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COMPLEX CENTRAL CONFIGURATIONS OF THE PLANAR FIVE-BODY PROBLEM AT SPECIAL MASSES

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Abstract. We study the complex solution sets of the polynomial system for planar central configurations of five bodies, modulo rotations, at the masses $(1,4,4,4,4)$, $(1,1,4,4,4)$ and $(9,36,36,36,4)$, where the proof of finiteness for all positive masses in a companion paper counts the real central configurations. At $(1,4,4,4,4)$ the solution set is the union of 36 explicit, pairwise disjoint irreducible curves and a finite set; the curves contain the modified Roberts continuum of Albouy and Kaloshin, up to relabelling the bodies, but none of their points is a positive central configuration. At $(1,1,4,4,4)$ the solution set is finite, and at $(9,36,36,36,4)$ the potential, which is positive at every central configuration, vanishes identically on every curve in the solution set, so these curves contain no positive central configuration. Finiteness at these masses, their positive multiples and their permutations therefore follows without a count. We also show that at positive masses two of the limit diagrams of Albouy and Kaloshin occur only for masses on finitely many explicit curves in the projective space of masses.

1. Introduction

A configuration of n distinct points in the plane with positive masses is a *central configuration* if the Newtonian force on each body points towards the centre of mass and is proportional to its distance from it. Smale's sixth problem asks whether, for every choice of positive masses, there are finitely many central configurations up to similarity [7]. Hampton and Moeckel proved this for four bodies [3], and Albouy and Kaloshin for five bodies in the plane when the masses lie outside a closed algebraic subset of codimension two [1]. The author's paper [8] proves finiteness for five bodies in the plane at all positive masses, and Chen and Tong [2] state the same theorem and give a different proof.

Albouy and Kaloshin consider the complexified equations. Infinitely many central configurations give an algebraic curve of complex solutions. Along such a curve the valuations of the mutual distances at the places at infinity sum to zero, and only the singular places of [8, §2.4] contribute to the sum. At a singular place the bodies approach one of the sixteen limit diagrams of [1, Fig. 11], and most diagrams force polynomial conditions on the masses. However, part of the exceptional set of [1] cannot be avoided by an argument on the complexified equations alone. Roberts found a continuum of central configurations for the masses $(1,1,1,1,-\frac{1}{4})$ [6]. Albouy and Kaloshin observe that changing the signs of m_5 and of the four distances r_{k5} turns it into a continuum of solutions of their system with the masses $(4,4,4,4,1)$ [1, p. 543 and Remark 8], the *modified Roberts continuum*. These solutions are not central configurations, since the distances r_{k5} are negative square roots. The separation by linear functionals of [8], described below, does not settle the masses on the rays through $(1,4,4,4,4)$, $(1,1,4,4,4)$ and $P_1 = (9,36,36,36,4)$, and [8] treats these masses and their permutations by counting the real central configurations, following Moczurad and Zgliczyński [4, 5].

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In this paper we study the complex solution sets at these masses. Let Y_m be the slice of [8, §2.2]. It is an affine variety whose points are the solutions of the system of [1] without the condition $y_1 = y_2$, which normalizes the rotations, taken modulo the complex rotations; each of its points corresponds to two solutions of the system with this condition. Let $U = \sum_{k<l} m_k m_l r_{kl}^{-1}$ be the potential, which is constant on every irreducible curve in Y_m [1, Lemma 2].

Theorem 1.1. *Let $m = (1, 4, 4, 4, 4)$. The slice Y_m is the union of 36 pairwise disjoint irreducible curves, 18 of genus 3 and 18 rational, and a finite set. Through the correspondence of* [8, Lemma 2.2(a)] *between the points of Y_m and the solutions of the system of* [1]*, the curves contain the modified Roberts continuum, carried to these masses by the exchange of bodies 1 and 5, and no point of the curves corresponds to a positive normalized central configuration.*

Theorem 1.2. *Let $m = (1, 1, 4, 4, 4)$. The slice Y_m is finite.*

Theorem 1.3. *Let m be a positive multiple of a permutation of $P_1 = (9, 36, 36, 36, 4)$. Then $U \equiv 0$ on every irreducible curve in Y_m.*

Corollary 1.4. *Let m be a positive multiple of a permutation of $(1, 4, 4, 4, 4)$, of $(1, 1, 4, 4, 4)$ or of P_1. Then the planar five-body problem with masses m has finitely many central configurations up to similarity.*

Proof. By [8, Lemma 2.2(c)] the number of central configurations up to similarity does not change under positive scaling and permutation of the masses. A central configuration gives two positive normalized central configurations [8, Lemma 2.1]. At $(1, 4, 4, 4, 4)$ these lie in the finite set of Theorem 1.1. At $(1, 1, 4, 4, 4)$ they lie in the finite set Y_m. At P_1, Theorem 1.3 says that P_1 is settled in the sense of [8, §2.3], and a settled mass vector has finitely many central configurations [8, Corollary 2.5]. □

Corollary 1.4 is a case of the finiteness theorem of [8], and of the one stated in [2]. In [8] it follows at these masses from the counts of [8, Theorems 10.1, 10.2 and 10.7], which give the numbers of central configurations. Here Theorems 1.1–1.3 give it without a count. Six of the 36 curves at $(1, 4, 4, 4, 4)$ have infinitely many real points, and only the signs of the distances exclude them (Proposition 3.2).

We also prove a statement on places of diagrams 1 and 4 of [1, Fig. 11] at arbitrary positive masses (Theorem 9.2). At such a place with triangle T, the torque identity of [1, §5.0] shows that the leading potential between T and the other two bodies vanishes. Together with the condition of [1] on the masses of T, this confines the masses at which such places occur to finitely many explicit curves in the projective space of masses, three up to relabelling of the bodies.

The method. We follow [1] and [8]. Let $C \subseteq Y_m$ be an irreducible curve. At each singular place P of C the exponents of the mutual distances form a vector $x(P) \in \mathbb{Q}^{10}$, the type of P, and a positive combination of the types vanishes. For each diagram and each assignment of the masses to its bodies, [8] determines the possible types. At the special masses we choose one or two linear functionals that are nonnegative on all types that can occur, and positive on each type at which the first one vanishes. Then no curve exists (Lemma 2.2). The types at which the functionals are negative must be excluded separately, in part by results of [8]. At $(1, 4, 4, 4, 4)$ the places of diagram 3 cannot be excluded: they occur on the 36 curves. We show by an expansion at the place that a curve with such a place is one of the 36 curves (Corollary 4.3). At $(1, 1, 4, 4, 4)$ we exclude the places of diagram 6 that the mass conditions allow, by an expansion to the sixth order (Theorem 5.1). At all three mass points we treat diagram 5 by case analyses of the exponents, and a lemma on triangles of equal masses excludes some places of diagram 7 (Lemma 6.2). The curves of Theorem 1.1 are explicit, and their real structure shows that none of their points is a positive central configuration (Proposition 3.2).

Theorems 1.1 and 1.2 concern all irreducible curves in Y_m, including those on which $U \equiv 0$. Lemma 2.1 extends to them the results of [8] on singular places that we need, apart from some ranges of the types and the exclusion of diagram 16; [8] states these results for curves with $U \not\equiv 0$. Our proofs are by hand, except for the exact polynomial computations of Lemma 9.1 (§10), although some of the results of [8] that we cite rest on computations.[1]

Organization. In §2 we recall the setting of [8], and in §3 we construct the 36 curves. Sections 4 and 5 contain the expansions at places of diagrams 3 and 6, and §6 proves two lemmas on triangles. Sections 7 and 8 prove Theorems 1.1–1.3, with diagram 5 at $(1, 1, 4, 4, 4)$ and at P_1 in Appendices A and B. Section 9 treats diagrams 1 and 4, and §10 describes the verification. Appendix C contains the proof of Lemma 2.1.

Relation to the work of Chen and Tong. Some statements of [2] on single diagrams overlap our results, and for diagrams 3 and 6 of [1, Fig. 11] their arguments use computations [2, Appendix A]. Chen and Tong write the modified Roberts continuum as an explicit curve and state that, because of its real points, curves with real points cannot be excluded at places of diagram 3 [2, Remark 7.14]. In their argument positivity enters through the identity $p_{15} = -2p_{13}$ between inverse distances; for our curves it is part of the definition of $\Lambda(\zeta_1, \zeta_2)$ in §3. Section 7 of [2] contains counterparts of Lemmas 4.1 and 4.2 for germs: at positive masses a germ of diagram 3 forces the masses $(4, 4, 4, 4, 1)$ up to scaling [2, Theorem 7.13], and up to symmetries every such germ is one of 36 explicit germs [2, Lemma 7.12 and p. 34]. These germs correspond to the branches of §4.1, up to the exchange of the two colours. For a curve with such a germ, [2, Theorem 7.13] gives the identities $p_{15} = p_{35} = -2p_{13}$ and $p_{25} = p_{45} = -2p_{24}$, and Corollary 4.3 identifies the curve itself. Theorem 5.1 is contained in the statement of Chen and Tong that at positive masses no curve has a place of diagram 6 [2, Theorem 9.10]. They also state the stronger result that no place of diagram 1 or 4 exists at positive masses [2, Theorems 5.13 and 8.2], which contains Theorem 9.2, and for diagram 4 part (a) of Theorem 9.2 appears in their argument (end of §9).

2. Setting

We use the notation and results of [8], cited by their numbers. Throughout, $m \in (\mathbb{R}_{>0})^5$.

2.1. **The slice.** A normalized central configuration is a solution (x, y, δ) of the system [8, (2.1)] of [1, p. 540]. We write $z = x + iy$, $w = x - iy$, $z_{kl} = z_k - z_l$, $w_{kl} = w_k - w_l$, $p_{kl} = \delta_{kl}$ and $r_{kl} = p_{kl}^{-1}$, so that $p_{kl}^2 z_{kl} w_{kl} = 1$. The *slice* is the affine variety $Y_m = X_m \cap \{u_1 = 1\}$, where $X_m \subset \mathbb{C}^{18}$ is the zero set of the eighteen rows

$$D_{kl} = p_{kl}^2 u_{kl} v_{kl} - 1, \qquad Z_k = F_k(u) - F_5(u), \qquad W_k = F_k(v) - F_5(v) \qquad (k \le 4)$$

of [8, (2.2)], in the unknowns $u = (u_1, \dots, u_4)$, $v = (v_1, \dots, v_4)$ and $p = (p_{kl})_{k<l}$, with $u_5 = v_5 = 0$, $u_k = z_k - z_5$, $v_k = w_k - w_5$ and $F_k(h) = h_k - \sum_{l \ne k} m_l p_{kl}^3 (h_k - h_l)$. The functions u_k, v_k, u_{kl}, v_{kl} and p_{kl} vanish nowhere on X_m. By [8, Lemma 2.2], $X_m \cong \mathbb{C}^* \times Y_m$, the system [8, (2.1)] has $2|Y_m|$ solutions, and positive scaling of the masses and relabelling of the bodies with their masses induce isomorphisms between the slices, which multiply U by nonzero constants. Let

$$U = \sum_{k<l} m_k m_l p_{kl}, \qquad I = \sum_k m_k u_k v_k - \frac{1}{M}\Big(\sum_k m_k u_k\Big)\Big(\sum_k m_k v_k\Big), \qquad V = U + \tfrac{1}{2} I,$$

with $M = \sum_k m_k$. The points of X_m are critical points of V on $\{D_{kl} = 0\}$ [8, Lemma 2.3]. Recall from [8, Lemma 2.4] that $U = I$ on X_m, that U is constant on every irreducible component of Y_m,

[1] The paper [8] cites Theorems 1.1–1.3 and 9.2 in its Remarks 5.43, 10.8 and 10.9, but its proofs do not use them.

which is [1, Lemma 2], and that $U > 0$ at the point of Y_m given by a real central configuration. A real central configuration corresponds to exactly two positive normalized central configurations [8, Lemma 2.1]. The mass vector m is *settled* if Y_m contains no irreducible curve on which some r_{kl}^2 is non-constant and $U \not\equiv 0$. A settled mass vector has finitely many central configurations up to similarity [8, Corollary 2.5].

The colour exchange. The substitution $(z, w) \mapsto (w, z)$ maps solutions of the first three equations of [8, (2.1)] to solutions. Followed by the rotation $(u, v) \mapsto (v_1^{-1}u, v_1 v)$ of [8, Lemma 2.2(a)], it defines the involution

$$\sigma_c(u, v, p) = (v/v_1,\ v_1 u,\ p)$$

of Y_m. It fixes every p_{kl}.

2.2. **Places and types.** Let $C \subseteq Y_m$ be an irreducible curve. We use the places at infinity of C, the singular places, the exponents $e(f)$ and the expansions of [8, §§2.4, 3.1]. For a pair kl put $\zeta_{kl} = e(z_{kl})$, $\omega_{kl} = e(w_{kl})$ and $x_{kl} = e(r_{kl}) = \frac{1}{2}(\zeta_{kl} + \omega_{kl})$. The *type* of a singular place P is $x(P) = (x_{kl})_{k<l} \in \mathbb{Q}^{10}$, with the pairs ordered $12, 13, \dots, 45$. The type is a positive multiple of $(-\operatorname{val}_P p_{kl})_{k<l}$, and the types satisfy the *balancing condition*

$$\sum_{P \text{ singular}} d_P\, x(P) = 0, \qquad d_P > 0 \tag{2.1}$$

[8, (3.1)]. A singular place has one of the sixteen diagrams of [1, Fig. 11], up to relabelling of the bodies and exchange of the colours [8, §2.5]. We use the inputs from [1] listed in [8, §2.6]. We use directly Proposition 1 of [1] and the mass conditions (23), (29), (32), (36) and (40) of [1, §8]. The others, among them the rules of [1, §3], Proposition 4 of [1] and the facts of [1, §§8.7 and 8.9], enter through the cited results of [8]. For a body j we put $Q_j = \{1, \dots, 5\} \setminus \{j\}$. For a triple T let P_T be the complementary pair, and for a set A of bodies let $\operatorname{tri}(A)$ be its set of pairs. We use the following points of $\mathbb{R}^{10}$ of [8, §3.2].

- $\mathsf{a}_1(T)$, the type of diagram 2: 0 on $\operatorname{tri}(T)$ and on P_T, -2 on the six other pairs.
- $\mathsf{h}(s; D_1|D_2)$, the type of diagram 3 with centre s and diagonals D_1, D_2, a partition of Q_s into two pairs: 0 on the four other pairs of Q_s (the sides), -2 on D_1, D_2 and on the four pairs ks.
- $\mathsf{v}(s; A|B)$ and $\mathsf{v}_7(a, b)$, the types of diagrams 6 and 7 with centre s and wings A, B, triples with $A \cap B = \{s\}$: $\mathsf{v}_7(a, b)$ is a on $\operatorname{tri}(A)$, b on $\operatorname{tri}(B)$ and $\frac{3}{2}(a+b) - 2$ on the four cross pairs, and $\mathsf{v}(s; A|B) = \mathsf{v}_7(0, 0)$.
- $\mathsf{u}_A(c)$, the type of diagram 10, with wings as above: 1 on $\operatorname{tri}(A)$, c on $\operatorname{tri}(B)$ and $\frac{3c-1}{2}$ on the cross pairs.
- $\mathsf{u}_c(j)$ and $\sigma_k(j)(t)$, the types of diagram 14 with isolated body j, and $\mathsf{b}_k(j)$, the type of diagram 15, for $k \in Q_j$: $\mathsf{u}_c(j)$ is 1 on $\operatorname{tri}(Q_j)$ and c on the four pairs ij; $\mathsf{b}_k(j)$ is 1 on $\operatorname{tri}(Q_j)$, 0 on kj and $-\frac{1}{2}$ on ij for $i \in Q_j \setminus k$; and $\sigma_k(j)(t) = (1-t)\,\mathsf{u}_0(j) + t\,\mathsf{b}_k(j)$.

At a place of diagram 5 with zw-triangle T, each of the other two bodies X has no stroke to T. In each colour, say z, either all ζ_{Xt}, $t \in T$, are equal and less than 1 (X is z-far), or all $\zeta_{Xt} \geq 1$ and at most one of them is larger than 1 (X is z-near), and X is not near in both colours [8, Lemma 3.4]. With $\bar\omega < 1$ the common far exponent in the other colour and s the exceptional exponent, the *modes* of X relative to T are F, with $x_{Xt} = c < 1$ for all t; Z1, with $x_{Xt} = \frac{1+\bar\omega}{2}$ for all t; Zs_{t_0}, with $x_{Xt_0} = \frac{s+\bar\omega}{2}$ and $x_{Xt} = \frac{1+\bar\omega}{2}$ for $t \neq t_0$, where $s > 1$; and W1 and Ws_{t_0}, the same with the colours exchanged. The parameters satisfy $s < 2$, $s + \bar\omega < 2$, $2s + \bar\omega < 3$ and $2s + 3\bar\omega < 5$. The colour exchange σ_c maps a singular place to a singular place with the same type, and exchanges the colours of its diagram.

By Lemma 2.1(c) below, the exponents at every singular place satisfy the rules of [8, Lemma 3.2], which we recall. The strokes, circles and zw-edges are those of the diagram [8, §2.5]. For every pair kl:

(O1) $\zeta_{kl}, \omega_{kl} \geq -2$, $\zeta_{kl} + 3\omega_{kl} \leq 4$ and $3\zeta_{kl} + \omega_{kl} \leq 4$. Hence $\zeta_{kl}, \omega_{kl} \leq 2$ and $-2 \leq x_{kl} \leq 1$.
(O2) There is a z-stroke kl if and only if $\zeta_{kl} + 3\omega_{kl} = 4$, and a w-stroke if and only if $3\zeta_{kl} + \omega_{kl} = 4$.
(O3) On a z-stroke, $\zeta_{kl} = 3x_{kl} - 2$, $\omega_{kl} = 2 - x_{kl}$ and $0 \leq x_{kl} \leq 1$. On a w-stroke, $\omega_{kl} = 3x_{kl} - 2$ and $\zeta_{kl} = 2 - x_{kl}$.
(O4) $x_{kl} = 1$ if and only if kl is a zw-edge. A z-stroke is maximal (its ends are not z-close, [1, Def. 6]) if and only if $x_{kl} = 0$.
(O5) In every triangle klm the minimum of $\zeta_{kl}, \zeta_{lm}, \zeta_{mk}$ is attained at least twice, and the same holds for ω.
(O6) If k is z-circled and l is not, then $\zeta_{kl} = -2$. If $\zeta_{kl} = 2$, then $\omega_{kl} = -2$ and kl is a w-stroke.

The results of [8] on singular places that we cite are stated there for *nontrivial* curves, on which $U \not\equiv 0$ and some r_{kl}^2 is non-constant [8, §2.3]. The second condition holds for every irreducible curve. Lemma 2.1 lists those that we apply to every irreducible curve, and Appendix C checks where U enters their proofs. In their proofs the first condition enters only through [8, Lemma 3.3], which says that at a singular place some x_{kl} is at most 0. We apply the other results of [8] on singular places only to curves with $U \not\equiv 0$, in §8 and Appendix B.

Lemma 2.1. *Let $C \subseteq Y_m$ be an irreducible curve.*

(a) *Some r_{kl}^2 is non-constant on C. A place at infinity at which some p_{kl} has a zero or a pole is singular. The balancing condition* (2.1) *holds, and C has a singular place.*
(b) *Let P be a singular place of C, with the bodies numbered as in* [1, Fig. 11]*. If the diagram of P is not* 5*, then $x(P)$ is as in* [8, Table 1]*, except for four diagrams. For diagram* 7*, $x(P) = \mathsf{v}_7(a, b)$ with $0 < a, b < 1$. For diagram* 10*, $x(P) = \mathsf{u}_A(c)$ with $0 < c < 1$. For diagram* 14*, $x(P) = \mathsf{u}_c(5)$ with $0 \leq c < 1$, or $\sigma_k(5)(t)$ with $k \in Q_5$ and $0 < t < 1$. For diagram* 16*, $x_{kl}(P) = 1$ for all pairs. If the diagram is* 5*, with zw-triangle T, then $x_{kl}(P) = 1$ on the pairs of T, and each of the other two bodies is in one of the modes of* [8, Lemma 3.4] *relative to T.*
(c) *The conclusions of* [8, Lemmas 3.1, 3.2, 3.4, 3.6 and 3.7, Theorem 3.9, Corollary 5.19, Theorem 5.37 and Proposition 7.1] *hold at every singular place of C, and so do the statements of the proof of* [8, Theorem 3.9] *in* [8, Appendix C]*. So does* [8, Theorem 3.13]*, with the point $\mathsf{u}_0(j)$ of part (a) replaced by $\mathsf{u}_c(j)$, $0 \leq c < 1$. In particular no singular place of C has type $\mathsf{b}_k(j)$, diagram* 9 *or diagram* 2*. At every singular place of C the mass conditions of* [1, §8] *given in the proof of* [8, Theorem 3.12] *hold for its diagram and labelling.*

Proof. See Appendix C. □

In particular, at a place of diagram 14 with isolated body j the leading coefficients of the cluster Q_j solve the system K14(m_{Q_j}) of [8, (3.2)]. By [8, Lemma 9.5], K14$(1, 1, 1, \nu)$ with $\nu > 0$ has a solution for exactly six values of ν, none of them in $\{1, \frac{1}{4}, \frac{1}{9}\}$.

