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THE RECIPROCAL CUBE SUMS OF LUCAS SEQUENCES OF QUADRATIC UNITS ARE IRRATIONAL

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ABSTRACT. Let $\eta > 1$ be a unit of a real quadratic order and $U_n = (\eta^n - \bar{\eta}^n)/(\eta - \bar{\eta})$ its Lucas sequence. We prove that $\sum_{n\geq 1} U_n^{-3}$ is irrational. Equivalently, $\sum_{n\geq 1} U_n(P,Q)^{-3} \notin \mathbb{Q}$ for the Lucas sequences with $Q = -1$, $P \geq 1$, and with $Q = 1$, $P \geq 3$. In particular the sums of the reciprocal cubes of the Pell numbers and of the numbers F_{kn}, for every $k \geq 1$, are irrational. The case of the Fibonacci numbers was treated in earlier work of the author; the proof here is the same Hankel-determinant construction, carried out for a general unit and for both signs of its norm. The constants of the argument do not depend on η once they are measured in units of $\log \eta$. The whole argument is formalised in Lean 4 with Mathlib.

1. INTRODUCTION

For integers P, Q let $U_n = U_n(P,Q)$ be the Lucas sequence $U_0 = 0$, $U_1 = 1$, $U_{n+1} = PU_n - QU_{n-1}$. When $Q = \pm 1$ and $D = P^2 - 4Q > 0$ is not a square, $U_n = (\eta^n - \bar{\eta}^n)/\sqrt{D}$ with $\eta = (P + \sqrt{D})/2$ a unit of the order $\mathbb{Z}[\eta]$ of norm $\eta\bar{\eta} = Q$. Conversely, every unit $\eta > 1$ of a real quadratic order arises in this way, with $P = \eta + \bar{\eta} \geq 1$ if $Q = -1$ and $P \geq 3$ if $Q = 1$. The Fibonacci numbers are $U_n(1,-1)$, the Pell numbers $\mathcal{P}_n$ are $U_n(2,-1)$, and $F_{kn}/F_k = U_n(L_k, (-1)^k)$, where L_k is the k-th Lucas number.

Theorem 1.1. *Let $Q = -1$ and $P \geq 1$, or $Q = 1$ and $P \geq 3$. Then $\displaystyle\sum_{n\geq 1} \frac{1}{U_n(P,Q)^3}$ is irrational.*

Corollary 1.2. *The numbers $\sum_{n\geq 1} \mathcal{P}_n^{-3}$ and, for every $k \geq 1$, $\sum_{n\geq 1} F_{kn}^{-3}$ are irrational.*

Proof. Theorem 1.1 with $(P,Q) = (2,-1)$, and with $(P,Q) = (L_k, (-1)^k)$, which is admissible: $L_k \geq 1$, and $L_k \geq 3$ for $k \geq 2$. Then $U_n = F_{kn}/F_k$ by Binet's formula, so $\sum_n F_{kn}^{-3} = F_k^{-3} \sum_n U_n^{-3}$. □

The case $(P,Q) = (1,-1)$, $\sum_n F_n^{-3} \notin \mathbb{Q}$, is the main result of the preprint [7]; we refer there for the history of the Fibonacci zeta function. Briefly:

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André-Jeannin proved that $\sum_n U_n^{-1}$ is irrational for a class of binary recurrences that includes every $U_n(P,\pm1)$, in particular $\sum_n F_n^{-1} \notin \mathbb{Q}$ [1]. For both signs of Q, Duverney, Ke. Nishioka, Ku. Nishioka and Shiokawa proved that the even powers $\sum_n U_n^{-2s}$ ($s \geq 1$) are transcendental, and for $Q=-1$ also the odd-index sums $\sum_n U_{2n-1}^{-s}$ [2]; Elsner, Shimomura and Shiokawa determined the algebraic relations among such numbers [3, 4, 5]. These sums are values of quasi-modular forms at algebraic points, to which Nesterenko's theorem [8] applies (see [13, §3.4] for the Fibonacci zeta function). The odd powers $\sum_n U_n^{-(2s+1)}$ of the full sums are not known to be of this kind. For the Pell numbers, Xu and Wang [14] evaluated the integer part of $\left(\sum_{k\geq n} \mathcal{P}_k^{-3}\right)^{-1}$; we know of no irrationality result for any full sum $\sum_n U_n^{-3}$ before the preprint [7].

The method. Put $t=\eta^{-1}$ and $w=Qt^2$. Since $\bar{\eta}/\eta = Qt^2$, we have $U_n = t^{-n}(1-w^n)/\sqrt{D}$, and so

$$\Xi := \frac{1}{D\sqrt{D}} \sum_{n\geq 1} \frac{1}{U_n^3} = \sum_{N\geq 1} \frac{t^{3N}}{(1-w^N)^3}. \tag{1.1}$$

We follow [7], which adapted to $\mathbb{Z}[\varphi]$ a construction of an unrefereed, AI-assisted research draft on the q-zeta value $\sum_n n^2/(p^n-1)$, $p\geq 2$ an integer [11, 10]; the construction is a moving-denominator Hankel determinant in the line of Uvarov [12], Krattenthaler [6], Prévost [9] and Zudilin [15, 16]. A linear functional L, a second Euler derivative summed against the weights $(Qt)^n$ at the nodes w^n, has moments $L(s^k) \in \mathbb{Q}(\sqrt{D})$, rational functions of t, and satisfies $L(1/(1-w^a s)) = \lambda_a(\Xi - S_a)$ with explicit λ_a and partial sums S_a of (1.1). The Hankel determinants $\Delta_K(X)$ of L against $D_K(s) = \prod_{a\leq K}(1-w^a s)$ satisfy $\Delta_K(\Xi) = \det[L(s^{i+j}/D_K)]$, and four estimates are combined:

	size (per $K^3 \log \eta$)	
real decay $\lvert\Delta_K(\Xi)\rvert$	$-2/9$	Proposition 3.1
conjugate $\lvert\sigma\Delta_K(-\Xi)\rvert$	-1	Proposition 5.1
denominators $\log\lvert \mathrm{N}\, C_K(t)\rvert$	$\leq c_* = 1.1954\ldots < 11/9$	Propositions 4.4, 4.6
nonvanishing	infinitely many K	Proposition 6.1

If $\sum_n U_n^{-3}$ were rational, $C_K(t)(mD^2)^K\Delta_K(\Xi)$, where m is the denominator of the sum, would be a nonzero element of $\mathbb{Z}[\eta]$ of norm <1 for large K.

