A Counterexample to the Realizability Conjecture for Surface Classes

Qiyue Tang

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Abstract

We give a counterexample to Huang–Huh–Michałek–Wang–Wang's conjecture on real realizability of universally pseudoeffective surface classes.

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1 Introduction

The algebraic Steenrod problem asks which homology classes are represented by irreducible algebraic subvarieties, possibly after multiplication by a positive integer. For a product of two projective spaces, Huh [3, Theorem 21] related this question to log-concavity of the multidegree. The multidegree records the degrees obtained by intersecting the subvariety with complementary numbers of hyperplanes from the two factors. Apart from the coordinate classes singled out in his theorem, the criterion is that the sequence of degrees be nonzero and nonnegative, with no zero between two positive entries, and that the square of each intermediate degree be at least the product of its neighbors.

For surface classes, the Hodge index theorem gives a restriction on the intersection form induced by any irreducible surface. Babaee and Huh [1, Proposition 5.9] used the persistence of this restriction under limits in their study of approximation by integration currents on smooth projective toric varieties. On the positivity side, Fulger and Lehmann [2, §2.3] introduced universal pseudoeffectivity for numerical cycle classes. This condition requires pseudoeffectivity after pullback along arbitrary projective test morphisms. Huang, Huh, Michałek, Botong Wang and Shouda Wang use its homological counterpart in their realization problem.

Huang et al. [4, Theorem 1.8] characterize rationally realizable surface classes on products of projective spaces. The entries of their matrix are the intersection degrees with pairs of hyperplane classes pulled back from the factors. Their criterion is that the matrix have nonnegative entries and at most one positive eigenvalue. The corresponding real statement follows by rational approximation as in [4, §7]. They propose a corresponding criterion on arbitrary smooth complex projective varieties in Conjecture 7.3 of the May 2025 preprint.

Let X be a smooth complex projective variety of dimension $n \geq 2$. We use Betti homology and cohomology. For a reduced irreducible subvariety $S \subset X$ of dimension k, write $[S]$ for its fundamental class pushed to $H_{2k}(X, \mathbb{R})$. For $0 \leq k \leq n$, let $B_k(X)$ be the real span of these classes. An effective class is a finite sum of such classes with nonnegative coefficients. Integral and rational effective classes have coefficients in $\mathbb{Z}$ and $\mathbb{Q}$, respectively. The pseudoeffective cone is the closure of the cone of effective classes in real homology.

Let $\mathcal{R}_{2,\mathbb{Q}}(X)$ be the set of classes $a[S]$ with $a \in \mathbb{Q}_{>0}$ and $S \subset X$ a reduced irreducible surface. Its closure $\mathcal{R}_2(X)$ in $B_2(X)$ consists of the classes realizable over $\mathbb{R}$ in the sense of [4, Definition 1.5 and §7]. In particular, $[S]/m \to 0$ as the positive integer m tends to infinity.

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For a smooth projective variety U, let PD_U denote the inverse of cap product with its fundamental class $[U]$. A lower star on a map denotes homological pushforward. For a morphism $\pi : Z \to X$ from a smooth projective variety of dimension m, the homological pullback of $\xi \in B_k(X)$ is

$$\pi^*\xi = \pi^*(\mathrm{PD}_X\, \xi) \frown [Z] \in H_{2(m-n+k)}(Z, \mathbb{R}).$$

It is zero in negative homological degrees. A class is universally pseudoeffective if this pullback is pseudoeffective for every such morphism. Pullback and duality are taken componentwise for a disconnected source.

Write $H^{1,1}(X)_{\mathbb{R}}$ for the real classes in the $(1,1)$ Hodge summand of $H^2(X, \mathbb{C})$. For a surface class ξ and $\alpha, \beta \in H^{1,1}(X)_{\mathbb{R}}$, set

$$q_\xi(\alpha, \beta) = \langle \alpha \smile \beta, \xi \rangle,$$

where $\smile$ is cup product and the brackets denote evaluation on homology. Write $p_+(q)$ for the positive index of a real symmetric form q, the maximal dimension of a positive-definite subspace. By Sylvester's law of inertia it counts the strictly positive eigenvalues of a matrix of the form.

