Essential enhancements of site percolation in every dimension and on periodic lattices

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Abstract

We prove Conjecture 1 of Balister, Bollobás and Riordan for site percolation on $\mathbb{Z}^d$ in every dimension $d \geq 2$: for every essential enhancement, activated at any positive density, the enhanced configuration percolates for all $p < p_c$ close enough to the critical probability p_c. Aizenman and Grimmett gave an argument for this in 1991, but Balister, Bollobás and Riordan found that its combinatorial step can fail. They proved the cases $d = 2, 3$, outlined proofs for bond percolation on $\mathbb{Z}^d$ and for site percolation on the triangular lattice, and left the cases $d \geq 4$ open. Near a site that is pivotal for a long connection, our proof reroutes the open path through a finite configuration in which the enhancement creates a connection, inside a ball that need not be centred at the pivotal site. A theorem valid in every dimension makes this possible unless the first and last sites of the path in the ball lie at a corner of the ball, and a mass-transport argument finds a ball where they do not. The method extends to site percolation on fourteen further periodic lattices, among them all Archimedean lattices other than the square lattice, and to bond percolation on the honeycomb lattice. The proof for $\mathbb{Z}^d$ uses no computer, parts of most lattice proofs are finite computer checks of local configurations, and the main theorems for $\mathbb{Z}^d$ and for nine of the lattices have been checked in the Lean proof assistant.

1. Introduction

1.1. Enhancements

In site percolation on $\mathbb{Z}^d$, $d \geq 2$, each site is open with probability p, independently of the others. We write ω for the random set of open sites. Following Balister, Bollobás and Riordan [BBR], an *enhancement* is a map E_0 from the set $\mathcal{P}(\mathbb{Z}^d)$ of configurations to itself for which there is an r such that $E_0(\omega)$ depends only on $\omega \cap B_r$ and $E_0(\omega) \subseteq B_r$. We then say that E_0 has *range* r. (On $\mathbb{Z}^d$ all balls are ℓ^1 balls; see Section 1.6.) Let α be a second random set, containing each site with probability s, independently of everything else. The *enhanced configuration* is

$$E(\omega, \alpha) = \omega \cup \bigcup_{v \in \alpha} \big(E_0(\omega - v) + v\big).$$

The enhancement is *activated* at the sites of α. The rule E_0 need not be monotone. An enhancement is *essential* if there is a configuration ω that contains no doubly infinite path while $\omega \cup E_0(\omega)$ does. Let $\theta(p, s)$ be the probability that the origin lies in an infinite cluster of $E(\omega, \alpha)$. On other lattices the same definitions apply with balls of the graph distance and with one rule for each orbit of sites under translation (Definition 6.2, which also says when such an enhancement is essential); there we write E for the enhancement, that is, for its family of rules, and $\theta_E(p, s)$ for the probability that a

fixed root site lies in an infinite cluster of the enhanced configuration. The main result of Aizenman and Grimmett [AG] is stated in [BBR] as a conjecture, for the reason explained below.

Conjecture 1 (Aizenman–Grimmett; [BBR, Conjecture 1]). *Let E_0 be an essential enhancement of site or bond percolation on a lattice $\mathcal{L}$ with critical probability $p_c > 0$. Then for any $s > 0$ there is a $\pi(s) < p_c$ such that $\theta(p, s) > 0$ for all p satisfying $\pi(s) < p < p_c$.*

Enhancements of bond percolation are defined in the same way, from configurations of open edges, and can be activated at edges or at vertices; Section 10.5 gives the definitions for the honeycomb lattice.

As [BBR] point out, the statement has many implications, such as strict inequalities between the critical probabilities of suitable pairs of lattices. Essential enhancements, and couplings that replace them, are also used to prove strict monotonicity of percolation thresholds under covering maps [MS, MPR]. For background on percolation see [BR, G].

We prove Conjecture 1 for site percolation on $\mathbb{Z}^d$ in every dimension $d \geq 2$ (Theorem 1.1), for site percolation on fourteen further periodic lattices (Theorems 1.2, 1.4 and 1.5) and for bond percolation on the honeycomb lattice (Corollary 1.3).

1.2. The gap and what was known

The argument of [AG] has a probabilistic part and a combinatorial part. The probabilistic part reduces Conjecture 1 to a deterministic statement, Claim 5 of [BBR] (Section 2.1): if a site v is pivotal for the event that the enhanced configuration connects the origin to a distant sphere, and no enhancement is activated near v, then a bounded modification near v produces a site near v at which activating the enhancement decides this event. Balister, Bollobás and Riordan accept this part [BBR, §2]. For the combinatorial part, [AG] reroute the open path through v inside a cube around v so that it passes through a copy of a fixed finite gadget, a configuration whose two terminals are joined only after the enhancement is activated at its centre ([BBR, Lemma 6]; Lemma 2.2), and they support the rerouting only by a figure. Balister, Bollobás and Riordan showed that such a rerouting need not exist: a path that first enters the cube at a corner cannot always be continued into the interior without becoming adjacent to the other half of the path, which would join the two halves without the gadget [BBR, §2].

They replaced the rerouting by Conjecture 7 of [BBR]. It states that for each m, the radius of the gadget, there is $r = r(d, m)$ such that the following holds: two induced paths, one red and one green, that start outside B_{r+2}, end at neighbours of the origin and have no red site adjacent to a green site, can be modified inside B_r so that they end at $\pm me_1$, contain no other site of B_m, and still have no red site adjacent to a green site. They showed that Conjecture 7 implies Conjecture 1 for site percolation on $\mathbb{Z}^d$, and they proved Conjecture 7 for $d = 2$ and $d = 3$ by a case analysis at the corners of ℓ^1 balls. This gives Conjecture 1 for site percolation on $\mathbb{Z}^2$ and $\mathbb{Z}^3$ [BBR, Theorem 2]. They also outlined the much easier cases of bond percolation on $\mathbb{Z}^d$, $d \geq 2$, and site percolation on the triangular lattice [BBR, Theorem 3, §6]. In bond percolation two open paths become connected only if they share a vertex, which is much easier to avoid than the adjacency of two sites [BBR, §6]. They left the general case open, even for site percolation on $\mathbb{Z}^d$, $d \geq 4$. For the cubes used in [AG] the situation is much worse in three or more dimensions, with problems along edges as well as at corners [BBR, p. 8].

For other lattices, [BBR, p. 3] recall the remark of [AG] that the choice of lattice is largely irrelevant to the arguments of [AG], and say that they themselves are slightly less confident of Conjecture 1 for arbitrary lattices. Strict inequalities between critical probabilities have been proved for particular enhancements by arguments adapted to them. Grimmett and Li [GL] compare a plane graph with its matching graph through the enhancement that adds the diagonals of its faces, by a path surgery specific to this enhancement, and Grimmett [Gri] completes the quasi-transitive case of this comparison with a combinatorial criterion. Martineau and Severo [MS] prove strict monotonicity of thresholds under covering maps with an enhancement for which the obstruction found in [BBR] does not arise, and for this enhancement they prove the differential inequality of [AG], between the derivatives of crossing probabilities in s and in p, on graphs of bounded degree. Martineau, Poudevigne-Auboiron and Rax [MPR] obtain the bond case of [MS] by couplings, and Grimmett and Stacey [GS] compare the critical probabilities of site and bond percolation. Each of these strict inequalities concerns one particular enhancement or comparison. We are not aware of later work that settles the cases $d \geq 4$, or that proves Conjecture 1 for all essential enhancements on a lattice other than those treated in [BBR].

1.3. Main results

Theorem 1.1. *Let $d \geq 2$. Conjecture 1 holds for site percolation on $\mathbb{Z}^d$. That is, for every essential enhancement E_0 of site percolation on $\mathbb{Z}^d$ and every $s > 0$ there is $\pi(s) < p_c^{\mathrm{site}}(\mathbb{Z}^d)$ such that $\theta(p, s) > 0$ for all p with $\pi(s) < p < p_c^{\mathrm{site}}(\mathbb{Z}^d)$.*

The cases $d = 2, 3$ were proved in [BBR, Theorem 2]. The new cases are $d \geq 4$, but the proof below treats all $d \geq 2$ in the same way. Like the proof in [BBR], ours uses the probabilistic part of [AG] in the form accepted in [BBR, §2]: Conjecture 1 follows from Claim 5 (Theorem 2.1; Appendix B gives a complete proof). We prove Claim 5 for every essential enhancement of site percolation on $\mathbb{Z}^d$ and every $d \geq 2$ (Theorem 2.13 together with Theorem 4.8), without proving Conjecture 7 for $d \geq 4$.

The method extends to several other *periodic lattices*, that is, connected, locally finite graphs on $\mathbb{Z}^n \times \{0, \dots, k-1\}$ on which $\mathbb{Z}^n$ acts by translations (Definition 6.1). The *honeycomb lattice* (also called the hexagonal lattice) is the graph of the hexagonal tiling of the plane, in which every site has degree 3. The *Kagomé lattice* is its line graph, a 4-regular graph whose faces are triangles and hexagons. The *triangular lattice* is the 6-regular graph of the tiling by equilateral triangles. The *elongated triangular*, *snub square*, *snub hexagonal* and *rhombitrihexagonal* lattices are the graphs of the Archimedean tilings $3^3.4^2$, $3^2.4.3.4$, $3^4.6$ and $3.4.6.4$. The *square lattice with diagonals* (the king's graph) is $\mathbb{Z}^2$ with an edge between any two sites at ℓ^∞ distance 1, and the *face-centred cubic lattice* is the graph on $\mathbb{Z}^3$ in which $x \sim y$ iff $y - x \in \{\pm e_1, \pm e_2, \pm e_3, \pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3)\}$. Sections 6.1 and 12.4–12.9 give coordinates for all of them, and Definition 6.2 defines enhancements on them.

Theorem 1.2. *Conjecture 1 holds for site percolation on each of the following lattices:*

1. *the honeycomb lattice;*
2. *the Kagomé lattice;*
3. *the triangular lattice;*
4. *the elongated triangular lattice;*
5. *the snub square lattice;*

6. *the snub hexagonal lattice;*
7. *the rhombitrihexagonal lattice;*
8. *the square lattice with diagonals;*
9. *the face-centred cubic lattice.*

That is, if G is one of these lattices and E is an essential enhancement of site percolation on G, then for every $s \in (0,1]$ there is $\pi(s) < p_c(G)$ such that $\theta_E(p,s) > 0$ for all p with $\pi(s) < p < p_c(G)$.

Here θ_E refers to a fixed root site. Automorphisms act transitively on the sites of each of these lattices, and the conclusion then holds at every root (Remark 6.4); the same is true for the lattices of Theorems 1.4 and 1.5 (Sections 12.12–12.16).

Case 3 is part of [BBR, Theorem 3], whose proof [BBR, §6] outline; we give a complete proof in Section 11, since this case shows the method in its simplest form. Case 2 is bond percolation on the honeycomb lattice in another guise: the Kagomé lattice is the line graph of the honeycomb lattice, and two sets of its sites touch (share a site, or contain adjacent sites, one in each set) exactly when the corresponding sets of edges of the honeycomb lattice share a vertex. The two-port theorem for the Kagomé lattice (Section 1.4) rests on Menger's theorem, as does the proof that [BBR, §6] sketch for bond percolation on $\mathbb{Z}^d$, a case not treated here; Section 10 gives a complete proof of Case 2 by verifying the hypotheses of the reduction (Theorem A, Section 1.4). Case 1 is the main new case: the corner obstruction of [BBR] returns at the six corners of the hexagonal balls, and a corner lemma (Section 1.4) is needed (Section 9). Cases 4–9 follow the scheme of Case 1, each with its own exceptional pairs (Sections 1.4 and 12); their complete proofs are in the supplementary file [S].

Through the line-graph identification above, Theorem 1.2 gives the following.

Corollary 1.3. *Conjecture 1 holds for bond percolation on the honeycomb lattice, both for enhancements activated at edges and for enhancements activated at vertices.*

The transfer to the Kagomé lattice is exact for enhancements, essentialness and the laws of the random configurations, and for ranges when the enhancement is activated at edges (Section 10.5); for essentialness it uses that the honeycomb lattice is 3-regular (Lemma 10.16).

The eleven Archimedean lattices are the triangular, square and honeycomb lattices, the Kagomé lattice, the four lattices in Cases 4–7, and the truncated square (4.8^2), truncated hexagonal (3.12^2) and truncated trihexagonal (4.6.12) lattices. Together with the square lattice (Theorem 1.1 for $d = 2$), Theorem 1.2 covers eight of them, and Theorem 1.5 below covers the truncated square, truncated hexagonal and truncated trihexagonal lattices.

The *body-centred cubic lattice* is the graph on $\mathbb{Z}^3$ in which $x \sim y$ iff $y - x \in \{\pm e_1, \pm e_2, \pm e_3, \pm(e_1 + e_2 + e_3)\}$. In the coordinates of Section 12.12, which differ from those just used, its graph balls are cubes.

Theorem 1.4. *Conjecture 1 holds for site percolation on the body-centred cubic lattice. That is, for every essential enhancement E of site percolation on this lattice and every $s \in (0,1]$ there is $\pi(s) < p_c$ such that $\theta_E(p,s) > 0$ for all p with $\pi(s) < p < p_c$.*

The *truncated square lattice* is the graph of the Archimedean tiling 4.8^2 by squares and octagons, the *truncated hexagonal lattice* that of the tiling 3.12^2 by triangles and dodecagons, and the *truncated trihexagonal lattice* that of the tiling 4.6.12 by squares, hexagons and dodecagons. The *diamond*

lattice is the graph on $\mathbb{Z}^3 \times \{0,1\}$ in which $(n,0) \sim (n-e_i,1)$ for $i = 0,1,2,3$, where $e_0 = 0$ and e_1, e_2, e_3 are the unit vectors of $\mathbb{Z}^3$. Coordinates are given in Sections 12.13, 12.14, 12.15 and 12.16.

Theorem 1.5. *Conjecture 1 holds for site percolation on the truncated square lattice, on the truncated hexagonal lattice, on the truncated trihexagonal lattice and on the diamond lattice. That is, if G is one of these lattices and E is an essential enhancement of site percolation on G, then for every $s \in (0,1]$ there is $\pi(s) < p_c(G)$ such that $\theta_E(p,s) > 0$ for all p with $\pi(s) < p < p_c(G)$.*

On the five lattices of Theorems 1.4 and 1.5 the families of exceptional pairs used in the two-port property grow with the radius, and the proofs use variants of the method (end of Section 1.4); Sections 12.12–12.16 describe these proofs, whose complete versions are in [S].

The proofs for $\mathbb{Z}^d$, of the general results of Sections 6–8 and for the triangular and Kagomé lattices (hence for bond percolation on the honeycomb lattice) and the square lattice with diagonals use no computer. In all other proofs finite computations enter some of the lemmas, and, where needed, transfer lemmas or arguments by hand extend their results to all radii; Section 13.2 and Table 5 list these computations. Theorems 1.1 and 1.2 and Corollary 1.3, with the probabilistic step and the bounds $0 < p_c < 1$, have been verified in the Lean proof assistant; the formal proofs do not use these computations but replace them by general arguments or by finite checks carried out by Lean's kernel (Section 14). Theorems 1.4 and 1.5 have not been formally verified.

1.4. Idea of the proof

Recentring, on $\mathbb{Z}^d$. Let v be pivotal for the event R_L that the enhanced configuration contains a path from 0 to the sphere S_L, with no site of α near v. The deletion step of [BBR] removes from a large ball $\Omega = B_K(v)$ every open site that is not needed for such a path. Afterwards the open sites in Ω are exactly the sites of $P \cap \Omega$ for a single induced path P from 0 to S_L, and every site of $P \cap \Omega$ is pivotal. Hence any ℓ^1 ball $D = B_\rho(c)$ with $B_{\rho+1}(c) \subseteq \Omega$ and $|P \cap D| \geq 3$ is a candidate host for the gadget of [BBR, Lemma 6], whatever its centre c. Only the first and last sites a, g of P in D, the *ports*, matter. After the open sites in D are replaced, the pieces of P outside D are still not connected to each other outside D (Lemma 2.6), so it suffices to join a and g to the two terminals of a gadget centred at c by a red set and a green set that lie in D and meet $S_\rho(c)$ only in a and g, with no red site adjacent to a green site (Theorem 2.10). [BBR] used only a ball centred at v.

Two ports. Two sites $a \neq g$ of a sphere $S_\rho(c)$ of $\mathbb{Z}^d$ can be wired in this way unless they form a *corner pair* $\{c+\rho\pi,\ c+(\rho-1)\pi+\delta\}$, where π and δ are orthogonal signed unit vectors; we call $c+\rho\pi$ the *apex* of the pair and π its *pole* (Corollary 3.3, for $\rho \geq \max(2d+1, m+13, 20)$). We continue the ports inward by at most three layers until a hyperplane separates them, and then finish with [BBR, Lemma 8] (Lemma 3.1), which wires two ports separated in this way and holds in every dimension. Corner pairs do fail, because the corner's only inner neighbour is adjacent to the other port.

The Corner Lemma. It remains to show that inside a large ball $B_K(v)$ some admissible ball of radius at least ρ_0, that is, some candidate host as above that is large enough for the two-port theorem, has ports that do not form a corner pair (the Corner Lemma CL(d), Theorem 4.8; ρ_0 is fixed in (2.2)). Suppose this fails for arbitrarily large K. Compactness and averaging over translates give a translation-invariant random pair $(V, \prec)$, where $V \subseteq \mathbb{Z}^d$ is nonempty and $\prec$ is a linear order on V, such that every site of V has exactly two neighbours in V, and every ball of radius at least ρ_0 containing at least three sites of V has its $\prec$-first and $\prec$-last sites forming a corner pair. Shrinking a

corner-paired ball towards its apex keeps its first and last sites, and the shrinking map is injective. Mass transport turns this into a strong statement: almost surely every ball of every radius $r \geq 2$ is corner-paired, and its apex is also the apex of corner-paired balls of every larger radius (Proposition 4.4). Two corner-paired balls with different poles and suitably placed apexes would contain each other's ports, and stationarity along lines then shows that no site is the apex of two balls with different poles (Lemma 4.6). Hence the apex map $c \mapsto c + r\sigma_r(c)$, where $\sigma_r(c)$ is the pole of $B_r(c)$, is injective, and mass transport forces $V = \mathbb{Z}^d$. This contradicts the degree condition. The degree condition cannot be dropped: a "parity double sweep" on $V = \mathbb{Z}^d$ satisfies all the other hypotheses (Example 5.1).

Periodic lattices: the reduction (Section 7). Let G be a periodic lattice; its balls are those of the graph distance (on $\mathbb{Z}^d$, the ℓ^1 balls). Theorem A derives Claim 5 for every essential enhancement of site percolation on G from five properties of G: a two-port property TP_X: for every large ball, every gadget centre near its centre and every pair of terminals that can be continued outward as in (B1) below, every pair of distinct, non-adjacent ports outside a prescribed *exceptional family* X can be joined to the terminals by a *design*, a red set and a green set in the ball that meet its sphere only in the ports and do not touch; a corner lemma CL_X, which provides a ball, meeting the path in at least three sites, whose ports are not exceptional; and three properties (B1)–(B3): (B1) supplies gadget radii at which every pair of distinct non-adjacent sites of a sphere can be continued outward by one step each without the two continuations touching, and (B2) and (B3) handle pivotal sites close to the root and close to S_L (Section 7.2). The proof of Section 2 carries over to G with the changes listed in Section 7, two of them substantial. The gadget is centred at a site of an essential orbit (Definition 6.2), close to the centre of the host ball, and its terminals lie wherever the two tails of a doubly infinite path first leave a ball, so the two-port property must hold for every pair of gadget terminals that can be continued outward, not only for a fixed pair. And the boundary cases of Theorem 2.13, which use the geometry of ℓ^1 balls, are replaced by arguments based on (B2) and (B3).

The abstract corner lemma (Section 8). Theorem C proves CL_X on every periodic lattice of minimum degree at least 3 for exceptional families X with the following structure: (X1) X is invariant under translations; (X2) every exceptional pair contains an *apex*, carrying a label from a finite set, and for each radius the apex and the label determine the centre of the ball; (X3) the other site of the pair, its *partner*, lies within bounded distance of the apex; (X4) two balls whose apexes carry different labels and are placed suitably along a line contain each other's apexes and partners. The proof of Theorem C is a double counting in a large box and gives an explicit value of the radius K in CL_X; for $\mathbb{Z}^d$ and corner pairs it yields the Corner Lemma $\mathrm{CL}(d)$ again (Corollary 8.15).

The instances (Sections 9–12). On the honeycomb lattice exactly twelve port pairs on each large sphere admit no design (Theorem B-HEX): each of the six corners of the hexagonal ball forms an exceptional pair with each of the two sites of the sphere that precede and follow it along the boundary polygon, both at distance 2 from it. We prove the two-port property with these exceptions by a construction from cycles ("tracks") around the centre of the ball and the centre of the gadget and geodesic rays through corners, verify (X1)–(X4) for the twelve pairs, and obtain the corner lemma from Theorem C. On the Kagomé and triangular lattices there are no exceptional pairs. On the six further lattices of Theorem 1.2 an explicit distance formula describes the balls as polygons (polytopes for the face-centred cubic lattice), the scheme of the honeycomb lattice applies, and the exceptional pairs sit at the corners of the balls; on the two snub lattices the metric notion of a corner

produces unboundedly many apexes, and smaller families, read off from the pairs that actually fail, are used (Section 12.1).

The lattices of Theorems 1.4 and 1.5 (Sections 12.12–12.16). Since the families of exceptional pairs grow with the radius on these five lattices (Sections 12.12–12.16), the abstract corner lemma does not apply, and the proofs use variants of the reduction. Many of these pairs can never be the first and last sites of the path in a ball $B_\rho(c)$ that the path enters two levels deep, that is, that contains a site of the path in $B_{\rho-2}(c)$. On the body-centred cubic and diamond lattices, for the pairs along the edges of the balls, and for most of the hexagon pairs on the truncated trihexagonal lattice (Section 12.15), the gadget is joined instead to the ports of a host set obtained from the ball by removing a few sites visited by the path. The pairs that remain are again of corner type, boundedly many per sphere and each containing an apex, and a version of the corner lemma deals with them. Table 1 in Appendix C lists, for the lattices of Theorems 1.2 and 1.5, the exceptional families and the numbers of port pairs without a design, and its caption says which of these numbers are only computed; the body-centred cubic lattice is described in Section 12.12.

1.5. Limits of the method

The method, in the form of Theorems A and C, needs a two-port theorem whose exceptional pairs are of corner type: boundedly many per sphere, each containing an apex from which, for a given radius and together with a label from a finite set, the centre can be recovered. We searched for designs on graph balls at finitely many radii, with a SAT solver, on sixteen lattices: $\mathbb{Z}^2$, $\mathbb{Z}^3$ and the fourteen lattices of Theorems 1.2, 1.4 and 1.5. In these searches the lengths of the paths were bounded, so a pair for which no design was found need not be a genuine failure; the searches are heuristic, and no proof uses them; the computations used in proofs are described in Section 13.2. On the lattices of Theorem 1.2 and on $\mathbb{Z}^2$ and $\mathbb{Z}^3$ the pairs without a design found were few, between 0 and 24 per sphere, and of corner type. On the five lattices of Theorems 1.4 and 1.5 the number grew with the radius overall, although not monotonically on the truncated hexagonal and truncated trihexagonal lattices, and on the body-centred cubic and diamond lattices it was also much larger:

lattice	**radii**	**port pairs without a design found**
body-centred cubic	5, 6	162, 168 (most along the edges of the ball)
diamond	6, 8	72, 84
truncated square 4.8^2	four radii from 10 to 16	11, 12, 13, 15
truncated hexagonal 3.12^2	five radii from 10 to 18	3, 6, 4, 8, 5
truncated trihexagonal 4.6.12	five radii from 10 to 18	16, 16, 18, 23, 22

(The counts are listed in increasing order of the radius. For the four three-dimensional lattices, $\mathbb{Z}^3$ and the face-centred cubic, body-centred cubic and diamond lattices, only port pairs at distance at most 4 were examined.) On the body-centred cubic lattice most of these pairs lie along the edges of the cubic balls, where their first inner step is blocked; [BBR] met difficulties along edges with the cubes of [AG] as well (Section 1.2). For families of this kind one apex serves many centres, and

the counting of Theorem C does not apply. The proofs for the five lattices of Theorems 1.4 and 1.5 handle such families by the arguments described at the end of Section 1.4.

1.6. Organization and notation

The paper has two parts. Sections 2–5 treat $\mathbb{Z}^d$. Section 2 recalls the setting and reduces Theorem 1.1 to the Corner Lemma CL(d). Section 3 proves the two-port theorem. Section 4 proves CL(d) and completes the proof of Theorem 1.1. Section 5 contains remarks on sharpness and constants. Sections 6–13 treat periodic lattices. Section 6 introduces periodic lattices, enhancements and the probabilistic step, whose proof is given in Appendix B, and shows $0 < p_c < 1$ for all lattices of Theorem 1.2 (for those of Theorems 1.4 and 1.5 these bounds are in Sections 12.12–12.16). Section 7 proves the reduction, Theorem A, by listing the changes to the proof of Section 2. Section 8 proves the abstract corner lemma, Theorem C. Section 9 treats site percolation on the honeycomb lattice, Section 10 the Kagomé lattice and bond percolation on the honeycomb lattice, and Section 11 the triangular lattice. Section 12 states the results for the six further lattices, describes the common scheme of their proofs and outlines the proof for each lattice, referring to [S] for the complete proofs; Section 12.12 treats the body-centred cubic lattice (Theorem 1.4), and Sections 12.13, 12.14, 12.15 and 12.16 treat the truncated square, truncated hexagonal, truncated trihexagonal and diamond lattices (Theorem 1.5). Section 13 contains remarks on constants, on the computations and on open problems, and Section 14 describes the formal verification. Appendix A lists computational cross-checks for $\mathbb{Z}^d$, Appendix C collects Tables 1, 5 and 6, and Appendix D gives the Lean names of the formal statements.

In Sections 2–5, and in Section 1 wherever $\mathbb{Z}^d$ is concerned, $|x| = \sum_i |x_i|$ is the ℓ^1 norm, $B_r(c) = \{x : |x-c| \le r\}$ and $S_r(c) = \{x : |x-c| = r\}$, with $B_r = B_r(0)$ and $S_r = S_r(0)$. Two sites are adjacent if $|x-y| = 1$. The *layer* of x relative to c is $|x-c|$, and adjacent sites lie in consecutive layers. We write $U = \{\pm e_1, \dots, \pm e_d\}$ for the signed unit vectors, and $\pi, \delta \in U$ are called orthogonal if $\delta \neq \pm\pi$. For $D \subseteq \mathbb{Z}^d$, $N[D]$ is the set of sites in D or adjacent to D, and $\operatorname{int} D$ is the set of sites of D all of whose neighbours lie in D. Thus $N[B_\rho(c)] = B_{\rho+1}(c)$ and $\operatorname{int} B_\rho(c) = B_{\rho-1}(c)$. A *path* is a finite sequence of distinct sites in which consecutive sites are adjacent. It is *induced* if no two non-consecutive sites are adjacent; infinite and doubly infinite paths are defined in the same way. A set $A \subseteq \mathbb{Z}^d$ *contains a path* from x to y if some path from x to y has all its sites in A. The components of A are those of the subgraph of $\mathbb{Z}^d$ induced by A. Where [BBR] write e_m for $(m, 0, \dots, 0)$, we write me_1. From Section 6 on, balls, spheres and distances are those of the graph distance of the lattice at hand, with the notation of Section 6.1; for $\mathbb{Z}^d$ they coincide with the ℓ^1 notions above.

