hexagon:2611.00472v2

NEW LOWER BOUNDS FOR ROMANOFF'S CONSTANT

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Abstract. Romanoff proved in 1934 that the integers of the form $p + 2^a$, with p prime and $a \geq 1$, have positive lower density d. We prove that $d \geq 0.110825$ and, conditionally on an upper bound for prime pairs that Johnston and Trudgian derive from Lichtman's work on primes in arithmetic progressions, that $d \geq 0.119178$; the best lower bound so far was 0.10788. The proof refines the method of Elsholtz and Schlage-Puchta, which bounds the second moment of the number of representations in each residue class modulo a fixed integer. Both bounds are certified by computer and also proved in the Lean proof assistant, with the analytic inputs as hypotheses.

1. Introduction

For a positive integer n let $r(n)$ be the number of pairs (p, a) with p prime, $a \geq 1$ and $n = p + 2^a$, and let

$$d = \liminf_{N\to\infty} \frac{1}{N} \#\{n \leq N : r(n) > 0\}.$$

Allowing $a = 0$ adds the integers $p + 1$, which have density 0, so it does not change d. Romanoff [19] proved that $d > 0$. Erdős [6] and van der Corput [20] showed that a positive proportion of the odd integers are not of this form. Habsieger and Roblot [8] showed that the upper density is less than 0.490941. Chen, Dai and Li [2] lowered this to 0.49035, and a certificate of Yoo [22] lowers it to 0.49019. Heuristic and numerical studies suggest a density of about 0.434 to 0.437 [18, 4].

Explicit lower bounds for d start from the second moment of $r(n)$. Chen and Sun [3] proved $d \geq 0.0868$. Habsieger and Roblot [8] improved this to 0.0933, Pintz [15] to 0.093626, and Habsieger and Sivak-Fischler [10] to 0.0936275. Elsholtz and Schlage-Puchta [5] split the integers into residue classes modulo $2^{24} - 1$, where $r(n)$ is distributed very unevenly, and proved $d \geq 0.107648$. Johnston and Trudgian [11, footnote 2] obtained the slightly better value 0.10788 by repeating these computations.[1] Under the bound (2.2) below, Johnston and Trudgian also obtained 0.10695 by Pintz's method.

Theorem 1.1. *We have $d \geq 0.110825$.*

Theorem 1.2. *Assume the upper bound* (2.2) *below for the number of prime pairs with a given difference. Then $d \geq 0.119178$.*

Johnston and Trudgian [11, Proposition 2] deduce (2.2) from Lichtman's Theorems 1.2 and 1.7 of [13]. Both papers are preprints; Theorem 1.2 uses them only through (2.2), and Theorem 1.1 uses neither them nor (2.2). With the sieve weights of Wu [21] in place of the constant 8 in (2.1), the method of Elsholtz and Schlage-Puchta would give 0.11011, but this extension of Wu's work has not been checked, and Elsholtz and Schlage-Puchta did not state it as a

2020 *Mathematics Subject Classification.* 11P32, 11N36, 11Y16; 90C05.

Key words and phrases. Romanoff's theorem, primes and powers of two, Selberg sieve, Bombieri–Vinogradov theorem, linear programming, computer-assisted proof, formal verification.

[1] They note that it is also stated at the end of page 721 of [5].

theorem [11, Section 4.2 and footnote 6]. Theorem 1.1 exceeds this value, although neither of our results uses Wu's weights.

The method. In the second-moment method one bounds $\sum_{n\le N} r(n)^2$ from above. The main term counts the pairs $p_1 + 2^{a_1} = p_2 + 2^{a_2}$ with $a_1 \neq a_2$. For a fixed shift $h = 2^{a_2} - 2^{a_1}$ this is a count of prime pairs $p_1 - p_2 = h$, which the Selberg sieve with the Bombieri–Vinogradov theorem bounds by four times the Hardy–Littlewood prediction (the constant 8 in (2.1) below, where the prediction has the constant 2). Elsholtz and Schlage-Puchta observed that the same bound holds with a congruence condition on p_2 modulo a fixed l, so that the second moment can be bounded in each residue class modulo l separately. Since the classes in which $r(n)$ is often large carry most of the pairs, the resulting bound for d is much better. Theorem 1.1 is this method with the modulus $l_3 = 29 \cdot 43 \cdot l_1$, where $l_1 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 37 \cdot 41 \cdot 73 \cdot 241$, for which the period of 2^a is 27720 instead of 24, and with sharper bounds for the singular series. For the part of the singular series that these bounds leave out we know only an upper bound for its total, not how it is distributed over the divisors of the period, and we take the worst case over all such distributions (Section 4).

For prime pairs without a congruence condition, the bound (2.2) below replaces four times the prediction by 3.3907 times the prediction. Johnston and Trudgian observed that their argument does not give the same constant with a congruence condition modulo l: sifting $p_2 + h$ over $p_2 \equiv k \pmod l$ needs residues that depend on the sifting modulus, but Lichtman's estimate allows only one residue for all moduli. For Theorem 1.2, with the same modulus, we therefore use the constant 8 in every class and (2.2) only for the total over all classes; both bounds apply to the same pair counts for each large N. Roughly speaking, since no class can then hold more than $8/6.7814 \approx 1.18$ times its share of the total, the worst allocation of the pairs to the classes gives a bound well above the one obtained with the constant 8 alone. Since Pintz's function is a maximum of linear functions, this worst case is the minimum of a linear program, which we bound from below by Lagrangian duality (Section 4).

Organization. Section 2 states the analytic inputs, Section 3 bounds the second moments in the residue classes, and Section 4 reduces the bound for d to a linear program and proves the two theorems with the bounds for the singular series of Section 5. Section 6 discusses how the bound depends on the two sieve constants. Appendix A describes the computation and the programs that check the certificates, and Appendix B the formal proofs in Lean.