Compared with [7] there are three changes. First, the sign: for $Q=1$ the nodes $w^n = t^{2n}$ are positive and the roots of unity at which the coefficients have poles are of a different type (Section 4); the local arguments are unchanged, only the classification of the roots changes, and the elementary count of the α-roots used in the proof is the same for both signs (the difference appears only in the sharp constant of Remark 4.7). Second, the unit: all analytic estimates use only $0<t\leq\varphi^{-1}$, which holds for every admissible (P,Q), and every constant is a nondecreasing function of t on that interval.

Third, the order $\mathbb{Z}[\eta]$ need not be maximal; the argument uses only that the norm of a nonzero element of $\mathbb{Z}[\eta]$ is a nonzero rational integer. Sections 2–7 follow [7] closely, often verbatim with φ^{-1} replaced by t and -1 by Q; we repeat the arguments so that the present paper is self-contained.

The same argument gives $\sum_n U_{kn}^{-3} \notin \mathbb{Q}$ for every admissible (P,Q) and $k \geq 1$, since $U_{kn} = U_k U'_n$ with U' the Lucas sequence of the unit η^k. The normalisation $\eta > 1$ does not cover the alternating sums $\sum_n (-1)^{n-1} U_n(P,-1)^{-3} = \sum_n U_n(-P,-1)^{-3}$.

Formal verification. The theorem and every step of its proof are formalised in Lean 4 with Mathlib (`https://github.com/telaviv/lucas-zeta3-irrationality`; Section 8). The analytic ingredients are formalised for every real t with $0 < t \leq \varphi^{-1}$ and both signs, and the integrality statement for every real $\tau \neq 0$ that is not a root of unity, with one integer polynomial for all such τ; only the final assembly uses that t is the inverse of a quadratic unit.

2. The functional and the determinants

Notation. Throughout, (P,Q) is admissible: $Q = -1$ and $P \geq 1$, or $Q = 1$ and $P \geq 3$. Put $D = P^2 - 4Q$, $\eta = (P + \sqrt{D})/2$, $\bar{\eta} = (P - \sqrt{D})/2 = Q/\eta$, $t = \eta^{-1}$, $\mathcal{O} = \mathbb{Z}[\eta]$, and let σ be the nontrivial automorphism of $\mathbb{Q}(\sqrt{D})$, so $\sigma t = 1/\bar{\eta} = Q\eta$. We write $\varphi = (1+\sqrt{5})/2$, Euler's function $\phi_{\mathrm{E}}(d)$ and the Möbius function $\mu(d)$. Put

$$\rho = Qt, \qquad w = Qt^2, \qquad \alpha_k = \rho w^k \ (k \geq 0),$$

$$\psi_N = \frac{t^{3N}}{(1-w^N)^3}, \qquad S_a = \sum_{N=1}^{a} \psi_N,$$

$A_K = K(K+1)/2$ and $\theta = s\,\frac{d}{ds}$. Then $\rho^2 = Qw$, $|w| = t^2$, $\Xi = \sum_N \psi_N$, and $w/\rho = t$. We write h_m for the complete homogeneous symmetric polynomial of degree m ($h_m = 0$ for $m < 0$), and $K \geq 2$.

Lemma 2.1. (i) *D is not a square, so $\sqrt{D} \notin \mathbb{Q}$.* (ii) *$\eta \geq \varphi$, so $0 < t \leq \varphi^{-1}$.* (iii) *t and σt are the roots of $\tau^2 - PQ\tau + Q$, and $t\,\sigma t = Q$.* (iv) *$\sqrt{D} = P - 2Qt$ and $P - 2Q\,\sigma t = -\sqrt{D}$.* (v) *$U_n = (\eta^n - \bar{\eta}^n)/\sqrt{D} = t^{-n}(1-w^n)/\sqrt{D}$, so $1/U_n^3 = D\sqrt{D}\,\psi_n$ and (1.1) holds.*

Proof. (i) For $Q = -1$, $P^2 < D < (P+2)^2$ and $D = (P+1)^2$ would give $2P = 3$; for $Q = 1$, $P \geq 3$, $(P-2)^2 < D < P^2$ and $D = (P-1)^2$ would give $2P = 5$. (ii) $\sqrt{D} \geq \sqrt{5}$ in both cases, and $P \geq 1$. (iii) From $\eta^2 = P\eta - Q$ we get $1 = Pt - Qt^2$, i.e. $t^2 - PQt + Q = 0$ (as $Q^2 = 1$); the other root is $Q/t = Q\eta = \sigma t$. (iv) $1/t = P - Qt$, so $\sqrt{D} = \eta - \bar{\eta} = 1/t - Qt = P - 2Qt$; and $P - 2Q^2\eta = P - 2\eta = -\sqrt{D}$. (v) Binet's formula, by induction from $\eta^2 = P\eta - Q$ and $\bar{\eta}^2 = P\bar{\eta} - Q$; then use $\eta^n = t^{-n}$, $\bar{\eta}^n = (Qt)^n = w^n t^{-n}$ and $\sqrt{D}^3 = D\sqrt{D}$. □

Definition 2.2. For g analytic on a neighbourhood of $|s| \le t^2$ put $L(g) = \sum_{n\ge1} \rho^n\,[\theta(\theta+1)g](w^n)$.

The series converges absolutely since $\theta(\theta+1)g = O(s)$ and $|\rho^n w^n| = t^{3n}$.

Lemma 2.3. (i) $L(s^k) = \nu_k := k(k+1)\alpha_k/(1-\alpha_k)$ *for* $k \ge 0$. (ii) *For* $a \ge 1$, $L\big(1/(1-w^a s)\big) = \lambda_a(\Xi - S_a)$ *with* $\lambda_a = 2\rho^{-a}$.

Proof. (i) $\theta(\theta+1)s^k = k(k+1)s^k$; sum the geometric series. (ii) $\theta(\theta+1)(1-cs)^{-1} = 2cs/(1-cs)^3$, so the left side is $\sum_n \rho^n\, 2w^{a+n}/(1-w^{a+n})^3 = 2\rho^{-a}\sum_{N>a}\rho^N w^N/(1-w^N)^3$, and $\rho^N w^N = Q^{2N}t^{3N} = t^{3N}$. □

Let $D_K(s) = \prod_{a=1}^K (1-w^a s)$, with poles $x_a = w^{-a}$, $|x_a| = \eta^{2a}$. For a polynomial P_0 of degree $< 2K-1$,

$$\frac{P_0}{D_K} = Q_0 + \sum_{a=1}^K \frac{\mathrm{res}_a(P_0)}{1-w^a s}, \qquad \mathrm{res}_a(P_0) = \frac{P_0(x_a)}{\prod_{b\ne a}(1-w^{b-a})},$$

with Q_0 a polynomial of degree $\le \deg P_0 - K$. Let L_X be the linear map with $L_X(s^k) = \nu_k$ and $L_X(1/(1-w^a s)) = \lambda_a(X - S_a)$, X an indeterminate, and

$$\Delta_K(X) = \det\big[L_X(s^{i+j}/D_K)\big]_{0\le i,j<K} \in \mathbb{Q}(\sqrt{D})[X], \qquad \deg\Delta_K \le K.$$

By Lemma 2.3, $\Delta_K(\Xi) = \det[L(s^{i+j}/D_K)]$.

