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UNRAMIFIED MULTIPLICITY ONE SPHERICAL PAIRS

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Abstract. Let p be an odd prime and let k be a finite extension of $\mathbb{Q}_p$. We study the distinction between multiplicity one for unramified representations and multiplicity one for irreducible unitary representations. We prove that $(\mathbf{G}_2, \mathbf{SL}_3)$ has the former property but not the latter. The unramified assertion uses spherical functions and the spherical Hecke module. For the failure of multiplicity one, we use Weil representations and theta lifting to obtain an irreducible unitarizable representation with at least two independent $\mathbf{SL}_3(k)$-invariant linear forms.

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1. Introduction

Throughout this paper, p is an odd prime and k is a finite extension of $\mathbb{Q}_p$. Write $\mathfrak{o}$ for its ring of integers, ϖ for a uniformizer, and $\mathbb{F}_q$ for its residue field, where $q = p^f$. No assumption on the ramification of $k/\mathbb{Q}_p$ is imposed. Let $\mathbf{G}$ be a split connected reductive group over k, with a Borel subgroup $\mathbf{B}$ and a maximal split torus $\mathbf{A} \subset \mathbf{B}$. Fix a split reductive model over $\mathfrak{o}$, and write

$$G = \mathbf{G}(k), \qquad B = \mathbf{B}(k), \qquad A = \mathbf{A}(k), \qquad K = \mathbf{G}(\mathfrak{o}).$$

Thus K is a hyperspecial maximal compact subgroup of G. Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a homogeneous affine spherical variety, and put $H = \mathbf{H}(k)$ and $X = \mathbf{X}(k)$. We distinguish X from the orbit $H\backslash G$ unless their equality has been established.

Unless otherwise stated, representations are smooth complex representations. A smooth representation is called *unitarizable* if it admits a positive-definite invariant Hermitian form. For a unitary representation Π on a Hilbert space, Π^∞ denotes its space of smooth vectors.

The formulation of [SV17, Conjecture 17.4.1] uses the following multiplicity-one assumption.

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Assumption 1.1. *For every irreducible unitary representation* Π *of* G*, we have*

$$\dim \mathrm{Hom}_G(\Pi^\infty, C^\infty(X)) \leq 1.$$

Definition 1.2. We call $(\mathbf{G}, \mathbf{H})$ a *spherical pair of multiplicity one* if Assumption 1.1 holds.

This condition can be difficult to check. We therefore consider an unramified analogue. A smooth representation π of G is *unramified* if $\pi^K \neq 0$.

Definition 1.3. We call $(\mathbf{G}, \mathbf{H})$ a *spherical pair of unramified multiplicity one* if, for every irreducible unramified representation π of G,

$$\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \leq 1.$$

The latter definition includes nonunitarizable representations; we do not infer a logical implication between these two conditions merely from their definitions.

Question 1.4. Does unramified multiplicity one imply Assumption 1.1?

The pair $(\mathbf{G}_2, \mathbf{SL}_3)$ provides a negative answer. More precisely, the main result is the following.

Theorem 1.5 (Main result)**.** *Let* p *be an odd prime and let* k *be a finite extension of* $\mathbb{Q}_p$*. Let* $\boldsymbol{G}_2$ *be the split group of type* G_2 *over* k*, let be its split subgroup* $\boldsymbol{H} = \boldsymbol{SL}_3$*, and put* $G = \boldsymbol{G}_2(k)$ *and* $H = \boldsymbol{SL}_3(k)$*.*

(1) For every irreducible unramified representation σ *of* G*,*

$$\dim \mathrm{Hom}_H(\sigma, \mathbb{C}) \leq 1.$$

(2) There is an irreducible unitarizable smooth representation π *of* G *such that*

$$\dim \mathrm{Hom}_H(\pi, \mathbb{C}) \geq 2.$$

2. Unramified multiplicity one criterion

2.1. **Formula for spherical functions.** In this section, let k be a p-adic local field of characteristic 0. We recall a formula for spherical functions.

Let $\mathbf{X}$ be a homogeneous affine spherical $\mathbf{G}$-variety. Write $\mathring{\mathbf{X}}$ for its open $\mathbf{B}$-orbit, $\mathcal{X}(\mathbf{X})$ for its lattice of rational $\mathbf{B}$-eigencharacters, and $\mathbf{A}_X$ for its associated Cartan quotient torus, as in [Sak13, §2.1]. Put

$$A_X = \mathbf{A}_X(k), \qquad \Lambda_X = \mathrm{Hom}(\mathbb{G}_m, \mathbf{A}_X).$$

We denote by Λ_X^+ the monoid of antidominant coweights and by A_X^+ its inverse image in A_X. The identification of the Cartan quotient with an orbit in $\mathbf{X}$ is understood when writing xJ or xK for $x \in A_X^+$.

Let A^* and A_X^* be the complex tori of unramified characters of A and A_X, respectively. Write Σ_X for the set of normalized simple spherical roots. Let W be the Weyl group of $\mathbf{G}$ and W_X the little Weyl group of $\mathbf{X}$. For a coweight $\check{\lambda}$ and a corresponding unramified character χ, set $e^{\check{\lambda}}(\chi) = \chi(\check{\lambda}(\varpi))$. We write $\mathcal{H}(G, K)$ for the spherical Hecke algebra, with Haar measure normalized by $\mathrm{vol}(K) = 1$.

Let $\mathbf{P}(\mathbf{X})$ be the parabolic stabilizing the open $\mathbf{B}$-orbit and let $\mathbf{L}(\mathbf{X})$ be an adapted Levi subgroup. We use the modular-character conventions of [Sak13, §1.7]: $\delta = \delta_B$, $\delta_{P(X)}$ is the modular character of $P(X) = \mathbf{P}(\mathbf{X})(k)$, and $\delta_{(X)}$ is the modular character of $B \cap \mathbf{L}(\mathbf{X})(k)$. The symbol $\rho_{P(X)}$ denotes the half-sum of the roots in the unipotent radical of $\mathbf{P}(\mathbf{X})$, and $\check{\rho}$ denotes the half-sum of the positive coroots of $\mathbf{G}$.

We recall two of the hypotheses needed below:

Axiom 2.1. *The set* A_X^+ *contains a complete set of representatives for the* K*-orbits on* X*. Moreover, elements of* A_X^+ *which map to distinct elements of* Λ_X^+ *belong to different* K*-orbits.*

Denote $J \subset K$ the inverse image of $\mathbf{B}(\mathbb{F}_q)$ under the reduction map $\mathbf{G}(\mathfrak{o}) \to \mathbf{G}(\mathbb{F}_q)$.

Axiom 2.2. *For $x \in A_X^+$, we have $xJ \subset x\boldsymbol{B}(\mathfrak{o})$.*

We record our assumptions as follows:

Assumption 2.3. *We assume:*

- *There is a single B-orbit in $\mathring{\boldsymbol{X}}(k)$.*
- *Axioms 2.1 and 2.2 hold.*
- *The local-field and Fourier-transform conventions of [Sak13, §1.1 and §5.3] are valid for the formulas being used; any extension of those conventions is justified.*
- *The integral-model and measure hypotheses of [Sak13, §1.7], including the integrality axiom in [Sak13, §2.2], hold for the chosen models.*
- *We assume that for every $\boldsymbol{Y}$ Borel orbits of maximal rank and α a simple root, we have $(\boldsymbol{Y}, \alpha)$ is not of type N.*

With the normalization of [Sak13, Equation (6.2)], the spherical functions admit the following expression.

Theorem 2.4 (Spherical-function formula)**.** *Under Assumption 2.3, for generic $\chi \in A_X^*$ the corresponding spherical eigenfunctions satisfy*

$$\Omega_{\delta_{(X)}^{1/2}\chi}(x) = \delta_{P(X)}^{-1/2}(x) \sum_{w \in W_X} B_w(\chi)\, {}^w\chi(x), \qquad x \in A_X^+.$$

Their Hecke eigencharacters are determined by the Satake parameters $\delta_{(X)}^{1/2}\chi$. The rational functions $B_w \in \mathbb{C}(A_X^*)$ satisfy the cocycle relation.

2.2. **Fourier normalization over a ramified local field.**

Lemma 2.5. *Let k be any characteristic-zero nonarchimedean local field. Choose a nontrivial additive character ψ_0 whose annihilator of $\mathfrak{o}$ is $\mathfrak{o}$, and give k the Haar measure dx with $\mathrm{vol}(\mathfrak{o}) = 1$. Then dx is self-dual. For an unramified multiplicative character μ, put $u = \mu(\varpi)$ and $d^\times x = dx/|x|$. With Fourier kernel $\psi_0(xy)$, one has, meromorphically in μ,*

$$\mathcal{F}(\mu(x)d^\times x) = \frac{1 - q^{-1}u^{-1}}{1 - u}\, \mu^{-1}(y)|y|\, d^\times y. \tag{2.1}$$

In particular, the unramified type U and split type T Fourier factors of [Sak13, Theorem 5.2.1] have the same form over ramified extensions with these conventions.

