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A LOWER BOUND FOR THE BOHR RADIUS OF THE BIDISC

TEJASVI SINGH TOMAR

Abstract. We prove that the Bohr radius K_2 of the bidisc, whose exact value is unknown, satisfies $K_2 > 0.301665$, improving Patel's bound $K_2 > 0.30158758$. As in his argument, we pass to functions with positive real part and value 1 at the origin; we bound the sum of the moduli of their coefficients of degree n by $2S_n$ instead of $2\sqrt{n}$, where S_n is the Sidon constant of $\{0, 1, \dots, n\}$, and we bound S_5, S_6 and S_7 with computer-assisted certificates. In Lean 4 we also prove $K_2 \geq 0.301665$, assuming Patel's inequality coupling the coefficients of degrees two and three.

1. Introduction

For $d \geq 1$ let $\mathbb{D}^d = \{z \in \mathbb{C}^d : |z_1|, \dots, |z_d| < 1\}$ be the unit polydisc and let the Schur class $\mathcal{S}(\mathbb{D}^d)$ be the set of holomorphic maps $f\colon \mathbb{D}^d \to \overline{\mathbb{D}}$. For $f(z) = \sum_\alpha c_\alpha z^\alpha$ and $r > 0$ write $\|f_r\|_1 = \sum_\alpha |c_\alpha| r^{|\alpha|}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The Bohr radius of $\mathbb{D}^d$ is

$$K_d = \sup\{r > 0 : \|f_r\|_1 \leq 1 \text{ for all } f \in \mathcal{S}(\mathbb{D}^d)\}.$$

Bohr [9] proved that $K_1 \geq 1/6$. The sharp value $K_1 = 1/3$ was found independently by Riesz, Schur and Wiener, and Bohr's paper includes Wiener's proof (see [8]). Boas and Khavinson [8] introduced K_d for $d \geq 2$, proved $1/(3\sqrt{d}) < K_d < 2\sqrt{(\log d)/d}$ and asked for the exact value of K_d when $d > 1$. Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip [11] showed that K_d has order $\sqrt{(\log d)/d}$, and Bayart, Pellegrino and Seoane-Sepúlveda [6] proved $K_d \sim \sqrt{(\log d)/d}$ as $d \to \infty$. No exact value of K_d is known for any $d \geq 2$.

For the bidisc, functions of one variable show $K_2 \leq 1/3$. Knese [14, Corollary 1.2] proved $K_2 \geq 0.3006$: by his Theorem 1.1 and Andô's inequality, K_2 is at least the root $0.3006282\ldots$ of $\sum_{k\geq 1} \sqrt{k}\, r^k = 1/2$. Baran, Pikul, Woerdeman and Wojtylak [5, Theorem 6.4] proved $K_2 < 0.3177$, and Griego [12] lowered this to $K_2 < 0.3174541$.

In an unrefereed solution to a problem on MathDB, Patel [20] proved $K_2 > 0.30158758$. He also reports the upper bound $K_2 < 0.302825279492$, from a rational inner function and an exact computation that his reproducibility package [21] carries out; we have not checked this bound, but if it holds, it and Theorem 1.1 confine K_2 to an interval of width about $1.2 \cdot 10^{-3}$.

Sidon constants enter through the coefficients. For a finite set $\Lambda \subset \mathbb{Z}$ the Sidon constant $S(\Lambda)$ is the least C with

$$\sum_{\lambda\in\Lambda} |a_\lambda| \leq C \max_{|w|=1} \Big|\sum_{\lambda\in\Lambda} a_\lambda w^\lambda\Big| \qquad (a_\lambda \in \mathbb{C}),$$

and we write

$$S_k = S(\{0, 1, \dots, k\}) \qquad (k \geq 1).$$

Bernal-González, Cabana, García, Maestre, Muñoz-Fernández and Seoane-Sepúlveda [7, Theorem 3.3] showed that K_d is at least the root of $\sum_{k\geq 1} S(k,d)\, r^k = 1/2$, where $S(k,d)$ is the

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Sidon constant of the k-homogeneous polynomials in d variables. For $d = 2$, $S(k, 2) = S_k$ (Remark 2.4). Newman's bound $S_k \le \sqrt{k}$, whose proof appears in Shapiro's thesis [23, Chapter 3] (see also [19, (1)]), turns this equation into Knese's, and Shapiro [23] showed that $S_k = \sqrt{k}$ if and only if $k \in \{1, 2, 4\}$. Few values of S_k are known; Table 1 lists them with their sources.

We give Patel's argument for the lower bound in full, confirm the exact rational computation behind it with our own programs, and improve the bound.

Theorem 1.1. *The Bohr radius of the bidisc satisfies* $K_2 > 0.301665$.

The gain over Patel's bound, about $8 \cdot 10^{-5}$, comes from the terms of degrees five, six and seven.

