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A LOWER BOUND FOR THE SMALLEST LIMIT POINT OF THE ABSOLUTE MAHLER MEASURE OF TOTALLY POSITIVE ALGEBRAIC INTEGERS

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Abstract. Let ℓ_+ be the smallest limit point of the absolute Mahler measures $\Omega(\alpha) = M(\alpha)^{1/\deg\alpha}$ of totally positive algebraic integers α. By work of Smyth, $\ell_+ \le 1.7274$, and the best lower bound was Flammang's $\ell_+ \ge 1.722069$. We prove that all but finitely many totally positive algebraic numbers α satisfy $\Omega(\alpha) > 1.7234615$, and that $\ell_+ > 1.7234615$. We also prove that all but finitely many totally real algebraic numbers β have absolute logarithmic Weil height $\log\Omega(\beta) > 0.2721673$. The proof adds to Smyth's method of auxiliary functions the logarithmic energy term that Orloski, Talebizadeh Sardari and Smith derived from the integrality of discriminants for the Schur–Siegel–Smyth trace problem. The resulting inequality is verified rigorously by computer, and the lower bounds are also proved in the Lean proof assistant.

1. Introduction

Let α be an algebraic number of degree n, with minimal polynomial $P(x) = a\prod_{i=1}^{n}(x-\alpha_i)$ in $\mathbb{Z}[x]$, primitive and with leading coefficient $a \ge 1$. The Mahler measure and the absolute Mahler measure of α are

$$M(\alpha) = a\prod_{i=1}^{n}\max(1,|\alpha_i|), \qquad \Omega(\alpha) = M(\alpha)^{1/n},$$

and $h(\alpha) = \log\Omega(\alpha)$ is the absolute logarithmic Weil height of α. We call α totally positive if all its conjugates α_i are real and positive. Let $\mathcal{L}_+$ be the set of the values $\Omega(\alpha)$, where α runs through the totally positive algebraic integers, and let ℓ_+ be the smallest limit point of $\mathcal{L}_+$.

Schinzel [19] proved that $\Omega(\alpha) \ge (1+\sqrt{5})/2$ for every totally positive algebraic integer $\alpha \ne 0, 1$, with equality for the roots of $x^2 - 3x + 1$. Smyth [22, Theorems 1 and 2] proved that the values $\Omega(\beta)$, for totally real algebraic integers β, have $\ell = 1.31427\ldots$ as a limit point and are dense in (ℓ,∞). If $\beta \ne 0$ is a totally real algebraic integer, then β^2 is a totally positive algebraic integer and $\Omega(\beta^2) = \Omega(\beta)^2$. Hence every number in $[\ell^2,\infty)$ is a limit point of $\mathcal{L}_+$, and $\ell_+ \le \ell^2 = 1.7273\ldots$.

Lower bounds for ℓ_+ come from the method of auxiliary functions. Suppose that

$$\log^+ x \;\ge\; m + \sum_j c_j \log|Q_j(x)| \qquad \text{for all } x > 0, \tag{1.1}$$

where $\log^+ x = \log\max(1,x)$, the Q_j are finitely many nonzero integer polynomials and $c_j \ge 0$. If α is a totally positive algebraic integer with $Q_j(\alpha) \ne 0$ for all j, then each product $\prod_i Q_j(\alpha_i)$ is a nonzero integer, since it is a resultant, and averaging (1.1) over the conjugates of α gives $\log\Omega(\alpha) \ge m$. Smyth [23] found such an inequality by linear programming and checked it in interval arithmetic. He stated the result for totally real algebraic integers β, as $\Omega(\beta) > 1.31040$ when β^2 is not in an explicit finite set, and his inequality gives $\Omega(\alpha) > 1.31040^2 > 1.7171$ for

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all totally positive algebraic integers α outside that set [23, (6) and (9)]. Flammang raised the bound to 1.720566 [6], Mu and Wu to 1.721916 [13], and Flammang to 1.722069 [7, Theorem 1]. In each case the bound holds for all but finitely many totally positive algebraic integers, so it is a lower bound for ℓ_+. In [7] the exceptions are the roots of seven listed polynomials, two of which, Q_6 and Q_9 in our Table 1, are reducible. Flammang conjectures that the next point of $\mathcal{L}_+$ is 1.722325, attained at the roots of $x^6 - 12x^5 + 44x^4 - 67x^3 + 44x^2 - 12x + 1$.

Theorem 1.1. *All but finitely many totally positive algebraic numbers α satisfy $\Omega(\alpha) > 1.7234615$. More precisely, for every $\varepsilon > 0$, all but finitely many totally positive algebraic numbers α satisfy $\log \Omega(\alpha) \geq m_0 - \varepsilon$, and $\ell_+ \geq e^{m_0} > 1.7234615$, where $m_0 = 0.5443347841\ldots$.*

Here m_0 is the rational number (3.1), the constant m of the auxiliary inequality that we verify in Section 3. With Smyth's upper bound, Theorem 1.1 gives $1.7234615 < \ell_+ \leq \ell^2 = 1.7273\ldots$. The bound for every $\varepsilon > 0$ in Theorem 1.1 is equivalent to saying that the essential minimum $\sup\{H : \{\alpha \text{ totally positive} : \Omega(\alpha) \leq H\} \text{ is finite}\}$ is at least e^{m_0}. Applied to the squares of totally real algebraic numbers, Theorem 1.1 gives the following.