Proof. Such a character always exists. More explicitly, if $\mathfrak{D}_{k/\mathbb{Q}_p} = d\mathfrak{o}$ is the different and ψ_p has conductor $\mathbb{Z}_p$, take $\psi_0(x) = \psi_p(\mathrm{Tr}_{k/\mathbb{Q}_p}(d^{-1}x))$. The inverse-different description of the trace-dual lattice shows that the annihilator of $\mathfrak{o}$ is exactly $\mathfrak{o}$. Consequently the Fourier transform of $1_\mathfrak{o}$ is $1_\mathfrak{o}$, and the specified measure is self-dual.

For generic μ, homogeneity determines the Fourier transform up to the scalar in (2.1). Pairing with $1_\mathfrak{o}$, the two geometric series are

$$\int_\mathfrak{o} \mu(x)d^\times x = \frac{1 - q^{-1}}{1 - u}, \qquad \int_\mathfrak{o} \mu^{-1}(x)dx = \frac{1 - q^{-1}}{1 - q^{-1}u^{-1}}.$$

They give the displayed scalar when $q^{-1} < |u| < 1$; meromorphic continuation gives the identity in general. The type U equations use this one-variable identity, and the split type T equation uses its product in two coordinates. Thus no factor depending on the different remains once the character is normalized as above. This addresses the simplifying convention in [Sak13, §1.1 and §5.3]. □

2.3. **Geometric data for the split pair.** In the remainder of this subsection, $\mathbf{G} = \mathbf{G}_2$ and $\mathbf{H} = \mathbf{SL}_3$ is the standard split subgroup. Let V_7 be the trace-zero subspace of the split octonion algebra, with $q_7 = -N$. In Zorn coordinates,

$$q_7(t, u, w) = t^2 + u_1w_1 + u_2w_2 + u_3w_3.$$

We use the right action $v \cdot g = g^{-1}v$.

Lemma 2.6. *The orbit map identifies $\boldsymbol{X} = \boldsymbol{SL}_3\backslash\boldsymbol{G}_2$ with the affine quadric $\{q_7 = 1\}$. Moreover,*

$$\boldsymbol{X}(k) = \boldsymbol{SL}_3(k)\backslash\boldsymbol{G}_2(k).$$

Proof. The homogeneous-quadric identification is [AHW19, Theorem 2.3.5(2) and its proof]. Applied to trace-zero octonions of norm -1, it identifies the geometric orbit of $v_0 = \mathrm{diag}(1, -1)$ with this quadric; its stabilizer is the standard $\mathbf{SL}_3$. Equivalently, $v \mapsto (1 + v)/2$ identifies it with the quadric of trace-one, norm-zero octonions. Thus the orbit map, defined over k, is an isomorphism after extension to $\bar{k}$ and hence over k.

The fiber of $\mathbf{G}_2 \to \mathbf{G}_2/\mathbf{SL}_3$ over a k-point is an $\mathbf{SL}_3$-torsor. The exact sequence

$$1 \longrightarrow \mathbf{SL}_3 \longrightarrow \mathbf{GL}_3 \xrightarrow{\det} \mathbb{G}_m \longrightarrow 1$$

and Hilbert's Theorem 90 give $H^1(k, \mathbf{SL}_3) = 1$: $\det : \mathrm{GL}_3(k) \to k^\times$ is surjective and $H^1(k, \mathbf{GL}_3) = 1$. Every such fiber therefore has a k-point. Passing from left to right quotients by inversion proves the assertion. □

Let α, β be the short and long simple roots and set $\gamma = 2\alpha + \beta$. The representation V_7 has highest weight γ and weights

$$0, \quad \pm\alpha, \quad \pm(\alpha + \beta), \quad \pm\gamma,$$

each with multiplicity one. Choose weight vectors $v_{-\gamma}$ and v_γ and a nonzero $c \in k$ so that $x_* = v_{-\gamma} + cv_\gamma$ has $q_7(x_*) = 1$.

Lemma 2.7. *The point x_* belongs to the open $\boldsymbol{B}$-orbit, and*

$$\boldsymbol{B}_{x_*} = \check{\beta}(\mathbb{G}_m) \ltimes \boldsymbol{U}_\beta.$$

In particular, $\mathring{\boldsymbol{X}}(k)$ is a single B-orbit. Both X and $\mathring{\boldsymbol{X}}(k)$ admit the invariant measures required in [Sak13, §1.7].

Proof. The torus $\ker\gamma = \check{\beta}(\mathbb{G}_m)$ and the root subgroup $\mathbf{U}_\beta$ fix x_*. The positive-root operators for

$$\alpha, \quad \alpha + \beta, \quad \gamma, \quad 3\alpha + \beta, \quad 3\alpha + 2\beta$$

send x_* to nonzero vectors of weights $-(\alpha + \beta), -\alpha, 0, \alpha, \alpha + \beta$, respectively. Nonvanishing follows from the corresponding root strings in V_7. Together with the tangent direction from $\mathbf{A}$, these give six independent tangent vectors. Hence the orbit is open and its stabilizer has dimension two.

The highest-weight linear function in V_7^* is nonzero at x_*, so the projection of this stabilizer to $\mathbf{A}$ lies in $\ker\gamma$. Its unipotent kernel is connected in characteristic zero, has dimension one, and contains $\mathbf{U}_\beta$. This proves the asserted equality of stabilizers, including connectedness. The stabilizer is split solvable, so its first Galois cohomology is trivial; the homogeneous-space exact sequence gives the assertion about B-orbits.

The groups $\mathbf{G}_2(k)$ and $\mathbf{SL}_3(k)$ are unimodular, so their quotient has an invariant measure. For the open orbit, write $S = \mathbf{B}_{x_*}(k)$. On $\check{\beta}(t)$ the modular character of S is $|t|^2$, since this torus acts on $\mathbf{U}_\beta$ by t^2. The modular character of B has the same restriction, because $2\rho = 10\alpha + 6\beta$ and $\langle\check{\beta}, 2\rho\rangle = 2$. Thus $\delta_B|_S = \delta_S$, which is the criterion for an invariant measure on $S\backslash B$. □

The rank-one data are $\Sigma_X = \{\gamma\}$, $W_X = \{1, w_\gamma\}$, and $\mathbf{P}(\mathbf{X}) = \mathbf{P}_\beta$; see [Sak21, §1.2, the G_2 entry]. Here the subscript β specifies the simple root of the Levi. The spherical root is of split type T for the split subgroup $\mathbf{SL}_3$ (see also [Sak21, §2.3]). The Levi $\mathbf{L}_\beta$ is adapted to x_*: its derived group fixes both v_γ and $v_{-\gamma}$. Consequently,

$$\mathbf{A}_X = \mathbf{A}/\check{\beta}(\mathbb{G}_m), \qquad \mathcal{X}(\mathbf{X}) = \mathbb{Z}\gamma, \qquad \Lambda_X = \mathbb{Z}\check{\theta}, \tag{2.2}$$

where $\check{\theta}$ is the image of $\check{\alpha}$ and $\langle\check{\theta}, \gamma\rangle = 1$. In particular, $\check{\gamma} = 2\check{\theta}$ in Λ_X. Furthermore,

$$\delta_{(X)} = |\beta|, \qquad \rho_{P(X)} = \tfrac{1}{2}(10\alpha + 5\beta) = \tfrac{5}{2}\gamma. \tag{2.3}$$

These are statements over k. The integral assertions needed for the spherical-function formula under our standing hypothesis on p are proved in Proposition 2.12.

Lemma 2.8. *There is a highest-weight linear function $f_\gamma \in V_7^*$ of weight γ such that $D = \{f_\gamma = 0\} \subset \boldsymbol{X}$ is the unique color. Its valuation is $\check{v}_D = \check{\theta}$. The weights of regular $\boldsymbol{B}$-eigenfunctions on $\boldsymbol{X}$ are exactly $\mathbb{Z}_{\geq 0}\gamma$.*

Proof. We work over $\bar{k}$ for the geometric assertions. Let $\mathcal{H}_n$ be the degree-n harmonic polynomials for q_7. The characteristic-zero harmonic decomposition gives

$$\mathrm{Sym}^n(V_7^*) = \bigoplus_{j=0}^{\lfloor n/2 \rfloor} q_7^j \mathcal{H}_{n-2j}, \qquad \bar{k}[\mathbf{X}] = \bigoplus_{n \geq 0} \mathcal{H}_n.$$

For completeness, this decomposition follows by repeatedly applying the Laplacian Δ: if h is harmonic of degree m and q_7 is written as a sum of seven squares over $\bar{k}$, then

$$\Delta(q_7^j h) = 2j(2m + 2j + 5)q_7^{j-1}h.$$

The displayed constants are nonzero for $j > 0$.

