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AN UPPER BOUND FOR THE SMALLEST LIMIT POINT OF THE ABSOLUTE MAHLER MEASURE OF TOTALLY POSITIVE ALGEBRAIC INTEGERS

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Abstract. Let ℓ_+ be the smallest limit point of the values of the absolute Mahler measure $\Omega(\alpha) = M(\alpha)^{1/\deg\alpha}$ on totally positive algebraic integers α. Smyth's work gives $\ell_+ \le \ell^2 = 1.72731\ldots$, where $\ell = 1.31427\ldots$ is the limit point that he found for totally real algebraic integers. We prove $\ell_+ \le 1.7239501$, and that infinitely many totally positive algebraic integers satisfy $\Omega(\alpha) < 1.7239501$. Together with the lower bound $\ell_+ > 1.7234615$ of a companion paper, this places ℓ_+ in an interval of length less than $4.9 \cdot 10^{-4}$. The proof exhibits a measure μ on $[0, 9]$ with a piecewise linear density on 3,481 rational nodes. Its logarithmic potential is bounded below by a combination of $\log|Q|$ for 21 irreducible polynomials Q, namely x and polynomials taken from Flammang's work. By a criterion of Smith, μ, normalized to mass 1, is a limit of distributions of conjugates of totally positive algebraic integers, and its integral of $\log^+ x$ gives the bound. Two separately written programs verify the conditions in ball arithmetic. Both results are also proved in the Lean proof assistant, assuming Smith's theorem and a lower bound for the potential of the arcsine measure, with the checks of the certificate run either as compiled code or by Lean's kernel.

1. Introduction

Let α be an algebraic number of degree n, with minimal polynomial $P(x) = a\prod_{i=1}^{n}(x-\alpha_i)$ in $\mathbb{Z}[x]$, primitive and with leading coefficient $a \ge 1$. The Mahler measure and the absolute Mahler measure of α are

$$M(\alpha) = a\prod_{i=1}^{n}\max(1, |\alpha_i|), \qquad \Omega(\alpha) = M(\alpha)^{1/n},$$

and $\log\Omega(\alpha)$ is the absolute logarithmic Weil height of α. We call α totally positive if all its conjugates α_i are real and positive. Let $\mathcal{L}_+$ be the set of the values $\Omega(\alpha)$, where α runs through the totally positive algebraic integers, let ℓ_+ be the smallest limit point of $\mathcal{L}_+$, and let

$$E_+ = \sup\{c : \text{ only finitely many totally positive algebraic integers } \alpha \text{ satisfy } \Omega(\alpha) \le c\}$$

be the essential minimum. Since every neighbourhood of ℓ_+ contains infinitely many points of $\mathcal{L}_+$, we have $E_+ \le \ell_+$.

Schinzel [13] proved that $\Omega(\alpha) \ge (1+\sqrt{5})/2$ for every totally positive algebraic integer $\alpha \ne 0, 1$. Smyth [15, Theorems 1 and 2] proved that the values $\Omega(\beta)$, for totally real algebraic integers β, have a limit point $\ell = 1.31427\ldots$ and are dense in (ℓ, ∞). Here $\ell = \lim_n \Omega(\beta_n)$, where $\beta_0 = 1$ and $\beta_n > 0$ satisfies $\beta_n - 1/\beta_n = \beta_{n-1}$; Smyth proved that β_n has degree 2^n, so that its conjugates are the 2^n solutions of $H^{\circ n}(x) = 1$ for $H(x) = x - 1/x$ [15, Theorem 1]. A floating-point computation of $\Omega(\beta_n)$ for $n \le 26$, in which the increments roughly halve from one n to the next, gives $\ell = 1.3142722\ldots$ and $\ell^2 = 1.7273115\ldots$. Since $\Omega(\beta^2) = \Omega(\beta)^2$ and β^2 is totally positive when $\beta \ne 0$ is totally real, every number in $[\ell^2, \infty)$ is a limit point

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of $\mathcal{L}_+$, and $\ell_+ \le \ell^2$. Lower bounds for E_+, and hence for ℓ_+, come from Smyth's method of auxiliary functions [16]. Flammang [4, Theorem 1] proved $\Omega(\alpha) \ge 1.722069$ for all totally positive algebraic integers outside an explicit finite set, and in a companion paper we prove that, for every $\varepsilon > 0$, all but finitely many totally positive algebraic numbers satisfy $\log \Omega(\alpha) \ge m_0 - \varepsilon$, where $e^{m_0} = 1.72346152\ldots > 1.7234615$ [17, Theorem 1.1]; in particular $E_+ \ge e^{m_0}$. As far as we know, no upper bound for ℓ_+ below ℓ^2 has been published.

