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VIETA INVARIANTS AND MARKOFF–HURWITZ ORBITS

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Abstract. We classify rational Vieta invariants for Guéritaud's family in characteristic zero, including every nonzero fiber. For each positive integer coefficient and each dimension residue class, we construct infinitely many Markoff–Hurwitz equations with infinitely many maxima shared by at least any prescribed number of distinct positive integer orbits.

1. Introduction

Markoff's study of the minima of indefinite binary quadratic forms led to the equation $x^2+y^2+z^2 = 3xyz$ [11]. Its positive integer solutions are generated from $(1,1,1)$ by permuting the coordinates and by moves such as $z \mapsto 3xy - z$. This move exchanges the roots of the quadratic in z. Frobenius asked whether the largest coordinate determines the sorted triple [8]. Zagier proved that the number of sorted triples with largest coordinate at most B grows asymptotically as a positive constant times $(\log B)^2$ as B tends to infinity [12]. Each triple is counted separately, even when two triples have the same maximum. McGinn studied the corresponding uniqueness question for three-variable equations with an additive parameter [7].

Hurwitz asked for positive integers whose sum of squares is divisible by their product [6]. This gives the equation

$$x_1^2 + \cdots + x_n^2 = kx_1 \cdots x_n, \qquad n \geq 3, \quad k \geq 1,$$

where n and k are integers. Replacing x_i by $k\prod_{j\neq i} x_j - x_i$ again exchanges two roots and preserves positive integrality. Descent in the sum of the coordinates reduces the solutions of a fixed equation to finitely many basic solutions, which have no decreasing Vieta move. Herzberg gave an algorithm for determining the coefficients that admit positive integer solutions in a prescribed dimension [10]. Baragar studied basic solutions, giving constructions with unbounded numbers of basic solutions. His construction using split primes already gives arbitrarily many basic solutions with $k = 3$ [1, Theorem 5], [2]. Baragar also studied the growth of the individual solution trees [9]. Gamburd, Magee and Ronan obtained an asymptotic formula for the number of solutions of the higher-dimensional equation in bounded height [4].

The root exchanges also act on rational functions. Guéritaud allowed arbitrary squarefree terms of degree less than n in the numerator of the sum of squares divided by the product. He proved that this ratio is preserved and that all iterated coordinates are Laurent polynomials [5, §5]. Chen and Huang determined the rational and Laurent invariants of the classical three-variable action and of several generalized three-variable actions [3, Theorems 5.8 and 6.4]. Theorem 1 gives the fixed field for the full Guéritaud family and proves that on each nonzero level of the ratio the fixed functions are constant. For the pure sum of squares, we also show that each positive integer Vieta orbit is Zariski dense in its level.

Theorem 2 concerns the largest coordinates shared by different integer orbits in the same equation. Fix a positive integer coefficient, a residue class of dimensions and a positive integer R. There are infinitely many dimensions in that class for which infinitely many

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largest coordinates are each shared by at least R distinct orbits. We do this by producing basic solutions with the same product and then synchronizing the growth of two coordinates in all of their orbits. The theorem gives a lower bound for the number of orbits that share each of the resulting maxima.

2. Fixed fields of Vieta involutions

Let F be a field of characteristic zero, let $n \geq 3$, and let $x_1, \ldots, x_n$ be algebraically independent over F. For every proper subset I of $\{1, \ldots, n\}$, choose $a_I \in F$. An empty product is understood to be 1. Put

$$H = \frac{\sum_{i=1}^{n} x_i^2 + \sum_{I \subsetneq \{1,\ldots,n\}} a_I \prod_{i \in I} x_i}{\prod_{i=1}^{n} x_i} \quad \text{in } F(x_1, \ldots, x_n).$$

For $h \in F$, let X_h be the affine hypersurface obtained by clearing the denominator in $H = h$. The Vieta involution in direction i exchanges the roots of this monic quadratic in x_i. On the ambient rational function field the involution fixes the other coordinates and sends x_i to

$$\frac{\displaystyle\sum_{j \neq i} x_j^2 + \sum_{I \subseteq \{1,\ldots,n\} \setminus \{i\}} a_I \prod_{j \in I} x_j}{x_i}.$$

The numerator is the product of the two roots in direction i and is a nonzero polynomial because it contains x_j^2 for each $j \neq i$. The substitution is an involution since its numerator is independent of x_i. Substituting the other root into the equation with $h = H$ shows that the new tuple has the same value of H. Write Γ for the group generated by all the root exchanges in each setting. Two roots that coincide are left fixed.

