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SCHUR NONNEGATIVITY OF DOUBLE-WIRING CLUSTER VARIABLES

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Abstract. We prove Cuenca's conjecture that cluster variables from double wiring diagrams evaluate to Schur nonnegative symmetric functions on generalized Jacobi–Trudi matrices. The same holds for cluster monomials.

1. Introduction

Fomin and Zelevinsky introduced double wiring diagrams to test total positivity [12]. The diagrams select families of matrix minors, one for each chamber. Local moves connect these families, and the determinantal identities for the moves express the product of the two exchanged minors as a sum of two products of common minors. Berenstein, Fomin and Zelevinsky placed these exchanges in the cluster structures of double Bruhat cells [14].

Haiman proved a Schur nonnegativity theorem for Kazhdan–Lusztig immanants [10, Theorem 1.5]. Rhoades and Skandera formulate the consequence for generalized Jacobi–Trudi matrices [9, Proposition 2.3]. Cuenca observed that this result would imply Schur nonnegativity for all double-wiring cluster variables if those variables belonged to the matrix dual canonical basis [1, Conjecture 1.1]. Cuenca called a variable Plücker-adjacent if it is obtained by one mutation from the seed of some double wiring diagram. His Conjecture 1.2 addresses those variables. Using Temperley–Lieb immanants, he proved Schur nonnegativity for a mutation at a chamber of degree at most six [1, Theorem 1.3 and §§5–6].

Geiss, Leclerc and Schroer realized quantum matrices as a quantum unipotent coordinate algebra with a cluster structure [2]. Grabowski and Launois describe its initial seed explicitly [3]. Kang, Kashiwara, Kim and Oh proved that quantum unipotent cluster monomials are upper global basis elements up to powers of the quantum parameter [4]. Chepuri and Sherman-Bennett formulate related questions for the big double Bruhat cell in the special linear group, including a conjectural characterization by permutation patterns [11, §4]. Here we work with polynomial matrix entries. We identify a double-wiring seed in these entries with the specialized quantum matrix

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seed, and compare the quantum unipotent and matrix bases in the same coordinates. This proves Cuenca's conjecture and its extension to cluster monomials.

Let $n \geq 1$, let $[n] = \{1, \ldots, n\}$, and let $X = (x_{ij})_{1 \leq i,j \leq n}$ be a matrix of commuting indeterminates. For subsets $I, J \subseteq [n]$ of equal cardinality, write $\Delta_{I,J}(X)$ for the minor with rows I and columns J, each taken in increasing order, with $\Delta_{\varnothing,\varnothing} = 1$. Let $\mathcal{A}_n$ be the polynomial matrix cluster algebra obtained from a double wiring diagram: its boundary chamber minors are frozen and are not inverted. The chamber with empty label has value 1 and is omitted from the seed. An extended cluster consists of its mutable and frozen variables.

A double wiring diagram comprises two families of n wires, red and blue, with each pair of wires of the same color crossing once. The blue wires are numbered increasingly from top to bottom at the left boundary, and the red wires at the right boundary. In each chamber the sets I of blue labels and J of red labels below the chamber give its variable $\Delta_{I,J}(X)$. A chamber with two crossing endpoints is mutable. The others are frozen. The level of a chamber is the number of blue wires below it, which is also the number of red wires below it. Cuenca's quiver is obtained by the following rules [1, §2.2.1]. Horizontal adjacency across a red crossing gives an arrow to the right, and adjacency across a blue crossing gives one to the left. For two chambers c, c' in adjacent levels, compare the horizontal intervals of the chambers. If the interval of c' is contained in that of c and c' has endpoints of different colors, the arrow is $c \to c'$ when the left endpoint of c' is blue and $c' \to c$ when it is red. If the intervals interlace with the left endpoint of c' inside c and the right endpoint of c inside c', and these endpoints have the same color, the arrow is $c \to c'$ for blue and $c' \to c$ for red. In the other interlacing order compare the right endpoint of c' with the left endpoint of c. If their colors agree the arrow is $c' \to c$ for blue and $c \to c'$ for red. Omit arrows between frozen vertices.