2. Setting and the reduction to a corner lemma

2.1. Crossing events and pivotal sites

As in [BBR, §3], all balls, spheres and distances are taken in the ℓ^1 metric, including the sphere $S_L = S_L(0)$ in the crossing event below. Let E_0 be an enhancement with range r_E (we write r_E for the range, which [BBR] call r, to free the letter r for radii). Let R_L be the event that the enhanced configuration contains a path from 0 to S_L. A site v is *p-pivotal* (for R_L, in the state (ω, α)) if $E(\omega \cup \{v\}, \alpha) \in R_L$ but $E(\omega \setminus \{v\}, \alpha) \notin R_L$. It is *s-pivotal* if $E(\omega, \alpha \cup \{v\}) \in R_L$ but

$E(\omega, \alpha \setminus \{v\}) \notin R_L$. The following are the two deterministic statements of [BBR, §2]. The dimension d is fixed throughout, and constants may depend on it.

Claim 4 ([BBR, Claim 4]). *Let E_0 be an essential enhancement. There are constants L_0 and R, depending only on E_0, such that whenever $L \geq L_0$ and a site v is p-pivotal for R_L in the state (ω, α), there is a state (ω', α') differing from (ω, α) only within distance R of v such that some site w within distance R of v is s-pivotal for R_L in the state (ω', α').*

Claim 5 ([BBR, Claim 5]). *The same statement, under the additional hypothesis $\alpha \cap B_R(v) = \emptyset$.*

2.2. The probabilistic reduction

Theorem 2.1 (Aizenman–Grimmett [AG, Theorem 1 and §3]; in the form of [BBR, pp. 4–6 and 9]). *Let $d \geq 2$ and let E_0 be an essential enhancement of site percolation on $\mathbb{Z}^d$. If Claim 5 holds for E_0, then the conclusion of Conjecture 1 holds for E_0.*

The formal theorem of [AG, Theorem 1] is stated for site percolation on $\mathbb{Z}^d$, which is exactly the case of Theorem 2.1 (see also [BBR, p. 3]). We recall the structure of the argument from [BBR, §2]. First, Claim 5 implies Claim 4 [BBR, pp. 5–6]. Let v be p-pivotal in (ω, α) and delete the sites of $\alpha \cap B_R(v)$ from α one at a time. The enhanced configuration is increasing in α. If at some stage $E(\omega \cup \{v\}, \alpha')$ no longer contains a path from 0 to S_L, the last deleted site w is s-pivotal in the state $(\omega \cup \{v\}, \alpha')$, which differs from (ω, α) only within $B_R(v)$. Otherwise v is still p-pivotal in $(\omega, \alpha \setminus B_R(v))$, and Claim 5 applies. Second, Claim 4 yields, through a finite-to-one modification map, an integer L_0 and a continuous positive function g on $(0,1)^2$ with $\mathbb{E}_{p,s} N_s(L) \geq g(p,s)\, \mathbb{E}_{p,s} N_p(L)$ for all $L \geq L_0$, where $N_p(L)$ and $N_s(L)$ count the p-pivotal and s-pivotal sites [BBR, (2)]. Third, a version of the Margulis–Russo formula gives $\partial_s \tau_L \geq g\, \partial_p \tau_L$ for $\tau_L(p,s) = \mathbb{P}_{p,s}(E(\omega, \alpha) \in R_L)$ [BBR, (3)]. Comparing τ_L along curves that do not depend on L and letting $L \to \infty$ gives the conclusion of Conjecture 1 for $s \in (0,1)$ [AG, pp. 822–823]. This comparison needs $p_c^{\mathrm{site}}(\mathbb{Z}^d) < 1$ (see Appendix B.6). For $s = 1$ one may take $\pi(1) = \pi(1/2)$, since $\theta(p,s)$ is non-decreasing in s. The proof of the cases $d = 2, 3$ in [BBR] rests on Theorem 2.1 in the same way [BBR, p. 9].

Theorem 2.1 is the case $G = \mathbb{Z}^d$ of Theorem 6.5: $\mathbb{Z}^d$ with nearest-neighbour adjacency is a periodic lattice with $n = d$ and $k = 1$ (Definition 6.1), and $p_c^{\mathrm{site}}(\mathbb{Z}^d) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2) < 1$ by Lemma 6.8 with $b = 1$ and $\beta(z) = \{((z_1, z_2, 0, \ldots, 0), 0)\}$, together with the classical bound for $\mathbb{Z}^2$ recalled after Proposition 6.6. So Appendix B, which proves Theorem 6.5, also gives a complete written proof of Theorem 2.1. The Lean formalization described in Section 14 proves Theorem 2.1 by the steps recalled above.

Theorem 1.1 thus reduces to Claim 5 for every essential enhancement of site percolation on $\mathbb{Z}^d$.

2.3. The gadget

Lemma 2.2 (the gadget; [BBR, Lemma 6] in ℓ^1 form). *Let E_0 be an essential enhancement of site percolation on $\mathbb{Z}^d$ with range r_E. There are an integer $m > r_E$ and a finite set $\omega_g \subseteq \mathbb{Z}^d$ such that*

1. *$\omega_g \subseteq B_m$ and $\omega_g \cap S_m = \{me_1, -me_1\}$;*
2. *ω_g contains no path from $-me_1$ to me_1;*
3. *$\omega_g \cup E_0(\omega_g)$ contains a path from $-me_1$ to me_1.*

Moreover $E_0(\omega_g) \subseteq B_{r_E} \subseteq B_{m-1}$.

Proof. Balister, Bollobás and Riordan prove the ℓ^∞ form [BBR, §4] and note that either form implies the other. We derive the ℓ^1 form. Write B_M^∞ and S_M^∞ for ℓ^∞ balls and spheres. Since $B_{r_E} \subseteq B_{r_E}^\infty$, the map E_0 also has range r_E with respect to the ℓ^∞ metric. The ℓ^∞ form of the lemma gives an integer $M > r_E$ and a finite set $\omega_\infty \subseteq B_M^\infty$ with $\omega_\infty \cap S_M^\infty = \{\pm Me_1\}$, such that ω_∞ contains no path from $-Me_1$ to Me_1 while $\omega_\infty \cup E_0(\omega_\infty)$ does. Put $m = dM$, let $\Sigma_\pm = \{\pm te_1 : M < t \le m\}$, and let $\omega_g = \omega_\infty \cup \Sigma_+ \cup \Sigma_-$.

(1) The sites of ω_∞ other than $\pm Me_1$ lie in $B_{M-1}^\infty \subseteq B_{d(M-1)} \subseteq B_{m-1}$. The sites $\pm te_1$ with $M \le t \le m$ have norm t, which equals m only for $t = m$. Hence $\omega_g \subseteq B_m$ and $\omega_g \cap S_m = \{\pm me_1\}$, and $m > M > r_E$.

(2) A site te_1 with $t > M$ has the neighbours $(t \pm 1)e_1$ and $te_1 \pm e_j$ $(j \ge 2)$. The latter have ℓ^∞ norm $t > M$, so they are not in ω_∞, and $(t-1)e_1 \in \omega_\infty$ only if $t = M+1$. Thus Σ_+ is a path attached to $\omega_\infty \cup \Sigma_-$ only through the edge between $(M+1)e_1$ and Me_1, and similarly for Σ_-. A path in ω_g from $-me_1$ to me_1 would therefore contain a path in ω_∞ from $-Me_1$ to Me_1, which does not exist.

(3) $E_0(\omega_g)$ depends only on $\omega_g \cap B_{r_E}$. Since $B_{r_E} \subseteq B_{M-1}^\infty$ and $\Sigma_\pm \cap B_M^\infty = \emptyset$, we have $\omega_g \cap B_{r_E} = \omega_\infty \cap B_{r_E}$, so $E_0(\omega_g) = E_0(\omega_\infty) \subseteq B_{r_E}$. A path in $\omega_\infty \cup E_0(\omega_\infty)$ from $-Me_1$ to Me_1, extended along Σ_- and Σ_+, joins $-me_1$ to me_1 in $\omega_g \cup E_0(\omega_g)$. □

From now on E_0, r_E, m and ω_g are fixed as in Lemma 2.2. Note that $E_0(\omega_g) \subseteq B_{m-1}$: activating the enhancement at the centre of a copy of the gadget changes nothing on or outside S_m, and it joins the two terminals $\pm me_1$, which ω_g alone does not join.

2.4. Two consecutive layers

We need the following elementary fact in Lemma 3.1 and in the boundary case of Theorem 2.13.

Lemma 2.3. *Let $d \ge 2$, $n \ge 1$, $i \in \{1, \dots, d\}$, and let $s < n$ be an integer. Then the set $A = (S_n \cup S_{n+1}) \cap \{x : x_i > s\}$ contains ne_i and is connected. In particular $S_n \cup S_{n+1}$ is connected.*

Proof. Let $x \in A$. We walk from x to ne_i inside $S_n \cup S_{n+1}$, changing one coordinate by one unit at each step. Write y for the current site.

Phase 1, while $y_i < 0$. If $|y| = n$, move some coordinate y_j with $j \ne i$ one unit away from 0 (this is possible since $d \ge 2$). The new site lies in S_{n+1}. If $|y| = n+1$, replace y_i by $y_i + 1$. As $y_i < 0$, the new site lies in S_n. Every two steps raise y_i by one, so eventually $y_i = 0$.

Phase 2, while $y_i \ge 0$. If $y = ne_i$, stop. If $y = (n+1)e_i$, step to ne_i and stop. If $|y| = n+1$ and $y \ne (n+1)e_i$, some coordinate y_j with $j \ne i$ is nonzero. Move it one unit towards 0, which leads to S_n. If $|y| = n$ and $y \ne ne_i$, replace y_i by $y_i + 1$, which leads to S_{n+1}. Two consecutive steps starting in S_n raise y_i by one and lower $\sum_{j \ne i} |y_j| = n - y_i$ by one, so the walk reaches ne_i.

Along the walk y_i never decreases, except in a final step from $(n+1)e_i$ to ne_i. Since $x_i > s$ and $n > s$, the walk stays in A. For the last assertion take $s = -n - 2$. □

2.5. The deletion step

In this subsection and the next we fix integers $K \ge m + 1$ and L, and a state (ω, α) in which a site v is p-pivotal for R_L, such that

$$K + 10 < |v| < L - K - 10 \qquad \text{and} \qquad \alpha \cap B_{K + r_E}(v) = \emptyset. \tag{2.1}$$

Put $\Omega = B_K(v)$. Then $0 \notin \Omega$, and every site of Ω has norm less than L.

Lemma 2.4. *There is a set $Y \subseteq \mathbb{Z}^d \setminus \Omega$ such that $E(\tilde{\omega}, \alpha) = \tilde{\omega} \cup Y$ for every configuration $\tilde{\omega}$ that agrees with ω outside Ω. For such $\tilde{\omega}$ and every $c \in \Omega$ we have $c \notin \alpha$ and $E(\tilde{\omega}, \alpha \cup \{c\}) = \tilde{\omega} \cup Y \cup (E_0(\tilde{\omega} - c) + c)$.*

Proof. Let $a \in \alpha$. Then $|a - v| > K + r_E$, so $B_{r_E}(a) \cap \Omega = \emptyset$. The set $E_0(\tilde{\omega} - a) + a$ depends only on $\tilde{\omega} \cap B_{r_E}(a)$, where $\tilde{\omega}$ agrees with ω, and it lies in $B_{r_E}(a)$. Hence $Y = \bigcup_{a \in \alpha} (E_0(\omega - a) + a)$ has the required properties. Finally $c \notin \alpha$ because $\Omega \subseteq B_{K + r_E}(v)$. □

Lemma 2.5 (the deletion step; [BBR, p. 9]). *There are a configuration ω' that agrees with ω outside Ω and an induced path $P = (p_0, \ldots, p_n)$ with $p_0 = 0$, $p_n \in S_L$ and $|p_j| < L$ for $j < n$, such that:*

1. *$P \subseteq E(\omega', \alpha)$ and $v \in P$;*
2. *$E(\omega', \alpha) \cap \Omega = \omega' \cap \Omega = P \cap \Omega$;*
3. *for every $x \in P \cap \Omega$, every path from 0 to S_L in $E(\omega', \alpha)$ passes through x;*
4. *both ends p_0 and p_n lie outside $B_{K+10}(v)$.*

Proof. Put $\omega_0 = \omega \cup \{v\}$. As long as some site $x \in \omega_t \cap \Omega$ satisfies $E(\omega_t \setminus \{x\}, \alpha) \in R_L$, delete one such site. The process stops at a configuration ω' that agrees with ω outside Ω, satisfies $E(\omega', \alpha) \in R_L$, and has $E(\omega' \setminus \{x\}, \alpha) \notin R_L$ for every $x \in \omega' \cap \Omega$. The site v is never deleted. Indeed, each $\omega_t \setminus \{v\}$ is contained in $\omega \setminus \{v\}$ and agrees with it outside Ω, so by Lemma 2.4

$$E(\omega_t \setminus \{v\}, \alpha) = (\omega_t \setminus \{v\}) \cup Y \subseteq (\omega \setminus \{v\}) \cup Y = E(\omega \setminus \{v\}, \alpha) \notin R_L.$$

Let P be a shortest path from 0 to S_L in $E(\omega', \alpha)$. A chord would shorten it, so P is induced. Since the norm changes by one at each step, P meets S_L only at its last site and all its other sites have norm less than L. By Lemma 2.4, $E(\omega' \setminus \{x\}, \alpha) = E(\omega', \alpha) \setminus \{x\}$ for $x \in \Omega$. Hence every path from 0 to S_L in $E(\omega', \alpha)$ passes through every site of $\omega' \cap \Omega$, and in particular $\omega' \cap \Omega \subseteq P$ and $v \in P$. As $Y \cap \Omega = \emptyset$ and $P \subseteq E(\omega', \alpha)$, this gives (2), and (3) follows. Finally $|p_0 - v| = |v| > K + 10$ and $|p_n - v| \geq L - |v| > K + 10$. □

From now on ω' and P are as in Lemma 2.5. We order P from $p_0 = 0$ to $z := p_n$, and for $x \in P \setminus \{0, z\}$ we write $\mathrm{pred}(x)$ and $\mathrm{succ}(x)$ for the sites before and after x on P.

Lemma 2.6 (the class property; [BBR, pp. 10–11]). *Let $D \subseteq \Omega$ meet P. Write $P = O_0 I_1 O_1 \cdots I_k O_k$ with $k \geq 1$, where $I_1, \ldots, I_k$ are the maximal runs of consecutive sites of P in D and $O_0, \ldots, O_k$ are the runs between them. ($O_0 \ni 0$ and $O_k \ni z$ are nonempty because $0, z \notin \Omega$.) Put $O_D = E(\omega', \alpha) \setminus D$, and let H_j be the component of O_D that contains O_j. Then $H_0, \ldots, H_k$ are pairwise distinct, and $H_j \cap S_L = \emptyset$ for every $j < k$.*

Proof. By Lemma 2.5(3), every path from 0 to S_L in $E(\omega', \alpha)$ meets each I_j. Suppose that $H_i = H_j$ with $i < j$, and let Q be a path in O_D from O_i to O_j. Follow P from 0 to O_i, then Q, then P from O_j to z. This walk lies in $E(\omega', \alpha)$ and avoids I_j, and it contains a path from 0 to S_L that avoids I_j, a contradiction. If H_j met S_L for some $j < k$, then P from 0 to O_j followed by a path in O_D from O_j to S_L would avoid I_k. □

For $D = B_r(v)$ the set O_D is the configuration ω^* of [BBR, (6)], but Lemma 2.6 needs only $D \subseteq \Omega$.

2.6. Admissible balls, ports and designs

Definition 2.7. A set $D \subseteq \mathbb{Z}^d$ is *admissible* if $N[D] \subseteq \Omega$ and $|P \cap D| \geq 3$. Its *ports* are the first site $a(D)$ and the last site $g(D)$ of P in D.

Recentring. A ball $D = B_\rho(c)$ is admissible if and only if $B_{\rho+1}(c) \subseteq \Omega$ and $|P \cap B_\rho(c)| \geq 3$. Here c is an arbitrary site of $\mathbb{Z}^d$: it need not lie on P, and D need not contain v. The Corner Lemma uses this freedom. For a ball $D = B_\rho(c)$ both ports lie on $S_\rho(c)$: the site $\mathrm{pred}(a(D))$ lies outside D, and the distance to c changes by exactly one at each step of P. The same holds for $g(D)$.

Proposition 2.8. *Let D be admissible, put $a = a(D)$ and $g = g(D)$, and use the notation of Lemma 2.6.*

1. *$a \neq g$, and a and g are not adjacent.*
2. *The sites of O_D adjacent to a are* $\mathrm{pred}(a) \in O_0$ *and, if $I_1 = \{a\}$, also* $\mathrm{succ}(a) \in O_1$. *The sites of O_D adjacent to g are* $\mathrm{succ}(g) \in O_k$ *and, if $I_k = \{g\}$, also* $\mathrm{pred}(g) \in O_{k-1}$.
3. *Let $\mathcal{A}$ consist of H_0, and of H_1 if $I_1 = \{a\}$. Let $\mathcal{G}$ consist of H_k, and of H_{k-1} if $I_k = \{g\}$. Then no member of $\mathcal{A}$ belongs to $\mathcal{G}$, and no member of $\mathcal{A}$ meets S_L.*

Proof. (2) The neighbours of a outside D lie in $N[D] \setminus D \subseteq \Omega \setminus D$, and $O_D \cap (\Omega \setminus D) = P \cap (\Omega \setminus D)$ by Lemma 2.5(2). Since P is induced and $a \notin \{0, z\}$, the only sites of P adjacent to a are $\mathrm{pred}(a)$ and $\mathrm{succ}(a)$. By the choice of a, $\mathrm{pred}(a) \notin D$, so it is the last site of O_0. The site $\mathrm{succ}(a)$ lies outside D exactly when $I_1 = \{a\}$, and then it belongs to O_1. The argument for g is symmetric.

(1) $a \neq g$ because $|P \cap D| \geq 3$. If a and g were adjacent, they would be consecutive on P because P is induced. All sites of $P \cap D$ lie between a and g on P, so we would get $P \cap D = \{a, g\}$.

(3) If $I_1 = \{a\}$ then $k \geq 2$, since otherwise $P \cap D = \{a\}$. Likewise $k \geq 2$ if $I_k = \{g\}$. As the H_j are pairwise distinct (Lemma 2.6), a common member of $\mathcal{A}$ and $\mathcal{G}$ would need equal indices. The index 0 differs from $k \geq 1$, and from $k-1$ when $I_k = \{g\}$. The index 1 (present only if $I_1 = \{a\}$) differs from k. It equals $k-1$ only if $k = 2$, and if moreover $I_2 = \{g\}$ then $P \cap D = I_1 \cup I_2 = \{a, g\}$, which is impossible. The members of $\mathcal{A}$ are components H_j with $j \in \{0, 1\}$ and $j < k$, so they do not meet S_L by Lemma 2.6. □

Definition 2.9 (interior design). Let $D = B_\rho(c)$ with $\rho \geq m + 1$, and let $a \neq g$ be sites of $S_\rho(c)$. An *interior design* for (a, g) in D is a set $\omega_1 = R \cup G \cup (\omega_g + c)$, where the *red set* R and the *green set* G are subsets of D such that:

- (D1) $a \in R$, $g \in G$, and R and G are connected;
- (D2) $R \setminus \{a\}$ and $G \setminus \{g\}$ lie in $B_{\rho-1}(c)$;
- (D3) $R \cap B_m(c) = \{c + me_1\}$ and $G \cap B_m(c) = \{c - me_1\}$, or the same with the two signs exchanged;
- (D4) no site of R equals or is adjacent to a site of G.

By (D3), $\omega_1 \cap B_m(c) = \omega_g + c$.

Theorem 2.10 (cf. [BBR, pp. 9–11]). *Let $D = B_\rho(c)$ be an admissible ball with $\rho \geq m + 1$, and let ω_1 be an interior design for $(a(D), g(D))$ in D. Put $\omega'' = (\omega' \setminus D) \cup \omega_1$. Then c is s-pivotal for R_L in the state (ω'', α). Moreover ω'' agrees with ω outside Ω, and $c \in \Omega$.*

Proof. Write $a = a(D)$ and $g = g(D)$, use the notation of Lemma 2.6 and Proposition 2.8, and suppose that $R \cap B_m(c) = \{c + me_1\}$ (the other case is symmetric). Let C_+ and C_- be the components of $\omega_g + c$ that contain $c + me_1$ and $c - me_1$. They are distinct by Lemma 2.2(2).

Step 1: the enhanced configurations. The configuration ω'' agrees with ω', hence with ω, outside Ω. By Lemma 2.4, $E(\omega'', \alpha) = \omega'' \cup Y = O_D \cup \omega_1$, because $O_D = (\omega' \cup Y) \setminus D = (\omega' \setminus D) \cup Y$. Also $c \notin \alpha$.

Since $r_E < m$ and $\omega'' \cap B_m(c) = \omega_g + c$, the enhancement at c sees $\omega'' \cap B_{r_E}(c) = (\omega_g + c) \cap B_{r_E}(c)$. Hence $E(\omega'', \alpha \cup \{c\}) = O_D \cup \omega_1 \cup X$ with $X = E_0(\omega_g) + c \subseteq B_{m-1}(c)$.

Step 2: the components of ω_1. A site of $R \setminus \{c + me_1\}$ lies outside $B_m(c)$, so its neighbours in $\omega_g + c$ lie in $S_m(c) \cap (\omega_g + c) = \{c \pm me_1\}$, and by (D4) it is not adjacent to $c - me_1 \in G$. Hence every site of $\omega_g + c$ that equals or is adjacent to a site of R is $c + me_1$ or a neighbour of $c + me_1$, and therefore lies in C_+. In the same way, every site of $\omega_g + c$ that equals or is adjacent to a site of G lies in C_-. Together with (D4) and $C_+ \neq C_-$, this shows that $R \cup C_+$ and $G \cup C_-$ are distinct components of ω_1. The other components of ω_1 are components of $\omega_g + c$.

Step 3: no crossing without activation. By (D2) and $\omega_g + c \subseteq B_m(c) \subseteq B_{\rho-1}(c)$, every site of ω_1 other than a and g lies in $\operatorname{int} D$, so all its neighbours lie in D, which O_D avoids. Hence a and g are the only sites of ω_1 with neighbours in O_D, and these neighbours are given by Proposition 2.8(2). Let Γ_0 be the union of $R \cup C_+$ and the members of $\mathcal{A}$. It is connected, since $\operatorname{pred}(a) \in H_0$ and, if $I_1 = \{a\}$, $\operatorname{succ}(a) \in H_1$. No site outside Γ_0 is adjacent to Γ_0: the members of $\mathcal{A}$ are components of O_D, they do not contain $\operatorname{pred}(g)$ or $\operatorname{succ}(g)$ by Proposition 2.8(3), and $R \cup C_+$ is a component of ω_1 whose only site with neighbours in O_D is a. So Γ_0 is the component of 0 in $E(\omega'', \alpha)$. It contains no site of S_L, because the members of $\mathcal{A}$ do not meet S_L (Proposition 2.8(3)) and $R \cup C_+ \subseteq D \subseteq \Omega$. Hence $E(\omega'', \alpha) \notin R_L$.

Step 4: a crossing with activation. By Lemma 2.2(3), $(\omega_g + c) \cup X$ contains a path from $c + me_1$ to $c - me_1$. The following walk lies in $E(\omega'', \alpha \cup \{c\})$: along P from 0 to $\operatorname{pred}(a)$ (these sites lie in $O_0 \subseteq O_D$), then to a, along a path in R to $c + me_1$, through $(\omega_g + c) \cup X$ to $c - me_1$, along a path in G to g, then to $\operatorname{succ}(g)$ and along P to $z \in S_L$ (these sites lie in $O_k \subseteq O_D$). Hence $E(\omega'', \alpha \cup \{c\}) \in R_L$. Together with Step 3 and $c \notin \alpha$, this shows that c is s-pivotal in (ω'', α). □

2.7. Corner pairs, the Corner Lemma, and the reduction

Definition 2.11. Let $c \in \mathbb{Z}^d$ and $\rho \geq 1$.

1. A *corner pair* of $S_\rho(c)$ is a set $\{c + \rho\pi,\ c + (\rho - 1)\pi + \delta\}$ with $\pi, \delta \in U$ orthogonal. If $\rho \geq 2$, the site $c + (\rho - 1)\pi + \delta$ has two nonzero coordinates relative to c, so the pair determines the corner $c + \rho\pi$ and hence π. We call π the *pole*, $c + \rho\pi$ the *apex* and $c + (\rho - 1)\pi + \delta$ the *partner*.
2. Two sites $x, y \in S_\rho(c)$ are *separated* if there are a coordinate i and an integer s with $|s| \leq \rho - 5$ lying strictly between $x_i - c_i$ and $y_i - c_i$.

A corner pair is never separated, and it admits no interior design. Indeed, by (D1)–(D3) the coloured set containing the apex $c + \rho\pi$ is connected, contains a site of $B_m(c)$ and has all its other sites in $B_{\rho-1}(c)$, so it contains a neighbour of the apex in $B_{\rho-1}(c)$. The only such neighbour is $c + (\rho - 1)\pi$, which is adjacent to the partner, contrary to (D4). This is the obstruction of [BBR, §2] in ℓ^1 form.

Corner Lemma CL(d). *For every integer $\rho_0 \geq 2$ there is an integer $K = K(d, \rho_0)$ with the following property. Let P be an induced path in $\mathbb{Z}^d$, ordered from one end to the other, that passes through a site v and has both end sites outside $B_K(v)$. Then there is a ball $B_\rho(c)$, with $c \in \mathbb{Z}^d$, $\rho \geq \rho_0$, $B_{\rho+1}(c) \subseteq B_K(v)$ and $|P \cap B_\rho(c)| \geq 3$, such that the first and the last site of P in $B_\rho(c)$ do not form a corner pair of $S_\rho(c)$.*

Reversing the order of P exchanges the first and the last site, so CL(d) does not depend on the orientation of P. If CL(d) holds with some K, it holds with every larger K. We prove CL(d) for

every $d \geq 2$ in Section 4 (Theorem 4.8). Here we show that it implies Claim 5.

For the given E_0 let m be as in Lemma 2.2, and put

$$\rho_0 = \max(2d+1,\ m+13,\ 20), \qquad K = K(d,\rho_0), \qquad R = 3K+100, \qquad L_0 = 100R. \tag{2.2}$$

Since CL(d) asks for a ball $B_{\rho+1}(c) \subseteq B_K(v)$ with $\rho \geq \rho_0$, we have $K \geq \rho_0 + 1 > m+1 > r_E + 1$.

Proposition 2.12. *Assume CL(d) and define ρ_0 and K by (2.2). Let (ω, α) be a state in which v is p-pivotal for R_L, and suppose that $K+10 < |v| < L-K-10$ and $\alpha \cap B_{K+r_E}(v) = \emptyset$. Then there are a configuration ω'' that agrees with ω outside $B_K(v)$ and a site $c \in B_K(v)$ that is s-pivotal for R_L in the state (ω'', α).*

Proof. The hypotheses (2.1) hold, with $K \geq m+1$. Let ω' and P be as in Lemma 2.5. By Lemma 2.5(1),(4), P is an induced path through v with both ends outside $B_K(v)$. CL(d) gives a ball $D = B_\rho(c)$ with $\rho \geq \rho_0$, $B_{\rho+1}(c) \subseteq \Omega$ and $|P \cap D| \geq 3$ whose ports do not form a corner pair. This ball is admissible, and its ports are distinct sites of $S_\rho(c)$ (Definition 2.7, Proposition 2.8). Since $\rho \geq \max(2d+1, m+13, 20)$, Corollary 3.3 below provides an interior design for $(a(D), g(D))$ in D. Theorem 2.10 now gives the claim. □

Theorem 2.13. *Let $d \geq 2$ and assume CL(d). Then Claim 5 holds for every essential enhancement E_0 of site percolation on $\mathbb{Z}^d$, with the constants R and L_0 of (2.2).*

Proof. Let $L \geq L_0$, let v be p-pivotal for R_L in the state (ω, α), and suppose that $\alpha \cap B_R(v) = \emptyset$. We find a state (ω'', α) that agrees with (ω, α) outside $B_R(v)$ and a site $c \in B_R(v)$ that is s-pivotal in it. Recall that $r_E < K$.

