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AN EXPLICIT BOUND BELOW $2/\pi$ FOR THE MAXIMUM TERM OF AN ENTIRE FUNCTION

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ABSTRACT. For a transcendental entire function f let $\mu(r)$ be its maximum term and $M(r)$ its maximum modulus on $|z| = r$. Erdős asked for the largest possible value B of $\liminf_{r\to\infty} \mu(r)/M(r)$. An argument of Clunie gives $B \le 2/\pi = 0.636619\ldots$, and Clunie and Hayman proved $B \le 2/\pi - c$ for an absolute constant $c > 0$ that they did not compute. We prove $B \le 0.63005$. We make the stability argument of Clunie and Hayman quantitative by testing f against trigonometric polynomials at one or both of two consecutive radii where the central index jumps. The proof is computer-assisted, and the theorem is also proved in the Lean proof assistant.

1. INTRODUCTION

Let $f(z) = \sum_{k\ge 0} a_k z^k$ be a transcendental entire function, and for $r > 0$ let

$$M(r) = \max_{|z|=r} |f(z)|, \qquad \mu(r) = \max_{k\ge 0} |a_k| r^k$$

be its maximum modulus and its maximum term. By Cauchy's inequality $\mu(r) \le M(r)$. The maximum term and the *central index*, the largest k with $|a_k| r^k = \mu(r)$, are basic quantities of the Wiman–Valiron theory of entire functions [5]. Erdős [3, p. 249] asked how large the ratio $\mu(r)/M(r)$ can stay, that is, for the value of

$$B := \sup_{f} \liminf_{r\to\infty} \frac{\mu(r)}{M(r)},$$

the supremum being over all transcendental entire functions (for polynomials the ratio tends to 1). This is Problem 2.14(c) in the Hayman–Lingham collection [6].[1]

Clearly $B \le 1$, and a suitable lacunary power series, all of whose terms except at most two are negligible compared with $\mu(r)$ at every large radius, shows that $B \ge 1/2$. An unpublished example of Kővári shows that $B > 1/2$ [2, p. 146]. Gray and Shah [4, Lemma 1] published an argument of Clunie that gives $B \le 2/\pi$. Clunie and Hayman showed that the same argument gives $\mu(r)/M(r) \le 2/\pi$ at every radius where the central index jumps, and that this is sharp for the upper limit over these radii [2, Theorem 1]. For the lower limit they proved $4/7 < B \le 2/\pi - c$ with an absolute constant $c > 0$ [2, Theorem 2]. Their constant comes from a stability argument, sketched below [2, Lemmas 2.2–2.4]. Its value is not computed there [2, p. 156], and as far as we know no explicit value has been recorded since then (see [1, 6]). Recently He and Tang [7] generalized the Clunie–Hayman construction and proved $B > 1/1.70919 > 0.5850724$ [7, Theorem 2.8 and Proposition 3.5], and a certified computation reported by Sothanaphan [11] gives $B > 0.5850788$.

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[1] It is also Problem #513 in Bloom's list of Erdős problems [1] and constant 51a in the repository of optimization constants [12].

Theorem 1.1. *For every transcendental entire function f,*

$$\liminf_{r\to\infty} \frac{\mu(r)}{M(r)} \le 0.63005.$$

Consequently $B \le 0.63005$.

The bound improves $2/\pi = 0.636619\ldots$ by more than 0.0065.

Idea of the proof. Clunie's argument is short. Write $c_k = a_k r^k$ and $g(\theta) = f(re^{i\theta}) = \sum_k c_k e^{ik\theta}$. At a radius where two indices $n < m$ both attain the maximum term, rotate θ so that c_n and c_m have the same argument. Then

$$2\mu(r) = |c_n + c_m| = \Big|\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\big(e^{-in\theta} + e^{-im\theta}\big)\,d\theta\Big| \le M(r)\cdot\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i(m-n)\theta}|\,d\theta = \frac{4}{\pi}M(r).$$

Since the central index tends to infinity, such radii are arbitrarily large, and $B \le 2/\pi$. Near equality forces g to be close to a multiple of the square wave $\operatorname{sgn}\cos((m-n)\theta/2)$, up to a phase and a frequency shift. The Fourier coefficients of that square wave decay only like $1/j$ along the progression $n + j(m-n)$. But the moduli $|a_k|$ are the same at every radius, and only the factor r^k changes. At the next radius where the central index jumps, a slowly decaying coefficient profile would produce a term larger than the two that tie there. Thus near equality at one jump forces the next jump to come soon, while near equality at the next jump forces it to come late. This is the mechanism of the Clunie–Hayman proof [2, §2], and we make it quantitative.

Method. We bound $M(r)$ from below by duality: at a radius r and for a finite set $S \subset \mathbb{Z}$ of frequencies, $M(r)$ is at least the smallest sup norm of a bounded function whose Fourier coefficients on S are the c_k, and trigonometric polynomials with frequencies in S give lower bounds for this quantity (Section 2). The radii where the central index jumps correspond to the edges of the Newton polygon of $\log|a_k|$, and two consecutive ones involve three consecutive vertices (Section 3). If the two gaps between these vertices are equal, we test at both radii, and a dilation reduces the problem to five consecutive coefficients with six real parameters. If they differ, we test at one radius, with five coefficients spaced by the larger gap and the coefficient of the remaining vertex. Averaging over roots of unity reduces this case to a family of problems in seven parameters, indexed by an integer $p \ge 2$ and covered by finitely many computations (Section 4). We bound both by a branch-and-bound computation with certified dual vectors (Sections 5 and 6), and Section 7 describes a formal proof of Theorem 1.1 in Lean 4. Numerically, the six-parameter problem has the value 0.630041, so this formulation cannot improve our bound by more than about 10^{-5}. Section 8 discusses the limits of this formulation.