Mutation at a mutable vertex k replaces its variable z_k by z'_k according to

$$z_k z'_k = \prod_{i \to k} z_i + \prod_{k \to j} z_j,$$

with one factor per arrow and empty products equal to 1. Mutation adds an arrow $i \to j$ for each two-arrow path $i \to k \to j$, reverses the arrows incident to k, cancels opposite pairs, and discards arrows joining frozen vertices. The cluster variables are the variables in the seeds reached by finite sequences of these operations.

Let $\Lambda_{\mathbb{Z}}$ be the ring of symmetric functions over $\mathbb{Z}$. For $t \geq 0$, let h_t be the complete symmetric function of degree t, with $h_0 = 1$. Set $h_t = 0$ for $t < 0$. For weakly decreasing sequences $\mu_1 \geq \cdots \geq \mu_n \geq 0$ and $\nu_1 \geq \cdots \geq \nu_n \geq 0$ of integers, define the generalized Jacobi–Trudi matrix $J(\mu, \nu)$ by

$$J(\mu, \nu)_{ij} = h_{\mu_i - \nu_j}.$$

Theorem 1. *Every cluster variable f of $\mathcal{A}_n$, and more generally every product of nonnegative integer powers of variables from one extended cluster, satisfies*

$$f(J(\mu,\nu)) \in \bigoplus_{\lambda} \mathbb{Z}_{\geq 0} s_\lambda,$$

where s_λ is the Schur function indexed by a partition λ.

Zero is included. Specialization to a finite alphabet gives the corresponding assertion for Schur polynomials.

2. External results

(1) **Quantum matrix seed.** Geiss–Leclerc–Schroer [2, §12.4, Corollary 12.10] realize quantum $n \times n$ matrices as a type-A quantum-unipotent cluster algebra. Grabowski–Launois [3, §5, Definition 5.1 and Remark 5.2] give its initial minors and exchange quiver.

(2) **Canonical membership.** Kang–Kashiwara–Kim–Oh [4, Theorem 11.2.7] place quantum cluster variables in the upper global basis up to powers of $q^{1/2}$. Their Theorem 11.2.3 and Corollary 7.1.4 give the corresponding statement for cluster monomials, and Corollary 11.2.8 gives the integral cluster structure. The Cartan matrix here is symmetric.

(3) **Basis characterizations.** Kimura [5, Theorem 4.26] gives upper-global-basis triangularity relative to dual PBW vectors. Brundan [6, §6, Theorem 16 and (6.2)] characterizes the intrinsic quantum-matrix basis by its bar and terminal monomials. We compare their normalizations below using [2, §§6–7].

(4) **Immanant realization.** The polynomial matrix dual canonical basis consists of nonzero Kazhdan–Lusztig immanants with repeated rows and columns [7, Theorem 2.1], [8, Theorem 6.4].

(5) **Schur positivity.** Rhoades–Skandera [9, Proposition 2.3], using Haiman [10, Theorem 1.5], prove Schur nonnegativity for every Kazhdan–Lusztig immanant on every generalized Jacobi–Trudi matrix.

3. The matrix seed

Proposition 2. *The cluster variables from any double wiring diagram are the specializations at $q = 1$ of the variables from the GLS quantum matrix seed, with the same correspondence of extended clusters.*

Proof. For $n = 1$, the seed consists of the frozen variable x_{11}, so the seed identification is immediate. Assume $n \geq 2$. For $1 \leq r, s \leq n$, define the integer intervals

$$R(r,s) = [n-r+1, n-r+s] \cap [n], \qquad C(r,s) = [n-s+1, n-s+r] \cap [n].$$

These sets have equal cardinality. The seed of [3, Definition 5.1] has variables $\Delta_{R(r,s),C(r,s)}$, frozen precisely when $r = n$ or $s = n$. Its arrows are

$$(r, s) \longrightarrow (r, s + 1), \qquad (r, s) \longrightarrow (r + 1, s), \qquad (r, s) \longrightarrow (r - 1, s - 1),$$

when the endpoints exist, with arrows between frozen vertices omitted.