Case 1: $K+10 < |v| < L-K-10$. Since $B_{K+r_E}(v) \subseteq B_R(v)$, Proposition 2.12 applies directly.

Case 2: $|v| \leq K+10$ (cf. [BBR, p. 11]). Let w be a site of S_{K+19} nearest to v, so that $|v-w| = K+19-|v| \leq K+19$. Let

$$\tilde{\omega} = (\omega \setminus B_{K+21}) \cup (B_{K+21} \setminus S_{K+19}) \cup \{w\}.$$

Thus every site of B_{K+21} is open in $\tilde{\omega}$, except the sites of S_{K+19} other than w. For $a \in \alpha$ we have $|a| \geq |a-v| - |v| > 2K+90$, so $B_{r_E}(a)$ misses B_{K+21}. As in Lemma 2.4, there is a set Y' disjoint from B_{K+21} such that $E(\hat{\omega}, \alpha) = \hat{\omega} \cup Y'$ for every $\hat{\omega}$ that agrees with ω outside B_{K+21}.

The configuration $E(\tilde{\omega}, \alpha)$ contains a path from 0 to S_L. Indeed, $E(\omega \cup \{v\}, \alpha)$ contains such a path Γ. Let x' be the last site of Γ in B_{K+21}, so that $|x'| = K+21$. The part of Γ after x' lies in $(\omega \cup \{v\} \cup Y') \setminus B_{K+21} = E(\tilde{\omega}, \alpha) \setminus B_{K+21}$. Inside B_{K+21}, the configuration $\tilde{\omega}$ contains the connected set B_{K+18}, which contains 0 and an inner neighbour of w; it contains w; and it contains $S_{K+20} \cup S_{K+21}$, which is connected by Lemma 2.3 and contains x' and an outer neighbour of w. Concatenation gives the path. On the other hand, every path from 0 to S_L meets S_{K+19}, and w is the only site of S_{K+19} in $E(\tilde{\omega}, \alpha)$. Therefore w is p-pivotal for R_L in $(\tilde{\omega}, \alpha)$. Moreover $K+10 < |w| = K+19 < L-K-10$ and $\alpha \cap B_{K+r_E}(w) \subseteq \alpha \cap B_{2K+19+r_E}(v) = \emptyset$. Proposition 2.12, applied at w, gives ω'' that agrees with $\tilde{\omega}$ outside $B_K(w)$ and a site $c \in B_K(w)$ that is s-pivotal in (ω'', α). The configuration ω'' agrees with ω outside $B_{K+21} \cup B_K(w) \subseteq B_{2K+31}(v)$, and $c \in B_{2K+19}(v)$.

Case 3: $|v| \geq L-K-10$ (cf. [BBR, pp. 11–12]). First, $|v| \leq L$. Otherwise take a path from 0 to S_L in $E(\omega \cup \{v\}, \alpha)$ and stop it at its first site on S_L. All its sites have norm at most L, so it avoids v.

Since $\alpha \cap B_{r_E}(v) = \emptyset$, no enhancement sees v, so $E(\omega \cup \{v\}, \alpha) \setminus \{v\} \subseteq E(\omega \setminus \{v\}, \alpha)$, and v would not be p-pivotal.

Choose w with $|w| = L - K - 20$ and $|v - w| = |v| - |w| \leq K + 20$, by moving v towards the origin one coordinate step at a time. Put $\Omega' = B_{K+30}(w)$. Then $0 \notin \Omega'$, and $\alpha \cap B_{K+30+r_E}(w) \subseteq \alpha \cap B_{2K+50+r_E}(v) = \emptyset$. So no enhancement reads or writes in Ω', and the proofs of Lemmas 2.4 and 2.5 apply verbatim with Ω' in place of Ω. They give a set Y disjoint from Ω', a configuration ω' that agrees with ω outside Ω', and an induced path P from 0 to S_L through v, meeting S_L only at its last site z, such that $P \subseteq E(\omega', \alpha)$ and $E(\omega', \alpha) \cap \Omega' = \omega' \cap \Omega' = P \cap \Omega'$. Let u be the first site of P in Ω'. Then $|u - w| = K + 30$. As $v \in B_{K+20}(w)$, the site u comes before v on P, so $u \neq z$ and $|u| < L$.

Following [BBR], we build a path P' from u to S_L through w. First go from u to w, reducing the distance to w by one at each step, and take all steps that reduce the absolute value of a coordinate before all steps that increase one. Let the last of these steps go from $w - e$ to w, where $e = \varepsilon e_i \in U$. Then step to $w + e$, and from there take steps that increase the absolute value of coordinates other than the i-th, until S_L is reached. The norm along the first segment is at most $\max(|u|, |w|) < L$. Since $|u| < L$ and $|u - w| = K + 30$, at least six of the steps from u to w increase the norm, so the last one does, and $|w + e| = |w| + 1$. Along the second segment the norm increases by one at each step, so P' meets S_L only at its last site, which is reached $K + 20$ steps after w. The distance to w decreases by one at each step of the first segment and increases by one at each step of the second, because each step after $w + e$ moves some coordinate x_j, $j \neq i$, away from 0, starting from w_j. Hence $P' \setminus \{u\} \subseteq B_{K+29}(w)$. The path P' is induced: sites whose distances to w differ by at least two are not adjacent, and a site x of the first segment other than w satisfies $\varepsilon(x_i - w_i) \leq -1$, while every site x of the second segment other than w satisfies $\varepsilon(x_i - w_i) = 1$.

Let $\hat{\omega} = (\omega' \setminus \Omega') \cup P'$, so that $E(\hat{\omega}, \alpha) = \hat{\omega} \cup Y$. The part of P before u lies in $(\omega' \setminus \Omega') \cup Y$, and followed by P' it gives a path from 0 to S_L in $E(\hat{\omega}, \alpha)$. Suppose that Γ is a path from 0 to S_L in $E(\hat{\omega} \setminus \{w\}, \alpha)$. The sites of $P' \setminus \{u\}$ lie in $B_{K+29}(w)$, so all their neighbours lie in Ω', where the open sites of $E(\hat{\omega} \setminus \{w\}, \alpha)$ are those of $P' \setminus \{w\}$. Since P' is induced, the second segment without w forms a component that does not contain 0, and the first segment without u and w is attached to the rest only through u and contains no site of S_L. Hence Γ avoids $\Omega' \setminus \{u\}$. As $u \in P \cap \Omega' \subseteq \omega'$ and $v \in \Omega' \setminus \{u\}$, the path Γ lies in $(\omega' \setminus \{v\}) \cup Y = E(\omega' \setminus \{v\}, \alpha) \subseteq E(\omega \setminus \{v\}, \alpha)$, contradicting the p-pivotality of v in (ω, α). Thus w is p-pivotal for R_L in $(\hat{\omega}, \alpha)$. Moreover $K + 10 < |w| < L - K - 10$ and $\alpha \cap B_{K+r_E}(w) = \emptyset$. Proposition 2.12, applied at w, gives ω'' that agrees with $\hat{\omega}$ outside $B_K(w)$ and a site $c \in B_K(w)$ that is s-pivotal in (ω'', α). The configuration ω'' agrees with ω outside $\Omega' \subseteq B_{2K+50}(v)$, and $c \in B_{2K+20}(v)$.

In each case the modification lies in $B_R(v)$ and $c \in B_R(v)$, which is the conclusion of Claim 5. □

Remark 2.14. In Case 2 we follow [BBR] with two precisions. First, [BBR] open the ball B_{r+20}. With ℓ^1 balls this does not suffice: no two sites of S_{r+20} are adjacent, so the outer neighbours of w can be cut off from the rest of the crossing. Opening B_{r+21} repairs this. Second, [BBR] take $w = (r + 19)e_1$, which can be at distance $2r + 29$ from v. Their main case at w needs $\alpha \cap B_{2r+2}(w) = \emptyset$, and $B_{2r+2}(w)$ is then only known to lie in $B_{4r+31}(v)$, which is not contained in $B_R(v)$, $R = 3r + 100$, once $r > 69$. Taking w nearest to v repairs this. Case 3 follows [BBR, pp. 11–12], with r replaced by K.

3. The two-port theorem in every dimension

In this section the centre is 0. By translation, all statements hold for balls $B_r(c)$, with coordinates taken relative to c and terminals $c \pm me_1$. For a site x with $x_i \neq 0$, the *inward step* of x in coordinate i is $x - \operatorname{sgn}(x_i)e_i$, and it lowers the layer by one.

3.1. Lemma 8 of [BBR] in every dimension

Lemma 3.1 ([BBR, Lemma 8]). *Let $d \geq 2$, $m \geq 1$ and $r \geq m + 10$. Let $a, g \in S_r$ be separated, that is, for some coordinate i there is an integer s with $|s| \leq r - 5$ strictly between a_i and g_i. Then there are a red path starting at a and a green path starting at g with the following properties: one of them ends at me_1 and the other at $-me_1$; all their sites other than a and g lie in B_{r-1}; all their sites other than these two end sites lie outside B_m; and no red site equals or is adjacent to a green site.*

[BBR] state and prove this lemma for general d. One step of their proof, the connectivity of the two halves of an annulus, is asserted without proof. Lemma 2.3 supplies it.

Proof. The statement is symmetric under exchanging a and g, so we may assume $a_i > s > g_i$. Put $w = (r-2)e_i$ and

$$A_+ = (S_{r-1} \cup S_{r-2}) \cap \{x : x_i > s\}, \qquad A_- = (S_{r-1} \cup S_{r-2}) \cap \{x : x_i < s\}.$$

Walks below can always be shortened to paths with the same ends and a subset of the sites, so it suffices to build walks.

The annulus. Since $s < r - 2$, Lemma 2.3 with $n = r - 2$ shows that A_+ is connected and contains w. Applying the same lemma to $-s$ and using the symmetry $x \mapsto -x$, the set A_- is connected and contains $-w$.

Entering the annulus. If $a_j \neq 0$ for some $j \neq i$, the inward step of a in coordinate j lies in S_{r-1} and has the same i-th coordinate as a, so it lies in A_+. Otherwise $a = \pm re_i$, and $a_i > s \geq -(r-5)$ forces $a = re_i$, whose neighbour $(r-1)e_i$ lies in A_+. In the same way g has a neighbour in A_-.

First stage. Join a by a red walk through A_+ to w, and continue it with $(r-3)e_i$ and $w' = (r-4)e_i$. Join g by a green walk through A_- to $-w$, and continue it with $-(r-3)e_i$ and $-w'$. Every red site x so far satisfies $x_i \geq s + 1$, and every green site satisfies $x_i \leq s - 1$: this holds on $A_\pm$ by definition, at a and g by hypothesis, and at $\pm(r-3)e_i$ and $\pm w'$ because $|s| \leq r - 5$. Hence no red site equals or is adjacent to a green site. Apart from a and g, all first-stage sites lie in layers $r-1, \dots, r-4$, and the only first-stage sites in layers $r-3$ and $r-4$ are $\pm(r-3)e_i$ and $\pm w'$.

Second stage. All further sites lie in layers at most $r - 4$. Such a site can be adjacent to a first-stage site only if that site is $\pm w'$, or if it lies in layer $r-4$ and is adjacent to $\pm(r-3)e_i$. The only site of layer $r-4$ adjacent to $(r-3)e_i$ is w', and similarly for $-(r-3)e_i$. So it suffices to join w' and $-w'$ to the terminals by a red and a green walk without red–green contacts among themselves. If $i = 1$, go along the axis from $w' = (r-4)e_1$ to me_1, and from $-w'$ to $-me_1$. If $i \neq 1$, we may assume $i = 2$ after permuting the coordinates $2, \dots, d$, which fixes $\pm me_1$. Join $w' = (r-4)e_2$ to $(r-4)e_1$ by the staircase

$$(r-4)e_2,\ (r-5)e_2,\ e_1 + (r-5)e_2,\ e_1 + (r-6)e_2,\ \dots,\ (r-5)e_1,\ (r-4)e_1,$$

whose sites x satisfy $x_1, x_2 \geq 0$ and $x_1 + x_2 \in \{r-5, r-4\}$, and then go along the axis to me_1. The green walk is the reflection of the red one through the origin. In both cases every red second-stage site has nonnegative coordinates and every green one has nonpositive coordinates, and all of them have norm at least m. Consequently a red site x and a green site y satisfy $|x-y| = |x|+|y| \geq 2m \geq 2$.

Location. First-stage sites other than a and g lie in layers $r-4, \dots, r-1$, and $r-4 \geq m+6$. Second-stage sites lie in layers at most $r-4$. The staircase lies in layers $r-5$ and $r-4$, which exceed m, and the axis segments meet B_m only at $\pm me_1$. □

3.2. Escaping from a non-corner pair

Theorem 3.2 (escape). *Let $d \geq 2$ and $r \geq \max(2d+1, 20)$. Let $a \neq g$ be sites of S_r that do not form a corner pair. Then there are $k \in \{0,1,2,3\}$ and walks γ_a from a to a site a^* and γ_g from g to a site g^* such that:*

- *(E1) $a^*, g^* \in S_{r-k}$, every site of γ_a and γ_g other than a and g lies in the layers $r-k, \dots, r-1$, and layer $r-k$ is met only by a^* and g^*;*
- *(E2) no site of γ_a equals or is adjacent to a site of γ_g;*
- *(E3) a^* and g^* are separated in S_{r-k}.*

For $k = 0$ the walks are $\gamma_a = (a)$ and $\gamma_g = (g)$. Together with Lemma 3.1, the theorem replaces Lemma 9 of [BBR], which is specific to $d = 3$, by a statement valid in every dimension. Its proof distinguishes five cases.

Notation. Put $\Delta = g - a$. Since $|a| = |g| = r$, the norm $|\Delta|$ is even and at least 2. We use the following bookkeeping rule throughout: when a site of one walk takes a step in coordinate j, its distance to a fixed site y of the other walk grows by one if the step moves its j-th coordinate away from y_j, and drops by one otherwise.

Step 0. If a and g are separated in S_r, take $k = 0$. From now on we assume that they are not. If $|\Delta_j| \geq 2$ for some j, the integers strictly between a_j and g_j form a nonempty interval that avoids $[-(r-5), r-5]$. Hence a_j and g_j have the same sign and $|a_j|, |g_j| \geq r-5$. Since $r \geq 20$, each of a and g has at most one coordinate of absolute value at least $r-7$.

Type II: every coordinate of a and of g has absolute value at most $r-7$. By Step 0, $|\Delta_j| \leq 1$ for every j, and neither a nor g is a corner of S_r. Call $j \in \operatorname{supp}\Delta$ *good* if a_j and g_j are both nonzero. They then have the same sign and differ by one.

Case II-A: a good j exists. Let X be the one of a, g with the smaller $|X_j|$ and Y the other, so that $Y_j = X_j + \operatorname{sgn}(X_j)$. Put $X' = X - \operatorname{sgn}(X_j)e_j$. Since Y is not a corner, it has a nonzero coordinate $i \neq j$, and we put $Y' = Y - \operatorname{sgn}(Y_i)e_i$. When $|\Delta| = 2$ and $\operatorname{supp}\Delta = \{j, l\}$, we choose $i \notin \{j, l\}$ if Y has such a nonzero coordinate, and $i = l$ otherwise. Let $\gamma_X = (X, X')$ and $\gamma_Y = (Y, Y')$, so $k = 1$.

The step from X to X' moves X_j away from Y_j, so $|X' - Y| = |\Delta| + 1$. The sites X' and Y' have j-th coordinates $X_j - \operatorname{sgn} X_j$ and $X_j + \operatorname{sgn} X_j$, so $|X' - Y'| \geq 2$. If $|\Delta| \geq 4$, then $|X - Y'| \geq |\Delta| - 1 \geq 3$. Let $|\Delta| = 2$ with $\operatorname{supp}\Delta = \{j, l\}$. If $i \notin \{j, l\}$, then $X_i = Y_i \neq 0$, the step moves Y_i away from X_i, and $|X - Y'| = 3$. Otherwise $\operatorname{supp} Y = \{j, l\}$ and $i = l$. If the step moved Y_l towards X_l, we would have $|X_l| = |Y_l| - 1$. Together with $|X_j| = |Y_j| - 1$ and $X = Y = 0$ outside $\{j, l\}$ this gives $|X| = |Y| - 2$, which is false. So again $|X - Y'| = 3$, and (E2) holds. Both X' and Y' lie in layer $r-1$, which gives (E1). For (E3), the integer $s = X_j$ lies strictly between X'_j and $Y'_j = Y_j$, and $|s| \leq r-7 \leq (r-1)-5$.

Case II-B: no good j exists. Then every $j \in \operatorname{supp}\Delta$ has $\{a_j, g_j\} = \{0, \pm 1\}$. If $|a_j| \le 1$ also held for every $j \notin \operatorname{supp}\Delta$, we would get $r = |a| \le d$, contrary to $r \ge 2d+1$. So some $j \notin \operatorname{supp}\Delta$ has $|a_j| \ge 2$, and there $a_j = g_j =: x$. Put $\sigma = \operatorname{sgn} x$ and $\gamma_a = (a,\, a - \sigma e_j,\, a - 2\sigma e_j)$. Let $\gamma_g = (g, g_1, g_2)$ consist of two successive inward steps of g in coordinates other than j, chosen as follows. If $|\Delta| = 2$, then $\operatorname{supp}\Delta = \{j_1, j_2\}$, where (a, g) takes the values $(0, \pm 1)$ in one of these coordinates and $(\pm 1, 0)$ in the other, because $|a| = |g|$. We take both steps in coordinates outside $\operatorname{supp}\Delta \cup \{j\}$, where g has total mass $r - |x| - 1 \ge 6$. If $|\Delta| \ge 4$, we take them in any coordinates other than j, where g has mass $r - |x| \ge 7$. Here $k = 2$.

Write $a_s = a - s\sigma e_j$. The steps of a move its j-th coordinate away from x, which is the j-th coordinate of every g_t, so $|a_s - g_t| = |a - g_t| + s$. If $|\Delta| = 2$, the steps of g are in coordinates where a and g agree and a never moves, so $|a - g_t| = 2 + t$. If $|\Delta| \ge 4$, then $|a - g_t| \ge |\Delta| - t \ge 2$. This proves (E2). Since $|x| \ge 2$, the sites a_1, g_1 lie in layer $r-1$ and a_2, g_2 in layer $r-2$, which gives (E1). For (E3), the j-th coordinates of a_2 and g_2 are $x - 2\sigma$ and x, and $s = x - \sigma$ satisfies $|s| = |x| - 1 \le r - 8 \le (r-2) - 5$.

Type I: some coordinate p has $\max(|a_p|, |g_p|) \ge r-6$. Then a_p and g_p have a common sign σ_p and $|a_p|, |g_p| \ge r-7$: this follows from Step 0 if $|\Delta_p| \ge 2$, and directly if $|\Delta_p| \le 1$. Write down $x = x - \sigma_p e_p$. Each down step below is applied to a site with $|x_p| \ge (r-7) - 3 \ge 10$, so it is an inward step. The coordinates $j \ne p$ are called *small*. The small parts of a and g have norm at most $7 < r - 5$, so $|a_j - g_j| \le 1$ for every small j, by Step 0. A small $j \in \operatorname{supp}\Delta$ is *good* if a_j and g_j are both nonzero.

Case I-A: a good small j exists. Define X, Y and X' as in Case II-A, put $Y' = \operatorname{down} Y$, and let $\gamma_X = (X, X')$ and $\gamma_Y = (Y, Y')$, so $k = 1$. As in Case II-A, $|X' - Y| = |\Delta| + 1$, and X' and Y' differ by two in coordinate j. The down step moves Y_p towards X_p only if $|X_p| < |Y_p|$. If $|\Delta| = 2$ and $\operatorname{supp}\Delta = \{j, p\}$, then $|X_j| = |Y_j| - 1$ forces $|X_p| = |Y_p| + 1$. If $|\Delta| = 2$ and $\operatorname{supp}\Delta = \{j, l\}$ with $l \ne p$, then $X_p = Y_p$. So $|X - Y'| = 3$ if $|\Delta| = 2$, and $|X - Y'| \ge |\Delta| - 1 \ge 3$ otherwise. This gives (E2), and (E1) is clear. (E3) holds in coordinate j with $s = X_j$, since $|s| \le 7 \le (r-1) - 5$.

Case I-B1: no good small coordinate exists, but some small j has $a_j = g_j$ and $|a_j| \ge 2$. Let X be one of a, g with $|X_p| \ge |Y_p|$, where Y is the other, and put $\sigma_j = \operatorname{sgn} a_j$. Let $\gamma_X = (X,\, X - \sigma_j e_j,\, X - 2\sigma_j e_j)$ and $\gamma_Y = (Y, \operatorname{down} Y, \operatorname{down}^2 Y)$, so $k = 2$. The walk γ_X moves only in coordinate j, where $X_j = Y_j$ and γ_Y never moves. The walk γ_Y moves only in coordinate p, away from X_p because $|X_p| \ge |Y_p|$, and γ_X never moves in p. By the bookkeeping rule, the s-th site of γ_X and the t-th site of γ_Y are at distance $|\Delta| + s + t \ge 2$. Both walks visit the layers $r, r-1, r-2$ in this order. (E3) holds in coordinate j with $s = a_j - \sigma_j$, since $|s| \le 6 \le (r-2) - 5$.

Case I-B2: all other cases. Every small coordinate takes values in $\{-1, 0, 1\}$ on a and g, and the two values agree whenever both are nonzero. Indeed, if $a_j \ne g_j$ then one of the values is 0 (as j is not good) and the other is ± 1. If $a_j = g_j$, then $|a_j| \le 1$ (as we are not in Case I-B1). Let Q be the number of small coordinates in which exactly one of a_j, g_j is nonzero. Then $Q \ge 1$, since otherwise the small parts agree, so $|a_p| = |g_p|$, $a_p = g_p$ and $a = g$. Choose $X \in \{a, g\}$ and a small i with $X_i =: \tau \ne 0 = Y_i$, where Y is the other port. Let

$$\gamma_X = (X,\ \operatorname{down} X,\ \operatorname{down}^2 X,\ X_3,\ \operatorname{down} X_3,\ \operatorname{down}^2 X_3), \qquad X_3 = \operatorname{down}^2 X + \tau e_i.$$

Its layers are $r, r-1, r-2, r-1, r-2, r-3$. The site X_3 is back in layer $r-1$ because $|2\tau| = |\tau| + 1$. We write its sites as $X_s = X - c_s \sigma_p e_p + \ell_s \tau e_i$ for $s = 0, \dots, 5$, with

$$(c_s, \ell_s) = (0,0),\ (1,0),\ (2,0),\ (2,1),\ (3,1),\ (4,1).$$

- If $Q \geq 2$, let $\gamma_Y = (Y, \operatorname{down} Y, \operatorname{down}^2 Y, \operatorname{down}^3 Y)$, with layers $r, r-1, r-2, r-3$. The small parts of X_s and of any site of γ_Y differ by one in each of the $Q-1$ coordinates other than i where exactly one value is nonzero, and by $1+\ell_s$ in coordinate i, which moves from τ to 2τ, away from $Y_i = 0$. Their distance is therefore at least $Q + \ell_s \geq 2$.
- If $Q = 1$, then the small part of X is that of Y plus τe_i, so $|X_p| = |Y_p| - 1$, that is, $X_p = Y_p - \sigma_p$. The site Y is not a corner of S_r: otherwise $Y = r\sigma_p e_p$ and $X = (r-1)\sigma_p e_p + \tau e_i$, and $\{X, Y\}$ would be a corner pair. (This is the only use of the hypothesis that $\{a, g\}$ is not a corner pair.) So Y has a small coordinate j with $Y_j \neq 0$, and then $j \neq i$, $|Y_j| = 1$ and $X_j = Y_j$. Let $\gamma_Y = (Y, Y_1, Y_2, Y_3)$ with $Y_t = Y - Y_j e_j - (t-1)\sigma_p e_p$ for $t \geq 1$. Its layers are $r, r-1, r-2, r-3$, and

$$|X_s - Y| = (1+\ell_s) + (1+c_s) \geq 2, \qquad |X_s - Y_t| = (1+\ell_s) + 1 + |t-2-c_s| \geq 2 \quad (t = 1, 2, 3).$$

In both sub-cases $k = 3$, (E2) holds, and (E1) follows from the lists of layers. (E3) holds in coordinate i: the end values are 2τ and 0, and $s = \tau$ satisfies $|s| = 1 \leq (r-3) - 5$.

Type I and Type II are complementary, Type II is split according to whether a good coordinate exists, and Type I into the three cases above. So the five cases are exhaustive, and Theorem 3.2 is proved. □

3.3. The two-port theorem

Corollary 3.3 (two-port theorem). *Let $d \geq 2$, $m \geq 1$, $r \geq \max(2d+1, m+13, 20)$ and $c \in \mathbb{Z}^d$, and let $a \neq g$ be sites of $S_r(c)$ that do not form a corner pair. Then there are sets $R, G \subseteq B_r(c)$ satisfying (D1)–(D4) of Definition 2.9 for $D = B_r(c)$. In particular (a, g) has an interior design in $B_r(c)$.*

Proof. We may take $c = 0$. Theorem 3.2 gives k, γ_a and γ_g. Since $r - k \geq r - 3 \geq m + 10$, Lemma 3.1 applies to the separated pair (a^*, g^*) in S_{r-k}. It gives a walk λ_a from a^* and a walk λ_g from g^* that end at the two terminals $\pm m e_1$. Let R be the set of sites of γ_a and λ_a, and G the set of sites of γ_g and λ_g.

(D1) is clear. (D2): the sites of γ_a, γ_g other than a, g lie in layers at most $r-1$ by (E1), and the sites of λ_a, λ_g other than a^*, g^* lie in B_{r-k-1}. (D3): the sites of γ_a and γ_g lie in layers at least $r - 3 \geq m + 10$, and the sites of λ_a, λ_g other than $\pm m e_1$ lie outside B_m. (D4): a contact between a red and a green site would be one of three kinds. A contact between γ_a and γ_g is excluded by (E2). A contact between λ_a and λ_g, including a^* and g^*, is excluded by Lemma 3.1. The remaining kind is between a site of $\gamma_a \setminus \{a^*\}$ and a site of $\lambda_g \setminus \{g^*\}$, or the same with the colours exchanged. The first lies in a layer at least $r - k + 1$ by (E1), the second in a layer at most $r - k - 1$, so they are neither equal nor adjacent. □

Remark 3.4.

1. The constants are not optimal: the hypothesis $r \geq 2d + 1$ is used only in Case II-B, where $r \geq d + 1$ suffices, and Lemma 3.1 holds for $r \geq m + 6$.
2. Corner pairs are the only exceptions, and they admit no interior design (Definition 2.11). For $d = 3$ they are exactly the pairs excluded in [BBR, Lemma 9], a corner of S_r and a site of S_r at distance 2 from it. The difficulty of [BBR, §5] for $d = 3$, and of Conjecture 7 for $d \geq 4$, is the case where the first sites of the two halves of the path in the fixed ball $B_r(v)$, that is, the ports of $B_r(v)$, form a corner pair. Section 4 shows instead that some other admissible ball avoids it.

4. Proof of the Corner Lemma

4.1. Ordered sets and corner-paired balls

Let $V \subseteq \mathbb{Z}^d$ and let $\prec$ be a strict linear order on V. For a ball B that meets V, let $a(B)$ and $g(B)$ be the $\prec$-first and $\prec$-last sites of $V \cap B$, and call

$$W(B) = \{z \in V : a(B) \preceq z \preceq g(B)\}$$

the *window* of B. It contains $V \cap B$, and it is an interval of $(V, \prec)$. A ball $B = B_r(c)$ with $r \geq 1$ is *corner-paired* if $|V \cap B| \geq 2$ and $\{a(B), g(B)\}$ is a corner pair of $S_r(c)$. For $r \geq 2$ its pole $\sigma_r(c)$ and its apex $c + r\sigma_r(c)$ are then well defined (Definition 2.11). Fix an integer $\rho_0 \geq 2$. We consider two properties of $(V, \prec)$:

- **(Deg)** every site of V has exactly two neighbours in V;
- **(Cor)** every ball $B_\rho(c)$ with $\rho \geq \rho_0$ and $|V \cap B_\rho(c)| \geq 3$ is corner-paired.