Lemma 2.2. *Let $C \subseteq Y_m$ be an irreducible curve, and let f, g be linear forms on $\mathbb{R}^{10}$. It is impossible that $f(x(P)) \geq 0$ at every singular place P of C and $g(x(P)) > 0$ at every singular place with $f(x(P)) = 0$.*

Proof. By (2.1), $\sum_P d_P f(x(P)) = 0$ is a sum of nonnegative terms, so $f(x(P)) = 0$ at every singular place. Then $\sum_P d_P g(x(P)) > 0$, since C has a singular place by Lemma 2.1(a). This contradicts (2.1). □

2.3. **Branches and their lifting.** In §§4 and 5 we will expand the coordinates at places of diagrams 3 and 6 in a weighted system. Let P be a singular place of an irreducible curve $C \subseteq Y_m$. After the rotation of [8, Lemma 3.1], each coordinate $q \in \{u_k, v_k, p_{kl}\}$ is $\vartheta^{d\,e(q)} y_q(\vartheta)$ with $y_q(0) \neq 0$. Suppose that the *weights* $w_q = \frac{1}{2}e(q)$ are integers, and put $T = \vartheta^{2d}$, so that $q = T^{w_q} y_q$. In each row substitute $q = T^{w_q} y_q$ for every coordinate and divide by the lowest power of T. This yields

$$G_r(T, y) = \sum_{i \geq 0} T^i G_{r,i}(y) \qquad (r = 1, \dots, 18),$$

with polynomials $G_{r,i}$. Put $e'_q = w_q - w_{u_1}$, $w_q + w_{u_1}$ or w_q for q a u_k, a v_k or a p_{kl}, and fix a coordinate f with $e'_f = -2$. A *leading solution* is a zero of $G_0 = (G_{r,0})_r$ with $y_{u_1} = y_f = 1$ and all coordinates nonzero. We write J_0 for the 18×16 Jacobian matrix of G_0 at a leading solution with respect to the sixteen coordinates other than y_{u_1} and y_f.

A *branch* is a vector $y(\xi)$ of convergent power series in the eighteen coordinates such that $G(\xi^L, y(\xi)) \equiv 0$ for an integer $L \geq 1$, $y_{u_1} \equiv y_f \equiv 1$, and $y(0)$ is a leading solution. We then write $T = \xi^L$.

Lemma 2.3. *After a rotation and a change of the local parameter, the coordinates near P are $q = T^{w_q} y_q(\xi)$, where y is a branch.*

Proof. This is the proof of [8, Lemma E.1], which treats diagram 2 with $f = v_1$. A rotation by $y_{u_1}^{-1}$ makes $y_{u_1} \equiv 1$. Replacing T by $T'\phi^L$ and rotating by $\phi^{-Lw_{u_1}}$ turns y_q into $\phi^{Le'_q} y_q$, and the equation $\phi^{-2L} y_f(\xi\phi) = 1$ has a convergent solution $\phi(\xi)$ with $\phi(0)^{2L} = y_f(0)$ by the implicit function theorem. The rows vanish on X_m, so $G(\xi^L, y(\xi)) \equiv 0$, and setting $\xi = 0$ we obtain $G_0(y(0)) = 0$. □

Let y_0 be a leading solution with $\operatorname{rank} J_0 = 16$, and let R be a set of sixteen rows such that $J_0[R]$ is invertible. Define $y_1, y_2, \dots \in \mathbb{C}^{16}$, vectors in the sixteen coordinates other than y_{u_1}, y_f, by

$$y_n = -J_0[R]^{-1} \beta_n[R], \qquad \beta_n = \big[T^n\big]\, G\Big(T,\ y_0 + \sum_{k<n} y_k T^k\Big),$$

where $[T^n]$ is the coefficient of T^n, and put $\eta_{R,n} = J_0[R^c]\, y_n + \beta_n[R^c] \in \mathbb{C}^2$, where R^c is the set of the two other rows.

Lemma 2.4. *Let y be a branch with $y(0) = y_0$. Then y is a power series in T, $y = y_0 + \sum_{n \geq 1} y_n T^n$, and $\eta_{R,n} = 0$ for every $n \geq 1$. In particular y is determined by y_0. Equivalently to $\eta_{R,n} = 0$, $\lambda \cdot \beta_n = 0$ for every λ in the left kernel of J_0.*

Proof. Write $y = y_0 + \sum_q c_q T^q$ over $q \in \frac{1}{L}\mathbb{Z}_{>0}$. The coefficient of T^q in $G(T, y)$ is $J_0 c_q$ plus a polynomial in the $c_{q'}$ with $q' < q$, each term of which has exponent $i + q'_1 + \cdots + q'_r = q$ with $i \in \mathbb{Z}_{\geq 0}$. If some c_q with $q \notin \mathbb{Z}$ is nonzero, take the least such q. Then every nonzero $c_{q'}$ with $q' < q$ has $q' \in \mathbb{Z}$, so the polynomial vanishes, and $J_0 c_q = 0$ implies $c_q = 0$, since J_0 is injective. Hence y is a power series in T, and the coefficient of T^n is $J_0 c_n + \beta_n$, with β_n evaluated at $c_1, \dots, c_{n-1}$. By induction on n, the rows R yield $c_n = y_n$, and then the rows R^c yield $\eta_{R,n} = 0$. For the last statement, $J_0 y_n + \beta_n$ vanishes on R and equals $\eta_{R,n}$ on R^c, so $\lambda \cdot \beta_n = \lambda[R^c] \cdot \eta_{R,n}$ for λ in the left kernel. The map $\lambda \mapsto \lambda[R^c]$ is injective on the left kernel, since $\lambda[R] J_0[R] = 0$ forces $\lambda[R] = 0$, and the left kernel has dimension 2. It is therefore bijective onto $\mathbb{C}^{R^c}$. □

3. The thirty-six curves

In this section and the next the masses are $A' = (4,4,4,4,1)$, the image of $(1,4,4,4,4)$ under the exchange of bodies 1 and 5, so that body 5 is light. For cube roots of unity ζ_1, ζ_2 let $\Lambda(\zeta_1,\zeta_2)$ be the affine subspace of the coordinate space of $Y_{A'}$ defined by

$$p_{12} = p_{34} = \tfrac{\zeta_1}{2}, \qquad p_{14} = p_{23} = \tfrac{\zeta_2}{2}, \qquad p_{15} = p_{35} = -2p_{13}, \qquad p_{25} = p_{45} = -2p_{24},$$
$$u_3 = -1, \qquad u_4 = -u_2, \qquad v_3 = -v_1, \qquad v_4 = -v_2,$$

and put $\mathcal{C}'(\zeta_1,\zeta_2) = Y_{A'} \cap \Lambda(\zeta_1,\zeta_2)$. On $\Lambda(\zeta_1,\zeta_2)$ we write $D_1 = p_{15}$ and $D_2 = p_{25}$. On $Y_{A'}$ the function $\tau = (1+u_2)/(1-u_2)$ is regular, since $u_{12} \neq 0$. Let

$$h_1(\tau) = \frac{\tau}{(\tau+1)(\zeta_1\tau+\zeta_2)}, \qquad h_2(\tau) = \frac{\tau}{(\tau-1)(\zeta_2-\zeta_1\tau)}.$$

Theorem 3.1. (a) *$\mathcal{C}'(\zeta_1,\zeta_2)$ is the set of points of $\Lambda(\zeta_1,\zeta_2)$ with*

$$u_2 = \frac{\tau-1}{\tau+1}, \qquad v_1 = \frac{(\tau+1)(\zeta_1\tau+\zeta_2)}{\tau}, \qquad v_2 = \frac{(\tau+1)(\zeta_2-\zeta_1\tau)}{\tau},$$
$$D_1^2 = h_1(\tau), \qquad D_2^2 = h_2(\tau)$$

and $\tau \in \mathbb{C}\backslash\{0,\pm1,\pm\zeta_2/\zeta_1\}$. Thus (τ, D_1, D_2) identifies it with an unramified four-sheeted cover of $\mathbb{C} \setminus \{0,\pm1,\pm\zeta_2/\zeta_1\}$.

(b) *If $\zeta_1 \neq \zeta_2$, then $\mathcal{C}'(\zeta_1,\zeta_2)$ is an irreducible curve of genus 3. Its smooth completion has twelve places at infinity, two over each of $\tau = 0, \infty, \pm1, \pm\zeta_2/\zeta_1$, each of ramification index 2 over the τ-line.*

(c) *If $\zeta_1 = \zeta_2 = \zeta$, then $\mathcal{C}'(\zeta,\zeta)$ is the disjoint union of two irreducible rational curves $\mathcal{C}^{\pm}(\zeta)$, defined by $D_1D_2 = \pm i\tau/(\zeta(\tau^2-1))$. Each is parametrized by $s = \sqrt{\zeta}\,(\tau+1)D_1$, which satisfies $s^2 = \tau$, and has six places at infinity, $s = 0, \infty, \pm1, \pm i$.*

(d) *These twelve curves and their images under the relabellings* (23) *and* (34)*, which fix A', are* 36 *pairwise disjoint irreducible curves in $Y_{A'}$,* 18 *of genus* 3 *and* 18 *rational.*

We write $\mathcal{C}_{A'}$ for the union of the 36 curves and $\mathcal{C}_A \subset Y_A$, $A = (1,4,4,4,4)$, for its image under the exchange of bodies 1 and 5 (§2.1).

Proof. (a) We first reduce the rows. On $\Lambda(\zeta_1,\zeta_2)$ the positions are $z = (1, u_2, -1, -u_2, 0)$ and $w = (v_1, v_2, -v_1, -v_2, 0)$, so $\sum_k m_k z_k = \sum_k m_k w_k = 0$. The ten rows D_{kl} reduce to four equations, since $p_{12}^2 = \zeta_1^2/4 = 1/(4\zeta_1)$ and likewise on the other sides:

$$(1-u_2)(v_1-v_2) = 4\zeta_1 \quad (12,\,34), \qquad (1+u_2)(v_1+v_2) = 4\zeta_2 \quad (14,\,23),$$
$$D_1^2 v_1 = 1 \quad (13,\,15,\,35), \qquad D_2^2 u_2 v_2 = 1 \quad (24,\,25,\,45).$$

The rows Z_k and W_k vanish identically on $\Lambda(\zeta_1,\zeta_2)$. Indeed, with $p_{kl}^3 = \frac{1}{8}$ on the four sides and $p_{13} = -D_1/2$, $p_{24} = -D_2/2$,

$$F_1(u) = 1 - \tfrac{1}{2}(1-u_2) - \tfrac{1}{2}(1+u_2) - 8p_{13}^3 - D_1^3 = 0,$$
$$F_2(u) = u_2 - \tfrac{1}{2}(u_2-1) - \tfrac{1}{2}(u_2+1) - u_2(8p_{24}^3 + D_2^3) = 0,$$
$$F_5(u) = 4\big(D_1^3(u_1+u_3) + D_2^3(u_2+u_4)\big) = 0,$$

and $F_3(u) = F_4(u) = 0$ by the symmetry $z \mapsto -z$, $1 \leftrightarrow 3$, $2 \leftrightarrow 4$ of $\Lambda(\zeta_1,\zeta_2)$. The functions $F_k(v)$ vanish in the same way. Hence $\mathcal{C}'(\zeta_1,\zeta_2)$ is defined in $\Lambda(\zeta_1,\zeta_2)$ by the four equations and by the nonvanishing of the coordinates and their differences. With $\tau = (1+u_2)/(1-u_2)$ we have $1-u_2 = 2/(\tau+1)$ and $1+u_2 = 2\tau/(\tau+1)$. Then $v_1 - v_2 = 2\zeta_1(\tau+1)$ and $v_1+v_2 = 2\zeta_2(\tau+1)/\tau$, which give the stated v_1 and v_2, and $D_1^2 = 1/v_1 = h_1$, $D_2^2 = 1/(u_2v_2) = h_2$. Since u_2 is finite and $u_{12} \neq 0$, the function τ takes neither the value -1 nor ∞. The values $\tau = 0$, 1, $-\zeta_2/\zeta_1$ and

ζ_2/ζ_1 mean $u_{23} = 0$, $u_2 = 0$, $v_1 = 0$ and $v_2 = 0$. For every other τ and every choice of D_1, D_2, all coordinates and their differences are nonzero. Since τ, D_1 and D_2 are regular functions on $Y_{A'}$, the map is an isomorphism onto its image. Neither h_1 nor h_2 has a zero or a pole over the base, so the cover is unramified.

(b) If $\zeta_1 \neq \zeta_2$, then ζ_2/ζ_1 is a primitive cube root of unity, and ± 1, $\pm\zeta_2/\zeta_1$ are four distinct points. None of h_1, h_2, h_1h_2 is a square in $\mathbb{C}(\tau)$: h_1 has a simple pole at -1, h_2 at 1, and h_1h_2 at ± 1 and $\pm\zeta_2/\zeta_1$. Hence $\mathbb{C}(\tau)(\sqrt{h_1}, \sqrt{h_2})$ has degree 4 over $\mathbb{C}(\tau)$, and the curve is irreducible. Over each of 0, ∞, ± 1, $\pm\zeta_2/\zeta_1$ at least one of h_1, h_2 has odd order: both have order 1 at 0 and at ∞, h_1 has a simple pole at -1 and at $-\zeta_2/\zeta_1$, where h_2 is a unit, and the reverse holds at 1 and ζ_2/ζ_1. The inertia group therefore has order 2, and there are two places of index 2 over each of these six points. By the Riemann–Hurwitz formula, $2g - 2 = 4 \cdot (-2) + 6 \cdot 2$, so $g = 3$.

(c) If $\zeta_1 = \zeta_2 = \zeta$, then $h_1 = \tau/(\zeta(\tau+1)^2)$, $h_2 = -\tau/(\zeta(\tau-1)^2)$ and $h_1h_2 = \big(i\tau/(\zeta(\tau^2-1))\big)^2$. Thus $D_1D_2 = \pm i\tau/(\zeta(\tau^2-1))$ on $\mathcal{C}'(\zeta,\zeta)$, and the two signs define disjoint closed subsets. On each of them D_2 is determined by D_1, and $s = \sqrt{\zeta}\,(\tau+1)D_1$ satisfies $s^2 = \zeta(\tau+1)^2h_1 = \tau$. Each of them is therefore isomorphic to $\mathbb{C} \setminus \{0, \pm 1, \pm i\}$.

(d) Distinct pairs (ζ_1, ζ_2) yield disjoint curves, since p_{12} or p_{14} differs, and $\mathcal{C}^+(\zeta) \cap \mathcal{C}^-(\zeta) = \emptyset$ by (c). The relabellings (23) and (34) fix bodies 1 and 5, so they act by $u_k \mapsto u_{\sigma(k)}$. The curves above satisfy $u_3 = -1$. Their images under (23) satisfy $u_2 = -1$, and those under (34) satisfy $u_4 = -1$. Two of these conditions together force a collision of two heavy bodies. □

The diagonals of the curves of Theorem 3.1(a)–(c) are the pairs 13 and 24; we say that they belong to the *split* 13|24. Their images under (23) and (34) belong to the splits 12|34 and 14|23.

3.1. **The real points.** Let σ be the composition of complex conjugation with the colour exchange σ_c of §2.1:

$$\sigma(u, v, p) = (\bar{v}/\bar{v}_1,\ \bar{v}_1\bar{u},\ \bar{p}),$$

an antiholomorphic involution of $Y_{A'}$. On configurations it is $(z, w, p) \mapsto (\bar{w}, \bar{z}, \bar{p})$.

Proposition 3.2. (a) *The fixed points of σ are the points given by real configurations, with real $p_{kl} = \pm 1/r_{kl}$. The positive normalized central configurations correspond to fixed points with all $p_{kl} > 0$.*

(b) *The thirty of the 36 curves that lie in a set $\mathcal{C}'(\zeta_1, \zeta_2)$ with $(\zeta_1, \zeta_2) \neq (1, 1)$, or in its image under (23) or (34), contain no fixed point of σ, and σ maps each of them onto another of the 36 curves.*

(c) *The other six curves, $\mathcal{C}^{\pm}(1)$ and their images, are mapped onto themselves. Their fixed points are the points with $|\tau| = 1$, $\tau \neq \pm 1$. At each of them $p_{13} = -p_{15}/2$, in the labelling of the split 13|24, so $p_{13}p_{15} < 0$.*

In particular no point of $\mathcal{C}_{A'}$, and no point of $\mathcal{C}_A$, corresponds to a positive normalized central configuration.

Proof. (a) Let $x = \sigma(x)$. Then p is real and $v_1 = \bar{v}_1\bar{u}_1 = \bar{v}_1$. The row D_{15} shows that $v_1 = 1/p_{15}^2 > 0$. With $c = \sqrt{v_1} > 0$, the rotated point $z = cu$, $w = c^{-1}v$ satisfies $w = c^{-1}v_1\bar{u} = c\bar{u} = \bar{z}$. This is a real configuration, with real $p_{kl} = \pm 1/r_{kl}$. Conversely, a real configuration rotated to $u_1 = 1$ is fixed by σ.

(b) The colour exchange maps $\Lambda(\zeta_1, \zeta_2)$ onto itself: $(v/v_1, v_1u)$ has again the form $(1, *, -1, -*)$, $(*, *, -*, -*)$, and p is unchanged. Hence σ_c maps $\mathcal{C}'(\zeta_1, \zeta_2)$ onto itself, by $\tau \mapsto \zeta_2/(\zeta_1\tau)$ and keeping D_1, D_2. Conjugation maps it onto $\mathcal{C}'(\bar{\zeta}_1, \bar{\zeta}_2)$. If $(\zeta_1, \zeta_2) \neq (1, 1)$, then p_{12} or p_{14} is not real, so the curve has no fixed point, and its image $\mathcal{C}'(\bar{\zeta}_1, \bar{\zeta}_2)$ is another curve. For each split this accounts for the six curves of genus 3 and the four curves $\mathcal{C}^{\pm}(\zeta)$ with $\zeta \neq 1$. The relabellings commute with σ.