The formal parameter. Every quantity above is the value at $\tau = t$ of a rational function of an indeterminate τ with rational coefficients, which depends on (P,Q) only through Q: $\rho = Q\tau$, $w = Q\tau^2$, $\alpha_k = Q^{k+1}\tau^{2k+1}$, $\psi_N = \tau^{3N}/(1-(Q\tau^2)^N)^3$. Write $\Delta_{K,\tau}(X) \in \mathbb{Q}(\tau)[X]$ for the resulting polynomial, and $w(\zeta)$, $\alpha_k(\zeta)$ for the values at $\tau = \zeta$. For $Q = -1$ it is the polynomial of [7]. Since $\sigma t = Q/t$, the coefficientwise conjugate $\sigma\Delta_K$ is the value of $\Delta_{K,\tau}$ at $\tau = Q/t$. Both values are defined because no denominator vanishes at a real number that is neither 0 nor a root of unity.

3. Real decay

Proposition 3.1. $\log|\Delta_K(\Xi)| \le -\frac{2}{9}(K^3+3K^2-4)\log\eta + O(K^2)$, *with an absolute implied constant.*

Proof. Let $V_j(s) = \prod_{u=1}^j (s - w^{\lceil u/3\rceil})$, $R_j = \lceil j/3\rceil$ and $a_j = \sum_{u\le j}\lceil u/3\rceil$. The change of basis from $1, s, \dots, s^{K-1}$ to $V_0, \dots, V_{K-1}$ is unitriangular, so $\Delta_K(\Xi) = \det[L(V_iV_j/D_K)]$.

Since $\theta(\theta+1)g(s_0)$ depends only on the 2-jet of g at s_0, the m-th term of $L(V_iV_jW)$, $W = 1/D_K$, vanishes for $m < R := \max(R_i, R_j)$: then V_iV_j has a triple zero at w^m. For $m \ge R$ and $s = w^m$ every root β of V_j satisfies $|s| \le |\beta|$, so $|s-\beta| \le 2|\beta|$ and $\theta(s-\beta) = s$; the signs of the nodes play no role. Hence $|V_j(s)| \le 2^j|w|^{a_j}$ and $|\theta^u V_j(s)| \le j^u 2^j |w|^{a_j}|s|/|w|^{R_j}$ for $u = 1, 2$. On $|s| \le t^2$ we have $|w^a s| \le t^{2a+2} \le \varphi^{-2a-2}$, so $|W|$, $|\theta W|/|s|$ and $|\theta^2 W|/|s|$ are

bounded by an absolute constant. Since $\theta(\theta+1)=\theta^2+\theta$ kills constants, by Leibniz every term of $\theta(\theta+1)(V_iV_jW)$ contains at least one θ on some factor, hence a factor $\le |s|/|w|^R$. With $|\rho|^m\le 1$ and $\sum_{m\ge R}|w|^{m-R}\le(1-\varphi^{-2})^{-1}$,

$$|L(V_iV_jW)|\le c_1(i+j+1)^2 4^{i+j}|w|^{a_i+a_j}$$

for an absolute constant c_1. Pull $|w|^{a_i}$ out of row i and column i and apply Hadamard's inequality: $|\Delta_K(\Xi)|\le |w|^{2\sum_{j<K}a_j}(c_1(2K)^2 4^{2K}\sqrt{K})^K$. Finally $2\sum_{j<K}a_j=(K^3+3K^2-\iota_K)/9$ with $\iota_K\in\{0,4,2\}$ according to K mod 3, and $|w|=\eta^{-2}$. □

4. Denominators

4.1. **The integrality ring and the root types.**

Lemma 4.1. *The coefficients of $\Delta_{K,\tau}(X)$ lie in $\mathbb{Z}[\tau,\tau^{-1},\Phi_d(\tau)^{-1}:d\in\mathcal{D}_K]$, where $\mathcal{D}_K$ is the set of orders of roots of unity ζ such that* (W) *$w(\zeta)=Q\zeta^2$ has order $e\le K$, or* (α) *$\alpha_k(\zeta)=1$ for some $1\le k\le K-2$.*

Proof. The inputs are the Laurent monomials w^a and $\lambda_a=2Q^a\tau^{-a}$; the factors $1/(1-w^{b-a})$ and $(1-w^N)^{-3}$ ($N\le K$), where $1-w^n=1-Q^n\tau^{2n}$ is $\pm$ a product of cyclotomic polynomials $\Phi_d(\tau)$ with $w(\zeta)^n=1$; and $\nu_k=k(k+1)\alpha_k/(1-\alpha_k)$, of which only $k\le K-2$ occur (the quotient has degree $\le 2K-2-K$), with $1-\alpha_k=1-Q^{k+1}\tau^{2k+1}=\pm\prod\Phi_d(\tau)$. Division by D_K is integral because its leading coefficient $\prod_a(-w^a)=\pm\tau^{K(K+1)}$ is a unit. No rational constant is inverted. □

For $d\in\mathcal{D}_K$ the type depends only on d: every primitive d-th root is a power ζ^j with j coprime to $2d$, and since $Q^j=Q$, $Q\zeta^{2j}=(Q\zeta^2)^j$ and $\alpha_k(\zeta^j)=\alpha_k(\zeta)^j$; powers coprime to the order preserve the order. If ζ has order d, the order $e(d)$ of $w(\zeta)$ and the least $k\ge 1$ with $\alpha_k(\zeta)=1$ are as follows.

$Q=-1$: $e(d)=2d$ (d odd), d ($d\equiv 2 \bmod 4$), $d/4$ ($d\equiv 4 \bmod 8$), $d/2$ ($d\equiv 0 \bmod 8$); $k_{\min}(d)=(d-1)/2$ ($d\equiv 3 \bmod 4$), $(3d-1)/2$ ($d\equiv 1 \bmod 4$), $(d-2)/4$ ($d\equiv 2 \bmod 8$, $d\ge 10$), $(3d-2)/4$ ($d\equiv 6 \bmod 8$), $k_{\min}(2)=2$, and none if $4\mid d$.