Theorem 1.1. *There are infinitely many totally positive algebraic integers α with $\Omega(\alpha) < 1.7239501$, and $E_+ \le \ell_+ \le 1.7239501$.*

Corollary 1.2. $1.7234615 < E_+ \le \ell_+ \le 1.7239501$.

The corollary combines Theorem 1.1 with [17, Theorem 1.1]. The bound of Theorem 1.1 lies more than 0.0033 below ℓ^2, and the interval of Corollary 1.2 has length less than $4.9 \cdot 10^{-4}$.

The method. Lower bounds for E_+ are found by averaging an inequality $\log^+ x \ge m + \sum_j c_j \log|Q_j(x)|$ over the conjugates. Upper bounds need algebraic integers with small $\Omega(\alpha)$, and we obtain them from measures. For a probability measure ν on $\mathbb{C}$ with compact support, let $U_\nu(z) = \int \log|z-y|\, d\nu(y)$ be its logarithmic potential. If $\alpha_1, \alpha_2, \ldots$ are distinct totally positive algebraic integers whose conjugates lie in a fixed interval $[0, X]$ and whose distributions of conjugates converge weakly to ν, then $\log \Omega(\alpha_k)$ converges to $\int \log^+ x\, d\nu$, and Serre showed that $\int \log|Q|\, d\nu \ge 0$ for every nonzero $Q \in \mathbb{Z}[x]$ [14, remark after Theorem 1.5]. Smith [14, Theorem 1.5] proved the converse when the interval has capacity greater than 1, that is, when $X > 4$, and Orloski and Talebizadeh Sardari [9, Theorem 1.2] proved a closely related characterization. Hence every probability measure ν on $[0, X]$ with $X > 4$ and $\int \log|Q|\, d\nu \ge 0$ for all nonzero $Q \in \mathbb{Z}[x]$ gives the bound $\ell_+ \le \exp \int \log^+ x\, d\nu$ (see the proof of Lemma 2.1). This way of bounding an essential minimum from above is described in [1, Section 1.3]. Smith used measures of this kind to show that the Schur–Siegel–Smyth trace constant is less than 1.89831 [14, Theorem 1.1 and Example 5.16], and Orloski and Talebizadeh Sardari [10] lowered the bound to 1.8216.

To verify the infinitely many conditions $\int \log|Q|\, d\nu \ge 0$ we use a criterion of Smith [14, Proposition 5.15]. It suffices that $U_\nu \ge \sum_j b_j \log|Q_j|$ on $\mathbb{C}$ for finitely many distinct primitive polynomials $Q_j \in \mathbb{Z}[x]$, irreducible over $\mathbb{Q}$, with weights $b_j \ge 0$ such that $\sum_j b_j \deg Q_j \le 1$, and that $\int \log|Q_j|\, d\nu \ge 0$ for these Q_j. When the degree condition holds strictly, the minimum principle reduces the first inequality to an interval that contains the support of ν (Lemma 2.1). Our measure has a continuous piecewise linear density, so that its potential has a closed form. We take as Q_j the polynomial x and the polynomials of Flammang's table [4, Table 1] or their irreducible factors, the same family that gives the lower bounds in [4] and [17]. We found the measure and the weights by linear programming in floating point. Two programs check the conditions in ball arithmetic [5, 3], with exact rational data, and prove (C3) on the whole interval $[e_0, e_N]$ (Section 3). Section 4 describes a proof in Lean of Theorem 1.1, apart from the inequality $E_+ \le \ell_+$, under two hypotheses from the literature.

2. The criterion

Let $X > 4$ be rational. Let $0 \le e_0 < e_1 < \cdots < e_N \le X$ be rational nodes and let $G\colon \mathbb{R} \to [0, \infty)$ be the continuous function that is linear on each interval $[e_{k-1}, e_k]$, takes rational values $G(e_k)$ with $G(e_0) = G(e_N) = 0$, and vanishes outside $[e_0, e_N]$. Let $\mu = G(x)\, dx$, let $s = \mu(\mathbb{R}) > 0$ be its mass, and let

$$U(z) = \int \log|z-y|\, G(y)\, dy \qquad (z \in \mathbb{C})$$

be its potential. Let $Q_1, \dots, Q_J \in \mathbb{Z}[x]$ be distinct primitive polynomials that are irreducible over $\mathbb{Q}$, let $b_1, \dots, b_J \geq 0$, and put

$$h(z) = U(z) - \sum_{j:\, b_j > 0} b_j \log |Q_j(z)|,$$

with $h(z) = +\infty$ at the roots of these Q_j.