Corollary 1.2. *For every $\varepsilon > 0$, all but finitely many totally real algebraic numbers β satisfy $h(\beta) \geq m_0/2 - \varepsilon$. In particular, all but finitely many totally real algebraic numbers β satisfy $h(\beta) > 0.2721673$ and $\Omega(\beta) > 1.3128067$.*

Proof. Let $\varepsilon > 0$, and let $\beta \neq 0$ be a totally real algebraic number with $h(\beta) < m_0/2 - \varepsilon$. Then β^2 is totally positive and $h(\beta^2) = 2h(\beta) < m_0 - 2\varepsilon$. By Theorem 1.1 only finitely many numbers occur as β^2, and each of them has at most two square roots. With $\beta = 0$ there is one more exception. □

For totally real algebraic integers, the theorem of Flammang [7] gives $h(\beta) \geq \frac{1}{2}\log 1.722069 = 0.271763\ldots$ with finitely many exceptions, and Smyth's construction shows that the lower limit of h is at most $\log \ell = 0.27328\ldots$. For totally real algebraic numbers $\beta \neq 0, \pm 1$, Schinzel's inequality $h(\beta) \geq \frac{1}{2}\log\frac{1+\sqrt{5}}{2} = 0.2406059\ldots$ holds without integrality [8, Theorem 1], [25, Section 6.1].

We add to (1.1) a term that comes from discriminants. For a probability measure ν on $\mathbb{C}$ with compact support, let $U_\nu(z) = \int \log|z - y|\, d\nu(y)$ be its logarithmic potential and $I(\nu) = \int U_\nu\, d\nu$ its logarithmic energy. The average of $2U_\nu - I(\nu)$ over the conjugates of α is bounded below by the energy of a slightly smoothed copy of the uniform measure on the conjugates, up to a small error. That energy is bounded below because the discriminant of the minimal polynomial is a nonzero integer. So it suffices to prove

$$\log^+ x \;\geq\; m + \sum_j c_j \log|Q_j(x)| + \lambda\big(2U_\nu(x) - I(\nu)\big) \qquad \text{for all } x > 0$$

for some $\lambda \geq 0$ and some ν that gives mass at most Kr to every disc of radius $r \leq 1$, for a constant K. A measure on $\mathbb{R}$ with a bounded density has this property, and for $\lambda = 0$ the inequality is (1.1). Lemma 2.2 below proves that this suffices, with an explicit error term for each degree. The leading coefficient of α enters through $M(\alpha)$, the resultants and the discriminant, and it is absorbed because $\sum_j c_j \deg Q_j + 2\lambda \leq 1$, a condition that the auxiliary inequality itself forces. This is why Theorem 1.1 holds for all totally positive algebraic numbers, not only for integers.

Orloski, Talebizadeh Sardari and Smith [14] introduced this energy condition for the Schur–Siegel–Smyth trace problem, the lower limit of $\mathrm{tr}(\alpha)/\deg(\alpha)$ over totally positive algebraic integers, where the left side is x instead of $\log^+ x$. For that problem, Smyth's method [24] and its refinements gave a sequence of improved lower bounds. Serre showed, in a letter published as an appendix to [1], that inequalities with x on the left and only polynomial terms on the

right cannot hold with a constant larger than 1.89830…, and Smith [21] later proved that the trace constant is less than 1.89831. Discriminants entered the trace problem already in Siegel's work [20]. Orloski and Talebizadeh Sardari [15] proved that every weak* limit μ of the uniform measures on the conjugates of a sequence of distinct algebraic integers, all of whose conjugates lie in a fixed compact set, satisfies $\int \log|Q(x_1,\dots,x_n)|\,d\mu(x_1)\cdots d\mu(x_n) \geq 0$ for every nonzero $Q \in \mathbb{Z}[x_1,\dots,x_n]$ [14, Section 1.1]. For $Q = x_1 - x_2$ this says that the energy of μ is nonnegative. For the Mahler measure of totally positive algebraic integers, Rhin [17] proved a version of Schinzel's inequality that improves as the discriminant grows [25, Section 9.2].

Our auxiliary function uses the polynomial x and the reciprocal polynomials $x^d U_j(x+1/x-2)$, where $d = \deg U_j$, built from eighteen of the twenty-seven polynomials U_j of Flammang's table [7, Table 1]. The reference measure ν has a continuous piecewise linear density with rational nodes, so that its potential and energy have closed forms (Lemma 3.1) and the auxiliary inequality can be verified on $(0,\infty)$ in ball arithmetic [10] (Section 3); the certificate, the programs and the Lean files are archived at doi:10.5281/zenodo.23715959. Numerically, without the energy term the best auxiliary function with these nineteen polynomials falls short of Flammang's bound (Section A.3).

Section 2 proves the lower bound from an auxiliary function with an energy term. Section 3 describes the certificate and its verification and proves Theorem 1.1. Section 4 describes a proof in Lean of Theorem 1.1 and Corollary 1.2. Section 5 lists open problems, and Appendix A explains how the certificate was found, and collects numerical observations.

2. Auxiliary functions with an energy term

For finite positive Borel measures θ, η on $\mathbb{C}$ with compact support we write

$$U_\eta(z) = \int \log|z-y|\,d\eta(y), \qquad I(\theta,\eta) = \int U_\eta\,d\theta, \qquad I(\theta) = I(\theta,\theta).$$

Since $\log|z-y|$ is bounded above on compact sets, these integrals exist in $[-\infty,\infty)$, and $I(\theta,\eta) = I(\eta,\theta)$ by Tonelli's theorem. We write $D(y,r) = \{z \in \mathbb{C} : |z-y| < r\}$ for the open disc and $\omega_{y,r}$ for the normalized arc length measure on the circle $|z-y| = r$. By Jensen's formula,

$$U_{\omega_{y,r}}(z) = \max\bigl(\log|z-y|, \log r\bigr) \qquad (z \in \mathbb{C}). \tag{2.1}$$

We use the following classical fact, see [18, Lemma I.1.8], where the energy is defined with the opposite sign. We include the short proof.