The covector f_γ is isotropic for the dual quadratic form: its nonzero weight forces its pairing with itself to vanish. Thus f_γ^n is harmonic and generates a copy of the irreducible highest-weight representation $V(n\gamma)$ in $\mathcal{H}_n$. The Weyl dimension formula gives

$$\dim V(n\gamma) = \frac{(n+1)(n+2)(n+3)(n+4)(2n+5)}{120} = \binom{n+6}{6} - \binom{n+4}{6} = \dim \mathcal{H}_n,$$

where binomial coefficients with upper entry less than 6 are zero. Hence $\mathcal{H}_n = V(n\gamma)$ and $\bar{k}[\mathbf{X}]^{\mathbf{U}} = \bar{k}[f_\gamma]$.

Choose hyperbolic coordinates so that $f_\gamma = u_1$. The ring

$$R = \bar{k}[t, u_1, u_2, u_3, w_1, w_2, w_3]/(q_7 - 1)$$

is normal because the quadric is smooth. Its localization $R[u_1^{-1}]$ is a Laurent polynomial ring (eliminate w_1), and u_1 is prime:

$$R/(u_1) = \bar{k}[t, u_2, u_3, w_1, w_2, w_3]/(t^2 + u_2w_2 + u_3w_3 - 1)$$

is a domain. Nagata's localization criterion therefore makes R a unique factorization domain. Also $R^\times = \bar{k}^\times$; indeed its degree filtration has domain $\mathrm{gr}\, R = \bar{k}[V_7]/(q_7)$, so degrees add under multiplication of nonzero elements.

Every $\mathbf{B}$-stable prime divisor is consequently defined by a prime element g whose line is $\mathbf{B}$-stable, since the units are constant. It is a $\mathbf{B}$-eigenfunction and lies in $\bar{k}[f_\gamma]$. The only prime eigenfunctions in that ring, up to scalar, are f_γ itself. Thus D is the unique color, and its equation is reduced, giving $v_D(f_\gamma) = 1$. The highest-weight line and this divisor are defined over the split field k, so the description and valuation descend to k. □

2.4. **Integral orbits in odd residue characteristic.** We now verify the integral inputs over $R = \mathbb{Z}[1/2]$. Thus every assertion in this subsection can be base-changed to $\mathfrak{o}$ when its residue characteristic is odd. This gives an explicit instance of the integral-chart approach to Cartan decomposition in [Sak12, §2.3].

Choose a Chevalley weight basis $e_1, \dots, e_7$ of $V_{7,R}$ with respective weights

$$\gamma, \quad \alpha+\beta, \quad \alpha, \quad 0, \quad -\alpha, \quad -(\alpha+\beta), \quad -\gamma.$$

Write $v = \sum x_i e_i$. We use the normalization

$$q_7(v) = x_4^2 + 4(x_1x_7 + x_2x_6 + x_3x_5). \tag{2.4}$$

It is an R-linear change of the Zorn coordinates above; all scalings involved are powers of 2. For transparency, the positive root operators in this basis can be chosen as follows, where $E_{ij}e_j = e_i$:

$$\begin{aligned} N_\alpha &= -E_{12} - E_{34} + 2E_{45} + E_{67}, & N_\beta &= -E_{23} + E_{56}, \\ N_{\alpha+\beta} &= E_{13} - E_{24} + 2E_{46} - E_{57}, & N_\gamma &= E_{14} + E_{25} - E_{36} - 2E_{47}, \\ N_{3\alpha+\beta} &= -E_{15} + E_{37}, & N_{3\alpha+2\beta} &= E_{16} - E_{27}. \end{aligned} \tag{2.5}$$

Here $x_\rho(t) = \exp(tN_\rho)$; the exponential has degree at most two. The negative simple operators are

$$N_{-\alpha} = -E_{21} - 2E_{43} + E_{54} + E_{76}, \qquad N_{-\beta} = -E_{32} + E_{65}.$$

These are the usual highest-weight-γ Chevalley operators. Their brackets give the G_2 Cartan and Serre relations; the remaining positive operators are obtained by the successive root brackets. All displayed root-group matrices preserve (2.4). The matrices and their quadratic terms are integral; the computations below divide only by 2, including in residue characteristic 3.

Lemma 2.9 (An integral toroidal chart)**.** *Let*

$$\overline{\mathcal{X}} = \{[v:s] \in \mathbb{P}(V_{7,R} \oplus R) : q_7(v) = s^2\}, \qquad \mathcal{C} = \overline{\mathcal{X}} \cap \{x_7 \neq 0\}.$$

Put $c(s) = [e_7 + (s^2/4)e_1 : s]$. *Then*

$$\mathbb{A}^1_R \times U_{P_\beta,R} \longrightarrow \mathcal{C}, \qquad (s,u) \longmapsto c(s)\cdot u$$

is an isomorphism. The open part $s \neq 0$ *is the integral open* B*-orbit of* $\mathcal{X} = \{q_7 = 1\}$.

Proof. The use of the right action replaces root parameters by those of u^{-1}, an automorphism of the unipotent group. It therefore suffices to check the left action. Apply, successively,

$$x_\alpha(t_1),\ x_{\alpha+\beta}(t_2),\ x_\gamma(t_3),\ x_{3\alpha+\beta}(t_4),\ x_{3\alpha+2\beta}(t_5)$$

to e_7. Matrix multiplication in (2.5) gives

$$\begin{aligned} &x_7 = 1, && x_6 = t_1, && x_5 = -t_2, \\ &x_4 = 2t_1t_2 - 2t_3, && x_3 = t_4 - t_1t_3, && x_2 = -t_1t_2^2 - t_2t_3 - t_5, \\ &x_1 = -t_3^2 + 2t_1t_2t_3 + t_2t_4 + t_1t_5. \end{aligned} \tag{2.6}$$

The five equations for x_6, x_5, x_4, x_3, x_2 are triangular and can be solved successively over R, dividing only by 2. The remaining coordinate is determined by $q_7 = 0$. Since all positive root groups fix e_1, applying the same product to $e_7 + (s^2/4)e_1$ simply adds $s^2/4$ to x_1. In the chart $x_7 = 1$, the equation $q_7 = s^2$ determines x_1 uniquely; the other five coordinates and s are free. This proves the claimed isomorphism over R.

For $s \neq 0$, rescale to the affine equation $q_7 = 1$. The torus acts transitively on the invertible coordinate s, since γ is primitive, and U_{P_β} acts transitively on each fiber. The open orbit is consequently smooth and has homogeneous geometric fibers over R. □

Proposition 2.10. *If the residue characteristic of k is odd, the Cartan and Iwahori axioms hold. Explicit representatives are*

$$x_n = \varpi^{-n} e_7 + \tfrac{1}{4}\varpi^n e_1, \qquad n \in \mathbb{Z}_{\geq 0}.$$

They correspond to $-n\check{\theta} \in \Lambda_X^+$.

Proof. The projective quadric $\overline{\mathcal{X}}$ is proper and smooth over $\mathfrak{o}$. Every point $[v:1]$ of X therefore has a primitive integral representative (v', s'). Its reduction can be moved to $\{x_7 \neq 0\}$ by an element of $G_2(\mathbb{F}_q)$ which lifts to K. Indeed, if any of the six nonzero-weight coordinates is nonzero, a Weyl representative moves it to x_7. If all six vanish, the reduction has $v' = x_4 e_4$ and $x_4^2 = (s')^2$; hence $x_4 \neq 0$. A short-root element moves this point to one with nonzero x_7: for example $N_{-\gamma} e_4$ is a nonzero unit multiple of e_7 in odd characteristic. Weyl representatives and these root elements have integral lifts.

After this K-translation, Lemma 2.9 writes the point as $c(s) \cdot u$ with $s \in \mathfrak{o} \setminus \{0\}$ and $u \in U_{P_\beta}(\mathfrak{o})$. The map $\gamma : A(\mathfrak{o}) \to \mathfrak{o}^\times$ is surjective, because $\langle \check{\alpha}, \gamma \rangle = 1$. It removes the unit part of s, leaving $s = \varpi^n$. Rescaling $c(\varpi^n)$ to the affine chart gives x_n.

The integer n is intrinsic: it is the valuation of the last coordinate s' of a primitive representative (v', s'). The group K preserves the lattice $V_7(\mathfrak{o}) \oplus \mathfrak{o}$ and fixes the last coordinate. It preserves primitivity and hence n. The representatives belong to distinct K-orbits.

