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THE SUM OF THE RECIPROCAL CUBES OF THE FIBONACCI NUMBERS IS IRRATIONAL

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ABSTRACT. We prove that $\zeta_F(3) = \sum_{n\geq 1} F_n^{-3}$ is irrational, where F_n are the Fibonacci numbers. To our knowledge the only odd value of the Fibonacci zeta function previously known to be irrational was $\zeta_F(1)$. The proof is a Hankel-determinant construction over $\mathbb{Z}[\varphi]$, $\varphi = (1+\sqrt{5})/2$, adapted from a recent unrefereed construction for the q-zeta value $\sum n^2/(p^n-1)$ at integers $p \geq 2$. Four estimates are combined: a real decay of size $\exp(-\frac{2}{9}K^3 \log\varphi)$, a smallness $\exp(-K^3\log\varphi)$ at the Galois conjugate, which replaces the base-prime content gain of the integer case, local cyclotomic denominator bounds at the twisted roots of unity that occur, and nonvanishing by rational interpolation. The whole argument is formalised in Lean 4 with Mathlib.

1. INTRODUCTION

Let F_n be the Fibonacci numbers and $\zeta_F(s) = \sum_{n\geq 1} F_n^{-s}$ the Fibonacci zeta function. André-Jeannin proved that $\zeta_F(1)$ is irrational [1], by Padé approximation of a Lambert series. Duverney, Ke. Nishioka, Ku. Nishioka and Shiokawa proved that the even values $\zeta_F(2s)$ are transcendental, and so are $\sum_n F_{2n-1}^{-s}$ for every $s \geq 1$ [2]; Elsner, Shimomura and Shiokawa determined the algebraic relations among these numbers [3, 4]. All of these sums are values of quasi-modular forms, to which Nesterenko's theorem [11] applies. The odd values $\zeta_F(2s+1)$, $s \geq 1$, are not of this kind, and as far as we know nothing was proved about their arithmetic (see the survey [16] and the introduction of [10]).

Theorem 1.1. $\zeta_F(3) = \displaystyle\sum_{n\geq 1} \frac{1}{F_n^3}$ *is irrational.*

With $t = \varphi^{-1} = (\sqrt{5}-1)/2$ and Binet's formula, $F_n = t^{-n}(1-(-t^2)^n)/\sqrt{5}$, so

$$\Xi := \frac{\zeta_F(3)}{5\sqrt{5}} = \sum_{N\geq 1} \frac{t^{3N}}{(1-(-t^2)^N)^3}. \tag{1.1}$$

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This is a q-series at the quadratic unit t, a relative of the q-zeta value $\zeta_q(3) = \sum_n n^2 q^n/(1-q^n)$ [9]. For $q = 1/p$ with $p \geq 2$ an integer, the irrationality of $\zeta_q(3)$ remains open in the refereed literature: Krattenthaler, Rivoal and Zudilin proved that one of $\zeta_q(3), \ldots, \zeta_q(11)$ is irrational [9], Jouhet and Mosaki reduced this to $\zeta_q(3), \ldots, \zeta_q(9)$ [7], and Fischler and Zudilin proved linear independence results of the same kind [6]. In October 2026 a proof that $\sum_{n\geq 1} n^2/(p^n-1)$ is irrational for every integer $p \geq 2$ was claimed in an unrefereed, AI-assisted research draft with a Lean formalisation, posted on GitHub under the pen name uplift-universal-inc [14, 13]. Its construction is a moving-denominator Hankel determinant in the line of Uvarov [15], Krattenthaler [8], Prévost [12], Zudilin [17, 18] and the unrefereed preprint of Fauzan [5].

Expanding (1.1) in Lambert series does not reduce Theorem 1.1 to a statement about $\zeta_q(3)$: one obtains a $\mathbb{Q}(\sqrt{5})$-linear combination of 1, odd parts of $\zeta_q(3)$ and $\zeta_q(1)$ at $q = \varphi^{-2}$, and values of holomorphic modular forms of weights 1 and 3 at $q = \varphi^{-1}, \varphi^{-2}$. We therefore treat the series (1.1) directly.

The method. We adapt the construction of [14] to the real quadratic field $\mathbb{Q}(\sqrt{5})$. A linear functional L, a second Euler derivative summed against the weights $(-t)^n$ at the nodes $(-t^2)^n$, has rational moments and pure simple-pole tails of Ξ (Lemma 2.2). The Hankel determinants $\Delta_K(X)$ of L against the moving denominators $D_K(s) = \prod_{a\leq K}(1-(-t^2)^a s)$ are polynomials of degree $\leq K$ in X with $\Delta_K(\Xi) = \det[L(s^{i+j}/D_K)]$. Four estimates are combined:

	size (per $K^3 \log\varphi$)	
real decay $\lvert\Delta_K(\Xi)\rvert$	$-2/9$	Proposition 3.1
conjugate $\lvert\sigma\Delta_K(-\Xi)\rvert$	-1	Proposition 5.1
denominators $\log\lvert\mathrm{N}\, C_K(t)\rvert$	$\leq c_* = 1.1954\ldots < 11/9$	Propositions 4.4, 4.6
nonvanishing	infinitely many K	Proposition 6.1

If $\zeta_F(3)$ were rational, $C_K(t)(25v)^K\Delta_K(\Xi)$ would be a nonzero element of $\mathbb{Z}[\varphi]$ of norm < 1 for large K.

Three ingredients are new compared with [14]. First, the setting: the Fibonacci kernel is a single q-series over $\mathbb{Z}[\varphi]$, handled with a formal parameter τ specialised at $\tau = t$ and at its conjugate $\tau = -\varphi$. Second, since φ is a unit there is no base prime and no coefficient content to extract. Its role is taken by the smallness of the conjugate determinant at the conjugate point $\sigma\Xi = -\Xi$ (Proposition 5.1). Its size, $K^3\log\varphi = \frac{1}{2}K^3\log\varphi^2$, is the size of the content gain $\frac{1}{2}K^3\log p$ of [14] at $p = \varphi^2$. Third, the poles of the coefficients sit at two kinds of twisted roots of unity, those ζ for which $-\zeta^2$ has small order and those for which $-\zeta(-\zeta^2)^k = 1$; we prove local denominator bounds of both types (Section 4). The present paper does not depend logically on [14]: every step is proved here, and the formal proof is self-contained.