2. Inputs

Throughout, p denotes a prime, l a fixed odd squarefree integer, φ Euler's function, and

$$C_0 = \prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big) = 0.66016\ldots, \qquad S(h) = \prod_{p\mid h,\ p>2} \frac{p-1}{p-2}.$$

For odd squarefree m put $f(m) = \prod_{p\mid m}(p-2)^{-1}$, so that $S(h) = \sum f(m)$ over the odd squarefree divisors m of h, and let o_m be the multiplicative order of 2 modulo m. For $N \ge 1$, an integer h with $0 < |h| < N$ and an integer k put

$$R(N,h) = \#\{(p_1,p_2) : p_1,p_2 \le N,\ p_1 - p_2 = h\},$$
$$R(N,h,k,l) = \#\{(p_1,p_2) : p_1,p_2 \le N,\ p_2 \equiv k \pmod l,\ p_1 - p_2 = h\}.$$

The per-class bound. Johnston and Trudgian [11, Section 4.2] quote from [5, (28)] the following consequence of the Selberg sieve and the Bombieri–Vinogradov theorem [1, 9]. We use it in the

form they state. For every $\varepsilon > 0$ there is $N_0(\varepsilon, l)$ such that for all $N \geq N_0$, all $0 < h < N$ and all k with $(k, l) = 1$,

$$R(N, h, k, l) \leq (8 + \varepsilon) \frac{C_0 N}{\varphi(l) \log^2 N} \prod_{p \mid lh,\ p > 2} \frac{p-1}{p-2}. \tag{2.1}$$

For $l = 1$ this is the classical upper bound for prime pairs. The congruence condition changes the moduli in the Bombieri–Vinogradov theorem from d to dl, with a residue that depends on d, which the maximum over the residues in that theorem allows. Since $R(N, h, k, l) = R(N, -h, k + h, l)$ and the right-hand side of (2.1) depends only on $|h|$, the bound (2.1) also holds for $-N < h < 0$ if $(k + h, l) = 1$.

The total bound. Johnston and Trudgian [11, Proposition 2] deduce from Theorems 1.2 and 1.7 of [13] that for every $\varepsilon > 0$ there is $N_1(\varepsilon)$ such that for all $N \geq N_1$ and all $0 < |h| < N$,

$$R(N, h) \leq (6.7814 + \varepsilon) \frac{C_0 N}{\log^2 N} S(h). \tag{2.2}$$

They state it for $h > 0$; the case $h < 0$ follows since $R(N, -h) = R(N, h)$.

For fixed even $h \neq 0$ the Hardy–Littlewood conjecture predicts $R(N, h) \sim 2C_0 N S(h)/\log^2 N$. The corresponding prediction for $R(N, h, k, l)$, when k and $k + h$ are coprime to l, is the right-hand side of (2.1) with 2 in place of $8 + \varepsilon$. So (2.1) is four times the prediction, and (2.2) is 3.3907 times the prediction.

Further inputs. We use the prime number theorem in progressions to the fixed modulus l, the bound

$$R_0 := \sum_{m \text{ odd squarefree}} \frac{f(m)}{o_m} < 1.93656 \tag{2.3}$$

which follows from Pintz's bound $C_0 R_0 < 1.2784421041$ [15, Lemma 2″] and $C_0 > 0.6601618158$, as stated in [17, p. 55] (see also [11, (17)]), and the following inequality of Pintz [15, Lemma 4′].

Lemma 2.1. *Let $b_1, \dots, b_K \geq 0$ be integers with $\sum b_i = M > 0$ and $\sum b_i^2 \leq DM$. Then $\#\{i : b_i > 0\} \geq M\psi(D)$, where*

$$\psi(D) = \max_{j \geq 1} \frac{2j + 1 - D}{j(j+1)}.$$

Proof. For integers $b \geq 1$ and $j \geq 1$ we have $(b-j)(b-j-1) \geq 0$, that is, $b^2 \geq (2j+1)b - j(j+1)$. So $b^2 \geq (2j+1)b - j(j+1)\,\mathbf{1}_{b>0}$ for all integers $b \geq 0$, with equality for $b = 0$. Summing over i gives $DM \geq (2j+1)M - j(j+1)\#\{i : b_i > 0\}$. □

On $[1, \infty)$ the function ψ is the piecewise linear interpolation of $1/D$ at the positive integers. It is convex and decreasing there.

3. Moments in residue classes

Let P be the order of 2 modulo l, so that $2^a \bmod l$ depends only on $a \bmod P$. For a residue $k \bmod l$ let

$$U_k = \{r \in \mathbb{Z}/P\mathbb{Z} : (k - 2^r, l) = 1\}, \qquad s_k = |U_k|.$$

Let $L = \lfloor \log_2 N \rfloor$. Throughout this section k is fixed and $N \to \infty$.

Lemma 3.1. $\sum_{n \leq N,\ n \equiv k} r(n) = \big(s_k/(P\varphi(l)\log 2) + o(1)\big)N$, *where $n \equiv k$ means $n \equiv k \pmod{l}$.*

Proof. The sum counts the pairs (p,a) with $a \le L$ and $p \le N - 2^a$, $p \equiv k - 2^a \pmod l$. If $a \bmod P \notin U_k$, there are $O(1)$ such primes. Otherwise the prime number theorem in progressions gives $(1+o(1))(N-2^a)/(\varphi(l)\log N)$ uniformly for $a \le L - \log_2 L$. The remaining $O(\log L)$ values of a contribute $O(N \log L/\log N) = o(N)$. Summing over the $(s_k/P + o(1))L$ admissible a gives the claim. □

The second moment is $\sum_{n\le N,\, n\equiv k} r(n)^2 = \sum_{n\le N,\, n\equiv k} r(n) + \sum_{a_1\ne a_2} X_k(a_1,a_2)$, where for $1 \le a_1, a_2 \le L$ with $a_1 \ne a_2$ and $h = 2^{a_2} - 2^{a_1}$,

$$X_k(a_1,a_2) = \#\{p_2 \le N - 2^{a_2} : p_2 \equiv k - 2^{a_2} \pmod l,\ p_2 + h \text{ prime}\}$$

counts the pairs of representations $p_1 + 2^{a_1} = p_2 + 2^{a_2} \le N$ in the class of k. It is $O(1)$ unless $a_1 \bmod P$ and $a_2 \bmod P$ lie in U_k. Both primes of such a pair are at most N, and $0 < |h| < N$, so

$$X_k(a_1,a_2) \le R(N,h,k-2^{a_2},l), \qquad \sum_{k \bmod l} X_k(a_1,a_2) \le R(N,h). \tag{3.1}$$

If $a_1 \bmod P$ and $a_2 \bmod P$ lie in U_k, then $k - 2^{a_2}$ and $k - 2^{a_2} + h = k - 2^{a_1}$ are coprime to l, so (2.1) bounds the first right-hand side, for either sign of h. The bound (2.2) bounds the second.