2. The window bound

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with the normalized measure $d\theta/2\pi$. The norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ refer to it, and $\widehat{h}(k) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)e^{-ik\theta}\,d\theta$. For a finite set $S \subset \mathbb{Z}$ and $\tau \in \mathbb{C}^S$ put

$$T_\tau(\theta) = \sum_{k\in S} \tau_k e^{-ik\theta}, \qquad \langle c, \tau\rangle = \sum_{k\in S} c_k\tau_k \quad (c \in \mathbb{C}^S).$$

Definition 2.1. For a finite $S \subset \mathbb{Z}$ and $c \in \mathbb{C}^S$ let $N_S(c) = \inf\{\|h\|_\infty :\ h \in L^\infty(\mathbb{T}),\ \widehat{h}(k) = c_k \text{ for all } k \in S\}$. For $S = \{-2,\ldots,2\}$ we write N_5 and list $c = (c_{-2}, c_{-1}, c_0, c_1, c_2)$.

Lemma 2.2 (duality)**.** *For every $\tau \in \mathbb{C}^S$ with $T_\tau \ne 0$, $N_S(c) \ge |\langle c,\tau\rangle|/\|T_\tau\|_1$.*

Proof. If $\widehat{h}(k) = c_k$ on S, then $\langle c,\tau\rangle = \sum_k \tau_k\widehat{h}(k) = \frac{1}{2\pi}\int_0^{2\pi} h\,T_\tau\,d\theta$, so $|\langle c,\tau\rangle| \le \|h\|_\infty\|T_\tau\|_1$. □

Remark 2.3. By the Hahn–Banach theorem the bound is sharp: $N_S(c) = \max|\langle c, \tau\rangle|$ over $\|T_\tau\|_1 \le 1$. Only Lemma 2.2 is used in the proofs.

Lemma 2.4. *Let $S \subset \mathbb{Z}$ be finite and $c, c' \in \mathbb{C}^S$.*

(a) $N_S(\lambda c) = |\lambda| N_S(c)$ *for* $\lambda \in \mathbb{C}$.
(b) *(rotation)* $N_S\big((c_k e^{ik\alpha})_k\big) = N_S(c)$ *for* $\alpha \in \mathbb{R}$.
(c) *(shift) for* $j \in \mathbb{Z}$, *the window* $S + j$ *with coefficients* c_{k-j} *has the same value* $N_S(c)$.
(d) *(restriction) if* $S' \subset S$ *then* $N_{S'}(c|_{S'}) \le N_S(c)$.
(e) *(Lipschitz)* $|N_S(c) - N_S(c')| \le \sum_{k\in S} |c_k - c'_k|$.
(f) *(dilation) for an integer* $d \ge 1$, *the window* $dS = \{dk : k \in S\}$ *with coefficient* c_k *at* dk *has the same value* $N_S(c)$.
(g) *(reversal and conjugation) the window* $-S$ *with coefficient* c_{-k} *at* k *has the value* $N_S(c)$, *and* $N_S(\bar{c}) = N_S(c)$.

Proof. Each statement follows by transforming admissible functions: (a) $h \mapsto \lambda h$; (b) $h \mapsto h(\cdot + \alpha)$; (c) $h \mapsto e^{ij\theta} h$; (d) restriction of the constraints; (e) $h \mapsto h + \sum_{k\in S}(c'_k - c_k)e^{ik\theta}$, which changes $\|h\|_\infty$ by at most $\sum_{k\in S}|c_k - c'_k|$; (g) $h \mapsto h(-\cdot)$ and $h \mapsto \overline{h(-\cdot)}$. For (f), if h is admissible for S then $h(d\,\cdot)$ is admissible for dS with the same norm. Conversely, if H is admissible for dS, the average $\widetilde{H}(\theta) = \frac{1}{d}\sum_{j=0}^{d-1} H(\theta + 2\pi j/d)$ has $\|\widetilde{H}\|_\infty \le \|H\|_\infty$, Fourier support in $d\mathbb{Z}$ and the same coefficients on dS, so $\widetilde{H} = h(d\,\cdot)$ with h admissible for S. □

For an entire function $f = \sum_{k\ge 0} a_k z^k$ and $r > 0$ put $c_k(r) = a_k r^k$ for $k \ge 0$ and $c_k(r) = 0$ for $k < 0$.

Lemma 2.5 (window bound)**.** *For every $r > 0$ and every finite $S \subset \mathbb{Z}$, $M(r) \ge N_S\big((c_k(r))_{k\in S}\big)$.*

Proof. The function $g(\theta) = f(re^{i\theta})$ satisfies $\|g\|_\infty = M(r)$ and $\widehat{g}(k) = c_k(r)$ for every $k \in \mathbb{Z}$. □

3. Vertices, events and the two reductions

3.1. **The Newton polygon.** The facts in this subsection are classical (see, e.g., [5]), but we include short proofs to keep the paper self-contained. Let f be transcendental and $L(t) = \log \mu(e^t) = \max_k \ell_k(t)$, where $\ell_k(t) = \log|a_k| + kt$ for $a_k \ne 0$. The maximum is attained because $|a_k|e^{kt} \to 0$ as $k \to \infty$. The *central index* $\nu(r)$ is the largest k with $|a_k|r^k = \mu(r)$. It is nondecreasing in r. Call k a *vertex* if $\ell_k(t) = L(t)$ for some $t \in \mathbb{R}$, and let $v_0 < v_1 < v_2 < \cdots$ be the vertices. For a vertex k the set $I_k = \{t : \ell_k(t) = L(t)\}$ is a closed interval, since $L - \ell_k$ is convex and nonnegative. If $k < k'$ are vertices, $t \in I_k$ and $t' \in I_{k'}$, then $t \le t'$: otherwise $\ell_{k'}(t) - \ell_k(t) = \ell_{k'}(t') - \ell_k(t') + (k' - k)(t - t') > 0$, contradicting $\ell_k(t) = L(t)$. The indices attaining L on a bounded interval $[a, b]$ are at most $\nu(e^b)$, so only finitely many I_k meet $[a, b]$. There are infinitely many vertices, since otherwise $|a_k|r^k \le \max_{j\le N}|a_j|r^j$ for all r and $k > N$ would force $a_k = 0$ for $k > N$. As the I_{v_j} cover $\mathbb{R}$, are ordered and are locally finite, consecutive ones meet:

$$t_j := \max I_{v_{j-1}} = \min I_{v_j} \qquad (j \ge 1),$$

so that $t_1 \le t_2 \le \cdots$, $t_j \to \infty$, and at $r = e^{t_j}$ both v_{j-1} and v_j attain the maximum term.