If P is a path as in CL(d), ordered from one end to the other, and the conclusion of CL(d) fails for P, then (Cor) holds for the balls with $B_{\rho+1}(c) \subseteq B_K(v)$. Moreover each site of $P \cap B_K(v)$ has exactly two neighbours in P, namely its predecessor and its successor, because P is induced and its ends lie outside $B_K(v)$. The proof of CL(d) shows that these two properties cannot hold on all of $\mathbb{Z}^d$ in a stationary way (Theorem 4.7).

4.2. The stationary limit

Let $\Xi = \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ with the product topology, a compact metrizable space. We encode a pair $(V, \prec)$ by the indicator functions of V and of $\{(x, y) : x \prec y\}$. For $z \in \mathbb{Z}^d$ let T_z be the translation by z: $T_z(V, \prec) = (V + z, \prec_z)$, where $x \prec_z y$ if and only if $x - z \prec y - z$. Let $\mathcal{X}_{\rho_0} \subseteq \Xi$ be the set of pairs for which $\prec$ is a strict linear order on V and (Deg) and (Cor) hold.

Each condition in this definition involves finitely many coordinates. The order axioms ($x \prec y$ only for $x, y \in V$; irreflexivity; transitivity; totality on V) involve at most three sites at a time. (Deg) at x involves $V \cap N[x]$. (Cor) at (c, ρ) can be stated without reference to first sites: if $|V \cap B_\rho(c)| \geq 3$, there are $x_-, x_+ \in V$ forming a corner pair of $S_\rho(c)$ with $x_- \preceq z \preceq x_+$ for all $z \in V \cap B_\rho(c)$. This involves only the coordinates indexed by $B_\rho(c)$ and $B_\rho(c)^2$. Each condition therefore defines a clopen set, and $\mathcal{X}_{\rho_0}$ is closed and translation-invariant.

Proposition 4.1. *Let $d \geq 2$. If CL(d) fails, then for some $\rho_0 \geq 2$ there is a translation-invariant Borel probability measure ν on Ξ with $\nu(\mathcal{X}_{\rho_0}) = 1$, under which V is almost surely infinite.*

Proof. Suppose that CL(d) fails for ρ_0. As CL(d) is invariant under translations, for every K there is an induced path P_K through 0, with both ends outside B_K and order $\prec_K$ from one end to the other, such that every ball $B_\rho(c)$ with $\rho \geq \rho_0$, $B_{\rho+1}(c) \subseteq B_K$ and $|P_K \cap B_\rho(c)| \geq 3$ is corner-paired for $(P_K, \prec_K)$.

Compactness. For each R the restriction of $(P_K, \prec_K)$ to B_R takes finitely many values. By a diagonal argument there is a sequence $K_j \to \infty$ along which the restriction to B_R is eventually constant, for every R. The eventual restrictions are consistent, and their union is a pair $(V_0, \prec_0)$ with $0 \in V_0$. Each order axiom involves at most three sites, all inside some B_R, so $\prec_0$ is a strict linear order on V_0.

(Deg) and (Cor) for $(V_0, \prec_0)$. Let $x \in V_0$. For large j we have $|x|+1 \le K_j$ and $P_{K_j} \cap B_{|x|+1} = V_0 \cap B_{|x|+1}$. Since $x \in B_{K_j}$ is not an end of P_{K_j}, it has a predecessor and a successor, and as P_{K_j} is induced these are its only neighbours in P_{K_j}. Hence x has exactly two neighbours in V_0. Next fix c and $\rho \ge \rho_0$ with $|V_0 \cap B_\rho(c)| \ge 3$. For large j we have $B_{\rho+1}(c) \subseteq B_{K_j}$, and the restrictions of $(P_{K_j}, \prec_{K_j})$ and $(V_0, \prec_0)$ to $B_\rho(c)$ coincide. By the choice of P_{K_j} the ball is corner-paired for P_{K_j}, hence for $(V_0, \prec_0)$. So $(V_0, \prec_0) \in \mathcal{X}_{\rho_0}$, and so are all its translates.

Averaging. Let $\mu_n = |B_n|^{-1} \sum_{x \in B_n} \delta_{T_{-x}(V_0, \prec_0)}$, and let μ be a weak limit point of (μ_n), along a subsequence (n_k). This is the averaging argument of Krylov and Bogolyubov [KB], with ℓ^1 balls as averaging sets. For continuous f on Ξ and $e \in U$ we have

$$\Big| \int f \circ T_e \, d\mu_n - \int f \, d\mu_n \Big| \le \|f\|_\infty \, \frac{|B_n \triangle (B_n - e)|}{|B_n|} \longrightarrow 0,$$

since $|B_n \triangle (B_n - e)| = O(n^{d-1})$. Hence μ is translation-invariant. As $\mathcal{X}_{\rho_0}$ is closed and $\mu_n(\mathcal{X}_{\rho_0}) = 1$ for all n, the portmanteau theorem gives $\mu(\mathcal{X}_{\rho_0}) = 1$.

Density. The event $\{0 \in V\}$ is clopen, and $\mu_n(0 \in V) = |V_0 \cap B_n|/|B_n|$. Hence $\mu(0 \in V) = \lim_k |V_0 \cap B_{n_k}|/|B_{n_k}|$. We bound this from below using (Deg) and (Cor) at the site $0 \in V_0$. Let $\rho \ge \rho_0$ and $|c| \le \rho - 1$. Then $B_\rho(c)$ contains 0 and the two neighbours of 0 in V_0, so it is corner-paired by (Cor). Its apex $c + \rho\sigma_\rho(c)$ is a site of V_0, and it lies in $B_{2\rho-1}$. The map $c \mapsto (c + \rho\sigma_\rho(c), \sigma_\rho(c))$ is injective, and there are $2d$ possible poles, so $|V_0 \cap B_{2\rho-1}| \ge |B_{\rho-1}|/(2d)$. For $n \ge 2\rho_0 - 1$ put $\rho = \lfloor (n+1)/2 \rfloor \ge \rho_0$. Then $2\rho - 1 \le n$, and

$$\frac{|V_0 \cap B_n|}{|B_n|} \;\ge\; \frac{|B_{\rho-1}|}{2d\,|B_n|} \;\longrightarrow\; \frac{2^{-d}}{2d} \qquad (n \to \infty).$$

So $\mu(0 \in V) \ge 2^{-d}/(2d) > 0$, and in particular $\mu(V \neq \emptyset) > 0$.

Conditioning. Let $\nu = \mu(\,\cdot \mid V \neq \emptyset)$. The event $\{V \neq \emptyset\}$ is invariant, so ν is invariant and $\nu(\mathcal{X}_{\rho_0}) = 1$. On the event $\{0 < |V| < \infty\}$ let ℓ be the lexicographically least site of V. Then $\ell \circ T_z = \ell + z$, so by invariance $\nu(\ell = z) = \nu(\ell = 0)$ for all z. These numbers sum to $\nu(0 < |V| < \infty) \le 1$, so they all vanish, and V is ν-almost surely infinite. □

4.3. Mass transport

A *covariant partial map* assigns to each $\xi \in \Xi$ a map φ_ξ from a set $\operatorname{dom}\varphi_\xi \subseteq \mathbb{Z}^d$ to $\mathbb{Z}^d$, measurably in ξ, such that $\operatorname{dom}\varphi_{T_z\xi} = \operatorname{dom}\varphi_\xi + z$ and $\varphi_{T_z\xi}(x+z) = \varphi_\xi(x) + z$ for all z and x.

Lemma 4.2 (mass transport on $\mathbb{Z}^d$). *Let ν be a translation-invariant probability measure on Ξ.*

1. *If $F : \mathbb{Z}^d \times \mathbb{Z}^d \times \Xi \to [0,\infty]$ is measurable and $F(x+z, y+z, T_z\xi) = F(x,y,\xi)$ for all x, y, z and ξ, then $\mathbb{E}_\nu \sum_y F(0,y,\xi) = \mathbb{E}_\nu \sum_x F(x,0,\xi)$.*
2. *If φ is a covariant partial map and φ_ξ is injective for ν-almost every ξ, then $\nu\big(0 \in \varphi_\xi(\operatorname{dom}\varphi_\xi)\big) = \nu(0 \in \operatorname{dom}\varphi_\xi)$.*

Proof. (1) For each y, covariance with $z = -y$ and invariance give $\mathbb{E}\,F(0,y,\xi) = \mathbb{E}\,F(-y,0,T_{-y}\xi) = \mathbb{E}\,F(-y,0,\xi)$. Sum over y. (2) Apply (1) to $F(x,y,\xi) = \mathbf{1}[x \in \operatorname{dom}\varphi_\xi,\ \varphi_\xi(x) = y]$. The left side is $\nu(0 \in \operatorname{dom}\varphi_\xi)$. The right side is the expected number of preimages of 0, which is almost surely at most one and equals the indicator of $0 \in \varphi_\xi(\operatorname{dom}\varphi_\xi)$. □

Part (1) is the mass-transport principle of Benjamini, Lyons, Peres and Schramm [BLPS] (see also [LP, Chapter 8]) in its simplest case, the group $\mathbb{Z}^d$.

4.4. Shrinking towards the apex

The next lemma is deterministic and holds for every pair $(V, \prec)$. For $R \geq 2$ let $\mathcal{D}_R$ be the set of sites x such that $B_R(x)$ is corner-paired. For $R \geq r \geq 2$ and $x \in \mathcal{D}_R$ put

$$\Psi_{R\to r}(x) = x + (R - r)\,\sigma_R(x).$$

Since $\mathcal{D}_R$ and $\sigma_R(x)$ are determined by the restriction of $(V, \prec)$ to $B_R(x)$, the map $\Psi_{R\to r}$ is a covariant partial map with domain $\mathcal{D}_R$.

Lemma 4.3. *Let $R \geq r \geq 2$ and $x \in \mathcal{D}_R$, with pole σ, apex $C = x + R\sigma$ and partner $N = x + (R-1)\sigma + \delta$. Put $y = \Psi_{R\to r}(x)$.*

1. *$B_r(y) \subseteq B_R(x)$, and $B_r(y)$ contains C and N. Consequently $B_r(y)$ has the same first and last sites and the same window as $B_R(x)$, and it is corner-paired with pole σ and apex C.*
2. *$\Psi_{R\to r}$ is injective on $\mathcal{D}_R$.*

Proof. (1) If $|z - y| \leq r$ then $|z - x| \leq r + (R - r) = R$. The vectors $C - y = r\sigma$ and $N - y = (r-1)\sigma + \delta$ have norm r. The first site of $V \cap B_R(x)$ belongs to the smaller set $V \cap B_r(y)$, so it is also the first site of $V \cap B_r(y)$, and the same holds for the last site. Hence the windows coincide, and $\{C, N\} = \{y + r\sigma,\ y + (r-1)\sigma + \delta\}$ is a corner pair of $S_r(y)$ with pole σ.

(2) If $\Psi_{R\to r}(x) = y$, then by (1) the ball $B_r(y)$ is corner-paired with pole $\sigma_R(x)$. As $r \geq 2$, the pole of $B_r(y)$ is unique, so $\sigma_R(x) = \sigma_r(y)$ and $x = y - (R - r)\sigma_r(y)$ is determined by y. □

For $r = 1$ the pole is not unique and injectivity can fail, which is why all radii below are at least 2.

Proposition 4.4. *Let ν be a translation-invariant probability measure on Ξ with $\nu(\mathcal{X}_{\rho_0}) = 1$ under which V is almost surely infinite. Then ν-almost surely, for every $y \in \mathbb{Z}^d$ and every $r \geq 2$, the ball $B_r(y)$ is corner-paired, and for every $R \geq r$ the ball $B_R(y - (R-r)\sigma_r(y))$ is corner-paired with the same window and the same apex $y + r\sigma_r(y)$ as $B_r(y)$.*

Proof. Fix $r \geq 2$, and for $R \geq r$ let $\mathcal{E}_R$ be the event $\{0 \in \Psi_{R\to r}(\mathcal{D}_R)\}$.

First let $R \geq \max(r, \rho_0)$, and let A_R be the set of sites x with $|V \cap B_R(x)| \geq 3$. By (Cor), $A_R \subseteq \mathcal{D}_R$ almost surely. Since V is almost surely infinite, $\beta_R := \nu(0 \in A_R)$ tends to 1 as $R \to \infty$. By Lemma 4.3(2) and Lemma 4.2(2), $\nu(\mathcal{E}_R) = \nu(0 \in \mathcal{D}_R) \geq \beta_R$.

Next, $\mathcal{E}_{R'} \subseteq \mathcal{E}_R$ whenever $R' \geq R \geq r$. Indeed, let $0 = \Psi_{R'\to r}(x)$ with $x \in \mathcal{D}_{R'}$ and pole σ. By Lemma 4.3(1), the site $x' = \Psi_{R'\to R}(x) = x + (R' - R)\sigma$ lies in $\mathcal{D}_R$ with pole σ, and $\Psi_{R\to r}(x') = x + (R' - r)\sigma = 0$.

Hence $\nu(\mathcal{E}_R) \geq \nu(\mathcal{E}_{R'}) \geq \beta_{R'}$ for every $R' \geq \max(R, \rho_0)$, so $\nu(\mathcal{E}_R) = 1$ for every $R \geq r$, and the events $\mathcal{E}_R$, $R \geq r$, hold simultaneously almost surely. On this event, fix $R \geq r$ and $x \in \mathcal{D}_R$ with $\Psi_{R\to r}(x) = 0$. By Lemma 4.3(1), $B_r(0)$ is corner-paired with pole $\sigma_R(x)$, and $\sigma_R(x) = \sigma_r(0)$ by uniqueness of the pole. Thus $x = -(R - r)\sigma_r(0)$, and $B_R(x)$ has the same window as $B_r(0)$ and the apex $x + R\sigma_r(0) = r\sigma_r(0)$. This proves the statement for $y = 0$ and the fixed r. By translation invariance it holds for each y almost surely, and a countable intersection over y and r completes the proof. □

4.5. One pole per apex

Lemma 4.5. *Let $(V, \prec)$ be any pair, and let $B_1 = B_{r_1}(c_1)$ and $B_2 = B_{r_2}(c_2)$ be corner-paired balls with $r_1, r_2 \geq 2$, poles $\pi_1 \neq \pi_2$, apexes C_1, C_2 and partners $N_1 = C_1 - \pi_1 + \delta_1$, $N_2 = C_2 - \pi_2 + \delta_2$. Then neither of the following holds:*

- *(perpendicular) $\pi_1 \perp \pi_2$ and $C_2 = C_1 + t(\pi_2 - \pi_1)$ for an integer t with $2 \leq t \leq \min(r_1, r_2)$;*
- *(antipodal) $\pi_2 = -\pi_1 =: -\sigma$ and $C_1 = C_2 + t\sigma$ for an integer t with $2 \leq t \leq \min(r_1, r_2) + 1$.*

Proof. We first show that in either situation each ball contains both ports of the other.

In the perpendicular situation $c_1 = C_1 - r_1\pi_1$ and $c_2 = C_2 - r_2\pi_2$, so

$$C_2 - c_1 = (r_1 - t)\pi_1 + t\pi_2, \qquad N_2 - c_1 = (r_1 - t)\pi_1 + (t-1)\pi_2 + \delta_2,$$

$$C_1 - c_2 = t\pi_1 + (r_2 - t)\pi_2, \qquad N_1 - c_2 = (t-1)\pi_1 + (r_2 - t)\pi_2 + \delta_1.$$

The first and third vectors have norms r_1 and r_2. Since $\delta_2 \perp \pi_2$, either $\delta_2 = -\pi_1$, or $\delta_2 = \pi_1$, or δ_2 is orthogonal to both poles. The norm of $N_2 - c_1$ is then $|r_1 - t - 1| + t - 1$, r_1 or r_1, and in all cases at most r_1. The fourth vector is treated in the same way and has norm at most r_2.

In the antipodal situation $c_1 = C_1 - r_1\sigma$, $c_2 = C_2 + r_2\sigma$, $N_1 = C_1 - \sigma + \delta_1$ and $N_2 = C_2 + \sigma + \delta_2$, so

$$C_2 - c_1 = (r_1 - t)\sigma, \quad N_2 - c_1 = (r_1 - t + 1)\sigma + \delta_2, \quad C_1 - c_2 = (t - r_2)\sigma, \quad N_1 - c_2 = (t - 1 - r_2)\sigma + \delta_1.$$

Since $2 \leq t \leq \min(r_1, r_2) + 1$, their norms $|r_1 - t|$, $r_1 - t + 2$, $|t - r_2|$ and $r_2 - t + 2$ are at most r_1, r_1, r_2 and r_2.

So $C_2, N_2 \in B_1$ and $C_1, N_1 \in B_2$. The ports of B_2 are sites of $V \cap B_1 \subseteq W(B_1)$, and $W(B_1)$ is an interval of $(V, \prec)$, so $W(B_2) \subseteq W(B_1)$. By symmetry $W(B_1) = W(B_2)$, so the two balls have the same first and last sites, and $\{C_1, N_1\} = \{C_2, N_2\}$. Since $t \neq 0$ we have $C_1 \neq C_2$, hence $C_1 = N_2$ and $C_2 = N_1$. In the perpendicular situation, $C_2 - C_1 = N_1 - C_1 = -\pi_1 + \delta_1$ has π_1-coefficient -1, while $t(\pi_2 - \pi_1)$ has π_1-coefficient $-t$, so $t = 1$, a contradiction. In the antipodal situation, $C_1 - C_2 = N_2 - C_2 = \sigma + \delta_2$ would equal $t\sigma$, which is impossible because $\delta_2 \perp \sigma$. □

The hypothesis $t \geq 2$ cannot be dropped in the perpendicular situation. For $t = 1$ the two balls can have the same port pair $\{C_1, C_2\}$. For $t = 0$ a single site can be the apex for both poles, with the common port pair $\{C, C - \pi_1 - \pi_2\}$. So Lemma 4.5 alone does not exclude two poles at one apex, and the next lemma uses stationarity.

Lemma 4.6. *Let ν be as in Proposition 4.4. Then ν-almost surely no site is the apex of two corner-paired balls, of radii at least 2, with different poles.*

Proof. We work on the full-measure event of Proposition 4.4. Let C be the apex of a corner-paired ball $B_r(y)$ with pole π and $r \geq 2$. Then for every $r' \geq 2$ the ball $B_{r'}(C - r'\pi)$ is corner-paired with pole π and apex C: for $r' \geq r$ this is Proposition 4.4, since $y - (r' - r)\pi = C - r'\pi$, and for $2 \leq r' < r$ it is Lemma 4.3(1). The set $A(\pi)$ of apexes with pole π therefore does not depend on the radius, and it is measurable.

Fix poles $\pi_1 \neq \pi_2$. If $\pi_1 \perp \pi_2$ put $u = \pi_2 - \pi_1$, and if $\pi_2 = -\pi_1 =: -\sigma$ put $u = \sigma$. For a line $\ell = q + \mathbb{Z}u$ and $k = 1, 2$ let $J_k(\ell) = \{t \in \mathbb{Z} : q + tu \in A(\pi_k)\}$. We claim that for all $t_1 \in J_1(\ell)$ and $t_2 \in J_2(\ell)$,

$$t_2 \leq t_1 + 1 \ \text{ in the perpendicular case}, \qquad t_1 \leq t_2 + 1 \ \text{ in the antipodal case}.$$

In the perpendicular case, suppose $t = t_2 - t_1 \geq 2$ and put $C_k = q + t_k u$. Then $C_2 = C_1 + t(\pi_2 - \pi_1)$, and the balls $B_t(C_k - t\pi_k)$ are corner-paired with poles π_k and apexes C_k. This contradicts the perpendicular case of Lemma 4.5 with $r_1 = r_2 = t$. In the antipodal case, $t = t_1 - t_2 \geq 2$ gives $C_1 = C_2 + t\sigma$, and the balls of radius t contradict the antipodal case of Lemma 4.5. The constraint is one-sided, and on a single line it can be met, for instance by $J_1 = [-1, \infty)$ and $J_2 = (-\infty, 0]$ in the perpendicular case. Translation invariance excludes it.

We treat the perpendicular case. The antipodal case is the same with J_1 and J_2 exchanged. Let $\ell_0 = \mathbb{Z}u$, and let $\mathcal{E}$ be the event that $J_1(\ell_0)$ and $J_2(\ell_0)$ are both nonempty. On $\mathcal{E}$ the claim gives $\min J_1(\ell_0) \geq t_2 - 1$ for any $t_2 \in J_2(\ell_0)$, so $J^* = \min J_1(\ell_0)$ is a finite integer. By covariance $A(\pi_k)$ is shifted by u under T_u, so $J_k(\ell_0) \circ T_u = J_k(\ell_0) + 1$. Hence $\mathcal{E}$ is T_u-invariant and $J^* \circ T_u = J^* + 1$. By invariance of ν the numbers $\nu(\mathcal{E} \cap \{J^* = j\})$, $j \in \mathbb{Z}$, are all equal. They sum to $\nu(\mathcal{E}) \leq 1$, so they vanish and $\nu(\mathcal{E}) = 0$. By translation invariance, almost surely no line $q + \mathbb{Z}u$ has both J_1 and J_2 nonempty. A site in $A(\pi_1) \cap A(\pi_2)$ would lie on such a line, with $0 \in J_1 \cap J_2$ when q is that site. A union over the finitely many pairs (π_1, π_2) completes the proof. □

4.6. Counting, and the proof of the Corner Lemma

Theorem 4.7. *Let $d \geq 2$ and $\rho_0 \geq 2$. There is no translation-invariant probability measure ν on Ξ with $\nu(\mathcal{X}_{\rho_0}) = 1$ under which V is almost surely nonempty.*

Proof. Suppose that ν is such a measure. The last step of the proof of Proposition 4.1 shows that V is ν-almost surely infinite, so Proposition 4.4 and Lemma 4.6 apply. Fix $r \geq 2$. On the full-measure event of Proposition 4.4 and Lemma 4.6, every site c is the centre of a corner-paired ball $B_r(c)$, whose apex $\mathcal{C}(c) = c + r\sigma_r(c)$ is one of its two ports and hence a site of V. The map $\mathcal{C}$ is injective. Indeed, if $\mathcal{C}(c_1) = \mathcal{C}(c_2)$ with $c_1 \neq c_2$, then $\sigma_r(c_1) \neq \sigma_r(c_2)$, because $c_k = \mathcal{C}(c_k) - r\sigma_r(c_k)$, and the common apex would carry two poles, contrary to Lemma 4.6. Defined on $\mathcal{D}_r$ by the same formula, $\mathcal{C}$ is a covariant partial map, and almost surely $\mathcal{D}_r = \mathbb{Z}^d$ and $\mathcal{C}$ is injective. Lemma 4.2(2) then gives $\nu(0 \in \mathcal{C}(\mathcal{D}_r)) = \nu(0 \in \mathcal{D}_r) = 1$. As $\mathcal{C}(\mathcal{D}_r) \subseteq V$, we get $\nu(0 \in V) = 1$, and by translation invariance $V = \mathbb{Z}^d$ almost surely. Then every site of V has $2d \geq 4$ neighbours in V, contradicting (Deg). □

Theorem 4.8 (the Corner Lemma). *CL(d) holds for every $d \geq 2$.*

Proof. If CL(d) failed for some ρ_0, Proposition 4.1 would give a measure that Theorem 4.7 excludes. □

4.7. Proof of Theorem 1.1

Let $d \geq 2$ and let E_0 be an essential enhancement of site percolation on $\mathbb{Z}^d$. Let m be as in Lemma 2.2 and $\rho_0 = \max(2d+1, m+13, 20)$. By Theorem 4.8, CL($d$) holds. By Theorem 2.13, Claim 5 holds for E_0, with $R = 3K(d, \rho_0) + 100$ and $L_0 = 100R$. By Theorem 2.1, the conclusion of Conjecture 1 holds for E_0. □

The formal verification of this proof is described in Section 14.

5. Remarks on the proof for $\mathbb{Z}^d$

5.1. What the proof of Section 4 uses

The degree condition (Deg) enters the proof of the Corner Lemma (Theorem 4.8) in exactly two places: at the single site 0 in Proposition 4.1, to make the limit measure nontrivial (without it, $V_0 = \{0\}$ would satisfy (Cor) vacuously, and the averages would converge to the point mass at $V = \emptyset$), and in the last line of the proof of Theorem 4.7. Proposition 4.4, Lemma 4.6 and the injectivity of the apex map use only translation invariance, (Cor), and the infinitude of V. So Sections 4.4–4.6 prove the following: *if ν is translation-invariant, V is almost surely nonempty, and almost surely $\prec$ is a strict linear order on V satisfying (Cor), then $V = \mathbb{Z}^d$ almost surely.* The degree condition cannot be omitted from Theorem 4.7, as the following example shows.

Example 5.1 (the parity double sweep). Let $V = \mathbb{Z}^d$ and $A = \{z : z_2 + \cdots + z_d \text{ even}\}$. Let $\prec$ list A first, by decreasing z_1, and then the complement A^c, by increasing z_1. Within a hyperplane $\{z_1 = \text{const}\}$, ties are broken by the lexicographic order of $(z_2, \dots, z_d)$. Then every ball $B_r(c)$ with $r \geq 1$ is corner-paired with pole e_1 (for $r = 1$, e_1 is one of the two admissible poles). Indeed, $c + re_1$ is the only site of $B_r(c)$ with first coordinate $c_1 + r$, and the sites with first coordinate $c_1 + r - 1$ are $c + (r-1)e_1$ and the $2(d-1)$ sites $c + (r-1)e_1 \pm e_j$, $j \geq 2$, whose parity class is opposite to that of $c + (r-1)e_1$ and $c + re_1$. If $c_2 + \cdots + c_d$ is even, then $c + re_1 \in A$ is the first site, and the sites of A^c with the largest first coordinate are the $c + (r-1)e_1 \pm e_j$, so one of them is the last site. If $c_2 + \cdots + c_d$ is odd, these $2(d-1)$ sites are the sites of A with the largest first coordinate, so one of them is first, and $c + re_1 \in A^c$ is last. In both cases the ports form a corner pair with pole e_1.

The order is invariant under translations by vectors u with $u_2 + \cdots + u_d$ even. Translation by e_2 exchanges A and A^c, and gives a second pair $(V, \prec')$ with the same property. The average $\nu = \frac{1}{2}(\delta_{(V,\prec)} + \delta_{(V,\prec')})$ is translation-invariant, satisfies (Cor) for every $\rho_0 \geq 2$ (indeed at every radius $r \geq 1$), and satisfies the conclusions of Proposition 4.4 and Lemma 4.6 (every apex has pole e_1), as well as $V = \mathbb{Z}^d$. Only (Deg) fails, since every site has $2d$ neighbours in V.

The argument uses the closeness of the two ports of a corner pair. For patterns in which the first and last sites of a ball lie at opposite poles of its sphere, the analogue of Lemma 4.3 fails, because a ball shrunk towards one pole no longer contains the opposite port.