Formal verification. The theorem and every step of its proof are formalised in Lean 4 with Mathlib (`https://github.com/telaviv/fibonacci-zeta3-irrationality`); Section 8 gives the statement, the map from the propositions below to Lean declarations, and what has to be trusted. The formal proof uses the elementary denominator constant $c_* = \frac{11}{8}\beta_7 + \frac{1}{3} = \frac{4393}{3675}$ of Proposition 4.6, where $\beta_7 = \prod_{p\le 7}(1-p^{-2})$; it needs no information on the distribution of Euler's function beyond a sieve by $2, 3, 5, 7$.

2. The functional and the determinants

Notation. Throughout, $\varphi = (1+\sqrt{5})/2$ is the golden ratio, $t = \varphi^{-1}$, $\mathcal{O} = \mathbb{Z}[\varphi]$, and σ is the nontrivial automorphism of $\mathbb{Q}(\sqrt{5})$, so $\sigma t = -\varphi$. Euler's function is written $\phi_{\mathrm{E}}(d)$ and the Möbius function $\mu(d)$. We put

$$\rho = -t, \qquad w = -t^2, \qquad \alpha_k = \rho w^k \ (k \ge 0),$$

$$\psi_N = \frac{t^{3N}}{(1-w^N)^3}, \qquad S_a = \sum_{N=1}^{a} \psi_N,$$

$A_K = K(K+1)/2$, and $\theta = s\,\frac{d}{ds}$. Then $\rho^2 = -w$, $|w| = t^2$ and $\Xi = \sum_N \psi_N$. We write h_m for the complete homogeneous symmetric polynomial of degree m ($h_m = 0$ for $m < 0$). Constants in $O(\cdot)$ are absolute and $K \ge 2$.

Definition 2.1. For g analytic on a neighbourhood of $|s| \le t^2$ put $L(g) = \sum_{n\ge 1} \rho^n\,[\theta(\theta+1)g](w^n)$.

The series converges absolutely since $\theta(\theta+1)g = O(s)$ and $|\rho^n w^n| = t^{3n}$.

Lemma 2.2. (i) $L(s^k) = \nu_k := k(k+1)\alpha_k/(1-\alpha_k)$ *for* $k \ge 0$. (ii) *For* $a \ge 1$, $L\big(1/(1-w^a s)\big) = \lambda_a(\Xi - S_a)$ *with* $\lambda_a = 2\rho^{-a}$.

Proof. (i) $\theta(\theta+1)s^k = k(k+1)s^k$; sum the geometric series. (ii) $\theta(\theta+1)(1-cs)^{-1} = 2cs/(1-cs)^3$, so the left side is $\sum_n \rho^n\, 2w^{a+n}/(1-w^{a+n})^3 = 2\rho^{-a}\sum_{N>a}\rho^N w^N/(1-w^N)^3$, and $\rho^N w^N = t^{3N}$. □

Let $D_K(s) = \prod_{a=1}^{K}(1-w^a s)$, with poles $x_a = w^{-a}$, $|x_a| = \varphi^{2a}$. For a polynomial P of degree $< 2K-1$,

$$\frac{P}{D_K} = Q + \sum_{a=1}^{K} \frac{\mathrm{res}_a(P)}{1-w^a s}, \qquad \mathrm{res}_a(P) = \frac{P(x_a)}{\prod_{b\ne a}(1-w^{b-a})},$$

with Q a polynomial of degree $\le \deg P - K$. Let L_X be the linear map with $L_X(s^k) = \nu_k$ and $L_X(1/(1-w^a s)) = \lambda_a(X - S_a)$, X an indeterminate, and

$$\Delta_K(X) = \det\big[L_X(s^{i+j}/D_K)\big]_{0\le i,j<K} \in \mathbb{Q}(\sqrt{5})[X], \qquad \deg \Delta_K \le K.$$

By Lemma 2.2, $\Delta_K(\Xi) = \det[L(s^{i+j}/D_K)]$.

The formal parameter. Every quantity above is the value at $\tau = t$ of a rational function of an indeterminate τ with rational coefficients: replace t by τ, so $\rho = -\tau$, $w = -\tau^2$, $\alpha_k = (-1)^{k+1}\tau^{2k+1}$ and $\psi_N = \tau^{3N}/(1-(-\tau^2)^N)^3$. Write $\Delta_{K,\tau}(X) \in \mathbb{Q}(\tau)[X]$ for the resulting polynomial, and $w(\zeta) = -\zeta^2$, $\alpha_k(\zeta)$ for the values at $\tau = \zeta$. Since $\sigma t = -\varphi$, the coefficientwise conjugate $\sigma\Delta_K$ is the value of $\Delta_{K,\tau}$ at $\tau = -\varphi$. Both values are defined because no denominator vanishes at a real number that is neither 0 nor a root of unity.

3. Real decay

Proposition 3.1. $\log|\Delta_K(\Xi)| \le -\frac{2}{9}(K^3+3K^2-4)\log\varphi + O(K^2)$.

Proof. Let $V_j(s) = \prod_{u=1}^{j}(s - w^{\lceil u/3\rceil})$, $R_j = \lceil j/3\rceil$ and $a_j = \sum_{u\le j}\lceil u/3\rceil$. The change of basis from $1, s, \dots, s^{K-1}$ to $V_0, \dots, V_{K-1}$ is unitriangular, so $\Delta_K(\Xi) = \det[L(V_iV_j/D_K)]$.