$Q=1$: $e(d)=d$ (d odd), $d/2$ (d even); $\alpha_k(\zeta)=\zeta^{2k+1}=1$ forces d odd, and then $k_{\min}(d)=(d-1)/2$ for $d\ge 3$, $k_{\min}(1)=1$.

In both cases $\sum_{e(d)=e}\phi_{\mathrm{E}}(d)=2\phi_{\mathrm{E}}(e)$: for each primitive e-th root ω the equation $Q\tau^2=\omega$ has two roots, and these $2\phi_{\mathrm{E}}(e)$ roots of unity are exactly those with $e(\tau)=e$. All $d\in\mathcal{D}_K$ satisfy $d\le 4K$: $w(\zeta)^e=1$ gives $\zeta^{4e}=1$, and $\alpha_k(\zeta)=1$ gives $\zeta^{4k+2}=1$.

Local setting. Fix ζ of order $d\in\mathcal{D}_K$, let $\varpi=\tau-\zeta$ and let $v=\mathrm{ord}_\varpi$ on $\mathbb{Q}(\zeta)((\varpi))$, extended to polynomials in X as the Gauss valuation (the minimum over the coefficients; it is multiplicative). Then:

(a) if $e(d) = e \le K$, then $x_a(\zeta) = x_b(\zeta)$ iff $e \mid a-b$, and then $v(x_a - x_b) = 1$: indeed $x_a - x_b = x_a(1 - w^{b-a})$, and $1 - w^n$ has only simple zeros, since $\frac{d}{d\tau} w^n \neq 0$ at $\tau \neq 0$;
(b) ψ_N has a pole of order exactly 3 if $e \mid N$ and is regular otherwise, by (a); so S_a has a pole of order ≤ 3, and is regular if $a < e$;
(c) $1 - \alpha_k = 1 - Q^{k+1}\tau^{2k+1}$ is separable, so it has at most a simple zero at ζ;
(d) λ_a and x_a are Laurent monomials, hence units.

We call ζ a *W-root* if $e(d) \le K$ and an *α-root* if $\alpha_k(\zeta) = 1$ for some $1 \le k \le K-2$; a *pure W-root* is a W-root that is not an α-root.

4.2. **Local bounds.** Put $c(m) = 2\sum_{j=0}^{m-2} \max(0, \frac{m+2}{2} - j)$ and $c'(m) = 2\sum_{j=0}^{m-1} \max(0, \frac{m+2}{2} - j)$; thus $c(1) = 0$, $c(m) = 4(m-1)$ for $m \le 6$, $c(m) \le (m+3)^2/4$, $c'(1) = 3$ and $0 \le c' - c \le 3$.

Proposition 4.2. *Let $e = e(d) \le K$ and partition $\{1, \dots, K\}$ into the clusters $I_r = \{a \equiv r \bmod e\}$ of sizes m_r, $1 \le r \le e$. Then*

$$-v(\Delta_{K,\tau}) \le \sum_{r=1}^{e-1} c(m_r) + c'(m_e) + \delta_\alpha \max(0, K - 1 - k_{\min}(d)),$$

with $\delta_\alpha = 1$ if ζ is also an α-root and $\delta_\alpha = 0$ otherwise.

Proof. The proof of [7, Proposition 4.2] uses only the facts (a)–(d) and the integrality of the quotient by D_K, which hold for both signs; we repeat it. Let $F(s) = \prod_a (s - x_a)$ and $F_r = \prod_{a \in I_r}(s - x_a)$. Fix a cluster r, write $m = m_r$ and $I_r = \{a_1 < \cdots < a_m\}$.

Newton cluster basis. For $0 \le j < m$ put $N_{r,j} = (F/F_r)\prod_{l=m-j+1}^{m}(s - x_{a_l})$; it vanishes at the top j nodes of the cluster. At $\varpi = 0$ these reduce to $(F/F_r)(s - y_r)^j \cdot (\text{unit})$, $y_r = x_r(\zeta)$, which form a basis of the polynomials of degree $< K$ by the Chinese remainder theorem (distinct residues r give distinct points y_r). So the change of basis from $1, \dots, s^{K-1}$ has unit determinant, and $v(\Delta_{K,\tau})$ is the valuation of $G = [L_X(NN'/D_K)]$ in this basis.

Entries. $G_{(r,j),(r',k)} = P^{\mathrm{pol}} + \sum_a \mathrm{res}_a(NN')\lambda_a(X - S_a)$, where P^{pol} is L_X of the polynomial part. (i) If $r \neq r'$ then $F \mid N_{r,j}N_{r',k}$ and all $\mathrm{res}_a = 0$. (ii) If $r = r'$, then $\mathrm{res}_{a_i} = 0$ unless $i \le m - \max(j,k)$, and for such i

$$\mathrm{res}_{a_i} = \frac{(F/F_r)(x_{a_i})^2 \prod_{l>m-j}(x_{a_i} - x_{a_l}) \prod_{l>m-k}(x_{a_i} - x_{a_l})}{\prod_{b \neq a_i}(1 - w^{b-a_i})}.$$

Here $(F/F_r)(x_{a_i})$ is a unit, each displayed difference has valuation exactly 1 by (a), and the denominator has valuation exactly $m-1$. So $v(\mathrm{res}_{a_i}) = j + k - (m-1)$. (iii) $v(\lambda_a) = 0$, and by (b) the Gauss valuation of $X - S_a$ is ≥ -3 if $a \ge e$ and ≥ 0 if $a < e$. (iv) For $r < e$ the node $a_1 = r$ is regular and a_i is singular for $i \ge 2$; for $r = e$ every node is singular. So the proper part of the (j,k) entry has $v \ge j + k - (m-1) - 3$ if some singular node

has $i \le m - \max(j,k)$, and $v \ge j + k - (m-1)$ otherwise. The second case occurs exactly when $r < e$ and $\max(j,k) = m-1$.