Lemma 2.1. *Let θ and η be probability measures on $\mathbb{C}$ with compact support such that $I(\theta)$, $I(\eta)$ and $I(\theta,\eta)$ are finite. Then $I(\theta) \leq 2I(\theta,\eta) - I(\eta)$.*

Proof. Let $\sigma = \theta - \eta$ and let R be such that both supports lie in $D(0,R)$. For z, w in the supports, $|\log|z-w|| = 2\log^+|z-w| - \log|z-w| \leq 2\log^+(2R) - \log|z-w|$, so $\iint |\log|z-w||\,d|\sigma|(z)\,d|\sigma|(w) < \infty$, because $|\sigma| \leq \theta + \eta$ and all mixed energies are finite. For $t > 0$ we have $-2\log t = \int_0^\infty (e^{-ut^2} - e^{-u})\,\frac{du}{u}$, and the integrand has constant sign, so its absolute value integrates to $2|\log t|$. A measure with an atom has energy $-\infty$, so θ and η have no atoms, and the diagonal $\{z = w\}$ is null for $|\sigma| \times |\sigma|$. By Fubini's theorem and $\sigma(\mathbb{C}) = 0$,

$$-2\iint \log|z-w|\,d\sigma(z)\,d\sigma(w) = \int_0^\infty \Bigl(\iint e^{-u|z-w|^2}\,d\sigma(z)\,d\sigma(w)\Bigr)\frac{du}{u} \;\geq\; 0,$$

since the Gaussian kernel $e^{-u|z-w|^2}$ is positive definite on $\mathbb{C} = \mathbb{R}^2$ (its Fourier transform is positive). The left side is $-2\bigl(I(\theta) - 2I(\theta,\eta) + I(\eta)\bigr)$ by bilinearity. □

Lemma 2.2 (Auxiliary function with an energy term)**.** *Let $Q_1,\dots,Q_J \in \mathbb{Z}[x]$ be nonzero, $c_1,\dots,c_J > 0$, $\lambda \geq 0$ and $m \in \mathbb{R}$. Let ν be a probability measure on $\mathbb{C}$ with compact support,*

and let $K > 0$ be such that $\nu(D(y,r)) \le Kr$ for all $y \in \mathbb{C}$ and $0 < r \le 1$. Let Z be the set of positive real roots of the polynomials Q_j. Assume that

$$F(x) = \log^+ x - m - \sum_{j=1}^{J} c_j \log|Q_j(x)| - \lambda\big(2U_\nu(x) - I(\nu)\big) \;\ge\; 0 \qquad \text{for all } x \in (0,\infty)\setminus Z. \tag{2.2}$$

Then $\sum_j c_j \deg Q_j + 2\lambda \le 1$. Moreover, every totally positive algebraic number α of degree $n \ge 1/(2K)$ that is not a root of one of the polynomials Q_j satisfies

$$\log \Omega(\alpha) \;\ge\; m - \frac{\lambda\,(1+\log(2Kn))}{n}. \tag{2.3}$$

Proof. For $y \in \mathbb{C}$ and $0 < r \le 1$ we have $\max(\log|x-y|, \log r) - \log|x-y| = \int_0^r \mathbf{1}_{\{|x-y|<u\}}\,\frac{du}{u}$. Integrating over x with respect to ν, we obtain

$$\int \max\big(\log|x-y|, \log r\big)\, d\nu(x) = U_\nu(y) + \int_0^r \nu\big(D(y,u)\big)\,\frac{du}{u} \;\le\; U_\nu(y) + Kr. \tag{2.4}$$

With $r = 1$ this gives $U_\nu(y) \ge \int \log^+|x-y|\, d\nu(x) - K \ge -K$. Since ν has compact support, U_ν is also bounded above on compact sets, so U_ν is finite everywhere and $I(\nu)$ is finite. As $x \to +\infty$ we have $U_\nu(x) = \log x + o(1)$ and $\log|Q_j(x)| = \deg(Q_j)\log x + O(1)$, so

$$F(x) = \Big(1 - \sum_j c_j \deg Q_j - 2\lambda\Big)\log x + O(1),$$

and (2.2) forces $\sum_j c_j \deg Q_j + 2\lambda \le 1$.

Now let α be as in the statement, with minimal polynomial $P(x) = a\prod_{i=1}^n (x - \alpha_i)$. All α_i are positive, so

$$n \log \Omega(\alpha) = \log a + \sum_{i=1}^{n} \log^+ \alpha_i. \tag{2.5}$$

First, let $1 \le j \le J$. Since $Q_j(\alpha) \ne 0$ and P is irreducible, Q_j vanishes at no α_i. In particular no α_i lies in Z, so every $F(\alpha_i)$ is a real number. The resultant $\mathrm{Res}(P, Q_j) = a^{\deg Q_j}\prod_i Q_j(\alpha_i)$ is a nonzero integer, hence

$$\sum_{i=1}^{n} \log|Q_j(\alpha_i)| \;\ge\; -\deg(Q_j)\log a. \tag{2.6}$$

Since P is irreducible, its discriminant $\mathrm{disc}(P) = a^{2n-2}\prod_{i<k}(\alpha_i - \alpha_k)^2$ is also a nonzero integer, and

$$\sum_{i\ne k} \log|\alpha_i - \alpha_k| \;\ge\; -(2n-2)\log a. \tag{2.7}$$

To smooth the conjugates over small circles, let $0 < r \le 1$ and $\tau_r = \frac{1}{n}\sum_{i=1}^n \omega_{\alpha_i, r}$. By (2.1), $I(\omega_{y,r}) = \log r$, and for $i \ne k$

$$I(\omega_{\alpha_i,r}, \omega_{\alpha_k,r}) = \int \max\big(\log|z-\alpha_k|, \log r\big)\, d\omega_{\alpha_i,r}(z) \;\ge\; \int \log|z-\alpha_k|\, d\omega_{\alpha_i,r}(z) \;\ge\; \log|\alpha_i - \alpha_k|,$$

where the last step is (2.1) again. With (2.7) we get

$$I(\tau_r) \;\ge\; \frac{\log r}{n} + \frac{1}{n^2}\sum_{i\ne k}\log|\alpha_i - \alpha_k| \;\ge\; \frac{\log r}{n} - \frac{(2n-2)\log a}{n^2}. \tag{2.8}$$