Theorem 1. *The ambient fixed field is $F(H)$. For every nonzero $h \in F$, the hypersurface X_h is geometrically integral and its function field $F(X_h)$ has fixed field F.*

Proof. Use the following argument over a base field L, taking $(L, \lambda) = (F, h)$ in the fixed-fiber case and $(L, \lambda) = (F(t), t)$ in the ambient case, with t an indeterminate. In the latter case the generic fiber identifies with the ambient function field by substituting H for t, since clearing the denominator gives a relation linear in t that solves uniquely for t on the coordinate torus.

For each $2 \leq j \leq n$, regard the coordinates outside directions $1, j$ as independent over L. Over their rational function field, the equation for the remaining coordinates $u = x_1$, $v = x_j$ has the form

$$u^2 + v^2 + cuv + au + bv + q = 0.$$

Here a, b, c are the coefficients of u, v, uv, respectively, and q is the constant term in u, v. As polynomials in the frozen variables, a, b, c have degree at most one in each variable and c has full product term $-\lambda \prod_{i \neq 1,j} x_i$. For any one frozen variable, the coefficient of its square in q is 1. The homogeneous quadratic form has matrix

$$\begin{pmatrix} 1 & c/2 & a/2 \\ c/2 & 1 & b/2 \\ a/2 & b/2 & q \end{pmatrix}.$$

Expanding along the first row gives determinant

$$q(1 - c^2/4) - (a^2 + b^2)/4 + abc/4.$$

The coefficient of the fourth power of any one frozen variable in this determinant comes only from $-c^2 q/4$ and is nonzero, since the coefficient of that variable in c contains $-\lambda$ times the product of the other frozen variables, which cannot cancel against a proper-subset term. Thus the conic is smooth and geometrically integral. The defining polynomial of the hypersurface is primitive over the polynomial ring in the frozen variables because the coefficient of u^2 is 1. Over an algebraic closure of L, the same determinant argument gives irreducibility over the rational function field in the frozen variables. Gauss's lemma then gives irreducibility over their polynomial ring.

The two Vieta involutions act on the conic by

$$(u, v) \longmapsto (-u - cv - a, v), \qquad (u, v) \longmapsto (u, -v - cu - b).$$

Applying the second map after the first gives linear part $\left(\begin{smallmatrix} -1 & -c \\ c & c^2-1 \end{smallmatrix}\right)$, with determinant 1 and trace $c^2 - 2$. A finite-order matrix in characteristic zero has roots of unity as eigenvalues, so its trace is algebraic over $\mathbb{Q}$. The trace here is transcendental over L. Moreover, an affine linear transformation acting as the identity on a geometrically integral conic is the identity, since the conic is contained in no line. The composition therefore has infinite order on the conic function field.

Write $E_j = L(x_i : i \neq 1, j)$. The conic function field has transcendence degree one over E_j, and E_j is algebraically closed in it. Indeed, a proper finite subextension would split as a product of at least two fields after tensoring with an algebraic closure of E_j, by separability. This product would inject into the scalar extension of the function field, which is a domain by geometric integrality. A function outside E_j is therefore transcendental over E_j, and the conic function field is a finite extension of the field it generates over E_j. Such a function cannot be fixed by the infinite-order composition, because a finite extension has only finitely many automorphisms over its base. Thus a function fixed by all Vieta involutions belongs to every E_j.

The coordinates $x_2, \ldots, x_n$ are algebraically independent over L, since the original equation is monic in x_1. The fields E_j lie inside $L(x_2, \ldots, x_n)$, and each omits one independent variable. Their intersection is L. For instance, a function in the intersection has zero derivative with respect to every one of these variables. For one variable, write a nonzero function in the kernel as a reduced fraction p/q over the field of the remaining variables. The identity $p'q = pq'$, where primes denote differentiation in the chosen variable, gives $p \mid p'$ and $q \mid q'$ by coprimality. In characteristic zero both polynomials must therefore be constant in that variable. All elements of L are fixed. This proves both statements. □

Corollary 1. *The ambient rational fixed field meets $F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ in $F[H]$. If $F = \mathbb{Q}$ and all $a_I \in \mathbb{Z}$, its intersection with $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is $\mathbb{Z}[H]$.*