(c) Let $\zeta_1 = \zeta_2 = 1$. The map σ_c acts by $\tau \mapsto 1/\tau$ and keeps $D_1 D_2$, while $i\tau/(\tau^2 - 1)$ changes sign. So σ_c exchanges $\mathcal{C}^+(1)$ and $\mathcal{C}^-(1)$. Conjugation exchanges them as well, so σ maps each of them onto itself and acts on τ by $\tau \mapsto 1/\bar{\tau}$, and a fixed point has $|\tau| = 1$. Write $\tau = e^{i\theta}$ with $\theta \notin \pi\mathbb{Z}$. Then

$$u = \big(1,\ i\tan\tfrac{\theta}{2},\ -1,\ -i\tan\tfrac{\theta}{2}\big),$$
$$D_1^2 = \frac{1}{4\cos^2(\theta/2)}, \qquad D_2^2 = \frac{1}{4\sin^2(\theta/2)}, \qquad D_1 D_2 = \pm\frac{1}{2\sin\theta}.$$

All p_{kl} are real, and one checks directly that these points are fixed. On each of $\mathcal{C}^{\pm}(1)$ they form four arcs, one for each sign pattern of (D_1, D_2). At each of them $p_{13} = -D_1/2 = -p_{15}/2$. □

The fixed points of Proposition 3.2(c) are rhombi with the light body at the centre and sides of length 2. On the arcs with $D_1, D_2 < 0$ the diagonal and side inverse distances are positive, and $p_{k5} < 0$ for $k \le 4$. Changing (m_5, p_{k5}) into $(-m_5, -p_{k5})$ preserves the rows D_{kl} and the terms $m_5 p_{k5}^3 (z_k - z_5)$ in the equations of bodies $k \le 4$. It changes the sign of the right side of the equation of body 5, which vanishes on these rhombi by their symmetry. This change maps these arcs onto Roberts' continuum of central configurations with the masses $(4, 4, 4, 4, -1)$ [6], in the normalization of [8, (2.1)]. The arcs are the continuum of solutions of [8, (2.1)] with the masses $(4, 4, 4, 4, 1)$ that Albouy and Kaloshin obtained from Roberts' continuum, the *modified Roberts continuum* [1, p. 543 and Remark 8], which Chen and Tong write explicitly [2, Remark 7.14]. The fixed points of σ on $\mathcal{C}_{A'}$ form an infinite set. Therefore the finiteness of the positive normalized central configurations at A' does not follow from the complex equations, even together with the real structure σ. It requires the signs of the p_{kl}, which enter here as $p_{13} p_{15} < 0$.

4. Places of diagram 3 at $(4, 4, 4, 4, 1)$

Let P be a singular place of an irreducible curve $C \subseteq Y_{A'}$ with type $\mathsf{h}(s; D_1|D_2)$. By [8, Theorem 3.9] and Lemma 2.1(c), $m_i m_j = m_k m_l$ for $D_1 = \{i, j\}$, $D_2 = \{k, l\}$. If s is heavy, one diagonal contains the light body and this fails, since $4 \ne 16$. So $s = 5$, and the type is $\mathsf{h}(5; 13|24)$, $\mathsf{h}(5; 12|34)$ or $\mathsf{h}(5; 14|23)$. The relabellings (23) and (34) fix A' and map the second and the third onto the first.

Let P have type $\mathsf{h}(5; 13|24)$. We call the pairs 12, 14, 23, 34 the *sides* and write k' for the body opposite to $k \le 4$: $1' = 3$, $2' = 4$. By [8, Appendix C] and Lemma 2.1(c) the exponents after the rotation are $e(u_k) = e(v_k) = -2$, $e(p_{kl}) = 0$ on the sides and $e(p_{kl}) = 2$ on the six other pairs. Thus §2.3 applies with $f = v_1$, and $e'_{v_1} = -2$. We write u_k, v_k for y_{u_k}, y_{v_k}, with $u_1 = v_1 = 1$, and k_e, q_e for y_{p_e} on the sides and on the other pairs. At A' the weighted rows are as follows.

- Sides kl: $k_{kl}^2 (u_k - u_l)(v_k - v_l) - T^2$. Diagonals 13, 24: $q_{kl}^2 (u_k - u_l)(v_k - v_l) - 1$. Pairs $k5$: $q_{k5}^2 u_k v_k - 1$.
- $Z_k = Z_k^0 + T^3 Z_k^3$ with $Z_k^0 = u_k - \sum_{l:\, kl \text{ a side}} 4k_{kl}^3 (u_k - u_l)$ and $Z_k^3 = -\big(4q_{kk'}^3 (u_k - u_{k'}) + q_{k5}^3 u_k + \Gamma\big)$, where $\Gamma = 4\sum_{l \le 4} q_{l5}^3 u_l$.
- W_k: the same with v in place of u.

So $G = G_0 + T^2 G_2 + T^3 G_3$, where G_2 is -1 on the side rows and 0 on the other rows. By [8, Appendix C] and Lemma 2.1(c) the leading solutions are as follows. In *case A*, $u = (1, -1, -1, 1)$ and $v = (1, 1, -1, -1)$; in *case B*, u and v are exchanged. In both cases $k_e^3 = \frac{1}{8}$ on the four sides, $q_{13}^2 = \frac{1}{4}$, $q_{24}^2 = -\frac{1}{4}$, $q_{15}^2 = q_{35}^2 = 1$ and $q_{25}^2 = q_{45}^2 = -1$, and every choice of the roots occurs. There are therefore $3^4 \cdot 2^6 = 5184$ leading solutions in each case. In both cases $u_{k'} = -u_k$ and $v_{k'} = -v_k$. The matrix J_0 has rank 16, and in case A its left kernel is spanned by $\lambda_Z = -Z_1 - Z_2 + Z_3 + Z_4$

and $\lambda_W = W_1 - W_2 - W_3 + W_4$, where a row stands for the corresponding unit vector. Moreover $\lambda_Z \cdot G_3 = O/16$ and $\lambda_W \cdot G_3 = -O/16$ at the leading solution, where

$$O = 16\,(4q_{13} + 4q_{24} + q_{15} + q_{25} + q_{35} + q_{45}).$$

The relabelling (24) and the colour exchange σ_c map places of type $\mathsf{h}(5; 13|24)$ at A' to places of the same type. On branches they act by permuting coordinates: (24) by $u_k \mapsto u_{\sigma(k)}$, $v_k \mapsto v_{\sigma(k)}$, $p_{kl} \mapsto p_{\sigma(k)\sigma(l)}$ with $\sigma = (24)$, and σ_c by exchanging u and v (as $u_1 = v_1 = 1$). Both map branches to branches. The colour exchange exchanges the two cases, and (24) followed by σ_c maps case A to case A.

The following condition mirrors the relations among the p_{kl} in the definition of $\Lambda(\zeta_1, \zeta_2)$. A leading solution is *balanced* if $k_{12} = k_{34}$, $k_{14} = k_{23}$, $q_{15} = q_{35} = -2q_{13}$ and $q_{25} = q_{45} = -2q_{24}$. There are 36 balanced leading solutions in each case.

Lemma 4.1. *Let y be a branch for the type $\mathsf{h}(5; 13|24)$ at A'. Then $y(0)$ is balanced.*

Proof. Since σ_c exchanges the cases and preserves balance, we may assume that $y_0 = y(0)$ is in case A. We show that the obstruction at order 3 vanishes only if $q_{15} = q_{35} = -2q_{13}$ and $q_{25} = q_{45} = -2q_{24}$, and that the even part of the expansion then forces $k_{12} = k_{34}$ and $k_{14} = k_{23}$ at y_0. By Lemma 2.4, $y = \sum_n y_n T^n$ is a power series in T and $\lambda \cdot \beta_n = 0$ for all n and all λ in the left kernel of J_0.

Orders 1 to 3. Since G has no term in T, $\beta_1 = 0$ and $y_1 = 0$. Then $\beta_3 = G_3(y_0)$, because $G_0(y_0 + y_2T^2 + \dots)$ and T^2G_2 contribute only even powers up to T^3. Hence $0 = \lambda_Z \cdot \beta_3 = O/16$. The numbers q_{13}, q_{15}, q_{35} are real and q_{24}, q_{25}, q_{45} are imaginary, with $|4q_{13}| = |q_{15}| + |q_{35}|$ and $|4q_{24}| = |q_{25}| + |q_{45}|$. Therefore $O = 0$ if and only if

$$q_{15} = q_{35} = -2q_{13}, \qquad q_{25} = q_{45} = -2q_{24}. \tag{4.1}$$

Then $\Gamma(y_0) = 4\big(q_{15}^3(u_1 + u_3) + q_{25}^3(u_2 + u_4)\big) = 0$ and $Z_k^3(y_0) = -u_k(8q_{kk'}^3 + q_{k5}^3) = 0$, and likewise $W_k^3(y_0) = 0$. So $\beta_3 = 0$ and $y_3 = 0$.

Parity. Since $y_1 = y_3 = 0$, we can write $y = Y(S) + T^5\Omega(S)$ with $S = T^2$ and power series Y, Ω. We write U, V, K for the parts of Y in u, v and the sides, and $\Delta U_{kl} = U_k - U_l$, $\Delta V_{kl} = V_k - V_l$. For a polynomial F, the even part of $F(y)$ in T is $F(Y) + O(S^5)$, and the even part of $T^3Z_k^3(y)$ is $O(S^4)$. The side rows, Z^0 and W^0 are polynomials in u, v, k and T^2. Taking even parts of $G(T, y) = 0$ we obtain

$$K_{kl}^2\,\Delta U_{kl}\,\Delta V_{kl} \equiv S \pmod{S^5} \text{ on the sides}, \qquad Z^0(U, K) \equiv W^0(V, K) \equiv 0 \pmod{S^4}. \tag{4.2}$$

The Laplacian. The rows Z^0 and W^0 are linear in U and V, with coefficients from the sides. Put $s_e = 4K_e^3$ and $a = s_{12}$, $b = s_{23}$, $c = s_{34}$, $d = s_{41}$. Let $\mathcal{L}$ be the weighted Laplacian of the cycle 1234, $(\mathcal{L}x)_k = \sum_{kl \text{ a side}} s_{kl}(x_k - x_l)$, so that $Z^0 = (I - \mathcal{L})U$ and $W^0 = (I - \mathcal{L})V$. In the orthonormal basis

$$e_0 = \tfrac{1}{2}(1,1,1,1), \quad e_1 = \tfrac{1}{2}(1,-1,-1,1), \quad e_2 = \tfrac{1}{2}(1,1,-1,-1), \quad e_3 = \tfrac{1}{2}(1,-1,1,-1)$$

write $x = \sum_i X_i e_i$ and $\bar{X} = (X_1, X_2, X_3)$. We write $\hat{U}_i$, $\hat{V}_i$ for the coefficients of U, V in this basis, and $\hat{U} = (\hat{U}_1, \hat{U}_2, \hat{U}_3)$, $\hat{V} = (\hat{V}_1, \hat{V}_2, \hat{V}_3)$. Then $\mathcal{L}e_0 = 0$, the side differences are $x_1 - x_2 = X_1 + X_3$, $x_2 - x_3 = X_2 - X_3$, $x_3 - x_4 = X_3 - X_1$ and $x_4 - x_1 = -X_2 - X_3$, and on $e_0^\perp$ the matrix of $\mathcal{L}$ is

$$M = \begin{pmatrix} a+c & 0 & a-c \\ 0 & b+d & d-b \\ a-c & d-b & a+b+c+d \end{pmatrix}.$$

By (4.2), $N\hat{U} \equiv N\hat{V} \equiv 0 \pmod{S^4}$, where $N = M - I$. At $S = 0$ we have $s_e = \frac{1}{2}$, so $M(0) = \operatorname{diag}(1,1,2)$, $\hat{U}(0) = (2,0,0)$ and $\hat{V}(0) = (0,2,0)$. Let $\varepsilon_3 = (0,0,1)$ and $n = N\varepsilon_3 =$

$(a-c,\ d-b,\ a+b+c+d-1)$, so that $n_3(0)=1$. The matrix $P=[\hat U\ \hat V\ \varepsilon_3]$ is invertible over $\mathbb{C}[[S]]$, since $\det P(0)=4$. From $NP\equiv[0\ 0\ n]$ we get $N\equiv n\,r^{\mathsf T}$, with $r^{\mathsf T}$ the last row of P^{-1}. Since N is symmetric and n_3 is a unit, comparing the last rows shows that $r\equiv n/n_3$. Hence

$$N\equiv\frac{nn^{\mathsf T}}{n_3}\pmod{S^4},\qquad n\cdot\hat U\equiv n\cdot\hat V\equiv 0\pmod{S^4}. \tag{4.3}$$

The entries $(1,1)$, $(1,2)$ and $(2,2)$ of (4.3) are

$$\text{(R1) } (a+c-1)(a+b+c+d-1)\equiv(a-c)^2,\qquad \text{(R2) } (a-c)(d-b)\equiv 0,$$
$$\text{(R3) } (b+d-1)(a+b+c+d-1)\equiv(d-b)^2.$$

Order S. For a series F write $F=\sum_j F^{(j)}S^j$, and let $\kappa_e=K_e(0)$, so that $\kappa_e^3=\frac18$. At $S=0$, $\Delta U_{12}=2$, $\Delta U_{34}=-2$, $\Delta V_{23}=2$, $\Delta V_{41}=-2$, and $\Delta V_{12}=\Delta V_{34}=\Delta U_{23}=\Delta U_{41}=0$. The order-$S$ part of the side rows in (4.2) yields

$$\Delta V_{12}^{(1)}=\frac{1}{2\kappa_{12}^2},\qquad \Delta V_{34}^{(1)}=-\frac{1}{2\kappa_{34}^2},\qquad \Delta U_{23}^{(1)}=\frac{1}{2\kappa_{23}^2},\qquad \Delta U_{41}^{(1)}=-\frac{1}{2\kappa_{14}^2}.$$

At order S, (R1) and (R3) give $c^{(1)}=-a^{(1)}$ and $b^{(1)}=-d^{(1)}$, and then $n\cdot\hat U$ and $n\cdot\hat V$ yield $\hat U_3^{(1)}=-4a^{(1)}$ and $\hat V_3^{(1)}=-4d^{(1)}$. Since $\Delta U_{23}+\Delta U_{41}=-2\hat U_3$ and $\Delta V_{12}+\Delta V_{34}=2\hat V_3$,

$$a^{(1)}=\frac{\kappa_{14}^2-\kappa_{23}^2}{16\,\kappa_{14}^2\kappa_{23}^2},\qquad d^{(1)}=\frac{\kappa_{12}^2-\kappa_{34}^2}{16\,\kappa_{12}^2\kappa_{34}^2}.$$

(R2) at order S^2 gives $4a^{(1)}d^{(1)}=0$. Since $\kappa_e=1/(8\kappa_e^2)$, $a^{(1)}=0$ if and only if $\kappa_{14}=\kappa_{23}$, and $d^{(1)}=0$ if and only if $\kappa_{12}=\kappa_{34}$.

The equality $\kappa_{12}=\kappa_{34}$ forces $\kappa_{14}=\kappa_{23}$. Suppose that $\kappa_{12}=\kappa_{34}=\kappa$ and $a^{(1)}\neq 0$. Then $d^{(1)}=b^{(1)}=0$. Since $a-c=2a^{(1)}S+O(S^2)$ is S times a unit, (R2) gives $d-b\equiv 0\pmod{S^3}$. From $\Delta V_{12}=\hat V_1+\hat V_3$ and $\Delta V_{34}=\hat V_3-\hat V_1$ we get $\hat V_1^{(1)}=1/(2\kappa^2)$ and $\hat V_3^{(1)}=0$. At order S^2, $n\cdot\hat V\equiv 0$ reads $n_1^{(1)}\hat V_1^{(1)}+\hat V_3^{(2)}=0$, since $n_2=O(S^3)$, $n_3^{(1)}=0$ and $\hat V_1(0)=\hat V_3(0)=\hat V_3^{(1)}=0$. Hence $\hat V_3^{(2)}=-a^{(1)}/\kappa^2$. At order S^2 the side rows 12 and 34 yield

$$\Delta V_{12}^{(2)}=-\frac{\Delta U_{12}^{(1)}}{4\kappa^2}-\frac{K_{12}^{(1)}}{\kappa^3},\qquad \Delta V_{34}^{(2)}=-\frac{\Delta U_{34}^{(1)}}{4\kappa^2}+\frac{K_{34}^{(1)}}{\kappa^3}.$$

We add the two, using $\Delta V_{12}+\Delta V_{34}=2\hat V_3$, $\Delta U_{12}+\Delta U_{34}=2\hat U_3$ and $\hat U_3^{(1)}=-4a^{(1)}$, and obtain $K_{12}^{(1)}-K_{34}^{(1)}=4\kappa a^{(1)}$. Hence the coefficient of S in $a-c=4(K_{12}^3-K_{34}^3)$ is $12\kappa^2(K_{12}^{(1)}-K_{34}^{(1)})=48\kappa^3a^{(1)}=6a^{(1)}$. But it is $2a^{(1)}$ by the order-S step. Thus $a^{(1)}=0$, a contradiction.

Conclusion. The relabelling (24) followed by σ_c fixes case A and the conditions $u_1=v_1=1$, and it exchanges $\kappa_{12}\leftrightarrow\kappa_{14}$ and $\kappa_{34}\leftrightarrow\kappa_{23}$. Applied to its image, the previous step shows that $\kappa_{14}=\kappa_{23}$ forces $\kappa_{12}=\kappa_{34}$. By the order-S step one of the two equalities holds, so both hold. Together with (4.1), y_0 is balanced. □

4.1. **The branches of the curves.** For $\gamma=(\zeta_1,\zeta_2,\varepsilon_1,\varepsilon_2)$, with cube roots of unity ζ_1, ζ_2 and signs ε_1, ε_2, let $Y_\gamma(t)$ be the point with the coordinates

$$u=\Big(1,\ \frac{t-1}{t+1},\ -1,\ -\frac{t-1}{t+1}\Big),\qquad v=(1,\ \varrho,\ -1,\ -\varrho),\qquad \varrho=\frac{\zeta_2-\zeta_1 t}{\zeta_2+\zeta_1 t},$$
$$k_{12}=k_{34}=\frac{\zeta_1}{2},\quad k_{14}=k_{23}=\frac{\zeta_2}{2},$$
$$q_{15}=q_{35}=\varepsilon_1,\quad q_{13}=-\frac{\varepsilon_1}{2},\quad q_{25}=q_{45}=\varepsilon_2 g,\quad q_{24}=-\frac{\varepsilon_2 g}{2},$$

which depends on t and on a further unknown g. Let $\Phi(T,y) = (y_u,\ T^{-2}y_v,\ k,\ Tq)$: the point with $u = y_u$, $v = T^{-2}y_v$, $p_e = k_e$ on the sides and $p_e = Tq_e$ on the other pairs. It is the rotation by T of the point with the coordinates $T^{w_q}y_q$, so $G(T,y) = 0$ if and only if $\Phi(T,y) \in X_{A'}$. If $T^2 = h_1(t)$ and $g^2 = h_2(t)/h_1(t)$, then $\Phi(T, Y_\gamma(t))$ is the point of $\mathcal{C}'(\zeta_1,\zeta_2)$ with $\tau = t$, $D_1 = T\varepsilon_1$ and $D_2 = T\varepsilon_2 g$, by Theorem 3.1(a), and so $G(T, Y_\gamma(t)) = 0$.

Since $h_1(0) = 0$ and $h_1'(0) = 1/\zeta_2$, the equation $S = h_1(t)$ has a convergent inverse $t(S) = \zeta_2 S + O(S^2)$ near 0. Since $h_2/h_1 = -1$ at $t = 0$, there is a convergent branch $g(S)$ of $\sqrt{h_2(t(S))/h_1(t(S))}$ with $g(0) = i$. Let $\tilde{Y}_\gamma(S)$ be $Y_\gamma(t(S))$ with this g. Then $\tilde{Y}_\gamma(T^2)$ is a convergent power series in T with $G(T, \tilde{Y}_\gamma(T^2)) \equiv 0$ and $u_1 = v_1 = 1$. Its value at $T = 0$ is the case A leading solution with $k_{12} = k_{34} = \zeta_1/2$, $k_{14} = k_{23} = \zeta_2/2$, $q_{13} = -\varepsilon_1/2$ and $q_{24} = -i\varepsilon_2/2$. The map $\gamma \mapsto \tilde{Y}_\gamma(0)$ is therefore a bijection onto the 36 balanced leading solutions of case A.

Lemma 4.2. *Let y be a branch for the type $\mathsf{h}(5;13|24)$ at A'. In case A, $y(\xi) = \tilde{Y}_\gamma(\xi^{2L})$ for the γ with $\tilde{Y}_\gamma(0) = y(0)$. In case B, $y(\xi)$ is the image of such a series under σ_c.*

Proof. By Lemma 4.1, $y(0)$ is balanced, so in case A it is $\tilde{Y}_\gamma(0)$ for a unique γ. Both y and $\tilde{Y}_\gamma(T^2)$ are branches with this leading term, so they agree by Lemma 2.4. Case B follows by σ_c, which maps branches to branches and case B to case A. □

Counterparts of Lemmas 4.1 and 4.2 for germs at positive masses are in [2, §7].