Weights. Let $\kappa_j = \max(0, \frac{m+2}{2} - j)$ for $j \le m-2$ and $\kappa_{m-1} = 0$ if $r < e$, and $\kappa_j = \max(0, \frac{m+2}{2} - j)$ for all $j < m$ if $r = e$. Then (A) if the (j,k) entry has a singular node, $\kappa_j + \kappa_k \ge m + 2 - j - k$, and (B) if $r < e$ and $\max(j,k) = m-1$, $\kappa_j + \kappa_k \ge 0 \ge m - 1 - j - k$. Indeed, in case (A) if both weights are positive the inequality is an equality; if $\kappa_k = 0$, then $k \ge \frac{m+2}{2}$, so $\kappa_j \ge \frac{m+2}{2} - j \ge m + 2 - j - k$; if both vanish, the right side is ≤ 0. Case (B) holds because $j + k \ge m - 1$. Scale row (r,j) by $\varpi^{\lceil \kappa_j \rceil}$ and column (r,j) by $\varpi^{\lfloor \kappa_j \rfloor}$. Since $a + b \ge n \in \mathbb{Z}$ implies $\lceil a \rceil + \lfloor b \rfloor \ge n$, every scaled proper entry is integral. Since $\kappa_j \in \frac{1}{2}\mathbb{Z}$, $\lceil \kappa_j \rceil + \lfloor \kappa_j \rfloor = 2\kappa_j$, so the total scaling is $2\sum \kappa_j$, which is $c(m)$ for $r < e$ and $c'(m)$ for $r = e$. All exponents are ≥ 0.

Polynomial part. Write $(\cdot)_+$ for the polynomial part (the quotient by D_K). The quotient of NN' has degree $\le K-2$ and coefficients in $\mathbb{Z}[\tau^{\pm 1}]$, so $P^{\mathrm{pol}} = \sum_{k \le K-2} [s^k](NN'/D_K)_+ \, \nu_k$ (with $\nu_0 = 0$). At a pure W-root every ν_k, $1 \le k \le K-2$, is regular, so the scaled matrix is integral; this gives the bound with $\delta_\alpha = 0$. At an α-root, put $P_\alpha = \{1 \le k \le K-2 : \alpha_k(\zeta) = 1\}$ and write $\nu_k = \varpi^{-1} u_k$ with u_k regular for $k \in P_\alpha$, by (c). Split $P^{\mathrm{pol}} = A^{\mathrm{pol}} + \varpi^{-1} B$, where $B(f,g) = \sum_{k \in P_\alpha} u_k(0)\,[s^k](fg/D_K)_+$ and A^{pol} collects $\varpi^{-1}(u_k - u_k(0))[s^k](\cdot)_+$ for $k \in P_\alpha$ and $\nu_k[s^k](\cdot)_+$ for $k \notin P_\alpha$; both have regular entries. If $\deg f \le k_{\min}$, then $\deg(fg) \le k_{\min} + K - 1$, the quotient of fg has degree $< k_{\min}$, and $B(f,\cdot) = 0$; so $\operatorname{rank} B \le \max(0, K - 1 - k_{\min})$. After scaling the matrix is $A + \varpi^{-1} B'$, where A (the scaled proper parts and the scaled A^{pol}) is integral and B' is B in the Newton basis, scaled; B' has regular entries and $\operatorname{rank} B' = \operatorname{rank} B$, since the basis change and the scalings are invertible over the fraction field. Expanding the determinant multilinearly in the columns, a term taking more than $\operatorname{rank} B$ columns from $\varpi^{-1} B'$ vanishes by linear dependence, and every other term has valuation $\ge -\max(0, K - 1 - k_{\min})$. □

Lemma 4.3. *If $e(d) > K$ and ζ is an α-root, then $-v(\Delta_{K,\tau}) \le \max(0, K - 1 - k_{\min}(d))$.*

Proof. Work in the monomial basis. Since $e > K$, no difference $|a - b| < K$ and no $N \le K$ is divisible by e, so no factor $1 - w^{b-a}$ and no $1 - w^N$ vanishes at ζ; hence all res_a, λ_a and S_a ($a \le K$) are regular at ζ, by (c), (d). Only the ν_k, $k \in P_\alpha$, carry poles, and the rank argument of Proposition 4.2 applies with no scaling. □

Proposition 4.4. *Let $b_{K,d}$ be the bound of Proposition 4.2 or Lemma 4.3 (0 if $d \notin \mathcal{D}_K$) and $C_K(\tau) = \prod_d \Phi_d(\tau)^{b_{K,d}}$. Then $C_K(\tau)\Delta_{K,\tau}(X) \in \mathbb{Z}[\tau^{\pm 1}][X]$; hence $C_K(t)\Delta_K(X) \in \mathcal{O}[X]$, and $C_K(\sigma t)\sigma\Delta_K(X) \in \mathcal{O}[X]$ is its conjugate.*

Proof. By Lemma 4.1 a coefficient is $P_1/(\tau^n \prod \Phi_d^{n_d})$ with $P_1 \in \mathbb{Z}[\tau]$. The bound $v_\zeta \ge -b_{K,d}$ at every primitive d-th root forces $\Phi_d^{n_d - b_{K,d}} \mid P_1$ in $\mathbb{Q}[\tau]$,

hence in $\mathbb{Z}[\tau]$ since Φ_d is monic. Specialise $\tau \mapsto t$, a unit of $\mathcal{O}$ ($t = Q\bar{\eta} = Q(P-\eta)$), and $\tau \mapsto \sigma t$. □

4.3. **The size of $C_K(t)$.**

Lemma 4.5. *For $d \geq 1$, $\big|\log|\Phi_d(t)\Phi_d(\sigma t)| - \phi_{\mathrm{E}}(d)\log\eta\big| \leq 2\mathcal{E}_1$, where $\mathcal{E}_1 = \sum_{a\geq 1}|\log(1-\varphi^{-a})| = 2.1136\ldots$.*

Proof. $\Phi_d(x) = \prod_{a|d}(x^a-1)^{\mu(d/a)}$ and $\sum_{a|d}\mu(d/a)a = \phi_{\mathrm{E}}(d)$. For $|x| = \eta$ write $x^a - 1 = x^a(1-x^{-a})$ and use $|\log|1-x^{-a}|| \leq |\log(1-\eta^{-a})| \leq |\log(1-\varphi^{-a})|$; for $x = t$ use $|1-t^a|$ in the same way. □

Let $\beta_7 = \prod_{p\leq 7}(1-p^{-2}) = \frac{768}{1225}$ and $c_* = \frac{11}{8}\beta_7 + \frac{1}{3} = \frac{4393}{3675} = 1.19537\ldots$.