Conjecture 7.3 in [4] asserts that, for universally pseudoeffective $\xi \in B_2(X)$,

$$\xi \in \mathcal{R}_2(X) \quad \Longleftrightarrow \quad p_+(q_\xi) \le 1.$$

Theorem 1. *There is a smooth complex projective fourfold X and an integral effective class $\eta \in B_2(X)$ which is universally pseudoeffective and for which q_η is nondegenerate with one positive and three negative eigenvalues, but $\eta \notin \mathcal{R}_2(X)$.*

The real statement for Grassmannians follows from the integral classification in [4, Theorem 7.7]. Their Example 7.4 shows that ordinary pseudoeffectivity is insufficient even with the Hodge index bound. Example 7.5 gives a universally pseudoeffective class satisfying that bound which is realizable over $\mathbb{R}$ but fails rational realizability. Example 7.6 asks about real realizability for the class of the Fano matroid, which satisfies both hypotheses and fails rational realizability.

The obstruction comes from a finite flat double cover of X. The inverse image of an irreducible surface has at most two irreducible components, which bounds the positive index of the pulled-back form even after passage to the real closure.

2 The positive-index bound for finite flat covers

Lemma 2. *For symmetric forms $q_1, \dots, q_r$ on a fixed finite-dimensional real vector space,*

$$p_+\left(\sum_{j=1}^{r} q_j\right) \le \sum_{j=1}^{r} p_+(q_j).$$

For a fixed integer $d \ge 0$, the set of symmetric forms with positive index at most d is closed.

Proof. Choose a subspace N_j of codimension $p_+(q_j)$ on which q_j is nonpositive, using a diagonalization of q_j. The intersection $N = \bigcap_j N_j$ has codimension at most the sum of these codimensions, and the sum form is nonpositive on N. A positive-definite subspace intersects N only in zero, proving the inequality. For closedness, if a limit form is positive definite on a $(d+1)$-dimensional subspace, its restriction has a positive minimum on the unit sphere of that subspace, so all sufficiently close approximants are positive definite there as well. This contradicts their positive-index bound. □

Lemma 3. *If T is an irreducible surface in a smooth complex projective variety Y, then $p_+(q_{[T]}) \leq 1$ on the entire space $H^{1,1}(Y)_{\mathbb{R}}$.*

Proof. Take a projective resolution $r : \widetilde{T} \to T \hookrightarrow Y$. It has $r_*[\widetilde{T}] = [T]$, so the projection formula gives

$$q_{[T]}(\alpha, \beta) = \int_{\widetilde{T}} r^*\alpha \smile r^*\beta.$$

On $H^{1,1}(\widetilde{T})_{\mathbb{R}}$ an ample class has positive square, and its orthogonal complement is negative definite by the Hodge–Riemann bilinear relations. The map r^* is injective on any subspace on which the pulled-back form is positive definite, and its image there is positive definite for the surface pairing. □

Lemmas 2 and 3 show that every real-realizable class satisfies the conjectured positive-index bound on X itself.

Proposition 4. *Let $f : Y \to X$ be a finite flat surjection of degree d between smooth complex projective varieties. Then*

$$\xi \in \mathcal{R}_2(X) \quad \Longrightarrow \quad p_+\big(q_{f^*\xi} \text{ on } H^{1,1}(Y)_{\mathbb{R}}\big) \leq d.$$

Proof. First let $S \subset X$ be an arbitrary reduced irreducible surface. Its scheme-theoretic inverse image $T = Y \times_X S$ is finite flat of rank d over S, and may be nonreduced. Each irreducible component of T dominates S: flat going-down says that the contraction of a minimal prime is a minimal prime, and S is integral. The algebra of the generic fiber has dimension d over the function field of S and is the product of its local Artinian factors. For each local factor, indexed by the generic point of an irreducible component j,, let m_j be its module length and e_j the degree of its residue field over the function field of S. Each composition factor of that local algebra is its residue field, so its vector-space dimension is $m_j e_j$. Thus $m_j, e_j \geq 1$ and $\sum_j m_j e_j = d$. Consequently T has $r \leq d$ reduced irreducible components T_j, all of dimension two.