Lemma 2.1. *Assume that*

(C1) $\sum_j b_j \deg Q_j < s$;
(C2) $\int \log |Q_j|\, d\mu \geq 0$ *for every* j *with* $b_j > 0$;
(C3) $h(x) \geq 0$ *for every* $x \in [e_0, e_N]$.

Let $v = \frac{1}{s}\int \log^+ x\, d\mu(x)$. *Then there are distinct totally positive algebraic integers* $\alpha_1, \alpha_2, \dots$ *with* $\Omega(\alpha_k) \to e^v$, *and* $\ell_+ \leq e^v$.

Proof. Step 1: $h \geq 0$ *on* $\mathbb{C}$. Since G is bounded, U is continuous on $\mathbb{C}$, so h is lower semi-continuous on $\mathbb{C}$. On the domain $D = \mathbb{C} \setminus [e_0, e_N]$ the function U is harmonic and each $\log |Q_j|$ is subharmonic, so h is superharmonic on D. As $z \to \infty$, $U(z) = s \log |z| + o(1)$ and $\log |Q_j(z)| = \deg Q_j \log |z| + O(1)$, so $h(z) \to +\infty$ by (C1). For large R, h is therefore positive on the circle $|z| = R$, and on $[e_0, e_N]$ we have $\liminf_{z\to\zeta} h(z) \geq h(\zeta) \geq 0$ by lower semi-continuity and (C3). By the minimum principle for superharmonic functions on the bounded domain $D \cap \{|z| < R\}$, $h \geq 0$ there, and hence on $\mathbb{C}$.

Step 2: $\int \log |P|\, d\mu \geq 0$ *for every nonzero* $P \in \mathbb{Z}[x]$. This is the argument of [14, Proposition 5.15]. It suffices to consider P primitive, irreducible over $\mathbb{Q}$ and of degree $d \geq 1$, since a nonzero integer c contributes $s \log |c| \geq 0$. If $P = \pm Q_j$ with $b_j > 0$, use (C2). Otherwise P has no common root with any Q_j with $b_j > 0$, because both are primitive and irreducible and $P \neq \pm Q_j$. Let a be the leading coefficient of P and $\rho_1, \dots, \rho_d$ its roots. By Fubini's theorem and Step 1,

$$\begin{aligned}\int \log |P|\, d\mu &= s \log |a| + \sum_{k=1}^{d} U(\rho_k) \;\geq\; s \log |a| + \sum_j b_j \sum_{k=1}^{d} \log |Q_j(\rho_k)| \\ &= s \log |a| + \sum_j b_j \big(\log |\operatorname{Res}(P, Q_j)| - \deg Q_j \log |a|\big),\end{aligned}$$

where the sums over j run over the j with $b_j > 0$. Each resultant is a nonzero integer, so the right side is at least $\big(s - \sum_j b_j \deg Q_j\big) \log |a| \geq 0$ by (C1).

Step 3: the sequence. The interval $\Sigma = [0, X]$ is compact, connected and has capacity $X/4 > 1$ [12, Example I.3.5]. By Step 2 and [14, Theorem 1.5], applied to the probability measure μ/s, there are distinct monic irreducible polynomials $R_1, R_2, \dots \in \mathbb{Z}[x]$ with all roots in Σ whose root distributions $\frac{1}{\deg R_k} \sum_{R_k(\rho)=0} \delta_\rho$ converge weakly to μ/s. At most one R_k equals x, and the roots of every other R_k lie in $(0, X]$. For these k let α_k be a root of R_k. Then the α_k are distinct totally positive algebraic integers, and since $\log^+$ is continuous on Σ,

$$\log \Omega(\alpha_k) = \frac{1}{\deg R_k} \sum_{R_k(\rho)=0} \log^+ \rho \;\longrightarrow\; \frac{1}{s}\int \log^+ x\, d\mu(x) = v.$$

Step 4: $\ell_+ \leq e^v$. Choose $X' \geq X$ with $\log(X'/4) > v$, and let ω be the equilibrium measure of $[0, X']$. Its potential satisfies $U_\omega(z) \geq \log(X'/4) > 0$ for all $z \in \mathbb{C}$ [12, (I.1.4) and Example I.3.5]. So every nonzero $P \in \mathbb{Z}[x]$ with leading coefficient a and roots ρ_k satisfies $\int \log |P|\, d\omega = \log |a| + \sum_k U_\omega(\rho_k) \geq 0$, and $w = \int \log^+ x\, d\omega \geq \int \log x\, d\omega = U_\omega(0) \geq \log(X'/4) > v$. For $t \in [0, 1]$ the probability measure $(1-t)\mu/s + t\omega$ on $[0, X']$ satisfies $\int \log |P| \geq 0$ for every nonzero $P \in \mathbb{Z}[x]$, because this condition is linear in the measure, and Step 3 with $[0, X']$ in place of $[0, X]$ gives

distinct totally positive algebraic integers α_k with $\log \Omega(\alpha_k) \to (1-t)v+tw$. So the closure of $\mathcal{L}_+$ contains the interval $[e^v, e^w]$, every point of this interval is a limit point of $\mathcal{L}_+$, and $\ell_+ \le e^v$. $\square$