Number adjacent crossings from top to bottom by $s_1, \dots, s_{n-1}$, and use the reduced sorting word

$$(s_{n-1} \cdots s_1)(s_{n-1} \cdots s_2) \cdots s_{n-1}$$

for each colour, performing the crossings in this word from left to right. Place all red crossings before all blue crossings. Number blue wires increasingly from top to bottom at the left boundary and red wires increasingly from top to bottom at the right boundary, following Cuenca's convention.

At chamber level k, meaning that k wires of each colour lie below the chamber, the red segment gives

$$\Delta_{[n-k+1,n],[j,j+k-1]}, \qquad 1 \le j \le n - k + 1,$$

and the blue segment gives

$$\Delta_{[i,i+k-1],[n-k+1,n]}, \qquad n - k + 1 \ge i \ge 1.$$

The central southeast minor is counted once. After the first a descending blocks the blue labels are $n, n - 1, \dots, n - a + 1, 1, \dots, n - a$ from top to bottom, since each block moves the largest remaining label to the front of the remaining segment. The red label order is obtained by replacing each label u in this list by $n + 1 - u$, because the initial red order is reversed. At level k, only the crossing s_{n-k} in each of the first $n - k$ blocks changes the bottom k labels. Reading those sets gives the two lists above. The nonunit chambers therefore number

$$\sum_{k=1}^{n} \big(2(n - k) + 1\big) = n^2.$$

They are exactly the minors indexed by (r, s) above, and their unbounded chambers give the stated frozen variables.

To compare arrows, set $N = n(n - 1)/2$. Within either sorting word, the a-th crossing at level k occurs at position

$$T(a, k) = (a - 1)n - \frac{a(a - 1)}{2} + k, \qquad 1 \le a \le n - k.$$

For $1 \le k < n$, put $m = n - k + 1$. The noncentral red chambers have labels $1 \le j < m$, with endpoints

$$\begin{cases} (-\infty, T(1, k)), & j = 1, \\ (T(j - 1, k), T(j, k)), & 2 \le j < m. \end{cases}$$

The central chamber has endpoints $(T(m-1,k), N+T(1,k))$. The non-central blue chambers have labels $1 \le i < m$, with endpoints

$$\begin{cases} (N+T(m-1,k), +\infty), & i=1, \\ (N+T(m-i,k), N+T(m-i+1,k)), & 2 \le i < m. \end{cases}$$

At $k=n$, the sole determinant chamber has endpoints $(-\infty, +\infty)$.

Cuenca's same-level rule directs the red edges right and the blue edges left. For adjacent levels $k, k+1$, write $m = n-k+1$ and index each minor by its least row index i and least column index j, so that its northwest corner is (i,j). Substitution in his three adjacent-level rules gives

$$\begin{aligned} (m-1,j) &\longrightarrow (m,j), & (m,j) &\longrightarrow (m-1,j-1), \\ (i,m-1) &\longrightarrow (i,m), & (i,m) &\longrightarrow (i-1,m-1), \\ (m,m) &\longrightarrow (m-1,m-1). \end{aligned}$$

In the first two rows of the display, $1 \le i,j < m$, and arrows with a nonexistent endpoint or two frozen endpoints are omitted. For the red comparisons, writing (L,R) for the lower interval and (L',R') for the upper interval, the upper column j gives $L < L' < R < R'$, and the upper column $j-1$ gives $L' < L < R' < R$. For blue comparisons, the upper row i gives $L' < L < R' < R$, and the upper row $i-1$ gives $L < L' < R < R'$. The same comparisons apply when an upper endpoint belongs to the central chamber: the two endpoints tested for color are both in the indicated red or blue segment. The lower central interval lies strictly inside the upper central interval, including when the upper chamber is the determinant chamber. At an infinite endpoint the strict interlacing comparison with two frozen chambers is discarded. Exhaustiveness follows by comparing the endpoint lists using

$$T(a,k+1) = T(a,k)+1, \qquad T(a+1,k) - T(a,k) = n-a > 1$$

whenever the two successive crossings occur in these comparisons. Together with the same-level arrows, the directions are east, south and northwest. Since

$$(r,s) = (n-i+1, n-j+1),$$

this is the global opposite of the quiver in [3]. Reversing all arrows interchanges the two numerator terms in each commutative exchange relation, and the rule for quiver mutation commutes with global reversal. Thus every mutation sequence produces the same commutative variables in the two seeds.

