

Then: (i) *for every $n \geq 1$ and $\varepsilon \in \{\pm 1\}$, $M_{n-1}^{-\varepsilon} \subset \overline{M_n^\varepsilon}$ (closure in $C^0(W,\mathbb{R}^3)$);* (ii) *$\mathcal{C}_+$ and $\mathcal{C}_-$ are connected;* (iii) *the mirror symmetry holds as an equality of sets, $\{r(-\omega) :\; r(\omega) \in \mathcal{C}_+\} = \mathcal{C}_-$ and vice versa;* (iv) *the total-limit points lie in the correct union: $r\big((-1)^n \varepsilon f\big) \in \overline{M_n^\varepsilon}$ (Theorem 24), consistently with* (i).

Proof. *(i)* Let $p = r(\omega_{n-1}(a,c,-\varepsilon f)) \in M_{n-1}^{-\varepsilon}$. By (4), $p = r\big(-\omega_{n-1}(a,-c,\varepsilon f)\big)$, and Proposition 11.7 applied to the prefix chain $\omega_{n-1}(a,-c,\varepsilon f)$ gives, as $a_n \to \infty$ (any c_n),

$$r\big(\omega_n((a,a_n),(-c,c_n),\varepsilon f)\big) \;\longrightarrow\; p,$$

with all terms in M_n^{ε}; the prefix parameters range over all of $(0,\infty)^{n-1}\times\mathbb{R}^{n-1}$, so *every* point of $M_{n-1}^{-\varepsilon}$ is obtained. *(ii)* Let $D_N := \bigcup_{0\le n\le N} M_n^{(-1)^n}$. Each D_N is connected: $D_0 = M_0^{+}$ is a point; and if D_{N-1} is connected, then $M_N^{(-1)^N} \cup M_{N-1}^{(-1)^{N-1}}$ is connected by *(i)* (a connected set together with a subset of its closure is connected), and it meets D_{N-1} in $M_{N-1}^{(-1)^{N-1}} \neq \varnothing$; hence D_N is connected. $\mathcal{C}_+ = \bigcup_N D_N$ is an increasing union of connected sets, hence connected; $\mathcal{C}_-$ follows by symmetry (replacing f with $-f$ exchanges the two). *(iii)* By (4), $\{r(-\omega) : r(\omega)\in M_n^{\varepsilon}\} = M_n^{-\varepsilon}$, and negation reverses the parity index: $M_n^{-((-1)^n)} = M_n^{(-1)^{n+1}}$. *(iv)* is Theorem 24. □

Corollary 45 (iterated trailing limits)**.** *Fix f and n. Sending $a_n\to\infty$, then $a_{n-1}\to\infty$, …, then $a_1\to\infty$ (each limit taken in the reduced chain, whose parameters are then finite), one obtains, after d steps,*

$$r\big(\omega_n(a,c,f)\big) \;\longrightarrow\; r\Big(\omega_{n-d}\big(a_1,\dots,a_{n-d},\,(-1)^d c,\,(-1)^d f\big)\Big),$$

and, for $d=n$, the value $r\big((-1)^n f\big)$ of Theorem 24. In particular the iterated trailing limit and the total limit agree.

Proof. Induction over d with Proposition 11.7 and (4): after one step, $r(\omega_n)\to r(-\omega_{n-1}) = r\big(\omega_{n-1}(a_1,\dots,a_{n-1},-c,-f)\big)$, which is a finite-parameter chain to which the next step applies verbatim; the sign and the constants flip once per step. □

Remark 11.12 (what remains open: exactly two components?)**.** Theorem 44 gives two *connected* unions; it does not claim that $\mathcal{C}_+\cap\mathcal{C}_- = \varnothing$, nor that these are the connected components of $\mathcal{C}_+\cup\mathcal{C}_-$. Both questions amount to possible *cross-parity coincidences* $r(\omega)=r(\omega')$ between immersions of chains with opposite index $\varepsilon(-1)^n$; nothing in this article excludes or establishes them. By Lemma 27 and Theorem 44(iii), $\mathcal{C}_- = \{rR : r\in\mathcal{C}_+\}$ with $R=\mathrm{diag}(1,-1,-1)$, so $\mathcal{C}_+\cap\mathcal{C}_-=\varnothing$ is equivalent to $\mathcal{C}_+\cap\mathcal{C}_+R=\varnothing$. Note the robustness of the classification: any gluing produced by the mechanism of Theorem 38 (replacing each of m infinite steps, wherever they sit in the chain, by the reflection $\omega\mapsto-\omega+2\pi k$) sends M_n^{ε} into $M_{n-m}^{\varepsilon(-1)^m}$ up to 2π shifts, and the index $\varepsilon(-1)^n$ is preserved because $(-1)^m\cdot(-1)^{n-m}=(-1)^n$; such gluings therefore organize $\mathcal{C}_\pm$ further but cannot merge them. Finally, if the seed itself is allowed to vary, the picture becomes trivially connected: $\mathcal{D}$ is affine, so (α,β) and $(-\alpha,-\beta)$ are joined by a straight segment, and Corollary 18 yields a continuous path of immersions from $r(f)$ to $r(-f)$—a path of a different nature than Bäcklund iteration.