Next we compare τ_r with ν. The potential of τ_r is at least $\log r$ by (2.1), and the potential of ν is at least $-K$. Both potentials are bounded above on compact sets, so $I(\tau_r)$, $I(\nu)$ and $I(\tau_r,\nu)$ are finite. By Lemma 2.1, $I(\tau_r) \le 2I(\tau_r,\nu) - I(\nu)$. By Fubini's theorem, (2.1) and (2.4),

$$I(\tau_r,\nu) = \frac{1}{n}\sum_{i=1}^{n}\int \max\bigl(\log|x-\alpha_i|, \log r\bigr)\,d\nu(x) \;\le\; \frac{1}{n}\sum_{i=1}^{n} U_\nu(\alpha_i) + Kr.$$

Together with (2.8),

$$\frac{1}{n}\sum_{i=1}^{n}\bigl(2U_\nu(\alpha_i) - I(\nu)\bigr) \;\ge\; I(\tau_r) - 2Kr \;\ge\; \frac{\log r}{n} - \frac{(2n-2)\log a}{n^2} - 2Kr. \tag{2.9}$$

Finally, we average (2.2) over the conjugates α_i and use (2.6) and (2.9):

$$\frac{1}{n}\sum_{i=1}^{n}\log^+\alpha_i \;\ge\; m - \frac{\log a}{n}\Bigl(\sum_j c_j \deg Q_j + 2\lambda\,\frac{n-1}{n}\Bigr) - \lambda\Bigl(2Kr + \frac{\log(1/r)}{n}\Bigr).$$

By (2.5),

$$\log\Omega(\alpha) \;\ge\; m + \frac{\log a}{n}\Bigl(1 - \sum_j c_j \deg Q_j - 2\lambda\,\frac{n-1}{n}\Bigr) - \lambda\Bigl(2Kr + \frac{\log(1/r)}{n}\Bigr).$$

The second term is nonnegative, because $\log a \ge 0$ and $\sum_j c_j \deg Q_j + 2\lambda \le 1$. Choosing $r = 1/(2Kn)$, which is admissible since $n \ge 1/(2K)$, we obtain (2.3). □

Corollary 2.3. *Under the assumptions of Lemma 2.2, for every $\varepsilon > 0$ only finitely many totally positive algebraic numbers α satisfy $\log\Omega(\alpha) \le m - \varepsilon$.*

Proof. Choose $n_0 \ge 1/(2K)$ such that $\lambda(1+\log(2Kn))/n < \varepsilon$ for all $n \ge n_0$. If α is totally positive and $\log\Omega(\alpha) \le m-\varepsilon$, then either α is one of the finitely many roots of the polynomials Q_j, or α has degree less than n_0 by (2.3). In the second case $h(\alpha) \le m$, and by Northcott's theorem [2, Theorem 1.6.8] there are only finitely many such α. □

Remark 2.4. For $\lambda = 0$, Lemma 2.2 is the classical argument of [23], and its conclusion is then $\log\Omega(\alpha) \ge m$ for every totally positive algebraic number α that is not a root of one of the Q_j, because the measure ν plays no role. For instance, the uniform measure on $[0,1]$ satisfies the hypothesis with $K = 2$, and then every $n \ge 1$ is admissible. Flammang [7, Section 2.1] uses auxiliary functions $\log^+ x - c_0\log x - \sum_j c_j \log|Q_j(x)|$ with reciprocal Q_j and $2c_0 + \sum_j c_j \deg Q_j = 1$, which are invariant under $x \mapsto 1/x$. Lemma 2.2 with $\lambda = 0$ applies to them, so bounds obtained with such functions, which have no energy term, also hold for all totally positive algebraic numbers outside a finite set.

Remark 2.5. For a fixed finite set of polynomials, the best bound that Lemma 2.2 can give is limited by a problem on measures. Let θ be a probability measure on $(0,\infty)$ with compact support, with $I(\theta) \ge 0$ and $\int \log|Q_k|\,d\theta \ge 0$ for all k. The condition $I(\theta) \ge 0$ excludes atoms. If (2.2) holds, then integrating it against θ and using Lemma 2.1 gives

$$\int \log^+ x\,d\theta \;\ge\; m + \lambda\bigl(2I(\theta,\nu) - I(\nu)\bigr) \;\ge\; m + \lambda I(\theta) \;\ge\; m.$$

So every value m that Lemma 2.2 gives with these polynomials is at most the infimum of $\int \log^+ x\,d\theta$ over such measures. For heights defined by a continuous Green function on $\mathbb{C}$, Burgos Gil, Menares, Qu and Sombra [3, Theorem D] proved that the best bound obtainable from auxiliary functions with arbitrary finite sets of integer polynomials equals the corresponding infimum over measures, which is the essential minimum of the height. Their setting includes neither the restriction to totally positive numbers nor an energy term (see Section 5).