Corollary 4.3. *Let $C \subseteq Y_A$, $A = (1,4,4,4,4)$, be an irreducible curve with a singular place of diagram* 3*. Then C is one of the* 36 *curves of $\mathcal{C}_A$. Conversely, each of the* 36 *curves has singular places of diagram* 3*.*

Proof. We transport C to $Y_{A'}$ by the exchange of bodies 1 and 5, and, by the beginning of this section and the relabellings (23) and (34), we may assume that the place P has type $\mathsf{h}(5;13|24)$. Let y be a branch at P as in Lemma 2.3. The point of C near P has $u_1 = 1$, so it is $\Phi(\xi^L, y(\xi))$. In case A, Lemma 4.2 and §4.1 show that this point lies in $\Lambda(\zeta_1,\zeta_2)$ over $\mathbb{C}((\xi))$. In case B the same holds, since $\sigma_c \circ \Phi = \Phi \circ \sigma_c$ on branches with $u_1 = v_1 = 1$ and σ_c maps $\Lambda(\zeta_1,\zeta_2)$ onto itself (proof of Proposition 3.2(b)). Consequently the affine linear functions that define $\Lambda(\zeta_1,\zeta_2)$ vanish on the expansion of the coordinates of C at P. As the change of parameter is an embedding of fields of Laurent series, they vanish near P, hence on C. Therefore $C \subseteq \mathcal{C}'(\zeta_1,\zeta_2)$. By Theorem 3.1, C is one of its components.

Conversely, on the curves of Theorem 3.1 the side values are constant, p_{13}, p_{15}, p_{35} are multiples of D_1, and p_{24}, p_{25}, p_{45} are multiples of D_2. At a place over $\tau = 0$ or $\tau = \infty$, D_1 and D_2 vanish to the first order in a local parameter, by the orders of h_1 and h_2. The vector $(-\operatorname{val} p_{kl})_{k<l}$ is therefore -1 on the diagonals and the pairs $k5$ and 0 on the sides, a positive multiple of $\mathsf{h}(5;13|24)$. It is not zero, so the place is singular [8, §2.4]. By Lemma 2.1(b) and [8, Table 1], the only types without a positive coordinate are the points $\mathsf{a}_1(T)$, $\mathsf{h}(s;D_1|D_2)$ and $\mathsf{v}(s;A|B)$ of diagrams 2, 3 and 6, and their zero coordinates form a triangle and a disjoint pair, a cycle of four pairs, and two triangles. So the place has diagram 3. □

At the places over $\tau = -1$ and $\tau = -\zeta_2/\zeta_1$, D_1 has a simple pole and D_2 is a unit, so the triangle 135 collapses. Over $\tau = 1$ and $\tau = \zeta_2/\zeta_1$ the triangle 245 collapses. The balancing condition (2.1) holds on each curve as an identity among these three vectors.

5. Places of diagram 6 on the rays through $(1,1,4,4,4)$

Theorem 5.1. *Let m be a positive multiple of a permutation of $(1,1,4,4,4)$. No singular place of an irreducible curve $C \subseteq Y_m$ has diagram* 6 *with a heavy centre and the two light bodies in one wing.*

The statement of [2, Theorem 9.10] covers all positive masses and labellings.

By scaling and relabelling (§2.1) we may take $m = B' = (4, 4, 1, 1, 4)$, with centre 5 and wings $\{1, 2, 5\}$ and $\{3, 4, 5\}$. After the colour exchange σ_c if necessary, 12, 15, 25 are z-edges and 34, 35, 45 are w-edges. The type is $\mathsf{v}(5; 125|345)$: $x = 0$ on the six edges and $x = -2$ on the four cross pairs 13, 14, 23, 24 [8, Table 1]. By (O3) and (O6), (ζ, ω) is $(-2, 2)$ on the z-edges, $(2, -2)$ on the w-edges and $(-2, -2)$ on the cross pairs. Hence the weights of §2.3 are $w(u) = (-1, -1, 1, 1)$, $w(v) = (1, 1, -1, -1)$, $w(p_{kl}) = 0$ on the edges and $w(p_{kl}) = 1$ on the cross pairs. We take $f = v_3$, with $e'_{v_3} = -2$. We write u_k, v_k, p_{kl} for the coordinates of y, with $u_1 = v_3 = 1$, and q_{kl} for $y_{p_{kl}}$ on the cross pairs.

5.1. **The rows.** We have $G = G^0 + T^2G^2 + T^3G^3 + T^4G^4 + T^5G^5$, where T^4 occurs only in the cross rows D_{kl}. We write X^j for the coefficient of T^j in a row X, and coefficients not listed are 0. The edge rows are

$$D_{12} = p_{12}^2(1 - u_2)(v_1 - v_2) - 1, \quad D_{15} = p_{15}^2 v_1 - 1, \quad D_{25} = p_{25}^2 u_2 v_2 - 1,$$
$$D_{34} = p_{34}^2(u_3 - u_4)(1 - v_4) - 1, \quad D_{35} = p_{35}^2 u_3 - 1, \quad D_{45} = p_{45}^2 u_4 v_4 - 1,$$

and the cross rows are

$$\begin{aligned}
D_{13} &= -q_{13}^2 - 1 + T^2 q_{13}^2(u_3 + v_1) - T^4 q_{13}^2 u_3 v_1,\\
D_{14} &= -q_{14}^2 v_4 - 1 + T^2 q_{14}^2(u_4 v_4 + v_1) - T^4 q_{14}^2 u_4 v_1,\\
D_{23} &= -q_{23}^2 u_2 - 1 + T^2 q_{23}^2(u_2 v_2 + u_3) - T^4 q_{23}^2 u_3 v_2,\\
D_{24} &= -q_{24}^2 u_2 v_4 - 1 + T^2 q_{24}^2(u_2 v_2 + u_4 v_4) - T^4 q_{24}^2 u_4 v_2.
\end{aligned}$$

The rows Z_k are

$$\begin{aligned}
Z_1^0 &= 1 - 4(1 - u_2)p_{12}^3 - 8p_{15}^3 - 4u_2 p_{25}^3, & Z_1^2 &= -(u_3 p_{35}^3 + u_4 p_{45}^3),\\
Z_1^3 &= -(q_{13}^3 + q_{14}^3), & Z_1^5 &= u_3 q_{13}^3 + u_4 q_{14}^3,\\
Z_2^0 &= u_2 + 4(1 - u_2)p_{12}^3 - 4p_{15}^3 - 8u_2 p_{25}^3, & Z_2^2 &= Z_1^2,\\
Z_2^3 &= -u_2(q_{23}^3 + q_{24}^3), & Z_2^5 &= u_3 q_{23}^3 + u_4 q_{24}^3,
\end{aligned}$$

$$Z_3^0 = Z_4^0 = -4(p_{15}^3 + u_2 p_{25}^3),$$
$$Z_3^2 = u_3 - (u_3 - u_4)p_{34}^3 - 5u_3 p_{35}^3 - u_4 p_{45}^3, \quad Z_3^3 = 4(q_{13}^3 + u_2 q_{23}^3), \quad Z_3^5 = -4u_3(q_{13}^3 + q_{23}^3),$$
$$Z_4^2 = u_4 + (u_3 - u_4)p_{34}^3 - u_3 p_{35}^3 - 5u_4 p_{45}^3, \quad Z_4^3 = 4(q_{14}^3 + u_2 q_{24}^3), \quad Z_4^5 = -4u_4(q_{14}^3 + q_{24}^3),$$

and the rows W_k are

$$W_1^0 = W_2^0 = -(p_{35}^3 + v_4 p_{45}^3),$$
$$W_1^2 = v_1 - 4(v_1 - v_2)p_{12}^3 - 8v_1 p_{15}^3 - 4v_2 p_{25}^3, \quad W_1^3 = q_{13}^3 + v_4 q_{14}^3, \quad W_1^5 = -v_1(q_{13}^3 + q_{14}^3),$$
$$W_2^2 = v_2 + 4(v_1 - v_2)p_{12}^3 - 4v_1 p_{15}^3 - 8v_2 p_{25}^3, \quad W_2^3 = q_{23}^3 + v_4 q_{24}^3, \quad W_2^5 = -v_2(q_{23}^3 + q_{24}^3),$$

$$\begin{aligned}
W_3^0 &= 1 - (1 - v_4)p_{34}^3 - 5p_{35}^3 - v_4 p_{45}^3, & W_3^2 &= -4(v_1 p_{15}^3 + v_2 p_{25}^3),\\
W_3^3 &= -4(q_{13}^3 + q_{23}^3), & W_3^5 &= 4(v_1 q_{13}^3 + v_2 q_{23}^3),\\
W_4^0 &= v_4 + (1 - v_4)p_{34}^3 - p_{35}^3 - 5v_4 p_{45}^3, & W_4^2 &= W_3^2,\\
W_4^3 &= -4v_4(q_{14}^3 + q_{24}^3), & W_4^5 &= 4(v_1 q_{14}^3 + v_2 q_{24}^3).
\end{aligned}$$

Expanding these expressions, we obtain the following identities.

(I0) $Z_4^0 \equiv Z_3^0$ and $W_2^0 \equiv W_1^0$.

(I1) $Z_1^0 + Z_2^0 - 3Z_3^0 = 1 + u_2$ and $W_3^0 + W_4^0 - 6W_1^0 = 1 + v_4$.

(I2) With $A = u_3 - u_4$, $K_2 = Ap_{34}^3 - 4u_4p_{45}^3$, $B = v_1 - v_2$ and $K_1 = 4Bp_{12}^3 - 4v_2p_{25}^3$,
$$\Phi := Z_4^2 - Z_3^2 = A(W_1^0 - W_3^0) - 4u_4W_1^0 + K_2(1 + v_4),$$
$$\Psi := W_2^2 - W_1^2 = B(Z_3^0 - Z_1^0) - v_2Z_3^0 + K_1(1 + u_2).$$

(I3) The rows Z_k^0, D_{12}, D_{15}, D_{25} involve only u_2, v_1, v_2, p_{12}, p_{15}, p_{25} (*wing 1*), and the rows W_k^0, D_{34}, D_{35}, D_{45} only u_3, u_4, v_4, p_{34}, p_{35}, p_{45} (*wing 2*). The parts Z_k^2, hence Φ, involve only the unknowns of wing 2, and the parts W_k^2, hence Ψ, only those of wing 1.

5.2. **Leading solutions, rank and kernel.** By (I0) and (I3), $G^0 = 0$ splits into three blocks. In wing 1, (I1) gives $u_2 = -1$. Then $Z_3^0 = 0$ forces $p_{25} = \varepsilon_1 p_{15}$ with $\varepsilon_1^3 = 1$, and we put $g_1 = p_{15}$. The rows D_{15} and D_{25} determine $v_1 = g_1^{-2}$ and $v_2 = -\varepsilon_1 g_1^{-2}$. With $h_1 = p_{12}/g_1$, the row D_{12} yields $h_1^2 = 1/(2(1+\varepsilon_1))$. Then $Z_1^0 = 1 - g_1^3(4 + 8h_1^3)$, so $g_1^3(4 + 8h_1^3) = 1$, and $Z_2^0 = 0$ follows from (I1). In wing 2, in the same way, $v_4 = -1$, $p_{45} = \varepsilon_2 g_2$ with $g_2 = p_{35}$ and $\varepsilon_2^3 = 1$, $u_3 = g_2^{-2}$, $u_4 = -\varepsilon_2 g_2^{-2}$, $p_{34} = h_2 g_2$ with $h_2^2 = 1/(2(1+\varepsilon_2))$, and $g_2^3(4 + 2h_2^3) = 1$. The cross rows force $q_{13}^2 = q_{24}^2 = -1$ and $q_{14}^2 = q_{23}^2 = 1$.

If $\varepsilon = 1$, then $h = \pm\frac{1}{2}$. If $\varepsilon \neq 1$, then $1 + \varepsilon = -\varepsilon^2$, $h^2 = -\varepsilon/2$ and $h^3 = \pm\frac{i}{2\sqrt{2}}$. So $4 + 8h_1^3 \in \{4 \pm 1,\ 4 \pm 2\sqrt{2}\,i\}$ and $4 + 2h_2^3 \in \{4 \pm \frac{1}{4},\ 4 \pm \frac{i}{\sqrt{2}}\}$ are nonzero, each wing has $3 \cdot 2 \cdot 3 = 18$ solutions (ε, h, g), and there are $18 \cdot 18 \cdot 16 = 5184$ leading solutions.

Rank. In wing 1 we use the unknowns (x, e, X_1, X_2, h, g) with $u_2 = x$, $p_{15} = g$, $p_{25} = eg$, $p_{12} = hg$, $v_1 = X_1/g^2$ and $v_2 = X_2/g^2$, and the functions
$$1 + x, \quad 1 + xe^3 = Z_3^0/(-4g^3), \quad X_1 - 1 = D_{15},$$
$$e^2xX_2 - 1 = D_{25}, \quad h^2(1-x)(X_1 - X_2) - 1 = D_{12}, \quad Z_1^0.$$
At a leading solution their Jacobian matrix is lower triangular with the diagonal entries 1, $-3e^2$, 1, $-e^2$, $2/h$ and $-3/g$, by $4h(1+\varepsilon) = 2/h$ and $-3g^2(4 + 8h^3) = -3/g$. These functions come from (Z_1^0, Z_2^0, Z_3^0) by a row operation of determinant 1 and the unit factor $-1/(4g^3)$. Wing 2 is the same, with (v_4, e, U_3, U_4, h, g), $u_3 = U_3/g^2$, $u_4 = U_4/g^2$, and the functions $W_3^0 + W_4^0 - 6W_1^0$, $W_1^0/(-g^3)$, D_{35}, D_{45}, D_{34}, W_3^0. The cross rows D_{13}^0, D_{14}^0, D_{23}^0, D_{24}^0 involve the cross unknowns, u_2 and v_4 only, with the diagonal entries $-2q_{13}$, $-2q_{14}v_4$, $-2q_{23}u_2$, $-2q_{24}u_2v_4$. So, for the set R of all rows except Z_4 and W_2, the matrix $J_0[R]$ is block lower triangular with invertible diagonal blocks, and $\operatorname{rank} J_0 = 16$.

Kernel. By (I0), $J_0[Z_4] = J_0[Z_3]$ and $J_0[W_2] = J_0[W_1]$, so the left kernel of J_0 is spanned by $\lambda_1 = Z_4 - Z_3$ and $\lambda_2 = W_2 - W_1$. Since $Z_3, W_1 \in R$, $\eta_{R,n} = (\lambda_1 \cdot \beta_n, \lambda_2 \cdot \beta_n)$. As $Z_4^0 - Z_3^0 \equiv 0$, with $Y = \sum_{k<n} y_k T^k$, $\Theta = Z_4^3 - Z_3^3$ and $\Xi = Z_4^5 - Z_3^5$,
$$\lambda_1 \cdot \beta_n = [T^{n-2}]\,\Phi(Y) + [T^{n-3}]\,\Theta(Y) + [T^{n-5}]\,\Xi(Y), \tag{5.1}$$
and $\lambda_2 \cdot \beta_n$ is the same with Ψ, $\Theta_W = W_2^3 - W_1^3$ and $\Xi_W = W_2^5 - W_1^5$.

5.3. **The obstruction.** We show that at every leading solution some $\eta_{R,n}$ with $n \leq 6$ is nonzero, so that by Lemma 2.4 no branch starts there. Let y_0 be a leading solution and $y_1, y_2, \ldots$ as in §2.3. Since G has no term in T, $y_1 = 0$. By (I2), $\eta_{R,2} = (\Phi(y_0), \Psi(y_0)) = 0$, as W_1^0, W_3^0, Z_1^0, Z_3^0, $1 + v_4$ and $1 + u_2$ vanish at y_0.

Multipliers. By (I1) and (I2), $\Phi = a_{W_1}W_1^0 + a_{W_3}W_3^0 + a_{W_4}W_4^0$ as functions, with $a_{W_1} = A - 4u_4 - 6K_2$, $a_{W_3} = -A + K_2$ and $a_{W_4} = K_2$. At y_0, using $1/g_2^2 = g_2(4 + 2h_2^3)$,
$$a_{W_1} = 4g_2(1 - \varepsilon_2)(1 - h_2^3), \qquad a_{W_3} + a_{W_4} = 4g_2(\varepsilon_2 - 1).$$
Likewise $\Psi = b_{Z_1}Z_1^0 + b_{Z_2}Z_2^0 + b_{Z_3}Z_3^0$, and at y_0
$$b_{Z_3} = 4g_1(1 - \varepsilon_1)(1 - h_1^3), \qquad b_{Z_1} + b_{Z_2} = 4g_1(\varepsilon_1 - 1).$$

Since the W_j^0 vanish at y_0 and $W_1, W_3, W_4 \in R$, $\nabla\Phi \cdot y_n = -\sum_j a_{W_j}\beta_n[W_j]$, and likewise $\nabla\Psi \cdot y_n = -\sum_j b_{Z_j}\beta_n[Z_j]$.

The vector $\beta_2 = G^2(y_0)$. Its entries are $\beta_2[W_1] = \beta_2[W_2] = -4g_1(1-\varepsilon_1)(1-h_1^3)$, $\beta_2[W_3] = \beta_2[W_4] = -4g_1(1-\varepsilon_1)$, $\beta_2[Z_1] = \beta_2[Z_2] = -g_2(1-\varepsilon_2)$, $\beta_2[Z_3] = \beta_2[Z_4] = -g_2(1-\varepsilon_2)(1-h_2^3)$, $\beta_2[D_{13}] = -(u_3+v_1)$, $\beta_2[D_{14}] = v_1-u_4$, $\beta_2[D_{23}] = u_3-v_2$, $\beta_2[D_{24}] = v_2+u_4$, and 0 on the edge rows.

Order 3. $\eta_{R,3} = (\Theta(y_0), \Theta_W(y_0))$. As $q_{13}^3 = -q_{13}$, $q_{14}^3 = q_{14}$, $q_{23}^3 = q_{23}$, $q_{24}^3 = -q_{24}$ and $u_2 = v_4 = -1$,

$$\eta_{R,3} = \Sigma\,(4,1), \qquad \Sigma = q_{13}+q_{14}+q_{23}+q_{24}.$$

Since $q_{13}, q_{24} \in \{\pm i\}$ and $q_{14}, q_{23} \in \{\pm 1\}$, $\Sigma = 0$ if and only if $q_{23} = -q_{14}$ and $q_{24} = -q_{13}$.

Order 4 ($\Sigma = 0$). Here $\eta_{R,4} = (\nabla\Phi \cdot y_2, \nabla\Psi \cdot y_2) = c\,(4,1)$ with

$$c = -4g_1g_2(1-\varepsilon_1)(1-\varepsilon_2)\big[h_1^3 + h_2^3 - h_1^3h_2^3\big].$$

If $\varepsilon_1, \varepsilon_2 \neq 1$, then $h_j^3 = s_j\frac{i}{2\sqrt{2}}$ with $s_j = \pm 1$, and the bracket is $\frac{i(s_1+s_2)}{2\sqrt{2}} + \frac{s_1s_2}{8} \neq 0$.

Order 5 ($\Sigma = 0$, exactly one $\varepsilon_j = 1$). Write $a = q_{13} \in \{\pm i\}$ and $b = q_{14} \in \{\pm 1\}$, so that $q_{23} = -b$, $q_{24} = -a$ and $a \pm b \neq 0$.