Definition 3.1. The *event* at $j \ge 1$ consists of the vertices $v_{j-1} < v_j < v_{j+1}$, the gaps $d = v_j - v_{j-1}$ and $d' = v_{j+1} - v_j$, the radii $r_0 = e^{t_j} \le r_1 = e^{t_{j+1}}$, and $s = t_{j+1} - t_j \ge 0$. It is an *equal-gap* event if $d = d'$ and a *mixed* event otherwise.

At an event we use only the following facts:

(E1) $|c_k(r)| \le \mu(r)$ for all $k \in \mathbb{Z}$ and $r > 0$;
(E2) $|c_{v_{j-1}}(r_0)| = |c_{v_j}(r_0)| = \mu(r_0)$ and $|c_{v_j}(r_1)| = |c_{v_{j+1}}(r_1)| = \mu(r_1)$;
(E3) $c_k(r_1) = c_k(r_0)e^{ks}$ for all k.

3.2. **Equal gaps.** At an equal-gap event we use the five coefficients at $v_j + dl$, $|l| \le 2$: the two pairs that tie at r_0 and at r_1, and one further coefficient on each side (wider windows are discussed in Remark 8.2). Let $K = [0,\infty) \times [0,1]^2 \times \mathbb{R}^3$, with points $\xi = (\sigma, x, y, \varphi_{-2}, \varphi_{-1}, \varphi_2)$, and

$$C_0(\xi) = \big(xe^{i\varphi_{-2}},\ e^{i\varphi_{-1}},\ 1,\ e^{-\sigma},\ ye^{-2\sigma}e^{i\varphi_2}\big),$$
$$C_1(\xi) = \big(xe^{-2\sigma}e^{i\varphi_{-2}},\ e^{-\sigma}e^{i\varphi_{-1}},\ 1,\ 1,\ ye^{i\varphi_2}\big), \qquad \Lambda(\xi) = \min_{i=0,1} \frac{1}{N_5(C_i(\xi))}.$$

($N_5(C_i(\xi)) \ge 1$ by Lemma 2.2 with $T_\tau = 1$.)

Proposition 3.2. *At an equal-gap event there is $\xi \in K$ with $\min\{\mu(r_0)/M(r_0),\ \mu(r_1)/M(r_1)\} \le \Lambda(\xi)$.*

Proof. Let $n = v_j$, $\lambda = c_n(r_0)$ and $b_l = c_{n+dl}(r_0)/\lambda$ for $l = -2, \dots, 2$; so $|\lambda| = \mu(r_0)$, $|b_{-1}| = |b_0| = 1$ and $x := |b_{-2}| \le 1$. By (E2) and (E3), $|b_1|e^{ds} = 1$, so $|b_1| = e^{-\sigma}$ with $\sigma = ds$, and (E1) at r_1 gives $y := |b_2|e^{2\sigma} \le 1$. Choose α with $d\alpha = -\arg b_1$ and replace b_l by $b_l e^{idl\alpha}$; then $b_0 = 1$, $b_1 = e^{-\sigma}$, and with $\varphi_l = \arg b_l$ ($l = -2, -1, 2$; arbitrary if $b_l = 0$) we have $b = C_0(\xi)$. Lemma 2.5 with $S = \{n + dl : |l| \le 2\}$ and Lemma 2.4(a)–(c),(f) give $M(r_0) \ge \mu(r_0)N_5(C_0(\xi))$. At r_1 we have $\mu(r_1) = |c_n(r_1)|$ and $c_{n+dl}(r_1)/c_n(r_1) = b_l e^{l\sigma}$, which after the same rotation is $C_1(\xi)$; so $M(r_1) \ge \mu(r_1)N_5(C_1(\xi))$. □

3.3. **Mixed gaps.** When the two gaps differ, the three vertices are not equally spaced. We then work at a single radius: the coefficients on the progression of the larger gap contain the two that tie there, and the coefficient of the remaining vertex, which lies off that progression, carries the information about the other jump. At a mixed event let $g = \gcd(d, d')$, $a = d/g$, $b = d'/g$ (so $\gcd(a,b) = 1$, $a \ne b$), and

$$p := \max(a,b) \ge 2, \qquad \sigma := gs.$$

For $p \ge 2$, $W \ge 0$ and $\varepsilon \ge 0$ let

$$R_p(W,\varepsilon) = \max_{\varphi\in\mathbb{R}} \frac{1}{p}\sum_{j=0}^{p-1} \big|W + \varepsilon e^{i(\varphi + 2\pi j/p)}\big| - W,$$

and for $\tau \in \mathbb{C}^5$ (indices $-2, \dots, 2$) and $\varepsilon \ge 0$ let

$$U_p(\tau,\varepsilon) = \|T_\tau\|_1 + \frac{1}{2\pi}\int_0^{2\pi} R_p\big(|T_\tau(X)|, \varepsilon\big)\, dX.$$

For $s \ge 0$ and $\kappa \ge 1$ let $Z(s,\kappa)$ be the set of vectors $z = (z_{-2}, z_{-1}, 1, 1, z_2)$ with $|z_{-2}| \le e^{-2\kappa s}$, $|z_{-1}| \le e^{-\kappa s}$ and $|z_2| \le 1$.

Proposition 3.3. *At a mixed event there are $s_1 \ge 0$ and $z \in Z\big(s_1, \frac{p}{p-1}\big)$ such that for every $\tau \in \mathbb{C}^5$ and $\varepsilon \ge 0$ with $\mathrm{Re}\langle z, \tau\rangle + \varepsilon e^{-s_1} > 0$,*

$$\min\Big\{\frac{\mu(r_0)}{M(r_0)}, \frac{\mu(r_1)}{M(r_1)}\Big\} \le \frac{U_p(\tau,\varepsilon)}{\mathrm{Re}\langle z,\tau\rangle + \varepsilon e^{-s_1}}.$$

Proof. Work on the progression $v_j + g\mathbb{Z}$ and write $w_k = c_{v_j+gk}(r_0)/c_{v_j}(r_0)$ and $w_k(\sigma) = c_{v_j+gk}(r_1)/c_{v_j}(r_1)$; by (E3), $w_k(\sigma) = w_k e^{k\sigma}$. By (E1) and (E2), $\mu(r_0) = |c_{v_j}(r_0)|$, $\mu(r_1) = |c_{v_j}(r_1)|$, $w_0 = w_0(\sigma) = 1$, $|w_{-a}| = 1$, $|w_b(\sigma)| = 1$, and $|w_k| \le 1$, $|w_k(\sigma)| \le 1$ for all k.