Since $\theta(\theta+1)g(s_0)$ depends only on the 2-jet of g at s_0, the m-th term of $L(V_iV_jW)$, $W = 1/D_K$, vanishes for $m < R := \max(R_i, R_j)$: then V_iV_j has a triple zero at w^m. For $m \ge R$ and $s = w^m$ every root β of V_j satisfies $|s| \le |\beta|$, so $|s-\beta| \le 2|\beta|$ and $\theta(s-\beta) = s$. Hence $|V_j(s)| \le 2^j|w|^{a_j}$ and $|\theta^u V_j(s)| \le j^u 2^j |w|^{a_j}|s|/|w|^{R_j}$ for $u = 1, 2$. On $|s| \le t^2$ we have $|w^a s| \le t^{2a+2}$, so $|W|$, $|\theta W|/|s|$ and $|\theta^2 W|/|s|$ are bounded by an absolute constant. Since $\theta(\theta+1) = \theta^2+\theta$ kills constants, by Leibniz every term of $\theta(\theta+1)(V_iV_jW)$ contains at least one θ on some factor, hence a factor $\le |s|/|w|^R$. With $|\rho|^m \le 1$ and $\sum_{m\ge R}|w|^{m-R} \le (1-t^2)^{-1}$,

$$|L(V_iV_jW)| \le c_1(i+j+1)^2 4^{i+j}|w|^{a_i+a_j}$$

for an absolute constant c_1. Pull $|w|^{a_i}$ out of row i and column i and apply Hadamard's inequality: $|\Delta_K(\Xi)| \le |w|^{2\sum_{j<K} a_j}(c_1(2K)^2 4^{2K}\sqrt{K})^K$. Finally $2\sum_{j<K} a_j = (K^3+3K^2-\iota_K)/9$ with $\iota_K \in \{0,4,2\}$ according to K mod 3. □

4. Denominators

4.1. The integrality ring and the root types.

Lemma 4.1. *The coefficients of $\Delta_{K,\tau}(X)$ lie in $\mathbb{Z}[\tau, \tau^{-1}, \Phi_d(\tau)^{-1} : d \in \mathcal{D}_K]$, where $\mathcal{D}_K$ is the set of orders of roots of unity ζ such that* (W) *$w(\zeta)$ has order $e \le K$, or* (α) *$\alpha_k(\zeta) = 1$ for some $1 \le k \le K-2$.*

Proof. The inputs are the Laurent monomials w^a and $\lambda_a = 2(-\tau)^{-a}$; the factors $1/(1-w^{b-a})$ and $(1-w^N)^{-3}$ ($N \le K$), where $1-w^n = 1-(-1)^n\tau^{2n}$ is $\pm$ a product of cyclotomic polynomials $\Phi_d(\tau)$ with $w(\zeta)^n = 1$; and $\nu_k = k(k+1)\alpha_k/(1-\alpha_k)$, of which only $k \le K-2$ occur (the quotient has degree $\le 2K-2-K$), with $1-\alpha_k = 1 \mp \tau^{2k+1} = \pm\prod\Phi_d(\tau)$. Division by D_K is integral because its leading coefficient $\prod_a(-w^a) = \pm\tau^{K(K+1)}$ is a unit. No rational constant is inverted. □

For $d \in \mathcal{D}_K$ the type depends only on d. If ζ has order d, then $w(\zeta)$ has order $e(d) = 2d$ (d odd), d ($d \equiv 2 \bmod 4$), $d/4$ ($d \equiv 4 \bmod 8$), $d/2$ ($d \equiv 0 \bmod 8$), and for each e the orders d with $e(d) = e$ satisfy $\sum_{e(d)=e} \phi_{\mathrm{E}}(d) = 2\phi_{\mathrm{E}}(e)$. The least $k \geq 1$ with $\alpha_k(\zeta) = 1$ is $k_{\min}(d) = (d-1)/2$ ($d \equiv 3 \bmod 4$), $(3d-1)/2$ ($d \equiv 1 \bmod 4$), $(d-2)/4$ ($d \equiv 2 \bmod 8$, $d \geq 10$), $(3d-2)/4$ ($d \equiv 6 \bmod 8$), and $k_{\min}(2) = 2$ (the formula $(d-2)/4$ would give 0); there is none if $4 \mid d$. (Every primitive d-th root is a power ζ^j with j coprime to $2d$; then $-\zeta^{2j} = (-\zeta^2)^j$ and $\alpha_k(\zeta^j) = \alpha_k(\zeta)^j$, and powers coprime to the order preserve the order.) All $d \in \mathcal{D}_K$ satisfy $d \leq 4K$: $w(\zeta)^e = 1$ gives $\zeta^{4e} = 1$, and $\alpha_k(\zeta) = 1$ gives $\zeta^{4k+2} = 1$.

Local setting. Fix ζ of order $d \in \mathcal{D}_K$, let $\varpi = \tau - \zeta$ and let $v = \mathrm{ord}_\varpi$ on $\mathbb{Q}(\zeta)((\varpi))$, extended to polynomials in X as the Gauss valuation (the minimum over the coefficients; it is multiplicative). Then:

(a) if $e(d) = e \leq K$, then $x_a(\zeta) = x_b(\zeta)$ iff $e \mid a-b$, and then $v(x_a - x_b) = 1$: indeed $x_a - x_b = x_a(1 - w^{b-a})$, and $1 - w^n$ has only simple zeros, since $\frac{d}{d\tau} w^n \neq 0$ at $\tau \neq 0$;
(b) ψ_N has a pole of order exactly 3 if $e \mid N$ and is regular otherwise, by (a); so S_a has a pole of order ≤ 3, and is regular if $a < e$;
(c) $1 - \alpha_k = 1 \mp \tau^{2k+1}$ is separable, so it has at most a simple zero at ζ;
(d) λ_a and x_a are Laurent monomials, hence units.

4.2. **Local bounds.** Put $c(m) = 2\sum_{j=0}^{m-2} \max(0, \frac{m+2}{2} - j)$ and $c'(m) = 2\sum_{j=0}^{m-1} \max(0, \frac{m+2}{2} - j)$; thus $c(1) = 0$, $c(m) = 4(m-1)$ for $m \leq 6$, $c(m) \leq (m+3)^2/4$, $c'(1) = 3$ and $0 \leq c' - c \leq 3$.