Remark 2.2. Only the weighted polynomials enter conditions (C2) and (C3). Reducible polynomials cannot carry weight: each irreducible factor needs its own condition (C2). Condition (C3) concerns only the interval $[e_0, e_N]$, so no estimates on the rest of the real line are needed. Smith writes the potential with the opposite sign; in our notation, the hypothesis of [14, Proposition 5.15] is $h \ge 0$ on all of $\mathbb{C}$ together with $\sum_j b_j \deg Q_j \le s$ and (C2), and Step 1 shows that (C1) and (C3) imply $h \ge 0$ on $\mathbb{C}$, while (C1) gives the degree condition.

3. The certificate and its verification

3.1. **The measure and the polynomials.** The certificate has $X = 9$ and 3,481 nodes $e_k = 0.1125 + k/400$ $(0 \le k \le N)$, from $e_0 = 0.1125$ to $e_N = 8.8125$. The support of μ consists of 53 intervals, and its mass is $s = 1 - \varepsilon$ with $0 < \varepsilon < 3 \cdot 10^{-15}$. Flammang's table [4, Table 1] lists polynomials in $y = x + 1/x - 2$; we write F_j for the reciprocal polynomial in x that she obtains from its j-th entry by this substitution and clearing the denominator [4, Section 2.1], and keep her numbering. The weighted polynomials are the 21 polynomials of Table 1: x, twelve of the F_j, and eight irreducible factors of five reducible F_j. All of them have only real roots except F_{17}, which has four non-real roots. All weights b_j are positive, and $\sum_j b_j \deg Q_j < 0.939667 < s$. The certificate stores the coefficients of the Q_j as integers and the other data used in the checks as exact rationals (its two floating-point entries are not used in the checks). It also states a rational v^* with $v \le v^*$, the claimed bound:

$$v^* = \frac{544618200701}{10^{12}} = 0.544618200701, \qquad e^{v^*} = 1.7239500537\ldots \le 1.7239501.$$

3.2. **Verification.** By Lemma 2.1, Theorem 1.1 follows once the certificate satisfies (C1)–(C3) and $v = \frac{1}{s}\int \log^+ x \, d\mu \le v^*$, since $E_+ \le \ell_+$. The first program, written in Python with python-flint [11], the Python interface of FLINT [3], checks these conditions with 200-bit balls [5]. It rebuilds μ from the nodes and the values of G. Since $G'' = \sum_i C_i \delta_{e_i}$ with rational slope jumps C_i satisfying $\sum_i C_i = \sum_i C_i e_i = 0$, the potential has the closed form

$$U(z) = \operatorname{Re} \sum_{i=0}^{N} C_i \Big(\tfrac{1}{2}(e_i - z)^2 \operatorname{Log}(e_i - z) - \tfrac{3}{4}(e_i - z)^2 \Big),$$

where Log is the principal logarithm; for non-real z all the numbers $e_i - z$ lie in one open half plane. For real x it reads $U(x) = \sum_i C_i (\frac{1}{2} t_i^2 \log|t_i| - \frac{3}{4} t_i^2)$ with $t_i = e_i - x$. The program checks:

(a) the data: integer coefficients, each Q_j primitive, irreducible over $\mathbb{Q}$ (factored by FLINT) and different from the others, $b_j \ge 0$, the nodes increasing with $e_0 \ge 0$ and $e_N \le X$, $G \ge 0$ with $G(e_0) = G(e_N) = 0$, and $X > 4$;
(b) condition (C1), in exact rational arithmetic;
(c) condition (C2), as $\int \log|Q_j| \, d\mu = s \log|a_j| + \sum_\rho U(\rho)$, the sum over the roots ρ of Q_j enclosed by FLINT, with the complex closed form at the non-real roots of F_{17}; the smallest of these integrals is about 10^{-7};
(d) condition (C3) on the whole interval $[e_0, e_N]$, as described below;
(e) $\frac{1}{s}\int \log^+ x \, d\mu \le v^*$, with the closed forms of $\int x^k \log x \, dx$ on each linear piece of G.