Case $a < b$ ($p = b$). The larger gap is the one between the two vertices that tie at r_1, so we work at r_1 on the progression $v_j + bg\mathbb{Z}$, which contains them. The third vertex, $v_{j-1} = v_j - ag$, lies off this progression and supplies the off-lattice coefficient. The vector $(w_{-2b}(\sigma), w_{-b}(\sigma), 1, w_b(\sigma), w_{2b}(\sigma))$ satisfies $|w_{-lb}(\sigma)| = |w_{-lb}|e^{-lb\sigma} \le e^{-lb\sigma}$ ($l = 1, 2$) and

$|w_{2b}(\sigma)| \le 1$. After a rotation (Lemma 2.4(b)) making $w_b(\sigma) = 1$, it is a vector $z \in Z(s_1, b/a) \subset Z(s_1, p/(p-1))$, where $s_1 = a\sigma$, because $b\sigma = (b/a)s_1$ and $b/a \ge b/(b-1)$. The off-lattice coefficient $w_{-a}(\sigma)$ has modulus $e^{-a\sigma} = e^{-s_1}$, and $-a$ is prime to b. Lemma 2.5 on the window $v_j + g(\{-2b, -b, 0, b, 2b\} \cup \{-a\})$, Lemma 2.4(a)–(c),(f), and Lemma 4.1 below with $S_L = \{-2b, \dots, 2b\}$ (step b) give $M(r_1) \ge \mu(r_1)\,(\mathrm{Re}\langle z, \tau\rangle + \varepsilon e^{-s_1})/U_b(\tau, \varepsilon)$.

Case $a > b$ ($p = a$). Now the larger gap is the one between the two vertices that tie at r_0, so we work at r_0 on the progression $v_j + ag\mathbb{Z}$, which contains them. The third vertex, $v_{j+1} = v_j + bg$, lies off this progression and supplies the off-lattice coefficient. Here $|w_{-2a}| \le 1$, $|w_{la}| \le e^{-la\sigma}$ ($l = 1, 2$; by (E1) at r_1), and the off-lattice coefficient w_b has modulus $e^{-b\sigma}$, with b prime to a. Reversing the window (Lemma 2.4(g)) moves the coefficient at k to $-k$. The rotation of Lemma 2.4(b) by an α with $e^{ia\alpha} w_{-a} = 1$ then makes the coefficient at a equal to 1 and keeps the one at 0, so that the coefficients at la, $l = -2, \dots, 2$, form the vector $z = (w_{2a}e^{-2ia\alpha},\, w_a e^{-ia\alpha},\, 1,\, 1,\, w_{-2a}e^{2ia\alpha})$. Since $|w_{2a}| \le e^{-2a\sigma}$, $|w_a| \le e^{-a\sigma}$ and $|w_{-2a}| \le 1$, this vector lies in $Z(s_1, a/b) \subset Z(s_1, p/(p-1))$ with $s_1 = b\sigma$, and the off-lattice coefficient, now at $-b$, has modulus $e^{-b\sigma} = e^{-s_1}$. Lemma 2.5 on the window $v_j + g(\{-2a, -a, 0, a, 2a\} \cup \{b\})$, Lemma 2.4(a)–(c),(f),(g), and Lemma 4.1 with $S_L = \{-2a, \dots, 2a\}$ (step a) give $M(r_0) \ge \mu(r_0)\,(\mathrm{Re}\langle z, \tau\rangle + \varepsilon e^{-s_1})/U_a(\tau, \varepsilon)$. □

4. Averaging over roots of unity

Lemma 4.1. *Let $p \ge 2$, let $e \in \mathbb{Z}$ with $\gcd(e, p) = 1$, let $S_L \subset p\mathbb{Z}$ be finite, $S = S_L \cup \{e\}$, and let $c \in \mathbb{C}^S$ with $|c_e| = \eta$. For $\tau \in \mathbb{C}^{S_L}$ and $\varepsilon \ge 0$ put $Q(X) = \sum_{k \in S_L} \tau_k e^{-i(k/p)X}$. Then*

$$N_S(c) \;\ge\; \frac{\mathrm{Re}\sum_{k \in S_L} c_k \tau_k + \varepsilon\eta}{\|Q\|_1 + \frac{1}{2\pi}\int_0^{2\pi} R_p\big(|Q(X)|, \varepsilon\big)\, dX}$$

whenever the numerator is positive.

Proof. Let $\chi = \arg c_e$ (any value if $\eta = 0$) and $T(\theta) = Q(p\theta) + \varepsilon e^{-i(e\theta + \chi)}$. This is $T_{\tau'}$ for the vector $\tau' \in \mathbb{C}^S$ that extends τ by $\tau'_e = \varepsilon e^{-i\chi}$, and $\langle c, \tau'\rangle = \sum_{S_L} c_k\tau_k + \varepsilon\eta$, so by Lemma 2.2 it suffices to bound $\|T\|_1$. Split $[0, 2\pi)$ into the arcs $[2\pi j/p, 2\pi(j+1)/p)$ and substitute $\theta = (X + 2\pi j)/p$ in the j-th arc. Since $k/p \in \mathbb{Z}$ for $k \in S_L$, $Q(p\theta) = Q(X)$, while $e\theta = eX/p + 2\pi ej/p$. Hence

$$\|T\|_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{p}\sum_{j=0}^{p-1} \big|Q(X) + \varepsilon e^{-i(eX/p + \chi)}\omega^j\big|\, dX, \qquad \omega = e^{-2\pi i e/p}.$$