Proposition 4.2. *Let $e = e(d) \leq K$ and partition $\{1, \dots, K\}$ into the clusters $I_r = \{a \equiv r \bmod e\}$ of sizes m_r, $1 \leq r \leq e$. Then*

$$-v(\Delta_{K,\tau}) \leq \sum_{r=1}^{e-1} c(m_r) + c'(m_e) + \epsilon \max(0, K - 1 - k_{\min}(d)),$$

with $\epsilon = 1$ if ζ is also an α-root and $\epsilon = 0$ otherwise.

Proof. Let $F(s) = \prod_a (s - x_a)$ and $F_r = \prod_{a \in I_r} (s - x_a)$. Fix a cluster r, write $m = m_r$ and $I_r = \{a_1 < \dots < a_m\}$.

Newton cluster basis. For $0 \leq j < m$ put $N_{r,j} = (F/F_r) \prod_{l=m-j+1}^{m} (s - x_{a_l})$; it vanishes at the top j nodes of the cluster. At $\varpi = 0$ these reduce to $(F/F_r)(s - y_r)^j \cdot (\text{unit})$, $y_r = x_r(\zeta)$, which form a basis of the polynomials of degree $< K$ by the Chinese remainder theorem (distinct residues r give distinct points y_r). So the change of basis from $1, \dots, s^{K-1}$ has unit determinant, and $v(\Delta_{K,\tau})$ is the valuation of $G = [L_X(NN'/D_K)]$ in this basis.

Entries. $G_{(r,j),(r',k)} = P + \sum_a \mathrm{res}_a(NN')\lambda_a(X - S_a)$, where P is L_X of the polynomial part. (i) If $r \neq r'$ then $F \mid N_{r,j} N_{r',k}$ and all $\mathrm{res}_a = 0$. (ii) If

$r = r'$, then $\mathrm{res}_{a_i} = 0$ unless $i \le m - \max(j,k)$, and for such i

$$\mathrm{res}_{a_i} = \frac{(F/F_r)(x_{a_i})^2 \prod_{l>m-j}(x_{a_i} - x_{a_l}) \prod_{l>m-k}(x_{a_i} - x_{a_l})}{\prod_{b \neq a_i}(1 - w^{b-a_i})}.$$

Here $(F/F_r)(x_{a_i})$ is a unit, each displayed difference has valuation exactly 1 by (a), and the denominator has valuation exactly $m-1$. So $v(\mathrm{res}_{a_i}) = j + k - (m-1)$. (iii) $v(\lambda_a) = 0$, and by (b) the Gauss valuation of $X - S_a$ is ≥ -3 if $a \ge e$ and ≥ 0 if $a < e$. (iv) For $r < e$ the node $a_1 = r$ is regular and a_i is singular for $i \ge 2$; for $r = e$ every node is singular. So the proper part of the (j,k) entry has $v \ge j + k - (m-1) - 3$ if some singular node has $i \le m - \max(j,k)$, and $v \ge j + k - (m-1)$ otherwise. The second case occurs exactly when $r < e$ and $\max(j,k) = m-1$.

Weights. Let $\kappa_j = \max(0, \frac{m+2}{2} - j)$ for $j \le m-2$ and $\kappa_{m-1} = 0$ if $r < e$, and $\kappa_j = \max(0, \frac{m+2}{2} - j)$ for all $j < m$ if $r = e$. We claim, following the two cases of (iv):

(A) if the (j,k) entry has a singular node (all pairs when $r = e$; the pairs with $\max(j,k) \le m-2$ when $r < e$), then $\kappa_j + \kappa_k \ge m + 2 - j - k$;
(B) if $r < e$ and $\max(j,k) = m-1$, then $\kappa_j + \kappa_k \ge 0 \ge m - 1 - j - k$.

In case (A) both indices carry the weight $\max(0, \frac{m+2}{2} - j)$ (for $r = e$ this includes $\kappa_{m-1} = \max(0, \frac{4-m}{2})$). If both weights are positive the inequality is an equality. If $\kappa_k = 0$, then $k \ge \frac{m+2}{2}$, so $\kappa_j \ge \frac{m+2}{2} - j \ge m + 2 - j - k$. If both vanish, then $j, k \ge \frac{m+2}{2}$ and the right side is ≤ 0. Case (B) holds because $j + k \ge m - 1$.

Scale row (r,j) by $\varpi^{\lceil \kappa_j \rceil}$ and column (r,j) by $\varpi^{\lfloor \kappa_j \rfloor}$. The required exponents are integers, and $a + b \ge n \in \mathbb{Z}$ implies $\lceil a \rceil + \lfloor b \rfloor \ge n$, so by (iv), (A) and (B) every scaled proper entry is integral. The total scaling is $\sum(\lceil \kappa_j \rceil + \lfloor \kappa_j \rfloor) = 2\sum \kappa_j$, which is $c(m)$ for $r < e$ and $c'(m)$ for $r = e$. All exponents are ≥ 0.

Polynomial part. The quotient of NN' by D_K has degree $\le K-2$ and coefficients in $\mathbb{Z}[\tau^{\pm 1}]$, so $P = \sum_{k \le K-2}(\mathrm{coeff})\nu_k$. At a pure W-root every ν_k, $k \le K-2$, is regular, so P is integral, and so is the scaled matrix; this gives the bound with $\epsilon = 0$.

At an α-root, put $P_\alpha = \{k \le K-2 : \alpha_k(\zeta) = 1\}$ and write $\nu_k = \varpi^{-1} u_k$ with u_k regular for $k \in P_\alpha$ (by (c)). Split $P = A^{\mathrm{pol}} + \varpi^{-1} B$, where $B(f,g) = \sum_{k \in P_\alpha} u_k(0)\,[s^k](fg/D_K)_+$ (here $(\cdot)_+$ is the polynomial part) and A^{pol} collects $\varpi^{-1}(u_k - u_k(0))[s^k](\cdot)_+$ for $k \in P_\alpha$ and $\nu_k[s^k](\cdot)_+$ for $k \notin P_\alpha$; both A^{pol} and B have regular entries. If $\deg f \le k_{\min}$, then $\deg(fg) \le k_{\min} + K - 1$, so the quotient has degree $< k_{\min}$ and $B(f,\cdot) = 0$. Since the polynomials of degree $\le k_{\min}$ form a space of dimension $k_{\min} + 1$, $\operatorname{rank} B \le \max(0, K - 1 - k_{\min})$. After scaling, the matrix is $A + \varpi^{-1} B'$. Here A is the sum of the scaled proper parts, integral by (A) and (B), and the scaled A^{pol}, integral because all exponents are ≥ 0. The matrix B' is B in the Newton basis, scaled; its entries are regular and its rank is that of B, since the scalings are invertible over the fraction field. Expand the determinant

multilinearly in the columns: a term taking more than $\max(0, K-1-k_{\min})$ columns from $\varpi^{-1}B'$ vanishes by linear dependence, and every other term has valuation $\geq -\max(0, K-1-k_{\min})$. □