We group the pairs (a_1,a_2) by $G = (a_1 \bmod P, a_2 \bmod P)$ and write $c = c(G) = a_2 - a_1 \bmod P$. For an odd prime p we have $p \mid h$ exactly when $\operatorname{ord}_p(2)$ divides $a_2 - a_1$. Hence $S(h) = Q(c)S'(h)$, where

$$Q(c) = \prod_{p\mid l,\ \operatorname{ord}_p(2)\mid c} \frac{p-1}{p-2}, \qquad S'(h) = \prod_{p\mid h,\ p\nmid 2l} \frac{p-1}{p-2},$$

and the product in (2.1) equals $P_l S'(h)$ with $P_l = \prod_{p\mid l}(p-1)/(p-2)$.

Lemma 3.2. *For $r_1, r_2 \in \mathbb{Z}/P\mathbb{Z}$ and $c = r_2 - r_1$,*

$$\sum_{\substack{1\le a_1\ne a_2\le L\\ a_1\equiv r_1,\ a_2\equiv r_2}} S'(2^{a_2} - 2^{a_1}) \le \Big(\frac{L}{P} + 1\Big)\frac{L}{P}\,T(c) + O(L\log L),$$

with

$$T(c) = \sum_{\substack{m \text{ odd squarefree}\\ (m,l)=1}} f(m)\,[g_m \mid c]\,\frac{g_m}{o_m},$$

where the congruences are modulo P, $g_m = \gcd(o_m, P)$, and $[\cdot]$ is 1 if the condition holds and 0 otherwise.

Proof. We have $S'(2^{a_2} - 2^{a_1}) = \sum f(m)$ over the squarefree m with $(m, 2l) = 1$ and $m \mid 2^{|a_2-a_1|} - 1$, that is, $o_m \mid a_2 - a_1$. For fixed m, the conditions $a_2 - a_1 \equiv c \pmod P$ and $o_m \mid a_2 - a_1$ are compatible only if $g_m \mid c$, and then they fix a_2 modulo $\operatorname{lcm}(P, o_m) = Po_m/g_m$ for given a_1. Since $0 < |a_2 - a_1| < L$, there is no such pair if $o_m \ge L$. So for each of the at most $L/P + 1$ values of a_1 there are at most $Lg_m/(Po_m) + [o_m < L]$ values of a_2. Summing over m gives the main term and an error at most $(L/P+1)\sum_{o_m<L} f(m)$. The primes p with $\operatorname{ord}_p(2) < L$ divide $M_L = \prod_{t<L}(2^t - 1)$, so

$$\sum_{o_m<L} f(m) \le \prod_{p\mid M_L} \frac{p-1}{p-2} \le \frac{1}{C_0}\,\frac{M_L}{\varphi(M_L)} \ll \log\log M_L \ll \log L. \qquad \square$$

4. The linear program

We measure counts in the unit $N/(P\varphi(l)\log 2)$. For a class k with $s_k > 0$ and a group $G = (r_1, r_2) \in U_k \times U_k$, let $Y_k(G)$ be the sum of $X_k(a_1, a_2)$ over the pairs in G, in this unit, and let $Y_k = \sum_G Y_k(G)$. Put $\kappa = C_0 P_l/(P\log 2)$, and let A_G be the number of classes $k \bmod l$ with $r_1, r_2 \in U_k$. A class k is counted in A_G exactly when, for every $p \mid l$, the residue $k \bmod p$ avoids 2^{r_1} and 2^{r_2}, which coincide modulo p when $\mathrm{ord}_p(2) \mid c$. Hence

$$A_G = \varphi(l) \prod_{p\mid l,\ \mathrm{ord}_p(2)\nmid c} \frac{p-2}{p-1}, \qquad P_l \frac{A_G}{\varphi(l)} = Q(c).$$

Lemma 4.1. *Let $T^+(c) \ge T(c)$ for all c, and let $\delta > 0$. For all large N, all k with $s_k > 0$ and all $G \in U_k \times U_k$,*

$$Y_k(G) \le (8+\delta)\,\kappa\, T^+(c) \tag{4.1}$$

and

$$\frac{1}{N}\,\#\{n \le N : n \equiv k,\ r(n) > 0\} \ge \frac{s_k - \delta}{P\varphi(l)\log 2}\,\psi\Big(1 + \frac{Y_k}{s_k} + \delta\Big). \tag{4.2}$$

If (2.2) holds, then for all large N and all G also

$$\sum_{k:\, G\in U_k\times U_k} Y_k(G) \le (6.7814+\delta)\,\kappa\, T^+(c)\, A_G. \tag{4.3}$$

Proof. By (3.1), (2.1) and Lemma 3.2, the sum of $X_k(a_1, a_2)$ over the pairs in G is at most $(8+\varepsilon)C_0 N P_l(\varphi(l)\log^2 N)^{-1}\big((L/P)^2 T(c) + O(L\log L)\big)$. Since $L^2/\log^2 N \to 1/\log^2 2$ and $T(c) \ge 1$ (the term $m = 1$), this is at most $(8+\delta)\kappa T(c)$ in the unit $N/(P\varphi(l)\log 2)$ for large N. For (4.2) we apply Lemma 2.1 to the numbers $r(n)$, $n \le N$, $n \equiv k$. By Lemma 3.1 their sum is $s_k + o(1)$ units. Their sum of squares exceeds their sum by the sum of $X_k(a_1, a_2)$ over all $a_1 \ne a_2$, which is at most $Y_k + o(1)$ units, since the pairs with $a_1 \bmod P$ or $a_2 \bmod P$ outside U_k contribute $O(L^2) = o(N)$. Since ψ is decreasing, (4.2) follows. Finally, in the same way as for (4.1), the bounds (3.1), (2.2) and Lemma 3.2 bound the left-hand side of (4.3) by $(6.7814+\delta)C_0 Q(c)T(c)\varphi(l)/(P\log 2)$, which is the right-hand side of (4.3) since $Q(c)\varphi(l) = P_l A_G$. □