- $\beta_3 = G^3(y_0)$ is $(a-b)(1,1,-4,-4)$ on the rows Z_k, $-(a+b)(1,1,-4,-4)$ on the rows W_k, and 0 on the rows D_{kl}.
- By (I1), $y_{2,u_2} = g_2(1-\varepsilon_2)(3h_2^3-1)$, $y_{2,v_4} = 8g_1(1-\varepsilon_1)(3h_1^3-2)$, $y_{3,u_2} = -14(a-b)$ and $y_{3,v_4} = -14(a+b)$.
- $\nabla\Phi \cdot y_3 = 4P$ with $P = (a+b)g_2(1-\varepsilon_2)(5-h_2^3)$, and $\nabla\Psi \cdot y_3 = 4Q$ with $Q = (a-b)g_1(1-\varepsilon_1)(5-4h_1^3)$.
- At y_0, $\nabla\Theta = 12(e_{q_{13}}+e_{q_{14}}+e_{q_{23}}+e_{q_{24}}) + 4(a+b)e_{u_2}$ and $\nabla\Theta_W = 3(e_{q_{13}}+e_{q_{14}}+e_{q_{23}}+e_{q_{24}}) + (a-b)e_{v_4}$, where e_q is the unit vector of the coordinate q. Moreover $\Xi(y_0) = -4(a+b)U$ and $\Xi_W(y_0) = -(a-b)V$, where $U = u_3+u_4 = (1-\varepsilon_2)g_2(4+2h_2^3)$ and $V = v_1+v_2 = (1-\varepsilon_1)g_1(4+8h_1^3)$. Hence $U - y_{2,u_2} = g_2(1-\varepsilon_2)(5-h_2^3)$ and $V - y_{2,v_4} = 4g_1(1-\varepsilon_1)(5-4h_1^3)$.
- The cross rows of R determine $y_{2,p_{13}} = a(u_3+v_1)/2$, $y_{2,p_{14}} = b(y_{2,v_4}+u_4-v_1)/2$, $y_{2,p_{23}} = b(u_3-v_2-y_{2,u_2})/2$ and $y_{2,p_{24}} = -a(y_{2,u_2}+y_{2,v_4}-v_2-u_4)/2$, whose sum is

$$S_2 = \tfrac{a+b}{2}(U - y_{2,u_2}) + \tfrac{a-b}{2}(V - y_{2,v_4}) = \tfrac{P}{2} + 2Q.$$

By (5.1), $\lambda_1 \cdot \beta_5 = \nabla\Phi\cdot y_3 + \nabla\Theta\cdot y_2 + \Xi(y_0) = 4P + 12S_2 - 4(a+b)(U - y_{2,u_2}) = 12S_2$, and in the same way $\lambda_2 \cdot \beta_5 = 3S_2$. Thus $\eta_{R,5} = 3S_2\,(4,1)$. If $\varepsilon_1 = 1 \neq \varepsilon_2$, then $Q = 0$ and $\eta_{R,5} = \frac{3}{2}P\,(4,1)$. If $\varepsilon_2 = 1 \neq \varepsilon_1$, it is $6Q\,(4,1)$. Both are nonzero, since $a \pm b \neq 0$, $|h_2^3| < 5$ and $|4h_1^3| < 5$.

Order 6 ($\Sigma = 0$, $\varepsilon_1 = \varepsilon_2 = 1$). Now β_2 vanishes on all wing rows, so the wing part of y_2 is 0, and $U = V = S_2 = 0$. By (I3), β_4 also vanishes on the wing rows, so $\nabla\Phi \cdot y_4 = \nabla\Psi \cdot y_4 = 0$, and the quadratic terms of Φ and Ψ in y_2 vanish. The cross rows determine $y_{3,p_{13}} = 0$, $y_{3,p_{14}} = by_{3,v_4}/2 = -7ab - 7$, $y_{3,p_{23}} = -by_{3,u_2}/2 = 7ab - 7$ and $y_{3,p_{24}} = -a(y_{3,u_2}+y_{3,v_4})/2 = -14$, with the sum $S_3 = -28$. So

$$\lambda_1 \cdot \beta_6 = \nabla\Theta \cdot y_3 = 12S_3 + 4(a+b)y_{3,u_2} = -224, \qquad \lambda_2 \cdot \beta_6 = 3S_3 + (a-b)y_{3,v_4} = -56,$$

and $\eta_{R,6} = -56\,(4,1)$.

Proof of Theorem 5.1. Every leading solution has $\eta_{R,n} \neq 0$ for some $n \leq 6$: for $n = 3$ at the 3888 leading solutions with $\Sigma \neq 0$, for $n = 4$ at the 576 with $\Sigma = 0$ and $\varepsilon_1, \varepsilon_2 \neq 1$, for $n = 5$ at the 576 with $\Sigma = 0$ and exactly one $\varepsilon_j = 1$, and for $n = 6$ at the remaining 144. By Lemma 2.4 no branch has such a leading solution, and by Lemma 2.3 a place of the stated kind would yield one. □

6. Two lemmas on triangles

Lemma 6.1. *Let P be a singular place of diagram 5 of an irreducible curve $C \subseteq Y_m$, with zw-triangle T. Let z_t°, w_t° be the coefficients of the positions of the bodies $t \in T$ at the exponent 1, defined up to a common translation. Then $w_{tt'}^\circ = b\, z_{tt'}^\circ$ for some $b \neq 0$, the leading coefficients of the $p_{tt'}$ are $p_{tt'}^\circ = s_{tt'} b^{-1/2}/z_{tt'}^\circ$ with signs $s_{tt'} = -s_{t't}$, and for every $t \in T$*

$$(C_t) \qquad \sum_{t' \in T \setminus t} \frac{m_{t'} s_{tt'}}{(z_{tt'}^\circ)^2} = 0.$$

If the masses of T are $(1,4,4)$, the light body is the midpoint of the other two, in z° and in w°. If the masses of $T = \{M,h,L\}$ are $(9,36,4)$, then $z_L^\circ - z_M^\circ = \frac{1}{3}(z_h^\circ - z_M^\circ)$, and likewise in w°.

Proof. The first assertion is proved as in [1, §5.3]. The pairs of T are zw-edges, with $\zeta = \omega = 1$, so $e(z_{tt'}) = e(w_{tt'}) = 1$ and $e(Z_{tt'}) = e(W_{tt'}) = -2$. No body is circled, and the only strokes are those of T. In the equation $z_t = \sum_{l \neq t} m_l Z_{tl}$ of a body $t \in T$, the left side and the terms with $l \notin T$ have exponents larger than -2. Comparing the coefficients of the exponent -2, we obtain $(\Psi z^\circ)_t = 0$, where $(\Psi\xi)_t = \sum_{t' \in T} m_{t'} (p_{tt'}^\circ)^3 (\xi_t - \xi_{t'})$. In the same way $\Psi w^\circ = 0$, and $\Psi \mathbb{1} = 0$. The matrix Ψ is not zero, so $\mathbb{1}$, z°, w° are linearly dependent in $\mathbb{C}^3$. Since $w_{tt'}^\circ \neq 0$, it follows that $w_{tt'}^\circ = b\, z_{tt'}^\circ$ with $b \neq 0$. The rows $D_{tt'}$ become $(p_{tt'}^\circ)^2 b\, (z_{tt'}^\circ)^2 = 1$, which proves the formula for $p_{tt'}^\circ$, with antisymmetric signs since $p_{tt'}$ is symmetric and $z_{tt'}^\circ$ is antisymmetric. Substituting it into $\Psi z^\circ = 0$, we obtain (C_t).

For the masses $(1,4,4)$ let l be the light body and h, h' the others, and normalize $z_{lh}^\circ = 1$, $z_{lh'}^\circ = \rho$. Then (C_l) reads $\rho^2 = \pm 1$, and (C_h) reads $(\rho - 1)^2 = \pm 4$. Among $\rho \in \{\pm 1, \pm i\}$ only $\rho = -1$ satisfies the second condition, since $(\pm i - 1)^2 = \mp 2i$ and $\rho = 1$ would give $z_{hh'}^\circ = 0$. So $z_{lh}^\circ = -z_{lh'}^\circ$. For the masses $(9,36,4)$ normalize $z_M^\circ = 0$, $z_h^\circ = 1$, $z_L^\circ = \rho$. Then (C_M) reads $\rho^2 = \pm\frac{1}{9}$, and (C_h) reads $(1-\rho)^2 = \pm\frac{4}{9}$. Among $\rho \in \{\pm\frac{1}{3}, \pm\frac{i}{3}\}$ only $\rho = \frac{1}{3}$ satisfies the second condition. In both cases the statement for w° follows from $w_{tt'}^\circ = b\, z_{tt'}^\circ$. □

Lemma 6.2. *No singular place of an irreducible curve $C \subseteq Y_m$ has diagram 7 with a z-triangle or a w-triangle of three equal masses.*

Proof. By the colour exchange σ_c we may assume that the triangle of equal masses is the z-triangle 125. By Lemma 2.1(b) the type is $\mathsf{v}_7(a,b)$ with $0 < a, b < 1$: $x = a$ on 125, $x = b$ on 345 and $x = \frac{3}{2}(a+b) - 2$ on the cross pairs, where $(\zeta, \omega) = (3a-2, 3b-2)$ [8, Appendix A]. By (O3), $(\zeta, \omega) = (3a-2, 2-a)$ on the pairs of 125, so there $e(Z) = -2$ and $e(W) = -\frac{1}{2}(3\zeta + \omega) = 2 - 4a$. On the cross pairs $e(Z) = 4 - \frac{3}{2}a - \frac{9}{2}b$ and $e(W) = 4 - \frac{9}{2}a - \frac{3}{2}b$, and

$$4 - \tfrac{3}{2}a - \tfrac{9}{2}b > -2, \qquad 4 - \tfrac{9}{2}a - \tfrac{3}{2}b > 2 - 4a, \tag{6.1}$$

because $a + 3b < 4$. There are no circles. By (6.1), the coefficients of the exponent -2 in the equations of bodies 1 and 2 therefore yield $Z_{12}^\circ : Z_{25}^\circ : Z_{51}^\circ = m_5 : m_1 : m_2$. We write the leading coefficients of w_{12}, w_{25}, w_{51} as $\kappa\alpha/m_5$, $\kappa\beta/m_1$, $\kappa\gamma/m_2$. From $p_{kl}^2 z_{kl} w_{kl} = 1$ we have $z_{kl} = 1/(w_{kl}^3 Z_{kl}^2)$ and $W_{kl} = 1/(z_{kl}^2 w_{kl}^2 Z_{kl})$. We obtain three relations.

- (L) $w_{12} + w_{25} + w_{51} = 0$ gives $\alpha/m_5 + \beta/m_1 + \gamma/m_2 = 0$.
- (C) $z_{12} + z_{25} + z_{51} = 0$ becomes $m_5/\alpha^3 + m_1/\beta^3 + m_2/\gamma^3 = 0$.
- (Q) In $w_{12} = (m_1 + m_2)W_{12} + m_5(W_{15} - W_{25}) + (\text{cross terms})$, the left side has the exponent $2 - a > 2 - 4a$, and the cross terms have larger exponents than $2 - 4a$ by (6.1). Hence $(m_1 + m_2)\alpha^4/m_5 = m_5(\beta^4/m_1 + \gamma^4/m_2)$.

Let the three masses be equal, and let e_1, e_2, e_3 be the elementary symmetric functions of α, β, γ. Then (L) gives $e_1 = 0$. Multiplied by $(\alpha\beta\gamma)^3$, (C) reads $(\alpha\beta)^3 + (\beta\gamma)^3 + (\gamma\alpha)^3 = e_2^3 - 3e_1e_2e_3 + 3e_3^2 = e_2^3 + 3e_3^2 = 0$. By Newton's identities with $e_1 = 0$, $\alpha^4 + \beta^4 + \gamma^4 = 2e_2^2$,

so (Q) reads $3\alpha^4 = 2e_2^2$. If $e_2 = 0$, then $e_3 = 0$ and $\alpha = \beta = \gamma = 0$, which is impossible. Otherwise, since the relations are homogeneous, we may take $e_2 = -3$. Then $e_3^2 = 9$ and $\alpha^4 = 6$. From $\alpha^3 = 3\alpha + e_3$ we get $3\alpha^2 + e_3\alpha = 6$, and multiplying by α and using the cubic again, $e_3\alpha^2 + 3\alpha + 3e_3 = 0$. Eliminating α^2, we obtain $(9 - e_3^2)\alpha + 15e_3 = 0$, so $e_3 = 0$, which contradicts $e_3^2 = 9$. □

7. Proofs of Theorems 1.1 and 1.2

In this section $A = (1, 4, 4, 4, 4)$ and $B = (1, 1, 4, 4, 4)$, up to positive scaling and permutation of the bodies (§2.1). At A we write L for the body of mass 1 and h, h' for heavy bodies. At B we write l, l' for the bodies of mass 1 and h, h', h'' for the heavy ones. We use the numbering of the bodies of [1, Fig. 11] for each diagram, so that a *labelling* assigns the masses to the bodies $1, \dots, 5$ of the figure.

7.1. **The mass conditions.** The conditions of [1, §8] that we use are (23) for diagrams 1, 4 and 5 on the triangle $T = \{3, 4, 5\}$, (29) for diagram 3, (32) for diagram 6, (36) for diagram 10 and (40) for diagrams 8, 12 and 13 [8, §2.6 and proof of Theorem 3.12] and Lemma 2.1(c). For diagram 11, [1, §§8.10–8.13] derive both (36) and (40), but we use only that one of them holds. They are evaluated directly from $1^{-1/2} = 1$ and $4^{-1/2} = \frac{1}{2}$.

Lemma 7.1. (a) *Condition* (23)*, one of* $m_3^{-1/2}, m_4^{-1/2}, m_5^{-1/2}$ *being the sum of the other two, holds if and only if the masses of* T *are* $1, 4, 4$*: at* A *if and only if* $L \in T$*, and at* B *if and only if* $T = \{l, h, h'\}$*. Condition* (36) *is the same condition on* $\{1, 2, 5\}$*.*

(b) *Condition* (29)*,* $m_1 m_3 = m_2 m_4$ *for diagram* 3 *with centre* 5 *and diagonals* 13*,* 24*, holds at* A *if and only if the centre is* L*, and at* B *if and only if the centre is heavy and the diagonals are* $\{l, h\}$ *and* $\{l', h'\}$*.*

(c) *Condition* (32)*,* $m_1 = m_2$ *and* $m_3 = m_4$ *for diagram* 6 *with centre* 5 *and wings* $\{1, 2, 5\}$*,* $\{3, 4, 5\}$*, holds at* A *if and only if the centre is* L*, and at* B *if and only if the centre is heavy and the two light bodies lie in one wing.*

(d) *Condition* (40)*,* $m_5^2(m_1 + m_2)^4 = m_1^3 m_2^3$*, holds if and only if body* 5 *is light and bodies* 1 *and* 2 *are heavy.*

Proof. (a) $\frac{1}{2} = \frac{1}{2} + \frac{1}{2}$ fails, $1 = \frac{1}{2} + \frac{1}{2}$ holds, and $1 = 1 + \frac{1}{2}$ fails. (b) and (c) follow by comparing the masses of the two diagonals or of the two wings. (d) The condition holds for $(m_5; m_1, m_2) = (1; 4, 4)$, since $8^4 = 4^6$, and fails for $(4; 4, 4)$, $(1; 1, 4)$, $(4; 1, 4)$ and $(4; 1, 1)$: $16 \cdot 8^4 \neq 4^6$, $5^4 \neq 4^3$, $16 \cdot 5^4 \neq 4^3$ and $16 \cdot 2^4 \neq 1$. □

7.2. **Two functionals.** The proofs of Theorems 1.1 and 1.2 apply Lemma 2.2 to the following functionals. At A let $s_1(x)$ be the sum of the coordinates x_{kl} over the four pairs that contain L and $s_2(x)$ the sum over the six heavy pairs, and put

$$f = 3s_1 - 2s_2.$$

At B let $c(x)$ be the sum over the six pairs of a light and a heavy body, $w(x)$ the sum over the three heavy pairs, and put

$$\varphi = c - w - 2x_{ll'}.$$

With these coefficients, f is negative on the types of diagram 3 with centre L, zero on the type of diagram 16 and positive on every other type that is not excluded, and φ is positive on every type that is not excluded. The next proposition lists their values on the types of singular places. It uses Lemma 2.1(b) for the types and [8, Lemma 3.6] for diagrams 1 and 4. In the table, a, b, c and t are the parameters of the types in Lemma 2.1(b), with $0 < a, b < 1$, $0 < c < 1$ for diagram 10, $0 \le c < 1$ for diagram 14, and $0 < t < 1$. For diagram 14, j is the isolated body,

and the three values are those of $\mathsf{u}_c(j)$, of $\sigma_k(j)(t)$ with $k = L$, resp. k light, and of $\sigma_k(j)(t)$ with k heavy.

Proposition 7.2. *Let C be an irreducible curve in Y_A, resp. Y_B, and P a singular place of C with a diagram other than 5. Then the diagram and the labelling of P are among those of Table 1, and $f(x(P))$, resp. $\varphi(x(P))$, is as listed. Diagrams 2, 9 and 15 do not occur.*

diagram and labelling	value	excluded by
at A, the value of f		
1; 4: $L \in T$	$a+4$; 5	
3: centre L	-16	Corollary 4.3, unless $C \subseteq \mathcal{C}_A$
6: centre L	16	
7: centre L	$16-8(a+b)$	
7: heavy centre		Lemma 6.2
8: centre L	$16-8a$	
10: $L \in \{1,2\}$; centre L	$3-3c$; $8-8c$	
11–13: $L \in \{1,2\}$ (diagram 11); centre L	3; 8	
14: j heavy	$3-3c$; $3+3t$; $3+\frac{t}{2}$	
14: $j = L$		Lemma 2.1(c), [8, Lemma 9.5], $\nu = 1$
16	0, and $\sum_{k<l} x_{kl} = 10$	
at B, the value of φ		
1; 4: $T = \{l, h, h'\}$	$2-\frac{a}{2}$; $\frac{3}{2}$	
3: heavy centre, diagonals $\{l,h\}$, $\{l',h'\}$		Lemma 2.1(c), [8, Theorem 3.9]
6: heavy centre, both light bodies in one wing		Theorem 5.1
7: heavy centre, a wing of three heavy bodies		Lemma 6.2
7: heavy centre, wings $\{h,l,h'\}$, $\{h,l',h''\}$	$2-\frac{a+b}{2}$	
7: light centre	a or b	
8: light centre	a	
10: heavy centre, $\{1,2\} = \{l,h'\}$; light centre	$\frac{3}{2}-\frac{c}{2}$; 1	
11–13: heavy centre, $\{1,2\} = \{l,h'\}$ (diagram 11); light centre	$\frac{3}{2}$; 1	
14: j heavy	1; $1+\frac{t}{2}$; $1-\frac{t}{2}$	
14: j light		Lemma 2.1(c), [8, Lemma 9.5], $\nu = \frac{1}{4}$
16	1	

TABLE 1. The values of f at A and of φ at B on the types of singular places, by diagram and labelling. The last column names the result that excludes the place.

Proof. By Lemma 2.1(c), Proposition 7.1, Theorem 5.37 and Corollary 5.19 of [8] exclude diagrams 2, 9 and 15. By Lemma 7.1 the labellings are those of the table, except for diagram 7, for which we use no condition. At A the wing of a diagram 7 with a heavy centre that does not contain L consists of three heavy bodies, and at B the wings of a diagram 7 with a heavy centre either are $\{h,l,h'\}$ and $\{h,l',h''\}$ or include a wing of three heavy bodies. We obtain the values by evaluating f and φ on the types. For example, diagram 1 at A has the type $x = a$ on $T = \{L,h,h'\}$, $x = 0$ on the other pair and $x = \frac{3}{2}a - 2$ on the six pairs between them, by [8, Lemma 3.6]. So $s_1 = 5a-4$, $s_2 = 7a-8$ and $f = a+4$. At a place of diagram 14 with $j = L$ at A, or with j light at B, the cluster has the masses $(4,4,4,4)$, resp. $(1,4,4,4)$, so its leading coefficients would solve K14$(1,1,1,\nu)$ with $\nu = 1$, resp. $\nu = \frac{1}{4}$, up to scaling and relabelling

(§2.2). For diagram 3 at B, [8, Theorem 3.9] and Lemma 2.1(c) require that one of $m_l^{-3/2} = 1$, $m_h^{-3/2} = \frac{1}{8}$ and $m_s^{-1} M_{Q_s}^{-1/2} = \frac{1}{4\sqrt{10}}$ be the sum of the other two, where s is the centre. None of them is. □

7.3. **Diagram 5.** At a place of diagram 5 the bodies A, B outside the zw-triangle T have no strokes. So each of them is in one of the modes of Lemma 2.1(b) relative to T: F with parameter c, Z1 with parameter $\bar\omega$, Zs_{t_0} with parameters s and $\bar\omega$, and their mirrors W1 and Ws_{t_0}, where $c < 1$, $\bar\omega < 1$, $1 < s < 2$, $s + \bar\omega < 2$, $2s + \bar\omega < 3$ and $2s + 3\bar\omega < 5$.