Patel reduces the problem to the coefficients of holomorphic functions H on $\mathbb{D}^2$ with $\operatorname{Re} H > 0$ and $H(0) = 1$ (Section 2). Carathéodory's inequality on slices bounds sums of moduli of the coefficients of degree two and, with the help of a positive semidefinite Toeplitz form, of degree three, and a third inequality (Lemma 3.4) couples the two degrees; Patel combines the three into a bound for the coefficients of degrees two and three together (Section 3). The third inequality comes from the unitary realization of Schur functions on the bidisc [1, 4], which rests on Andô's inequality [3]. In the degrees $n \ge 4$ Patel uses the bound $2\sqrt{n}$ for the sum of the absolute values of the coefficients of degree n of H, which he derives from Knese's estimate [14, Lemma 2.1].

Our improvement starts from a simple observation: Carathéodory's inequality on slices also bounds this sum by $2S_n$ (Lemma 2.3). By Newman's bound and Shapiro's theorem, $S_n < \sqrt{n}$ for every $n \ge 5$, while $S_4 = 2$. We prove $S_5 \le 2.187$, $S_6 \le 2.363$ and $S_7 \le 2.552$ with computer-assisted certificates built from positive definite Toeplitz matrices (Section 4); for S_5 our bound is slightly above the value $S_5 = 2.186529771892569\ldots$ that Ye et al. determined [24]. In the degrees $n \ge 8$ we keep Newman's bound $S_n \le \sqrt{n}$, since the size of the certificates grows quickly with n (Table 2). With rational upper bounds for $\sqrt{n}$ and an explicit tail, an exact rational computation shows that Patel's majorant, with these bounds, is less than 1 at $r = 0.301665$, and by continuity also at slightly larger r, which proves Theorem 1.1 (Section 5). Section 6 describes the computer checks and a formal proof in Lean, conditional on Lemma 3.4, of the non-strict bound $K_2 \ge 0.301665$, which needs only the computation at $r = 0.301665$.

2. The Carathéodory normalization and slices

Let the Carathéodory class $\mathcal{P}_2$ be the set of holomorphic functions H on $\mathbb{D}^2$ with $\operatorname{Re} H > 0$ and $H(0) = 1$. We write

$$H(z) = 1 + \sum_{n \ge 1} H_n(z), \qquad H_n(z) = \sum_{j+k=n} h_{jk} z_1^j z_2^k,$$

and set $\|H_n\|_1 = \sum_{j+k=n} |h_{jk}|$ and $M_r(H) = \sum_{n \ge 1} \|H_n\|_1 \, r^n$.

We use the following reduction.

Lemma 2.1. *Let* $0 < r < 1$. *If* $M_r(H) \le 1$ *for every* $H \in \mathcal{P}_2$, *then* $K_2 \ge r$.

Proof. Constant functions satisfy $\|f_r\|_1 \le 1$. Let $f \in \mathcal{S}(\mathbb{D}^2)$ be nonconstant. By the maximum principle $|f| < 1$ on $\mathbb{D}^2$. Multiplying f by a unimodular constant does not change $\|f_r\|_1$, so we may assume that $t = f(0)$ lies in $[0, 1)$. Then $H = (1 - f)/(1 - t)$ lies in $\mathcal{P}_2$. Its coefficients of positive degree are those of f multiplied by $-1/(1-t)$. Hence $\|f_r\|_1 = t + (1-t) M_r(H) \le 1$. □

Remark 2.2. The normalization loses nothing: if $r < K_2$, then $M_r(H) \le 1$ for every $H \in \mathcal{P}_2$. Indeed, for $0 < s < 1$ the function $z \mapsto H(sz)$ maps $\overline{\mathbb{D}}^2$ into a compact subset of the right half-plane, hence into a disc $\{w : |w - \rho| < \rho\}$ with $\rho \ge 1$. Then $f(z) = 1 - H(sz)/\rho$ lies in $\mathcal{S}(\mathbb{D}^2)$ with $f(0) = 1 - 1/\rho$, and $\|f_r\|_1 \le 1$ becomes $\sum_n \|H_n\|_1 \, s^n r^n \le 1$. Let $s \to 1$. Thus $K_2 = \sup\{r : M_r(H) \le 1 \text{ for all } H \in \mathcal{P}_2\}$.

Lemma 2.3. *Let $H \in \mathcal{P}_2$ and $n \geq 1$. Then $|H_n(\zeta)| \leq 2$ for $\zeta \in \mathbb{T}^2$, and $\|H_n\|_1 \leq 2S_n$.*