Put $\nu = (l/\varphi(l))/(P\log 2)$ and

$$F(Y) = \frac{\nu}{l} \sum_{k:\, s_k > 0} s_k\, \psi\Big(1 + \frac{Y_k}{s_k}\Big). \tag{4.4}$$

Let V_8 be the minimum of F over all $Y \ge 0$ that satisfy (4.1) with $\delta = 0$, and let V be its minimum over all $Y \ge 0$ that satisfy (4.1) and (4.3) with $\delta = 0$; we call the constraints (4.1) the caps and (4.3) the budget rows. Summing (4.2) over the classes k with $s_k > 0$ shows that $d \ge V_8$, and that $d \ge V$ if (2.2) holds. Indeed, for large N the counts $Y_k(G)$ satisfy the constraints of Lemma 4.1 with δ, and dividing them by $1+\delta$ gives a point that satisfies them with $\delta = 0$. Since ψ is $\frac{1}{2}$-Lipschitz and the Y_k are bounded by (4.1), this division, the replacement of $s_k - \delta$ by s_k and the removal of the $+\delta$ in the argument of ψ change the sum of the right-hand sides of (4.2) by $O(\delta)$. So $d \ge V_8 - O(\delta)$ for every $\delta > 0$, and in the same way $d \ge V - O(\delta)$. Since the Y_k are bounded, only finitely many j in the definition of ψ matter, so V_8 and V are values of linear programs.

Reduction to rotation orbits. We call U_k the pattern of the class k. For a rotation orbit u of nonempty patterns $U \subset \mathbb{Z}/P\mathbb{Z}$, let K_u be the set of classes k with $U_k \in u$, and put $m_u = |K_u|$,

$\rho_u = m_u/l$, $s_u = |U|$, $w_u = \nu\rho_u s_u$ and $n_u(c) = \#\{r \in U : r + c \in U\}$. These do not depend on the choice of $U \in u$. For a feasible point Y put

$$y_u(c) = \frac{1}{m_u} \sum_{k \in K_u} \sum_{G \in U_k \times U_k,\ c(G)=c} Y_k(G).$$

For each $k \in K_u$ there are $n_u(c)$ groups $G \in U_k \times U_k$ with $c(G) = c$, so (4.1) gives $0 \le y_u(c) \le y_u^{\max}(c)$ with $y_u^{\max}(c) = 8\kappa\, n_u(c)\, T^+(c)$. Summing (4.3) over the P groups with difference c and dividing by l gives $\sum_u \rho_u\, y_u(c) \le 6.7814\,\kappa\, T^+(c)\,\mathcal{N}(c)$, where

$$\mathcal{N}(c) = \frac{1}{l} \sum_{c(G)=c} A_G = \sum_u \rho_u\, n_u(c).$$

Since ψ is convex, Jensen's inequality gives $\sum_{k \in K_u} s_u \psi(1 + Y_k/s_u) \ge m_u s_u \psi\big(1 + \sum_c y_u(c)/s_u\big)$. Hence V is at least the minimum of

$$\sum_u w_u\, \psi\Big(1 + \frac{\sum_c y_u(c)}{s_u}\Big)$$

subject to

$$0 \le y_u(c) \le y_u^{\max}(c), \qquad \sum_u \rho_u\, y_u(c) \le 6.7814\,\kappa\, T^+(c)\,\mathcal{N}(c),$$

and V_8 is at least the minimum subject to the first family of constraints alone.

A bound affine in the uncomputed part. For Theorem 1.1 we use no budget rows, and we take into account that a part of the singular series is known only through an upper bound for its total. Let $\vartheta \ge 0$, and let $\mathcal{S}$ be the simplex of the vectors $\alpha = (\alpha_g)$, indexed by the positive divisors g of P, with $\alpha_g \ge 0$ and $\sum_g \alpha_g \le \vartheta$. Its vertices are 0 and the points ϑe_g.

Proposition 4.2. *Let T_0 be a function on $\mathbb{Z}/P\mathbb{Z}$, and suppose that there is an $\alpha \in \mathcal{S}$ with $T(c) \le T_0(c) + \sum_{g|P,\, g|c} g\,\alpha_g$ for all c. For each class k with $s_k > 0$ let $j_k \ge 1$ be an integer, and put*

$$\mathcal{Q}_k = \sum_{r_1, r_2 \in U_k} T_0(r_2 - r_1), \qquad S_{k,g} = \#\{(r_1, r_2) \in U_k \times U_k : r_1 \equiv r_2 \pmod g\}.$$

Then $d \ge \min_\beta M(\beta)$, where β runs through the vertices of $\mathcal{S}$ and

$$M(\beta) = \frac{\nu}{l} \sum_{k:\, s_k > 0} \Big(\frac{2 s_k}{j_k + 1} - \frac{8\kappa}{j_k(j_k+1)} \Big(\mathcal{Q}_k + \sum_{g|P} g\,\beta_g\, S_{k,g} \Big) \Big).$$

Proof. The function $T^+(c) = T_0(c) + \sum_{g|c} g\alpha_g$ is an upper bound for T, so $d \ge V_8$ for the caps (4.1) with this T^+. Since ψ is decreasing, F is smallest when every $Y_k(G)$ equals its cap, and then $Y_k = 8\kappa\big(\mathcal{Q}_k + \sum_g g\alpha_g S_{k,g}\big)$, because $g \mid c(G)$ exactly when the two entries of G are congruent modulo g. By its definition, $\psi(D) \ge (2j + 1 - D)/(j(j+1))$ for every integer $j \ge 1$. With $D = 1 + Y_k/s_k$ and $j = j_k$ this gives $V_8 \ge M(\alpha)$. Since M is affine, its minimum over $\mathcal{S}$ is attained at a vertex. □

The choice of the j_k affects only the quality of the bound. On $[j, j+1]$ the linear function in the proof equals ψ, so $j_k = \lfloor 1 + Y_k/s_k \rfloor$ is the best choice for a given α. The numbers s_k, $\mathcal{Q}_k$ and $S_{k,g}$ depend only on the rotation orbit of U_k. In Section 5 we show that the hypothesis of Proposition 4.2 holds with $\vartheta = 1.93656 - R^-_{0,\mathrm{a}} - R^-_{0,\mathrm{b}}$ for every upper bound T_0 of $T_\mathrm{a} + T_\mathrm{b}$, and we define the function T_0 that we use. The prime factors of l_3 beyond those of $2^{24} - 1$ and 11 and 31, namely 19, 23, 29, 37, 41, 43 and 73, were chosen with the help of floating-point models of the bound, which are in the archive (see Data availability); 29 and 43, modulo which 2 has the orders 28 and 14, bring the factor 7 into the period. For $l = l_3$, whose period is $P = 27720$,

P has 96 positive divisors, and the smallest of the 97 values $M(\beta)$ is at least 0.1108257388, at $\beta = \vartheta e_{27720}$ (Appendix A). This proves Theorem 1.1.