For the Iwahori assertion, J preserves $\mathcal{C}(\mathfrak{o})$: modulo ϖ it acts through $B(\mathbb{F}_q)$, and this chart is B-stable. If $c(s)j = c(s')u$ in the integral chart, the unnormalized last coordinate is unchanged and the new x_7 is a unit. Thus $s'/s \in \mathfrak{o}^\times$. Absorb that unit by $A(\mathfrak{o})$ and use $u \in U_{P_\beta}(\mathfrak{o})$ to obtain $x_n J \subset x_n B(\mathfrak{o})$. Finally, the torus action on the affine representatives gives $x_n = x_0 \check{\alpha}(\varpi^{-n})$, which identifies their coweights as asserted. □

Lemma 2.11 (The integral rank-one quotients)**.** *Over R, the maximal-rank B-orbits in $\mathcal{X}$ are*

$$Y_0 = \{x_7 \neq 0\}, \quad Y_1 = \{x_7 = 0,\ x_6 \neq 0\}, \quad Y_2 = \{x_7 = x_6 = 0,\ x_5 \neq 0\}.$$

Their dimensions are $6, 5, 4$. The remaining two B-orbits have dimension 3 and rank zero. The nontrivial links between maximal-rank orbits are

$$Y_0 \overset{\alpha,U}{\longleftrightarrow} Y_1 \overset{\beta,U}{\longleftrightarrow} Y_2, \qquad (Y_2, \alpha) \text{ is split type } T.$$

The pair (Y_0, β) is of type G. These descriptions and the rank-one quotient isomorphisms hold over R, so the integrality axiom of [Sak13, §2.2] holds over $\mathfrak{o}$ in odd residue characteristic.

Proof. The description of Y_0 is Lemma 2.9. For Y_1, apply successively the root elements for $\beta, \alpha+\beta, \gamma, 3\alpha+2\beta$, with parameters a, b, c, d, to $e_6 + \frac{1}{4}e_2$. The resulting coordinates are

$$x_7 = 0, \quad x_6 = 1, \quad x_5 = a, \quad x_4 = 2b, \quad x_3 = -c, \quad x_2 = \tfrac{1}{4} - b^2 + ac, \quad x_1 = 2bc + d.$$

They give a triangular chart on the slice $x_6 = 1$; the torus scales x_6 through the primitive character $\alpha + \beta$. This proves that Y_1 is a smooth homogeneous B-scheme.

The P_α-saturation of Y_0 is $\{(x_7, x_6) \neq (0,0)\}$. Projection to $[x_7 : x_6]$ identifies its quotient by $\mathcal{R}(P_\alpha)$ with $\mathbb{P}^1_R$. To check the fibers, on $x_7 \neq 0$ the ratio x_6/x_7 is the first root coordinate in (2.6); the other four root groups lie in U_{P_α}. The center of L_α scales the nonzero coordinate primitively:

$$\langle \check{\alpha} + 2\check{\beta}, \gamma \rangle = 1.$$

Thus the radical is transitive on each fiber. The other affine chart is obtained using n_α. This proves the quotient isomorphism over R, not just on generic points. The two Borel orbits on $\mathbb{P}^1_R$ give the first type U link.

Likewise, the P_β-saturation of Y_1 is $\{x_7 = 0,\ (x_6, x_5) \neq (0,0)\}$ and projection to $[x_6 : x_5]$ gives $\mathbb{P}^1_R$. Use the preceding four-parameter chart, with its first parameter now the β-coordinate, and

$$\langle 2\check{\alpha} + 3\check{\beta}, \alpha + \beta \rangle = 1.$$

The other three root groups are in U_{P_β}, and n_β supplies the other chart. This proves the second type U link.

The P_α-saturation of Y_2 is $Z = \{x_7 = x_6 = 0\}$. Its projection to (x_3, x_4, x_5) has base

$$Q_2 = \{x_4^2 + 4x_3x_5 = 1\}$$

and fibers with coordinates (x_1, x_2). The radical root groups for $\beta, \alpha+\beta, \gamma, 3\alpha+\beta$ act on these fibers by translations whose columns are

$$\begin{pmatrix} 0 & x_3 & x_4 & -x_5 \\ -x_3 & -x_4 & x_5 & 0 \end{pmatrix}. \tag{2.7}$$

Its maximal minors generate the unit ideal: they include x_3x_5 up to sign and $x_4^2 + x_3x_5$, and $x_4^2 + 4x_3x_5 = 1$. Hence the translations are surjective locally over the base; the quotient is Q_2. Under

$$(x_3, x_4, x_5) \longmapsto \begin{pmatrix} x_4 & 2x_3 \\ 2x_5 & -x_4 \end{pmatrix},$$

this is the split semisimple orbit with eigenvalues $1, -1$, namely $T\backslash\mathrm{PGL}_2$ for a split torus T. Its open Borel orbit is $x_5 \neq 0$ and the two boundary orbits are $x_5 = 0, x_4 = \pm 1$. Their equations are reduced over R. This proves the split type T assertion and gives the two remaining dimension-three, rank-zero orbits in Z.

Finally P_β preserves Y_0, with trivial radical quotient, giving type G. The displayed charts exhaust the strata and all pairs of a maximal-rank orbit with a simple root. Their defining equations, open conditions, and reduced boundary components commute with base change and have no vertical components. This also verifies the closure condition in the cited integrality axiom. □

Proposition 2.12. *For odd residue characteristic, $\boldsymbol{X} = \boldsymbol{SL}_3\backslash\boldsymbol{G}_2$ satisfies Assumption 2.3.*

Proof. The quotient model is the smooth affine quadric $q_7 = 1$ over $\mathfrak{o}$, by [AHW19, Theorem 2.3.5 and its proof] and the invertibility of 2. Its open orbit has smooth homogeneous special fiber by Lemma 2.9; its k-points form one B-orbit by Lemma 2.7. Proposition 2.10 and Lemma 2.11 give the three orbit axioms. There are no type N links. The group is split and simply connected, and the variety is affine and homogeneous.

The invariant measures were established in Lemma 2.7. They have compatible integral normalizations: the smooth quotient of the Chevalley models by $\mathbf{SL}_3$ has an invariant integral top form, since the determinant character of the stabilizer on the tangent quotient is trivial. The root charts above and the adjacent-orbit factorizations of [Sak13, §3.3] choose the remaining forms up to units. Root coordinates have additive measure $\mathrm{vol}(\mathfrak{o}) = 1$, and the common scalar may be fixed by $\mathrm{vol}(x_0 J) = 1$. Lemma 2.5 supplies the Fourier convention for every ramification degree. □

2.5. **An unramified multiplicity one criterion.** We formulate a criterion for varieties with a single spherical root of split type T, and apply it to $(\mathbf{G}_2, \mathbf{SL}_3)$. For G_2, let α and β be the short and long simple roots, respectively. The symbol $\check{\theta}$ denotes the valuation of the color appearing in the Brion path below.

Statement 2.13. *Let $\mathfrak{g}$ be a Brion path for the simple spherical reflection ω_γ, where γ is a normalized spherical root of split type T. Let $\check{\theta} \in \mathrm{Hom}(\mathcal{X}(\boldsymbol{X}), \mathbb{Z})$ be the valuation defined by the color in this path. Then $\langle\check{\theta}, \gamma\rangle = 1$ and*

$$\begin{aligned} B_{\omega_\gamma}(\chi) = & -e^{\check{\gamma}}(\chi)\,\frac{1 - q^{-\langle\check{\theta},\rho_{P(X)}\rangle}e^{-\check{\theta}}(\chi)}{1 - q^{\langle\check{\theta},\rho_{P(X)}\rangle - \langle\check{\rho},\gamma\rangle}e^{\check{\theta}}(\chi)} \\ & \times \frac{1 - q^{\langle\check{\theta},\rho_{P(X)}\rangle - \langle\check{\rho},\gamma\rangle}e^{\omega_\gamma\check{\theta}}(\chi)}{1 - q^{-\langle\check{\theta},\rho_{P(X)}\rangle}e^{-\omega_\gamma\check{\theta}}(\chi)}. \end{aligned} \tag{2.8}$$

Lemma 2.14. *For $\eta = \delta_{(X)}^{1/2}\chi$ and $z = e^{\check\theta}(\chi)$, the shifted Cartan torus satisfies*

$$e^{\check\alpha}(\eta) = q^{3/2}z, \qquad e^{\check\beta}(\eta) = q^{-1}.$$

Proof. By (2.2), $\check\alpha$ maps to $\check\theta$ and $\check\beta$ maps to zero. Thus $\chi(\check\alpha(\varpi)) = z$ and $\chi(\check\beta(\varpi)) = 1$. By (2.3), $\delta_{(X)}^{1/2} = |\beta|^{1/2}$. Finally,

$$\langle\check\alpha, \beta\rangle = -3, \qquad \langle\check\beta, \beta\rangle = 2, \qquad |\varpi| = q^{-1},$$

which gives the two factors $q^{3/2}$ and q^{-1}. □

Proposition 2.15. *Assume that the residue characteristic is odd. Statement 2.13 holds for $\boldsymbol{X} = \boldsymbol{SL}_3\backslash\boldsymbol{G}_2$ and its type T spherical root $\gamma = 2\alpha + \beta$.*

Proof. We first calculate $B_{\omega_\gamma}(\chi)$ from the expression given in Statement 2.13. We have $\langle\check\theta, \gamma\rangle = 1$ and $\rho_{P(X)} = \frac{5}{2}\gamma$, hence

$$\langle\check\theta,\ \rho_{P(X)}\rangle = \langle\check\theta, \frac{5}{2}\gamma\rangle = \frac{5}{2}$$

and $\langle\check\rho, \gamma\rangle = \langle\check\rho, 2\alpha + \beta\rangle = 2\langle\check\rho, \alpha\rangle + \langle\check\rho, \beta\rangle = 3$.