Flat pullback of cycles, compatible with the homological pullback defined above, gives

$$f^*[S] = \sum_{j=1}^{r} m_j [T_j].$$

The multiplicities are positive; they equal one in the étale case. Lemmas 2 and 3 imply $p_+(q_{f^*[S]}) \leq r \leq d$. Positive scalar multiplication leaves this bound unchanged. See [6, Tag 02RA] for the flat cycle pullback used here.

Finally, in any fixed basis of $H^{1,1}(Y)_{\mathbb{R}}$, every matrix entry of $q_{f^*\xi}$ is linear and continuous in $\xi \in B_2(X)$. Apply the closedness in Lemma 2 to a sequence from $\mathcal{R}_{2,\mathbb{Q}}(X)$ converging to ξ. □

Lemma 5. *On a smooth projective variety, a product of first Chern classes of globally generated line bundles is Poincar'e dual to the homology class of an effective cycle, possibly zero. The same assertion holds after arbitrary pullback to a smooth projective variety.*

Proof. Choose sections successively. At each stage, global generation allows a section that vanishes identically on none of the finitely many components of the preceding intersection. The resulting divisors intersect properly, or the intersection is empty. Their intersection cycle is effective and represents the product of their first Chern classes. If there are more factors than the dimension, the class is zero. Pullback preserves global generation, so the same argument applies to each pullback. Disconnected test varieties can be treated componentwise. □

3 The fourfold counterexample

3.1 Descent of universal pseudoeffectivity

Lemma 6. *Let $f : Y \to X$ be finite flat and surjective of degree d, with X, Y smooth projective, and let $\xi \in B_2(X)$. Suppose that every pullback of $f^*\xi$ to a smooth projective test variety is a rational effective homology class. Then the same is true of ξ. If all those upstairs pullbacks are integral effective, then $d\pi^*\xi$ is integral effective for every test morphism $\pi : Z \to X$.*

Proof. Fix a test morphism $\pi : Z \to X$ and work on one connected component of the smooth source at a time. The fiber product $W = Z \times_X Y$ is finite flat of rank d over Z, but need not be smooth or reduced. Resolve its reduced irreducible components projectively. Denote the resulting maps by

$$g_j : \widetilde{W}_j \to Z, \qquad h_j : \widetilde{W}_j \to Y, \qquad f h_j = \pi g_j.$$

Every g_j is generically finite and surjective. Let m_j be the length of the local ring of W at the generic point of the jth component and e_j the degree of g_j. Resolution preserves the function field, so the generic-fiber dimension count in Proposition 4 gives $\sum_j m_j e_j = d$. Functoriality of cohomological pullback, interpreted in homology by Poincaré duality, and the projection formula between smooth varieties of equal dimension give

$$h_j^* f^* \xi = g_j^* \pi^* \xi, \qquad (g_j)_* g_j^* \pi^* \xi = e_j \pi^* \xi.$$

Indeed $(g_j)_*[\widetilde{W}_j] = e_j[Z]$, which proves the second formula by taking the cap product with $\mathrm{PD}_Z(\pi^*\xi)$. Consequently

$$d\pi^*\xi = \sum_j m_j (g_j)_* h_j^* f^* \xi.$$

Every summand is rational effective, or integral effective under the stronger hypothesis. Proper pushforward preserves effectivity, and division by d proves the assertion. □

Proof of Theorem 1. Let V be the blowup of $\mathbb{P}^2$ at $p_1 = [1 : 0 : 0]$ and $p_2 = [0 : 1 : 0]$. Denote by H the pullback of the hyperplane class and by E_1, E_2 the classes of the exceptional curves. We write a divisor and its first Chern class with the same symbol when taking cohomological products. On V, a dot between divisor classes denotes the degree of their product. The intersection matrix of H, E_1, E_2 is $\mathrm{diag}(1, -1, -1)$: a general line misses both blown-up points, the exceptional curves are disjoint, and each has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$.