As $\gcd(e, p) = 1$, ω^j runs over all p-th roots of unity. For fixed X write $Q(X) = We^{i\alpha}$. The inner average equals $\frac{1}{p}\sum_j |W + \varepsilon e^{i(\phi + 2\pi j/p)}|$ for some ϕ, which is at most $W + R_p(W, \varepsilon)$. No cancellation between the lattice part and the off-lattice term is needed. □

For $\varepsilon > 0$ we have $R_p(W, \varepsilon) = \varepsilon r_p(W/\varepsilon)$ with

$$r_p(u) := R_p(u, 1) = \max_{\varphi} \frac{1}{p}\sum_{j=0}^{p-1} |u + e^{i(\varphi + 2\pi j/p)}| - u,$$

and $R_p(W, 0) = 0$. Let also $r_\infty(u) = \frac{1}{2\pi}\int_0^{2\pi} |u + e^{it}|\, dt - u = \frac{2}{\pi}(u+1)E\big(\frac{4u}{(u+1)^2}\big) - u$, where $E(m) = \int_0^{\pi/2}(1 - m\sin^2 x)^{1/2} dx$. The formula follows from $|u + e^{it}|^2 = (u+1)^2 - 4u\sin^2(t/2)$.

Lemma 4.2. *Let $p \ge 2$.*

(i) *r_p is nonincreasing on $[0, \infty)$.*
(ii) *$r_p(u) \le \sqrt{u^2 + 1} - u \le 1/(2u)$.*

(iii) *For $p_0 \geq 2$, $p \geq p_0$ and $u \geq 0$, $r_p(u) \leq r_\infty(u) + \mathrm{err}(p_0, u)$, where $\mathrm{err}(p_0, 1) = 2\pi/(3p_0^2)$ and for $u \neq 1$*

$$\mathrm{err}(p_0, u) = \min\Big\{ \frac{2\pi \min(u,1)}{3p_0^2},\ \frac{4\max(u,1)\,|\binom{1/2}{p_0}|\,q^{p_0}}{1-q^{p_0}} \Big\}, \qquad q = u\ (u<1), \quad q = 1/u\ (u>1).$$

Proof. (i) Write $\psi_j = \varphi + 2\pi j/p$. For fixed φ the derivative in u of $\frac{1}{p}\sum_j |u + e^{i\psi_j}| - u$ is $\frac{1}{p}\sum_j \mathrm{Re}(u+e^{i\psi_j})/|u+e^{i\psi_j}| - 1 \leq 0$ wherever it exists, and the function is Lipschitz. A maximum of nonincreasing functions is nonincreasing.

(ii) By the Cauchy–Schwarz inequality, $\frac{1}{p}\sum_j |u + e^{i\psi_j}| \leq (\frac{1}{p}\sum_j |u + e^{i\psi_j}|^2)^{1/2} = (u^2+1)^{1/2}$, because $\sum_j e^{i\psi_j} = 0$ for $p \geq 2$.

(iii) Let $F(t) = |u + e^{it}|$. Its Fourier series converges absolutely, so by Poisson summation $\frac{1}{p}\sum_j F(\varphi + 2\pi j/p) = \sum_{m\in\mathbb{Z}} \widehat{F}(mp) e^{imp\varphi}$ and $r_p(u) - r_\infty(u) \leq \sum_{m\neq 0} |\widehat{F}(mp)|$. We bound the right-hand side in two ways.

First bound. We have $F'(t) = -u\sin t/F(t)$. For $u \neq 1$, the function $u\sin t/F(t)$ is unimodal on $[0,\pi]$; indeed its derivative has the sign of $\frac{B_u}{2}c^2 + A_u c + \frac{B_u}{2}$ with $c = \cos t$, $A_u = u^2+1$, $B_u = 2u$, which has exactly one root in $(-1,0)$. Its maximum, attained at $c = -u$ for $u < 1$ and at $c = -1/u$ for $u > 1$, equals $\min(u,1)$. So F' has total variation $4\min(u,1)$ over a period. This also holds for $u = 1$, where $F = 2|\cos(t/2)|$. Two integrations by parts give $|\widehat{F}(n)| \leq 4\min(u,1)/(2\pi n^2)$, and $2\sum_{m\geq 1} 4\min(u,1)/(2\pi m^2 p^2) = 2\pi\min(u,1)/(3p^2)$.

Second bound ($u \neq 1$). We have $F(t) = \max(u,1)\,|1 + qe^{it}|$ (for $u < 1$ use $|u + e^{it}| = |1 + ue^{-it}|$ and the evenness of F). With $(1+w)^{1/2} = \sum_{j\geq 0}\gamma_j w^j$, $\gamma_j = \binom{1/2}{j}$, we get $\widehat{F}(n) = \max(u,1)\sum_{j\geq 0}\gamma_{j+n}\gamma_j q^{2j+n}$ for $n \geq 0$. Since $|\gamma_j|$ decreases for $j \geq 1$ and $\sum_{j\geq 1}|\gamma_j|x^j = 1 - \sqrt{1-x}$, this gives $|\widehat{F}(n)| \leq \max(u,1)|\gamma_n| q^n (2 - \sqrt{1-q^2}) \leq 2\max(u,1)|\gamma_n| q^n$, and then $\sum_{m\neq 0}|\widehat{F}(mp)| \leq 4\max(u,1)|\gamma_p| q^p/(1-q^p)$. Both bounds are nonincreasing in p. □

5. The two finite-dimensional problems

Throughout, $\beta_0 = 12601/20000 = 0.63005$.