Proof. Fix $\zeta \in \mathbb{T}^2$. The function $\varphi(\lambda) = H(\lambda\zeta) = 1 + \sum_{m\geq 1} H_m(\zeta)\lambda^m$ is holomorphic on $\mathbb{D}$ with $\operatorname{Re}\varphi > 0$. For $0 < \rho < 1$,

$$H_n(\zeta)\rho^n = \frac{1}{\pi}\int_0^{2\pi} \operatorname{Re}\varphi(\rho e^{i\theta})\, e^{-in\theta}\, d\theta,$$

so $|H_n(\zeta)|\rho^n \leq 2\operatorname{Re}\varphi(0) = 2$. Letting $\rho \to 1$, we obtain Carathéodory's inequality $|H_n(\zeta)| \leq 2$. Now write $H_n(\zeta_1, \zeta_2) = \zeta_2^n\, q_n(\zeta_1/\zeta_2)$ with $q_n(w) = \sum_{j=0}^n h_{j,n-j} w^j$. As ζ runs over $\mathbb{T}^2$, the quotient ζ_1/ζ_2 runs over $\mathbb{T}$ and $|H_n(\zeta)| = |q_n(\zeta_1/\zeta_2)|$. Hence $\max_{\mathbb{T}} |q_n| \leq 2$, and by the definition of S_n,

$$\|H_n\|_1 = \sum_{j=0}^n |h_{j,n-j}| \leq S_n \max_{\mathbb{T}} |q_n| \leq 2S_n. \qquad \square$$

Remark 2.4. The same identification shows that $S(k, 2) = S_k$: on $\mathbb{T}^2$ a k-homogeneous polynomial in two variables is, up to the factor ζ_2^k, a polynomial in ζ_1/ζ_2 of degree at most k. With this identification, the next corollary is the case $d = 2$ of [7, Theorem 3.3].

Corollary 2.5. *If $0 < r < 1$ and $\sum_{k\geq 1} S_k r^k \leq 1/2$, then $K_2 \geq r$.*

Proof. By Lemma 2.3, $M_r(H) \leq 2\sum_{k\geq 1} S_k r^k \leq 1$ for every $H \in \mathcal{P}_2$. Apply Lemma 2.1. $\square$

In [7] the corresponding estimate is proved for f itself. There Wiener's inequality takes the place of Carathéodory's: the Taylor coefficients $\hat{g}(k)$ of a holomorphic $g\colon \mathbb{D} \to \overline{\mathbb{D}}$ satisfy $|\hat{g}(k)| \leq 1 - |g(0)|^2$ for $k \geq 1$. Patel derives the bound $\|H_n\|_1 \leq 2\sqrt{n}$ for all $H \in \mathcal{P}_2$ from Knese's estimate [14, Lemma 2.1], by a limiting argument with $(1 - \varepsilon H)/(1 + \varepsilon H)$ [20]. Lemma 2.3 with Newman's bound gives it directly.

With the values of S_k for $k \leq 5$ in Table 1 and $S_k \leq \sqrt{k}$ for $k \geq 6$, Corollary 2.5 gives $K_2 \geq 0.301339$. If the value of Ye et al. for S_5 and Newman's bounds for S_6 and S_7 are replaced by the bounds of Proposition 4.2, it gives $K_2 \geq 0.301370$. Patel's coupling of degrees two and three goes beyond this.

3. The coefficients of degrees two and three

Throughout this section $H \in \mathcal{P}_2$, and

$$A_2 = |h_{20}| + |h_{02}|, \quad B_2 = |h_{11}|, \qquad A_3 = |h_{30}| + |h_{03}|, \quad B_3 = |h_{21}| + |h_{12}|,$$

so that $\|H_2\|_1 = A_2 + B_2$ and $\|H_3\|_1 = A_3 + B_3$. Inner products are linear in the first argument, $\langle x, y\rangle = y^* x$.

Lemma 3.1 (Patel [20])**.** $A_2^2 + B_2^2 \leq 4$.

Proof. Choose $\zeta_0 \in \mathbb{T}$ such that $h_{20}\zeta_0^2$ and h_{02} have the same argument, and put $a = h_{20}\zeta_0^2 + h_{02}$ and $b = h_{11}\zeta_0$. Then $|a| = A_2$, $|b| = B_2$ and $H_2(\pm\zeta_0, 1) = a \pm b$. By Lemma 2.3, $|a \pm b| \leq 2$, and therefore $|a|^2 + |b|^2 = \frac{1}{2}(|a+b|^2 + |a-b|^2) \leq 4$. $\square$

Lemma 3.2. *Let R be a positive semidefinite Hermitian Toeplitz matrix of size $k+1$, and let $q(\zeta) = \sum_{j=0}^k c_j \zeta^j$. Then $c^* R c \leq R_{00} \max_{\mathbb{T}} |q|^2$, where $c = (c_0, \ldots, c_k)^{\mathsf{T}}$.*

Proof. Let $e_0, \ldots, e_k$ be vectors in $\mathbb{C}^{k+1}$ with $\langle e_l, e_j\rangle = R_{jl}$, for example the columns of $R^{1/2}$. Since R is Toeplitz, the tuples $(e_0, \ldots, e_{k-1})$ and $(e_1, \ldots, e_k)$ have the same Gram matrix. Hence $e_j \mapsto e_{j+1}$ $(0 \leq j < k)$ extends to a well-defined linear isometry from the span of $e_0, \ldots, e_{k-1}$

onto the span of $e_1, \dots, e_k$, and then to a unitary operator Ω on the span of all the e_j. Then $e_j = \Omega^j e_0$, so $\sum_j c_j e_j = q(\Omega)e_0$. As Ω is unitary, $\|q(\Omega)\| \le \max_{\mathbb{T}} |q|$, and

$$c^* R c = \Big\| \sum_j c_j e_j \Big\|^2 = \|q(\Omega)e_0\|^2 \le \max_{\mathbb{T}} |q|^2 \, \|e_0\|^2 = R_{00} \max_{\mathbb{T}} |q|^2. \qquad \square$$

Lemma 3.2 is a consequence of the Carathéodory–Toeplitz theorem. Ye et al. prove it in the same way [24, Lemma 3.1].