Lemma 4.3. *If $e(d) > K$ and ζ is an α-root, then $-v(\Delta_{K,\tau}) \leq \max(0, K-1-k_{\min}(d))$.*

Proof. Work in the monomial basis. Since $e > K$, no difference $|a-b| < K$ is divisible by e, so by (a)–(d) the coefficients res_a, the λ_a and the S_a ($a \leq K < e$) are regular at ζ. Only the ν_k, $k \in P_\alpha$, carry poles, and the rank argument of Proposition 4.2 applies with no scaling. □

Proposition 4.4. *Let $b_{K,d}$ be the bound of Proposition 4.2 or Lemma 4.3 (0 if $d \notin \mathcal{D}_K$) and $C_K(\tau) = \prod_d \Phi_d(\tau)^{b_{K,d}}$. Then $C_K(\tau)\Delta_{K,\tau}(X) \in \mathbb{Z}[\tau^{\pm 1}][X]$; hence $C_K(t)\Delta_K(X) \in \mathcal{O}[X]$, and $C_K(-\varphi)\sigma\Delta_K(X) \in \mathcal{O}[X]$ is its conjugate.*

Proof. By Lemma 4.1 a coefficient is $P/(\tau^n \prod \Phi_d^{n_d})$ with $P \in \mathbb{Z}[\tau]$. The bound $v_\zeta \geq -b_{K,d}$ at every primitive d-th root forces $\Phi_d^{n_d - b_{K,d}} \mid P$ in $\mathbb{Q}[\tau]$, hence in $\mathbb{Z}[\tau]$ since Φ_d is monic. Specialise $\tau \mapsto t$, a unit of $\mathcal{O}$, and $\tau \mapsto -\varphi$. □

4.3. **The size of $C_K(t)$.**

Lemma 4.5. *For $d \geq 1$, $\left|\log|\Phi_d(t)\Phi_d(-\varphi)| - \phi_{\mathrm{E}}(d)\log\varphi\right| \leq 2\mathcal{E}_1$, where $\mathcal{E}_1 = \sum_{a\geq 1} |\log(1-\varphi^{-a})| = 2.1136\ldots$.*

Proof. $\Phi_d(x) = \prod_{a|d}(x^a-1)^{\mu(d/a)}$ and $\sum_{a|d}\mu(d/a)a = \phi_{\mathrm{E}}(d)$. For $|x| = \varphi$ write $x^a - 1 = x^a(1-x^{-a})$, and $|\log|1-x^{-a}|| \leq |\log(1-\varphi^{-a})|$; for $x = t$ use $|1-t^a|$ in the same way. □

Let $\beta_7 = \prod_{p\leq 7}(1-p^{-2}) = \frac{768}{1225}$ and $c_* = \frac{11}{8}\beta_7 + \frac{1}{3} = \frac{4393}{3675} = 1.19537\ldots$.

Proposition 4.6. $\log|\mathrm{N}\, C_K(t)| \leq c_* K^3 \log\varphi + O(K^2 \log K)$.

Proof. By Lemma 4.5, $\log|\mathrm{N}\, C_K(t)| \leq \log\varphi \sum_d b_{K,d}\phi_{\mathrm{E}}(d) + 2\mathcal{E}_1 \sum_d b_{K,d}$. Here $\sum_d b_{K,d} = O(K^2\log K)$: for each $e \leq K$ at most four d have $e(d) = e$, each with a W part $\leq 4K + K^2/(4e) + 3$ (see below), and the at most $4K$ orders $d \leq 4K$ have α parts $\leq K$. Split $b_{K,d}$ into its W part and its α part.

W part. Since $m_r \geq 1$, $\sum_r m_r = K$ and $m_r < K/e + 1$, and since $c(m) = 4(m-1) + \delta(m)$ with $0 \leq \delta(m) \leq \max(0, m-5)^2/4$ (for $m \geq 6$, $c(m) = (m+2)(m+4)/4$ or $(m+3)^2/4$ according to the parity of m),

$$\sum_{r<e} c(m_r) + c'(m_e) \leq 4(K-e) + \frac{\max(0, K-4e)^2}{4e} + 3.$$

Summing over the orders d with $e(d) = e$ gives the weight $\sum_{e(d)=e}\phi_{\mathrm{E}}(d) = 2\phi_{\mathrm{E}}(e)$. Bound $\phi_{\mathrm{E}}(e) \leq e\,\eta(e)$ with $\eta(e) = \prod_{p\leq 7,\, p|e}(1-1/p)$, a function of period $P = 210$ with $0 \leq \eta \leq 1$ and mean β_7. So the W part is at most $\sum_{e\leq K}\eta(e) f_K(e)$, where

$$f_K(y) = 8y(K-y) + \tfrac{1}{2}\max(0, K-4y)^2 + 6y.$$

We use the elementary estimate: if η has period P, $0 \le \eta \le 1$ and mean β, and f is a function on $[0,K]$ with $|f| \le M$ and total variation V, then

$$\Big| \sum_{1 \le e \le K} \eta(e) f(e) - \beta \int_0^K f(y)\, dy \Big| \le 2P(M+V).$$