Lemma 7.3. *At a place of diagram* 5 *at* A, $f(x(P)) > 0$.

Proof. By Lemma 7.1(a), $T = \{L, h, h'\}$, and the other two bodies a, b are heavy. Then

$$f = 4 + g_a + g_b - 2x_{ab}, \qquad g_X = 3x_{XL} - 2x_{Xh} - 2x_{Xh'}.$$

In the modes F, Z1, Zs_L and Zs_h, g_X equals $-c$, $-\frac{1+\bar\omega}{2}$, $\frac{3s-4-\bar\omega}{2}$ and $\frac{1-2s-\bar\omega}{2}$, and the mirror modes give the same values. Each of these is larger than -1, by $c < 1$, $\bar\omega < 1$, $s > 1$ and $2s + \bar\omega < 3$. Since $x_{ab} \le 1$ by (O1), $f > 0$. □

Lemma 7.4. *At a place of diagram* 5 *at* B, $\varphi(x(P)) > 0$.

The proof, a case analysis over the modes of the two outer bodies, is given in Appendix A.

7.4. **The proofs.**

Proof of Theorem 1.1. Let $C \subseteq Y_A$ be an irreducible curve that is not one of the 36 curves of $\mathcal{C}_A$. By Corollary 4.3, no singular place of C has diagram 3. By Proposition 7.2 and Lemma 7.3, $f(x(P)) \ge 0$ at every singular place P, with equality only for diagram 16, where $\sum_{k<l} x_{kl} = 10$. This contradicts Lemma 2.2, applied to $(f, \sum_{k<l} x_{kl})$. Hence every irreducible curve in Y_A is one of the 36 curves.

Let Z be an irreducible component of Y_A of dimension at least 2. The 36 curves meet Z in a proper closed subset. Through a point of Z outside them there is an irreducible curve in Z, which is not one of the 36, a contradiction. Therefore every irreducible component of Y_A is one of the 36 curves or a point, and $Y_A = \mathcal{C}_A \cup F$ with F finite. By Theorem 3.1 the curves are pairwise disjoint, 18 of genus 3 and 18 rational. By §3.1 they contain the modified Roberts continuum, and by Proposition 3.2 no point of them corresponds to a positive normalized central configuration. □

Proof of Theorem 1.2. Let $C \subseteq Y_B$ be an irreducible curve. By Proposition 7.2 and Lemma 7.4, $\varphi(x(P)) > 0$ at every singular place P of C. This contradicts Lemma 2.2. Hence Y_B contains no irreducible curve, and it is finite. □

8. The mass point P_1

In this section $m = P_1 = (9, 36, 36, 36, 4)$, which is enough by scaling and relabelling (§2.1). We write M for the body of mass 9, h, h', h'' for the bodies of mass 36, and L for the body of mass 4. Let $C \subseteq Y_m$ be an irreducible curve on which $U \not\equiv 0$. Then all results of [8] apply to C, in particular [8, Lemma 3.3], the types of [8, Table 1] with the ranges given there, and [8, Theorem 3.12] with [8, Table 2] for diagram 7. We show that no such curve exists. We use the labels of [8, §3.4] for the conditions of [1, §8]: $\mathsf{E}(T)$ is condition (23) on a triangle T, $\mathsf{H}(D_1|D_2)$ is the product condition (29) for the diagonals D_1, D_2, $\mathsf{D6}(ab|cd)$ is condition (32), $m_a = m_b$ and $m_c = m_d$, and $\mathsf{R}(ab; s)$ is the condition $m_s^2(m_a + m_b)^4 = m_a^3 m_b^3$ of (40).

Lemma 8.1. *With* $x_i = m_i^{-1/2}$, *that is* $x_M = \frac{1}{3}$, $x_h = \frac{1}{6}$, $x_L = \frac{1}{2}$:

(a) $\mathsf{E}(T)$ *holds if and only if* $M \in T$;

(b) $\mathsf{H}(D_1|D_2)$ *holds for no centre and no diagonals, and* $\mathsf{D6}(ab|cd)$ *for no disjoint pairs;*
(c) $\mathsf{R}(ab;s)$ *holds if and only if* $s = M$ *and* a, b *are heavy.*

Proof. (a) $\frac{1}{3} = \frac{1}{6} + \frac{1}{6}$ on $\{M, h, h'\}$ and $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ on $\{M, h, L\}$. On $\{h, h', h''\}$, $\frac{1}{6} \neq \frac{1}{3}$, and on $\{h, h', L\}$, $\frac{1}{2} \neq \frac{1}{3}$ and $\frac{1}{6} \neq \frac{2}{3}$. (b) For the centre L the products of the pairs are $9 \cdot 36$ and 36^2, for the centre M they are $36 \cdot 4$ and 36^2, and for a heavy centre $9 \cdot 36$ and $36 \cdot 4$, or $9 \cdot 4$ and 36^2. Only the heavy bodies have equal masses, and they do not form two disjoint pairs. (c) $\mathsf{R}(hh'; M)$ holds, since $81 \cdot 72^4 = 36^6$. Up to permutations of the heavy bodies the other labels are $\mathsf{R}(hh'; L)$, $\mathsf{R}(Mh; L)$, $\mathsf{R}(Mh'; h)$, $\mathsf{R}(h'h''; h)$, $\mathsf{R}(h'L; h)$, $\mathsf{R}(ML; h)$ and $\mathsf{R}(hL; M)$, and they fail: $16 \cdot 72^4 \neq 36^6$, $16 \cdot 45^4 \neq 9^3 \cdot 36^3$, $36^2 \cdot 45^4 \neq 9^3 \cdot 36^3$, $36^2 \cdot 72^4 \neq 36^6$, $36^2 \cdot 40^4 \neq 36^3 \cdot 4^3$, $36^2 \cdot 13^4 \neq 9^3 \cdot 4^3$ and $81 \cdot 40^4 \neq 36^3 \cdot 4^3$. □

As in §7, we apply Lemma 2.2, now to the functionals

$$f_1 = s_{MH} - s_{HH}, \qquad g = \tfrac{3}{2} x_{ML} - s_{HH},$$

where s_{MH} is the sum of the coordinates x_{kl} over the three pairs Mh and s_{HH} the sum over the three pairs of heavy bodies.

Proposition 8.2. *Let* P *be a singular place of* C *with a diagram other than* 5. *Then* $f_1(x(P)) \geq 0$, *with equality only at places of diagram* 10 *or* 11 *whose* zw*-triangle is* $\{h, M, L\}$, *with centre* h. *At these places* $g(x(P)) = \frac{3}{2} - 3c \geq \frac{1}{2}$ *for diagram* 10 *and* $g(x(P)) = \frac{3}{2}$ *for diagram* 11.

Proof. Diagrams 2, 9 and 15 do not occur by [8, Proposition 7.1, Theorem 5.37 and Corollary 5.19], and diagram 16 does not occur by [8, Lemma 3.3]. By Lemma 8.1(b) and [1, §8], diagrams 3 and 6 do not occur. A place of diagram 14 with isolated body M, resp. L, would give a solution of K14 at the cluster masses $(36, 36, 36, 4)$, resp. $(9, 36, 36, 36)$, that is, of $\mathrm{K14}(1, 1, 1, \nu)$ with $\nu = \frac{1}{9}$, resp. $\nu = \frac{1}{4}$, up to scaling and relabelling [8, Theorem 3.13]. By [8, Lemma 9.5] neither exists. For diagram 7 with centre h and wings $\{h, M, L\}$, $\{h, h', h''\}$, Lemma 6.2 applies. On the remaining types, f_1 takes the following values, by Lemma 8.1, [8, Table 1, Lemma 3.6 and Table 2] and the conditions of §7.1. We write a', b' for the parameters of diagram 7, LA for the line $4a' + 3b' = 4$ and LB for $3a' + 4b' = 4$.

- Diagram 1: $f_1 = \frac{4-a}{2}$ for $T = \{M, h, h'\}$ and $f_1 = a$ for $T = \{M, h, L\}$, with $0 < a < 1$. Diagram 4: $f_1 = \frac{3}{2}$, resp. 1.
- Diagram 7 with centre M and wings $\{M, h, h'\}$, $\{M, h'', L\}$, where we write A for the z-wing and B for the w-wing: the labels $\mathsf{E}(A)$, $\mathsf{E}(B)$ hold, and R holds on the wing of three bodies M, h, h' only. By [8, Table 2] the type lies on or above LA, resp. LB, if that wing is the z-wing, resp. the w-wing, and $f_1 = 4 - 2(a' + b') \geq \frac{4}{3}$, as $a' + b' \leq \frac{4}{3}$.
- Diagram 7 with centre L and wings $\{L, M, h\}$, $\{L, h', h''\}$, or with centre h and wings $\{h, M, h'\}$, $\{h, h'', L\}$: only the label E of the wing that contains M holds. If it is the z-wing, the type lies on LB strictly above LA, so $a' > \frac{4}{7}$ and $f_1 = a' - b' = \frac{7}{4}a' - 1 > 0$. If it is the w-wing, the mirror statement shows that $f_1 = b' - a' > 0$.
- Diagram 8: by (40) the centre is M and bodies 1 and 2 are heavy, and $f_1 = 4 - 2a > 0$.
- Diagrams 10 and 11: by (36), resp. (36) or (40), the zw-triangle $\{1, 2, 5\}$ contains M. With $0 \leq c \leq \frac{1}{3}$, and $c = 0$ for diagram 11, f_1 is 0 if the zw-triangle is $\{h, M, L\}$ with centre h, $1 - c$ if the centre is L or the zw-triangle is $\{M, h, h'\}$ with a heavy centre, and $2 - 2c$ if the centre is M. In the first case $x_{ML} = 1$ and $s_{HH} = 3c$, so $g = \frac{3}{2} - 3c$.
- Diagrams 12 and 13: by (40) the centre is M and bodies 1 and 2 are heavy, and $f_1 = 2$.
- Diagram 14 with a heavy isolated body: $f_1 \geq 1$ on $\mathsf{u}_0(h)$ and on every $\sigma_k(h)(t)$.

These values are nonnegative, and zero only in the stated case. □

Lemma 8.3. *At a place of diagram* 5 *of* C, $f_1(x(P)) > 0$.

The proof is a case analysis over the modes of the two outer bodies, like that of Lemma 7.4, and we give it in Appendix B.

Proof of Theorem 1.3. By scaling and relabelling we may take $m = P_1$. Suppose that $C \subseteq Y_m$ is an irreducible curve with $U \not\equiv 0$. By Proposition 8.2 and Lemma 8.3, $f_1(x(P)) \geq 0$ at every singular place P of C, and $g(x(P)) > 0$ where $f_1(x(P)) = 0$. This contradicts Lemma 2.2. Hence $U \equiv 0$ on every irreducible curve in Y_m. □

Theorem 1.3 says that P_1 is settled. The points of Y_m with $U \neq 0$ form a finite set: every irreducible component of Y_m of positive dimension is a union of irreducible curves, on each of which $U \equiv 0$. The positive normalized central configurations lie in this finite set, since $U > 0$ there.

9. Places of diagrams 1 and 4

In this section $m \in (\mathbb{R}_{>0})^5$ is arbitrary and $C \subseteq Y_m$ is an irreducible curve. At a place of diagram 1 or 4 with triangle T, condition (23) of [1] is $\mathsf{E}(T)$, a hypersurface in the masses. The torque identity of [1, §5.0] yields a second relation, which also involves the masses of the other two bodies.

We use the polynomials

$$A_a = s^6(s+2)(1-s)(1+s)^6, \qquad A_b = (2s+1)(1-s)(1+s)^6, \qquad A_c = s^6(s+2)(2s+1),$$

and $P_{14} = A_c - A_a - A_b$ and $P_{28} = (A_b - A_a - A_c)^2 - 4A_aA_c$, which lie in $\mathbb{Z}[s]$ and have degrees 14 and 28. Let

$$\Xi(X) = \sum_{k=0}^{28} c_{2k} X^{2k}, \qquad c_{56-2k} = c_{2k},$$

with

$$\begin{aligned}(c_0, c_2, \dots, c_8) &= (1,\ 40,\ -926,\ -23224,\ 512795),\\ (c_{10}, c_{12}, \dots, c_{18}) &= (-2049680,\ -31884,\ 8113392,\ -28533847,\ -13730312),\\ (c_{20}, c_{22}, \dots, c_{28}) &= (36197566,\ 13354904,\ 15851835,\ -139882848,\ -47991080).\end{aligned}$$

Lemma 9.1. †

(a) *In* $(0,1)$ *the polynomial* P_{14} *has exactly one root* s_1*,* P_{28} *has exactly two roots* $s_2 < s_3$*, and* $A_a - A_b - A_c$ *has none. The roots lie in the intervals*

$$\begin{aligned}s_1 &\in [0.976893651953910,\ 0.976893651953911],\\ s_2 &\in [0.837339342761276,\ 0.837339342761277],\\ s_3 &\in [0.988353917708555,\ 0.988353917708556].\end{aligned}$$

(b) $\operatorname{Res}_s\big(P_{14},\ A_b(X^2-1)^2 - 4X^2A_a\big) = -81\,\Xi$*, and* Ξ *is irreducible over* $\mathbb{Q}$*.*

(c) *Let* $y_1 = A_a(s_1)/A_b(s_1) = s_1^6(s_1+2)/(2s_1+1)$ *and* $\xi_1 = \sqrt{y_1} + \sqrt{1+y_1}$*. Then* ξ_1 *is the only root of* Ξ *in* $(1,\infty)$*, and* $\xi_1 \in [2.3055544026389899254,\ 2.3055544026389899255]$*. In particular,* ξ_1 *is algebraic of degree* 56*.*

Proof. (c) Since $1/\xi_1 = \sqrt{1+y_1} - \sqrt{y_1}$, we have $(\xi_1 - 1/\xi_1)^2 = 4y_1$, so $A_b(s_1)(X^2-1)^2 - 4X^2A_a(s_1)$ vanishes at $X = \xi_1$. As P_{14} is monic, the resultant of (b) is the product of the polynomials $A_b(\sigma)(X^2-1)^2 - 4X^2A_a(\sigma)$ over the roots σ of P_{14}. Hence $\Xi(\xi_1) = 0$, and $\xi_1 > 1$. The other statements are exact computations over $\mathbb{Q}$: the resultant and the factorization of Ξ over $\mathbb{Q}$; the numbers of roots of P_{14}, P_{28} and $A_a - A_b - A_c$ in $(0, \frac{9}{10})$ and in $(\frac{9}{10}, 1)$, and of Ξ in $(1,\infty)$, counted by Sturm sequences; and the signs at the ends of the stated intervals. □

Theorem 9.2. *Let P be a singular place of C of diagram 1 or 4, with triangle T, and let $\{d,e\}$ be the complement of T.*

(a) *The leading coefficients of the p_{kt} satisfy $\sum_{k\in\{d,e\}}\sum_{t\in T} m_k m_t \hat{p}_{kt} = 0$.*

(b) *$\mathsf{E}(T)$ holds. Write $T=\{a,b,c\}$ with $x_c = x_a + x_b$, where $x_t = m_t^{-1/2}$, and $m_a \le m_b$, and put $s = x_b/x_a$. Then $s<1$, and either $s = s_1$ and $m_e/m_d \in \{\xi_1^{2/3}, \xi_1^{-2/3}\}$, or $s \in \{s_2, s_3\}$ and $m_d = m_e$.*

In terms of the masses, the conditions of (b) say that $(m_a, m_b, m_c, m_d, m_e)$ is a positive multiple of

$$(\lambda,\ \lambda s^{-2},\ \lambda(1+s)^{-2},\ 1,\ \theta), \qquad \lambda>0, \quad (s,\theta) \in \big\{(s_1,\xi_1^{2/3}),\ (s_1,\xi_1^{-2/3}),\ (s_2,1),\ (s_3,1)\big\},$$

where $\xi_1^{2/3} = 1.745220274547\ldots$. Thus at positive masses the places of diagrams 1 and 4 occur only for masses on finitely many curves in the projective space of masses, three up to relabelling of the bodies, while $\mathsf{E}(T)$ defines a hypersurface. In particular, no such place exists if two masses of T are equal. This is the case at $(1,4,4,4,4)$ and $(1,1,4,4,4)$, where $\mathsf{E}(T)$ holds only for triangles with the masses 1, 4, 4.

Proof. We write a' for the parameter a of the type of a place of diagram 1 [8, Table 1], to keep a for a body, and put $a'=1$ for diagram 4. Let $\mu = 3a'-2$, so that $\mu \le 1$. After an exchange of the colours if necessary, de is a z-edge with z-circles d and e, and T is a w-triangle (diagram 1) or a zw-triangle (diagram 4) without circles. By Lemma 2.1(b), [8, Lemma 3.6] and (O3), (O6), (ζ_{kl},ω_{kl}) is $(2-a',\mu)$ on the pairs of T, $(-2,\mu)$ on the six pairs kt with $k\in\{d,e\}$ and $t\in T$, and $(-2,2)$ on de. Since d and e are z-circled and no body of T is, $e(z_d) = e(z_e) = -2 < e(z_t)$ for $t\in T$ [8, Lemma 3.1]. As in the proof of [8, Lemma 3.6], Proposition 1 of [1] with the colours exchanged shows that $e(w_d), e(w_e) \ge 2$, since $\omega_{de} = 2 > \mu = \omega_{te}$ and $\zeta_{de} = \zeta_{te} = -2$. Hence $e(w_t) = \mu$ and $\hat{w}_t = -\hat{w}_{dt} \ne 0$ for $t\in T$, the leading coefficients of z_{kt} and w_{kt} are $\hat{z}_k$ and $-\hat{w}_t$ for $k\in\{d,e\}$, and the rows D_{kt} become

$$\hat{p}_{kt}^2\, \hat{z}_k\, \hat{w}_t = -1 \qquad (k\in\{d,e\},\ t\in T). \tag{9.1}$$

(a) We multiply the equations $z_k = \sum_{l\ne k} m_l p_{kl}^3 z_{kl}$ and $w_k = \sum_{l\ne k} m_l p_{kl}^3 w_{kl}$ of [8, (2.1)] by $m_k w_k$ and by $m_k z_k$, subtract, and sum over $k\in\{d,e\}$. The terms with $l\in\{d,e\}$ cancel in pairs, and we obtain the torque identity of [1, §5.0],

$$\textstyle\sum_{k\in\{d,e\}}\sum_{t\in T} m_k m_t p_{kt}^3\,(z_k w_t - w_k z_t) = 0.$$

The terms $p_{kt}^3 z_k w_t$ have the exponent $\frac{3}{2}(2-\mu)+\mu-2 = 1-\frac{\mu}{2}$ and, by (9.1), the leading coefficients $-\hat{p}_{kt}$. The terms $p_{kt}^3 w_k z_t$ have larger exponents, as $e(w_k z_t) > 0 > \mu - 2 = e(z_k w_t)$. Comparing the coefficients of the exponent $1-\frac{\mu}{2}$, we obtain (a).