3. The certificate

3.1. **The polynomials.** The certificate uses the set $\mathcal{Q}$ of the 19 polynomials $Q_0, \dots, Q_{18}$ of Table 1. For $1 \le k \le 18$, the polynomial Q_k is the reciprocal polynomial $Q_k(x) = x^d U(x + 1/x - 2)$ of degree $2d$, where U is the polynomial of degree d listed in [7, Table 1] under the index j given in the table. The weights c_k are dyadic rationals. The polynomial $Q_1 = (x-1)^2$ vanishes at the corner $x = 1$ of $\log^+ x$, so the auxiliary function tends to $+\infty$ there. The polynomials Q_6, Q_9 and Q_{15} are products of a polynomial and its reciprocal, and Q_{15} is the only Q_k whose leading coefficient is not 1. All roots of the Q_k are real and nonnegative, except for two pairs of complex roots of Q_{16}.

Table 1. The polynomials and their weights, rounded to four significant digits; the certificate in the archive gives the exact weights. For $1 \le k \le 18$, $Q_k(x) = x^{\deg U} U(x + 1/x - 2)$, where U is the polynomial U_j of [7, Table 1], given by its coefficients from the leading one down to the constant term.

k	j	U_j, or Q_0	$\deg Q_k$	c_k
0	–	$Q_0 = x$	1	$1.932 \cdot 10^{-1}$
1	1	$1, 0$	2	$1.558 \cdot 10^{-1}$
2	2	$1, -1$	2	$3.860 \cdot 10^{-2}$
3	3	$1, -2$	2	$4.143 \cdot 10^{-7}$
4	4	$1, -3, 1$	4	$1.295 \cdot 10^{-2}$
5	5	$1, -4, 1$	4	$2.372 \cdot 10^{-4}$
6	6	$1, -5, 6, -1$	6	$1.686 \cdot 10^{-3}$
7	7	$1, -6, 5, -1$	6	$5.191 \cdot 10^{-4}$
8	8	$1, -7, 13, -7, 1$	8	$4.143 \cdot 10^{-3}$
9	10	$1, -7, 14, -8, 1$	8	$7.517 \cdot 10^{-4}$
10	11	$1, -8, 14, -7, 1$	8	$9.505 \cdot 10^{-5}$
11	13	$1, -13, 61, -131, 136, -66, 14, -1$	14	$1.237 \cdot 10^{-5}$
12	14	$1, -13, 61, -132, 138, -67, 14, -1$	14	$1.946 \cdot 10^{-6}$
13	16	$1, -13, 62, -137, 147, -73, 15, -1$	14	$8.026 \cdot 10^{-7}$
14	18	$1, -15, 83, -220, 303, -220, 83, -15, 1$	16	$8.049 \cdot 10^{-4}$
15	20	$2, -26, 128, -308, 391, -265, 94, -16, 1$	16	$3.204 \cdot 10^{-7}$
16	21	$1, -21, 182, -853, 2386, -4151, 4545, -3109,$ $1298, -315, 40, -2$	22	$4.052 \cdot 10^{-7}$
17	24	$1, -25, 266, -1582, 5817, -13835, 21682,$ $-22449, 15233, -6667, 1833, -302, 27, -1$	26	$3.650 \cdot 10^{-7}$
18	25	$1, -31, 413, -3141, 15261, -50187, 115410,$ $-189036, 222621, -189036, 115410, -50187,$ $15261, -3141, 413, -31, 1$	32	$5.251 \cdot 10^{-8}$

The roots of the Q_k, which Lemma 2.2 excludes, include totally positive algebraic integers with $\Omega(\alpha) < e^{m_0}$, for instance 1, the roots of $x^2 - 3x + 1$, and the roots of the sextic Q_7, where $\Omega = 1.72232\ldots$. They are among the finitely many exceptions that Theorem 1.1 allows.

3.2. **The reference measure.** Let $0 = e_0 < e_1 < \cdots < e_N = 11.2$ be the $N+1 = 5{,}473$ nodes of the certificate, which are rational numbers. Let G be the continuous function on $\mathbb{R}$ that is linear on each interval $[e_k, e_{k+1}]$, takes rational values $G(e_k) = G_k \ge 0$ at the nodes, with $G_0 = G_N = 0$, and vanishes outside $[0, 11.2]$. Let ρ be the measure $G(x)\,dx$, let $s = \rho(\mathbb{R}) = 0.05288541\ldots$, and put

$$\nu = \rho/s, \qquad \lambda = s.$$

Then $\lambda(2U_\nu - I(\nu)) = 2U_\rho - I(\rho)/s$. With

$$m_0 = \frac{19152077587359311629}{35184372088832000000} = 0.5443347841\ldots, \tag{3.1}$$

the function (2.2) with $m = m_0$ becomes

$$F(x) = \log^+ x - m_0 - \sum_{k=0}^{18} c_k \log|Q_k(x)| - 2U_\rho(x) + \frac{I(\rho)}{s}. \tag{3.2}$$

The function G vanishes outside $[0.08525, 11.2]$ and on 401 of the intervals $[e_k, e_{k+1}]$ inside it, which form nine gaps, each containing one root of a Q_k. A disc of radius r meets $\mathbb{R}$ in an interval of length at most $2r$, so ν satisfies the hypothesis of Lemma 2.2 with $K = 2\max G/s$. Computed exactly from the node values, $\max G/s$ is less than 2.44, so we may take $K = 4.88$, a value that enters only the error term of Remark 3.4.