5.2. Constants

The constants $\rho_0 = \max(2d + 1, m + 13, 20)$, $R = 3K + 100$ and $L_0 = 100R$ are explicit in terms of m and $K = K(d, \rho_0)$, and m is explicit in terms of the gadget of [BBR, §4]. The proof of Section 4 obtains K from a compactness argument and gives no explicit value. Claim 5 only asks for constants that depend on E_0 and d, so this suffices for Theorem 1.1. The finitary counting argument of Section 8, which is of the kind used in the formal proof of the Corner Lemma (Section 14), gives an explicit but very large K: by Corollary 8.15, for $d = 2$ and $\rho_0 \leq 55\,489$ one may take $K = 120\,799\,558$. A suitable K is also computable in principle. Call a pair $(V, \prec)$ with $V \subseteq B_K$ and $0 \in V$ a *K-window* if (Deg) holds at every site of V whose neighbours all lie in B_K, and (Cor) holds for every ball $B_\rho(c)$ with $\rho \geq \rho_0$ and $B_{\rho+1}(c) \subseteq B_K$. If CL($d$) failed at K for a path P through $v = 0$, then $P \cap B_K$ with its path order would be a K-window. Conversely, if K-windows existed for every K, a diagonal limit would be a pair on $\mathbb{Z}^d$ with $0 \in V$, (Deg) and (Cor), and the averaging step of Proposition 4.1 would produce a measure excluded by Theorem 4.7. So K-windows cease to exist for some finite K, and

any such K works in CL(d). For each K the existence of a K-window is a finite question. The values relevant here, with $\rho_0 \geq 20$, are far beyond exhaustive search, and the cross-checks in Appendix A only treat $d = 2$ and $\rho_0 \leq 4$.

6. Periodic lattices, enhancements and the probabilistic step

6.1. Periodic lattices

Definition 6.1. A *periodic lattice* is a simple graph G on the vertex set $V = \mathbb{Z}^n \times \{0, \dots, k-1\}$, with $n \geq 2$ and $k \geq 1$, whose adjacency relation $\sim$ is symmetric, irreflexive, invariant under the translations $\tau_z(x, i) = (x + z, i)$ ($z \in \mathbb{Z}^n$), locally finite and connected.

The elements of V are called *sites*. The *root* is $o = (0, 0)$, the *orbit* of (x, i) is $\mathbb{Z}^n \times \{i\}$, and $x_i = (0, i)$ is the representative of the orbit i. Translations act on sets by $\tau_z A = \{\tau_z v : v \in A\}$. Each τ_z is a graph automorphism. By translation invariance and local finiteness the degree of (z, i) equals that of x_i, so the maximum degree Δ and the minimum degree δ are finite.

Notation. $d(x, y)$ is the graph distance, $|x| = d(o, x)$, $B_r(c) = \{x : d(x, c) \leq r\}$ and $S_r(c) = \{x : d(x, c) = r\}$, with $B_r = B_r(o)$ and $S_r = S_r(o)$. From this section on, all balls, spheres and distances are those of the graph. A site $x \in S_r(c)$ is a *pocket* of $S_r(c)$ if it has no neighbour in $S_{r+1}(c)$. For $A \subseteq V$, $N[A]$ is the set of sites that lie in A or are adjacent to a site of A, and $\operatorname{int} A$ is the set of sites of A all of whose neighbours lie in A. Two sets A, B *touch* if some site of A equals or is adjacent to some site of B. A *walk* is a finite sequence of sites in which consecutive sites are adjacent. A *path* is a walk with distinct sites, and it is *induced* if no two non-consecutive sites are adjacent. A set A *contains a path* from x to y if some path from x to y has all its sites in A, and A is *connected* if it contains a path between any two of its sites. The *components* of A are its maximal connected subsets. A *doubly infinite path* is an injective map $f : \mathbb{Z} \to V$ with $f(j) \sim f(j + 1)$ for all j.

Two features of $\mathbb{Z}^d$ are lost in general. Along an edge the distance to a fixed site changes by *at most* one, and by exactly one only on bipartite lattices. And a sphere may have pockets.

The lattices of Sections 9–11 (canonical coordinates, $n = 2$, $e_1 = (1, 0)$, $e_2 = (0, 1)$; the coordinates of the lattices of Section 12 are given in the subsections of Section 12).

- *The triangular lattice* TRI: $k = 1$, and $(z, 0) \sim (z + e, 0)$ for $e \in \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$. Degree 6.
- *The honeycomb lattice* HEX: $k = 2$. We write $A(z) = (z, 0)$ and $B(z) = (z, 1)$. The edges are $A(z) \sim B(w)$ for $z - w \in \{0, e_1, e_2\}$. Degree 3; HEX is bipartite.
- *The Kagomé lattice* KAG: $k = 3$. We write $a(z) = (z, 0)$, $b(z) = (z, 1)$, $c(z) = (z, 2)$. The edges are the sides of the *up triangles* $\{a(z), b(z), c(z)\}$ and of the *down triangles* $\{a(z), b(z - e_1), c(z - e_2)\}$, $z \in \mathbb{Z}^2$. Degree 4.

All three are periodic lattices, and all three are vertex-transitive (Lemma 9.2, Section 10.1 and, for the triangular lattice, $k = 1$). For a periodic lattice put

$$\kappa(G) = \max_{c \in V} \max_{0 \leq i < k} d\big(c, \mathbb{Z}^n \times \{i\}\big),$$

the largest distance from a site to an orbit. It is finite, because $d(c, \mathbb{Z}^n \times \{i\})$ depends only on the

orbit of c. Every site of HEX is adjacent to a site of the other orbit, and every site of KAG is adjacent to sites of both other orbits (it lies in an up triangle), so $\kappa(\mathrm{TRI}) = 0$ and $\kappa(\mathrm{HEX}) = \kappa(\mathrm{KAG}) = 1$.

6.2. Enhancements, crossings and pivotal sites

Definition 6.2. An *enhancement* of site percolation on G is a family $E = (E_i)_{0 \le i < k}$ of maps $E_i : \mathcal{P}(V) \to \mathcal{P}(V)$ with a common *range* $r_E \in \mathbb{N}$: for every i, $E_i(\omega)$ depends only on $\omega \cap B_{r_E}(x_i)$, and $E_i(\omega) \subseteq B_{r_E}(x_i)$. For a site $v = (z, i)$ put $E_v(\omega) = \tau_z E_i(\tau_{-z}\omega)$. The *enhanced configuration* is

$$E(\omega, \alpha) = \omega \cup \bigcup_{v \in \alpha} E_v(\omega) \qquad (\omega, \alpha \subseteq V).$$

The enhancement E is *essential* if for some i there is a configuration ω that contains no doubly infinite path such that $\omega \cup E_i(\omega)$ contains one. Such an i is an *essential orbit*.

The maps E_i need not be monotone. As in [BBR], enhancements are covariant under the full translation group $\mathbb{Z}^n$ of G. For $k = 1$ and $G = \mathbb{Z}^d$ this is the definition of Section 1.1, with $E_v(\omega) = E_0(\omega - v) + v$, since the graph balls of $\mathbb{Z}^d$ are its ℓ^1 balls.

For $A \subseteq V$ and $L \in \mathbb{N}$ let $R_L(A)$ be the statement that A contains a path from o to a site of S_L. A *state* is a pair (ω, α) of subsets of V. In the state (ω, α) a site v is *p-pivotal* (for R_L) if $R_L(E(\omega \cup \{v\}, \alpha))$ holds and $R_L(E(\omega \setminus \{v\}, \alpha))$ does not. It is *s-pivotal* if $R_L(E(\omega, \alpha \cup \{v\}))$ holds and $R_L(E(\omega, \alpha \setminus \{v\}))$ does not. Two states agree outside a set D if both coordinates agree outside D. Claims 4 and 5 of Section 2.1, the deterministic statements of [BBR, §2], read as follows on G.

Claim 4 (on G). *There are $L_0, R \in \mathbb{N}$ such that for all $L \ge L_0$, all states (ω, α) and all sites v that are p-pivotal for R_L in (ω, α), there are a state (ω', α') that agrees with (ω, α) outside $B_R(v)$ and a site $w \in B_R(v)$ that is s-pivotal for R_L in (ω', α').*

Claim 5 (on G). *The same statement, under the additional hypothesis $\alpha \cap B_R(v) = \emptyset$.*

6.3. Percolation and Conjecture 1 on a periodic lattice

For $p, s \in [0, 1]$ let ω and α be independent random subsets of V, containing each site independently with probability p and s respectively. Put

$$\theta_E(p, s) = \mathbb{P}_{p,s}\big(o \text{ lies in an infinite component of } E(\omega, \alpha)\big), \qquad \theta(p) = \mathbb{P}_p\big(o \text{ lies in an infinite component of } \omega\big),$$

and $p_c(G) = \sup\{p : \theta(p) = 0\}$. We say that *Conjecture 1 holds for E* if for every $s \in (0, 1]$ there is $\pi(s) < p_c(G)$ such that $\theta_E(p, s) > 0$ for all $p \in (\pi(s), p_c(G))$. Since $E(\omega, \alpha)$ is increasing in α, $\theta_E(p, \cdot)$ is non-decreasing, so the case $s = 1$ follows from $s = \frac{1}{2}$.

Remark 6.3. The definition of $p_c(G)$ does not depend on the root: by finite energy, $\theta_x(p) > 0$ for one site x if and only if it holds for every site. For a single non-monotone enhancement the positivity of θ_E can depend on the root on general periodic lattices, so Conjecture 1 is stated at the root o.

Remark 6.4 (all roots). Let ϕ be an automorphism of G with $\phi \circ \tau_y = \tau_{Ay} \circ \phi$ for some $A \in GL(n, \mathbb{Z})$ and all y, and put $E^\phi_v(\omega) = \phi\big(E_{\phi^{-1}v}(\phi^{-1}\omega)\big)$. Then E^ϕ is an enhancement with range r_E, it is essential if and only if E is, and $E^\phi(\phi\omega, \phi\alpha) = \phi E(\omega, \alpha)$, so the probability that $\phi(x)$ lies in an infinite component of $E^\phi(\omega, \alpha)$ equals that for x and $E(\omega, \alpha)$. If such automorphisms act transitively on V, then Conjecture 1 for all essential enhancements at the root o gives the same conclusion at

every root. This applies to TRI (translations), to HEX ($\phi(z,i) = (-z, 1-i)$ with the translations) and to KAG (the automorphism $a(z) \mapsto b(Mz)$, $b(z) \mapsto c(Mz)$, $c(z) \mapsto a(Mz)$, where $Me_1 = -e_1 + e_2$ and $Me_2 = -e_1$, with the translations). The verification is routine and is given in Appendix B.7. The same applies to the lattices of Section 12: on those with $k = 1$ the translations act transitively, and on the others the translations together with one further automorphism ϕ with $\phi \circ \tau_y = \tau_{Ay} \circ \phi$ act transitively (Sections 12.4–12.7).

6.4. The probabilistic step

Theorem 6.5 (the probabilistic step; [AG, §3], in the form of [BBR, pp. 4–6]). *Let G be a periodic lattice with $p_c(G) < 1$, and let E be an enhancement of site percolation on G. If Claim 5 holds for E, then Conjecture 1 holds for E.*

The proof follows the four steps recalled in Section 2.2, with $N_p(L)$ and $N_s(L)$ as there. Claim 5 implies Claim 4 because the enhanced configuration is increasing in α [BBR, pp. 5–6]; Claim 4 and a finite-to-one modification map give a continuous positive function g on $(0,1)^2$ with $\mathbb{E}_{p,s} N_s(L) \geq g(p,s)\, \mathbb{E}_{p,s} N_p(L)$ for $L \geq L_0$ [BBR, (2)]; a Margulis–Russo formula for the polynomial $\tau_L(p,s) = \mathbb{P}_{p,s}(R_L(E(\omega,\alpha)))$ turns this into $\partial_s \tau_L \geq g\, \partial_p \tau_L$ [BBR, (3)]; and comparing τ_L along curves independent of L, as $L \to \infty$, gives the conclusion [AG, pp. 822–823]. On a periodic lattice the only new points are a bound $M_R = \max_c |B_R(c)| \leq (\Delta+1)^R$ on the size of balls that is uniform over all sites, which makes g independent of L, and the locality of enhancements and crossing events for graph balls. The differential inequality on graphs of bounded degree is also proved, for the enhancement used there, in [MS, §6]. A complete proof is given in Appendix B, with the explicit value

$$\pi(s) = p_c - \min\Big(\frac{p_c}{2},\ \frac{g_0 s_0}{2},\ \frac{1-p_c}{2}\Big), \qquad s_0 = \min\big(s, \tfrac{1}{2}\big), \qquad g_0 = \frac{1}{M_R 4^{M_R}} \Big(\frac{\min(p_c, 1-p_c)\, s_0}{4}\Big)^{M_R},$$

where $p_c = p_c(G)$ and R is the radius in Claim 5.

6.5. Critical points

Proposition 6.6. *Let G be one of the lattices of Theorem 1.2. Then $0 < p_c(G) < 1$. More precisely, $p_c(G) \geq 1/(\Delta-1)$ and $p_c(G) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/b}$, with the maximum degree Δ and the integer b of the following table.*

G	**TRI**	**HEX**	**KAG**	**KING**	**ELTRI**	**SNUBSQ**	**SNUBHEX**	**RTH**	**FCC**
Δ	6	3	4	8	5	5	5	4	12
$1/(\Delta-1)$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{11}$
b	1	2	3	1	2	4	6	6	1

(The abbreviations of the lattices of Section 12 are those of Sections 12.4–12.9.) Here $p_c^{\mathrm{site}}(\mathbb{Z}^2) < 1$ is classical (see [G, §1.4], [BR, Ch. 1]; for instance $p_c^{\mathrm{site}}(\mathbb{Z}^2) \leq \frac{127}{128}$ by a Peierls argument). The proposition follows from the next two lemmas.

Lemma 6.7 (path counting). *For every periodic lattice G and every $p < 1/(\Delta-1)$ we have $\theta(p) = 0$. Hence $p_c(G) \geq 1/(\Delta-1)$.*

Proof. We have $\Delta \geq 2$, because G is infinite and connected. Let $\ell \geq 1$. If o lies in an infinite component of ω, then ω contains a path from o to a site of S_ℓ. Such a path has at least $\ell+1$ sites, because $|\cdot|$ changes by at most one along an edge, and its first $\ell+1$ sites form a path in ω starting at o. There are at most $\Delta(\Delta-1)^{\ell-1}$ paths $(y_0,\dots,y_\ell)$ with $y_0=o$, and each lies in ω with probability $p^{\ell+1}$. Hence $\theta(p) \leq \Delta p^2\big(p(\Delta-1)\big)^{\ell-1}$ for every ℓ, which tends to 0 when $p(\Delta-1)<1$. □

Lemma 6.8 (the minor lemma). *Let G be a periodic lattice and $b \geq 1$, and suppose that there are pairwise disjoint finite connected sets $\beta(z) \subseteq V$ ($z \in \mathbb{Z}^2$) with $|\beta(z)| \leq b$ and $o \in \beta(0)$, such that for every z and $j \in \{1,2\}$ some site of $\beta(z)$ is adjacent to some site of $\beta(z+e_j)$. Then $\theta_G(p) \geq \theta_{\mathbb{Z}^2}(p^b)$ for every $p \in [0,1]$, where $\theta_{\mathbb{Z}^2}$ refers to site percolation on the square lattice. Consequently $p_c(G) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/b}$.*

Proof. Let $p \in (0,1]$ (the case $p=0$ is trivial). Let $(U_z)_{z\in\mathbb{Z}^2}$ be independent uniform random variables on $[0,1]$, independent of ω, and put $\xi' = \{z : \beta(z) \subseteq \omega,\ U_z \leq p^{\,b-|\beta(z)|}\}$. The sets $\beta(z)$ are disjoint, so the events $\{z \in \xi'\}$ are independent, each of probability $p^{|\beta(z)|}p^{b-|\beta(z)|} = p^b$. If 0 lies in an infinite component C' of ξ', then $U = \bigcup_{z\in C'} \beta(z)$ is contained in ω, contains o, is infinite, and is connected: consecutive sites z_l, z_{l+1} of a path in C' give adjacent sites of $\beta(z_l)$ and $\beta(z_{l+1})$, and each $\beta(z_l)$ is connected. So o lies in an infinite component of ω. If $q > p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/b}$, then $\theta_{\mathbb{Z}^2}(q^b) > 0$, hence $\theta_G(q) > 0$. □

Proof of Proposition 6.6. The lower bounds are Lemma 6.7. For the upper bounds we apply Lemma 6.8 with the following branch sets. For TRI take $\beta(z) = \{(z,0)\}$ (adjacent to $\beta(z+e_j)$ since e_j is a lattice step), for HEX $\beta(z) = \{A(z), B(z)\}$ (connected as $A(z) \sim B(z)$; $B(z) \sim A(z+e_j)$), and for KAG $\beta(z) = \{a(z), b(z), c(z)\}$ (an up triangle; $b(z) \sim a(z+e_1)$ and $c(z) \sim a(z+e_2)$ by the down triangles at $z+e_1$ and $z+e_2$). For KING take $\beta(z) = \{(z,0)\}$ ($\mathbb{Z}^2$ is a subgraph), and for FCC $\beta(z) = \{((z_1,z_2,0),0)\}$ ($\pm e_1, \pm e_2$ are lattice steps). For ELTRI, SNUBSQ, SNUBHEX and RTH take for $\beta(z)$ the k sites of the cell z, $k = 2,4,6,6$. By the neighbour tables of Sections 12.4–12.7 they span a connected subgraph: an edge for ELTRI, a 4-cycle $(z,0),(z,1),(z,2),(z,3)$ for SNUBSQ and a 6-cycle $(z,0),\dots,(z,5)$ for SNUBHEX and RTH. Some site of $\beta(z)$ is adjacent to some site of $\beta(z+e_j)$: for ELTRI $((z),0) \sim ((z+e_1),0)$ and $((z),1) \sim ((z+e_2),0)$; for SNUBSQ $(z,0) \sim (z+e_1,1)$ and $(z,3) \sim (z+e_2,0)$; for SNUBHEX $(z,0) \sim (z+e_1,2)$ and $(z,0) \sim (z+e_2,4)$; for RTH $(z,0) \sim (z+e_1,2)$ and $(z,0) \sim (z+e_2,4)$. Lemma 6.8 applies with $b = |\beta(z)|$. □

The bounds are far from the true values, but only the strict inequalities matter here.

7. The reduction on periodic lattices

Throughout this section G is a periodic lattice and E is an essential enhancement of site percolation on G with range r_E and essential orbit i. Red and green sets are denoted R and G, to avoid a clash with the lattice G and with the radius R of Claim 5. We prove Theorem A below by going through the proof of Theorem 2.13 (Sections 2.3–2.7) and listing what changes; the steps that change in substance (the gadget, the ports, the main case and the two boundary cases) are proved in full.

7.1. Elementary facts

The following facts hold on every periodic lattice. They are the only properties of balls used in this section.

- **(F1)** *Balls are finite and connected,* $B_{r+1}(c) = N[B_r(c)]$*, and* $S_r(c) \neq \emptyset$ *for all* c *and* r*.* A site at distance $r+1$ from c is adjacent to the second-to-last site of a geodesic, which lies at distance r; finiteness follows by induction from local finiteness. A geodesic from c to a site of $B_r(c)$ stays in $B_r(c)$. Since V is infinite, some site has distance at least r from c, and a geodesic to it meets $S_r(c)$.
- **(F2)** *If* $x \sim y$*, then* $|d(x,c) - d(y,c)| \leq 1$ *for every site* c*.*
- **(F3)** $N[B_\rho(c)] \subseteq B_{\rho+1}(c)$ *and* $B_{\rho-1}(c) \subseteq \operatorname{int} B_\rho(c)$ *for* $\rho \geq 1$*.*
- **(F4)** *Each* τ_z *is an automorphism of* G*.* Hence it preserves distances, balls, spheres, paths, induced paths, connected sets, pockets and escapable pairs (Definition 7.1).
- **(F5)** *If* $d(x,c) = r \geq 1$*, then* x *has a neighbour in* $S_{r-1}(c)$*.*
- **(F6)** (discrete intermediate values) *Let* $(y_0, \ldots, y_q)$ *be a walk and* s *an integer with* $d(y_0,c) \leq s \leq d(y_q,c)$*. Then the least* j *with* $d(y_j,c) \geq s$ *satisfies* $d(y_j,c) = s$*.* In particular, for $0 \leq n \leq L$ every path from o to a site of S_L meets S_n.
- **(F7)** *Every walk from* x *to* y *contains a path from* x *to* y *all of whose sites are sites of the walk.*
- **(F8)** (locality) $E_v(\omega) \subseteq B_{r_E}(v)$*, and* $E_v(\omega) = E_v(\tilde{\omega})$ *whenever* $\omega \cap B_{r_E}(v) = \tilde{\omega} \cap B_{r_E}(v)$*.* This follows from Definition 6.2 and (F4).
- **(F9)** *For every site* c *and every orbit* j *there is a site* c' *of the orbit* j *with* $d(c,c') \leq \kappa(G)$*.*
- **(F10)** R_L *is increasing, and* $\alpha \mapsto E(\omega,\alpha)$ *is increasing.* (The map $\omega \mapsto E(\omega,\alpha)$ need not be.)

7.2. The lattice hypotheses and Theorem A

Definition 7.1 (escapable pair). Let $c' \in V$ and $m \geq 1$. An ordered pair (t_+, t_-) of distinct sites of $S_m(c')$ is *escapable* if there are $u_+, u_- \in S_{m+1}(c')$ with $u_+ \sim t_+$ and $u_- \sim t_-$ such that $\{t_+, u_+\}$ and $\{t_-, u_-\}$ do not touch.

Definition 7.2 (design). Let $c, c' \in V$ and $\rho, m \in \mathbb{N}$ with $B_m(c') \subseteq B_{\rho-1}(c)$, let $a \neq g$ be sites of $S_\rho(c)$, and let (t_+, t_-) be a pair of sites of $S_m(c')$. A *design* for $(a, g; t_+, t_-)$ in $B_\rho(c)$ is a pair of sets $\mathsf{R}, \mathsf{G} \subseteq B_\rho(c)$ such that:

- (D1) $a \in \mathsf{R}$, $g \in \mathsf{G}$, and R and G are connected;
- (D2) $\mathsf{R} \setminus \{a\} \subseteq B_{\rho-1}(c)$ and $\mathsf{G} \setminus \{g\} \subseteq B_{\rho-1}(c)$;
- (D3) $\mathsf{R} \cap B_m(c') = \{t_+\}$ and $\mathsf{G} \cap B_m(c') = \{t_-\}$, or $\mathsf{R} \cap B_m(c') = \{t_-\}$ and $\mathsf{G} \cap B_m(c') = \{t_+\}$;
- (D4) R and G do not touch.

An *exceptional family* is a family $X = (X_\rho(c))_{c \in V, \rho \geq 2}$, where $X_\rho(c)$ is a set of unordered pairs of distinct sites of $S_\rho(c)$. We consider the following properties of G. All constants are integers.

- **(B1)** (gadget radii) There is an infinite set $M(G)$ of radii such that for every $m \in M(G)$ and every orbit representative x_j, the sphere $S_m(x_j)$ has no pockets and every pair of distinct non-adjacent sites of $S_m(x_j)$ is escapable.
- **(B2)** (inner boundary) There are $n_0, t_0, c_2 \geq 1$ and an infinite set $M'(G) \subseteq \mathbb{N}$ whose consecutive elements differ by at most c_2, such that for every $n \in M'(G)$ with $n \geq n_0$: S_n has no pockets, the annulus $S_{n+1} \cup \cdots \cup S_{n+t_0}$ is connected, and every site v with $|v| \leq n$ satisfies $d(v, S_n) \leq n - |v| + c_2$.
- **(B3)** (outer boundary) There are $0 \leq c_3 < c_4$ and $K_3 \geq 1$ such that for all $K \geq K_3$, all $L \geq 4K + 4c_4$ and every site v with $L - K - 10 \leq |v| \leq L$, there is a site w with $K + c_4 < |w| <$

$L-K-10$ and $d(v,w) \le K+c_3$, such that for every site u with $d(u,w) = K+c_4$ and $|u| < L$ there is an induced path $P' = (u = y_0, \dots, y_q)$ with $y_q \in S_L$, $|y_j| < L$ for $j < q$, $w \in P' \setminus \{u, y_q\}$ and $P' \setminus \{u\} \subseteq B_{K+c_4-1}(w)$.

- **TP**$_X$ (two ports) TP$_X$ holds with constants ρ_{TP} and $C_0 \ge \kappa(G)+1$ if for all $c \in V$, all $\rho \ge \rho_{\mathrm{TP}}$ and $m \ge 1$ with $\rho \ge m + C_0$, all c' with $d(c,c') \le \kappa(G)$, every escapable pair (t_+, t_-) of $S_m(c')$ and every pair of distinct non-adjacent sites $a, g \in S_\rho(c)$ that are not pockets of $S_\rho(c)$ and satisfy $\{a,g\} \notin X_\rho(c)$, there is a design for $(a, g; t_+, t_-)$ in $B_\rho(c)$. (Here $B_m(c') \subseteq B_{m+\kappa(G)}(c) \subseteq B_{\rho-1}(c)$.) We write TP$_\emptyset$ when X is empty.
- **CL**$_X$ (corner lemma) For every $\rho_0 \ge 2$ there is K such that for every induced path P, ordered from one end to the other, that passes through a site v and has both end sites outside $B_K(v)$, there are $c \in V$ and $\rho \ge \rho_0$ with $B_{\rho+1}(c) \subseteq B_K(v)$ and $|P \cap B_\rho(c)| \ge 3$ such that the first and last sites a, g of P in $B_\rho(c)$ satisfy $\{a,g\} \notin X_\rho(c)$. We write $K_{\mathrm{CL}}(\rho_0)$ for such a K.

Reversing the order of P exchanges a and g, and $X_\rho(c)$ consists of unordered pairs, so CL$_X$ does not depend on the orientation of P. CL$_\emptyset$ holds with $K_{\mathrm{CL}}(\rho_0) = \rho_0 + 1$ and the ball $B_{\rho_0}(v)$, which contains v and its two neighbours on P.

Theorem A. *Let G be a periodic lattice that satisfies (B1), (B2), (B3), TP$_X$ and CL$_X$ for some exceptional family X. Then Claim 5 holds for every essential enhancement E of site percolation on G, with the constants $R = 3K + C$ and $L_0 = 100R$ of (7.1) and (7.2) below. If moreover $p_c(G) < 1$, then Conjecture 1 holds for E.*

The last sentence follows from the first and Theorem 6.5. The proof of the first sentence occupies Sections 7.3–7.6. It follows the proof of Theorem 2.13 in Sections 2.3–2.7, read on G with graph balls; the following table lists the changes.

step for $\mathbb{Z}^d$	**where ℓ^1 geometry is used**	**on a periodic lattice**
Section 2.1: R_L, pivotal sites, Claims 4 and 5	S_L is an ℓ^1 sphere	$S_L = S_L(o)$ is a graph sphere (Section 6.2)
Theorem 2.1	$\mathbb{Z}^d$	Theorem 6.5, proved in Appendix B
Lemma 2.2 (gadget)	the ℓ^∞ gadget of [BBR, §4], its conversion to ℓ^1 balls and the axis rays $\pm te_1$	Lemma G, with a direct proof: the two tails of the doubly infinite path are cut where they first leave $B_{m-1}(x_i)$; the terminals are arbitrary distinct non-adjacent sites of $S_m(x_i)$, and (B1) makes them escapable
Lemma 2.3 (two layers)	a coordinate walk in $S_n \cup S_{n+1}$	not used; its role in Case 2 is taken by the annulus clause of (B2), and Lemma 3.1 is part of the two-port theorem, that is, of TP$_X$
setting (2.1)	$K+10 < \lvert v\rvert < L-K-10$	any finite Ω with (H1), (H2) (Section 7.4); $K < \lvert v\rvert < L-K$ suffices in the main case
Lemma 2.4	translates $E_0(\omega - a) + a$; ℓ^1 triangle inequality	Lemma 7.3: $E_a = \tau_z E_j(\tau_{-z}\,\cdot\,)$ for $a = \tau_z x_j$ (one rule per orbit); (F4), (F8)

step for $\mathbb{Z}^d$	**where ℓ^1 geometry is used**	**on a periodic lattice**
Lemma 2.5	the norm changes by exactly one along an edge, to see that P meets S_L only at its end	Lemma 7.4: minimality of P and (F6)
Lemma 2.6	none	Lemma 7.5, same proof
Definition 2.7, recentring	the distance to c changes by exactly one along P	(F2) suffices; Lemma 7.7 adds that ports are not pockets, as TP_X requires
Proposition 2.8	none (only $N[D] \subseteq \Omega$)	Proposition 7.8, same proof
Definition 2.9	gadget centred at c, terminals $c \pm me_1$	Definitions 7.2, 7.9: gadget at a site c' of the essential orbit with $d(c,c') \le \kappa(G)$, terminals $t_\pm^{c'}$, $B_m(c') \subseteq B_{\rho-1}(c)$, (D3) relative to $B_m(c')$
Theorem 2.10	$N[B_\rho(c)] = B_{\rho+1}(c)$ and $\operatorname{int} B_\rho(c) = B_{\rho-1}(c)$	Theorem 7.10: only the inclusions (F3) are used; the s-pivotal site is c'
Definition 2.11, $\mathrm{CL}(d)$	corner pairs $\{c+\rho\pi, c+(\rho-1)\pi+\delta\}$	exceptional family X, CL_X
(2.2)	$\rho_0 = \max(2d+1, m+13, 20)$, $R = 3K+100$	(7.1), (7.2): $\rho_0 = \max(\rho_{\mathrm{TP}}, m+C_0)$, $R = 3K+C$
Proposition 2.12	Corollary 3.3 (terminals $\pm me_1$)	Proposition 7.13: TP_X, for the escapable terminal pair of Lemma G
Theorem 2.13, Case 2	the site of S_{K+19} nearest to v (ℓ^1 geodesics), no pockets, Lemma 2.3, opening B_{K+21}	Proposition 7.14: a radius $n \in M'(G)$ from (B2); open B_{n+t_0} except $S_n \setminus \{w\}$
Theorem 2.13, Case 3	the coordinate path P' and a deletion step in $B_{K+30}(w)$	Proposition 7.15: (B3) supplies w and P'

The proof of Theorem 2.13 and the proof of Theorem A, step by step.