For (C3) the program covers $[e_0, e_N]$ by windows around the real roots of the weighted Q_j and by intervals between them, with exact rational ends, and checks that the cover is complete. It brackets each real root in $[e_0, e_N]$ by an exact sign change and compares the number of brackets with a Sturm count. On a window $[L, R]$ with centre c and radius r it uses the mean value theorem, $h \ge U(c) - r \max|U'| - \sum_j b_j \log(|Q_j(c)| + r \max|Q_j'|)$, with the maxima bounded in

TABLE 1. The weighted polynomials of the certificate, with their place in Flammang's table [4, Table 1] and the weights b_j, rounded to four significant digits. For the polynomials of degree 12 and more only the index is given.

polynomial	degree	in [4, Table 1]	b_j
x	1	–	0.06166
$x-1$	1	factor of F_1	0.08517
x^2-3x+1	2	F_2	0.04562
x^3-6x^2+5x-1	3	factor of F_6	0.007028
x^3-5x^2+6x-1	3	factor of F_6	0.006941
$x^4-7x^3+13x^2-7x+1$	4	F_4	0.02576
$x^4-8x^3+15x^2-8x+1$	4	F_5	0.006898
$x^4-8x^3+14x^2-7x+1$	4	factor of F_{10}	0.006098
$x^4-7x^3+14x^2-8x+1$	4	factor of F_{10}	0.006738
$x^6-12x^5+44x^4-67x^3+44x^2-12x+1$	6	F_7	0.003420
$x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1$	8	F_8	0.01546
$x^8-16x^7+91x^6-244x^5+337x^4-244x^3+91x^2-16x+1$	8	F_9	0.003620
$x^8-16x^7+90x^6-239x^5+329x^4-239x^3+90x^2-16x+1$	8	F_{11}	0.002133
$x^8-15x^7+84x^6-228x^5+323x^4-240x^3+91x^2-16x+1$	8	factor of F_{19}	$2.313\cdot10^{-8}$
$x^8-16x^7+94x^6-265x^5+391x^4-308x^3+128x^2-26x+2$	8	factor of F_{20}	0.002978
$2x^8-26x^7+128x^6-308x^5+391x^4-265x^3+94x^2-16x+1$	8	factor of F_{20}	0.003152
	12	F_{12}	$2.934\cdot10^{-5}$
	14	F_{13}	0.005300
	14	F_{17}	$8.394\cdot10^{-5}$
	16	F_{18}	0.004701
	32	F_{25}	0.002733

ball arithmetic over $[L,R]$; windows are shrunk until this bound is positive. On an interval $[p,q]$ that contains no node in its interior and no root of a weighted Q_j, the function h is continuously differentiable, twice continuously differentiable on (p,q), and has the integrable second derivative

$$h''(x)=\sum_i C_i\log|e_i-x|+\sum_j b_j\sum_\rho \operatorname{Re}\frac{1}{(x-\rho)^2},$$

and the program uses the following lemma.

Lemma 3.1. *Let $f\colon[p,q]\to\mathbb{R}$ be continuously differentiable, twice continuously differentiable on (p,q), with f'' integrable, and let $\delta=q-p$. Then $f(x)\ge\min\bigl(f(p),f(q)\bigr)-\frac{\delta}{4}\int_p^q(f'')_+$ for all $x\in[p,q]$.*

Proof. The derivative f' is continuous on $[p,q]$ and has an integrable derivative on (p,q), so it is absolutely continuous, and integration by parts twice gives $f(x)=\Lambda(x)-\int_p^q k(x,t)f''(t)\,dt$, where Λ is the linear function with $\Lambda(p)=f(p)$ and $\Lambda(q)=f(q)$ and $k(x,t)=(\min(x,t)-p)(q-\max(x,t))/\delta$. Since $0\le k\le\delta/4$, the integral is at most $\frac{\delta}{4}\int_p^q(f'')_+$, and $\Lambda(x)\ge\min(f(p),f(q))$. □

The program evaluates $h(p)$ and $h(q)$ with the differences e_i-p, e_i-q and the values $Q_j(p)$, $Q_j(q)$ computed exactly. It bounds $\int_p^q(h'')_+$ by ball arithmetic over $[p,q]$. A node $e_i\in\{p,q\}$ contributes the logarithmic singularity $C_i\log|e_i-x|$, which the program integrates in closed form if $C_i<0$ and bounds by $C_i\log\delta$ if $C_i>0$, with $\delta\le1$. If the resulting lower bound for h on $[p,q]$ is not positive, the interval is bisected. The program fails if a lower bound at an end of an interval is not positive, or after 50 successive bisections. For the certificate all checks

pass. Condition (C3) needed 161 root windows and 3,640 initial intervals, which bisection turned into 30,058 checked intervals, and the smallest lower bound for h at an end of an interval was $1.51 \cdot 10^{-6}$. The run took 59 seconds with 40 processes.