For $t \in \mathbb{R}$ let $H(t) = \frac{t^2}{2}\log|t| - \frac{3}{4}t^2$ and $L(t) = \frac{t^4}{24}\log|t| - \frac{25}{288}t^4$, with $H(0) = L(0) = 0$. Let C_i be the jump of the slope of G at e_i, that is, $C_0 = g_0$, $C_N = -g_{N-1}$ and $C_i = g_i - g_{i-1}$ for $0 < i < N$, where $g_k = (G_{k+1} - G_k)/(e_{k+1} - e_k)$. Then $G'' = \sum_i C_i\delta_{e_i}$ in the sense of distributions, and $\sum_i C_i = \sum_i C_i e_i = 0$ because G vanishes outside $[e_0, e_N]$.

Lemma 3.1. *For $x \in \mathbb{R}$,*

$$U_\rho(x) = \sum_{i=0}^{N} C_i H(e_i - x), \qquad I(\rho) = \sum_{i,k=0}^{N} C_i C_k L(e_i - e_k).$$

Proof. A direct computation gives $H''(t) = \log|t|$ and $L'' = H$ for $t \neq 0$. Both functions are even, H is continuously differentiable and L is three times continuously differentiable. Since G is Lipschitz with compact support, two integrations by parts give

$$U_\rho(x) = \int H''(y-x)\,G(y)\,dy = \int H(y-x)\,dG'(y) = \sum_i C_i H(e_i - x),$$

with no boundary terms. Here the second integral is a Stieltjes integral with respect to the step function G', which jumps by C_i at e_i. In the same way $\int H(e_i - x)G(x)\,dx = \sum_k C_k L(e_k - e_i)$, and integrating the first formula against G gives the second. □

Since H is continuously differentiable with $H'' = \log|t|$, the potential U_ρ is continuously differentiable on $\mathbb{R}$, and $U_\rho''(x) = \sum_i C_i \log|x - e_i|$ for x different from the nodes. With m_0, c_k, e_i and G_i exact, all terms of (3.2) at a rational point x are elementary functions of rational numbers.

3.3. **Verification.** Two programs verify the following statement.

Proposition 3.2. *For the data of the certificate, $\sum_k c_k \deg Q_k + 2s < 1$, and the function* (3.2) *satisfies $F(x) \geq 0$ for every $x > 0$ that is not a root of one of the polynomials Q_k.*

Both programs use Arb ball arithmetic [10] through python-flint [16]. In ball arithmetic every computed quantity is an interval, given by a midpoint and a radius, that is guaranteed to contain the exact value. Both programs read the data as exact rationals, check that they have the required form (distinct polynomials with nonzero leading coefficients, among them x with a positive weight, nonnegative weights and values G_k, increasing nodes, $G_0 = G_N = 0$ and $s > 0$), and compute $\sum_k c_k \deg Q_k + 2s$ exactly. Its value is $0.80677179\ldots < 1$. All nineteen weights are positive, as Lemma 2.2 requires. Apart from the certified root isolation and factorization of FLINT [5], acceptance depends only on exact comparisons of rationals and on the positivity of lower ends of balls. Heuristic choices, such as window widths, cut points and limits on bisection, therefore affect the running time but not the validity of an accepted bound.

The first program checks $\sum_i C_i = \sum_i C_i e_i = 0$ exactly, evaluates U_ρ and $I(\rho)$ by Lemma 3.1, and uses three kinds of bound. On an interval $(0, x_{\mathrm{lo}}]$ it bounds F from below by a positive constant. This bound uses $\log^+ x = 0$, the inequality $-c_0 \log x \geq -c_0 \log x_{\mathrm{lo}}$, upper bounds for $|Q_k|$ with $k \geq 1$, and the inequality $U_\rho(x) \leq U_\rho(0)$, which holds because U_ρ decreases to the left of the support of G. A Sturm sequence shows that no Q_k with $k \geq 1$ vanishes on $(0, x_{\mathrm{lo}}]$. On an interval $[x_{\mathrm{hi}}, \infty)$ the bounds $|Q_k(x)| \leq \|Q_k\|_1 x^{\deg Q_k}$, where $\|Q_k\|_1$ is the sum of the absolute values of the coefficients, and $U_\rho(x) \leq s \log x$ give $F(x) \geq \kappa \log x - b$. Here $\kappa = 1 - \sum_k c_k \deg Q_k - 2s > 0$ and $b = m_0 - I(\rho)/s + \sum_k c_k \log \|Q_k\|_1$, and $\kappa \log x_{\mathrm{hi}} - b > 0$. On $[x_{\mathrm{lo}}, x_{\mathrm{hi}}]$ the program puts each root of a Q_k in a short window W. The roots are located by the certified root isolation of FLINT and confirmed by Sturm counts and exact signs. Let x_c be the rational centre of W. On W, each $\log |Q_k|$ is bounded from above, roots included, by the logarithm of $|Q_k(x_c)| + \frac{|W|}{2} \max_W |Q_k'|$, and U_ρ is bounded from above by $U_\rho(x_c) + \frac{|W|}{2} \max_W |U_\rho'|$. The function $\log^+ x$ is bounded from below on W by its value at the left end of W. The nodes, the windows and 1 cut the rest into intervals $[p, q]$. On each of them the program applies Lemma 3.3 below to F, with a rigorous upper bound for $\int_p^q (F'')_+$, and it checks that the windows and the intervals cover $[x_{\mathrm{lo}}, x_{\mathrm{hi}}]$.