Proof. As a Laurent polynomial in x_1, the extreme exponents of H are 1 and -1. The coefficient at 1 is $1/\prod_{i=2}^n x_i$, and the coefficient at -1 is nonzero because it contains the contributions of the squares of the other variables. Thus a polynomial of degree $d > 0$ in H has extreme exponents d and $-d$. Such a polynomial cannot be a unit of the Laurent ring, whose units are scalar monomials. If a reduced fraction $p(H)/q(H)$ is Laurent, Bézout applied to $p, q \in F[t]$ implies $q(H)$ is a Laurent unit. Hence q is constant. In the integral setting, the coefficient of the highest power of x_1 in a polynomial of degree d in H is its leading coefficient times $(x_2 \cdots x_n)^{-d}$. If the Laurent expansion has integral coefficients, this forces the leading coefficient to be integral. Subtract the highest power of H with this coefficient and iterate, ending with an integer constant. □

Corollary 2. *For $H = (x_1^2 + \cdots + x_n^2)/(x_1 \cdots x_n)$, the involution in direction i sends x_i to $\left(\sum_{j \neq i} x_j^2\right)/x_i$ and fixes H. The rational fixed field over $\mathbb{Q}$ is $\mathbb{Q}(H)$, and its intersections with the rational and integral Laurent rings are $\mathbb{Q}[H]$ and $\mathbb{Z}[H]$, respectively.*

Proof. The product of the two roots in direction i is $\sum_{j \neq i} x_j^2$, which gives the involution formula. Root exchange preserves the value of H. Take all $a_I = 0$ in Theorem 1 and in the preceding corollary. □

3. Density of positive integral orbits

Proposition 1. *Suppose all $a_I = 0$, and let k be a positive integer. Every orbit of a positive integral point of X_k under the Vieta involutions is Zariski dense in X_k over $\mathbb{C}$.*

Proof. At a positive integral point, replacement of x_i gives

$$x_i' = k\prod_{\ell\neq i} x_\ell - x_i = \frac{\sum_{\ell\neq i} x_\ell^2}{x_i},$$

which is again positive and integral. If another coordinate is at least x_i, the replacement is strictly greater than the largest unchanged coordinate. In fact the numerator is strictly greater than the square of that largest unchanged coordinate, since $n \geq 3$. After relabelling the directions we may take x_1 to be maximal at the starting point. For each $j = 2, \ldots, n$, in that order, perform the pair of mutations in directions $j, 1$ a freely chosen positive number of times. At the beginning of each pair, x_1 is maximal. The first mutation makes x_j the strict maximum, and the second makes x_1 the strict maximum again. Hence, with all earlier choices fixed, the j-th stage gives infinitely many values of x_j. Later stages leave that coordinate unchanged.

If a polynomial in $x_2, \ldots, x_n$ vanishes at all these points, fix the choices through stage $n-1$ and vary the final stage. The polynomial has infinitely many roots in x_n, so each of its coefficients in that variable vanishes for all the earlier choices. Repeating the argument in reverse stage order proves that the polynomial is zero. Thus the projection of the orbit to these $n-1$ coordinates is Zariski dense. The projection of X_k is finite because its equation is monic in x_1. A proper closed subset of the geometrically integral X_k has dimension less than $n-1$, and its image under a finite morphism is closed of at most that dimension. The closure of the orbit therefore equals X_k. □

4. Common maximal coordinates in distinct integer orbits

For integers $n \geq 3$ and $k \geq 1$, a positive solution is a tuple $x = (x_1, \ldots, x_n)$ of positive integers satisfying

$$\sum_{i=1}^{n} x_i^2 = k\prod_{i=1}^{n} x_i.$$

In this section, an orbit is an equivalence class of positive solutions under coordinate permutations and the Vieta involutions. The involution in direction i replaces x_i by

$$x_i' = k\prod_{j\neq i} x_j - x_i = \frac{\sum_{j\neq i} x_j^2}{x_i},$$

and fixes the other coordinates. The first expression gives integrality, and the second gives positivity. The two values of the coordinate are the roots of the quadratic obtained by holding the other coordinates fixed. The maximal coordinate of x is $\max_{1\leq i\leq n} x_i$. A positive integer m is realized as a maximal coordinate in an orbit if some solution in the orbit has maximal coordinate m.

Theorem 2. *Fix a positive integer k, a positive integer M, and an integer a. For every real number c with $0 < c < \log(2)/3$, there are infinitely many dimensions $n \geq 3$ with $n \equiv a \pmod{M}$ such that infinitely many integers are each realized as maximal coordinates in at least*

$$\exp\left(c\frac{\log n}{\log\log n}\right)$$

distinct orbits. Here log *denotes the natural logarithm.*

Proof. Call a positive solution fundamental if no Vieta involution decreases its coordinate sum. A coordinate that is not strictly larger than all the others cannot decrease. Indeed, the sum of the squares of the other coordinates then exceeds its square. A solution therefore has at most one decreasing move. To see that distinct sorted fundamental solutions belong to distinct orbits, consider the graph on sorted solutions in which an edge between distinct vertices is a Vieta move followed by sorting. Along each edge the coordinate sums differ. A simple path between two fundamental solutions would have an interior vertex of maximal coordinate

sum, with two distinct decreasing neighbors. Thus such a path does not exist. Sorting after each move represents the orbit under moves and permutations because permuting coordinates conjugates each Vieta involution to the one in the permuted direction.