(b) Let $t\in T$. In the equations $z_t = \sum_{l\ne t} m_l Z_{tl}$ and $w_t = \sum_{l\ne t} m_l W_{tl}$, where $Z_{tl} = p_{tl}^3 z_{tl}$ and $W_{tl} = p_{tl}^3 w_{tl}$, the terms with $l\in T$ have the exponents $2-4a'$ and -2, and those with $l\in\{d,e\}$ the larger exponents $4-\frac{9}{2}a'$ and $4-\frac{3}{2}a'$. Also $e(w_t) = \mu > -2$ and $e(z_t) > 2-4a'$. For the latter, let z_T be the centre of mass of T. Then $M_T(z_t - z_T) = \sum_{t'\in T} m_{t'} z_{tt'}$ has an exponent at least $2-a'$, and $M_T z_T = \sum_{t'\in T}\sum_{k\in\{d,e\}} m_{t'} m_k Z_{t'k}$ has an exponent at least $4-\frac{9}{2}a'$. Comparing the coefficients of the exponents $2-4a'$ and -2, we obtain

$$\textstyle\sum_{t'\in T\setminus\{t\}} m_{t'}\hat{Z}_{tt'} = \sum_{t'\in T\setminus\{t\}} m_{t'}\hat{W}_{tt'} = 0,$$

which are the relations (21) and (22) of [1]. We conclude as in [1, §5.3]. Writing $T=\{a,b,c\}$, these relations say that $\hat{Z}_{ab}/m_c = \hat{Z}_{bc}/m_a = \hat{Z}_{ca}/m_b$, and the same for $\hat{W}$. As $\hat{Z}_{tt'}/\hat{W}_{tt'} = \hat{z}_{tt'}/\hat{w}_{tt'}$, there is $\rho\ne 0$ with $\hat{w}_{tt'} = \rho\hat{z}_{tt'}$ on the three pairs. Then $\hat{Z}_{tt'}^2 = \hat{p}_{tt'}^6\hat{z}_{tt'}^2 = \rho^{-3}\hat{z}_{tt'}^{-4}$, so the

numbers $m_{t''}\hat{z}_{tt'}^2$, with t'' the third body, agree up to sign. Hence $\hat{z}_{bc} = \lambda\epsilon_a x_a$, $\hat{z}_{ca} = \lambda\epsilon_b x_b$ and $\hat{z}_{ab} = \lambda\epsilon_c x_c$ with $\lambda \neq 0$ and $\epsilon_t^4 = 1$, and $\epsilon_a x_a + \epsilon_b x_b + \epsilon_c x_c = 0$. If $\epsilon_b/\epsilon_a = \pm i$, the imaginary part of this relation, divided by ϵ_a, forces $\epsilon_c/\epsilon_a = \mp i$ and $x_b = x_c$, and its real part gives $x_a = 0$. Hence ϵ_b/ϵ_a and likewise ϵ_c/ϵ_a are ± 1, and one of x_a, x_b, x_c is the sum of the other two. This is $\mathsf{E}(T)$.

We now name the bodies of T as in (b). Since $x_c = x_a + x_b$, the relation $\epsilon_a x_a + \epsilon_b x_b + \epsilon_c x_c = 0$ forces $\epsilon_a = \epsilon_b = -\epsilon_c$. After replacing λ by $\lambda\epsilon_a$, $\hat{z}_{tt'} = \lambda(z_t^0 - z_{t'}^0)$ with $z_a^0 = -x_b$, $z_c^0 = 0$ and $z_b^0 = x_a$. The centre of mass gives $\sum_{t\in T} m_t\hat{w}_t = 0$, as $e(w_d), e(w_e) > \mu$. Hence $\hat{w}_t = \rho\lambda r_t$ with $r_t = z_t^0 - \sum_{t'\in T} m_{t'} z_{t'}^0/M_T$, and $r_t \neq 0$ as $\hat{w}_t \neq 0$. The relations below are homogeneous in (x_a, x_b), and we take $x_a = 1$ and $x_b = s$, so that $m_a = 1$, $m_b = s^{-2}$, $m_c = (1+s)^{-2}$ and

$$r_a = -\frac{2s+1}{s^2+s+1}, \qquad r_b = \frac{s^2(s+2)}{s^2+s+1}, \qquad r_c = \frac{(s-1)(s+1)^2}{s^2+s+1}.$$

So $s \neq 1$, that is, $s < 1$, and then $r_a, r_c < 0 < r_b$.

The centre of mass also gives $m_d\hat{z}_d + m_e\hat{z}_e = 0$, as $e(z_t) > -2$, so $\hat{z}_d = m_e\hat{c}$ and $\hat{z}_e = -m_d\hat{c}$ with $\hat{c} \neq 0$. For real $r \neq 0$ we write $r^{-1/2}$ for $|r|^{-1/2}$ if $r > 0$ and for $i|r|^{-1/2}$ if $r < 0$. Let K be a square root of $-1/(\hat{c}\rho\lambda)$. By (9.1), $\hat{p}_{dt} = \epsilon_{dt} K m_e^{-1/2} r_t^{-1/2}$ and $\hat{p}_{et} = \epsilon_{et} K m_d^{-1/2}(-r_t)^{-1/2}$ with signs $\epsilon_{kt} = \pm 1$, and (a), divided by $K m_d m_e^{-1/2}$, becomes

$$\textstyle\sum_{t\in T} m_t\big(\epsilon_{dt}\, r_t^{-1/2} + \xi\, \epsilon_{et}(-r_t)^{-1/2}\big) = 0, \qquad \xi = (m_e/m_d)^{3/2}.$$

With $g_t = m_t|r_t|^{-1/2}$, its real and imaginary parts are

$$\epsilon_{db} g_b + \xi(\epsilon_{ea} g_a + \epsilon_{ec} g_c) = 0, \qquad \epsilon_{da} g_a + \epsilon_{dc} g_c + \xi\epsilon_{eb} g_b = 0.$$

Their product gives $(\epsilon_{da} g_a + \epsilon_{dc} g_c)(\epsilon_{ea} g_a + \epsilon_{ec} g_c) = \sigma_b g_b^2$, where $\sigma_t = \epsilon_{dt}\epsilon_{et}$. If $\sigma_a = \sigma_c$, the left side is $\sigma_a(g_a \pm g_c)^2$, so $\sigma_b = \sigma_a$ and $g_b = |g_a \pm g_c|$, and the second equation implies $\xi = 1$. If $\sigma_a = -\sigma_c$, the left side is $\sigma_a(g_a^2 - g_c^2)$, so $g_a^2 = g_b^2 + g_c^2$ or $g_c^2 = g_a^2 + g_b^2$, and in the second case the second equation gives $\xi = (g_c \pm g_a)/g_b$.

Finally, the formulas for m_t and r_t yield $g_t^2 = (s^2+s+1)A_t(s)/D(s)$ with $D = s^6(1-s)(1+s)^6(2s+1)(s+2) > 0$. The three cases are therefore: $P_{28}(s) = 0$ and $\xi = 1$; $A_a(s) - A_b(s) - A_c(s) = 0$; and $P_{14}(s) = 0$ and $\xi^{\pm 1} = (\sqrt{A_c} + \sqrt{A_a})/\sqrt{A_b} = \sqrt{y} + \sqrt{1+y}$ with $y = A_a(s)/A_b(s)$. By Lemma 9.1, the first case leads to $s \in \{s_2, s_3\}$ and $m_d = m_e$, the second does not occur, and the third leads to $s = s_1$ and $\xi^{\pm 1} = \xi_1$. □

For diagram 4, part (a) is the condition $f_1 = f_2$ of [2, Lemma 8.9(a)]. In their notation the bodies d, e are 1, 2, and the leading coefficient $\hat{p}_{ik}$ corresponds to D_{ik}. With $S_i = \sum_{k\in T} m_k D_{ik}$ and [2, Lemma 8.7(a)], the condition $f_1 = f_2$ reads $m_1 S_1 + m_2 S_2 = 0$, which is (a).

10. Verification

Only the proof of the statement marked †, Lemma 9.1, includes computations. The computations on which some of the cited results of [8] rest are described and checked in [8, §12]. Table 2 lists the computations of this paper, the software on which they depend, and the computer checks of the hand proofs. The software is SymPy, FLINT through python-flint and mpmath, in Python; the supplementary material gives the versions. NumPy and SciPy are used only in floating-point steps that propose certificates or approximate values, which are then checked in exact arithmetic. The programs, their outputs and the checksums of all files are in the supplementary material.

statement	computation and trusted software	checks
Lemma 9.1†	resultant (SymPy); factorization over $\mathbb{Q}$ (FLINT); root counts by Sturm sequences and signs at the ends of the intervals in exact rational arithmetic (SymPy)	the root counts again by Descartes' rule of signs with bisection, in a separate implementation; an enclosure of ξ_1 in ball arithmetic from the interval of s_1 meets the interval of the root of Ξ; the displayed coefficients of Ξ are compared with the computed ones
Theorem 3.1	hand proof	the twenty equations of the system of [1] vanish on the nine sets $\mathcal{C}'(\zeta_1, \zeta_2)$ of §3 and their images under (23) and (34), 27 sets in all, carried to $(1, 4, 4, 4, 4)$ by the exchange of bodies 1 and 5 (SymPy)
Lemma 4.1	hand proof	the weighted rows recomputed from the equations; the identities of the proof checked in SymPy; all leading solutions lifted in exact arithmetic over $\mathbb{Q}(\zeta_{12})$ to order 7, with a nonzero obstruction at order 3, 4 or 6 at all but the 72 balanced ones
§4.1, Lemma 4.2	hand proof	the 18 weighted rows vanish identically on the 36 branches; $\gamma \mapsto \tilde{Y}_\gamma(0)$ is the stated bijection (SymPy, exact in $\mathbb{Q}(\zeta_{12})$)
Theorem 5.1	hand proof	the weighted rows recomputed from the equations by two programs; the identities (I0)–(I3) and the rank in exact arithmetic (SymPy); the leading solutions and the obstructions of orders 3 to 6 at all 5184 leading solutions, computed modulo a prime that splits completely in their field of definition, where a nonzero value proves that the obstruction does not vanish, and compared with the closed forms
Propositions 7.2 and 8.2	hand proof	the values of f, φ and f_1, g on the types, for every labelling that satisfies the conditions used, in exact rational arithmetic, with the minima over the closures of the parameter ranges
Lemma 8.3	hand proof	a branching over the alternatives of the relations of the place, in which every closed node carries a dual certificate verified in exact rational arithmetic

TABLE 2. The computation used in the proofs, and the checks of the hand proofs.

APPENDIX A. DIAGRAM 5 AT $(1, 1, 4, 4, 4)$

We prove Lemma 7.4. By Lemma 7.1(a) the zw-triangle is $T = \{l, h, k\}$ with h, k heavy, and the outer bodies are $A = l'$, of mass 1, and $B = h''$, of mass 4. Then

$$\varphi = 1 + g_A + g'_B + x_{AB}, \qquad g_A = x_{Ah} + x_{Ak} - 2x_{Al}, \qquad g'_B = x_{Bl} - x_{Bh} - x_{Bk}.$$

The bodies A and B have no strokes. Each is in one of the modes of Lemma 2.1(b) relative to T, and subscripts A, B mark the parameters c and s of each body. In a mode in which a body is near T in z, we write $\bar{\alpha}$, resp. $\bar{\beta}$, for its far w-exponent. The values of g_A and g'_B are as follows; the mode Zs_k gives the values of Zs_h, and the modes W1, Ws_t give those of Z1, Zs_t:

mode	g_A	g'_B
F	0	$-c_B$
Z1	0	$-\frac{1+\bar\beta}{2}$
Zs_l	$1-s_A$	$\frac{s_B-2-\bar\beta}{2}$
Zs_h	$\frac{s_A-1}{2}$	$-\frac{s_B+\bar\beta}{2}$

The colour exchange σ_c preserves φ and exchanges the modes Z and W. It therefore suffices to treat the classes F/F, F/Z, Z/F, Z/Z and Z/W, where we write the mode of A first and Z stands for Z1 or Zs.

Coordinates. Let z_c be the centre of mass of T, $a = z_A - z_c$ and $b = z_B - z_c$, with $\alpha = e(a)$ and $\beta = e(b)$, and let $\bar a$, $\bar b$, $\bar\alpha$, $\bar\beta$ be the analogues in w. In a mode near in z, $\bar\alpha$ is the far w-exponent, in agreement with the notation above. By Lemma 6.1, l is the midpoint of h and k at the leading order, so $e(z_l - z_c) > 1$, while $z_h - z_c$ and $z_k - z_c$ have the coefficients h_1 and $-h_1$ at the exponent 1. Let a_1 be the coefficient of a at the exponent 1. In terms of z, the mode F means $\alpha < 1$, Z1 means $a_1 \notin \{0, \pm h_1\}$, Zs_l means $a_1 = 0$, and Zs_h means $a_1 = h_1$.

Tools. A function that vanishes identically on C has the exponent $+\infty$; the relations below hold with this convention.

(R) The equations of A and B: $z_A = T_{zA} + 4Z_{AB}$, $w_A = T_{wA} + 4W_{AB}$, $z_B = T_{zB} - Z_{AB}$, $w_B = T_{wB} - W_{AB}$, where $T_{zA} = \sum_{t\in T} m_t Z_{At}$, and likewise for the others. In each equation the least exponent of the three terms is attained at least twice.

(TS) In the modes Zs one term dominates, so $e(T_{zA}) = -\frac{s_A+3\bar\alpha}{2}$ and $e(T_{wA}) = -\frac{3s_A+\bar\alpha}{2}$. In the mode F the three terms have the same leading coefficient up to sign and the factor m_t, and $\pm1 \pm 4 \pm 4 \neq 0$, so the exponents of T_{zA}, T_{wA} are again those of their terms. In Z1 only the lower bounds $e(T_{zA}) \geq -\frac{1+3\bar\alpha}{2}$ and $e(T_{wA}) \geq -\frac{3+\bar\alpha}{2}$ hold. The same holds for B.

(CM) $14z_c = -(a+4b)$, $14z_A = 13a - 4b$ and $14z_B = -a + 10b$, and the same in w.

(OE) The six forms a, b, $a-b$, $a+4b$, $13a-4b$ and $-a+10b$ are pairwise linearly independent. Hence at most one of them has an exponent larger than $\min(\alpha, \beta)$.

(LF) If $\alpha < \beta$, then $e(z_A) = e(z_B) = e(z_c) = \zeta_{AB} = \alpha$, by (CM) and (OE).

(FC) If $\alpha < \min(1, \beta)$, then $\alpha \geq -\max_y \omega_{Ay}$, the maximum over $y \in T \cup \{B\}$. Indeed, by (LF) and the far mode, $\zeta_{Ay} = \alpha$ for all y, so every term of the equation of z_A has an exponent at least $-\frac{1}{2}(\alpha + 3\max_y \omega_{Ay})$, and so has z_A.

Class F/F. Here $\varphi = 1 - c_B + x_{AB}$, with $c_B = \frac{1}{2}(\beta + \bar\beta)$.

- If $\alpha \geq \beta$ and $\bar\alpha \geq \bar\beta$, then $\zeta_{AB} \geq \beta$ and $\omega_{AB} \geq \bar\beta$, so $\varphi \geq 1$.
- If $\alpha < \beta$ and $\bar\alpha \geq \bar\beta$, then $\zeta_{AB} = \alpha$ by (LF). If $\omega_{AB} \leq \bar\alpha$, then (FC) shows that $\alpha \geq -\bar\alpha$, and $\varphi \geq 1 - \frac{\bar\alpha+\beta}{2} > 0$. If $\omega_{AB} > \bar\alpha$, then $\bar\alpha = \bar\beta$ by (O5), and (FC) gives $\alpha \geq -\omega_{AB}$, so $x_{AB} \geq 0$ and $\varphi \geq 1 - c_B > 0$.
- If $\alpha < \beta$ and $\bar\alpha < \bar\beta$, then (LF) and (FC) in both colours give $x_{AB} = c_A \geq 0$, so $\varphi \geq 1 - c_B > 0$.
- The case $\alpha \geq \beta$, $\bar\alpha < \bar\beta$ is the mirror of the second.

Class F/Z. The body B is near in z, so $\beta \geq 1 > \alpha$, and $\zeta_{AB} = \alpha$ by (LF).

- If $\bar\alpha \leq \bar\beta$, then $x_{AB} \geq 0$ by (FC), so $\varphi \geq 1 + g'_B > 0$.
- If $\bar\alpha > \bar\beta$, then $\omega_{AB} = \bar\beta$. In Z1, $\varphi = \frac{1+\alpha}{2}$, and $\alpha \geq -\bar\alpha > -1$ by (FC). In Zs, $e(w_B) = \bar\beta$ by (LF) in w, and $e(T_{wB}) = -\frac{3s_B+\bar\beta}{2} < e(W_{AB})$, so (R) for w_B forces $\bar\beta = -s_B$. Then (R) for z_B shows that $e(z_B) = e(T_{zB}) = s_B \neq \alpha$, contradicting (LF).

Class Z/F. Here $\zeta_{AB} = \beta$ and $\varphi = 1 + g_A + \frac{\omega_{AB}-\bar\beta}{2}$.

- If $\bar\alpha \ge \bar\beta$, then $\varphi \ge 1 + g_A \ge 2 - s_A > 0$, with $s_A = 1$ in Z1.
- If $\bar\alpha < \bar\beta$, then $\omega_{AB} = \bar\alpha = e(w_A)$. In Z1, if $\bar\alpha < -1$, then $e(T_{wA}) \ge -\frac{3+\bar\alpha}{2} > \bar\alpha$ and $e(W_{AB}) = -\frac{3\beta+\bar\alpha}{2} > \bar\alpha$, contradicting (R). So $\bar\alpha \ge -1$ and $\varphi > \frac{1+\bar\alpha}{2} \ge 0$. In Zs, (R) for w_A forces $\bar\alpha = -s_A$, and then (R) for z_A yields $e(z_A) = s_A \ne \beta$, a contradiction.

Class Z/Z. Put $\zeta = \zeta_{AB}$. Then $\varphi = 1 + \frac{\Psi}{2} + \frac{\omega_{AB}-\bar\beta}{2}$, where $\Psi = 2g_A + 2g'_B + \zeta + \bar\beta$ is

	A/B	Ψ		A/B	Ψ
1	Z1/Z1	$\zeta - 1$	6	$\mathrm{Zs}_l/\mathrm{Zs}_h$	$3 - 2s_A - s_B$
2	Z1/Zs_l	$s_B - 1$	7	Zs_h/Z1	$s_A - 1$
3	Z1/Zs_h	$1 - s_B$	8	$\mathrm{Zs}_h/\mathrm{Zs}_l$	$s_A + s_B - 2$
4	Zs_l/Z1	$2 - 2s_A$	9	$\mathrm{Zs}_h/\mathrm{Zs}_h$	$s_A - 1 - s_B + \zeta$
5	$\mathrm{Zs}_l/\mathrm{Zs}_l$	$s_B - 2s_A + \zeta$	10	$\mathrm{Zs}_h/\mathrm{Zs}_k$	$s_A - s_B$

In the combinations in which A and B are near different bodies, or exactly one of them is in Z1, the coefficients $a_1 \ne b_1$ give $\zeta = 1$. In the combinations 5 and 9, $\zeta \ge \min(s_A, s_B)$, with equality if $s_A \ne s_B$, by (O5).

(a) $\bar\alpha \ge \bar\beta$. Then $\omega_{AB} \ge \bar\beta$, and $\Psi > -2$ in every combination except 6, so $\varphi > 0$. In combination 6, $e(z_A) = e(z_B) = 1$ by (OE). If $\bar\alpha > \bar\beta$, then (R) for w_B forces $\bar\beta = -s_B$ as in class F/Z, and then $e(T_{zB}) = s_B$ and $e(Z_{AB}) = \frac{3s_B-1}{2}$ are both larger than $e(z_B) = 1$, contradicting (R). Let $\bar\alpha = \bar\beta = \nu$ and $u = e(Z_{AB})$.

- If $u > 1$, then (R) for z_A and z_B forces $e(T_{zA}) = e(T_{zB}) = 1$, so $s_A = s_B = s$ and $\nu = -\frac{2+s}{3}$. Then $e(T_{wA}) = -\frac{4s-1}{3} < \nu \le e(w_A)$, so in (R) for w_A we have $e(W_{AB}) = e(T_{wA})$, that is, $\omega_{AB} = \frac{8s-11}{3}$, and $u = 5 - 4s < 1$, a contradiction.
- If $u = 1$ and $\omega_{AB} = \nu$, then $\nu = -1$, and T_{wA}, with the exponent $-\frac{3s_A-1}{2} < -1$, is the only term of least exponent in (R) for w_A, a contradiction.
- If $u = 1$ and $\omega_{AB} > \nu$, then $\omega_{AB} = -1$, and (OE) in w gives $e(w_A) = e(w_B) = \nu$. By (R) for z_A and z_B, $\nu \le -\frac{2+s_A}{3} < -1$ and $\nu \le -\frac{2+s_B}{3}$. Then (R) for w_A and w_B forces $\nu = -s_A = -s_B$, and $\varphi = 2 - s > 0$ with $s = s_A = s_B$.
- If $u < 1$, then (R) for z_A and z_B yields $u = e(T_{zA}) = e(T_{zB})$, so $s_A = s_B = s$ and $\omega_{AB} = \nu + \frac{s-1}{3} > \nu$. By (OE) in w, $e(w_A) = \nu$. If $\nu > -s$, then $e(T_{wA}) < \nu$, and (R) for w_A forces $e(T_{wA}) = e(W_{AB})$, which means $s = 1$. If $\nu < -s$, then (R) for w_A gives $e(W_{AB}) = \nu$, which means $\nu = -1 - \frac{s-1}{9} > -s$. If $\nu = -s$, then $u = s > 1$. Each case is a contradiction.