(Cut $[0,K]$ into blocks of length P. On a full block $(e_0, e_0+P]$ both $\sum \eta(e)f(e)$ and $\beta \int f$ differ from $P\beta f(e_0)$ by at most P times the oscillation of f on the block; the oscillations add up to at most V, and the last, incomplete block costs at most $2PM$.) For f_K we have $M, V = O(K^2)$, so the W part is at most

$$\beta_7 \int_0^K f_K(y)\, dy + O(K^2) = \Big(\tfrac{4}{3} + \tfrac{1}{24}\Big)\beta_7 K^3 + O(K^2) = \tfrac{11}{8}\beta_7 K^3 + O(K^2).$$

α part. $\max(0, K-1-k_{\min})$ counts the $k \in [k_{\min}, K-2]$, so

$$\begin{aligned} \sum_d \max(0, K-1-k_{\min}(d))\, \phi_{\mathrm{E}}(d) &\le \sum_{k=1}^{K-2} \sum_{j=1}^{k} \#\{\zeta : \alpha_j(\zeta) = 1\} \\ &= \sum_{k=1}^{K-2} (k^2 + 2k) \le \tfrac{1}{3} K^3 + O(K^2), \end{aligned}$$

since $\alpha_j(\zeta) = 1$ means $\zeta^{2j+1} = (-1)^{j+1}$, which has $2j+1$ solutions. □

Remark 4.7. The following is a sketch and is not used in the proof. Using $\sum_{n \le y} \phi_{\mathrm{E}}(n) = 3y^2/\pi^2 + O(y \log y)$ in residue classes modulo 8 and the exact $k_{\min}$, the same bound holds with c_* replaced by $(12I + 80/27)/\pi^2 \le 1.130790$, where $I = \int_0^1 \ell(x) x\, dx \in [0.6822834, 0.6831236]$ and ℓ is the piecewise linear function with $\ell(1/m) = c(m)/m$. Numerically, $\sum_d b_{K,d} \phi_{\mathrm{E}}(d)/K^3$ is 1.19 at $K = 20$ and 1.16 at $K = 40$, decreasing towards 1.1308.

5. Smallness at the conjugate

Proposition 5.1. $\log |\sigma\Delta_K(-\Xi)| \le -K^2(K+1)\log\varphi + K\log(\sqrt{K}\, M_K)$, *where* $M_K = C_0\big(K^2/((1-t^3)(1-t^2)) + \varphi^2\big)$ *and* $C_0 = \prod_{i \ge 1}(1-t^{2i})^{-1} = 2.0770906\ldots$.

Proof. At $\tau = -\varphi$ the data become: $w \mapsto \sigma w = w^{-1}$, $\lambda_a \mapsto \sigma\lambda_a = 2t^a$, $S_a \mapsto \sigma S_a = -S_a$ (as $\sigma\psi_N = -\psi_N$) and $\nu_k \mapsto \sigma\nu_k = -k(k+1)/(1+\alpha_k)$ (as $\sigma\alpha_k = -1/\alpha_k$). Let L_X^σ be the corresponding formal functional: $L_X^\sigma(s^k) = \sigma\nu_k$ and $L_X^\sigma(1/(1-w^{-a}s)) = 2t^a(X + S_a)$. The conjugate poles are $r_a = w^a$, $|r_a| = t^{2a}$, and $\sigma D_K(s) = \prod_a (1 - w^{-a}s) = (-1)^K w^{-A_K} F^\sigma(s)$ with $F^\sigma(s) = \prod_a (s - r_a)$. Hence

$$\sigma\Delta_K(X) = (-1)^{K^2} w^{A_K K} \det[Y_{i+j}(X)], \qquad Y_k(X) = L_X^\sigma(s^k/F^\sigma).$$

Partial fractions give $s^k/F^\sigma = \sum_{j=0}^{k-K} h_{k-K-j}(r) s^j + \sum_a r_a^k / \big((F^\sigma)'(r_a)(s - r_a)\big)$.

The pole moments at $X = -\Xi$. Since $1/(1-w^{-a}s) = -r_a/(s-r_a)$, we have $L_X^\sigma(1/(s-r_a)) = -r_a^{-1}\,2t^a(X+S_a)$, and at $X=-\Xi$ this is $2t^a r_a^{-1}(\Xi - S_a) = 2t^a r_a^{-1}\sum_{N>a}\psi_N$. Writing $N = a+M$, $t^{3N} = t^{3a}t^{3M}$, $t^{4a} = r_a^2$ and $w^{a+M} = r_a w^M$, this equals $\Gamma(r_a)$, where

$$\Gamma(r) = 2r\sum_{M\ge1}\frac{t^{3M}}{(1-rw^M)^3} = \sum_{n\ge1}\gamma_n r^n, \qquad \gamma_n = n(n+1)\sum_{M\ge1}t^{3M}w^{M(n-1)},$$

analytic on $|r|<\varphi^2$, with $|\gamma_n| \le n(n+1)t^{2n+1}/(1-t^{2n+1}) \le n(n+1)t^{2n+1}/(1-t^3)$. Since $\sum_a r_a^m/(F^\sigma)'(r_a) = h_{m-K+1}(r)$,

$$Y_k(-\Xi) = \sum_{j=0}^{k-K} h_{k-K-j}(r)\,\sigma\nu_j + \sum_{n\ge1}\gamma_n h_{k+n-K+1}(r).$$

Bounds. As $h_m(r_1,\dots,r_K) = w^m h_m(1,w,\dots,w^{K-1})$ and the coefficients of $\prod_{i\ge0}(1-|w|^i u)^{-1} = \sum_m u^m\prod_{i=1}^m(1-t^{2i})^{-1}$ are at most C_0, we get $|h_m(r)| \le C_0 t^{2m}$. For $j\ge1$, $|\sigma\nu_j| \le j(j+1)/(1-t^3)$, so the first sum is at most $C_0K^2/((1-t^3)(1-t^2))$ for $k\le 2K-2$. For the second, $\sum_n n(n+1)t^{2n+1} = 2t^3/(1-t^2)^3 = 2$ (as $1-t^2=t$) and $1/(1-t^3) = \varphi^2/2$, so $\sum_n|\gamma_n| \le \varphi^2$. Hence $|Y_k(-\Xi)| \le M_K$ for $0\le k\le 2K-2$. Hadamard's inequality and $|w|^{A_K K} = \varphi^{-K^2(K+1)}$ finish the proof. □

Remark 5.2. Coefficientwise, the factor $2t^a = 2|r_a|^{1/2}$ is not analytic in r_a; only at the point $X = \sigma\Xi = -\Xi$ does it combine with the tail $\sum_{N>a}\psi_N \approx t^{3a}$ into r_a^2 times an analytic function. This is why the conjugate point, rather than the coefficients of $\sigma\Delta_K$, carries the gain.