For Theorem 1.2 we add the budget rows (4.3), summed over all groups G, with one multiplier μ, and we relax the caps of each class with a multiplier λ_k.

Proposition 4.3. *Assume* (2.2)*, and let* T_0*,* α*,* $\mathcal{Q}_k$ *and* $S_{k,g}$ *be as in Proposition 4.2. Let* $\mu \ge 0$*, and for each class* k *with* $s_k > 0$ *let* $\lambda_k \ge 0$ *with* $\mu+\lambda_k > 0$*. Put* $h_k = \min_{i\ge1}\bigl(\nu/i+(\mu+\lambda_k)(i-1)\bigr)$*. Then* $d \ge \min_\beta M_\mu(\beta)$*, where* β *runs through the vertices of* $\mathcal{S}$ *and*

$$M_\mu(\beta) = \frac{1}{l}\sum_{k:\,s_k>0}\Bigl(s_k h_k - (8\lambda_k + 6.7814\mu)\,\kappa\Bigl(\mathcal{Q}_k + \sum_{g|P} g\,\beta_g\,S_{k,g}\Bigr)\Bigr).$$

Proof. Let T^+ be as in the proof of Proposition 4.2 and $X_k = \mathcal{Q}_k + \sum_g g\alpha_g S_{k,g}$, so that the caps (4.1) give $Y_k \le 8\kappa X_k$. Since A_G is the number of classes k with $G \in U_k \times U_k$, the sum of (4.3) over all G is $\sum_k Y_k \le 6.7814\,\kappa \sum_k X_k$. For every Y that satisfies both constraints, therefore,

$$F(Y) \ge F(Y) + \frac{\mu}{l}\Bigl(\sum_k Y_k - 6.7814\,\kappa\sum_k X_k\Bigr) + \frac{1}{l}\sum_k \lambda_k\bigl(Y_k - 8\kappa X_k\bigr).$$

With $D_k = 1 + Y_k/s_k$, the right-hand side is the sum over k of $l^{-1}s_k\bigl(\nu\psi(D_k) + (\mu+\lambda_k)(D_k - 1)\bigr) - l^{-1}(8\lambda_k + 6.7814\mu)\kappa X_k$. On $[1,\infty)$ the function $\nu\psi(D) + (\mu+\lambda_k)(D-1)$ is convex and piecewise linear with breakpoints at the integers, so its minimum is attained at an integer, and it is h_k. Hence $V \ge M_\mu(\alpha)$. Since M_μ is affine, its minimum over $\mathcal{S}$ is attained at a vertex. □

With $\mu = 0$ and $\lambda_k = \nu/(j_k(j_k+1))$ we have $h_k = 2\nu/(j_k+1)$, and M_0 is the function M of Proposition 4.2. For Theorem 1.2 we take $l = l_3$ again, $\mu = 6.87217\cdot10^{-7}$, about $3.676\cdot10^{-3}\nu$, and $\lambda_k = \max\bigl(0, \nu^-/(j_k(j_k+1)) - \mu\bigr)$, where ν^- is the lower bound for ν that the computation uses, with the same j_k as for Theorem 1.1, chosen at $\alpha = \vartheta e_{27720}$ (Appendix A). The smallest of the 97 values $M_\mu(\beta)$ is at least 0.1191789096, at $\beta = \vartheta e_{27720}$. This proves Theorem 1.2. Table 1 summarizes the bounds for the three moduli.

TABLE 1. The lower bounds for d given by the certificates for the three moduli. Theorems 1.1 and 1.2, the two bounds for l_0 and the unconditional bound for l_1 are also proved in Lean (Appendix B).

Modulus	Period	Bound	Bound under (2.2)
l_0	120	0.109406	0.117953
l_1	3960	0.110449	0.118859
l_3	27720	0.110825	0.119178

In both theorems the smallest value is at $\beta = \vartheta e_P$. In fact $M(\beta) \ge M(\vartheta e_P)$ and $M_\mu(\beta) \ge M_\mu(\vartheta e_P)$ for every $\beta \in \mathcal{S}$. Each class modulo a divisor g of P contains P/g residues modulo P, so $gS_{k,g} \le Ps_k = PS_{k,P}$. Hence $\sum_g g\beta_g S_{k,g} \le \vartheta P s_k$ for $\beta \in \mathcal{S}$, with equality at $\beta = \vartheta e_P$, and this sum enters $M(\beta)$ and $M_\mu(\beta)$ with a nonpositive coefficient. The programs of Appendix A nevertheless evaluate all vertices.

In the checker, κ is replaced by an upper bound and ν by a lower bound, computed from the bounds for C_0 and $\log 2$ in Appendix A. This is allowed, since the values $M(\beta)$, when they are positive, do not increase when κ or T^+ increases and do not decrease when ν increases. The same holds for the values $M_\mu(\beta)$ with fixed λ_k, and for the right-hand side of (4.5) below.