Let w_0 be the reversing permutation in the symmetric group on $[n]$. Double reduced words for (w_0, w_0) are connected by braid, commutation and colour-interchange moves. The corresponding chamber seeds are mutation equivalent [12, proof of Theorem 17 and Lemma 18], [13, Definition 2.4.1

and Proposition 2.4.3]. In the generic matrix exchange identities of [12], the full determinant is retained as a common chamber minor. Thus the mutation equivalence identifies the polynomial matrix variables for every double-wiring diagram with the determinant kept as a frozen variable. In the integral quantum cluster algebra of [4, Corollary 11.2.8], specialize by sending $q^{1/2}$ to 1. To check each exchange, assign the value 1 to all variables of the algebraically independent initial commutative cluster. The ordinary exchange rule then gives a positive value at every step, so each ordinary variable is a nonzero rational function. Inductively the specialized quantum identity determines the new variable uniquely, since the variable being replaced has a nonzero specialization. This identity is the ordinary exchange identity because its parameter powers specialize to 1. □

4. The matrix dual canonical basis

Proposition 3. *In the GLS quantum matrix realization, the upper global basis and the intrinsic matrix dual canonical basis have the same specialization at $q = 1$ in the matrix coordinate integral form.*

Proof. Let q be an indeterminate and work first over $\mathbb{Q}(q)$. The quantum matrix algebra has generators x_{ij} and relations

$$\begin{aligned} x_{ij}x_{ik} &= qx_{ik}x_{ij} && (j<k),\\ x_{ij}x_{kj} &= qx_{kj}x_{ij} && (i<k),\\ x_{ij}x_{k\ell} &= x_{k\ell}x_{ij} && (i<k,\ j>\ell),\\ x_{ij}x_{k\ell} - x_{k\ell}x_{ij} &= (q-q^{-1})x_{i\ell}x_{kj} && (i<k,\ j<\ell). \end{aligned}$$

Let ϕ fix each generator, reverse products, and satisfy $\phi(a(q)x) = a(q^{-1})\phi(x)$ for $a(q) \in \mathbb{Q}(q)$. These rules define a semilinear anti-automorphism of the matrix algebra.

Let s_a, for $1 \le a < 2n$, be the simple reflection exchanging positions a and $a+1$ in the type A_{2n-1} Weyl group. Take

$$w = (s_n \cdots s_1)(s_{n+1} \cdots s_2) \cdots (s_{2n-1} \cdots s_n).$$

Write $w = s_{i_{n^2}} \cdots s_{i_1}$ as in [2, (7.6)]. Let $\varepsilon_1, \ldots, \varepsilon_{2n}$ be the standard orthonormal basis of $\mathbb{R}^{2n}$, and let $\alpha_a = \varepsilon_a - \varepsilon_{a+1}$ be the simple root of s_a. The roots in order are $s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t})$, for $1 \le t \le n^2$, so read the displayed word from right to left. The first block moves label n successively past $n+1, \ldots, 2n$, and the subsequent blocks do the same for $n-1, \ldots, 1$. Thus the dual PBW generators are the matrix entries ordered by decreasing row and decreasing column within each row. Their weights are

$$\beta_{ij} = \varepsilon_i - \varepsilon_{2n+1-j} = \alpha_i + \cdots + \alpha_{2n-j}.$$

The proof of Proposition 12.1 and the matrix realization in §12.4 of [2] identify each entry x_{ij} with the dual PBW root vector of weight β_{ij}.