Proposition 4.6. $\log|\mathrm{N}\, C_K(t)| \leq \big(c_* K^3 + O(K^2\log K)\big)\log\eta$, *with an absolute implied constant.*

Proof. By Lemma 4.5, $\log|\mathrm{N}\, C_K(t)| \leq \log\eta\sum_d b_{K,d}\phi_{\mathrm{E}}(d) + 2\mathcal{E}_1\sum_d b_{K,d}$, and $2\mathcal{E}_1 \leq (2\mathcal{E}_1/\log\varphi)\log\eta$. Moreover $\sum_d b_{K,d} = O(K^2\log K)$: for each $e \leq K$ at most four orders d have $e(d) = e$ (two if $Q = 1$), each with a W part $\leq 4K + K^2/(4e) + 3$ by the cluster bound below, and the at most $4K$ orders $d \leq 4K$ have α parts $\leq K$; summing $K^2/(4e)$ over $e \leq K$ gives $O(K^2\log K)$. Split $b_{K,d}$ into its W part and its α part.

W part. This part depends only on the cluster sizes and on the identity $\sum_{e(d)=e}\phi_{\mathrm{E}}(d) = 2\phi_{\mathrm{E}}(e)$, which are the same for both signs, so the estimate of [7, Proposition 4.6] applies verbatim. Since $m_r \geq 1$, $\sum_r m_r = K$, $m_r < K/e + 1$ and $c(m) = 4(m-1) + \delta(m)$ with $0 \leq \delta(m) \leq \max(0, m-5)^2/4$,

$$\sum_{r<e} c(m_r) + c'(m_e) \leq 4(K-e) + \frac{\max(0, K-4e)^2}{4e} + 3.$$

Bound $\phi_{\mathrm{E}}(e) \leq e\, g_7(e)$ with $g_7(e) = \prod_{p\leq 7,\, p|e}(1-1/p)$, a function of period 210 with $0 \leq g_7 \leq 1$ and mean β_7. Summing over the orders d with $e(d) = e$ gives the weight $\sum_{e(d)=e}\phi_{\mathrm{E}}(d) = 2\phi_{\mathrm{E}}(e) \leq 2e\, g_7(e)$, so the W part is at most $\sum_{e\leq K} g_7(e) f_K(e)$ with $f_K(y) = 8y(K-y) + \frac{1}{2}\max(0, K-4y)^2 + 6y$. We use the elementary averaging estimate: if g has period ℓ, $0 \leq g \leq 1$ and mean β, and f is a function on $[0, K]$ with $|f| \leq M$ and total variation V, then

$$\Big|\sum_{1\leq e\leq K} g(e)f(e) - \beta\int_0^K f(y)\,dy\Big| \leq 2\ell(M+V).$$

(Cut $[0, K]$ into blocks of length ℓ. On a full block $(e_0, e_0+\ell]$ both $\sum g(e)f(e)$ and $\beta\int f$ differ from $\ell\beta f(e_0)$ by at most ℓ times the oscillation of f on the block; the oscillations add up to at most V, and the last, incomplete block costs at most $2\ell M$.) For f_K we have $M, V = O(K^2)$, so the W part is at most $\beta_7\int_0^K f_K + O(K^2) = \frac{11}{8}\beta_7 K^3 + O(K^2)$.

α part. $\max(0, K-1-k_{\min})$ counts the $k \in [k_{\min}, K-2]$, so

$$\sum_d \max(0, K-1-k_{\min}(d))\,\phi_{\mathrm{E}}(d) \le \sum_{k=1}^{K-2}\sum_{j=1}^{k} \#\{\zeta : \alpha_j(\zeta)=1\} = \sum_{k=1}^{K-2}(k^2+2k) \le \tfrac{1}{3}K^3 + O(K^2),$$

since $\alpha_j(\zeta)=1$ means $\zeta^{2j+1}=Q^{j+1}$, which has $2j+1$ solutions, for either sign. □

Remark 4.7. The following is a sketch and is not used in the proof. Using $\sum_{n\le y}\phi_{\mathrm{E}}(n) = 3y^2/\pi^2 + O(y\log y)$ in residue classes and the exact $k_{\min}$, the same bound holds with c_* replaced by $(12I+80/27)/\pi^2 \le 1.130790$ if $Q=-1$ and by $(12I+8/3)/\pi^2 \le 1.100769$ if $Q=1$. Here $I = \int_0^1 \ell(x)x\,dx \in [0.6822834, 0.6831236]$, ℓ is the piecewise linear function with $\ell(1/m) = c(m)/m$, and $8/(3\pi^2)$ is the contribution of the odd d with $k_{\min}=(d-1)/2$.

5. Smallness at the conjugate

Proposition 5.1. $\log|\sigma\Delta_K(-\Xi)| \le -K^2(K+1)\log\eta + K\log(\sqrt{K}\,M_K)$, *where* $M_K = C_0\big(K^2/((1-\varphi^{-3})(1-\varphi^{-2}))+\varphi^2\big)$ *and* $C_0 = \prod_{i\ge1}(1-\varphi^{-2i})^{-1} = 2.0770906\ldots$.

Proof. At $\tau=\sigma t = Q/t$ the data become: $w\mapsto w^{-1}$, $\lambda_a \mapsto 2(\sigma\rho)^{-a} = 2t^a$ (as $\sigma\rho = Q\sigma t = \eta$), $S_a \mapsto -S_a$ (as $\sigma\psi_N = -\psi_N$: U_N is rational and $\sigma\sqrt{D} = -\sqrt{D}$), and $\nu_k \mapsto \sigma\nu_k = Qk(k+1)/(\alpha_k - Q)$ (as $\sigma\alpha_k = Q/\alpha_k$). Let L_X^σ be the corresponding formal functional. The conjugate poles are $r_a = w^a$, $|r_a| = t^{2a}$, and $\sigma D_K(s) = \prod_a(1-w^{-a}s) = (-1)^K w^{-A_K}F^\sigma(s)$ with $F^\sigma(s) = \prod_a(s-r_a)$. Hence

$$\sigma\Delta_K(X) = (-1)^{K^2} w^{A_K K}\det[Y_{i+j}(X)], \qquad Y_k(X) = L_X^\sigma(s^k/F^\sigma).$$