For a positive integer h, choose one of the two triples

$$(2, 13, 24), \qquad (3, 8, 26)$$

independently for each $j = 1, \dots, h$, and multiply the chosen triple by $78j + 1$. Both triples have product 624 and sum of squares 749. Append the common coordinate $78M$, together with r copies of 2 and s copies of 3. Choose r, s in $\{0, \dots, M-1\}$ so that

$$3r + 8s \equiv 1 - a - \sum_{j=1}^{h} \big(749(78j+1)^2 - 3\big) \pmod{M}.$$

This is possible because 3 and 8 are relatively prime. Write P for the common product of the chosen coordinates and D for the common sum of the quantities obtained by subtracting one from the square of each coordinate. Explicitly,

$$P = (78M)\, 2^r 3^s\, 624^h \prod_{j=1}^{h} (78j+1)^3, \qquad D = (78M)^2 - 1 + 3r + 8s + \sum_{j=1}^{h} \big(749(78j+1)^2 - 3\big).$$

Define $n = kP - D$. Because M divides P and $78M$, reducing $kP - D$ modulo M gives $n \equiv a \pmod{M}$.

For fixed k, M and sufficiently large h, we have

$$n \geq 3h + 1 + r + s + 2, \qquad kP > 2 \max\{78M, 26(78h+1)\}^2.$$

Indeed, D is bounded by a constant depending on M times h^3, and $P \geq 624^h$. Pad each tuple with ones to reach dimension n. The resulting tuple has square sum $n + D = kP$ and product P, so it is a positive solution. For any coordinate x_i, the second inequality gives $kP > 2x_i^2$, hence $x_i' = (kP - x_i^2)/x_i > x_i$. These solutions are fundamental.

The appended coordinates and the padding ones contribute the same multiset to every tuple. Cancel this common multiset to compare two tuples. In fact, removing all factors of $2, 3, 13$ from any entry in the triple at index j recovers $78j + 1$. Thus the entries from each index can be read off from the multiset of entries, and division by $78j + 1$ recovers the chosen triple. The sorted fundamental solutions therefore belong to 2^h distinct orbits.

Select two of the padding coordinates equal to 1 in each solution. Apply the Vieta moves alternately to these coordinates, keeping every other coordinate fixed. Their values follow the same sequence in all the orbits. Specifically, define

$$u_0 = u_1 = 1, \qquad u_{\ell+1} = kP\, u_\ell - u_{\ell-1} \quad (\ell \geq 1).$$

After ℓ moves the two selected coordinates have values $u_\ell, u_{\ell+1}$ in some order, since the product of the remaining coordinates is P. We have $u_2 = kP - 1 > \max\{78M, 26(78h+1)\}$. For $\ell \geq 2$,

$$u_{\ell+1} - u_\ell = (kP - 2)u_\ell + (u_\ell - u_{\ell-1}),$$

so the sequence is strictly increasing from u_1 onward. Consequently every $u_{\ell+1}$ with $\ell \geq 1$ is a maximal coordinate in all 2^h orbits.

As h tends to infinity with k, M fixed, the bounded adjustments r, s and the polynomial growth of D give

$$\log n = 3h \log h + O(h), \qquad \lim_{h \to \infty} \frac{h \log\log n}{\log n} = \frac{1}{3}.$$

Indeed, $D/(kP) \to 0$, so $\log n = \log(kP) + o(1)$. The first relation then follows from $\log(78j + 1) = \log j + O(1)$ for $j \geq 1$ and $\log(h!) = h \log h + O(h)$. In these estimates the constants are independent of h. Thus n tends to infinity and 2^h is at least the asserted bound for all sufficiently large h. □

Corollary 3. *For every fixed positive integer k and every residue class of dimensions, there are infinitely many dimensions n such that, for every real B, there are two distinct sorted positive solutions with the same largest coordinate exceeding B.*

Proof. In Theorem 2, restrict to dimensions for which the lower bound on the orbit count exceeds 1. There are infinitely many such dimensions since the bound tends to infinity. Each common largest coordinate given by the theorem is realized by at least two distinct sorted solutions, and the set of these coordinates is unbounded. □

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