We put $z = e^{\check\theta}(\chi) = \chi(\check\theta(\varpi))$ here $\check\theta$ is viewed as a cocharacter of A_X, then $\omega_\gamma\check\theta = -\check\theta$, we have $e^{\omega_\gamma\check\theta}(\chi) = z^{-1}$. From $\check\gamma = 2\check\theta$, we get $e^{\check\gamma}(\chi) = z^2$.

Hence the right hand side of (2.8) is

$$-z^2\frac{1 - q^{-\frac{5}{2}}z^{-1}}{1 - q^{-\frac{1}{2}}z}\frac{1 - q^{-\frac{1}{2}}z^{-1}}{1 - q^{-\frac{5}{2}}z}$$

To verify the cocycle, consider the reduced expression $w_\gamma = s_\alpha s_\beta s_\alpha s_\beta s_\alpha$ and the following Brion path, established over $\mathfrak{o}$ in Lemma 2.11:

$$\mathring{X} \xrightarrow{(\alpha,U)} Y_1 \xrightarrow{(\beta,U)} Y_2 \xrightarrow{(\alpha,T)} Y_2 \xrightarrow{(\beta,U)} Y_1 \xrightarrow{(\alpha,U)} \mathring{X}.$$

Put $\eta = \delta_{(X)}^{1/2}\chi$. The cocycle relation is

$$b^Y_{w_1w_2}(\eta) = b^{{}^{w_2}Y}_{w_1}({}^{w_2}\eta)b^Y_{w_2}(\eta).$$

With $w = w_\gamma$, this gives

$$\begin{aligned} b_w(\eta) = b_{s_\alpha}(\eta)\, b^{Y_1}_{s_\beta}({}^{s_\alpha}\eta) \\ \times b^{Y_2}_{s_\alpha}({}^{s_\beta s_\alpha}\eta)\, b^{Y_2}_{s_\beta}({}^{s_\alpha s_\beta s_\alpha}\eta) \\ \times b^{Y_1}_{s_\alpha}({}^{s_\beta s_\alpha s_\beta s_\alpha}\eta). \end{aligned}$$

Here are compatible representatives. Put $v_0 = e_4$ and identify the α-string with trace-zero 2×2 matrices as in Lemma 2.11. The element

$$g_\alpha = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o})$$

takes $\mathrm{diag}(1,-1)$ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by conjugation. Thus $\xi_2 = g_\alpha v_0 = \frac{1}{2}(e_3 + e_5)$ lies in Y_2. For integral Chevalley Weyl representatives, take

$$\xi_1 = \xi_2 \cdot n_\beta, \qquad \xi_0 = \xi_1 \cdot n_\alpha.$$

For the choices $n_\rho = x_\rho(1)x_{-\rho}(-1)x_\rho(1)$, the displayed matrices give

$$\xi_1 = \tfrac{1}{2}(e_2 + e_6), \qquad \xi_0 = \tfrac{1}{2}(e_1 + e_7).$$

They lie in Y_1 and Y_0, respectively, and $\xi_0 = x_0\check{\alpha}(1/2)$ for the base point in Proposition 2.10. Thus the two open-orbit normalizations differ by an element of $A(\mathfrak{o})$. These are the compatible base points for the type U links in [Sak13, §3.3]; changing any root coordinate or eigenfunction by a unit has no effect on the unramified factors.

We now use the root system and the local equations of [Sak13, Theorem 5.2.1]. In the basis $(\check{\alpha}, \check{\beta})$, the simple reflections act by

$$s_\alpha(a,b) = (-a+b, b), \qquad s_\beta(a,b) = (a, 3a-b).$$

For the five successive factors in the displayed cocycle product, the coroot pulled back to the original character and its value are

factor	pulled-back coroot	value on η
1	$\check{\alpha}$	$q^{3/2}z$
2	$\check{\alpha}+\check{\beta}$	$q^{1/2}z$
3	$2\check{\alpha}+3\check{\beta}$	z^2
4	$\check{\alpha}+2\check{\beta}$	$q^{-1/2}z$
5	$\check{\alpha}+3\check{\beta}$	$q^{-3/2}z$

If x is the value of the relevant simple coroot, the type U formulas simplify to

$$b_{\mathrm{lower}}(x) = \frac{1-q^{-1}x^{-1}}{1-x}, \qquad b_{\mathrm{raise}}(x) = \frac{1-x^{-1}}{1-q^{-1}x}.$$

Consequently the four type U factors are

1	$\dfrac{1-q^{-5/2}z^{-1}}{1-q^{3/2}z}$
2	$\dfrac{1-q^{-3/2}z^{-1}}{1-q^{1/2}z}$
4	$\dfrac{1-q^{1/2}z^{-1}}{1-q^{-3/2}z}$
5	$\dfrac{1-q^{3/2}z^{-1}}{1-q^{-5/2}z}$

It remains to calculate the middle factor from the split torus quotient, rather than infer it from the final answer. Write $U_P = U_{P_\alpha}$ and $H = \mathrm{Stab}(v_0)$. The subgroup H is generated by the long roots, so

$$H \cap U_P = U_\beta U_{3\alpha+\beta} U_{3\alpha+2\beta}.$$

Its determinant character on A is the sum of these three roots,

$$\zeta = 6\alpha + 4\beta.$$

Since $\langle \check{\alpha}, \zeta \rangle = 0$, this character extends to L_α. Conjugating the stabilizer by g_α therefore gives the same extended modular character $\delta_{(U_P)_{\xi_2}} = |\zeta|$.

Let $\eta' = {}^{s_\beta s_\alpha}\eta$. The coroot calculation gives

$$e^{\check{\alpha}}(\eta') = z^2, \qquad e^{\check{\beta}}(\eta') = q^{-1/2}z^{-1}.$$

Since $\rho = 5\alpha + 3\beta$, the character used in the split type T formula is

$$\lambda = \eta' \delta^{1/2} \delta^{-1}_{(U_P)_{\xi_2}} = \eta' |-\alpha - \beta|.$$

The extension of ζ to L_α makes this a genuine character of the Cartan quotient of Y_2. More explicitly,

$$e^{\check{\alpha}}(\lambda) = q^{-1}z^2, \qquad e^{\check{\beta}}(\lambda) = q^{1/2}z^{-1},$$

so it is trivial on $\ker \alpha = (\check{\alpha} + 2\check{\beta})(\mathbb{G}_m)$.

The rank-one quotient has equation $x_4^2+4x_3x_5=1$. Its two boundary divisors are $x_5=0, x_4=1$ and $x_5=0, x_4=-1$. The eigenfunction x_5, of weight α, vanishes to order one on each. Thus both valuations are the cocharacter $\check{\theta}_\alpha$ dual to α. The integral cocharacter $-\check{\beta}$ maps to $\check{\theta}_\alpha$, because $\langle -\check{\beta}, \alpha\rangle = 1$. Hence both evaluations in the split type T formula are, without choosing a square root,

$$e^{\check{v}_D}(\lambda) = e^{\check{v}_{D'}}(\lambda) = e^{-\check{\beta}}(\lambda) = q^{-1/2}z.$$

The two divisors are distinct and reduced over $\mathfrak{o}$, and g_α is integral with determinant one. Together with Lemma 2.5 and the compatible integral forms, this checks the normalization as well as the valuations. Theorem 5.2.1 of [Sak13] now gives

$$b_{s_\alpha}^{Y_2}({}^{s_\beta s_\alpha}\eta) = \left(\frac{1-q^{-1/2}z^{-1}}{1-q^{-1/2}z}\right)^2. \tag{2.9}$$

Multiplying all five factors yields

$$b_{w_\gamma}(\eta) = z^{-4}\frac{1-q^{-5/2}z^{-1}}{1-q^{-1/2}z}\frac{1-q^{-1/2}z^{-1}}{1-q^{-5/2}z}.$$

This multiplication is an identity of rational functions, so no exclusion of special values of z is being asserted at this step. Finally, w_γ fixes $\check{\beta}$ and inverts precisely the five positive coroots in the table. Their sum is $6\check{\alpha}+9\check{\beta}$, whose image in Λ_X is $6\check{\theta}$. Thus the normalization in [Sak13, Equation (6.1)] is

$$\prod_{\substack{\check{\nu}\in\check{\Phi}^+\\ w_\gamma\check{\nu}<0}} (-e^{\check{\nu}}(\chi)) = -z^6,$$

and gives the required expression

$$B_{w_\gamma}(\chi) = -z^2\frac{1-q^{-5/2}z^{-1}}{1-q^{-1/2}z}\frac{1-q^{-1/2}z^{-1}}{1-q^{-5/2}z}.$$

This is exactly (2.8), and proves the proposition.