The pole moments at $X=-\Xi$. Since $1/(1-w^{-a}s) = -r_a/(s-r_a)$, $L_X^\sigma(1/(s-r_a)) = -r_a^{-1}2t^a(X+S_a)$, and at $X=-\Xi$ this is $2t^a r_a^{-1}\sum_{N>a}\psi_N = \Gamma(r_a)$, using $t^{4a} = w^{2a} = r_a^2$ (for both signs) and $w^{a+M} = r_a w^M$, where

$$\Gamma(r) = 2r\sum_{M\ge1}\frac{t^{3M}}{(1-rw^M)^3} = \sum_{n\ge1}\gamma_n r^n, \qquad \gamma_n = n(n+1)\sum_{M\ge1} t^{3M}w^{M(n-1)},$$

analytic on $|r|<\eta^2$, with $|\gamma_n| \le n(n+1)t^{2n+1}/(1-t^{2n+1}) \le n(n+1)t^{2n+1}/(1-t^3)$. Partial fractions give

$$\frac{s^k}{F^\sigma(s)} = \sum_{j=0}^{k-K} h_{k-K-j}(r)\,s^j + \sum_{a=1}^{K}\frac{r_a^k}{(F^\sigma)'(r_a)(s-r_a)},$$

and with $\sum_a r_a^m/(F^\sigma)'(r_a) = h_{m-K+1}(r)$ we get

$$Y_k(-\Xi) = \sum_{j=0}^{k-K} h_{k-K-j}(r)\,\sigma\nu_j + \sum_{n\ge1}\gamma_n h_{k+n-K+1}(r).$$

Bounds. As $h_m(r_1,\dots,r_K) = w^m h_m(1,w,\dots,w^{K-1})$, $|h_m(r)| \le C_0 t^{2m}$, using $\prod_{i\ge1}(1-t^{2i})^{-1} \le C_0$ for $t \le \varphi^{-1}$. For $j \ge 1$, $|\sigma\nu_j| \le j(j+1)/(1-t^3)$, so the first sum is at most $C_0K^2/((1-t^3)(1-t^2))$. For the second, $\sum_n n(n+1)t^{2n+1} = 2t^3/(1-t^2)^3 \le 2$ and $1/(1-t^3) \le \varphi^2/2$ for $t \le \varphi^{-1}$, so $\sum_n |\gamma_n| \le \varphi^2$. All these quantities increase with t, so $|Y_k(-\Xi)| \le M_K$ for $0 \le k \le 2K-2$. Hadamard's inequality and $|w|^{A_K K} = \eta^{-K^2(K+1)}$ finish the proof. □

6. Nonvanishing

Proposition 6.1. $\Delta_K(\Xi) \ne 0$ *for infinitely many* K.

Proof. The proof of [7, Proposition 6.1] uses only $0 < |w| < 1$, $\rho \ne 0$ and the distinctness of the nodes; we repeat it. Let $\mathcal{C}(z) = L(1/(z-s)) = \sum_n \rho^n\, 2w^n z/(z-w^n)^3$ for $|z| > t^2$. It continues to $\mathbb{C}\setminus\{0\}$ with a pole of order exactly 3 at each w^n (leading coefficient $2\rho^n w^{2n} \ne 0$), so it is not rational; it is analytic at ∞ and $O(1/z)$. For $\deg R \le n-1$ put $H_{n,R}(z) = L\big(R(s)/((z-s)D_n(s))\big)$, analytic for $|z| > t^2$ and in particular at every x_a, and

$$T_{n,R}(z) = L\Big(\frac{R(z)D_n(s) - D_n(z)R(s)}{(z-s)D_n(s)}\Big),$$

a polynomial in z of degree $\le n-1$. Then $\mathcal{C}R - T_{n,R} = D_n H_{n,R}$.

If $\Delta_n(\Xi) = 0$, any kernel vector gives $R_n \ne 0$ of degree $\le n-1$ with $L(s^j R_n/D_n) = 0$ for $j < n$. Put $T_n = T_{n,R_n}$ and $E_n = \mathcal{C}R_n - T_n$. Expanding $(z-s)^{-1} = \sum_j s^j z^{-j-1}$, uniformly with two s-derivatives on $|s| \le t^2$ for $|z| > t^2+\epsilon_0$ ($\epsilon_0 > 0$), gives $H_{n,R_n}(z) = O(z^{-n-1})$, so $E_n = O(z^{-1})$ and $E_n(x_a) = 0$ for $a \le n$. If $\Delta_n(\Xi) = \Delta_{n+1}(\Xi) = 0$, then $V_0 = T_{n+1}R_n - T_nR_{n+1} = R_{n+1}E_n - R_nE_{n+1}$ is a polynomial, $O(z^{n-1})$, vanishing at $x_1,\dots,x_n$; so $V_0 = 0$ and $T_n/R_n = T_{n+1}/R_{n+1}$. If $\Delta_n(\Xi) = 0$ for all $n \ge N$, all T_n/R_n equal one reduced fraction T/R_0, so $R_n = U_n'R_0$, $T_n = U_n'T$, and $E := \mathcal{C}R_0 - T \ne 0$. Then $U_n'E = E_n = O(1/z)$ forces $E = c_0 z^{-q}(1+o(1))$ with $c_0 \ne 0$, $q \ge 1$, and $\deg U_n' \le q-1$; but $E(x_a) \ne 0$ for all large a while $E_n(x_a) = 0$, so $U_n'(x_a) = 0$ for all large $a \le n$, too many zeros once n is large. □

7. Proof of Theorem 1.1

Suppose $\sum_n U_n^{-3} = u/m$ with $u \in \mathbb{Z}$, $m \in \mathbb{Z}_{>0}$. By (1.1) and Lemma 2.1(iv), $\Xi = u\sqrt{D}/(mD^2) = u(P-2Qt)/(mD^2)$ and $\sigma\Xi = -\Xi$. For K in the infinite set of Proposition 6.1 put $y_K = C_K(t)(mD^2)^K\Delta_K(\Xi)$. By Proposition 4.4, $y_K \in \mathcal{O}$, and $y_K \ne 0$ because $\Delta_K(\Xi) \ne 0$ and $C_K(t) \ne 0$ (t is not a root of unity). Moreover $\sigma y_K = C_K(\sigma t)(mD^2)^K\sigma\Delta_K(-\Xi)$. The

norm $y_K\,\sigma y_K$ of the nonzero algebraic integer y_K is a nonzero rational integer, so $|y_K\,\sigma y_K| \ge 1$, and by Propositions 3.1, 5.1 and 4.6

$$0 \le \log|\,\mathrm{N}\,C_K(t)| + 2K\log(mD^2) + \log|\Delta_K(\Xi)| + \log|\sigma\Delta_K(-\Xi)| \\ \le \big(c_* - \tfrac{2}{9} - 1\big)K^3\log\eta + O(K^2\log K).$$

Since $c_* - \frac{11}{9} = \frac{11}{8}\beta_7 - \frac{8}{9} = -0.02684\ldots < 0$, this fails for large K. □

Remark 7.1. Not used in the proof. With the constants of Remark 4.7 the margin is at least 0.0914 ($Q=-1$) and 0.1214 ($Q=1$) per $K^3\log\eta$. At finite K, computed with exact determinants over $\mathbb{Q}(\sqrt{D})$ and interval arithmetic, the left side above without the term $2K\log(mD^2)$, divided by $K^3\log\eta$, is about -0.12 for the Pell numbers at $K=20$ and $K=25$ and about -0.14 for $(P,Q)=(3,1)$, i.e. $\sum_n F_{2n}^{-3}$, at $K=20$. With the true denominator, the norm of the ideal generated by the coefficients of Δ_K, in place of $C_K(t)$, these numbers become about -0.17 and -0.21. For $Q=1$ the local bounds of Section 4 agree with the exact multiplicities of the cyclotomic factors in $\mathbb{Q}(\tau)[X]$ for $K \le 13$, with equality in most cases. The scripts are available from the author.