Lemma 3.3 (Patel [20])**.** *Let $p(\zeta) = \sum_{j=0}^{3} c_j \zeta^j$ satisfy $\max_{\mathbb{T}} |p| \le 1$. Then*

$$35\big(|c_0| + |c_3|\big)^2 + 28\big(|c_1| + |c_2|\big)^2 \le 44.$$

Consequently $35A_3^2 + 28B_3^2 \le 176$.

Proof. Replacing $p(\zeta)$ by $p(e^{i\theta}\zeta)$ for a suitable θ, we may assume that c_1 and c_2 have the same argument. Choose $\xi \in \mathbb{T}$ with $|\bar{\xi} c_0 + c_3| = |c_0| + |c_3|$, and let

$$u = (3, 4, -4, -3)^{\mathsf{T}}, \qquad v = (0, 1, 1, 0)^{\mathsf{T}}, \qquad w = (\xi, 0, 0, 1)^{\mathsf{T}}, \qquad R = uu^* + 28vv^* + 35ww^*.$$

The matrix R is positive semidefinite. Since $|\xi| = 1$, it is also Toeplitz, with first row $(44, 12, -12, -9 + 35\xi)$. By Lemma 3.2, with $c = (c_0, \dots, c_3)^{\mathsf{T}}$,

$$|u^* c|^2 + 28|v^* c|^2 + 35|w^* c|^2 = c^* R c \le 44.$$

Here $|v^* c| = |c_1 + c_2| = |c_1| + |c_2|$ and $|w^* c| = |\bar{\xi} c_0 + c_3| = |c_0| + |c_3|$, which proves the first claim. For the second, apply it to $p = q_3/2$ from the proof of Lemma 2.3. Then $|c_0| + |c_3| = A_3/2$ and $|c_1| + |c_2| = B_3/2$. $\square$

The proof above extends to a one-parameter family. For $-1 \le t \le 2$ and $u_t = (t, 1, -1, -t)^{\mathsf{T}}$, the matrix $u_t u_t^* + (t+1)vv^* + (2 + t - t^2)ww^*$ is positive semidefinite and Toeplitz, with first row $(2 + t, t, -t, -t^2 + (2 + t - t^2)\xi)$, so the same argument shows

$$(2 + t - t^2)\big(|c_0| + |c_3|\big)^2 + (t+1)\big(|c_1| + |c_2|\big)^2 \le 2 + t.$$

Lemma 3.3 is the case $t = 3/4$. The case $t = 1/2$ is the inequality $9(|c_0| + |c_3|)^2 + 6(|c_1| + |c_2|)^2 \le 10$, from which Yu et al. obtain $S_3 \le 5/3$ by the Cauchy–Schwarz inequality [25, (6)]. On its own, Lemma 3.3 gives only $\|H_3\|_1 \le \sqrt{396/35} \approx 3.364$, more than $2S_3 = 10/3$. It is used together with the next inequality, which bounds A_3 in terms of B_2.

Lemma 3.4 (Patel [20])**.** $A_3 + \frac{1}{2}B_2^2 \le 2$.

Proof. We use the transfer-function realization of Schur functions on the bidisc. The Schur–Agler class consists of the $g \in \mathcal{S}(\mathbb{D}^2)$ with $\|g(X_1, X_2)\| \le 1$ for every pair of commuting strict contractions X_1, X_2 on a Hilbert space. By Andô's inequality [3], every $g \in \mathcal{S}(\mathbb{D}^2)$ belongs to this class. By Agler's theorem [1] (see also [4, 2]) there are a Hilbert space $\mathcal{K}$, complementary orthogonal projections P and Q on $\mathcal{K}$, and an isometry $V = \left(\begin{smallmatrix} \kappa & x^* \\ y & W \end{smallmatrix}\right)$ of $\mathbb{C} \oplus \mathcal{K}$, the colligation, such that

$$g(z) = \kappa + x^* Z (I - WZ)^{-1} y, \qquad Z = z_1 P + z_2 Q \qquad (z \in \mathbb{D}^2).$$

For $z \in \mathbb{D}^2$ we have $\|Z\| \le \max_j |z_j| < 1$, so $I - WZ$ and the other operators $I - ZW$, $I - ZU$ inverted below are invertible. We may take V unitary, which matters if the realization is given with an isometric colligation, as in [14, Section 2]. Indeed, the operator $\left(\begin{smallmatrix} V & I - VV^* \\ 0 & V^* \end{smallmatrix}\right)$ on $(\mathbb{C} \oplus \mathcal{K}) \oplus (\mathbb{C} \oplus \mathcal{K})$ is unitary and agrees with V on the first summand. We use it as the new colligation, with state space $\mathcal{K} \oplus (\mathbb{C} \oplus \mathcal{K})$ and projections $P \oplus I$ and $Q \oplus 0$. Since the new y is the old one, the new W and x^* agree with the old ones on $\mathcal{K}$, and the new projections map $\mathcal{K}$

into itself, each term $x^*Z(WZ)^m y$ of the expansion of g is unchanged. From now on $\mathcal{K}$, V, x, y, W, P and Q denote the new data.

