An Irredundant Test for Circular Positivity

Qiyue Tang

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Abstract

We give an inclusion-minimal circular-minor positivity test on five boundary vertices that is not weakly separated. This refutes the inclusion-minimal version of Conjecture 6.3.4 of Alman, Lian and Tran.

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1 Introduction

A circular planar electrical network is a resistor network drawn in a disk with its boundary vertices on the circumference. Its response matrix represents the linear map from prescribed boundary voltages to boundary currents obtained by imposing Kirchhoff's current law at the interior vertices. We use the sign convention of Alman, Lian and Tran [4], in which off-diagonal response entries are nonnegative.

Let M be a real symmetric matrix with zero row sums, indexed by vertices labeled in clockwise order. A circular pair $(P;Q)$ consists of disjoint lists $P = (p_1, \dots, p_k)$ and $Q = (q_1, \dots, q_k)$ of distinct vertices, where $k \geq 1$, such that $p_1, \dots, p_k, q_k, \dots, q_1$ occur in that cyclic order. Its circular minor is the determinant with rows P and columns Q, in the indicated orders. Transposition identifies the corresponding minor for the reversed lists $(q_k, \dots, q_1; p_k, \dots, p_1)$ with the original minor.

Curtis, Ingerman and Morrow [1] characterized response matrices of circular planar networks by symmetry, zero row sums and nonnegativity of the circular minors, with the corresponding choice of response sign. Circular minors also detect disjoint connections in the network: the minor for $(P;Q)$ is positive precisely when there are vertex-disjoint paths from p_i to q_i for all i. Strict positivity of all circular minors thus corresponds to well-connectedness.

For n boundary vertices, Kenyon and Wilson [2] gave a test for well-connectedness using $\binom{n}{2}$ central circular minors. They also expressed contiguous minors by Laurent polynomials obtained from weighted domino tilings. Lai [3] extended the tiling formulas to semicontiguous minors. These results give specific families and formulas for testing positivity. Alman, Lian and Tran [4, Section 6] asked for a classification of minimal tests and related the question to determinant exchange relations.

A positivity test is a family of circular minors whose simultaneous strict positivity implies strict positivity of every circular minor for every real symmetric zero-row-sum matrix.

Two sets of vertices are weakly separated if there are no four cyclically ordered vertices alternating between their two set differences. For two circular pairs $(P;Q)$ and $(R;S)$, weak separation

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requires that $P \cup R$ be weakly separated from $Q \cup S$ and that $P \cup S$ be weakly separated from $Q \cup R$. The lists are regarded as sets in these comparisons.

In [4, Conjecture 6.3.4], the authors proposed that minimal positivity tests are precisely the inclusion-maximal pairwise weakly separated collections, and that these collections coincide with the clusters of their Laurent phenomenon algebra consisting of circular minors. The following example refutes the equivalence when minimality means that no test inequality can be deleted.

2 The counterexample

Theorem 1. *There is an eleven-member inclusion-minimal circular-minor positivity test on five vertices containing the incompatible minors M_{12} and M_{34}.*

Note. *In the version of Conjecture 6.3.4 considered here, minimal means minimal under inclusion: no proper subfamily is a positivity test. Minimum cardinality asks for the fewest members among all positivity tests.*

Proof. For five vertices write

$$(M_{12}, M_{13}, M_{14}, M_{15}, M_{23}, M_{24}, M_{25}, M_{34}, M_{35}, M_{45}) = (a, b, c, d, e, f, g, h, i, j).$$

For $1 \le r \le 5$, set $M_{rr} = -\sum_{1 \le s \le 5,\ s \ne r} M_{rs}$, with $M_{sr} = M_{rs}$ for $s > r$. Thus each real tuple defines a matrix in the test domain. Besides the ten entries above, the circular minors, up to equality by symmetry, are

$$ce - bf,\ de - bg,\ df - cg,\ dh - ci,\ ah - bf,\ ai - bg,\ aj - cg,\ bj - ci,\ gh - fi,\ ej - fi.$$

Indeed, for four labels $u < v < w < z$, the two circular pairs $(u, v; z, w)$ and $(v, w; u, z)$, up to reversal and transposition, give the minors $M_{uz}M_{vw} - M_{uw}M_{vz}$ and $M_{uv}M_{wz} - M_{uw}M_{vz}$. Three rows and three disjoint columns would require six vertices.

Consider the family

$$\{a, b, c, g, h,\ ce - bf,\ df - cg,\ dh - ci,\ ai - bg,\ bj - ci,\ gh - fi\}.$$

If all eleven members are positive, the inequalities $ai > bg$ and $bj > ci$ give $i, j > 0$. Next $dh > ci$, $df > cg$, and $ce > bf$ give $d, f, e > 0$, in that order. The other four minors are positive by the identities

$$\begin{aligned} c(de - bg) &= d(ce - bf) + b(df - cg), \\ i(ah - bf) &= h(ai - bg) + b(gh - fi), \\ i(aj - cg) &= j(ai - bg) + g(bj - ci), \\ c(ej - fi) &= j(ce - bf) + f(bj - ci). \end{aligned}$$

For each member in the left column, substitution of the tuple on the right makes that member

negative and the other ten strictly positive.

Deleted minor	$(a, b, c, d, e, f, g, h, i, j)$
a	$(-2, 1, 1, 2, 2, 1, 1, 1, -2, 1)$
b	$(1, -1, 1, 2, 1, 1, 1, 1, 0, -1)$
c	$(1, 1, -1, 0, 0, -1, 1, 1, 2, 1)$
g	$(1, 2, 1, 1, 5, 2, -1, 1, -1, 1)$
h	$(1, 1, 1, -3, 0, -1, 1, -1, 2, 3)$
$ce - bf$	$(3, 1, 1, 2, 0, 1, 1, 2, 1, 2)$
$df - cg$	$(2, 1, 1, 3, 0, -1, 1, 1, 1, 2)$
$dh - ci$	$(2, 1, 1, -2, 0, -1, 1, 1, 1, 2)$
$ai - bg$	$(1, 1, 1, 2, 2, 1, 1, 1, -1, 1)$
$bj - ci$	$(2, 1, 1, 2, 2, 1, 1, 2, 1, 0)$
$gh - fi$	$(2, 1, 1, 2, 3, 2, 1, 1, 1, 2)$

The strict signs persist on an open neighborhood of each tuple, so witnesses can also be taken outside any prescribed proper algebraic exceptional set. Every proper subfamily omits at least one member and fails on its witness. Finally, for the pairs $(1; 2)$ and $(3; 4)$, the sets $\{1, 3\}$ and $\{2, 4\}$ alternate in cyclic order. Thus the test is not pairwise weakly separated. □

References

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School of Mathematical Sciences, Tsinghua University, Beijing, China
Email address: tangqy24@mails.tsinghua.edu.cn