The smaller moduli. For $l = l_1$, whose period is 3960, the smallest of the 49 values $M(\beta)$ is at least 0.1104497986, at $\beta = \vartheta e_{3960}$, which gives the weaker bound $d \geq 0.110449$. For the modulus $l_0 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 41 \cdot 241$, whose period is 120, there are 10,815,600 nonempty patterns, in 92,206 rotation orbits, and one empty pattern, and we bound the relaxed program directly. For multipliers $\mu_c \geq 0$, weak duality for the Lagrangian relaxation of the budget rows gives

$$V \geq \sum_u \min_{0 \leq y \leq y_u^{\max}} \Big(w_u \, \psi\Big(1 + \frac{\sum_c y(c)}{s_u}\Big) + \rho_u \sum_c \mu_c \, y(c) \Big) - \sum_c \mu_c \, 6.7814 \, \kappa \, T^+(c) \, \mathcal{N}(c). \tag{4.5}$$

For a fixed total $z = \sum_c y(c)$ the linear part is smallest when the $y(c)$ with the smallest μ_c are filled first. So each inner minimum is the minimum of a convex piecewise linear function of z, which is attained at one of its breakpoints and can be computed exactly.

Without budget rows, the same bound with all $\mu_c = 0$ holds for V_8. Since ψ is decreasing, each inner minimum is then attained at $y = y_u^{\max}$, so every class carries its full cap, and

$$d \geq V_8 \geq \sum_u w_u \, \psi\Big(1 + \frac{\sum_c y_u^{\max}(c)}{s_u}\Big) \geq 0.10940622$$

for $l = l_0$ and the bounds T^+ of Section 5.

For the same modulus we take all μ_c equal to $\mu^* \approx 1.302046 \cdot 10^{-4}$ (the certificate gives it exactly), found by maximizing the right-hand side of (4.5) over a common value of the μ_c. If (2.2) holds, then $d \geq V$, and (4.5) gives $V \geq 0.11795319$. Since (4.5) holds for all $\mu \geq 0$, the bound does not depend on how the multipliers were found. The value μ^* exceeds $\nu/272$ by a relative $2 \cdot 10^{-12}$, where $1/272 = 1/(16 \cdot 17)$ is the slope of $-\psi$ on $[16, 17]$. So with these multipliers the orbits with $1 + \sum_c y_u^{\max}(c)/s_u \leq 16$ are filled to their cap, and the others stop at $1 + z/s_u = 16$.

5. The singular series

We need upper bounds $T^+(c)$ for the sums $T(c)$ of Lemma 3.2. Let $\mathcal{A}$ be the set of the primes p with $\mathrm{ord}_p(2) \leq 300$; there are 721 of them. They divide the cyclotomic values $\Phi_t(2)$, $t \leq 300$, whose complete factorizations are known. Let $\mathcal{M}$ be the set of the 3231 primes $q < 10^{100}$ with $300 < \mathrm{ord}_q(2) \leq 1200$ that occur in the known factorizations of $\Phi_t(2)$, $301 \leq t \leq 1200$. We split the odd squarefree m into three classes:

(a) every prime factor of m lies in $\mathcal{A}$;
(b) $m = m'q$, where m' is in class (a) and $q \in \mathcal{M}$;
(c) all other m.

We do not need $\mathcal{M}$ to contain all primes of these orders: a missing prime only moves terms from class (b) to class (c).

Let $T_{\mathrm{a}}(c)$, $T_{\mathrm{b}}(c)$ and $T_{\mathrm{c}}(c)$ be the parts of $T(c)$ from the three classes, and $R_{0,\mathrm{a}}$, $R_{0,\mathrm{b}}$, $R_{0,\mathrm{c}}$ those of R_0. The sums T run over the m coprime to l, and the sums R_0 over all odd squarefree m. For the m that contribute to $T_{\mathrm{c}}(c)$ we have $g_m \mid c$ and $g_m \mid P$, so $g_m \leq \gcd(c, P)$, and therefore

$$T_{\mathrm{c}}(c) \leq \gcd(c, P) \, R_{0,\mathrm{c}} \leq \gcd(c, P) \big(1.93656 - R_{0,\mathrm{a}}^- - R_{0,\mathrm{b}}^-\big),$$

where $R_{0,\mathrm{a}}^-$ and $R_{0,\mathrm{b}}^-$ are lower bounds for $R_{0,\mathrm{a}}$ and $R_{0,\mathrm{b}}$.

For Theorems 1.1 and 1.2 we keep more information about class (c). Grouping the terms of $T_{\mathrm{c}}(c)$ by the value of g_m, a divisor of P, gives

$$T_{\mathrm{c}}(c) = \sum_{g \mid P,\ g \mid c} g\,\alpha_g, \qquad \alpha_g = \sum_{\substack{m \text{ in class (c)} \\ (m,l)=1,\ g_m = g}} \frac{f(m)}{o_m}.$$

The α_g are nonnegative, and their sum is at most $R_{0,\mathrm{c}} \le \vartheta := 1.93656 - R^{-}_{0,\mathrm{a}} - R^{-}_{0,\mathrm{b}}$. So the hypothesis of Proposition 4.2 holds for every upper bound T_0 of $T_{\mathrm{a}} + T_{\mathrm{b}}$. For a single c, the largest value of this sum over $\mathcal{S}$ is $\gcd(c,P)\vartheta$, the bound above; Proposition 4.2 uses that the same α serves for all c.

We compute $T_{\mathrm{a}}(c)$ and $R_{0,\mathrm{a}}$ by a dynamic program over the primes of $\mathcal{A}$ (for T_{a} only those not dividing l) that keeps, for each value of the least common multiple of the orders, the sum of $f(m)$, in ball arithmetic. A branch is stopped when this multiple exceeds $3 \cdot 10^6$. For $R_{0,\mathrm{a}}$ the stopped branches are dropped, which only lowers the bound. For $T_{\mathrm{a}}(c)$ a term with $o_m > 3 \cdot 10^6$ is at most $f(m)P/(3 \cdot 10^6)$, since $g_m \le P$. So the stopped branches and all their extensions contribute at most

$$E = \Delta \prod_{p \in \mathcal{A},\ p \nmid l} \Big(1 + \frac{1}{p-2}\Big) \frac{P}{3 \cdot 10^6},$$