Apply this to $g = (H-1)/(H+1)$, which lies in $\mathcal{S}(\mathbb{D}^2)$ and vanishes at 0. Then $\kappa = 0$, and since V is unitary,

$$\|x\| = \|y\| = 1, \qquad Wx = 0, \qquad W^*y = 0, \qquad xx^* + W^*W = I, \qquad yy^* + WW^* = I.$$

Put $U = W + yx^*$. These identities imply $U^*U = UU^* = I$. Let $G = I - ZW$. Then $Gx = x$ and $x^*G^{-1}Zy = x^*Z(I-WZ)^{-1}y = g$. Since $I - ZU = G - Zyx^*$ is a rank-one perturbation of G and $1 - g \neq 0$,

$$x^*(I-ZU)^{-1}x = x^*G^{-1}x + \frac{(x^*G^{-1}Zy)(x^*G^{-1}x)}{1 - x^*G^{-1}Zy} = 1 + \frac{g}{1-g} = \frac{1}{1-g}.$$

Hence $x^*(I+ZU)(I-ZU)^{-1}x = 2x^*(I-ZU)^{-1}x - 1 = (1+g)/(1-g) = H$, and expanding the resolvent,

$$H_n(z) = 2x^*(ZU)^n x \qquad (n \geq 1).$$

Relative to $\mathcal{K} = P\mathcal{K} \oplus Q\mathcal{K}$ write

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \qquad x = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad Ux = \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix},$$

so that $\|a\|^2 + \|b\|^2 = \|\tilde{a}\|^2 + \|\tilde{b}\|^2 = 1$. Collecting the monomials z_1z_2, z_1^3 and z_2^3 in $2x^*(ZU)^n x$, we find

$$\tfrac{1}{2}h_{11} = a^*U_{12}\tilde{b} + b^*U_{21}\tilde{a}, \qquad \tfrac{1}{2}h_{30} = a^*U_{11}^2\tilde{a}, \qquad \tfrac{1}{2}h_{03} = b^*U_{22}^2\tilde{b}.$$

Put $\sigma = \|U_{11}^*a\|^2 + \|U_{22}^*b\|^2$ and $\tau = \|U_{11}\tilde{a}\|^2 + \|U_{22}\tilde{b}\|^2$. Since U^* is an isometry, $\|U_{11}^*a\|^2 + \|U_{12}^*a\|^2 = \|a\|^2$ and $\|U_{21}^*b\|^2 + \|U_{22}^*b\|^2 = \|b\|^2$, so $\|U_{12}^*a\|^2 + \|U_{21}^*b\|^2 = 1 - \sigma$. Since U is an isometry, $\|U_{12}\tilde{b}\|^2 + \|U_{21}\tilde{a}\|^2 = 1 - \tau$ in the same way. Applying the Cauchy–Schwarz inequality to $\frac{1}{2}h_{11} = \langle \tilde{b}, U_{12}^*a\rangle + \langle \tilde{a}, U_{21}^*b\rangle = \langle U_{12}\tilde{b}, a\rangle + \langle U_{21}\tilde{a}, b\rangle$ in the two groupings, we obtain

$$\tfrac{1}{4}B_2^2 \leq 1 - \sigma \qquad \text{and} \qquad \tfrac{1}{4}B_2^2 \leq 1 - \tau.$$

Finally $\frac{1}{2}|h_{30}| = |\langle U_{11}\tilde{a}, U_{11}^*a\rangle| \leq \frac{1}{2}(\|U_{11}^*a\|^2 + \|U_{11}\tilde{a}\|^2)$, and likewise for h_{03}, so $A_3 \leq \sigma + \tau \leq 2 - \frac{1}{2}B_2^2$. □

Remark 3.5. Lemmas 3.1 and 3.4 are sharp. Equality holds in both for $H = (1+z_1)/(1-z_1)$, where $A_2 = A_3 = 2$ and $B_2 = 0$, and for $H = (1+z_1z_2)/(1-z_1z_2)$, where $A_2 = A_3 = 0$ and $B_2 = 2$.

Lemma 3.6 (Patel [20])**.** *Let $\alpha, \beta > 0$, $\gamma \geq 0$ and $r > 0$, and put*

$$\mathcal{B}(r) = 4\alpha + 176\beta + 2\gamma + \frac{1}{4\alpha} + \frac{1}{4\alpha + 2\gamma} + \frac{(r-\gamma)^2}{140\beta} + \frac{r^2}{112\beta}.$$

Then $\|H_2\|_1 + r\|H_3\|_1 \leq \mathcal{B}(r)$.