5.1. **Equal gaps.**

Lemma 5.1 (symmetries)**.** *For $\xi = (\sigma, x, y, \varphi_{-2}, \varphi_{-1}, \varphi_2) \in K$ let $\rho(\xi) = (\sigma, y, x, \varphi_2 + 2\varphi_{-1}, \varphi_{-1}, \varphi_{-2} - 2\varphi_{-1})$ and $\bar{\xi} = (\sigma, x, y, -\varphi_{-2}, -\varphi_{-1}, -\varphi_2)$. Then $N_5(C_0(\rho\xi)) = N_5(C_1(\xi))$, $N_5(C_1(\rho\xi)) = N_5(C_0(\xi))$ and $N_5(C_i(\bar{\xi})) = N_5(C_i(\xi))$; hence $\Lambda(\rho\xi) = \Lambda(\bar{\xi}) = \Lambda(\xi)$.*

Proof. Reversing $C_1(\xi)$ (Lemma 2.4(g)) gives $(ye^{i\varphi_2}, 1, 1, e^{-\sigma}e^{i\varphi_{-1}}, xe^{-2\sigma}e^{i\varphi_{-2}})$, and the rotation $c_k \mapsto c_k e^{-ik\varphi_{-1}}$ turns it into $C_0(\rho\xi)$. Reversing and rotating $C_0(\xi)$ in the same way gives $C_1(\rho\xi)$. Finally $C_i(\bar{\xi}) = \overline{C_i(\xi)}$. □

Given $\xi \in K$, reduce φ_{-1} modulo 2π into $[-\pi,\pi]$, apply $\xi \mapsto \bar{\xi}$ if $\varphi_{-1} < 0$, then apply ρ if $x > y$ (it does not change φ_{-1}), and finally reduce φ_{-2} and φ_2 modulo 2π. By Lemma 5.1, Λ is unchanged, and the result lies in

$$D_K = [0,\infty) \times \{(x,y) \in [0,1]^2 : x \leq y\} \times [-\pi,\pi] \times [0,\pi] \times [-\pi,\pi].$$

Lemma 5.2 (tail)**.** *For all $x \in [0,1]$ and $\psi \in \mathbb{R}$ there is $\tau \in \mathbb{C}^5$ with $T_\tau \neq 0$ and $\mathrm{Re}\langle (xe^{i\psi}, 1, 1, 0, 0), \tau\rangle \geq 1.61\,\|T_\tau\|_1$; in particular $N_5\big((xe^{i\psi}, 1, 1, 0, 0)\big) \geq 1.61$. Consequently $\Lambda(\xi) \leq 1/N_5(C_0(\xi)) < 0.6285$ whenever $\sigma \geq 4$.*

Proof. The first statement is a certified computation (Section 6); the second follows by Lemma 2.2. For the third, the rotation $c_k \mapsto c_k e^{ik\varphi_{-1}}$ maps $(xe^{i\varphi_{-2}}, e^{i\varphi_{-1}}, 1, 0, 0)$ to $(xe^{i\psi}, 1, 1, 0, 0)$ with $\psi = \varphi_{-2} - 2\varphi_{-1}$, and Lemma 2.4(e) gives $N_5(C_0(\xi)) \geq 1.61 - e^{-\sigma} - e^{-2\sigma} \geq 1.61 - e^{-4} - e^{-8} > 1.5913$. □

Theorem 5.3 (equal gaps)**.** $\Lambda(\xi) \leq \beta_0$ *for every* $\xi \in K$.

Proof. By Lemma 5.1 it suffices to treat $\xi \in D_K$, and by Lemma 5.2 only $\sigma \leq 4$. The certificate of Section 6 splits $[0,4] \times [0,1]^2 \times [-P,P] \times [0,P] \times [-P,P]$ into finitely many boxes. The boxes that lie entirely in $\{x > y\}$ do not meet D_K; every other box carries an index i and a vector τ with $T_\tau \neq 0$ and $\mathrm{Re}\langle C_i(\xi), \tau\rangle \geq \|T_\tau\|_1/\beta_0$ on the box, and Lemma 2.2 gives $N_5(C_i(\xi)) \geq 1/\beta_0$. □

5.2. **Mixed gaps.** For $p \geq 2$, $s_1 \geq 0$ and $z \in Z(s_1, p/(p-1))$ let

$$\Phi_p(z, s_1) = \inf\Big\{ \frac{U_p(\tau,\varepsilon)}{\mathrm{Re}\langle z,\tau\rangle + \varepsilon e^{-s_1}} : \ \tau \in \mathbb{C}^5,\ \varepsilon \geq 0,\ \mathrm{Re}\langle z,\tau\rangle + \varepsilon e^{-s_1} > 0 \Big\}.$$

Theorem 5.4 (mixed gaps)**.** $\Phi_p(z, s_1) \leq \beta_0$ *for all* $p \geq 2$, $s_1 \geq 0$ *and* $z \in Z(s_1, p/(p-1))$.

Proof. Tail $s_1 \geq 4$. Take $\varepsilon = 0$, so that $U_p(\tau, 0) = \|T_\tau\|_1$. Let τ be given by Lemma 5.2 for $xe^{i\psi} = z_2$, and let $\tau'_k = \tau_{-k}$, so that $T_{\tau'}(\theta) = T_\tau(-\theta)$ and $\|T_{\tau'}\|_1 = \|T_\tau\|_1$. Then $\langle z, \tau'\rangle = \langle (z_2, 1, 1, 0, 0), \tau\rangle + z_{-1}\tau_1 + z_{-2}\tau_2$. Since $|\tau_k| \leq \|T_\tau\|_1$ (τ_k is a Fourier coefficient of T_τ) and $|z_{-1}| \leq e^{-s_1}$, $|z_{-2}| \leq e^{-2s_1}$ (as $p/(p-1) \geq 1$), we get $\mathrm{Re}\langle z, \tau'\rangle \geq (1.61 - e^{-4} - e^{-8})\|T_{\tau'}\|_1 > 1.5913\,\|T_{\tau'}\|_1$. Hence $\Phi_p(z, s_1) < 0.6285$.