Lemma 3.3. *Let f be continuously differentiable on $[p, q]$ with f' absolutely continuous, and let $\delta = q - p$. Then for $p \leq x \leq q$,*

$$f(x) \;\geq\; \min(f(p), f(q)) - \frac{\delta}{4} \int_p^q \big(f''(t)\big)_+ \, dt.$$

Proof. Let Λ be the linear function with $\Lambda(p) = f(p)$ and $\Lambda(q) = f(q)$. Then $f(x) = \Lambda(x) - \int_p^q k(x,t) f''(t)\, dt$ with $k(x,t) = (\min(x,t) - p)(q - \max(x,t))/\delta$. Since $0 \leq k \leq \delta/4$, the integral is at most $\frac{\delta}{4} \int_p^q (f'')_+$, and $\Lambda(x) \geq \min(f(p), f(q))$. □

On each interval $[p, q]$ of the first program, F satisfies the hypotheses of Lemma 3.3. Indeed, $[p, q]$ contains no root of a Q_k, and 1 and the nodes are not interior points, so $\log^+ x$ and the polynomial terms are smooth on $[p, q]$. The function $U_\rho'(x) = -\sum_i C_i H'(e_i - x)$ is absolutely continuous, since $H'(t) = t \log |t| - t$ is an indefinite integral of the locally integrable function $\log |t|$.

A second program checks Proposition 3.2 in a different way, with direct bounds near the irrational roots of the Q_k and second-order Taylor bounds instead of Lemma 3.3; it shares no code with the first and was written from a description of the certificate format, which also gives the closed forms of Lemma 3.1 and the tail bounds of the first program. Table 2 lists the data of the certificate and the smallest lower bounds for F that the two programs obtain, and the archive describes both programs in more detail, with their cut points and other parameters.

TABLE 2. The certificate and the smallest lower bounds for F in the two checks.

Bound m_0	(3.1)	
$\sum_k c_k \deg Q_k$; s	$0.70100096\ldots$; $0.05288541\ldots$	
$\sum_k c_k \deg Q_k + 2s$	$0.80677179\ldots < 1$	
Nodes; $G = 0$ outside; K	5,473 in $[0, 11.2]$; $[0.08525, 11.2]$; 4.88	
$I(\rho)$	$2.8781306519\ldots \cdot 10^{-11}$	
	first program	second program
Smallest bound	$2.884 \cdot 10^{-6}$ (lower bound for F at the ends of the intervals)	$1.57 \cdot 10^{-9}$ (lower bound on an interval)

Proof of Theorem 1.1. By Proposition 3.2, the function (3.2) is nonnegative on $(0,\infty)$ outside the roots of the Q_k. It is the function (2.2) with $\nu = \rho/s$, $\lambda = s$ and $m = m_0$, and ν satisfies the hypothesis of Lemma 2.2 with $K = 4.88$. The bound for every $\varepsilon > 0$ follows from Corollary 2.3. Since $e^{m_0} = 1.72346152\ldots$, Corollary 2.3 with $\varepsilon = m_0 - \log 1.7234615 > 0$ shows that only finitely many totally positive algebraic numbers satisfy $\Omega(\alpha) \le 1.7234615$. Finally, by Corollary 2.3, for every $\varepsilon > 0$ only finitely many values $\Omega(\alpha)$ with α a totally positive algebraic integer lie below $e^{m_0-\varepsilon}$, so $\ell_+ \ge e^{m_0}$. □

Remark 3.4. The proof is effective. With $\lambda = s < 0.0529$ and $K = 4.88$, inequality (2.3) gives

$$\log \Omega(\alpha) \;\ge\; m_0 - \frac{0.0529\,(3.28 + \log n)}{n}$$

for every totally positive algebraic number α of degree n that is not a root of one of the Q_k. For instance $\Omega(\alpha) > 1.72334$ for $n \ge 10^4$.

4. Formal verification

We also prove Theorem 1.1 and Corollary 1.2 in the Lean 4 proof assistant [12] with the Mathlib library [11]. The formal statements are built from Mathlib's notions of irreducibility, integrality, complex roots, Mahler measure and accumulation point. An integer polynomial P is a minimal polynomial of $\alpha \in \mathbb{C}$ if it is irreducible in $\mathbb{Z}[x]$, has positive degree and vanishes at α; then $\Omega(\alpha) = M(P)^{1/\deg P}$, where $M(P)$ is Mathlib's Mahler measure, which equals $|a| \prod_i \max(1, |\alpha_i|)$. The number α is totally positive if all complex roots of P are real and positive, and totally real if they are all real. Formally, we prove the first two assertions of Theorem 1.1 as stated, as the propositions `TpOmega` and `TpMain`, with m_0 the rational number (3.1). We prove the third as `TpLimitPoint` and `ExpM0Gt`: every accumulation point y of $\mathcal{L}_+$, in Mathlib's sense, satisfies $y \ge e^{m_0}$, and $e^{m_0} > 1.7234615$; this form does not presuppose that ℓ_+ exists. The three assertions of Corollary 1.2 are `CorReal`, `CorRealDec` and `CorRealOmega`. These propositions can be read without the proofs, and a check after the build confirms that each formal theorem proves the proposition of the same name.

The formal theorems have no hypotheses. Their proofs cover the resultant and discriminant bounds (2.6) and (2.7), the smoothing of the conjugates over small circles and the energy inequality of the proof of Lemma 2.2, with the instance of Lemma 2.1 that it uses, the averaging over the conjugates with $r = 1/(2Kn)$, Corollary 2.3 with Northcott's theorem in the form of Mathlib's finiteness of integer polynomials of bounded degree and Mahler measure, Lemma 3.1, the reduction of Theorem 1.1 to Proposition 3.2, and Proposition 3.2 itself, with the help of the checks described next. Corollary 1.2 is derived from Theorem 1.1 applied to β^2, as in Section 1.