Proof. Expanding the squares verifies the identity

$$\begin{aligned}\mathcal{B}(r) - (A_2 + B_2 + r(A_3 + B_3)) = {}& \alpha(4 - A_2^2 - B_2^2) + \beta(176 - 35A_3^2 - 28B_3^2) + \gamma(2 - A_3 - \tfrac{1}{2}B_2^2) \\ & + \alpha\Big(A_2 - \frac{1}{2\alpha}\Big)^2 + \Big(\alpha + \frac{\gamma}{2}\Big)\Big(B_2 - \frac{1}{2\alpha+\gamma}\Big)^2 \\ & + 35\beta\Big(A_3 - \frac{r-\gamma}{70\beta}\Big)^2 + 28\beta\Big(B_3 - \frac{r}{56\beta}\Big)^2.\end{aligned}$$

By Lemmas 3.1, 3.3 and 3.4, each term on the right is nonnegative. □

Thus $\mathcal{B}(r)$ is the Lagrangian bound, with multipliers α, β and γ, for the maximum of $A_2 + B_2 + r(A_3 + B_3)$ under the three inequalities of Lemmas 3.1, 3.3 and 3.4.

4. Sidon constants and certificates for S_5, S_6 and S_7

The proof of Theorem 1.1 uses upper bounds for S_1 and for S_k with $k \geq 4$. Degrees two and three are covered by Lemma 3.6.

Table 1. Sidon constants $S_k = S(\{0, 1, \dots, k\})$.

k	S_k	source
1	1	elementary
2	$\sqrt{2}$	Newman (see [18])
3	$5/3$	Yu et al. [25]; lower bound Neuwirth [19]
4	2	Newman's bound; equality by Shapiro [23]
5	$2.186529771892569\ldots$; ≤ 2.187	Ye et al. [24]; Proposition 4.2
6	≤ 2.363	Proposition 4.2
7	≤ 2.552	Proposition 4.2
$k \geq 8$	$\leq \sqrt{k}$	Newman [23, Chapter 3]

The proof of Theorem 1.1 uses neither the value $S_3 = 5/3$ nor the value of S_5 below, so that it does not rely on these recent preprints. Ye et al. [24, Theorem 1.1] show that S_5 is the root in $(2.18, 2.19)$ of $675X^4 - 1026X^3 - 936X^2 - 414X + 677$. The proof of Proposition 4.2 rests on the following lemma.

Lemma 4.1. *Let $k \geq 1$. For $w \in \mathbb{C}^{k+1}$ put*

$$\nu(w) = \max\Big\{\operatorname{Re}\sum_{j=0}^{k} \bar{w}_j c_j : \max_{\mathbb{T}}\Big|\sum_{j=0}^{k} c_j z^j\Big| \leq 1\Big\}.$$

(a) $S_k = \max\{\nu(w) : w_0 = w_1 = 1,\ |w_j| = 1\ (2 \leq j \leq k)\}$.
(b) *ν is convex.*
(c) *Let $y \in \mathbb{C}^k$, and let $T(y)$ be the Hermitian Toeplitz matrix with diagonal entries 1 and entries $T(y)_{jl} = y_{l-j}$ for $l > j$. If $T(y)$ is positive definite, then $\nu(w)^2 \leq w^* T(y)^{-1} w$.*
(d) *For $0 < \delta \leq \pi/4$, the arc $\{e^{i\varphi} : |\varphi - \varphi_0| \leq \delta\}$ lies in the triangle with vertices $e^{i(\varphi_0 - \delta)}$, $e^{i(\varphi_0 + \delta)}$ and $e^{i\varphi_0}/\cos\delta$.*
(e) *$\nu(\bar{w}) = \nu(w)$, where $\bar{w} = (\bar{w}_0, \dots, \bar{w}_k)$.*

Proof. (a) Write $q(z) = \sum_j c_j z^j$. If $|w_j| = 1$ for all j, then $\nu(w) \leq \max\{\sum_j |c_j| : \max_{\mathbb{T}} |q| \leq 1\} = S_k$. Equality holds when c attains this maximum and $w_j = c_j/|c_j|$ (any unimodular number if $c_j = 0$). The substitution $c_j \mapsto c_j e^{i(\psi + j\theta)}$ replaces $q(z)$ by $e^{i\psi} q(e^{i\theta} z)$ and w_j by $w_j e^{-i(\psi + j\theta)}$ without changing ν, and a choice of ψ and θ makes $w_0 = w_1 = 1$. (b) ν is a supremum of real-linear functions of w. (c) By Lemma 3.2, $c^* T(y) c \leq \max_{\mathbb{T}} |q|^2$ for $c = (c_0, \dots, c_k)^{\mathsf{T}}$. Hence, by the Cauchy–Schwarz inequality,

$$\operatorname{Re} w^* c \leq \big(w^* T(y)^{-1} w\big)^{1/2} \big(c^* T(y) c\big)^{1/2} \leq \big(w^* T(y)^{-1} w\big)^{1/2} \max_{\mathbb{T}} |q|.$$