Compact part $s_1 \leq 4$. The periods are split into the groups $\{2\}$, $\{3\}$, $\{4\}$, $\{5, 6\}$, $\{7, \ldots, 10\}$, $\{11, \ldots, 16\}$, $\{17, \ldots, 24\}$ and $\{p \geq 25\}$. For a group G let $\kappa_G = \inf_{p \in G} p/(p-1)$ (so $\kappa_G = 1$ for the last group) and let $\bar{r}_G$ be the step function of Section 6, which satisfies $\bar{r}_G \geq r_p$ for all $p \in G$. Let $U_G(\tau, \varepsilon)$ be $U_p(\tau, \varepsilon)$ with $R_p(W, \varepsilon)$ replaced by $\varepsilon \bar{r}_G(W/\varepsilon)$ (and by 0 if $\varepsilon = 0$). For $p \in G$ we have $Z(s_1, p/(p-1)) \subset Z(s_1, \kappa_G)$ and $U_p \leq U_G$. The certificate covers, for each group, all pairs (s_1, z) with $s_1 \in [0, 4]$ and $z \in Z(s_1, \kappa_G)$, in the coordinates

$$z = \big(m_{-2}e^{-2\kappa_G s_1}e^{i\vartheta_{-2}},\ m_{-1}e^{-\kappa_G s_1}e^{i\vartheta_{-1}},\ 1,\ 1,\ m_2 e^{i\vartheta_2}\big),$$

$$(s_1, m_{-2}, m_{-1}, m_2, \vartheta_{-2}, \vartheta_{-1}, \vartheta_2) \in [0,4] \times [0,1]^3 \times [-\pi, \pi] \times [0, \pi] \times [-\pi, \pi],$$

by boxes carrying (τ, ε) with $T_\tau \neq 0$ and $\mathrm{Re}\langle z, \tau\rangle + \varepsilon e^{-s_1} \geq U_G(\tau, \varepsilon)/\beta_0$ on the box (so the left side is positive). The range $\vartheta_{-1} \in [-\pi, 0]$ follows by conjugation: $(\bar{\tau}, \varepsilon)$ works for $\bar{z}$, because $|T_{\bar{\tau}}(X)| = |T_\tau(-X)|$. □

5.3. **Proof of Theorem 1.1.** Let f be transcendental. By Section 3 it has events at every $j \geq 1$, with $r_0 = e^{t_j} \to \infty$. At an equal-gap event, Proposition 3.2 and Theorem 5.3 give a radius $r \in \{r_0, r_1\}$ with $\mu(r)/M(r) \leq \beta_0$. At a mixed event, Proposition 3.3 and Theorem 5.4 give the same. Hence $\mu(r)/M(r) \leq \beta_0$ for arbitrarily large r. □

6. The certificates

6.1. **Branch and bound.** Each certificate is a list of boxes, products of closed intervals with rational endpoints, obtained by repeated bisection of a fixed initial grid. A box for Theorem 5.3 carries an index $i \in \{0, 1\}$ and a vector $\tau \in \mathbb{Q}(i)^5$; a box for Theorem 5.4 carries $\tau \in \mathbb{Q}(i)^5$ and a rational $\varepsilon \geq 0$; a box for Lemma 5.2 carries τ. The phases range over $[-P, P]$ (or $[0, P]$) with the rational $P = 3.14159265358980 > \pi$. We obtained the vectors from a floating-point optimization of the dual problem at the centre of the box and rounded them to multiples of 2^{-24}. A box that cannot be certified is bisected, and the vectors of its parent are tried first. A box of the equal-gap domain lying entirely in $\{x > y\}$ is recorded as covered by the symmetry ρ.

A leaf is valid if a lower bound of the linear form $\mathrm{Re}\langle C_i(\xi), \tau\rangle$ (respectively $\mathrm{Re}\langle z, \tau\rangle + \varepsilon e^{-s_1}$) over the box is positive and at least $1/\beta_0$ times an upper bound of $\|T_\tau\|_1$ (respectively of $U_G(\tau, \varepsilon)$). A leaf for Lemma 5.2 is valid if a lower bound of $\mathrm{Re}\langle (xe^{i\psi}, 1, 1, 0, 0), \tau\rangle$ over the box is positive and at least 1.61 times an upper bound of $\|T_\tau\|_1$. Upper bounds for $\|T_\tau\|_1$ come from an adaptive subdivision of $[0, 2\pi]$. For the correction term in U_G, on each piece the integrand $\varepsilon \bar{r}_G(|T_\tau(X)|/\varepsilon)$ is at most $\varepsilon \bar{r}_G(w/\varepsilon)$ for a rigorous lower bound w of $|T_\tau|$ on the piece, because $\bar{r}_G$ is nonincreasing. The first program takes w from a ball enclosure of T_τ. The second uses a chord lower bound of $|T_\tau|^2$ along the piece, integrates the step function exactly, and uses the last table value beyond the grid.

6.2. **The tables.** For $p = 2, \dots, 24$ there is a table of integers n_j with $n_j 2^{-40} \geq r_p(u_j)$ on a grid $0 = u_0 < u_1 < \cdots < u_N \approx 1.02 \cdot 10^4$ of 1421 points. For all $p \geq 25$ a single table holds rounded-up upper bounds $n_j 2^{-40} \geq r_\infty(u_j) + \mathrm{err}(25, u_j)$ (Lemma 4.2(iii)). For a group G let n_j^G be the maximum of the entries of its tables and

$$\bar{r}_G(u) = n_j^G 2^{-40} \quad (u_j \leq u < u_{j+1}), \qquad \bar{r}_G(u) = \min\bigl(n_N^G 2^{-40}, 1/(2u)\bigr) \quad (u \geq u_N).$$

Every table is nonincreasing in j, so $\bar{r}_G$ is nonincreasing, and $\bar{r}_G \geq r_p$ for $p \in G$ by Lemma 4.2(i),(ii). Some entries exceed the quantity they bound by less than 10^{-15}, so checking them needs more than double precision.

6.3. **Sizes and checking.** Table 1 lists the three certificates.

TABLE 1. The certificates, with one row for each group of periods p of the mixed certificate. The margin is the smallest relative margin $\beta_0 L/U - 1$ of a leaf, as bounded from below by the second program and rounded down, where L is the lower bound of the linear form on the leaf and U the upper bound of $\|T_\tau\|_1$ (or of U_G). For the tail certificate, 1.61 replaces $1/\beta_0$.