where Δ is the sum of f over the branches at the moment they are stopped, and we add E to the bound for every $T_{\mathrm{a}}(c)$. For class (b) we have $f(m) = f(m')/(q-2)$ and $o_m = \mathrm{lcm}(o_{m'}, \mathrm{ord}_q(2))$. We bound $T_{\mathrm{b}}(c)$ from above with the weights of the dynamic program for m', adding $E \sum_{q \in \mathcal{M}} (q-2)^{-1}$, and $R_{0,\mathrm{b}}$ from below. These sums of positive terms are evaluated in floating-point arithmetic, with every weight rounded outward and a relative margin of 10^{-9}. Each sum has fewer than $3 \cdot 10^6 + 10^3$ terms, so its relative rounding error is below $4 \cdot 10^{-10}$. This gives $R_{0,\mathrm{a}} \ge 1.9352319$, $R_{0,\mathrm{b}} \ge 0.0011322$, and a mass of at most $1.959 \cdot 10^{-4}$ for class (c). With the resulting upper bounds $T^{+}_{\mathrm{a}}(c)$ and $T^{+}_{\mathrm{b}}(c)$ we take $T^{+}(c) = T^{+}_{\mathrm{a}}(c) + T^{+}_{\mathrm{b}}(c) + \gcd(c,P)\vartheta$ for the two weaker bounds. For Theorems 1.1 and 1.2 we group the terms that the dynamic program keeps by the value of g_m, as for class (c). For each divisor g of P let B_g be an upper bound for g times the sum of $f(m)/o_m$ over the m with $(m,l) = 1$ and $g_m = g$ that lie in class (a) and are kept by the dynamic program, or lie in class (b) with m' kept, and add $E + E \sum_{q \in \mathcal{M}} (q-2)^{-1}$ to B_1. The stopped branches and their extensions contribute at most this amount to $T_{\mathrm{a}}(c) + T_{\mathrm{b}}(c)$ for every c, and B_1 occurs in $T_0(c) = \sum_{g \mid P,\, g \mid c} B_g$ for every c, so T_0 is an upper bound for $T_{\mathrm{a}} + T_{\mathrm{b}}$. Since g divides $r_2 - r_1$ exactly when $r_1 \equiv r_2 \pmod g$, with this T_0 in Propositions 4.2 and 4.3 we have $\mathcal{Q}_k = \sum_{g \mid P} B_g S_{k,g}$.

6. Dependence on the constants

The bound responds strongly to the constant for the total. Table 2 gives floating-point evaluations of (4.5) for the modulus l_0 with equal multipliers, computed by a script in the archive, for hypothetical pairs of constants; they are not certified bounds. The first row is the certified bound $d \ge 0.117953$ of Section 4. Johnston and Trudgian [11, Table 2] list the total constant 7.8209 as implied by the work of Wu [21]; the last row assumes that this holds uniformly in the shift h.

A per-class form of (2.2) would also raise the bound: with the constant 6.7814 for every class and for the total, the linear program of Table 3 gives 0.125340 already for the modulus $3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$.

TABLE 2. The bound for the modulus l_0 for other values of the two constants. Only the first row is a certified bound; the other values are floating-point evaluations.

Per-class constant	Total constant	Bound
8	6.7814	0.117953 (certified)
8	6.5	0.121009
8	6.0	0.127623
8	4.0	0.174945
7.8209	6.7814	0.118891
8	7.8209	0.110119

APPENDIX A. COMPUTATION AND CHECKS

A program in Python with python-flint [7] and numpy checks a certificate, which lists the prime factors of l, the external constants below and the multipliers. It reads the factorizations of $\Phi_t(2)$ for $t \le 300$, checks by multiplication that they are complete, and proves with FLINT that every factor is prime. It reads the known factorizations of $\Phi_t(2)$ for $301 \le t \le 1200$, checks that the product of the factors is $\Phi_t(2)$, and for every factor $q < 10^{100}$ coprime to t proves that q is prime and that $\mathrm{ord}_q(2) = t$; it does not use the other factors. It then computes the bounds of Section 5, the patterns U_k, their orbits and the data of the relaxed program, and evaluates (4.5) in exact rational arithmetic. The external inputs are the per-class constant 8, the total constant 6.7814 (for the conditional bounds) and the bounds

$$C_0 < 0.66016181584686958, \qquad R_0 < 1.93656,$$
$$0.693147180559945309 < \log 2 < 0.693147180559945310.$$

For l_1 and l_3 two further programs, built on the code of the first, combine the bounds of Section 5 into the numbers B_g, rounded up, and evaluate $M(\beta)$ of Proposition 4.2, or $M_\mu(\beta)$ of Proposition 4.3, at all vertices of $\mathcal{S}$ in exact rational arithmetic, with $j_k = \lfloor 1 + Y_k/s_k \rfloor$ chosen in floating point at $\alpha = \vartheta e_P$. They check in exact arithmetic that the smallest value is at least the claimed bound of Table 1. Since $l_1 > 10^{15}$, they do not treat the classes one at a time; two reductions make the computation feasible.

Splitting the modulus. Write $l = l'l''$ with $l' = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 241$, so that $l'' = 19 \cdot 23 \cdot 37 \cdot 41 \cdot 73$ for $l = l_1$, and write $k = (k', k'')$ by the Chinese remainder theorem. Then $U_k = U'_{k'} \cap U''_{k''}$, where $U'_{k'}$ and $U''_{k''}$ are defined with l' and l'' in place of l, as subsets of $\mathbb{Z}/P\mathbb{Z}$. For a prime $p \mid l''$, a residue $k \bmod p$ that is not a power of 2 excludes no r, and $k \equiv 2^i \pmod p$ excludes the $r \equiv i \pmod{\mathrm{ord}_p(2)}$. The patterns $U'_{k'}$ have period 120; the nonempty ones fall into 7,870 rotation orbits, and there is one empty pattern. If $U'_{k'} = U + t$, then $U_{(k',k'')} = (U \cap U''_{2^{-t}k''}) + t$. Since s_k, $\mathcal{Q}_k$ and $S_{k,g} = \sum_{a \bmod g} |U_k \cap (a + g\mathbb{Z})|^2$ are invariant under rotation, each orbit therefore contributes the number of classes k' with $U'_{k'}$ in the orbit times the sum over k'' of the terms for $U \cap U''_{k''}$. The $o + 1$ different exclusions by one prime $p \mid l''$, where $o = \mathrm{ord}_p(2)$, are evaluated together. If a pattern V arises from a pattern U by removing the elements $r \equiv i \pmod o$, then $|V \cap (a + g\mathbb{Z})| = |U \cap (a + g\mathbb{Z})| - |U \cap (a + g\mathbb{Z}) \cap (i + o\mathbb{Z})|$, and the last set is empty or the intersection of U with one class modulo $\mathrm{lcm}(g, o)$, which divides P. So the numbers $S_{k,g}$ for all $o + 1$ exclusions follow from the counts of one pattern in the classes modulo the divisors of P; the program for l_1 uses this for the added prime that its certificate lists last.