For a matrix $A = (a_{ij})$ of nonnegative integers, let

$$r_i = \sum_j a_{ij}, \qquad c_j = \sum_i a_{ij}, \qquad d = \sum_{i,j} a_{ij}, \qquad \beta = \sum_{i,j} a_{ij}\beta_{ij}.$$

Let P_A be the monomial with exponents A in this root order. Let M_A instead use decreasing rows and increasing columns within each row. Thus M_A is Brundan's initial-double-index monomial. Write $E^*(A)$ for the dual PBW vector indexed by the tuple of exponents of A in this root order. Same-row commutation and [2, (7.11)–(7.13)] give

$$\begin{aligned} E^*(A) &= q^{-\sum_{i,j}\binom{a_{ij}}{2}} P_A \\ &= q^{-\sum_{i,j}\binom{a_{ij}}{2} - \sum_i \sum_{j<k} a_{ij}a_{ik}} M_A \\ &= q^{-\sum_i \binom{r_i}{2}} M_A. \end{aligned} \tag{1}$$

For a root with multiplicity a, GLS give the dual divided power as $\{a\}!/[a]_q!$ times the a-th power of the corresponding root generator. Here $[a]_q! = \prod_{u=1}^{a}[u]_q$, with $[u]_q = (q^u - q^{-u})/(q - q^{-1})$, and $\{a\}! = \prod_{u=1}^{a}(1-q^{-2u})/(1-q^{-2})$. Both products are 1 if $a = 0$. Each factor in the second product is $q^{1-u}[u]_q$, so the ratio is $q^{-\binom{a}{2}}$. Multiplication over the roots gives the first equality.

We specify how Kimura's conventions transfer. The GLS map φ from (6.14) is an antilinear anti-isomorphism taking q to q^{-1} and the positive Chevalley generator e_i to the negative Chevalley generator f_i. Their Lemma 6.1 identifies the bilinear forms, and hence the dual bases, under this map. Their Lemma 7.2 identifies the braid operators. The GLS operator is Kimura's T_i^{-1}. Consequently φ sends the GLS PBW vector to Kimura's $e = -1$ PBW vector, whose product order is reversed by definition [5, §4.3.2]. Kimura's $e = -1$ triangularity also follows from the $e = 1$ case by the star involution [5, Proposition 4.7]. Applying φ^{-1} changes the coefficients $q\mathbb{Z}[q]$ in [5, Theorem 4.26] to $q^{-1}\mathbb{Z}[q^{-1}]$.

Use $B < A$ for the lexicographic order on PBW exponent tuples in the specified root order. The GLS dual canonical element b_A indexed by A is homogeneous of weight β. Its expansion in dual PBW vectors of that weight is

$$b_A = E^*(A) + \sum_{B<A} h_{A,B}(q)E^*(B), \qquad h_{A,B}(q) \in q^{-1}\mathbb{Z}[q^{-1}]. \tag{2}$$

To pass from Kimura's expansion to (2), invert its unitriangular matrix on each homogeneous component. These components are finite: equality of the weight β, whose positive and negative coordinates lie in disjoint blocks, fixes all row and column sums of B. In particular, $\sum_i \binom{r_i}{2}$ is constant throughout (2). Thus (1) gives

$$q^{\sum_i \binom{r_i}{2}} b_A = M_A + \sum_{B<A} h_{A,B}(q) M_B. \tag{3}$$

For a root-lattice element $\sum_a t_a \alpha_a$, define its height as $\sum_a t_a$. Let σ be the GLS antilinear anti-automorphism from §6.2. Use $(\cdot,\cdot)$ for the Euclidean inner product on $\mathbb{R}^{2n}$. Their equation (6.7) gives

$$\sigma(b_A) = q^{(\beta,\beta)/2 - \operatorname{ht}\beta} b_A, \qquad \sigma(x_{ij}) = q^{1-\operatorname{ht}\beta_{ij}} x_{ij}.$$

Since degree and height are additive, on this homogeneous component

$$\phi = q^{\operatorname{ht}\beta - d}\sigma, \qquad (\beta,\beta) = \sum_i r_i^2 + \sum_j c_j^2.$$