7.3. The gadget

Lemma G (the gadget). *Suppose (B1). There is an integer m_E such that for every $m \in M(G)$ with $m \ge m_E$ there is a finite set $\omega_g \subseteq B_m(x_i)$ with the following properties:*

1. *$m > r_E$, and $\omega_g \cap S_m(x_i) = \{t_+, t_-\}$ with (t_+, t_-) an escapable pair of $S_m(x_i)$;*
2. *ω_g contains no path from t_+ to t_-;*
3. *$\omega_g \cup E_i(\omega_g)$ contains a path from t_+ to t_-;*
4. *$E_i(\omega_g) \subseteq B_{r_E}(x_i) \subseteq B_{m-1}(x_i)$.*

Moreover t_+ and t_- are distinct and non-adjacent. Without (B1) the same construction works for every integer $m \ge m_E$, with "escapable" in 1 replaced by "distinct and non-adjacent".

Proof. Step 1: a thin essential configuration. Since i is an essential orbit, there are a configuration

ω^0 that contains no doubly infinite path and a doubly infinite path $\Gamma : \mathbb{Z} \to V$ in $\omega^0 \cup E_i(\omega^0)$. Put $X_0 = E_i(\omega^0)$ and

$$\omega_1 = \big(\omega^0 \cap B_{r_E}(x_i)\big) \cup \big(\Gamma(\mathbb{Z}) \setminus B_{r_E}(x_i)\big).$$

(a) $\omega_1 \subseteq \omega^0$, because $X_0 \subseteq B_{r_E}(x_i)$ gives $\Gamma(\mathbb{Z}) \setminus B_{r_E}(x_i) \subseteq \omega^0$. So ω_1 contains no doubly infinite path. (b) $\omega_1 \cap B_{r_E}(x_i) = \omega^0 \cap B_{r_E}(x_i)$, so $E_i(\omega_1) = X_0$. (c) $\Gamma(\mathbb{Z}) \subseteq \omega_1 \cup X_0$.

Step 2: the two tails. Let $J = \{j : \Gamma(j) \notin \omega_1\}$. By (c), $\Gamma(J) \subseteq X_0$, which is finite, so J is finite, and $J \neq \emptyset$ by (a). Let $j_- = \min J$, $j_+ = \max J$, $T_+ = \{\Gamma(j) : j > j_+\}$ and $T_- = \{\Gamma(j) : j < j_-\}$. Both lie in ω_1, and they are disjoint.

Claim: ω_1 contains no path from a site of T_+ to a site of T_-. Suppose $Q = (q_0, \dots, q_\ell)$ is such a path. Let i_1 be the largest index with $q_{i_1} \in T_+$ and i_2 the least index $i > i_1$ with $q_i \in T_-$, and write $q_{i_1} = \Gamma(p)$, $q_{i_2} = \Gamma(p')$. The sites q_i with $i_1 < i < i_2$ lie in neither tail. Then $f(j) = \Gamma(p' + j)$ for $j \leq 0$, $f(j) = q_{i_2 - j}$ for $0 \leq j \leq i_2 - i_1$, and $f(j) = \Gamma(p + j - (i_2 - i_1))$ for $j \geq i_2 - i_1$ defines a doubly infinite path in $T_- \cup Q \cup T_+ \subseteq \omega_1$, contrary to (a). In particular no site of T_+ is adjacent to a site of T_-.

Step 3: the radius. Let $\mu = \max\{d(x_i, \Gamma(j)) : j_- \leq j \leq j_+\}$ and $m_E = \max(r_E + 1, \mu + 2)$. Let $m \geq m_E$. Let $t_+ = \Gamma(p_+)$ be the site of T_+ outside $B_{m-1}(x_i)$ with the least index. By (F2), $d(x_i, \Gamma(j_+ + 1)) \leq \mu + 1 \leq m - 1$, so $p_+ > j_+ + 1$, $\Gamma(p_+ - 1) \in B_{m-1}(x_i)$ and $t_+ \in S_m(x_i)$; all $\Gamma(j)$ with $j_+ < j < p_+$ lie in $B_{m-1}(x_i)$. Define $t_- = \Gamma(p_-)$ symmetrically. Put

$$\omega_g = \big(\omega_1 \cap B_{m-1}(x_i)\big) \cup \{t_+, t_-\}.$$

(1) $m > r_E$, and $\omega_g \cap S_m(x_i) = \{t_+, t_-\}$. The sites $t_\pm$ are distinct, and non-adjacent by the claim. By (B1), (t_+, t_-) is escapable if $m \in M(G)$. (2) $\omega_g \subseteq \omega_1$, so a path in ω_g from t_+ to t_- would contradict the claim. (4) As $r_E \leq m - 1$ and $t_\pm \notin B_{m-1}(x_i)$, we have $\omega_g \cap B_{r_E}(x_i) = \omega_1 \cap B_{r_E}(x_i)$, so $E_i(\omega_g) = X_0 \subseteq B_{r_E}(x_i)$ by (b). (3) The sites of $(\Gamma(p_-), \dots, \Gamma(p_+))$ lie in $\omega_g \cup X_0$: the ends are $t_\pm$; for $p_- < j < j_-$ and $j_+ < j < p_+$ the site $\Gamma(j)$ lies in $\omega_1 \cap B_{m-1}(x_i)$; for $j_- \leq j \leq j_+$ it lies in $B_{m-2}(x_i)$ and, by (c), in ω_1 or in X_0. □

From now on we fix $m \in M(G)$ with $m \geq m_E$ and the gadget $\omega_g, t_\pm$ of Lemma G. For a site $c' = \tau_z x_i$ of the essential orbit put $\omega_g^{c'} = \tau_z \omega_g$ and $t_\pm^{c'} = \tau_z t_\pm$. By (F4), $\omega_g^{c'} \subseteq B_m(c')$, $\omega_g^{c'} \cap S_m(c') = \{t_+^{c'}, t_-^{c'}\}$, the pair $(t_+^{c'}, t_-^{c'})$ is escapable, and $\omega_g^{c'}$ contains no path between its terminals. If a configuration ω satisfies $\omega \cap B_m(c') = \omega_g^{c'}$, then by (F8) and $r_E < m$,

$$E_{c'}(\omega) = \tau_z E_i(\omega_g) \subseteq B_{r_E}(c') \subseteq B_{m-1}(c'), \tag{7.0}$$

and $\omega_g^{c'} \cup \tau_z E_i(\omega_g)$ contains a path from $t_+^{c'}$ to $t_-^{c'}$.

7.4. Deletion, the class property and ports

In this subsection and the next we fix $L \geq 1$, a state (ω, α), a site v that is p-pivotal for R_L in (ω, α), and a finite set $\Omega \subseteq V$ such that

- **(H1)** $v \in \Omega$, $o \notin \Omega$ and $\Omega \subseteq B_{L-1}$;
- **(H2)** $d(a, \Omega) > r_E$ for every $a \in \alpha$.

If $K < |v| < L - K$ and $\alpha \cap B_{K+r_E}(v) = \emptyset$, then $\Omega = B_K(v)$ satisfies (H1) and (H2), by the triangle inequality. The setting (2.1) of Section 2.5 is a special case.

Lemma 7.3 (locality; Lemma 2.4 on G). *Let $\Omega \subseteq V$ and let (ω, α) satisfy (H2). There is a set $Y \subseteq V \setminus \Omega$ such that $E(\tilde{\omega}, \alpha) = \tilde{\omega} \cup Y$ for every $\tilde{\omega}$ that agrees with ω outside Ω. For such $\tilde{\omega}$ and every $c \in \Omega$ we have $c \notin \alpha$ and $E(\tilde{\omega}, \alpha \cup \{c\}) = \tilde{\omega} \cup Y \cup E_c(\tilde{\omega})$.*

Proof. We repeat the proof of Lemma 2.4 with $E_a(\omega)$ instead of $E_0(\omega - a) + a$. For $a \in \alpha$, (H2) gives $B_{r_E}(a) \cap \Omega = \emptyset$, so $E_a(\tilde{\omega}) = E_a(\omega) \subseteq V \setminus \Omega$ by (F8), and $Y = \bigcup_{a \in \alpha} E_a(\omega)$ has the required properties. If $c \in \Omega$, then $c \notin \alpha$ by (H2). □

Lemma 7.4 (the deletion step; Lemma 2.5 on G). *There are a configuration $\omega' \subseteq \omega \cup \{v\}$ that agrees with ω outside Ω and an induced path $P = (p_0, \dots, p_N)$ with $p_0 = o$, $p_N \in S_L$ and $|p_j| < L$ for $j < N$, such that:*

1. *$P \subseteq E(\omega', \alpha)$ and $v \in P$;*
2. *$E(\omega', \alpha) \cap \Omega = \omega' \cap \Omega = P \cap \Omega$;*
3. *for every $x \in P \cap \Omega$, every path from o to a site of S_L in $E(\omega', \alpha)$ passes through x;*
4. *$p_0, p_N \notin \Omega$.*

Proof. The proof of Lemma 2.5 applies, with Lemma 7.3 in place of Lemma 2.4 and one change: on G we take for P a path from o to S_L in $E(\omega', \alpha)$ with the least number of sites; by (F6) its first site of norm at least L has norm L, so by minimality only the last site of P has norm at least L. The deletion process removes sites of $\omega \cup \{v\}$ only, so $\omega' \subseteq \omega \cup \{v\}$, and (4) follows from (H1). □

Conclusions (1)–(3) use only that Ω is finite, $v \in \Omega$ and (H2). From now on ω' and P are fixed. We order P from o to $z := p_N$ and write $\mathrm{pred}(x)$ and $\mathrm{succ}(x)$ for the sites before and after $x \in P \setminus \{o, z\}$.

Lemma 7.5 (the class property; Lemma 2.6 on G). *Let $D \subseteq \Omega$ meet P. Write $P = O_0 I_1 O_1 \cdots I_k O_k$ with $k \geq 1$, where $I_1, \dots, I_k$ are the maximal runs of consecutive sites of P in D and $O_0, \dots, O_k$ are the runs between them. Put $O_D = E(\omega', \alpha) \setminus D$, and let H_j be the component of O_D containing O_j. Then $H_0, \dots, H_k$ are pairwise distinct, and $H_j \cap S_L = \emptyset$ for every $j < k$.*

Proof. As for Lemma 2.6, using Lemma 7.4(3) instead of Lemma 2.5(3) (the runs $O_0 \ni o$ and $O_k \ni z$ are nonempty because $o, z \notin \Omega$ by Lemma 7.4(4)) and (F7) to extract a path from a walk. □

Definition 7.6. A set $D \subseteq V$ is *admissible* if $N[D] \subseteq \Omega$ and $|P \cap D| \geq 3$. Its *ports* are the first site $a(D)$ and the last site $g(D)$ of P in D.

As in Section 2.6, the centre c of an admissible ball $D = B_\rho(c)$ is an arbitrary site. It need not lie on P, and D need not contain v.

Lemma 7.7. *Let $D = B_\rho(c)$ be admissible. Then $a(D), g(D) \in S_\rho(c)$, the sites $\mathrm{pred}(a(D))$ and $\mathrm{succ}(g(D))$ lie in $S_{\rho+1}(c)$, and neither port is a pocket of $S_\rho(c)$.*

Proof. Let $a = a(D)$. As $a \in \Omega$ and $o \notin \Omega$, $\mathrm{pred}(a)$ exists, and it lies outside D by the choice of a. By (F2), $d(a, c) \geq \rho$, so $a \in S_\rho(c)$ and $\mathrm{pred}(a) \in S_{\rho+1}(c)$, and a is not a pocket. The argument for $g(D)$ is symmetric. □

Proposition 7.8 (Proposition 2.8 on G). *Let D be admissible, put $a = a(D)$ and $g = g(D)$, and use the notation of Lemma 7.5.*

1. *$a \neq g$, and a and g are not adjacent.*

2. *The sites of O_D adjacent to a are* $\mathrm{pred}(a) \in O_0$ *and, if* $I_1 = \{a\}$, *also* $\mathrm{succ}(a) \in O_1$. *The sites of O_D adjacent to g are* $\mathrm{succ}(g) \in O_k$ *and, if* $I_k = \{g\}$, *also* $\mathrm{pred}(g) \in O_{k-1}$.
3. *Let $\mathcal{A}$ consist of H_0, and of H_1 if $I_1 = \{a\}$. Let $\mathcal{G}$ consist of H_k, and of H_{k-1} if $I_k = \{g\}$. Then no member of $\mathcal{A}$ belongs to $\mathcal{G}$, and no member of $\mathcal{A}$ meets S_L.*

Proof. Proposition 2.8 carries over, with Lemma 7.4(2) and Lemma 7.5 replacing Lemma 2.5(2) and Lemma 2.6; besides these lemmas its proof uses only $N[D] \subseteq \Omega$, that P is induced, and $o, z \notin D$. □

7.5. Wiring with an off-centre gadget

Definition 7.9. Let $D = B_\rho(c)$ be admissible and let $c' = \tau_z x_i$ be a site of the essential orbit with $B_m(c') \subseteq B_{\rho-1}(c)$. If (R, G) is a design for $(a(D), g(D); t_+^{c'}, t_-^{c'})$ in $B_\rho(c)$, then $\omega_1 = \mathsf{R} \cup \mathsf{G} \cup \omega_g^{c'}$ is an *interior design for D at c'*. By (D3), $\omega_1 \cap B_m(c') = \omega_g^{c'}$, and $\omega_1 \subseteq D$.

Theorem 7.10 (wiring; Theorem 2.10 on G). *Let $D = B_\rho(c)$ be admissible, let c' be a site of the essential orbit with $B_m(c') \subseteq B_{\rho-1}(c)$, and let $\omega_1 = \mathsf{R} \cup \mathsf{G} \cup \omega_g^{c'}$ be an interior design for D at c'. Put $\omega'' = (\omega' \setminus D) \cup \omega_1$. Then c' is s-pivotal for R_L in (ω'', α). Moreover ω'' agrees with ω outside Ω, and $c' \in \Omega$.*

Proof. We follow the proof of Theorem 2.10, with the gadget $\omega_g + c$ and its terminals $c \pm me_1$ replaced by $\omega_g^{c'}$ and $t_\pm := t_\pm^{c'}$, the set $E_0(\omega_g) + c$ replaced by $X := \tau_z E_i(\omega_g)$, and Lemmas 7.3–7.5, Proposition 7.8 and Lemma G in place of Lemmas 2.4–2.6, Proposition 2.8 and Lemma 2.2. Suppose that $\mathsf{R} \cap B_m(c') = \{t_+\}$; the other case is the same with t_+ and t_- exchanged, since properties 2 and 3 of Lemma G are symmetric in $t_\pm$. Let $C_\pm$ be the components of $\omega_g^{c'}$ containing $t_\pm$; they are distinct by Lemma G(2). The four steps of the proof of Theorem 2.10 change as follows.

Step 1. As there, $E(\omega'', \alpha) = \omega'' \cup Y = O_D \cup \omega_1$ by Lemma 7.3, since $O_D = (\omega' \setminus D) \cup Y$. Now $c' \in B_m(c') \subseteq D \subseteq \Omega$, so $c' \notin \alpha$. As $\omega'' \cap B_m(c') = \omega_g^{c'}$ and $r_E < m$, (7.0) gives $E_{c'}(\omega'') = X \subseteq B_{m-1}(c')$, and $E(\omega'', \alpha \cup \{c'\}) = O_D \cup \omega_1 \cup X$.

Step 2. A site $y \in \mathsf{R} \setminus \{t_+\}$ lies outside $B_m(c')$ by (D3), so by (F2) its neighbours in $\omega_g^{c'}$ lie in $S_m(c') \cap \omega_g^{c'} = \{t_+, t_-\}$, and by (D4) not at t_-. The rest of Step 2 is unchanged: $\mathsf{R} \cup C_+$ and $\mathsf{G} \cup C_-$ are distinct components of ω_1, and the other components of ω_1 are components of $\omega_g^{c'}$.

Step 3. By (D2) and $\omega_g^{c'} \subseteq B_m(c') \subseteq B_{\rho-1}(c)$, every site of $\omega_1 \setminus \{a, g\}$ lies in $B_{\rho-1}(c)$, which is contained in $\operatorname{int} D$ by (F3). The rest of Step 3 goes through as before, and the component Γ_0 of o in $E(\omega'', \alpha)$ misses S_L because the members of $\mathcal{A}$ do and $\mathsf{R} \cup C_+ \subseteq \Omega \subseteq B_{L-1}$ by (H1).

Step 4. As before, with a path in $\omega_g^{c'} \cup X$ from t_+ to t_- (it exists by the end of Section 7.3) in place of a path in $(\omega_g + c) \cup (E_0(\omega_g) + c)$, and with (F7). With Step 3 and $c' \notin \alpha$, the site c' is s-pivotal. Finally ω'' agrees with ω' outside D, hence with ω outside Ω. □

7.6. Proof of Theorem A

Lemma 7.11. *(a) If CL_X holds for ρ_0 with K_*, it holds for ρ_0 with every $K \geq K_*$. (b) Under (B2), for every integer $N > \min M'(G)$ there is $n \in M'(G)$ with $N \leq n \leq N + c_2 - 1$. (c) If (B3) holds with (c_3, c_4, K_3), then for every integer $j \geq 0$ it holds with $(c_3 + j, c_4 + j, K_3)$.*

Proof. (a) If the ends of P lie outside $B_K(v)$, they lie outside $B_{K_*}(v)$, and the ball found for K_* lies in $B_{K_*}(v) \subseteq B_K(v)$. (b) Let n be the least element of $M'(G)$ with $n \geq N$. It has a predecessor

$n^- < N$ in $M'(G)$, so $n \le n^- + c_2 \le N - 1 + c_2$. (c) Given $K \ge K_3$, $L \ge 4K + 4(c_4 + j)$ and v with $L - K - 10 \le |v| \le L$, apply (B3) at $K' = K + j$: then $L \ge 4K' + 4c_4$ and $L - K' - 10 \le |v|$, and $K' + c_4 = K + (c_4 + j)$, $K' + c_3 = K + (c_3 + j)$, $L - K' - 10 \le L - K - 10$. □

Lemma 7.12. *Let v be p-pivotal for R_L in (ω, α) with $\alpha \cap B_{r_E}(v) = \emptyset$, and let $Q = (q_0, \dots, q_N)$ be a path in $E(\omega \cup \{v\}, \alpha)$ from o to a site of S_L with $|q_j| < L$ for $j < N$ (such a path exists by (F6)). Then $v \in Q$. In particular $|v| \le L$.*

Proof. Lemma 7.3 with $\Omega = \{v\}$ gives $Y_0 \not\ni v$ with $E(\omega \cup \{v\}, \alpha) = \omega \cup \{v\} \cup Y_0$ and $E(\omega \setminus \{v\}, \alpha) = (\omega \setminus \{v\}) \cup Y_0$. If $v \notin Q$, then $Q \subseteq E(\omega \setminus \{v\}, \alpha)$, contradicting p-pivotality. □

Constants. Let G satisfy (B1), (B2), (B3), TP_X and CL_X, with the constants named in Section 7.2, and fix m and the gadget as at the end of Section 7.3. Put

$$\rho_0 = \max(\rho_{\mathrm{TP}},\, m + C_0), \quad c_4^* = \max(c_4, 10), \quad c_3^* = c_3 + c_4^* - c_4, \quad n_0^* = \max\bigl(n_0, \min M'(G)\bigr),$$

$$K = \max\bigl(K_{\mathrm{CL}}(\rho_0),\ \rho_0 + 1,\ K_3,\ n_0^*\bigr), \tag{7.1}$$

$$C = 20 + 2c_2 + t_0 + 2c_4^*, \qquad R = 3K + C, \qquad L_0 = 100R. \tag{7.2}$$

Then CL_X holds for ρ_0 with K (Lemma 7.11(a)); (B3) holds with (c_3^*, c_4^*, K_3) (Lemma 7.11(c)); $r_E < m < \rho_0 < K$; $K + r_E < R$; and $L_0 \ge 4K + 4c_4^*$.

Proposition 7.13 (main case). *Let $L \ge 1$, let v be p-pivotal for R_L in (ω, α), and suppose that $K < |v| < L - K$ and $\alpha \cap B_{K + r_E}(v) = \emptyset$. Then there are a configuration ω'' that agrees with ω outside $B_K(v)$ and a site $c' \in B_K(v)$ of the essential orbit that is s-pivotal for R_L in (ω'', α).*

Proof. $\Omega = B_K(v)$ satisfies (H1) and (H2). Let ω' and P be as in Lemma 7.4; P is an induced path through v with both ends outside $B_K(v)$. CL_X gives c and $\rho \ge \rho_0$ with $B_{\rho+1}(c) \subseteq B_K(v)$ and $|P \cap B_\rho(c)| \ge 3$ whose first and last sites a, g satisfy $\{a, g\} \notin X_\rho(c)$. By (F3) the ball $D = B_\rho(c)$ is admissible with ports a, g, which by Proposition 7.8(1) and Lemma 7.7 are distinct, non-adjacent, non-pocket sites of $S_\rho(c)$. By (F9) there is a site c' of the essential orbit with $d(c, c') \le \kappa(G)$. As $\rho \ge \rho_0 \ge m + C_0 \ge m + \kappa(G) + 1$, we have $B_m(c') \subseteq B_{\rho-1}(c)$. Since $(t_+^{c'}, t_-^{c'})$ is escapable and $\rho \ge \rho_{\mathrm{TP}}$, TP_X gives a design for $(a, g; t_+^{c'}, t_-^{c'})$ in D, and Theorem 7.10 concludes. □

Proposition 7.14 (pivotal sites near the root). *Let $L \ge L_0$, let v be p-pivotal for R_L in (ω, α) with $\alpha \cap B_R(v) = \emptyset$, and suppose $|v| \le K + 10$. Then there are ω'' that agrees with ω outside $B_R(v)$ and $c' \in B_R(v)$ that is s-pivotal for R_L in (ω'', α).*

Proof. Since $K + 11 > \min M'(G)$, Lemma 7.11(b) gives $n \in M'(G)$ with $K + 11 \le n \le K + 10 + c_2$, and $n \ge n_0$. By (B2), S_n has no pockets, $A = S_{n+1} \cup \cdots \cup S_{n+t_0}$ is connected, and there is $w \in S_n$ with

$$d(v, w) \le n - |v| + c_2 \le K + 10 + 2c_2. \tag{7.3}$$

Put $\Lambda = B_{n+t_0}$. For $a \in \alpha$ and $x \in \Lambda$ we have $d(a, x) \ge |a| - |x| > (R - K - 10) - (n + t_0) \ge K > r_E$, so Lemma 7.3 gives Y' with $Y' \cap \Lambda = \emptyset$ and $E(\hat\omega, \alpha) = \hat\omega \cup Y'$ whenever $\hat\omega$ agrees with ω outside Λ. Let

$$\tilde\omega = (\omega \setminus \Lambda) \cup (\Lambda \setminus S_n) \cup \{w\}.$$

$E(\tilde\omega, \alpha)$ satisfies R_L. Let Γ be a path from o to $y \in S_L$ in $E(\omega \cup \{v\}, \alpha) \subseteq \omega \cup \{v\} \cup Y'$. As $L > n + t_0$, the last site x' of Γ in Λ lies in $S_{n+t_0} \subseteq A$, and the part after it lies in $(\omega \setminus \Lambda) \cup Y'$. Inside Λ, the connected set $B_{n-1} \ni o$, a neighbour $w^- \in S_{n-1}$ of w (F5), the site w, a neighbour $w^+ \in S_{n+1}$ of

w (no pockets) and the connected set $A \ni w^+, x'$ are open in $\tilde{\omega}$. Concatenation and (F7) give the claim. $E(\tilde{\omega} \setminus \{w\}, \alpha)$ *does not satisfy* R_L, because it contains no site of S_n, and every path from o to S_L meets S_n (F6). So w is p-pivotal in $(\tilde{\omega}, \alpha)$.

Now $K < |w| = n \le K + 10 + c_2 < L - K$, and by (7.3), $B_{K+r_E}(w) \subseteq B_{3K+9+2c_2}(v) \subseteq B_R(v)$, so $\alpha \cap B_{K+r_E}(w) = \emptyset$. Proposition 7.13 at w gives ω'' that agrees with $\tilde{\omega}$ outside $B_K(w)$ and $c' \in B_K(w)$ that is s-pivotal. Since $\Lambda \subseteq B_{2K+20+c_2+t_0}(v)$ and $B_K(w) \subseteq B_{2K+10+2c_2}(v)$, both lie in $B_R(v)$. □

Proposition 7.15 (pivotal sites near S_L). *Let $L \ge L_0$, let v be p-pivotal for R_L in (ω, α) with $\alpha \cap B_R(v) = \emptyset$, and suppose $|v| \ge L - K - 10$. Then there are ω'' that agrees with ω outside $B_R(v)$ and $c' \in B_R(v)$ that is s-pivotal for R_L in (ω'', α).*

Proof. Step 1. Let $Q = (q_0, \dots, q_N)$ be a path in $E(\omega \cup \{v\}, \alpha)$ from o to S_L with $|q_j| < L$ for $j < N$. By Lemma 7.12, $v \in Q$ and $L - K - 10 \le |v| \le L$.

Step 2. By (B3) with (c_3^*, c_4^*, K_3) there is w with $K + c_4^* < |w| < L - K - 10$ and $d(v, w) \le K + c_4^* - 1$, with the path property of (B3). Put $\Omega' = B_{K+c_4^*}(w)$. Then $v \in \Omega'$ and $o \notin \Omega'$. For $a \in \alpha$ and $x \in \Omega'$, $d(a, x) > R - (K + c_4^* - 1) - (K + c_4^*) > r_E$, so Lemma 7.3 gives Y with $Y \cap \Omega' = \emptyset$ and $E(\hat{\omega}, \alpha) = \hat{\omega} \cup Y$ whenever $\hat{\omega}$ agrees with ω outside Ω'.

Step 3. Let $u = q_k$ be the first site of Q in Ω'. By (F2), $d(u, w) = K + c_4^*$, so $u \ne v$, u comes before v on Q, $|u| < L$, and $u \in (\omega \cup \{v\}) \cap \Omega'$ gives $u \in \omega$.