The second program was written from the criterion of Lemma 2.1 and the description of the certificate format, before its author had read the first program, and shares no code with it. It uses the closed forms above, which are stated in that description, but a different method for (C3). It covers $[e_0, e_N]$ by intervals I and bounds h from below on each of them as follows. If $[e_{k-1}, e_k]$ is the interval between consecutive nodes that contains I, the nodes at distance more than 0.05 from $[e_{k-1}, e_k]$ enter through a Taylor model of order 6 with a rigorous remainder. The terms of nodes with $C_i > 0$ within distance $4|I|$ of I are concave on I and are bounded by their chords. All other terms are bounded by Taylor's formula with integral remainder, $R(x) \geq R(m) + R'(m)(x-m) + \frac{1}{2} \inf_I R''\,(x-m)^2$, where m is the midpoint of I. The real roots of the weighted polynomials are cut out by brackets of width $2 \cdot 10^{-40}$ with a logarithmic lower bound, and intervals are bisected until the bound is positive. The program also compares its closed form for U with a separate formula at random points. On the certificate it confirms all conditions, with 13,356 interval tests and 161 root brackets for (C3). For the polynomial F_{17}, whose non-real roots enter (C2) through the complex closed form, the value $\int \log|F_{17}|\, d\mu = 0.1857\ldots$ agrees with direct numerical quadrature to 18 digits.

3.3. **Tests of the programs.** To see that the programs detect violations, we ran both of them on the modified certificates of Table 2. Those in the first group violate a condition of Lemma 2.1 or the bound $v \leq v^*$. The effect of those in the second group is not clear in advance; that both programs reject them shows that the certificate has little slack. The third group adds a narrow bump to the density, with v^* raised by 10^{-6}, so that (C1), (C2) and $v \leq v^*$ still hold. The first is a tent of mass $1.2 \cdot 10^{-7}$ and width $4 \cdot 10^{-6}$ between two points of the grid of spacing $2 \cdot 10^{-5}$ on which the search checked h (Section 5). The other two are bumps of masses $1.1 \cdot 10^{-7}$ and $2.0 \cdot 10^{-7}$, one inside the support at a point where h is small and one next to a node with $C_i > 0$. Each makes h negative on a short interval while h stays positive at the nearby points of the grid and at the nodes, and the two programs reject each of them at the same point.

Table 2. Modified certificates. Both programs reject all of them except the last one.

group	modifications
violates a condition	v^* lowered by $2 \cdot 10^{-9}$ or by 10^{-6}; G multiplied by 0.99 or 0.5; the heaviest or the second heaviest polynomial removed; $\sum_j b_j \deg Q_j > s$; a reducible weighted polynomial; a non-primitive polynomial; two nodes swapped; $G(e_N) \neq 0$; $X = 4$
effect not clear in advance	G multiplied by 1.01; the heaviest weight multiplied by 1.01 or 0.99; one coefficient changed; F_{17} removed; its weight multiplied by 0.5, 1.5 or 10; one coefficient of F_{17} changed
narrow bumps	the three bumps described in the text
valid	v^* raised by 10^{-6}

4. Formal verification

We also prove Theorem 1.1, except for the inequality $E_+ \leq \ell_+$, which follows from the definitions, in the Lean 4 proof assistant [7] with the Mathlib library [6], under two hypotheses taken from the literature. The formal statements use the definitions of the Lean formalization of the companion paper [17, Section 4]. An integer polynomial P is a minimal polynomial of $\alpha \in \mathbb{C}$

if it is irreducible in $\mathbb{Z}[x]$, has positive degree and vanishes at α; then $\Omega(\alpha) = M(P)^{1/\deg P}$, where $M(P)$ is Mathlib's Mahler measure, and α is totally positive if all complex roots of P are real and positive. The theorem `tpup_infinite` states that the set of totally positive algebraic integers α with $\Omega(\alpha) < 1.7239501$ is infinite. The theorem `tpup_limit_point` states that some accumulation point y of $\mathcal{L}_+$, in Mathlib's sense, satisfies $y \le 1.7239501$; this is the inequality $\ell_+ \le 1.7239501$, in a form that does not presuppose that ℓ_+ exists. The theorem `exp_v0_lt` states $e^{v^*} < 1.7239501$. These three statements are written in one file with `sorry` in place of the proofs, so that they can be read apart from the proofs. The proved theorems have the same names in two namespaces, described below, and a check run after the build confirms that their types are equal to those of the statements.