6. Nonvanishing

Proposition 6.1. *$\Delta_K(\Xi)\neq 0$ for infinitely many K.*

Proof. Let $\mathcal{C}(z) = L(1/(z-s)) = \sum_n \rho^n\,2w^n z/(z-w^n)^3$ for $|z|>t^2$. It continues to $\mathbb{C}\setminus\{0\}$ with a pole of order exactly 3 at each w^n (leading coefficient $2\rho^n w^{2n}\neq 0$), so it is not rational; it is analytic at ∞ and $O(1/z)$. For $\deg Q \le n-1$ put $H_{n,Q}(z) = L\big(Q(s)/((z-s)D_n(s))\big)$, analytic for $|z|>t^2$ and in particular at every x_a, and

$$T_{n,Q}(z) = L\Big(\frac{Q(z)D_n(s) - D_n(z)Q(s)}{(z-s)D_n(s)}\Big),$$

a polynomial in z of degree $\le n-1$. Then $\mathcal{C}Q - T_{n,Q} = D_n H_{n,Q}$.

If $\Delta_n(\Xi) = \det[L(s^{i+j}/D_n)] = 0$, a kernel vector gives $Q_n\neq 0$ of degree $\le n-1$ with $L(s^jQ_n/D_n) = 0$ for $j<n$. Put $T_n = T_{n,Q_n}$ and $E_n = \mathcal{C}Q_n - T_n$. For $|z|>t^2+\varepsilon$ the series $(z-s)^{-1} = \sum_j s^j z^{-j-1}$ converges, together with its first two s-derivatives, uniformly on $|s|\le t^2$, so L may be applied termwise; hence $H_{n,Q_n}(z) = \sum_{j\ge n}L(s^jQ_n/D_n)z^{-j-1} = O(z^{-n-1})$. So $E_n = O(z^{-1})$ and $E_n(x_a) = 0$ for $a\le n$.

If $\Delta_n(\Xi) = \Delta_{n+1}(\Xi) = 0$, then $V = T_{n+1}Q_n - T_nQ_{n+1} = Q_{n+1}E_n - Q_nE_{n+1}$ is a polynomial, $O(z^{n-1})$, vanishing at $x_1, \dots, x_n$; so $V = 0$ and $T_n/Q_n = T_{n+1}/Q_{n+1}$. If $\Delta_n(\Xi) = 0$ for all $n \geq N$, all T_n/Q_n equal one reduced fraction T/Q, so $Q_n = U_nQ$ and $T_n = U_nT$, and $E := \mathcal{C}Q - T \neq 0$ since $\mathcal{C}$ is not rational. Then $U_nE = E_n = O(1/z)$ forces $E = c_0 z^{-q}(1+o(1))$ with $c_0 \neq 0$, $q \geq 1$, and $\deg U_n \leq q-1$. But $E(x_a) \neq 0$ for all large a, while $E_n(x_a) = 0$; so $U_n(x_a) = 0$ for all large $a \leq n$, too many zeros for a polynomial of degree $< q$ once n is large. □

7. Proof of Theorem 1.1

Suppose $\zeta_F(3) = u/v$ with $u \in \mathbb{Z}$, $v \in \mathbb{Z}_{>0}$. Since $\sqrt{5} = 1 + 2t$, (1.1) gives $\Xi = u(1+2t)/(25v)$ and $\sigma\Xi = -\Xi$. For K in the infinite set of Proposition 6.1 put $y_K = C_K(t)(25v)^K\Delta_K(\Xi)$. By Proposition 4.4, $y_K \in \mathcal{O}$, and $y_K \neq 0$ because $\Delta_K(\Xi) \neq 0$ and $C_K(t) \neq 0$ (t is not a root of unity, so $\Phi_d(t) \neq 0$). Moreover $\sigma y_K = C_K(-\varphi)(25v)^K\sigma\Delta_K(-\Xi)$, so $|\,\mathrm{N}(y_K)| = |y_K\,\sigma y_K| \geq 1$, and by Propositions 3.1, 5.1 and 4.6

$$0 \leq \log|\,\mathrm{N}\,C_K(t)| + 2K\log(25v) + \log|\Delta_K(\Xi)| + \log|\sigma\Delta_K(-\Xi)| \\ \leq \big(c_* - \tfrac{2}{9} - 1\big)K^3\log\varphi + O(K^2\log K).$$

Since $c_* - \frac{11}{9} = \frac{11}{8}\beta_7 - \frac{8}{9} = -0.02684\ldots < 0$, this fails for large K. □

Remark 7.1. Not used in the proof. With the sharp constant of Remark 4.7 the margin is $\frac{11}{9} - 1.130790 \geq 0.0914$ per $K^3\log\varphi$. At finite K, computed with exact determinants and interval arithmetic, the sum $\log|\,\mathrm{N}\,C_K(t)| + \log|\Delta_K(\Xi)| + \log|\sigma\Delta_K(-\Xi)|$ (without the term $2K\log(25v)$) is already negative, about $-0.12K^3\log\varphi$ for $8 \leq K \leq 26$; and the local bounds of Section 4 agree with the exact multiplicities of the cyclotomic factors in $\mathbb{Q}(\tau)[X]$ for small K. The scripts are available from the author.

8. Formal verification

The Lean development is the repository

`https://github.com/telaviv/fibonacci-zeta3-irrationality`;

all file names below refer to it. It (library `FibZeta`, Lean 4.35.0, Mathlib v4.35.0, about 8500 lines) proves

`theorem FibZeta.irrational_zetaF3 : Irrational FibZeta.zetaF3`,

where `zetaF3` is the series $\sum_{n\geq 0} 1/F_{n+1}^3$, written with Mathlib's `Nat.fib` and a `tsum` (`FibZeta/Final.lean`; `hasSum_zetaF3` shows that the series converges to it). The proof uses only the axioms `propext`, `Classical.choice` and `Quot.sound` (`Axioms.lean`); there is no `sorry`, no `native_decide` and no added axiom.