For indices in $\{1, 2, 3\}$, put

$$C_1 = H, \qquad C_2 = H - E_1, \qquad C_3 = H - E_2.$$

The pullback of the line system gives global generation of C_1. The line pencils through p_1 and p_2 give global generation of C_2 and C_3 after blowing up their respective base points. The intersection matrix gives $C_i \cdot C_j = 1$ for $i \neq j$. Choose curves L_1, L_2, L_3 of these classes by taking strict transforms of the lines in homogeneous coordinates $[x : y : z]$ given by $x + y + z = 0$, $y + z = 0$, and $x + 2z = 0$, respectively. The pairwise intersections in the projective plane are $[0 : -1 : 1]$, $[-2 : 1 : 1]$, and $[-2 : -1 : 1]$. They avoid the centers, so the strict transforms meet transversely at the corresponding points.

Let $\mathrm{pr}_1, \mathrm{pr}_2 : V \times V \to V$ be the projections and let $\Delta \subset V \times V$ be the diagonal. Blow up Δ to obtain $b : Y \to V \times V$ with exceptional divisor D. The factor swap lifts to an involution σ of Y. Its projective quotient $f : Y \to X$ is the Hilbert square of V, the scheme of length-two subschemes of V [5, §2.1].

The quotient is smooth and f is finite flat of degree two. Indeed, in common local analytic coordinates $x, y \in \mathbb{C}^2$ on the two factors near the diagonal, use $u = (x+y)/2$ and $v = (x-y)/2$.

In one blowup chart put $v_1 = r$ and $v_2 = rw$. The other chart interchanges v_1 and v_2. The action fixes u, w and sends r to $-r$. The quotient has coordinates (u, r^2, w), and the local ring upstairs is free over the invariant ring with basis $1, r$. This description also holds for the completed local rings, so flatness follows at the branch points by faithful flatness of completion. Away from D the involution acts freely and the quotient map is étale of degree two.

For $i \neq j$ let T_{ij} be the strict transform of $L_i \times L_j$, and for $i < j$ put $S_{ij} = f(T_{ij})$. The inverse image of $L_i \times L_j$ meets D only over the point where L_i meets L_j, with exceptional fiber $\mathbb{P}^1$. Thus the two pulled-back divisors have a proper intersection with sole surface component T_{ij}. Its generic multiplicity is one because b is an isomorphism away from D and the two divisors intersect transversely there. Consequently

$$\mathrm{PD}_Y[T_{ij}] = b^*(\mathrm{pr}_1^* C_i \ \mathrm{pr}_2^* C_j).$$

The two strict transforms T_{ij} and T_{ji} are distinct and exchanged by the involution. Both dominate S_{ij}. The generic fiber has dimension two as an algebra over the function field of S_{ij}, so these two components exhaust it and both have residue-field degree and multiplicity one. Set

$$\eta = \sum_{i<j} [S_{ij}], \qquad f^*\eta = \sum_{i \neq j} [T_{ij}].$$

This gives integral effectivity of η. Lemma 5 applied to the displayed products gives integral effectivity of every smooth-source pullback of $f^*\eta$. Lemma 6 now gives integral effectivity of $2\pi^*\eta$ for every smooth projective test morphism $\pi : Z \to X$. Hence η is universally pseudoeffective.

The basis dual to C_1, C_2, C_3 under the surface intersection pairing is

$$D_1 = H - E_1 - E_2, \qquad D_2 = E_1, \qquad D_3 = E_2.$$

Indeed, the intersection matrix of H, E_1, E_2 gives $D_i \cdot C_j = 1$ for $i = j$ and zero otherwise. The point blowup formula gives $H^1(V, \mathbb{R}) = 0$ and $H^2(V, \mathbb{R}) = \mathbb{R}H \oplus \mathbb{R}E_1 \oplus \mathbb{R}E_2$. By Künneth and the codimension-two blowup formula, the classes $A_i = b^* \mathrm{pr}_1^* D_i$, $B_i = b^* \mathrm{pr}_2^* D_i$ and D constitute a basis of $H^2(Y, \mathbb{R})$. They are divisor classes, so the whole space is of type $(1, 1)$.