The two hypotheses are results that the proof of Lemma 2.1 cites; they are arguments of the theorems, not axioms. The first, `SmithTheorem`, is one direction of [14, Theorem 1.5], for the intervals $\Sigma = [0, X]$ with $X > 4$: if ν is a probability measure on $\mathbb{R}$ with $\nu(\mathbb{R} \setminus [0, X]) = 0$ such that, for every nonzero $P \in \mathbb{Z}[x]$, ν gives no mass to the zeros of P, $\log|P|$ is ν-integrable and $\int \log|P|\, d\nu \ge 0$, then there are pairwise distinct monic irreducible polynomials $R_k \in \mathbb{Z}[x]$, with all roots real and in $[0, X]$, such that $\frac{1}{\deg R_k} \sum_{R_k(\rho)=0} f(\rho) \to \int f\, d\nu$ for every continuous $f\colon \mathbb{R} \to \mathbb{R}$. The three conditions on P together say that $\int \log|P|\, d\nu \ge 0$ with $\log 0 = -\infty$, which is Smith's hypothesis; the first of them is needed because Mathlib defines $\log 0 = 0$. The second hypothesis, `ArcsinePotential`, states that for $X > 0$ the arcsine measure $dx/(\pi\sqrt{x(X-x)})$ on $(0, X)$ is a probability measure whose potential is defined and at least $\log(X/4)$ at every point of $\mathbb{C}$ [12, (I.1.4) and Example I.3.5]. It enters Step 4 of the proof of Lemma 2.1, and only `tpup_limit_point` assumes it. Everything else is proved. This includes Steps 1 to 4 of that proof. Step 1 uses a minimum principle for functions with the super-mean-value property, and Steps 3 and 4 are proved from the two hypotheses. It also includes the closed form of U on the real line, conditions (C1)–(C3), the bound $v \le v^*$ and the irreducibility of the 21 polynomials.

The conditions on the certificate are checked by programs written in Lean's core language, which work with integers only and compute logarithms with proved error bounds; some of their routines are those of the Lean checker of [17, Section 4]. For (C3) a program covers $[e_0, e_N + 1/400]$, one node spacing beyond e_N, by 11,393 intervals and checks that they tile this interval. The intervals are computed in advance and are part of its input. On 11,232 of them it bounds h from below by a linear function. For this it uses a variant of Lemma 3.1: if $f'' \le M$ on (p, q), then f is at least its chord on $[p, q]$ minus $\max(M, 0)(q-p)^2/8$. It applies the variant to h without two kinds of terms. The first are some terms $-b_j \log|Q_j|$ that are convex on $[p, q]$; the program verifies their convexity and bounds them from below by their tangent lines at p or at q. The second are the terms of the nodes $e_i \in \{p, q\}$ with $C_i < 0$, which are nonnegative on $[p, q]$. The other 161 intervals are windows around the 161 real roots of the weighted Q_j in this range, and on them it bounds the terms of h separately. For (C2) with the twenty weighted polynomials that have only real roots, the input gives $\deg Q_j$ disjoint brackets for the roots of Q_j, each with a sign change or an exact rational root, which the program verifies; it then bounds U at each root from below by its value at one end of the bracket minus a Lipschitz term. For F_{17} (`Q_15` in the Lean files), which has non-real roots, it bounds $\int \log|F_{17}|\, d\mu$ from below by quadrature over pieces: on a piece without a zero of F_{17} it uses a lower bound for $|F_{17}|$ there, and on a piece around a real root ρ it uses $|F_{17}(t)| \ge m|t - \rho|$, where m is a lower bound for $|F_{17}'|$ on the piece. Further programs check (C1), check the bound $v \le v^*$ with the closed forms of item (e) of Section 3, and check the irreducibility of the Q_j. For each Q_j the input gives a prime p_j that does not divide its leading coefficient and, for $1 \le k \le \deg Q_j/2$, polynomials that show that $Q_j \bmod p_j$ is coprime to $x^{p_j^k} - x$ over $\mathbb{F}_{p_j}$, so that $Q_j \bmod p_j$ is irreducible, and hence

so is Q_j, which is primitive. A Lean proof shows that if the programs accept, the certificate satisfies the hypotheses of the formal version of Lemma 2.1 and $v \le v^*$.