8. Formal verification

The Lean development is the repository

`https://github.com/telaviv/lucas-zeta3-irrationality`;

all file names below refer to it. It (library `LucasU`, Lean 4.35.0, Mathlib v4.35.0, about 9200 lines) generalises the formalisation of [7] and proves

```
theorem LucasU.irrational_zetaU (hA : Admissible a E) :
    Irrational (zetaU a E),
```

where `Admissible a E` means $E=1$, $a \ge 1$ or $E=-1$, $a \ge 3$, and `zetaU a E` is the series $\sum_{n\ge0} 1/U_{n+1}^3$ for the sequence `Useq a E` with $U_{n+2} = aU_{n+1} + EU_n$, written with a `tsum`; so $(a,E) = (P,-Q)$. The proof uses only the axioms `propext`, `Classical.choice` and `Quot.sound`; there is no `sorry`, no `native_decide` and no added axiom.

What has to be trusted. For the theorem itself, only the definitions of `Useq`, `zetaU` and `Admissible` (and Mathlib's `tsum` and `Irrational`) have to be read. Since a divergent `tsum` has the value 0, which is rational, the statement cannot hold for a degenerate reason; moreover `hasSum_zetaU` shows that the series converges (to $D\sqrt{D}\,\Xi$). The corollaries are `LucasU.irrational_pell`, that the `tsum` of $1/\mathcal{P}_{n+1}^3$ over $n \ge 0$ is irrational, with `pell` = `Useq 2 1`, and `LucasU.irrational_fib_mul`, that for $k \ge 1$ the `tsum` of $1/F_{k(n+1)}^3$ over $n \ge 0$, written with `Nat.fib`, is irrational; the identity $F_{kn} = F_k U_n(L_k, (-1)^k)$ is `LucasU.fib_mul_eq`, stated as $F_{kn} = F_k\,U_n$ for `Useq (lucasNum k) ((-1)^(k+1))`, i.e. $(a,E) = (L_k, (-1)^{k+1})$, with the Lucas numbers `lucasNum` (all in `LucasU/Corollaries.lean` and `LucasU/Final.lean`).

Definitions and the chain. `LucasU/Defs.lean` defines the construction of Section 2 over an arbitrary field with a sign $\varepsilon = Q$ and a parameter τ (`wpar` $= \varepsilon\tau^2$, `alpha`, `nu`, `psi`, `S`, `node`, `F`, `res`, `mConst`, `mLin`, `nrm`, `Delta`), the real functional `Lfun` and `XiR` as functions of (ε, t), and the clearing polynomial `CKpoly` with exponents `nExp`. `LucasU/Statements.lean` states the ingredients as propositions in (ε, t) (and `Integrality` in ε alone, for every real $\tau \neq 0$ that is not a root of unity); the proofs below assume `Good` ε t, i.e. $\varepsilon = \pm 1$ and $0 < t \leq \varphi^{-1}$. The conjugate point is $\tau = \varepsilon/t$. `LucasU/Unit.lean` contains Lemma 2.1, and `LucasU.irrational_zetaU_of` (in `LucasU/Main.lean`) derives the theorem from the ingredients at $\varepsilon = -E$, $t = 1/\eta$, using that t and ε/t are the two roots of $\tau^2 + aE\tau - E$ and that $\sqrt{D}$ is irrational (`norm_ne_zero`).

Paper	Lean statement	Lean proof
Lemma 2.3, entries	`EntryIdentity`	`LucasU.Analytic.entryIdentity`
Prop. 3.1	`DecayBound`	`LucasU.Analytic.decayBound`
Lemma 4.1–Prop. 4.4	`Integrality`	`LucasU.Denom.integrality`
Lemma 4.5, Prop. 4.6	`CostBound`	`LucasU.Cost.costBound`
Prop. 5.1	`ConjBound`	`LucasU.Conj.conjBound`
Prop. 6.1	`Nonvanishing`	`LucasU.NonVan.nonvanishing`
Section 7		`LucasU.irrational_zetaU_of`

`EntryIdentity` is used only inside the proof of `DecayBound`, not by the assembly. The formal statements are the propositions in the form the assembly needs, with constants that may depend on (ε, t): the uniformity in (P, Q) claimed in Propositions 3.1, 4.6 and the explicit M_K of Proposition 5.1 are not expressed by the formal statements and are not needed, since the theorem concerns one fixed (P, Q) (the formal proof of `DecayBound` does give the uniform constant $C = 4096$). `DecayBound` is $|\Delta_K(\Xi)| \leq C^{K^2} t^{2b_K}$ with $b_K = 2\sum_{j<K} a_j$ (proved with $C = 4096$), `ConjBound` is $|\sigma\Delta_K(-\Xi)| \leq C^{K^2} t^{K^2(K+1)}$, and `CostBound` is $\log|C_K(t)C_K(\varepsilon/t)| \leq \frac{121}{100}K^3 \log t^{-1} + CK^2(1 + \log K)^2$; its proof follows Proposition 4.6 with the cubic constant $c_* = \frac{4393}{3675}$ for both signs, and $c_* < \frac{121}{100} < \frac{11}{9}$. The proof of `Integrality` works in the local rings of $\mathbb{C}(\tau)$ at the roots of unity, and uses the root types only through `LucasU/Roots.lean`; the proof of `Nonvanishing` works with the Cauchy transform $\mathcal{C}$ directly.

Provenance. The formal development generalises that of [7], parts of which are adapted from the Lean formalisation QZeta3 [10] of [11] (Apache License 2.0); every adapted file says so in its header.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop and check the proof, to write the Lean formalisation

and to draft this text. The author directed the project and is responsible for its content. The formal proof is checked by the Lean kernel.

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