Remark 11.13 (reflection chamber: informal picture)**.** The structure exhibited above is a chamber picture for the group $\mathbb{Z}/2$: the open chamber is $(0,\infty)^n$ (the levels), the walls are $a_j=\infty$, the wall reflection is $g\mapsto -g$ with $R^2=\mathrm{id}$ (Lemma 43), and crossing the *trailing* wall $a_n=\infty$ freezes the immersion on the mirror of the previous level, while the solution itself in general has no C^0 limit (Proposition 11.1; degenerate "stall" data, for which the layer disappears and ω_n does converge, are not excluded)—the "stop of the seed change" is the generic behaviour. Two honest caveats. First, the walls $a_j=\infty$, $j<n$, in the middle of the chain are now glued as well: Theorem 38 and Corollary 39(ii) give the continuous extension to the whole closed orthant $\varepsilon\in[0,\infty)^n$, with the same reflection mechanism as for

the trailing wall (Proposition 11.7); what is *not* covered is the mixed regime $a_j \to 0$ together with $a_k \to \infty$ (Remark 11.7) and the uniformity in finite parameters that grow together with the infinite ones (Remark 11.10). Remark 11.12 shows that all these gluings respect the same two unions. Second, "two mirror chambers" is a statement about the *parametrized* orbit; at the level of immersion sets the two unions may a priori meet (Remark 11.12).

12 Numerical illustrations (not part of the proof)

The experiments below illustrate the behavior of the estimates of §§5–8; they are not used in any step of the proof.

Method. The seed Y is taken from an ODE with a series start; the chain ω_j, the frames, and r for both maps are integrated along the axis in x (DOP853), then along the columns in both directions in y (LSODA, rtol $10^{-10} \dots 10^{-12}$); the grid in y has 20001–30001 points and 7–9 columns.

Results.

- Lemma 8: the ratio "left-hand side/right-hand side" is < 1 everywhere, with maximum 0.95.
- Lemma 10: $\Lambda/(\text{estimate}) \le 0.69$.
- Lemma 11: the maximum ratio "error/estimate" is 0.996 on three seeds (Amsler $s = 0.5$; kink $4\arctan(e^{kx+y/k})$, $k = 0.7$; breather with parameter 0.6), $n = 1, 2, 3$, random $c_j \in [-6, 6]$, $L = 2$.
- Realistic scale: $\sup|\Delta r| \approx 2\text{–}10 \cdot S$ for the Amsler seed at $n = 1, \dots, 4$, $a_j \in [20, 100]$, $s \in \{0.05; 0.3; 0.5; 0.7; 0.95\}$, $L \in \{2; 2.5; 3\}$.
- Control test with a deliberately wrong reference ($+f$ instead of $-f$ at $n = 1$): $\sup|\Delta r| = 3.97$ versus 0.13 with the correct reference—the computational scheme distinguishes the references.
- The constant K_n of Theorem 1 is far from attained: for the kink seed $4\arctan(e^{0.7x+y/0.7})$, $L = 2$, $a_j \in [20, 60]$, random $c_j \in [-6, 6]$, the ratio $\sup|\Delta r|/(K_n S)$ is ≈ 0.04 for $n = 1$ and ≈ 0.01 for $n = 2$ (while the intermediate bound of Lemma 11 is attained up to a factor ≈ 0.96). Only the order $1/a$ is sharp.

An interactive demonstration of the estimates of Theorem 1 for $n = 1, 2$ (Amsler seed) and of the second Bäcklund iteration has been published in [11].

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ORCID

Alexander Timmermans (ORCID: 0009-0007-1942-3934)
Anton Y. Kalmykov (ORCID: 0009-0001-7185-4367)

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A The interactive demonstration screenshot