Primes whose order is divisible by 7 *or* 11. For $l_3 = 29 \cdot 43 \cdot l_1$ the period is $P = 27720 = 360 \cdot 7 \cdot 11$, and the orders 14, 28 and 11 of 2 modulo 43, 29 and 23 do not divide 360, while the

orders modulo the other prime factors of l_3 do. Writing r as $(r \bmod 360, r \bmod 7, r \bmod 11)$, the indicator function of U_k is therefore a function of $r \bmod 360$ times, for $q = 7$ and for $q = 11$, a function of $r \bmod 360$ and $r \bmod q$. Since every divisor of P is $g_c g_7 g_{11}$ with $g_c \mid 360$ and $g_q \in \{1, q\}$, the sums over $r \bmod 7$ and $r \bmod 11$ in $S_{k,g}$ can be evaluated in closed form. A permutation of $\mathbb{Z}/q\mathbb{Z}$ applied to $r \bmod q$ preserves every congruence modulo a divisor of P and permutes the classes, so s_k and $S_{k,g}$ need to be evaluated for only one combination of exclusions by 43, 29 and 23 in each orbit of these permutations. The program for l_3 runs through the exclusions by the other added primes one at a time, and it checks that every floating-point sum that enters the bound has nonnegative integer terms and a total below 2^{53}, so that it is exact; floating point without this guarantee only chooses the j_k.

In cross-checks, the program for l_3 gives the same values as the program for l_1 on the certificate of l_1, to 12 digits, separately written bounds for the singular series and an independently written exact evaluator of (4.5) give the two bounds for l_0 to within $7 \cdot 10^{-9}$, and a separately written solver of the linear program of Section 4 agrees to within $2.1 \cdot 10^{-5}$ with the values that Johnston and Trudgian [11] report for the method of Elsholtz and Schlage-Puchta (Table 3). That solver uses the modulus $3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$, the radical of $2^{24} - 1$, whose period is 24, and a coarser bound for the singular series, and each of its values is an exact lower bound for this linear program. The README of the archive gives the details of the programs and lists further tests of them.

TABLE 3. The linear program for the modulus $3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$ and the values of Johnston and Trudgian [11].

Constant for the classes and for the total	Our solver	[11]
8	0.107896	0.10788
6.7814 (a hypothetical case)	0.125340	0.12532
7.8209, Wu's constant	0.110125	0.11011

APPENDIX B. FORMAL VERIFICATION

We prove Theorems 1.1 and 1.2 in the Lean 4 proof assistant [12] with the Mathlib library [14]. The analytic inputs of Section 2 are hypotheses of the formal theorems: the bound (2.1) and the prime number theorem in progressions for the modulus l_3, the bound (2.3), stated for every finite partial sum of R_0, and, for Theorem 1.2, the bound (2.2). In (2.1) and (2.2) the hypotheses have the upper bound 0.66016181584686958 in place of C_0, which only weakens them. Lemma 2.1 is proved, not assumed. The density is formalized as in the introduction: for $N \geq 1$ let d_N be N^{-1} times the number of $n \leq N$ of the form $p + 2^a$ with p prime and $a \geq 1$. The formal theorems state $\liminf_{N\to\infty} d_N \geq 0.110825$ and $\liminf_{N\to\infty} d_N \geq 0.119178$.

The proof follows Sections 3 to 5 for l_3 and Propositions 4.2 and 4.3. Since $M(\beta) \geq M(\vartheta e_P)$ and $M_\mu(\beta) \geq M_\mu(\vartheta e_P)$ for every $\beta \in \mathcal{S}$ (Section 4), it needs only the two values at $\beta = \vartheta e_P$. For the sums $T(c)$ of Lemma 3.2 it computes its own upper bounds by a variant of the method of Section 5, in fixed-point integer arithmetic with lower and upper bounds, from a list of primes and orders of 2 that it certifies: primes above 2^{16} by Pratt certificates [16], smaller ones by trial division, and each order t by checking $2^t \equiv 1$ and $2^{t/r} \not\equiv 1$ for the prime factors r of t. A few large primes of the factorizations are left out, which only moves their terms to class (c). The stopped branches and their extensions are bounded with $g_m \leq \gcd(c, P)$ instead of $g_m \leq P$, so that their bound is spread over the numbers B_g of Section 5, rounded up, in proportion to $\varphi(g)$ instead of being added to B_1. A program written in Lean then evaluates both bounds at $\beta = \vartheta e_P$ in exact rational arithmetic, with the reductions that the program for l_3 uses (Appendix A), except that

it groups the residues modulo $43 \cdot 29$ and modulo 23 by the counts that enter $S_{k,g}$ instead of by orbits; the formulas behind these reductions are proved in Lean. It obtains $0.1108276292\ldots$ and $0.1191809294\ldots$, a little more than the smallest values of Section 4 because of the different bound for the stopped branches.

Every computation of the proof is checked by Lean's kernel, and the two theorems depend only on the standard axioms `propext`, `Classical.choice` and `Quot.sound`. For this the evaluation uses a further factorization of $S_{k,g}$, proved correct in Lean, and runs in a separate module for each rotation orbit of the patterns modulo l' and for the empty pattern. The README of the archive gives the directories, the names of the theorems and the versions of Lean and Mathlib, and describes a second version of the proof, with compiled checks, and the earlier formal proofs of the weaker bounds named in the caption of Table 1.

Data availability

The checking programs, the certificates, the factorization data and the programs that produced them, and the Lean files of Appendix B, are archived at doi:10.5281/zenodo.23715955 and available at `https://github.com/tejasvi/romanoff-constant/tree/v11` (commit `92b5fbd2aa23`).

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