What has to be trusted. For the theorem itself, only the definition of `zetaF3` (and Mathlib's `Nat.fib`, `tsum` and `Irrational`) has to be read: the kernel checks everything else. Since a divergent `tsum` has the value 0, which is rational, the statement cannot hold for a degenerate reason. To

see that the formal proof follows the argument of this paper one also reads the definitions in `FibZeta/Defs.lean` and the intermediate statements in `FibZeta/Statements.lean`; an internal comparison of these with an earlier draft of this section is in the repository (`FIDELITY_REFEREE.md`).

Definitions. `FibZeta/Defs.lean` defines the construction of Section 2 over an arbitrary field $\mathbb{F}$ with a parameter $\tau \in \mathbb{F}$: `alpha`, `nu`, `psi`, `S`, the poles `node` $= w^{-a}$, the monic polynomial `F` $= \prod_a (s - x_a)$ (so $D_K = (-1)^K w^{A_K} F$), the residues `res`, the two coefficients `mConst`, `mLin` of $L_X(s^k/F)$, the normalisation `nrm`, and the determinant `Delta`. Specialised at $\tau = t$ (`tR`) and at $\tau = -\varphi$ it gives Δ_K and $\sigma\Delta_K$; at $\tau =$ `RatFunc.X` it gives $\Delta_{K,\tau}$. The real functional is `Lfun`, with $\theta(\theta+1)g = s^2 g'' + 2sg'$ written with `deriv`, and `XiR` is Ξ. The clearing polynomial is `CKpoly` $= \prod_{d \le 4K} \Phi_d^{b_{K,d}}$, with exponents `nExp` $=$ `nW` $+$ `nAlpha`: the W part (cluster costs `cW`, `cW'` and cluster sizes `clusterSize`) and the α part. The root types are defined through the chosen root $\exp(2\pi i/d)$ (`zetaC`, `eW`, `IsAlphaRoot`), and `FibZeta/Roots.lean` proves that they are the same for every primitive d-th root.

The chain. `FibZeta/Statements.lean` states the ingredients as propositions, and `FibZeta.irrational_zetaF3_of` (in `FibZeta/Main.lean`) derives the theorem from them by the argument of Section 7. Instead of norms in $\mathcal{O}$ it uses that t and $-\varphi$ are the two roots of $\tau^2 + \tau - 1$: for $Q \in \mathbb{Z}[X]$ the product $Q(t)Q(-\varphi)$ is an integer, nonzero if $Q(t) \neq 0$ (`norm_ne_zero`). The docstrings of the Lean files refer to the numbering of the author's notes; the header of `FibZeta/Statements.lean` translates it to the numbering of this paper.

Paper	Lean statement	Lean proof
Lemma 2.2, entries	`EntryIdentity`	`FibZeta.Analytic.entryIdentity`
Prop. 3.1	`DecayBound`	`FibZeta.Analytic.decayBound`
Lemma 4.1–Prop. 4.4	`Integrality`	`FibZeta.Denom.integrality`
Lemma 4.5, Prop. 4.6	`CostBound`	`FibZeta.Cost.costBound`
Prop. 5.1	`ConjBound`	`FibZeta.Conj.conjBound`
Prop. 6.1	`Nonvanishing`	`FibZeta.NonVan.nonvanishing`
Section 7		`FibZeta.irrational_zetaF3_of`

`EntryIdentity` is used only in the proof of `DecayBound`, not by the assembly. The formal statements are equivalent to, or slightly weaker than, the propositions, in the form the assembly needs. `DecayBound` is $|\Delta_K(\Xi)| \le C^{K^2} t^{2b_K}$ with $b_K = 2\sum_{j<K} a_j$ (proved with $C = 4096$). `ConjBound` is $|\sigma\Delta_K(-\Xi)| \le C^{K^2} t^{K^2(K+1)}$; its proof bounds the complete homogeneous sums $h_m(r)$ by φ^K instead of $C_0 t^{2m}$, which still gives entries of size $\le D^K$. `CostBound` is $\log|C_K(t)C_K(-\varphi)| \le \frac{121}{100}K^3 \log\varphi + CK^2(1+\log K)^2$; its proof follows Proposition 4.6, with cubic constant $c_* = \frac{4393}{3675}$ and the constant 12 in place of $2\mathcal{E}_1$ in the upper half of Lemma 4.5, and $c_* < \frac{121}{100} < \frac{11}{9}$. `Integrality` is Proposition 4.4 for every real $\tau \neq 0$ that is not a root of

unity, with one integer polynomial for all such τ; this covers t and $-\varphi$ at once. Its proof works in the local rings of $\mathbb{C}(\tau)$ at the roots of unity: the Newton cluster basis at W-roots, the monomial basis elsewhere, and the rank correction at α-roots (`FibZeta.Denom.local_bound_cluster`, `FibZeta.Denom.local_bound_monomial`, `FibZeta.Denom.integrality_of_local_bounds`). The proof of `Nonvanishing` uses the Cauchy transform $\mathcal{C}$ directly: its values at the interpolation nodes, $\mathcal{C}(x_a) = 2t^a(\Xi - S_a)$, and its expansion $\mathcal{C}(1/u) = \sum_k \nu_k u^{k+1}$ at ∞ replace the functional L in the interpolation identities.

Provenance. Parts of the formal development are adapted from the Lean formalisation QZeta3 [13] of [14] (Apache License 2.0): the localisation, specialisation and descent machinery, the weighted and rank determinant lemmas, the Cauchy-transform and normality arguments for nonvanishing, the decay estimates, the totient sieve and the cyclotomic size bounds. Every adapted file says so in its header, and the repository's `NOTICE` file records it.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop and check the proof, to write the Lean formalisation and to draft this text. The author directed the project and is responsible for its content. The formal proof is checked by the Lean kernel.

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