Certificate	Periods p	κ_G	Leaves	Margin
Theorem 5.3 (equal gaps)			176,240*	$2.1 \cdot 10^{-7}$
Lemma 5.2 (tail)			18	$3.7 \cdot 10^{-3}$
Theorem 5.4 (mixed gaps)	2	2	3,465	$1.4 \cdot 10^{-3}$
	3	3/2	2,820	$1.1 \cdot 10^{-3}$
	4	4/3	3,136	$3.6 \cdot 10^{-4}$
	5, 6	6/5	3,762	$1.2 \cdot 10^{-4}$
	$7, \dots, 10$	10/9	4,627	$4.5 \cdot 10^{-4}$
	$11, \dots, 16$	16/15	5,524	$4.3 \cdot 10^{-4}$
	$17, \dots, 24$	24/23	6,157	$2.7 \cdot 10^{-4}$
	≥ 25	1	7,707	$4.1 \cdot 10^{-5}$

*and 15,276 boxes covered by the symmetry ρ.

Two programs check the three certificates and the tables in ball arithmetic [8]. They bound the linear forms, the norms $\|T_\tau\|_1$ and the table entries by different methods and check the coverage of the domains in different ways, and both check that the groups cover all $p \geq 2$ with admissible κ_G. Each program reports every leaf, every table entry and the coverage as verified. The two programs implement the same specification of the checks, so their agreement would not detect a misreading of it. They are written in Python and call the Arb library, now part of FLINT, through python-flint. With seven processes each, the first took about 18 minutes and the second about 7. The certificates, both programs and their outputs are available at `https://github.com/tejasvi/maximum-term-bound/tree/v4` and archived at doi:10.5281/zenodo.23716683.

7. Formal verification

We also prove Theorem 1.1 and the bound $B \le \beta_0$ in the Lean 4 proof assistant [10], with the Mathlib library [9]. The lower bound quoted in the introduction is not formalized. The formal statements use Mathlib's definitions: for $f\colon \mathbb{C} \to \mathbb{C}$, $a_k = f^{(k)}(0)/k!$, $M(r)$ is the supremum of $|f|$ on the circle $|z| = r$, $\mu(r)$ is the supremum of $|a_k| r^k$ over $k \ge 0$, and f is called transcendental entire if it is complex differentiable on $\mathbb{C}$ and differs from every polynomial function. With these definitions the formal theorems state Theorem 1.1, the bound $B \le \beta_0$, and that $\mu(r) \le \beta_0 M(r)$ holds for arbitrarily large r. The README of the repository gives their names and explains why the formal limit inferior and supremum agree with the usual ones.

The formal proof follows Sections 2 to 5 in a dual form: it does not use N_S but the bound $|\langle c(r), \tau\rangle| \le M(r)\|T_\tau\|_1$, which combines Lemmas 2.2 and 2.5. In this form it proves that every transcendental entire function has, beyond every radius, two radii $r_0 \le r_1$ and three indices with the properties (E1)–(E3). It also proves Propositions 3.2 and 3.3, Lemma 4.1, Lemma 5.1 in the form used in the proof of Theorem 5.3, and all parts of Lemma 4.2, where the first bound in (iii) is proved by the trapezoid rule with a Peano-kernel estimate instead of Poisson summation. Lemma 5.2 and Theorems 5.3 and 5.4 are proved from the certificates and the tables of Section 6: for each kind of data a checking program written in Lean carries out the checks of Section 6 again, in exact rational arithmetic with enclosures of the exponential function, the cosine, the sine, π and square roots that are proved from Mathlib, and a Lean proof shows that whenever the program accepts its input, the inequalities hold on the region that the input covers. The data are converted into Lean source by our scripts and decoded into Lean terms when the files are elaborated. Neither step needs to be trusted, since the correctness proofs hold for every input that the programs accept and the kernel checks every statement about the decoded terms. The runs of the programs are evaluated by the Lean kernel itself, without compiled code. Building the modules that contain the data and the checks takes about 150 hours of computation. The three formal theorems depend only on the three standard axioms of Lean. The Lean files are in the directory `lean/` of the repository.

8. Remarks

Remark 8.1 (the optimal configuration). The value 0.630041 of $\sup_K \Lambda$ quoted in Section 1 was obtained by numerical optimization. At the optimum, a primal-dual computation gives 0.6300408 to seven digits. Most leaves of the equal-gap certificate lie near the optimal configuration, where $\beta_0 - \Lambda$ is only about $9 \cdot 10^{-6}$. This configuration is real: $\sigma \approx 1.536$, $x = y \approx 0.417$ and $\varphi_{-2} = \varphi_{-1} = \varphi_2 = \pi$. The signs $(-,-,+,+,-)$ are those of the coefficients $(-1)^{q_n}$, $n = -2, \dots, 2$, of the Laurent series used by Clunie and Hayman for $B > 4/7$ [2, pp. 156–158] ($q_n = 0$ for $n \equiv 0, 1$ and $q_n = 1$ for $n \equiv 2, 3 \pmod 4$), but the moduli differ from those of their series. The leaf of the equal-gap certificate with the smallest margin (Table 1) lies near this configuration. It carries $i = 1$ and, to three decimals,

$$\tau = (-0.146 - 0.003i,\ -0.006 - 0.026i,\ 0.830 - 0.020i,\ 0.741 + 0.023i,\ -0.027 + 0.020i),$$

for which $\|T_\tau\|_1 \approx 1$ and $\mathrm{Re}\langle C_1(\xi), \tau\rangle \ge \|T_\tau\|_1/\beta_0$ on the leaf. For the mixed problems, the largest value we found is about 0.6281.

Remark 8.2 (wider windows). Wider windows may lower the limit of the five-coefficient formulation. With seven coefficients centred at the middle of five consecutive vertices with equal gaps, tested at the four jump radii between them, the best configuration we know for the equal-gap problem has value about 0.6059 (numerically). It is again real, with the Clunie–Hayman sign pattern. The exact value of that problem is not known: it is at least this number and, by Lemma 2.4(d) applied at the two middle radii, at most the value of the six-parameter problem.

A proof along these lines would also need a different treatment of mixed events: numerically, the five-coefficient single-radius bound of Proposition 3.3 tends to $(\sqrt{5}-1)/2 \approx 0.618$ as $s_1 \to \infty$ (the off-lattice coefficient becomes negligible), so that argument alone cannot go below 0.618.

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