The checks are run in two ways, and each gives its own copy of the three theorems, in the namespaces `TPUpper.N` and `TPUpper.K`. In the first, they are evaluated by `native_decide`, which executes compiled code; the trusted base therefore includes the Lean compiler, the C compiler that it uses and the Lean runtime. Besides the standard axioms `propext`, `Classical.choice` and `Quot.sound`, the theorems `tpup_infinite` and `tpup_limit_point` of this version depend on 65 auxiliary axioms, one for each check, each asserting that this check evaluates to true; such an axiom is added after the compiled code has returned true. There are 40 checks for the blocks of intervals of (C3), 21 for the irreducibility certificates, and one each for (C1), for (C2) with the real-rooted polynomials, for the quadrature and for $v \le v^*$. In the second version, every check is decided by Lean's kernel, so that the compiler and the runtime are not part of the trusted base. The kernel checks the 11,393 intervals one at a time, (C2) separately for each real-rooted polynomial, and the quadrature in 32 parts. The three theorems of this version depend only on the standard axioms. With a prebuilt Mathlib and after the first version, building the second took about 35 core-hours. The theorem `exp_v0_lt` uses only the standard axioms in both versions. In separate checks, the development version of the Lean files, which differs from the shipped files only in comments, was rebuilt from its sources on other machines. Lean's kernel then checked again, in a single environment, the 108 modules of the project on which the theorems of the first version depend. Every declaration on which the theorems of the second version depend was checked by `leanchecker`, which runs Lean's kernel. It was also checked by two further implementations of the Lean kernel, nanoda [8] and lean4lean [2]. The theorems of 11,447 of the modules with computations were checked in batches, and the rest of the proof was checked with these theorems as axioms. To confirm that the checks can fail, we ran them on modified inputs. They reject, for example, the first interval of the tiling with its right end moved by $1/800$ or extended to the end of the tiling, the last interval checked against an end point moved by $2^{-64}/400$, the bound v^* lowered by $2 \cdot 10^{-9}$, the quadrature with one piece removed, and the irreducibility certificate of $x^2 - 3x + 1$ applied to $x^2 + 1$, which is $(x+1)^2$ modulo 2. The Lean files, for Lean `v4.35.0-rc3` and Mathlib commit `c55e6e786f49`, are in the directory `lean/` of the archive.

5. Remarks

The statements of this section are numerical observations, computed in floating point; the proof does not use them.

The search. We solved a linear program by a cutting-plane method, with an interior-point solver of HiGHS through SciPy. The unknowns are a measure $\mu = \sum_k p_k \phi_k$, where the ϕ_k are unit-mass hat functions with spacing $1/400$ on $[0, 12]$, and weights $b_j \ge 0$. The program minimizes $\int \log^+ x \, d\mu$ subject to (i) $\mu(\mathbb{R}) = 1$, (ii) $\sum_j b_j \deg Q_j \le 1 - 10^{-6}$, (iii) $\int \log |Q_j| \, d\mu \ge 10^{-7}$ for each of 32 irreducible polynomials, namely x and the irreducible factors of the F_j, and (iv) $h \ge 3 \cdot 10^{-6}$ on a grid. The method refines this grid until $h \ge 1.5 \cdot 10^{-6}$ at all points of a finer check grid, of spacing $2 \cdot 10^{-5}$ near the support. Its value was 0.5446182. Of the 32 polynomials, 21 received positive weight. The support of the density consists of 53 intervals, on which the inequality $h \ge 0$ is nearly an equality. We converted the solution to exact rationals, trimmed the nodes to the support, and chose v^*, which exceeds the exact value of $\frac{1}{s} \int \log^+ x \, d\mu$ by $9.95 \cdot 10^{-10}$.

The gap. The interval of Corollary 1.2 has length less than 0.00049. Both of its ends come from finite-dimensional problems. The lower bound comes from an auxiliary function with x, eighteen

of Flammang's polynomials and an energy term [17]; the upper bound comes from the measure and the polynomials above. Finer discretizations should narrow the interval. For heights defined by a continuous Green function on $\mathbb{C}$, the best bounds of the two kinds are optimal values of mutually dual linear programs, and they coincide [1, Theorem D]; the restriction to totally positive algebraic integers is not covered by that theorem.

Data availability

The certificate, the two checking programs, their logs, the modified certificates with their logs, the programs that produced the certificate, the computation of ℓ, and the Lean files of Section 4 are archived at doi:10.5281/zenodo.23544557 and available at `https://github.com/tejasvi/totally-positive-mahler-upper-v2`. The SHA-256 checksum of the certificate is

`c34e6e5378a48b6dffca0cf80d2b362c531655dec4af8ccbbc510527a6b038d7`.

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