Denote Brundan's bar by $\overline{\ }_B$. By [6, Remark 17], on the component with row sums $r = (r_1, \ldots, r_n)$ and column sums $c = (c_1, \ldots, c_n)$ it satisfies

$$\overline{x}_B = q^{(\sum_i r_i^2 - \sum_j c_j^2)/2} \phi(x).$$

Combining these formulas yields

$$\overline{\overline{b_A}_B} = q^{\sum_i r_i^2 - d} b_A = q^{2\sum_i \binom{r_i}{2}} b_A,$$

so $q^{\sum_i \binom{r_i}{2}} b_A$ is invariant under Brundan's bar. Apply that bar to (3). By [6, Theorem 16], each initial monomial becomes the terminal monomial with the same exponent matrix: its factors have increasing columns and decreasing rows within each column. The coefficients become elements of $q\mathbb{Z}[q]$. The resulting expression differs from the terminal monomial of A by a linear combination of terminal monomials of the same row and column sums with coefficients in $q\mathbb{Z}[q]$. It therefore satisfies Brundan's uniqueness characterization (6.2) for the intrinsic matrix dual canonical element L_A. Consequently

$$q^{\sum_i \binom{r_i}{2}} b_A = L_A, \tag{4}$$

Finally, (1) identifies the $\mathbb{Z}[q, q^{-1}]$-span of the dual PBW vectors with the integral matrix algebra spanned by ordered monomials. The unitriangular transition identifies the same integral form with the span of the b_A. Specialization at $q = 1$ is therefore specialization of this coordinate integral form, which gives $\mathbb{Z}[x_{ij}]$. The scalar in (4) specializes to 1. Extending scalars to include $q^{1/2}$, as required for quantum cluster variables, leaves this conclusion unchanged. Thus the two canonical bases have identical classical specializations. □

5. Proof of the theorem

Proof. Fix a mutation sequence from a double-wiring seed to an extended cluster containing the variable f, or containing all factors if f is a cluster monomial. The seed identification realizes this sequence in the specialization of the GLS quantum matrix seed, reversing the commutative quiver if necessary. Perform the corresponding quantum mutations. By [4, Theorem 11.2.7], the resulting variable is a power of $q^{1/2}$ times an upper-global-basis element in the nonlocalized quantum-unipotent coordinate ring. For a cluster monomial, use [4, Theorem 11.2.3 and Corollary 7.1.4]. Their integral structure [4, Corollary 11.2.8] and (4) permit specialization at $q = 1$. By Proposition 2, specialized variables are the ordinary variables. A product from one extended cluster specializes factor by factor, and its overall parameter power specializes to 1. Thus the resulting element is f, which belongs to the polynomial matrix dual canonical basis.

We now apply the ring homomorphism sending each commuting matrix entry to the corresponding entry of $J(\mu, \nu)$. This evaluation is made on the polynomial obtained after specialization at $q = 1$.

Write S_m for the symmetric group on $\{1, \dots, m\}$. By the immanant realization, there are an integer $m \geq 0$, weakly increasing sequences $R = (r_1, \dots, r_m)$ and $C = (c_1, \dots, c_m)$ in $[n]$, and a permutation $v \in S_m$, such that

$$f(X) = \mathrm{Imm}_v(X(R, C)), \qquad X(R, C)_{ab} = x_{r_a, c_b}.$$

For an $m \times m$ matrix $Y = (y_{ab})$, this Kazhdan–Lusztig immanant is

$$\mathrm{Imm}_v(Y) = \sum_{u \in S_m} (-1)^{\ell(u)-\ell(v)} P_{w_0 u, w_0 v}(1) \prod_{a=1}^{m} y_{a,u(a)}.$$

Here ℓ is inversion number, w_0 is the longest permutation in S_m, and $P_{a,b}(t)$ is the type-A Kazhdan–Lusztig polynomial, taken to be zero unless $a \leq b$ in Bruhat order. For $m = 0$, the immanant is 1.

After substitution, the repeated-row-and-column matrix has entries

$$J(\mu, \nu)(R, C)_{ab} = h_{\mu_{r_a} - \nu_{c_b}}.$$

The sequences $(\mu_{r_a})_{a=1}^m$ and $(\nu_{c_b})_{b=1}^m$ are weakly decreasing and nonnegative. This is a generalized Jacobi–Trudi matrix of size m, including when rows or columns repeat. By [9, Proposition 2.3], its immanant, which equals $f(J(\mu, \nu))$, is Schur nonnegative. Its Schur coefficients are integers because the displayed immanant lies in $\Lambda_{\mathbb{Z}}$ and the Schur functions form a $\mathbb{Z}$-basis. □

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