For Proposition 3.2, a checking program written in Lean's core language works with integers only and computes logarithms with proved error bounds. It checks that a list of intervals, computed in advance, tiles $[2^{-30}, 21]$, and it bounds F from below on most of them by a variant of Lemma 3.3 and on short windows around the roots of the Q_k term by term. It also checks the two tails $(0, 2^{-30}]$ and $[21, \infty)$, a lower bound for $I(\rho)$ and the inequality $\sum_k c_k \deg Q_k + 2s < 1$. A Lean proof shows that whenever the program accepts, the conclusion of Proposition 3.2 holds. Every check is decided by Lean's kernel, so that the trusted base is Lean's kernel, including its built-in arithmetic on natural numbers, and all the formal theorems depend only on the standard axioms. The archive prints the propositions, gives the versions of Lean and Mathlib, the build commands and further checks of the proof, and describes a second version of the proof of Theorem 1.1, in which the checks are evaluated as compiled code.

5. Open problems

We end with three questions that the results leave open.

(1) Is $\ell_+ = \ell^2$, the upper bound that Smyth's construction gives?
(2) Which totally positive algebraic integers have $\Omega(\alpha) < e^{m_0}$, and which points of $\mathcal{L}_+$ lie below e^{m_0}, apart from the values at the roots of the Q_k (Section 3)?
(3) Do auxiliary functions with an energy term reach the essential minimum of Ω on totally positive algebraic numbers, as auxiliary functions do for the heights considered by Burgos Gil, Menares, Qu and Sombra (Remark 2.5)? With the polynomials of our search (Appendix A) they cannot go much beyond Theorem 1.1 (Section A.2), so larger bounds need further polynomials.

Appendix A. Search and numerical observations

The statements of this appendix are numerical observations, computed in floating point, and the proof does not use them.

A.1. **Search.** We computed the infimum of Remark 2.5 approximately over measures with piecewise linear densities, using an interior-point solver [9] through a modeling language [4], fixed the reference measure ν to the computed optimal measure, and solved the linear program for m, the weights $c_k \geq 0$ and $\lambda \geq 0$, with (2.2) imposed on a fine grid. The certificate comes from the last of several rounds of these steps: its nodes are decimal fractions, its weights c_k and values G_k are the exact values of the double-precision solution, and m_0 is the value 0.54433778 of the linear program minus $3 \cdot 10^{-6}$, which is why the smallest value of F at the ends of the intervals in Table 2 is close to $3 \cdot 10^{-6}$. The set $\mathcal{Q}'$ of polynomials used in the search consists of x, the twenty-seven polynomials of [7, Table 1], and the quartic $x^4 - 8x^3 + 14x^2 - 7x + 1$ and the cubic $x^3 - 6x^2 + 5x - 1$, which divide Q_9 and Q_6. We screened 246,207 monic irreducible polynomials Q of degree at most 8, with all roots in $[0, 10]$ and small trace, by the values of $\int \log|Q|\, d\theta$ for approximate optimal measures θ. The quartic and the cubic had negative values in a first screen, which covered only the degrees up to 5 and used an early approximate measure. The linear program over $\mathcal{Q}'$ gives positive weight to the polynomials of $\mathcal{Q}$ and to no others. For the reference measure ν of the certificate, the only screened polynomials outside $\mathcal{Q}'$ with $\int \log|Q|\, d\nu < 0$ are the reciprocals of the cubic and the quartic, which also divide Q_6 and Q_9, and the factor $x - 1$ of Q_1. The search evaluates $\log|Q_k|$ by Horner's rule in double precision, which is inaccurate near $x = 1.95$ for the polynomials of degree 32 to 38; with $\log|Q_k|$ computed from the roots, a linear program of the same kind has the value 0.544338.

A.2. **The discretization gap.** For $\mathcal{Q}'$, the measure program over piecewise linear densities on the nodes of the certificate has the approximate value 0.5443536. So no certificate with these polynomials gives more than about $m = 0.5443536$, that is, $\Omega \approx 1.72349$. The difference to the linear program with the fixed reference measure is $0.5443536 - 0.54433778 \approx 1.6 \cdot 10^{-5}$.

A.3. **The energy term.** If the energy term is removed, the linear program over $\mathcal{Q}$ has the value 0.543251, that is, $\Omega \approx 1.72159$. With x and all twenty-seven polynomials of [7, Table 1] it gives $\Omega \approx 1.72207$, in agreement with [7, Theorem 1]. So the energy term raises the bound obtainable from $\mathcal{Q}$ by about $1.1 \cdot 10^{-3}$ in $\log\Omega$, and Theorem 1.1 improves on [7] by about $8.1 \cdot 10^{-4}$ in $\log\Omega$. With the weights printed in [7, Table 1] (HAL version) and $c_0 = \frac{1}{2}(1 - \sum_j c_j \deg Q_j)$, the minimum of Flammang's auxiliary function is 0.543273, that is, $\Omega \approx 1.72163$, short of her

constant, apparently because of a misprint in one weight.[1] For her polynomials of degree 32 to 38, these values need the same care with precision as in Section A.1.

Data availability

The certificate, the two verification programs and a short auxiliary program with their output, modified certificates with the results of both programs on them, the programs that produced the certificate and those behind the numbers of Appendix A, and the Lean files of Section 4 are archived at doi:10.5281/zenodo.23715959 and available at `https://github.com/tejasvi/totally-positive-mahler-measure/tree/v4` (commit `d03717788979`). The certificate is the file `tp_certificate.json`, with SHA-256

`9f1e51157fbe30ce8ab436c8552d7d6a9b5aaaa5483449de00ec68716eac3864`.

The two verification programs are the files `check_certificate.py` and `check_independent.py`, with SHA-256

`de4434350b21bbe8e831ced9729a8592f6496f671a256d18285004a28b4cd4ec` and
`33bce343517e35399626348d0e07b690215576cd81bdc0d58b98b01f97e0f5ad`.

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