□

Remark 2.16. Theorem 6.3.2 of [Sak13] assumes that the Levi subgroup $\mathbf{L}_\gamma = \mathbf{L}_{\mathrm{supp}(\gamma)}$ is classical. That hypothesis does not hold here, since $\mathrm{supp}(\gamma) = \{\alpha, \beta\}$ and $\mathbf{L}_\gamma$ is of type G_2. Consequently, the explicit verification above is needed to justify Statement 2.13 in this case.

Following [Sak13, §7.1], let Θ be the W_X-invariant multiset of triples $(\check{\theta}, \sigma, r)$ generated by the virtual weighted colors and closed under $\check{\theta} \mapsto -\check{\theta}$. Here $\check{\theta} \in \mathrm{Hom}(\mathcal{X}(\mathbf{X}), \mathbb{Z})$, $\sigma \in \{+1, -1\}$, and $r \in \frac{1}{2}\mathbb{Z}$. Let Θ^+ consist of those triples for which $\langle \check{\theta}, \eta\rangle \geq 0$ for every weight η of a regular $\mathbf{B}$-eigenfunction on $\mathbf{X}$. Write $\check{\theta} > 0$ for membership in Θ^+ and $\check{\theta} < 0$ if the triple with first component $-\check{\theta}$ belongs to Θ^+.

Statement 2.17. *For every $(\check{\theta}, \sigma, r) \in \Theta$, we have either $\check{\theta} > 0$ or $\check{\theta} < 0$. For every $\gamma \in \Sigma_X$, the set $\{(\check{\theta}, \sigma, r) \in \Theta^+ \mid \omega_\gamma\check{\theta} < 0\}$ consists precisely of the virtual weighted colors belonging to γ.*

Remark 2.18. This statement is conjectured for homogeneous affine spherical varieties in [Sak13, §7.1].

Proposition 2.19. *Statement 2.17 holds for the set of virtual weighted colors for $\boldsymbol{X} = \boldsymbol{SL}_3\backslash\boldsymbol{G}_2$.*

Proof. By Lemma 2.8, $D = \{f_\gamma = 0\}$ is the unique color and $\check{v}_D = \check{\theta}$. Any Brion path for the unique spherical reflection has a codimension-one orbit on its first descent from the open orbit. Its closure must therefore be D. Thus D is relevant and belongs to γ. The compatible representatives used to calculate the cocycle are given in Proposition 2.15.

Apply the definition of virtual weighted colors in [Sak13, §7.1]. The actual color supplies $(\check{\theta}, +, 5/2)$ by (2.3). Since γ is of split type T, the definition requires one additional element, with triple

$$(-w_\gamma\check{\theta}, +, \langle\check{\rho}, \gamma\rangle - \langle\check{\theta}, \rho_{P(X)}\rangle) = (\check{\theta}, +, 1/2).$$

There are no other colors or spherical roots. These two distinct triples already satisfy the defining requirements, and the definition is minimal, so the list is complete. Its W_X-invariant closure is

$$\Theta = \{(\check{\theta}, +, 5/2), (\check{\theta}, +, 1/2), (-\check{\theta}, +, 5/2), (-\check{\theta}, +, 1/2)\}.$$

The regular eigenweights are exactly $\mathbb{Z}_{\geq 0}\gamma$ by Lemma 2.8. Hence

$$\Theta^+ = \{(\check{\theta}, +, 5/2), (\check{\theta}, +, 1/2)\}.$$

Every element of Θ is either positive or negative, and w_γ takes both positive elements to negative ones. Both positive elements belong to γ, which proves the statement.

□

We consider the restriction map induced by the shifted Cartan torus:

$$\text{Res}: \quad \mathbb{C}[A^*]^W \longrightarrow \mathbb{C}[\delta_{(X)}^{1/2} A_X^*]^{W_X} \tag{2.10}$$

Proposition 2.20. *The restriction map is surjective for* $\boldsymbol{X} = \boldsymbol{SL}_3\backslash\boldsymbol{G}_2$.

Proof. We define $z \in \mathbb{C}[\delta_{(X)}^{1/2} A_X^*]$ via

$$z(\eta) = e^{\check{\theta}}(\chi), \quad \eta = \delta_{(X)}^{1/2}\chi$$

Since $\mathbf{A}_X$ is a one-dimensional torus, we have

$$\mathbb{C}[\delta_{(X)}^{1/2} A_X^*] \cong \mathbb{C}[z, z^{-1}]$$

and $\mathbb{C}[\delta_{(X)}^{1/2} A_X^*]^{W_X} \cong \mathbb{C}[z + z^{-1}]$.

The Weyl orbit of $\check{\beta}$ is

$$W\check{\beta} = \{\pm\check{\beta},\ \pm(\check{\alpha} + \check{\beta}),\ \pm(\check{\alpha} + 2\check{\beta})\}$$

Hence the orbit sum

$$F = e^{\check{\beta}} + e^{-\check{\beta}} + e^{\check{\alpha}+\check{\beta}} + e^{-\check{\alpha}-\check{\beta}} + e^{\check{\alpha}+2\check{\beta}} + e^{-\check{\alpha}-2\check{\beta}}$$

is W-invariant. Lemma 2.14 gives its restriction to the shifted torus.

Using these restrictions, we obtain

$$\text{Res}(F) = q + q^{-1} + (q^{1/2} + q^{-1/2})(z + z^{-1})$$

and hence

$$\text{Res}\left(\frac{F - (q + q^{-1})}{q^{1/2} + q^{-1/2}}\right) = z + z^{-1}$$

We conclude Res is surjective.

□

Theorem 2.21 (A rank-one unramified multiplicity-one criterion)**.** *Let* $\boldsymbol{X}$ *be a homogeneous affine spherical variety with a* G*-invariant measure. Assume Assumption 2.3 and the eigenfunction formula of Theorem 2.4 with its stated normalization. Suppose that* $\mathcal{X}(\boldsymbol{X}) = \mathbb{Z}\gamma$*, where* γ *is a normalized spherical root of split type* T*, and that Statement 2.13 holds. If the restriction map* (2.10) *is surjective, then every irreducible unramified representation* π *satisfies*

$$\dim \text{Hom}_G(\pi, C^\infty(X)) \leq 1.$$

Proof. Here is a direct rank-one proof of the Hecke-module step underlying [Sak13, Theorem 8.0.2 and Corollary 8.0.4]. Let $\check{\theta}$ be dual to γ and put

$$r = \langle \check{\theta}, \rho_{P(X)} \rangle, \quad s = \langle \check{\rho}, \gamma \rangle, \quad a = q^{-r}, \quad b = q^{r-s}.$$

In particular $ab = q^{-s} \neq 1$, since $s > 0$. Write $z = e^{\check{\theta}}(\chi)$ and $\eta = \delta_{(X)}^{1/2}\chi$. The spherical coroot is $2\check{\theta}$, and the assumed cocycle is

$$B(z) = -z^2 \frac{(1 - az^{-1})(1 - bz^{-1})}{(1 - az)(1 - bz)}.$$

Set

$$d(z) = \frac{1 - z^2}{(1 - az)(1 - bz)}.$$

Then $B(z) = d(z)/d(z^{-1})$. By the Cartan axiom, the K-orbits are indexed by $x_n = x_{-n\check{\theta}}$, $n \geq 0$. Dividing the eigenfunction formula by $d(z)$ gives

$$\frac{\Omega_\eta(x_n)}{d(z)} = \delta_{P(X)}^{-1/2}(x_n) Q_n(z), \tag{2.11}$$

where

$$Q_n(z) = \frac{z^{-n}(1 - az)(1 - bz) - z^{n+2}(1 - az^{-1})(1 - bz^{-1})}{1 - z^2}.$$

These are invariant Laurent polynomials. Explicitly, set $S_n(z) = z^n + z^{n-2} + \cdots + z^{-n}$ for $n \geq 0$. Then

$$\begin{aligned} Q_0 &= 1 - ab, \\ Q_1 &= z + z^{-1} - (a + b), \\ Q_n &= S_n - (a + b)S_{n-1} + abS_{n-2} \quad (n \geq 2). \end{aligned}$$

Since S_n is monic of degree n in $z + z^{-1}$, the Q_n, $n \geq 0$, form a basis of $\mathbb{C}[z + z^{-1}]$.