(b) $\bar\alpha < \bar\beta$. Then $\omega_{AB} = \bar\alpha = e(w_A)$, and (R) for w_A gives $\bar\alpha \ge -\max(s_A, \zeta)$, with $s_A = 1$ in Z1, and with equality in Zs if $s_A \ne \zeta$. The combinations give:

- 1 and 7: $\varphi \ge \frac{1-\bar\beta}{2}$;
- 2 and 8: $\varphi \ge \frac{s_B-\bar\beta}{2}$;
- 3 and 10: $\varphi \ge 1 - \frac{s_B+\bar\beta}{2}$;
- 4 and 6: (OE) gives $e(z_A) = 1$, while $\bar\alpha = -s_A$ and (R) for z_A give $e(z_A) = s_A$, a contradiction;
- 5 with $\zeta = s_A \le s_B$: then $\bar\alpha \ge -s_A$, and in (R) for w_B, $e(T_{wB}) = -\frac{3s_B+\bar\beta}{2} < -\frac{3s_A+\bar\alpha}{2} = e(W_{AB}) \le \bar\alpha = e(w_B)$, a contradiction;
- 5 with $s_A = s_B = s < \zeta$: then $\bar\alpha = -\zeta$, and $\varphi = 1 - \frac{s+\bar\beta}{2} > 0$;
- 5 with $\zeta < s_A$: then $\zeta = s_B < s_A$ and $\bar\alpha = -s_A$. The equations (R) give $e(z_A) = s_A$, $\bar\beta = 2s_A - 3s_B$ and $e(z_B) = 4s_B - 3s_A$. With $e(z_A - z_B) = s_B$ these are three distinct exponents among the forms of (OE), a contradiction;
- 9: $\varphi \ge \frac{1+s_A-s_B-\bar\beta}{2} > \frac{s_A-1}{2}$ if $\zeta \ge s_A$, and otherwise $\zeta = s_B$, $\bar\alpha = -s_A$ and $\varphi = \frac{1-\bar\beta}{2}$.

All these values are positive, by $\bar\beta < 1$, $s_B + \bar\beta < 2$ and $s > 1$.

Class Z/W. Here $\beta < 1 \le \alpha$ and $\bar\alpha < 1 \le \bar\beta$, so $\zeta_{AB} = e(z_A) = \beta$ and $\omega_{AB} = e(w_A) = \bar\alpha$ by (LF) in both colours. If A is in Zs, then (R) for w_A gives $\bar\alpha = -s_A$, and (R) for z_A gives $e(z_A) = s_A \ne \beta$. If B is in Ws, then in the same way (R) for z_B gives $\beta = -s_B$, and (R) for w_B gives $e(w_B) = s_B \ne \bar\alpha$. In Z1/W1, $\varphi = \frac{1+\bar\alpha}{2}$, and (R) for z_A gives $\bar\alpha \ge -\beta$ or $\bar\alpha \ge -\frac{1+2\beta}{3}$, so $\bar\alpha \ge -\frac{1+2\beta}{3} > -1$.

Thus $\varphi > 0$ in every class, which proves Lemma 7.4.

Appendix B. Diagram 5 at P_1

We prove Lemma 8.3. Let P be a place of diagram 5 of a curve $C \subseteq Y_{P_1}$ with $U \not\equiv 0$. By Lemma 8.1(a) the zw-triangle T contains M.

The triangle $T = \{M, h, h'\}$, with the outer bodies h'' and L. Then $f_1 = 1 + x_{h''M} - x_{h''h} - x_{h''h'}$. In the modes F, Z1, Zs_M and Zs_h of h'' this is $1-c$, $\frac{1-\bar\omega}{2}$, $\frac{s-\bar\omega}{2}$ and $\frac{2-s-\bar\omega}{2}$, and the mirror modes give the same values. All are positive, by $c, \bar\omega < 1$, $s > 1$ and $s + \bar\omega < 2$.

The triangle $T = \{M, h, L\}$, with the outer bodies $A = h'$ and $B = h''$, both of mass 36. Then

$$f_1 = 1 + g_A + g_B - x_{AB}, \qquad g_X = x_{XM} - x_{Xh}.$$

The value of g_X is 0 in the modes F, Z1, W1, Zs_L and Ws_L, $\frac{s_X-1}{2}$ in Zs_M and Ws_M, and $-\frac{s_X-1}{2}$ in Zs_h and Ws_h. Since $x_{AB} < 1$, $f_1 > 0$ unless an outer body is near h. By the symmetry $A \leftrightarrow B$ and the colour exchange we may take B in Zs_h. Then $f_1 = 1 + g_A - \frac{s_B-1}{2} - x_{AB}$. We write $\sigma_B = e(T_{wB}) = -\frac{3s_B+\bar\beta}{2}$ and $\tau_B = e(T_{zB}) = -\frac{s_B+3\bar\beta}{2}$.

Tools. We use the notation a, b, α, β, $\bar a$, $\bar b$, $\bar\alpha$, $\bar\beta$ and z_c of Appendix A, now with the masses of P_1.

- By Lemma 6.1 the bodies M, L, h are collinear at the leading order, with $z_t^\circ - z_c^\circ = \lambda_t \alpha_0$, $(\lambda_M, \lambda_L, \lambda_h) = \frac{1}{21}(-16, -9, 5)$ and $\alpha_0 \ne 0$, and the same in w. A body X near t_0 in z has $e(z_X - z_c) = 1$, with the coefficient $\lambda_{t_0}\alpha_0$ at the exponent 1.
- (R): $z_A = T_{zA} + 36Z_{AB}$, $z_B = T_{zB} - 36Z_{AB}$, and $49z_c = -36(T_{zA} + T_{zB})$, and the same in w. In each of these equations the least exponent is attained at least twice.
- (TS): the exponents of T_{zX} and T_{wX} are those of their dominant terms in the modes Zs and Ws, and in F, since $\pm 9 \pm 36 \pm 4 \ne 0$. In Z1 and W1 they are bounded below by $-\frac{1+3\bar\omega}{2}$ and $-\frac{3+\bar\omega}{2}$.
- (CM), (OE): $121z_c = -36(a+b)$, $121z_A = 85a - 36b$ and $121z_B = -36a + 85b$. The six forms a, b, $a-b$, $a+b$, $85a-36b$ and $-36a+85b$ are pairwise linearly independent, so all but at most one of them have the exponent $\min(\alpha, \beta)$. In particular, (LF): if $\alpha < \beta$, then $e(z_A) = e(z_B) = e(z_c) = \zeta_{AB} = \alpha$. For two bodies near the same t_0 in the same colour, $e(a-b) > 1$ and the other five forms have the exponent 1. For bodies near $t_0 \ne t_1$, all six have the exponent 1, since $\lambda_{t_0} \ne \lambda_{t_1}$, $\lambda_{t_0} + \lambda_{t_1} \ne 0$ and $85\lambda_{t_0} \ne 36\lambda_{t_1}$.
- (O1), (O5), the absence of strokes at A and B, and [8, Lemma 3.3].

A in F ($\alpha, \bar\alpha < 1$, $g_A = 0$, $x_{At} = c_A$). By (LF), $\zeta_{AB} = \alpha$.

- $\bar\alpha < \bar\beta$: then $x_{AB} = c_A$. By [8, Lemma 3.3], $c_A \le 0$ or $\bar\beta \le -1$, and the latter also gives $c_A < 0$. Hence $f_1 \ge \frac{3-s_B}{2} > 0$.
- $\bar\alpha > \bar\beta$: then $x_{AB} = \frac{\alpha+\bar\beta}{2} < \frac{1+\bar\beta}{2}$, so $f_1 > \frac{2-s_B-\bar\beta}{2} > 0$.
- $\bar\alpha = \bar\beta = \nu$: if $\omega_{AB} = \nu$, the previous bound holds. If $\omega_{AB} > \nu$, then $e(w_A) = e(w_B) = \nu$ by (OE) in w. If $\nu > -s_B$, then $\sigma_B < \nu$, so (R) for w_B gives $e(W_{AB}) = \sigma_B$, and (R) for w_A needs $e(T_{wA}) = -\frac{3\alpha+\nu}{2} = \sigma_B$, that is, $\alpha = s_B > 1$, which is impossible. If $\nu < -s_B$, then (R) for w_B gives $e(W_{AB}) = \nu$, that is, $\omega_{AB} = -2\nu - 3\alpha$, and in (R) for z_A the term Z_{AB}, of exponent $4\alpha + 3\nu < \alpha < e(T_{zA})$, is the only one of least exponent, a

contradiction. If $\nu = -s_B$, then $\tau_B = s_B > \alpha$, so (R) for z_B gives $e(Z_{AB}) = \alpha$, that is, $\omega_{AB} = -\alpha$ and $x_{AB} = 0$, and $f_1 = \frac{3-s_B}{2} > 0$.

A in W1 ($\alpha < 1 \le \bar{\alpha}$). (LF) in both colours gives $x_{AB} = \frac{\alpha+\bar{\beta}}{2}$, so $f_1 = \frac{3-s_B-\alpha-\bar{\beta}}{2} > 0$.

A in Z1, Zs_M or Zs_L ($g_A \ge 0$). The coefficients of a and b at the exponent 1 differ, so $\zeta_{AB} = 1$ and $f_1 \ge \frac{2-s_B-\omega_{AB}}{2}$. If $\omega_{AB} = \min(\bar{\alpha}, \bar{\beta})$, then $f_1 > 0$. Otherwise $\bar{\alpha} = \bar{\beta} = \nu < \omega_{AB}$ and $e(w_A) = e(w_B) = \nu$.

- $\nu > -s_B$: (R) for w_B gives $e(W_{AB}) = \sigma_B$. In Z1 this is impossible, since $e(T_{wA}) \ge -\frac{3+\nu}{2} > \sigma_B$. In Zs, (R) for w_A forces $s_A = s_B = s$ and $\omega_{AB} = 3s + \nu - 3$. Then in (R) for z_A, $e(Z_{AB}) - e(T_{zA}) = 4(1-s) < 0$ forces $e(Z_{AB}) = 1$, that is, $\nu = 2 - 3s < -s$, a contradiction.
- $\nu < -s_B$: (R) for w_B yields $\omega_{AB} = -2\nu - 3$, so $e(Z_{AB}) = 4 + 3\nu < 1$, and (R) for z_A forces $e(T_{zA}) = 4 + 3\nu$. In Z1 this is impossible, since $e(T_{zA}) \ge -\frac{1+3\nu}{2}$ needs $\nu \ge -1$. In Zs it gives $\nu = -\frac{8+s_A}{9}$ and $x_{AB} = \frac{s_A-1}{9}$, so $f_1 > 1 - \frac{1}{2} - \frac{1}{9} > 0$.
- $\nu = -s_B$: then $e(W_{AB}) \ge \nu$, that is, $\omega_{AB} \le 2s_B - 3$. In Z1, (R) for z_A forces $e(Z_{AB}) \ge 1$, so $\omega_{AB} \le -1$ and $f_1 \ge \frac{3-s_B}{2}$. In Zs, either $e(Z_{AB}) \ge 1$, with the same bound, or $e(Z_{AB}) = e(T_{zA}) = \frac{3s_B-s_A}{2}$, which gives $f_1 = \frac{7}{6} + g_A - \frac{s_A}{6} > 0$.

A in Ws_{t_0}. (LF) in both colours gives $\zeta_{AB} = \alpha$ and $\omega_{AB} = \bar{\beta}$. In (R) for z_B, $\tau_B < e(Z_{AB})$, so $\alpha = \tau_B$. In (R) for z_A, $e(Z_{AB}) > \alpha$, so $\alpha = e(T_{zA}) = -\frac{3s_A+\alpha}{2}$, that is, $\alpha = -s_A$, and then $3\bar{\beta} = 2s_A - s_B$. Hence

$$f_1 = \tfrac{3}{2} + g_A - \tfrac{s_B}{3} + \tfrac{s_A}{6} \ge 2 - \tfrac{s_A+s_B}{3} > 0.$$

A in Zs_h (both outer bodies near h in z). Here $\zeta = \zeta_{AB} > 1$, $e(z_A) = e(z_B) = e(z_c) = 1$, and

$$f_1 = \frac{4 - s_A - s_B - \zeta - \omega_{AB}}{2}.$$

By (O5) on $\{A, B, h\}$, $\zeta \ge \min(s_A, s_B)$, with equality if $s_A \ne s_B$.

- $\bar{\alpha} < \bar{\beta}$, after exchanging A and B if necessary. Then $\omega_{AB} = \bar{\alpha} = e(w_A)$. The least exponent in (R) for z_A is attained by T_{zA} and Z_{AB}, which gives $\zeta = s_A$, or by z_A and T_{zA}, which gives $\bar{\alpha} = -\frac{2+s_A}{3}$ and $\zeta \le s_A$, or by z_A and Z_{AB}, which gives $\bar{\alpha} = -\frac{2+\zeta}{3}$ and $\zeta \ge s_A$.
  - $\zeta = s_A$: by (O5), $s_B \ge s_A$, so $\tau_B < e(Z_{AB}) \le 1$ and T_{zB} is the only term of least exponent in (R) for z_B, a contradiction.
  - $\bar{\alpha} = -\frac{2+s_A}{3}$, $\zeta \le s_A$: if the least exponent in (R) for z_B is attained by Z_{AB}, then $e(Z_{AB}) \le 1$, so $\zeta \ge s_A$, hence $\zeta = s_A$, and the previous case applies. Otherwise $\tau_B = 1$, so $\bar{\beta} = -\frac{2+s_B}{3}$, and $\bar{\alpha} < \bar{\beta}$ gives $s_B < s_A$, so $\zeta = s_B$ by (O5). Since $\bar{\alpha} > -s_A$, $\sigma_A = -\frac{3s_A+\bar{\alpha}}{2} < \bar{\alpha} = e(w_A)$, and $e(W_{AB}) = -\frac{3s_B+\bar{\alpha}}{2} > \sigma_A$. Hence T_{wA} is the only term of least exponent in (R) for w_A, a contradiction.
  - $\bar{\alpha} = -\frac{2+\zeta}{3}$, $\zeta \ge s_A$: (R) for z_B needs $\tau_B \ge 1$, so $\bar{\beta} \le -\frac{2+s_B}{3}$. With $\bar{\alpha} < \bar{\beta}$ this gives $s_B < \zeta$, hence $s_A = s_B = s < \zeta$ by (O5). Then $f_1 = \frac{7}{3} - s - \frac{\zeta}{3}$. Since AB is not a stroke, $3\zeta + \bar{\alpha} < 4$, so $\zeta < \frac{7}{4}$ and $f_1 > \frac{7}{3} - \frac{4}{3}\zeta > 0$.
- $\bar{\alpha} = \bar{\beta} = \nu$. If $s_A < s_B$, then $\zeta = s_A$, and the centre equation in z gives $e(T_{zB}) = 1$, that is, $\nu = -\frac{2+s_B}{3}$. Then (R) for z_A forces $e(Z_{AB}) = 1$, that is, $\omega_{AB} = -\frac{2+s_A}{3} > \nu$, so $e(w_A) = e(w_B) = \nu$ by (OE). Then (R) for w_A gives $e(W_{AB}) = \nu$, and (R) for w_B needs $\sigma_B \ge \nu$, that is, $\nu \le -s_B$, so $s_B \le 1$, which is impossible. The case $s_B < s_A$ is the same. If $s_A = s_B = s$, the centre equation in z gives $-\frac{s+3\nu}{2} \le 1$.
  - If $\omega_{AB} = \nu$, then (R) for z_A forces $\zeta = s$, and $f_1 = \frac{4-3s-\nu}{2} > 0$, because Ah is not a w-stroke, so $3s + \nu < 4$.

– If $\omega_{AB} > \nu$, then $e(w_A) = e(w_B) = e(w_c) = \nu$ by (OE), and $\zeta \geq s$. Therefore $e(W_{AB}) < \sigma$, where $\sigma = e(T_{wA}) = e(T_{wB}) = -\frac{3s+\nu}{2}$. In (R) for w_A the least exponent is then attained by w_A and W_{AB}, so $\nu = e(W_{AB}) < \sigma$, that is, $\nu < -s$. The centre equation in w, with the exponents ν, σ, σ, gives $\sigma \leq \nu$, that is, $\nu \geq -s$, a contradiction.

Thus $f_1 > 0$ at every place of diagram 5, which proves Lemma 8.3.

Appendix C. Proof of Lemma 2.1

Proof of Lemma 2.1. In [8] the condition $U \not\equiv 0$ is part of the standing assumption of [8, §§2.4 and 3] that C is nontrivial, and a hypothesis of [8, Theorem 3.12, Corollary 5.19, Theorem 5.37 and Proposition 7.1]. We check where U enters the proofs.

(a) The fibres of the map $(u, v, p) \mapsto (r_{kl}^2)_{k<l}$ on Y_m have at most 2^{13} points [8, proof of Corollary 2.5], so some r_{kl}^2 is non-constant on C. The rest is derived in [8, §2.4] from three facts: the p_{kl} are units on C, one of them is non-constant, and the bounds [8, (2.4)] hold on Y_m. The form (2.1) of the balancing condition follows, since $x(P)$ is a positive multiple of $(-\operatorname{val}_P p_{kl})_{k<l}$ [8, §3.1]. These steps do not use U.

(b) In the derivation of [8, Table 1] in [8, §3.2 and Appendix A], the condition $U \not\equiv 0$ enters only through [8, Lemma 3.3], whose proof uses that $U = I$ is a nonzero constant on C. The lemma is used only for $a + b \leq \frac{4}{3}$ in diagram 7, $c \leq \frac{1}{3}$ in diagram 10, $c = 0$ in diagram 14 and the exclusion of diagram 16. Without it the derivations give the stated types. For diagram 14 this is [8, Lemma 3.7], case (A) with $0 \leq c < 1$ and case (B) with $a = 1 + t < 2$. The derivation for diagram 9 uses that U is constant on C [8, Lemma 2.4(b)], which holds for every irreducible curve. No other step of these derivations uses U. For diagram 5 we do not use the hulls of [8, §3.3]. Their models use [8, Lemma 3.3] and, in the relations (I) and (V) of [8, §B.1], that $U = I$ is a nonzero constant on C. We use only (O4) on the zw-triangle and [8, Lemma 3.4], which applies since the other two bodies have no strokes.

(c) None of these proofs uses [8, Lemma 3.3]. The proofs of [8, Lemmas 3.1, 3.2, 3.4, 3.6 and 3.7, Theorems 3.9 and 3.13 and Proposition 7.1], with [8, Appendices C and E], do not use U, V or I. The rows of [8, Table 1] that they use, for diagrams 2 and 3, are covered by (b). In the proof of [8, Theorem 3.12] the mass conditions follow from [8, Lemma 3.1] and the inputs of [8, §2.6], without U. The proofs of [8, Corollary 5.19 and Theorem 5.37] expand V at the place [8, Lemmas 5.3 and 5.39]. Through [8, Lemmas 5.1 and 5.38] they use two facts: the gradient of V vanishes on X_m [8, Lemma 2.3], and V is constant on C [8, Lemma 2.4]. Both hold for every irreducible curve. The identity [8, (5.1)] is used only for $q = 3$, 5 and $3 + \alpha$ with $0 < \alpha < 2$, where it does not involve the value of V on C. The proof of [8, Theorem 5.37] also uses the type of diagram 9, which (b) covers. In the proof of [8, Theorem 3.13(a)], case (A) of [8, Lemma 3.7] includes every $c \in [0, 1)$. □

Data availability

The supplementary material, with the programs, their outputs and the checksums of all files, is available at `https://doi.org/10.5281/zenodo.23444951`. The DOI `https://doi.org/10.5281/zenodo.23444948` refers to all versions of this paper and resolves to the latest one.

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