The invariant subspace has basis $A_i + B_i$ for $1 \leq i \leq 3$ and D. Each sum descends since D_i has a prime representative (the strict transform of the line through the two centers for $i = 1$, and an exceptional curve otherwise). The two corresponding prime divisors on Y are distinct, exchanged by the swap, and their sum is the flat pullback of their common image. Also $f^* c_1(\mathcal{O}_X(f(D))) = 2D$, with $f(D)$ taken as the reduced branch divisor,, by the local equation $r^2 = 0$ of the inverse image of the branch. Thus the invariant subspace is the image of f^* on degree-two cohomology. This map is injective: $f_*[Y] = 2[X]$ and the cap-product projection formula give $f_*(f^*\alpha \frown [Y]) = 2\alpha \frown [X]$ for every $\alpha \in H^2(X, \mathbb{R})$. The descended classes are real divisor classes, so they span the entire $H^{1,1}(X)_{\mathbb{R}}$.

In the ordered basis of the three A_i, the three B_i, and D, the pairing is computed by

$$q_{f^*\eta}(U, W) = \sum_{k \neq l} \int_Y b^*(\mathrm{pr}_1^* C_k \ \mathrm{pr}_2^* C_l)\, UW$$

for degree-two classes U, W, where integration means evaluation on $[Y]$. The A_i, A_j entries vanish because each term contains three degree-two classes from the first factor, and the B_i, B_j entries vanish by the same degree count on the second factor. The mixed entry is

$$q_{f^*\eta}(A_i, B_j) = \sum_{k \neq l} (D_i \cdot C_k)(D_j \cdot C_l) = \begin{cases} 1 & i \neq j, \\ 0 & i = j. \end{cases}$$

Here pullback by b leaves the top-degree integral unchanged since b has degree one.

For the exceptional entries, cohomological pushforwards are defined by homological pushforward via Poincaré duality. Let $j : D \hookrightarrow Y$ be the inclusion and $p : D \to \Delta$ the exceptional projection. The exceptional divisor parametrizes lines in the normal bundle of Δ and its normal line bundle in Y is the tautological line. Write ζ for the first Chern class of the dual tautological line. Then $D = j_*1$ and $D^2 = -j_*\zeta$ by the projection formula and $j^*D = -\zeta$. Integration on the $\mathbb{P}^1$ fibers gives $p_*1 = 0$ and $p_*\zeta = 1$. Therefore $b_*D = 0$ and b_*D^2 is the negative of the cohomology class of Δ. By the projection formula, the entries pairing D with A_i or B_i vanish since these classes and $\mathrm{PD}_Y(f^*\eta)$ are pulled back by b and $b_*D = 0$. The identity for b_*D^2 gives

$$q_{f^*\eta}(D, D) = -\sum_{i \neq j} C_i \cdot C_j = -6.$$

Let K be the 3×3 matrix with zero diagonal and all off-diagonal entries one. The upstairs matrix is

$$\begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \oplus [-6].$$

For $\alpha, \beta \in H^{1,1}(X)_{\mathbb{R}}$, the cap-product projection formula gives $f_*f^*\eta = 2\eta$. Evaluating against $\alpha \smile \beta$ gives $q_{f^*\eta}(f^*\alpha, f^*\beta) = 2q_\eta(\alpha, \beta)$. The invariant basis $A_i + B_i, D$ has restricted matrix $2K \oplus [-6]$, so the matrix downstairs in the corresponding basis is $K \oplus [-3]$. The eigenvalues of K are 2 on the all-ones line and -1 on the plane of coordinate sum zero. If v is an eigenvector of K with eigenvalue λ, then (v, v) and $(v, -v)$ have eigenvalues λ and $-\lambda$ in the six-dimensional block. Thus the upstairs form has three positive and four negative eigenvalues, while the downstairs form has one positive and three negative eigenvalues. Proposition 4 with degree two now excludes η from $\mathcal{R}_2(X)$.

□

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[6] The Stacks Project Authors, *The Stacks Project*, Tag 02RA.

tangqy24@mails.tsinghua.edu.cn