(d) After a rotation, $\varphi_0 = 0$. The triangle is $\{u \in \mathbb{C} : \operatorname{Re} u \geq \cos\delta,\ \operatorname{Re}(u e^{-i\delta}) \leq 1,\ \operatorname{Re}(u e^{i\delta}) \leq 1\}$, and $u = e^{i\varphi}$ with $|\varphi| \leq \delta$ satisfies $\cos\varphi \geq \cos\delta$ and $\cos(\varphi \mp \delta) \leq 1$. (e) The map $c \mapsto \bar{c}$ maps the set of admissible c onto itself, since $\sum_j \bar{c}_j z^j = \overline{q(\bar{z})}$, and $\operatorname{Re}\sum_j w_j \bar{c}_j = \operatorname{Re}\sum_j \bar{w}_j c_j$. Hence $\nu(\bar{w}) = \max \operatorname{Re}\sum_j w_j c_j = \max \operatorname{Re}\sum_j w_j \bar{c}_j = \nu(w)$. □

Parts (a) and (c) are variants of the reduction to phase functionals and of the Toeplitz domination bound used by Ye et al. [24, (14) and Proposition 3.2], who normalize the phases differently [24, (17)]. Proposition 4.2 also differs from their proof in the covering of the phase space.

Proposition 4.2. *$S_5 \le 2.187$, $S_6 \le 2.363$ and $S_7 \le 2.552$.*

Proof (computer-assisted). Let (k, b_k) be $(5, 2.187)$, $(6, 2.363)$ or $(7, 2.552)$. By Lemma 4.1(a) it suffices to show that $\nu(1, 1, e^{i\varphi_2}, \dots, e^{i\varphi_k}) \le b_k$ for $(\varphi_2, \dots, \varphi_k) \in [-\pi, \pi]^{k-1}$. By Lemma 4.1(e), which replaces $(\varphi_2, \dots, \varphi_k)$ by $(-\varphi_2, \dots, -\varphi_k)$, we may also assume $\varphi_2 \ge 0$. For $k = 5$ and $k = 6$ the certificate covers $[-\pi, \pi]^{k-1}$ by boxes, obtained from a uniform grid of 8^{k-1} boxes with half-widths $\pi/8$ by repeated bisection. For $k = 7$ it covers $[0, \pi] \times [-\pi, \pi]^5$ in the same way, starting from a uniform grid of $2 \cdot 4^5$ boxes with half-widths $\pi/4$. By Lemma 4.1(d), the vectors $(1, 1, e^{i\varphi_2}, \dots, e^{i\varphi_k})$ with φ in the box lie in a product of $k - 1$ triangles. By Lemma 4.1(b), the maximum of ν over this product is attained at one of its 3^{k-1} vertices. For each vertex v that occurs, the certificate gives $y \in \mathbb{C}^k$ with $T(y)$ positive definite and $v^* T(y)^{-1} v < b_k^2$, so that $\nu(v) < b_k$ by Lemma 4.1(c). A vertex with a coordinate -1 occurs with two labels, phase $-\pi$ and phase π, and has a certificate for each. Table 2 gives the sizes of the three certificates, and Section 6 describes the check. □

Table 2. The certificates of Proposition 4.2.

	$k = 5$	$k = 6$	$k = 7$
bound b_k	2.187	2.363	2.552
initial grid	8^4	8^5	$2 \cdot 4^5$
boxes	8,912	61,118	210,960
distinct vertex labels	269,769	4,333,585	46,344,804
smallest margin $b_k^2 - v^* T(y)^{-1} v$	$1.49 \cdot 10^{-5}$	$8.48 \cdot 10^{-8}$	$8.88 \cdot 10^{-8}$

Numerical maximization, which is not part of the proof, gives $S_6 \approx 2.36256$ and $S_7 \approx 2.55166$, so the bounds 2.363 and 2.552 are close to the true values. Further gains along these lines need bounds for S_n with $n \ge 8$ or sharper inequalities in the degrees two and three.

5. Proof of Theorem 1.1

Throughout the paper, lower bounds are rounded down and upper bounds up. Let $N = 40$. Let $s_4 = 2$, $s_5 = 2.187$, $s_6 = 2.363$, $s_7 = 2.552$, and $s_n = \lceil 10^8 \sqrt{n} \rceil / 10^8$ for $8 \le n \le N$. By Table 1 and Proposition 4.2, $S_n \le s_n$ for $4 \le n \le N$, and $S_n \le \sqrt{n} \le n$ for all n. Let $H \in \mathcal{P}_2$, let $0 < r < 1$, and fix $\alpha, \beta > 0$ and $\gamma \ge 0$. Lemma 2.3 bounds $\|H_n\|_1$ by 2 for $n = 1$, by $2s_n$ for $4 \le n \le N$ and by $2n$ for $n > N$, and Lemma 3.6 bounds $\|H_2\|_1 + r\|H_3\|_1$. Summing the series $\sum_{n>N} n r^n = r^{N+1}(N + 1 - Nr)/(1 - r)^2$, we obtain the majorant

$$M_r(H) \le E(r) := 2r + r^2 \mathcal{B}(r) + 2 \sum_{n=4}^{N} s_n r^n + \frac{2r^{N+1}(N + 1 - Nr)}{(1 - r)^2}.$$