Normalize the characteristic function of each $x_n K$ to have integral one. Equation (2.11) shows that

$$\mathcal{F}(f)(z) = \int_X f(x) \frac{\Omega_\eta(x)}{d(z)}\, dx$$

is a vector-space isomorphism

$$C_c^\infty(X)^K \longrightarrow \mathbb{C}[z + z^{-1}].$$

The integral is first taken for generic z; the formula on the orbit basis extends it as a polynomial. Invariant integration and the Hecke eigenfunction property give

$$\mathcal{F}(h * f) = \widehat{h^\iota}(\eta)\mathcal{F}(f), \qquad h^\iota(g) = h(g^{-1}),$$

with the usual convolution convention. The involution ι is an automorphism of the commutative spherical Hecke algebra. Surjectivity of (2.10) therefore implies that $M = C_c^\infty(X)^K$ is cyclic over $\mathcal{H}(G, K)$.

The integration pairing identifies $C^\infty(X)^K$ with the full algebraic dual of M: the space X/K is discrete, and elements of M have finite support on it. A character eigenspace in that dual has dimension at most one, because a functional with a prescribed Hecke eigencharacter is determined by its value on a cyclic generator (the adjoint action again uses ι). Finally, $\dim \pi^K = 1$ for an irreducible unramified representation, and π is generated by this line. Restriction to π^K is therefore injective on $\mathrm{Hom}_G(\pi, C^\infty(X))$ and takes values in one such Hecke eigenspace. This proves the bound. □

Remark 2.22. For the present pair, $a = q^{-5/2}$, $b = q^{-1/2}$ and $Q_0 = 1 - q^{-3} \neq 0$. The proof above is independent of the classical-Levi hypothesis in [Sak13, Theorem 6.3.2]. The hypotheses of the criterion have been verified for this pair: Proposition 2.12 establishes the integral and measure assumptions, Proposition 2.15 computes the cocycle, and Proposition 2.20 proves the surjectivity of the restriction map.

Proposition 2.23. *If the residue characteristic of k is odd, the pair $(\boldsymbol{G}_2, \boldsymbol{SL}_3)$ is a spherical pair of unramified multiplicity one.*

Proof. Proposition 2.12 verifies all the hypotheses of the spherical-function formula, allowing arbitrary ramification of $k/\mathbb{Q}_p$. Proposition 2.15 gives its normalized cocycle, and Proposition 2.20 gives the surjectivity of the Hecke restriction map. The Cartan lattice is $\mathbb{Z}\gamma$ on the character side by (2.2). Theorem 2.21 therefore gives

$$\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \leq 1$$

for every irreducible unramified π. By Lemma 2.6, $X = H\backslash G$, and smooth Frobenius reciprocity identifies this space with $\mathrm{Hom}_H(\pi, \mathbb{C})$. □

3. Failure of multiplicity one

We construct an irreducible unitarizable representation of $G_2(k)$ with at least two independent $\mathrm{SL}_3(k)$-invariant linear forms. This proves the second assertion of Theorem 1.5; the orbit identification needed below was established in Lemma 2.6.

3.1. **G_2 and its spherical subgroup SL_3.** We describe the split group G_2 using the split octonion algebra, and its subgroup SL_3 as a stabilizer. For the stabilizer description, see [GG06, §5.6].

Let $\mathbb{O}$ be the split octonion algebra over k, we use Zorn's model of $\mathbb{O}$,

$$\mathbb{O} = \{\begin{pmatrix} a & u \\ w & b \end{pmatrix} \ : \ a, b \in k, \quad u, w \in k^3\}$$

Choose the convention,

$$\mathrm{Tr}\begin{pmatrix} a & u \\ w & b \end{pmatrix} = a + b, \quad N\begin{pmatrix} a & u \\ w & b \end{pmatrix} = ab - u \cdot w$$

The split group $G_2 = \mathrm{Aut}(\mathbb{O})$ preserves the trace and norm and hence it acts on

$$V_7 = \mathbb{O}_0 = \ker(\mathrm{Tr}) = \{v(t, u, w) = \begin{pmatrix} t & u \\ w & -t \end{pmatrix}\}$$

Define $q_7(v) = -N(v)$, in these coordinates

$$q_7(t, u, w) = t^2 + u \cdot w = t^2 + u_1 w_1 + u_2 w_2 + u_3 w_3$$

We consider the k-points of the six-dimensional affine quadric:

$$Y = \{(t, u, w) \in k^7 \ : \ q_7(t, u, w) = t^2 + u_1 w_1 + u_2 w_2 + u_3 w_3 = 1\}$$

For $v_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have $q_7(v_0) = 1$ and $\mathrm{Stab}_{G_2(k)}(v_0) = \mathrm{SL}_3(k)$. We use the right-action convention $v \cdot g = g^{-1}v$, so the orbit of v_0 is $\mathrm{SL}_3(k)\backslash G_2(k)$.

Lemma 2.6 proves transitivity on all of Y, including over fields with residue characteristic 2. Hence

$$Y \simeq \mathrm{SL}_3(k)\backslash G_2(k).$$

3.2. **Weil representations.** We recall the Weil-representation conventions of [GG06, §3]. Fix a nontrivial continuous additive character $\psi : k \to \mathbb{C}^\times$.

Let $(W, \langle -, - \rangle)$ be a two-dimensional symplectic space with a basis $\{e, f\}$ satisfying $\langle e, f \rangle = 1$. This basis allows us to identify $Sp(W)(k)$ with $SL_2(k)$ and thus we have a 2-fold cover $\widetilde{SL}(W)$ of $SL(W)$.

Let (V, q) be a nondegenerate quadratic space of odd dimension, and put $b_q(v, v') = q(v + v') - q(v) - q(v')$. Then $W \otimes V$ has the alternating form $(-, -) = \langle -, - \rangle \otimes b_q$. Then we can consider the Heisenberg group $H(W \otimes V)$ associated with $W \otimes V$. Now the group $SL(W) \times SO(V, q)$ acts naturally on $H(W \otimes V)$ as group automorphisms:

$$g \cdot (x, t) = (g \cdot x, t)$$

thus we may form the semi-direct product $\mathcal{G} = (SL(W) \times SO(V, q)) \ltimes H(W \otimes V)$.

We choose the Lagrangian subspace $L = f \otimes V \cong V$. We use the Schrödinger model for the Weil representation $\omega_{\psi,q}$ of $\tilde{\mathcal{G}} = (\widetilde{SL}(W) \times SO(V, q)) \ltimes H(W \otimes V)$ associated with a nontrivial additive character ψ of k and the isomorphism class of the quadratic space (V, q). This model is realized on the space $\mathcal{S}(f \otimes V) = C_c^\infty(f \otimes V)$ of Schwartz functions on $f \otimes V$. The center of $H(W \otimes V)$ acts by the character ψ and the action of the unipotent elements $n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ are given by:

$$\omega_{\psi,q}(n(b))\phi(x) = \psi(bq(x))\phi(x) \tag{3.1}$$

the above realization of the representation $\omega_{\psi,q}$ will be called the Schrödinger model.

We now consider a special choice of q, let's consider the seven-dimensional quadratic vector space

$$V_7 = \mathbb{O}_0, \quad q_7 = -N_{\mathbb{O}}|_{\mathbb{O}_0}$$

with associated symmetric bilinear form

$$b_7(v, v') = q_7(v + v') - q_7(v) - q_7(v')$$

The tensor product

$$\mathbb{W} = W \otimes_k V_7$$

has a symplectic form defined by

$$\langle w \otimes v, w' \otimes v' \rangle_{\mathbb{W}} = \langle w, w' \rangle_W b_7(v, v')$$

this is a nondegenerate alternating form.

Restricting the Weil representation ω_{ψ,q_7} to $\widetilde{SL}_2(k) \times G_2(k)$ gives Ω_{ψ,q_7} on $\widetilde{SL}_2(k) \times G_2(k)$ and Ω_{ψ,q_7} is realized on $\mathcal{S}(V_7)$.