Step 4. (B3) gives an induced path $P' = (u = y_0, \dots, y_q)$ with $y_q \in S_L$, $|y_j| < L$ for $j < q$, $w = y_s$ with $1 \le s \le q - 1$, and $P' \setminus \{u\} \subseteq B_{K+c_4^*-1}(w)$. Let $F = \{y_0, \dots, y_{s-1}\}$ and $S = \{y_{s+1}, \dots, y_q\}$; they do not touch. Put $\hat{\omega} = (\omega \setminus \Omega') \cup P'$.

Step 5: w is p-pivotal in $(\hat{\omega}, \alpha)$. The walk $(q_0, \dots, q_{k-1}, y_0, \dots, y_q)$ lies in $\hat{\omega} \cup Y$, so $R_L(E(\hat{\omega}, \alpha))$. Let Γ be a path from o to S_L in $(\hat{\omega} \setminus \{w\}) \cup Y$. Its sites in Ω' lie in $P' \setminus \{w\} = F \cup S$, because $Y \cap \Omega' = \emptyset$, and all neighbours of sites of $P' \setminus \{u\}$ lie in Ω' (F3). If Γ met S, the site before its first visit to S (which exists, as $o \notin \Omega'$) would lie in $\Omega' \cap \Gamma \subseteq F \cup S$, hence in F, and would be adjacent to a site of S: impossible. If Γ met $F \setminus \{u\}$, take a maximal run of consecutive sites of Γ in $F \setminus \{u\}$. It neither starts nor ends Γ, because $o \notin \Omega'$ and the sites of F have norm less than L. The sites of Γ just before and just after the run are adjacent to sites of $F \setminus \{u\}$, so they lie in $\Omega' \cap \Gamma$, hence in F, hence, by maximality, both equal u, which contradicts the injectivity of Γ. So $\Gamma \cap \Omega' \subseteq \{u\}$, and $\Gamma \subseteq (\omega \setminus \{v\}) \cup Y = E(\omega \setminus \{v\}, \alpha)$, contradicting the p-pivotality of v.

Step 6. $K + 10 \le K + c_4^* < |w| < L - K - 10$ and $B_{K+r_E}(w) \subseteq B_{3K+c_4^*-2}(v) \subseteq B_R(v)$. Proposition 7.13 at w gives ω'' that agrees with $\hat{\omega}$ outside $B_K(w) \subseteq \Omega'$ and $c' \in B_K(w)$ that is s-pivotal. Finally $\Omega' \subseteq B_{2K+2c_4^*-1}(v) \subseteq B_R(v)$. □

Proof of Theorem A. Let $L \ge L_0$ and let v be p-pivotal for R_L in (ω, α) with $\alpha \cap B_R(v) = \emptyset$. If $K + 10 < |v| < L - K - 10$, Proposition 7.13 applies, since $B_{K+r_E}(v) \subseteq B_R(v)$, and gives the conclusion with $B_K(v) \subseteq B_R(v)$. If $|v| \le K + 10$, use Proposition 7.14, and if $|v| \ge L - K - 10$, use Proposition 7.15. In each case the state (ω'', α) agrees with (ω, α) outside $B_R(v)$, which is Claim 5. □

7.7. Remarks

Remark 7.16. 1. *Use of (B1)–(B3), TP_X and CL_X.* Each hypothesis is used in one place: (B1) through Lemma G(1), at the single radius m and for the essential orbit; (B2), with all three of its

clauses, in Proposition 7.14; (B3) in Proposition 7.15, which uses only the listed properties of the path P', and its inducedness only to guarantee that F and S do not touch; TP_X and CL_X in Proposition 7.13. In particular, Theorem 7.10 uses none of (B1)–(B3), TP_X and CL_X, nor the escapability of the terminals. The distance clause of (B2) could be dropped at the price of $R = 4K + C$, because $d(v, S_n) \le |v| + n$ in every connected graph. 2. *Comparison with Theorem 2.13.* For $\mathbb{Z}^d$, (B2) holds with $M' = \{n \ge 1\}$, $n_0 = 1$, $t_0 = 2$, $c_2 = 1$ (by Lemma 2.3), and (B3) holds with $c_3 = 20$, $c_4 = 30$ by the construction in Case 3 of the proof of Theorem 2.13. The boundary cases of Theorem 2.13 thus correspond to Propositions 7.14 and 7.15. In Proposition 7.15 the crossing Q is used directly, without a deletion step, as in the formal proof of Case 3 (Section 14). The gadget is the one point where Theorem A and Theorem 2.13 differ in form: Section 2 normalizes the terminals to $\pm me_1$, by attaching straight rays to a gadget in ℓ^∞ form (Lemma 2.2), and proves its two-port theorem only for these. On a general lattice there are no preferred terminals: they are wherever the two tails of the doubly infinite path Γ of the proof of Lemma G first leave $B_{m-1}(x_i)$. This is why TP_X must accept every escapable terminal pair, and why (B1) is needed. 3. *Dependence of the constants.* R and L_0 depend on E only through m, and on G through ρ_{TP}, C_0, $K_{\mathrm{CL}}(\cdot)$, K_3, c_3, c_4, n_0, t_0, c_2 and $\min M'(G)$. For $X = \emptyset$ all constants are explicit, and Theorem C below gives an explicit K_{CL} in the other case whenever $\delta(G) \ge 3$ and X satisfies (X1)–(X4).

8. The abstract corner lemma

8.1. Statement

Let G be a periodic lattice and X an exceptional family (Section 7.2). For a site $x = (z, i)$ we write $\bar{x} = z$ for its *cell*, and on $\mathbb{Z}^n$ we use $\|z\|_\infty = \max_s |z_s|$. The constants $\rho_X \ge 2$ and $p_0 \ge 0$ below are integers.

- **(X1)** *Covariance:* $X_\rho(\tau_z c) = \tau_z X_\rho(c)$ for all z, c and ρ.
- **(X2)** *Apex labelling:* there are a finite set Π and partial maps $A_\rho : V \times \Pi \rightharpoonup V$ ($\rho \ge 2$) with $A_\rho(\tau_z c, \pi) = \tau_z A_\rho(c, \pi)$, such that $A_\rho(c, \pi) \in S_\rho(c)$ whenever it is defined, every pair in $X_\rho(c)$ contains $A_\rho(c, \pi)$ for some π, and $A_\rho(c_1, \pi) = A_\rho(c_2, \pi)$ implies $c_1 = c_2$.
- **(X3)** *Bounded partners:* $N_\rho(c, \pi) := \{y : \{A_\rho(c, \pi), y\} \in X_\rho(c)\} \subseteq B_{p_0}(A_\rho(c, \pi))$.
- **(X4)** *Mutual containment:* for all $\pi_1 \ne \pi_2$ there are $u \in \mathbb{Z}^n \setminus \{0\}$, an integer $t_0 \ge 1$ and a real $\lambda > 0$ such that for all $\rho \ge \rho_X$ and all c_1, c_2 for which $A_j := A_\rho(c_j, \pi_j)$ is defined ($j = 1, 2$) and $A_2 = \tau_{tu} A_1$ for an integer t with $t_0 \le t \le \lambda\rho$, we have $d(A_1, A_2) > p_0$, $B_\rho(c_1) \supseteq \{A_2\} \cup N_\rho(c_2, \pi_2)$ and $B_\rho(c_2) \supseteq \{A_1\} \cup N_\rho(c_1, \pi_1)$.

For every ordered pair (π_1, π_2) of distinct labels we fix one triple $(u, t_0, \lambda) = (u(\pi_1, \pi_2), t_0(\pi_1, \pi_2), \lambda(\pi_1, \pi_2))$ as in (X4), its *data*. The site $A_\rho(c, \pi)$ is an *apex* of $B_\rho(c)$, and the sites of $N_\rho(c, \pi)$ are its *partners*. On $\mathbb{Z}^d$ the corner pairs of Definition 2.11 satisfy (X1)–(X4) with apexes $c + \rho\pi$ (Lemma 8.14).

Theorem C. *Let G be a periodic lattice with $\delta(G) \ge 3$, and let X be an exceptional family satisfying (X1)–(X4) with constants ρ_X, p_0. Then CL_X holds. More precisely, for every integer $\rho_0 \ge \rho_X$ the conclusion of CL_X holds with the constant $K = K(\rho_0)$ of (8.1) below. The covariance (X1) and the covariance clause of (X2) are not used.*

The conclusion for $\rho_0 \geq \rho_X$ implies it for every $2 \leq \rho_0 < \rho_X$, since the ball found has radius at least $\rho_X > \rho_0$; so CL_X holds for every $\rho_0 \geq 2$. If the conclusion holds with some K, it holds with every larger K.

Idea of the proof. Suppose that the conclusion fails for a path P through v. Then every ball of radius at least ρ_0 well inside $B_K(v)$ that contains at least three sites of P has an exceptional port pair, one of whose sites is an apex, and the centre is determined by the apex and its label. Fix a large radius r and a box around v that is much larger than r, and compare three counts. *Density:* the balls of a radius proportional to the box, centred in the box, all contain v and its two neighbours on P; by the injectivity in (X2), P occupies a fixed proportion of a larger box, independent of r. *Degree:* every site of P has a neighbour off P, and this neighbour is the centre of a ball of radius r with at least three sites of P; so the set $\mathcal{W}_r$ of such centres exceeds P by a fixed proportion. *Apexes:* the centres in $\mathcal{W}_r$ map injectively to pairs (apex, label), and apexes are sites of P; a site that is an apex for two labels is rare, because by (X3) and (X4) two such sites cannot lie between t_0 and λr steps apart along a line in direction u. So $|\mathcal{W}_r|$ exceeds $|P|$ in the box only by a boundary term and a term of relative order $1/r$. This contradicts the degree count when r is large and the box is large compared with r.

The argument is a finitary counting, like the formal proof of the Corner Lemma (Section 14), and it replaces the stationary limit and the mass transport of Sections 4.2–4.6. It needs neither the shrinking map of Lemma 4.3 nor the uniqueness of the pole at an apex (Lemma 4.6).

8.2. Boxes and two lattice constants

For $q \in \mathbb{Z}^n$ and an integer $h \geq 0$ let $Q_h(q) = \{x \in V : \|\bar{x} - q\|_\infty \leq h\}$, so that $|Q_h(q)| = k(2h+1)^n$. Put

$$\ell = \max\{\|\bar{x} - \bar{y}\|_\infty : x \sim y\}, \qquad \gamma = \max\{d(x,y) : \|\bar{x} - \bar{y}\|_\infty \leq 1\}.$$

Lemma 8.1. *(1) ℓ and γ are positive integers. (2) $\|\bar{x} - \bar{y}\|_\infty \leq \ell\, d(x,y)$. (3) $d(x,y) \leq \gamma \max\{1, \|\bar{x} - \bar{y}\|_\infty\}$. (4) If $x \in Q_h(q)$ and $d(x,y) \leq s$, then $y \in Q_{h+\ell s}(q)$. If $h \geq 1$, then $Q_h(\bar{v}) \subseteq B_{\gamma h}(v)$ for every site v.*

Proof. (1) By translation invariance the maxima are over finitely many pairs, involving the sites $(0,i)$ and their neighbours, resp. the sites (q,j) with $q \in \{-1,0,1\}^n$; they are finite by local finiteness and connectedness. If $\ell = 0$, the finite set $\{0\} \times \{0,\dots,k-1\}$ would be a union of components of G. Also $\gamma \geq d((0,0),(e_1,0)) \geq 1$. (2) Sum along a geodesic. (3) If $m = \|\bar{x} - \bar{y}\|_\infty \geq 2$, move from $\bar{x}$ to $\bar{y}$ through cells $q_0, \dots, q_m$ with $\|q_j - q_{j-1}\|_\infty \leq 1$ (changing every coordinate that is not yet correct by one unit at each step), and use sites $(q_j, 0)$ for $0 < j < m$; each step costs at most γ. (4) follows from (2) and (3). □

8.3. Labelled balls and the two-label exclusion

In this subsection $P = (x_0, \dots, x_N)$ is any path. For $D \subseteq V$ meeting P let $a_P(D)$ and $g_P(D)$ be the first and last sites of P in D, at indices $i^-(D) \leq i^+(D)$, and write $a(c,\rho)$, $g(c,\rho)$ for the ports of $B_\rho(c)$.

Definition 8.2. Let $\rho \geq 2$. The ball $B_\rho(c)$ is *X-paired with label π* if $P \cap B_\rho(c) \neq \emptyset$, $\{a(c,\rho), g(c,\rho)\} \in X_\rho(c)$, $A_\rho(c,\pi)$ is defined, and $A_\rho(c,\pi) \in \{a(c,\rho), g(c,\rho)\}$.

Lemma 8.3. *Let $\rho \geq 2$ and $P \cap B_\rho(c) \neq \emptyset$. (1) If $\{a(c,\rho), g(c,\rho)\} \in X_\rho(c)$, then $B_\rho(c)$ is X-paired with some label. (2) If $B_\rho(c)$ is X-paired with label π, then $A_\rho(c,\pi)$ is a site of P, and the ports are $A_\rho(c,\pi)$ and some $N \in N_\rho(c,\pi)$, with $N \neq A_\rho(c,\pi)$ and $d(A_\rho(c,\pi), N) \leq p_0$.*

Proof. (1) is (X2). (2) The ports are sites of P, the pairs of $X_\rho(c)$ consist of distinct sites, and the other port lies in $N_\rho(c,\pi)$ by definition; apply (X3). □

Lemma 8.4 (equal windows). *Let D_1, D_2 meet P. If both ports of D_2 lie in D_1 and both ports of D_1 lie in D_2, then $a_P(D_1) = a_P(D_2)$ and $g_P(D_1) = g_P(D_2)$.*

Proof. From $a_P(D_2) \in D_1$ and $g_P(D_2) \in D_1$ we get $i^-(D_1) \leq i^-(D_2)$ and $i^+(D_2) \leq i^+(D_1)$, and symmetrically. □

Lemma 8.5 (two-label exclusion). *Let $\pi_1 \neq \pi_2$ have data (u, t_0, λ), let $\rho \geq \rho_X$, and let $B_\rho(c_j)$ be X-paired with label π_j, $j = 1, 2$. Put $A_j = A_\rho(c_j, \pi_j)$. Then $A_2 \neq \tau_{tu} A_1$ for every integer t with $t_0 \leq t \leq \lambda\rho$.*

Proof. Suppose $A_2 = \tau_{tu} A_1$. By Lemma 8.3(2) the ports of $B_\rho(c_j)$ are A_j and some $N_j \in N_\rho(c_j, \pi_j)$. By (X4) each ball contains both ports of the other, so by Lemma 8.4, $\{A_1, N_1\} = \{A_2, N_2\}$. By (X4), $d(A_1, A_2) > p_0 \geq 0$, so $A_1 = N_2$, and then $d(A_1, A_2) = d(N_2, A_2) \leq p_0$, a contradiction. □

The same argument applies to two labellings of one ball, so exceptional pairs whose two sites are both apexes need no separate treatment.

8.4. Consequences of a failure

In this subsection $\rho_0 \geq \rho_X$ and $K \geq 0$ are integers, $P = (x_0, \dots, x_N)$ is an induced path, v is a site of P with $x_0, x_N \notin B_K(v)$, and we assume that the conclusion of CL_X fails:

> **(Cor)** for every $c \in V$ and every $\rho \geq \rho_0$ with $B_{\rho+1}(c) \subseteq B_K(v)$ and $|P \cap B_\rho(c)| \geq 3$, we have $\{a(c,\rho), g(c,\rho)\} \in X_\rho(c)$.

We write $Q_h = Q_h(\bar{v})$. For $r \geq 1$ let $\mathcal{W}_r = \{y \in V : |P \cap B_r(y)| \geq 3\}$, and for $r \geq 2$ and $\pi \in \Pi$ let $\mathcal{A}_r(\pi)$ be the set of sites $A_r(c,\pi)$ such that $B_r(c)$ is X-paired with label π. By Lemma 8.3(2), $\mathcal{A}_r(\pi) \subseteq P$.

Lemma 8.6 (interior sites). *Let $x = x_j \in P \cap B_K(v)$. Then $0 < j < N$, the neighbours of x on P are exactly x_{j-1} and x_{j+1}, x has at least $\delta - 2$ neighbours that are not sites of P, and $|P \cap B_r(y)| \geq 3$ for every $r \geq 1$ and every site y with $d(y, x) \leq r - 1$.*

Proof. The ends of P lie outside $B_K(v)$, and P is induced. The three sites x_{j-1}, x_j, x_{j+1} lie in $B_r(y)$. □

Lemma 8.7 (density). *Let $h \geq 1$ with $\gamma h + 1 \geq \rho_0$ and $2\gamma h + 2 \leq K$. Then $|Q_h| \leq |\Pi| \cdot |P \cap Q_{\ell(2\gamma h+1)}|$.*

Proof. Put $R' = \gamma h + 1 \geq \rho_0$ and let $c \in Q_h$. By Lemma 8.1(4), $d(c, v) \leq R' - 1$, so $|P \cap B_{R'}(c)| \geq 3$ by Lemma 8.6, and $B_{R'+1}(c) \subseteq B_{2R'}(v) \subseteq B_K(v)$. By (Cor) and Lemma 8.3(1) there is a label π_c such that $B_{R'}(c)$ is X-paired with label π_c. The apex $A_c = A_{R'}(c, \pi_c)$ lies in $P \cap S_{R'}(c)$, so $d(v, A_c) \leq 2\gamma h + 1$ and $A_c \in Q_{\ell(2\gamma h+1)}$. The map $c \mapsto (A_c, \pi_c)$ is injective by (X2). □

Lemma 8.8 (non-path neighbours). *Let $r \geq 2$ and $H \geq 1$ with $\gamma H \leq K$. Then $(\delta - 2)|P \cap Q_H| \leq \Delta\,|(\mathcal{W}_r \setminus P) \cap Q_{H+\ell}|$.*

Proof. Each $x \in P \cap Q_H$ lies in $B_K(v)$ and has at least $\delta - 2$ neighbours $y \notin P$ (Lemma 8.6), each of which lies in $\mathcal{W}_r$ (Lemma 8.6 with $d(y,x) = 1 \le r-1$) and in $Q_{H+\ell}$. Each such y is counted at most Δ times. □

Lemma 8.9. *Let $r \ge 1$ and $H \ge 1$ with $\gamma H \le K$. Then $P \cap Q_H \subseteq \mathcal{W}_r$, so $|(\mathcal{W}_r \setminus P) \cap Q_H| = |\mathcal{W}_r \cap Q_H| - |P \cap Q_H|$.* (Lemma 8.6 with $y = x$.)

Lemma 8.10 (centres and apexes). *Let $r \ge \rho_0$ and $H \ge 1$ with $\gamma H + r + 1 \le K$. Then $|\mathcal{W}_r \cap Q_H| \le \sum_{\pi \in \Pi} |\mathcal{A}_r(\pi) \cap Q_{H+\ell r}|$.*

Proof. For $y \in \mathcal{W}_r \cap Q_H$ we have $B_{r+1}(y) \subseteq B_K(v)$, so by (Cor) and Lemma 8.3(1) there is a label π_y such that $B_r(y)$ is X-paired with label π_y. Its apex lies in $\mathcal{A}_r(\pi_y) \cap Q_{H+\ell r}$, and $y \mapsto (\pi_y, A_r(y, \pi_y))$ is injective by (X2). □

Lemma 8.11 (several labels). *For $r \ge 2$ and finite $F \subseteq V$,*

$$\sum_{\pi} |\mathcal{A}_r(\pi) \cap F| \le |P \cap F| + \sum_{\pi_1 \ne \pi_2} |\mathcal{A}_r(\pi_1) \cap \mathcal{A}_r(\pi_2) \cap F|.$$

Proof. With $n_z = |\{\pi : z \in \mathcal{A}_r(\pi)\}|$ the two sides are $\sum_{z \in F} n_z$ and $\sum_{z \in F} (\mathbf{1}[z \in P] + n_z(n_z - 1))$, and $n_z \le \mathbf{1}[z \in P] + n_z(n_z - 1)$ because $n_z \ge 1$ forces $z \in P$ and $(n_z - 1)^2 \ge 0$. □

Lemma 8.12 (two labels along lines). *Let $\pi_1 \ne \pi_2$ have data (u, t_0, λ), let $r \ge \rho_X$ and let J be an integer with $0 \le J \le \lambda r$. Then for every $H \ge 0$,*

$$(J+1)\, |\mathcal{A}_r(\pi_1) \cap \mathcal{A}_r(\pi_2) \cap Q_H| \le t_0\, |Q_{H + J\|u\|_\infty}|.$$

Proof. Let $S = \mathcal{A}_r(\pi_1) \cap \mathcal{A}_r(\pi_2) \cap Q_H$. The map $(z, j) \mapsto \tau_{ju} z$ sends $S \times \{0, \dots, J\}$ into $Q' = Q_{H + J\|u\|_\infty}$. For $w \in Q'$ let $I_w = \{j : \tau_{-ju} w \in S\}$. If $i < j$ lie in I_w, then $A_1 = \tau_{-ju} w \in \mathcal{A}_r(\pi_1)$ and $A_2 = \tau_{-iu} w \in \mathcal{A}_r(\pi_2)$ satisfy $A_2 = \tau_{(j-i)u} A_1$ with $1 \le j - i \le \lambda r$, so $j - i < t_0$ by Lemma 8.5. Hence $|I_w| \le t_0$, and $(J+1)|S| = \sum_w |I_w| \le t_0 |Q'|$. □

Lemma 8.13 (boundary). *For $0 \le H \le H'$, $|P \cap Q_{H'}| - |P \cap Q_H| \le |Q_{H'}| - |Q_H| \le 2nk(H' - H)(2H' + 1)^{n-1}$.* (Use $y^n - x^n \le n(y - x) y^{n-1}$ for $0 \le x \le y$.)

8.5. Proof of Theorem C

If $\Pi = \emptyset$, then X is empty by (X2). If $|\Pi| \ge 2$ put

$$T = \max_{\pi_1 \ne \pi_2} t_0(\pi_1, \pi_2), \qquad U = \max_{\pi_1 \ne \pi_2} \|u(\pi_1, \pi_2)\|_\infty, \qquad \lambda_* = \min\Big\{1,\ \min_{\pi_1 \ne \pi_2} \lambda(\pi_1, \pi_2)\Big\},$$

over ordered pairs of distinct labels, and if $|\Pi| \le 1$ put $T = 1$, $U = 0$, $\lambda_* = 1$. For $\rho_0 \ge \rho_X$ define

$$\begin{aligned} a &= 4\ell\gamma + 6\ell + 2U + 1, \qquad r = \max\Big\{\rho_0,\ \Big\lfloor \frac{2\Delta |\Pi|^3\, T\, a^n}{(\delta - 2)\, 2^n\, \lambda_*} \Big\rfloor + 1\Big\}, \\ h &= \max\Big\{r,\ \Big\lceil \frac{4n\ell\Delta |\Pi|\, a^{n-1}\, r}{(\delta - 2)\, 2^n} \Big\rceil\Big\}, \qquad K(\rho_0) = 2\ell\gamma(\gamma h + 1) + r + 1. \end{aligned} \tag{8.1}$$

These depend only on $n, \Delta, \delta, \ell, \gamma$, on $|\Pi|$, on the data through T, U, λ_*, and on ρ_0.

Proof of Theorem C. Let P be an induced path through $v = x_s$ with both ends outside $B_K(v)$, $K = K(\rho_0)$. If $\Pi = \emptyset$, take $c = v$ and $\rho = r$: then $B_{r+1}(v) \subseteq B_K(v)$, $|P \cap B_r(v)| \geq 3$ by Lemma 8.6, and $X_r(v) = \emptyset$. Let $\Pi \neq \emptyset$ and suppose the conclusion fails, so that (Cor) holds. Put

$$M_0 = \ell(2\gamma h + 1), \quad M_1 = M_0 + \ell = 2\ell(\gamma h + 1), \quad M_2 = M_1 + \ell r, \quad J = \lfloor \lambda_* r \rfloor, \quad M_3 = M_2 + JU.$$

Since $\gamma M_1 + r + 1 = K$, the side conditions of Lemmas 8.7 (with h), 8.8 (with $H = M_0$), 8.9 and 8.10 (with $H = M_1$) and 8.12 (with $H = M_2$, for every ordered pair, using $J \leq \lambda_* r \leq \lambda(\pi_1, \pi_2) r$, $Q_{M_2 + J\|u\|_\infty} \subseteq Q_{M_3}$ and $t_0 \leq T$) hold. Hence

$$\begin{aligned}
\frac{\delta - 2}{|\Pi|}\, k(2h+1)^n &\leq (\delta - 2)\, |P \cap Q_{M_0}| && \text{(Lemma 8.7)} \\
&\leq \Delta\, |(\mathcal{W}_r \setminus P) \cap Q_{M_1}| && \text{(Lemma 8.8)} \\
&= \Delta \big(|\mathcal{W}_r \cap Q_{M_1}| - |P \cap Q_{M_1}| \big) && \text{(Lemma 8.9)} \\
&\leq \Delta \Big(\sum_{\pi} |\mathcal{A}_r(\pi) \cap Q_{M_2}| - |P \cap Q_{M_1}| \Big) && \text{(Lemma 8.10)} \qquad (8.2) \\
&\leq \Delta \Big(|P \cap Q_{M_2}| - |P \cap Q_{M_1}| + \sum_{\pi_1 \neq \pi_2} |\mathcal{A}_r(\pi_1) \cap \mathcal{A}_r(\pi_2) \cap Q_{M_2}| \Big) && \text{(Lemma 8.11)} \\
&\leq \Delta \Big(2nk\, \ell r\, (2M_2 + 1)^{n-1} + |\Pi|(|\Pi| - 1)\, \frac{T}{J+1}\, k(2M_3 + 1)^n \Big) && \text{(Lemmas 8.13, 8.12).}
\end{aligned}$$

Since $h \geq r \geq 1$ and $J \leq r$, we have $2M_2 + 1 \leq 2M_3 + 1 \leq ah$. With $J + 1 > \lambda_* r$, $|\Pi|(|\Pi| - 1) \leq |\Pi|^2$ and $(2h+1)^n \geq 2^n h^n$, dividing by k gives

$$\frac{(\delta - 2)\, 2^n}{|\Pi|}\, h^n \leq 2n\ell\Delta\, r\, a^{n-1} h^{n-1} + \frac{\Delta |\Pi|^2\, T\, a^n}{\lambda_*\, r}\, h^n. \tag{8.3}$$

By the choice of h the first term on the right is at most $\frac{(\delta-2)2^n}{2|\Pi|} h^n$, and by the choice of r the second is strictly less than $\frac{(\delta-2)2^n}{2|\Pi|} h^n$, a contradiction. □

8.6. Remarks and the case of $\mathbb{Z}^d$

1. *Use of $\delta \geq 3$ and (X1)–(X4).* $\delta \geq 3$ is used only in Lemma 8.8. The clause $A_\rho(c, \pi) \in S_\rho(c)$ of (X2) locates the apexes, the clause "every pair contains an apex" gives Lemma 8.3(1), and the injectivity is used at the two radii $\gamma h + 1$ (Lemma 8.7) and r (Lemma 8.10). (X3) and (X4) are used only in Lemma 8.5, at the single radius r. In particular (X4) needs to hold only for an unbounded set of radii, with r replaced by the least such radius above the value in (8.1) and with h and $K(\rho_0)$ then computed from this r by (8.1), and data for one of the two orders of each pair of labels suffice.
2. *The degree condition.* It cannot be dropped from the argument in the form used here without a replacement, since it is the only source of the excess of $\mathcal{W}_r$ over P. On a periodic lattice, sites of degree 2 occur in chains of bounded length L_2, and replacing each path site by the first site of degree at least 3 that follows it on P gives $|P \cap Q_H| \leq (L_2 + 1)\Delta |(\mathcal{W}_r \setminus P) \cap Q_{H + \ell(L_2+1)}|$ in place of Lemma 8.8. The same proof then gives CL_X without the hypothesis $\delta \geq 3$, with a larger constant.
3. *Size of K.* K is of order $2\ell\gamma^2 h$ with h and r as in (8.1). The constants are not optimized.