We take

$$r_* = \frac{60333}{200000}, \qquad \alpha = \frac{20674053}{62500000}, \qquad \beta = \frac{3901457}{1562500000}, \qquad \gamma = \frac{101990031}{10^9}.$$

All quantities in $E(r_*)$ are then rational, and an exact computation shows $E(r_*) < 0.99999556 < 1$. Since α, β and γ are fixed, E is continuous on $(0, 1)$, so there is $r' > r_*$ with $E(r') < 1$. By Lemma 2.1, $K_2 \ge r' > r_*$, which proves Theorem 1.1.

Remark 5.1 (Patel's computation). With Patel's parameters $\alpha = 264631/800000$, $\beta = 249627/10^8$ and $\gamma = 10197021/10^8$, with $N = 20$ and with $s_n = \lceil 10^8\sqrt{n}\,\rceil/10^8$ for $4 \le n \le 20$, the function E is the rational majorant that he evaluates in [20]; its value at his radius $r_P = 15079379/(5 \cdot 10^7)$ is less than 0.99999997857, below his bound $1 - 1/(5 \cdot 10^7)$.

6. Verification

6.1. **Computer checks.** The certificates of Proposition 4.2 are checked in the ball arithmetic of Arb [13], through python-flint [22]. The checker reads the numbers of the certificates as binary64 numbers, exactly, rebuilds the boxes from the recorded bisections and verifies at every vertex the condition stated in the proof of Proposition 4.2; for S_6 and S_7, whose certificates are split into files, a further program checks, from the outputs of the file checks, that the files together cover the phase domain. The majorant $E(r_*)$ of Section 5 is evaluated in exact rational arithmetic, and so is Patel's majorant in Remark 5.1. The programs and their output are archived with the certificates (see Data availability).

6.2. **Formal proof.** The formal proof is written in the Lean 4 proof assistant [15] with the Mathlib library [16]. The formal statements use Mathlib's definitions. A family $c\colon \mathbb{N}^2 \to \mathbb{C}$ is admissible if the series $\sum_\alpha c_\alpha z^\alpha$ converges absolutely at every point of $\mathbb{D}^2$ and its value there has modulus less than 1. Bohr's inequality holds at r if $\sum_{\alpha\in F} |c_\alpha| r^{|\alpha|} \le 1$ for every admissible family c and every finite set $F \subset \mathbb{N}^2$, and K_2 is the supremum of the radii $r > 0$ at which it holds. It agrees with the definition in the introduction: a nonconstant $f \in \mathcal{S}(\mathbb{D}^2)$ maps $\mathbb{D}^2$ into $\mathbb{D}$, and a constant of modulus 1 satisfies Bohr's inequality with equality. The main theorem is the following.

Theorem 6.1 (formalized in Lean)**.** *Suppose that $A_3 + \frac{1}{2}B_2^2 \le 2$, the inequality of Lemma 3.4, holds for every family h with $h_{00} = 1$ whose series converges absolutely on $\mathbb{D}^2$ to values with positive real part, that is, for the coefficients of every $H \in \mathcal{P}_2$. Then $K_2 \ge 0.301665$.*

The proof of the hypothesis rests on Andô's inequality and Agler's realization, which are not in Mathlib. The formal proof follows Sections 2 to 5 and checks the certificates of Proposition 4.2 in one of two ways.

(1) A checking program written in Lean validates the certificates, converted into Lean source files, in exact rational arithmetic and ball arithmetic, and a Lean proof shows that every part it accepts implies the bound of Lemma 4.1(c) on the boxes of that part, so that the converted files need not be trusted. These checks run as compiled code, through `native_decide`, so the trusted base includes Lean's compiler and runtime; the rest of the proof uses only the standard axioms.
(2) A second formalization proves the three bounds of Proposition 4.2 with every check run by Lean's kernel. With the step of the first proof that derives the main theorem from them, the main theorem follows from four theorems in which every check is run by the kernel; these are not combined into a single Lean theorem.

Data availability

The certificates of Proposition 4.2, the programs of Section 6, the programs that produced the certificates, the controls and the output of all runs, and the Lean files of Section 6.2, are archived at doi:10.5281/zenodo.23716440. The same files, except the 48 files of the certificate for S_6, the 576 files of the certificate for S_7 and an archive of generated Lean files of the kernel-only proof of $S_7 \le 2.552$, are also in the repository `https://github.com/tejasvi/bidisc-bohr-radius/tree/v6`. The README of the archive lists the Lean theorems and projects, with the versions

of Lean and Mathlib, and gives the commands that run the checks and build the proofs. The certificate for S_5 has the SHA-256 checksum

`ce3e9dd41a0dbd9c3a8ea49b04aa7aaf21f0fe557291a2084c7739461a460a14`,

and the archive lists the checksums of the other files.

Use of AI tools

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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ORCID: `https://orcid.org/0000-0003-4668-3639`