Write $\Omega = \Omega_{\psi,q_7}$ and let $N_2 = \{n(b) : b \in k\}$ be the upper unipotent subgroup, identified with its canonical lift to $\widetilde{\mathrm{SL}}_2(k)$. For $\Phi \in \mathcal{S}(V_7)$, equation (3.1) gives

$$\Omega(n(b))\Phi(v) = \psi(bq_7(v))\Phi(v) \tag{3.2}$$

For $(V_1, q_1) = (k, x \mapsto x^2)$, the Weil representation $\omega_1 = \omega_{\psi,q_1}$ has Schrödinger model $\mathcal{S}(k)$. We define

$$\mathcal{S}(k)^+ = \{f \in \mathcal{S}(k) : f(-x) = f(x)\},$$
$$\mathcal{S}(k)^- = \{f \in \mathcal{S}(k) : f(-x) = -f(x)\}.$$

and $\omega_1^+ = \omega_1|_{\mathcal{S}(k)^+}$, $\omega_1^- = \omega_1|_{\mathcal{S}(k)^-}$

$$\omega_1 = \omega_1^+ \oplus \omega_1^-$$

The unipotent group action is given by

$$\omega_1^\pm(n(b))f(x) = \psi(bx^2)f(x)$$

we have

$$(\omega_1^{\pm})_{N_2,\psi} \cong \mathbb{C} \tag{3.3}$$

The isomorphism is given by $[f] \mapsto f(1)$. Indeed, the twisted coinvariants are obtained by restriction to the fiber $x^2 = 1$, and its even and odd function spaces are each one-dimensional.

3.3. **An irreducible unitary representation of G_2 with two SL_3-periods.** We retain the character ψ and the subgroup N_2 fixed above.

Fix a split torus $T \subset B$, we write α for the short simple root and β for the long simple root. We introduce two maximal parabolic subgroups $P_1 = \langle B, U_{-\beta}\rangle$ and $P_2 = \langle B, U_{-\alpha}\rangle$.

Choose the identifications of the Levi subgroups $L_i(k)$ with $\mathrm{GL}_2(k)$ as in [GG06, §7.1]. For an irreducible tempered representation σ of $\mathrm{GL}_2(k)$ and $s > 0$, set

$$I_{P_i}(\sigma, s) = \mathrm{Ind}_{P_i(k)}^{G_2(k)}\big(\delta_{P_i}^{1/2}\sigma|\det|^s\big).$$

Here Ind denotes unnormalized smooth induction. Its unique irreducible quotient is denoted by $J_{P_i}(\sigma, s)$. Put

$$\tau = J_{P_2}\big(\mathrm{St}_{\mathrm{GL}_2}, \tfrac{1}{2}\big), \qquad \pi = \tau^{\vee}.$$

The representation τ is unitarizable: by [GG06, §7.9 and Proposition 7.10(i)], it occurs as a direct summand in the restriction to $G_2(k)$ of a unitarizable minimal representation of the split group $\mathrm{Spin}_8(k)$. Restricting the invariant positive-definite Hermitian form to this summand proves the assertion. Its smooth contragredient π is also irreducible and unitarizable; equivalently, $\tau^{\vee} \simeq \overline{\tau}$ via this Hermitian form.

Proposition 3.1. *([GG06, Proposition 8.1(ii)]) We have*

$$\Omega_{N_2,\psi} \cong \mathcal{S}(Y)$$

here $Y = \{v \in V_7 \ : \ q_7(v) = 1\}$.

Proof. The action of N_2 is given by

$$\Omega(n(b))\Phi(v) = \psi(bq_7(v))\Phi(v) \tag{3.4}$$

on the subset $Y = \{v \in V_7 \ : \ q_7(v) = 1\}$, this becomes multiplication by $\psi(b)$. Therefore the restriction map

$$r : \ \mathcal{S}(V_7) \longrightarrow \mathcal{S}(Y)$$

factors through

$$\Omega_{N_2,\psi} \longrightarrow \mathcal{S}(Y)$$

The full identification, including the kernel and surjectivity assertions, is the case $a = 1$ of [GG06, Proposition 8.1(ii)], using $Y \simeq \mathrm{SL}_3(k)\backslash G_2(k)$. Thus

$$\Omega_{N_2,\psi} \simeq \mathcal{S}(Y).$$

□

Write $M_{\pm} = \omega_1^{\pm} \otimes \tau$. For an irreducible genuine representation σ of $\widetilde{\mathrm{SL}}_2(k)$, its big theta lift $\theta(\sigma)$ is defined by the maximal σ-isotypic quotient of Ω:

$$q_\sigma : \ \Omega \twoheadrightarrow \sigma \otimes \theta(\sigma)$$

Proposition 3.2. *([GG06, Theorem 9.1(c)]) Let $\sigma = \omega_1^+$ or $\sigma = \omega_1^-$ be the even and odd Weil representations of $\widetilde{SL}_2(k)$ and we write the maximal σ-isotypic quotient of Ω_{ψ,q_7} as $\sigma \otimes \theta(\sigma)$. Then*

$$\theta(\omega_1^+) \cong \pi_+ \oplus \tau$$
$$\theta(\omega_1^-) \cong \tau \oplus \pi_-$$

here π_+ and π_- are irreducible representations of G_2.

Composing $q_{\omega_1^\pm}$ with projection onto τ gives,

$$Q_\pm :\ \Omega \twoheadrightarrow \omega_1^\pm \otimes \tau$$

we define

$$Q = (Q_+, Q_-) :\ \Omega \longrightarrow M_+ \oplus M_-$$

Proposition 3.3. *Q is a surjection*

$$Q :\quad \Omega \twoheadrightarrow (\omega_1^+ \otimes \tau) \oplus (\omega_1^- \otimes \tau)$$

Proof. Let z be a lift of $-I$ to $\widetilde{SL}_2(k)$. By the torus action in [GG06, §3.3], z acts on ω_1 by $(\omega_1(z)f)(x) = c_\psi f(-x)$ for a nonzero scalar c_ψ. Thus z acts on $M_\pm$ by $\pm c_\psi$ and the operators $\frac{1}{2}(1 \pm c_\psi^{-1} z)$ project onto $M_\pm$. By equivariance, $\mathrm{im}(Q)$ is stable under these operators. Each $Q_\pm$ is surjective, so $\mathrm{im}(Q)$ contains both summands and hence Q is surjective.

□

Right exactness of (N_2, ψ)-coinvariants, together with Proposition 3.1 and the two identities $(\omega_1^\pm)_{N_2,\psi} \simeq \mathbb{C}$ in (3.3), gives

$$\mathcal{S}(Y) \twoheadrightarrow \tau \oplus \tau$$

Proposition 3.4. *The irreducible unitarizable representation $\pi = \tau^\vee$ satisfies*

$$\dim \mathrm{Hom}_{\mathrm{SL}_3(k)}(\pi, \mathbb{C}) \geq 2.$$

Proof. Taking smooth contragredients of $\mathcal{S}(Y) \twoheadrightarrow \tau \oplus \tau$ gives an injection $\pi \oplus \pi \hookrightarrow \mathcal{S}(Y)^\vee$. Since $H = \mathrm{SL}_3(k)$ and $G = G_2(k)$ are unimodular, an invariant measure on $Y = H\backslash G$ identifies this smooth dual with the smooth induced representation $C^\infty(Y)$. Thus we have two G-equivariant embeddings $i_\pm : \pi \to C^\infty(Y)$ whose direct sum is injective. Define

$$\ell_\pm(v) = i_\pm(v)(v_0) \qquad (v \in \pi).$$

These are H-invariant. If $a\ell_+ + b\ell_- = 0$, equivariance and transitivity on Y imply $a\, i_+(v)(v_0 g) + b\, i_-(v)(v_0 g) = 0$ for all $g \in G$ and $v \in \pi$. Hence $a i_+ + b i_- = 0$, and injectivity of the direct-sum map forces $a = b = 0$. Therefore

$$\dim \mathrm{Hom}_{\mathrm{SL}_3(k)}(\pi, \mathbb{C}) \geq 2.$$

□

Proof of Theorem 1.5. Proposition 2.23 proves the first assertion for every finite extension of $\mathbb{Q}_p$ with p odd. Proposition 3.4 proves the second assertion for the irreducible unitarizable representation

$$\pi = J_{P_2}\big(\mathrm{St}_{\mathrm{GL}_2}, \tfrac{1}{2}\big)^\vee.$$

As an irreducible admissible unitarizable representation, π is the smooth-vector representation of its irreducible unitary Hilbert completion. Thus the second assertion also contradicts Assumption 1.1, as required. □

Declaration of generative AI use

The negative example was obtained using ChatGPT-6 Astra. Subsequently, ChatGPT-6 Astra and the OSVerify verification framework [NYL+26], with ChatGPT-6 Astra, were used to review the output and improve the writing. Further revisions, including assistance with proof development, reference checking, and LaTeX editing, were prepared with OpenAI Codex. The author is responsible for reviewing the final arguments and for the correctness of this paper.

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