We now check that Theorem C contains the Corner Lemma CL(d) of Section 2.7. Regard $\mathbb{Z}^d$ as a periodic lattice with $n = d$, $k = 1$ and $x \sim y$ iff $\|x - y\|_1 = 1$. Then $d(x, y) = \|x - y\|_1$, $\Delta = \delta = 2d$,

$\ell = 1$ and $\gamma = d$. Let $\mathcal{U} = \{\pm e_1, \dots, \pm e_d\}$, write $\pi \perp \eta$ if $\eta \notin \{\pm\pi\}$, and let

$$X_\rho^{\mathrm{cor}}(c) = \big\{\{c + \rho\pi,\ c + (\rho-1)\pi + \eta\} :\ \pi, \eta \in \mathcal{U},\ \pi \perp \eta\big\} \qquad (\rho \geq 2),$$

the corner pairs of Definition 2.11. Then $\mathrm{CL}_{X^{\mathrm{cor}}}$ is, word for word, the Corner Lemma CL(d) of Section 2.7.

Lemma 8.14. *X^{cor} satisfies (X1)–(X4) with $\Pi = \mathcal{U}$, $A_\rho(c,\pi) = c + \rho\pi$, $\rho_X = 2$, $p_0 = 2$, and the data $u = \pi_2 - \pi_1$, $t_0 = 2$, $\lambda = 1$ if $\pi_1 \perp \pi_2$, and $u = -\pi_1$, $t_0 = 3$, $\lambda = 1$ if $\pi_2 = -\pi_1$. Moreover $N_\rho(c,\pi) = \{c + (\rho-1)\pi + \eta : \eta \perp \pi\}$.*

Proof. (X1) and (X2) are clear. For (X3): the vector $(\rho-1)\pi' + \eta'$ has two nonzero coordinates while $\rho\pi$ has one, so a corner pair containing $c + \rho\pi$ has apex $c + \rho\pi$ and partner $c + (\rho-1)\pi + \eta$ with $\eta \perp \pi$, at distance $\|\eta - \pi\|_1 = 2$. For (X4), write $N_j = A_j - \pi_j + \eta_j$ with $\eta_j \perp \pi_j$, and let $t_0 \leq t \leq \rho$. *Perpendicular poles:* $d(A_1, A_2) = 2t \geq 4$; $A_2 - c_1 = (\rho - t)\pi_1 + t\pi_2$ has norm ρ; $N_2 - c_1 = (\rho-t)\pi_1 + (t-1)\pi_2 + \eta_2$ has norm $|\rho - t - 1| + t - 1 \leq \rho$ if $\eta_2 = -\pi_1$, and ρ otherwise; the other direction is symmetric. *Antipodal poles* ($\pi_1 = \sigma$, $u = -\sigma$): $d(A_1, A_2) = t \geq 3$; $A_2 - c_1 = (\rho - t)\sigma$ and $N_2 - c_1 = (\rho - t + 1)\sigma + \eta_2$ have norms $\rho - t$ and $\rho - t + 2$, at most ρ; the other direction is symmetric. □

For antipodal poles the data differ from a literal transcription of Lemma 4.5. The direction must be $u = -\pi_1$ (with $u = \pi_1$ the apex A_2 leaves $B_\rho(c_1)$), and $t_0 = 2$ fails because then $d(A_1, A_2) = 2 = p_0$: for $d = 2$, $\rho = 5$, $c_1 = 0$, $A_1 = (5,0)$, $A_2 = (3,0)$ and $c_2 = (8,0)$ the two port pairs $\{(5,0),(4,1)\}$ and $\{(3,0),(4,1)\}$ share the partner $(4,1)$. The proof of Lemma 4.5 reaches $t = 2$ by a finer argument that (X4) does not encode.

Corollary 8.15 (Theorem 4.8 with an explicit constant). *Let $d \geq 2$ and $\rho_0 \geq 2$. Then CL(d) holds with*

$$r = \max\Big\{\rho_0, \Big\lfloor \frac{48d^4(4d+9)^d}{(d-1)2^d} \Big\rfloor + 1\Big\}, \quad h = \max\Big\{r, \Big\lceil \frac{8d^3(4d+9)^{d-1}r}{(d-1)2^d} \Big\rceil\Big\}, \quad K = 2d(dh+1) + r + 1.$$

For $d = 2$ and $\rho_0 \leq 55\,489$ this gives $r = 55\,489$, $h = 15\,093\,008$ and $K = 120\,799\,558$.

Proof. Apply Theorem C with Lemma 8.14: $|\Pi| = 2d$, $T = 3$, $U = 1$, $\lambda_* = 1$, so $a = 4d + 9$, and (8.1) gives the stated values. For $d = 2$: $48 \cdot 16 \cdot 17^2/4 = 55\,488$, $h = 272r$ and $K = 4(2h+1) + r + 1$. □

9. The honeycomb lattice

Recall that HEX has the sites $A(z) = (z,0)$ and $B(z) = (z,1)$, $z \in \mathbb{Z}^2$, and the edges $A(z) \sim B(w)$ for $z - w \in E := \{0, e_1, e_2\}$. The root is $o = A(0)$, and we call $|v| = d(o,v)$ the *level* of v. For a site c let

$$X_\rho^{\mathrm{hex}}(c) = \big\{\{C,N\} :\ C,N \in S_\rho(c),\ d(C,N) = 2,\ C \text{ has exactly one neighbour } C' \text{ in } S_{\rho-1}(c),\ \text{and } C' \text{ has a neighbour in } S_\rho(c) \text{ other than } C\big\}.$$

A site C with the property in this definition is called a *corner* of $S_\rho(c)$.

Theorem B-HEX. *The honeycomb lattice satisfies (B1) with $M(\mathrm{HEX}) = \{m \geq 3\}$, (B2) with $M'(\mathrm{HEX}) = \{n \geq 1\}$, $n_0 = 1$, $t_0 = 3$, $c_2 = 1$, (B3) with $c_3 = 43$, $c_4 = 50$, $K_3 = 1$, and $TP_{X^{\mathrm{hex}}}$ with $\rho_{\mathrm{TP}} = 200$ and $C_0 = 30$; and X^{hex} satisfies (X1)–(X4) with $\rho_X = 2$ and $p_0 = 2$. Moreover,*

for $\rho \geq \max(200, m+30)$, every $c' \in B_1(c)$ and every escapable pair of $S_m(c')$, the pairs of distinct, non-adjacent, non-pocket sites of $S_\rho(c)$ that admit no design are exactly the twelve pairs of $X_\rho^{\mathrm{hex}}(c)$.

The twelve exceptional pairs sit at the six corners of the hexagonal ball: each corner forms an exceptional pair with each of its two neighbours along the boundary polygon of the sphere (Lemma 9.4). They admit no design for the reason found in [BBR, §2]: the corner has a unique inner neighbour, and every way into the ball from the corner comes too close to the partner (Proposition 9.9). Two layers do not suffice in (B2): $S_q \cup S_{q+1}$ is disconnected for every $q \geq 1$ (see after Proposition 9.8).

The honeycomb lattice is the main new case of Theorem 1.2. Theorem B-HEX, Theorem C, Theorem A and Proposition 6.6 give Conjecture 1 for site percolation on it (Section 12.11). The computer enters only through the three local lemmas of Section 9.5, at finitely many radii, and a transfer lemma extends them to all radii (Lemma 9.17; Section 13.2).

9.1. Cube coordinates, distances and symmetries

For $x \in \mathbb{Z}^3$ write $s(x) = x_1 + x_2 + x_3$ and $\|x\|_1 = |x_1| + |x_2| + |x_3|$, and let f_1, f_2, f_3 be the unit vectors of $\mathbb{Z}^3$. For a site $v = ((z_1, z_2), i)$ put $q(v) = (-z_1, -z_2, z_1 + z_2 + i)$, let $i(v) = i$, and let $\sigma(v) = 1 - 2i(v) \in \{\pm 1\}$.

Lemma 9.1 (cube coordinates; distance formula). *(a) q is a bijection from V onto $\{x \in \mathbb{Z}^3 : s(x) \in \{0,1\}\}$ with $s(q(v)) = i(v)$, and $q(\tau_y v) = q(v) + T(y)$ with $T(y) = (-y_1, -y_2, y_1 + y_2)$. (b) The three neighbours of v are the sites $v^{(j)}$ with $q(v^{(j)}) = q(v) + \sigma(v) f_j$, $j = 1, 2, 3$. (c) $d(v, v') = \|q(v) - q(v')\|_1$; that is, for $v = ((a, b), i)$ and $v' = ((a', b'), i')$,*

$$d(v, v') = |a - a'| + |b - b'| + |(a + b + i) - (a' + b' + i')|.$$

(d) With $w = q(v) - q(c)$ and $\varepsilon = s(w) \in \{-1, 0, 1\}$ we have $d(c, v) = \max_j |2w_j - \varepsilon|$.

Proof. (a) is immediate. (b) $q((w - \nu, 1)) - q((w, 0))$ equals f_1, f_2, f_3 for $\nu = e_1, e_2, 0$. (c) Images of adjacent sites differ by a unit vector, so $\|q(v) - q(v')\|_1 \leq d(v, v')$. Conversely, let $x = q(v) \neq y = q(v')$. If $i(v) = 0$ and $y_j \leq x_j$ for all j, then $s(x - y) = -s(y) \leq 0$ forces $x = y$; so some $y_j > x_j$, and the neighbour with image $x + f_j$ is closer to y. The case $i(v) = 1$ is symmetric. Induction on $\|x - y\|_1$ gives (c). (d) $\|w\|_1 = \max_{e \in \{\pm 1\}^3} \langle e, w \rangle = \max(|\varepsilon|, \max_j |\varepsilon - 2w_j|)$, and $|2w_j - \varepsilon| \geq |\varepsilon|$ by parity. □

With $\lambda_1(x, y) = x$, $\lambda_2(x, y) = y$, $\lambda_3(x, y) = x + y$, put $h(z) = \max_l |\lambda_l(z)|$ and $g(z) = \max_l \max(\lambda_l(z), -\lambda_l(z) - 1)$. By Lemma 9.1(d), applied with $c = o$ and $\varepsilon = 0$, resp. $\varepsilon = 1$, and the identity $|2t + 1| = 2\max(t, -t - 1) + 1$,

$$|A(z)| = 2h(z), \qquad |B(z)| = 2g(z) + 1. \tag{9.1}$$

Let $U_6 = \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$.

Lemma 9.2 (automorphisms). *The following maps are automorphisms of HEX: the translations τ_w; $\iota(z, i) = (-z, 1 - i)$; $\sigma_0(z, i) = ((z_2, z_1), i)$; and $\varrho(A(z)) = A(Mz)$, $\varrho(B(z)) = B(Mz - e_1)$, where $M(x, y) = (-x - y, x)$. The maps σ_0 and ϱ fix o, $\varrho^3 = \mathrm{id}$, and the translations together with ι act transitively on V.*

Proof. For ι, $A(-w) \sim B(-z)$ iff $z - w \in E$. For σ_0, E is invariant under exchanging coordinates. For ϱ, the affine map $t \mapsto Mt + e_1$ permutes E ($0 \mapsto e_1 \mapsto e_2 \mapsto 0$), and $z - w \in E$ iff $Mz - (Mw - e_1) \in E$. Finally $M^3 = I$ and $(M^2 + M + I)e_1 = 0$. □

Corollary 9.3. *(a) HEX is bipartite with classes $\{A(z)\}$ and $\{B(z)\}$; along every edge the distance to a fixed site changes by exactly one; all sites of a sphere have the same type, and no two sites of a sphere are adjacent. (b) Two distinct sites have at most one common neighbour. (c) $d(A(z),A(z')) = d(B(z),B(z')) = 2h(z-z')$; two sites of the same type at distance 2 differ by a vector of U_6 and have exactly one common neighbour.*

Proof. (a) follows from Lemma 9.1 and $s(q(v)) = i(v)$. (b) Common neighbours of $A(z) \neq A(z')$ are the $B(w)$ with $w \in (z-E)\cap(z'-E)$, and the six differences $e-e'$ ($e \neq e' \in E$) are distinct; similarly for B-sites. (c) follows from Lemma 9.1(c) and (b). □

9.2. Spheres around the root

Lemma 9.4 (shape of the spheres). *Let $k \geq 1$ and $u_0 = (-1,1)$, $u_1 = (-1,0)$, $u_2 = (0,-1)$, $u_3 = (1,-1)$, $u_4 = (1,0)$, $u_5 = (0,1)$ (indices mod 6).*

(a) $S_{2k} = \{A(z) : z \in \partial H_k\}$, where ∂H_k is the closed lattice polygon with vertices $a_0 = (k,0)$, $a_1 = (0,k)$, $a_2 = (-k,k)$, $a_3 = (-k,0)$, $a_4 = (0,-k)$, $a_5 = (k,-k)$ and sides $\{a_j + tu_j : 0 \leq t \leq k\}$. $|S_{2k}| = 6k$.

(b) $S_{2k+1} = \{B(z) : z \in \partial G_k\}$, where ∂G_k has vertices $b_0 = (k,0)$, $b_1 = (0,k)$, $b_2 = (-k-1,k)$, $b_3 = (-k-1,0)$, $b_4 = (0,-k-1)$, $b_5 = (k,-k-1)$ and sides $\{b_j + tu_j : 0 \leq t \leq \ell_j\}$, with $\ell_j = k$ for even j and $\ell_j = k+1$ for odd j. $|S_{2k+1}| = 6k+3$.

(c) Vertices of the polygons have two outer neighbours and one inner neighbour. Interior points of a side j of ∂H_k have one outer and two inner neighbours if j is even, and two outer and one inner if j is odd; for ∂G_k it is the other way round. In particular no sphere S_n, $n \geq 1$, has a pocket.

(d) The corners of S_{2k} are the six sites $A(a_j)$, and the corners of S_{2k+1} are the six sites $B(b_j)$.

(e) For $n \geq 4$ the set $X_n^{\mathrm{hex}}(o)$ consists of the twelve pairs formed by a corner and one of its two neighbours along the polygon: $\{A(a_j), A(a_j+u_j)\}$, $\{A(a_j), A(a_j-u_{j-1})\}$ for $n = 2k$, and $\{B(b_j), B(b_j+u_j)\}$, $\{B(b_j), B(b_j-u_{j-1})\}$ for $n = 2k+1$.

Proof. For (a) and (b), by (9.1), $\{h \leq k\}$ is the hexagon $|x|,|y|,|x+y| \leq k$ and $\{g \leq k\}$ is given by $x,y,x+y \leq k$ and $x,y,x+y \geq -k-1$; their boundaries are the stated polygons. (c) For $A(z) \in S_{2k}$ let $J^{\pm} = \{l : \lambda_l(z) = \pm k\}$. Since $g(z-e)$ is the maximum of $\lambda_l(z) - \lambda_l(e)$ and $-\lambda_l(z) + \lambda_l(e) - 1$, the neighbour $B(z-e)$ lies in S_{2k+1} iff some $l \in J^+$ has $\lambda_l(e) = 0$ or some $l \in J^-$ has $\lambda_l(e) = 1$, and in S_{2k-1} otherwise. Similarly, for $B(z) \in S_{2k+1}$ with $K^+ = \{l : \lambda_l(z) = k\}$, $K^- = \{l : \lambda_l(z) = -k-1\}$, the neighbour $A(z+e)$ lies in S_{2k+2} iff some $l \in K^+$ has $\lambda_l(e) = 1$ or some $l \in K^-$ has $\lambda_l(e) = 0$. We use $(\lambda_l(e))_l = (0,0,0),(1,0,1),(0,1,1)$ for $e = 0, e_1, e_2$. At the vertices $a_0,\dots,a_5$ the pair (J^+,J^-) is $(\{1,3\},\emptyset)$, $(\{2,3\},\emptyset)$, $(\{2\},\{1\})$, $(\emptyset,\{1,3\})$, $(\emptyset,\{2,3\})$, $(\{1\},\{2\})$, and in the interior of the sides $0,\dots,5$ it is $(\{3\},\emptyset)$, $(\{2\},\emptyset)$, $(\emptyset,\{1\})$, $(\emptyset,\{3\})$, $(\emptyset,\{2\})$, $(\{1\},\emptyset)$; the same list gives (K^+,K^-) at the b_j and on the sides of ∂G_k. Hence the outer neighbours $B(z-e)$ of $A(z) \in S_{2k}$ are given by $e \in \{0,e_2\},\{0,e_1\},\{0,e_1\},\{e_1,e_2\},\{e_1,e_2\},\{0,e_2\}$ at $a_0,\dots,a_5$ and by $e \in \{0\},\{0,e_1\},\{e_1\},\{e_1,e_2\},\{e_2\},\{0,e_2\}$ on the sides $0,\dots,5$; the outer neighbours $A(z+e)$ of $B(z) \in S_{2k+1}$ are given by $e \in \{e_1,e_2\},\{e_1,e_2\},\{0,e_2\},\{0,e_2\},\{0,e_1\},\{0,e_1\}$ at $b_0,\dots,b_5$ and by $e \in \{e_1,e_2\},\{e_2\},\{0,e_2\},\{0\},\{0,e_1\},\{e_1\}$ on the sides $0,\dots,5$. Every site has three neighbours, in the two adjacent spheres, which gives the inner counts. For $S_1 = \{B(0),B(-e_1),B(-e_2)\}$ each site has the inner neighbour o and two outer neighbours. (d) By (c), the sites with exactly one inner neighbour are the vertices and the interior points of the odd sides of ∂H_k, resp. of the even

sides of ∂G_k. At $A(a_0)$ the inner neighbour $B(k-1,0)$ is adjacent to $A(k-1,1) \in S_{2k}$, while at an interior point $A(k,y)$ of side 5 ($-k < y < 0$) the inner neighbour $B(k-1,y)$ has its other neighbours $A(k-1,y)$, $A(k-1,y+1)$ in S_{2k-2}. At $B(b_0)$ the inner neighbour $A(k,0)$ is adjacent to $B(k,-1) \in S_{2k+1}$, while at an interior point $B(p,k-p)$ of side 0 ($0 < p < k$) the inner neighbour $A(p,k-p)$ has its other neighbours $B(p-1,k-p)$, $B(p,k-p-1)$ in S_{2k-1}. The automorphisms ϱ, σ_0 fix o, act affinely on each type, map the six vertices of each polygon onto each other (ϱ maps vertex j to vertex $j+2$, σ_0 exchanges vertices 0 and 1) and sides onto sides. This transfers the computation to all vertices and sides. (e) The sites at distance 2 from $A(a_0)$ are the $A(a_0+\delta)$, $\delta \in U_6$, and $h(a_0+\delta) = k+1, k-1, k+1, k, k+1, k$ for $\delta = e_1, -e_1, e_2, -e_2, e_1-e_2, -e_1+e_2$; similarly for $B(b_0)$. For $n \geq 4$ the partners are interior points of sides, so the twelve pairs are distinct. □

At the corner $A(a_0)$ the inner neighbour $C' = B(k-1,0)$ is adjacent to the partner $A(a_0+u_0)$ (the *corner partner*), while the partner $A(a_0-u_5) = A(k,-1)$ shares with $A(a_0)$ the outer neighbour $B(k,-1)$ (the *depth-two partner*). At $B(b_0)$, $C' = A(k,0)$ is adjacent to $B(b_0-u_5)$, and $B(b_0+u_0)$ shares with $B(b_0)$ the outer neighbour $A(k,1)$.

Remark 9.5 (sectors). Let $\Sigma_0 = \{x, y \geq 0\}$, $\Sigma_1 = \{x \leq 0 \leq x+y\}$ and $\Sigma_5 = \{y \leq 0 \leq x+y\}$. By (9.1): on Σ_0, $|A(x,y)| = 2(x+y)$ and $|B(x,y)| = 2(x+y)+1$; on Σ_1, $|A(x,y)| = 2y$, and $|B(x,y)| = 2y+1$ whenever $x \leq 0 \leq x+y+1$ and $y \geq 0$; on Σ_5, $|A(x,y)| = 2x$ and $|B(x,y)| = 2x+1$. In the interior of Σ_0 an A-site has one outer neighbour and a B-site two, and in the interiors of Σ_1 and Σ_5 it is the other way round. (A formula "level $= 2\mu(z) + i$" with one linear form μ for both types fails on the other three sectors, because ϱ acts on B-sites by $z \mapsto Mz - e_1$; for instance $B(-3,3)$ and $B(-3,2)$ have levels 7 and 5. Only $\Sigma_0, \Sigma_1, \Sigma_5$ are used below.)

Lemma 9.6. *Let $q \geq 1$. Two distinct sites of S_q are at distance 2 if and only if they are consecutive in the cyclic order of Lemma 9.4.*

Proof. For $q = 1$ the three sites of S_1 are pairwise consecutive and have the common neighbour o. For $q \geq 2$, consecutive sites differ by a vector of U_6. Conversely, of the six sites $(z+\delta, \iota)$, $\delta \in U_6$, at distance 2 from $(z,\iota) \in S_q$, at most two lie in S_q. By ϱ, σ_0 it suffices to check a vertex, which is done in the proof of Lemma 9.4(e), an interior point $z = (p, k-p)$ of side 0, and an interior point $z = (k,y)$ of side 5. On side 0, $\delta \in \{e_1, e_2\}$ gives $\lambda_3(z+\delta) = k+1$, and $\delta = -e_1$ gives $(p-1, k-p)$ with $h = g = k-1$; the case $\delta = -e_2$ follows by σ_0, so only $\delta = \pm u_0$ remain. On side 5, $\delta \in \{e_1, e_1-e_2\}$ gives $\lambda_1(z+\delta) = k+1$, and $\delta \in \{-e_1, -e_1+e_2\}$ gives a point $(k-1, y')$ with $h = g = k-1$, so only $\delta = \pm u_5$ remain. □

9.3. The properties (B1) and (B2)

Proposition 9.7 ((B1)). *For every $m \geq 3$ and each orbit representative $x_0 = A(0)$, $x_1 = B(0)$, the sphere $S_m(x_i)$ has no pockets and every pair of distinct sites of $S_m(x_i)$ is escapable.*

Proof. As ι maps x_0 to x_1, take $x_0 = o$. The sphere S_m has no pockets by Lemma 9.4(c). Distinct sites t, t' of S_m are not adjacent. Let $\mathrm{out}(s)$ be the set of neighbours of s in S_{m+1}. *Step 1: if $\mathrm{out}(t) = \{u\}$, then u has no neighbour in S_m other than t.* By Lemma 9.4(c) and the symmetry ϱ it suffices to check $t = A(p,k-p)$, $0 < p < k$ ($m = 2k$), where $u = B(p,k-p)$ has its other neighbours at $h = k+1$, and $t = B(x,k)$, $-k-1 < x < 0$ ($m = 2k+1$), where $u = A(x,k+1)$ has its other neighbours at $g = k+1$. *Step 2.* If t, t' have no common neighbour, any choice of outer neighbours $u \in \mathrm{out}(t)$, $u' \in \mathrm{out}(t')$ works ($u \neq u'$, $u \not\sim t'$, $u' \not\sim t$, and $u \not\sim u'$ as they have the same type). Otherwise the

common neighbour is unique; if it lies in S_{m+1}, Step 1 lets us choose outer neighbours of t, t' different from it, and these work as before. □

Proposition 9.8 ((B2)). *(B2) holds with $M' = \{n \geq 1\}$, $n_0 = 1$, $t_0 = 3$ and $c_2 = 1$.*

Proof. S_n has no pockets, and following outer neighbours from v reaches S_n in $n - |v|$ steps. Let $W = S_{n+1} \cup S_{n+2} \cup S_{n+3}$ and $j = n + 2$. Every site of S_{n+1} and of S_{n+3} has a neighbour in S_j. By Lemma 9.4, S_j is a cyclic sequence of sites in which consecutive sites differ by a vector of U_6, so they have a common neighbour in $S_{j-1} \cup S_{j+1} \subseteq W$. Hence W is connected. □

The value $t_0 = 3$ is optimal: by Lemma 9.4(c) there are $9k + 3$ edges between S_{2k} and S_{2k+1}, which have $12k + 3$ sites, and $9k + 6$ edges between S_{2k+1} and S_{2k+2}, which have $12k + 9$ sites; since there are no edges inside spheres, $S_q \cup S_{q+1}$ is disconnected for $q \geq 1$.

9.4. The exceptional pairs fail

Proposition 9.9. *Let $\rho \geq 4$, $m \geq 1$ with $\rho \geq m + 4$, $d(c, c') \leq 1$, and let t_+, t_- be any two sites of $S_m(c')$. For every $\{a, g\} \in X^{\mathrm{hex}}_\rho(c)$ there is no design for $(a, g; t_+, t_-)$, nor for $(g, a; t_+, t_-)$, in $B_\rho(c)$.*

Proof. We may take $c = o$ and, by ϱ, σ_0, the corner $C = A(k, 0)$ ($\rho = 2k$) or $C = B(k, 0)$ ($\rho = 2k+1$). Note that $B_m(c') \subseteq B_{\rho-3}(o)$. Suppose (R, G) is a design, let Q be the colour containing C and Q' the other, which contains the partner N. By (D1)–(D3), Q contains a path $C = q_0, q_1, q_2, \dots$ to $B_m(c')$, with $q_1 = C'$ the inner neighbour of C and q_2 a neighbour of C' in $B_{\rho-1}(o)$ other than C. Likewise Q' contains an inner neighbour of N. For $C = A(k, 0)$: $C' = B(k-1, 0)$, whose other neighbours are $A(k-1, 1) \in S_\rho$ and $A(k-1, 0) \in S_{\rho-2}$, so $A(k-1, 0) \in Q$. If $N = A(k-1, 1)$, then $N \sim C'$, contrary to (D4). If $N = A(k, -1)$, its only inner neighbour is $B(k-1, -1)$, which lies in Q' and is adjacent to $A(k-1, 0) \in Q$. For $C = B(k, 0)$: $C' = A(k, 0)$, $q_2 = B(k-1, 0)$; the corner partner $B(k, -1)$ is adjacent to C', and the depth-two partner $B(k-1, 1)$ has the only inner neighbour $A(k-1, 1)$, which is adjacent to $B(k-1, 0)$. □

9.5. The two-port property

In this subsection $c = o$; this is no loss of generality, since automorphisms preserve balls, pockets, escapable pairs, designs and X^{hex}. For a site c' let $\psi_{c'}$ be τ_w if $c' = A(w)$ and $\tau_w \iota$ if $c' = B(w)$; it maps o to c'. Write $|v|_{c'} = d(c', v)$; if $d(o, c') \leq 1$, then $\big||v| - |v|_{c'}\big| \leq 1$.

Overview. The design is assembled from four cycles around the two centres, the *tracks* T_1, T_2 a few levels inside $S_\rho(o)$ and T_3, T_4 a few levels outside $B_m(c')$. One port descends to T_1, runs along T_1 to a geodesic ray of c' through corners (an *apex ray*), descends along it to T_3, and runs along T_3 to the top of a short path climbing from one terminal. The other port uses T_2, a second apex ray and T_4. Near the ports and near the terminals the colours are separated by local escapes (Lemmas 9.14 and 9.15). Each arc on a track has to avoid one *block* of that track created by the other colour, and all other contacts are excluded by comparing levels.

A set Q of sites of a cycle Γ is a *block* of Γ if $Q = \emptyset$, or Q is the vertex set of a subpath of Γ, or Q is all of Γ. If $Q \neq \Gamma$ is a block, any two sites of $\Gamma \setminus Q$ are joined by a path in Γ avoiding Q.

Definition 9.10 (tracks). Let $n \geq 1$ and let $s_0, \dots, s_{N-1}$ be the sites of $S_{n+1}(o)$ in the cyclic order of Lemma 9.4. Consecutive sites have a unique common neighbour c_i (Corollary 9.3(c)). The *track* $C_n(o)$ is the cyclic sequence $s_0, c_0, s_1, c_1, \dots, s_{N-1}, c_{N-1}$, and $C_n(c') = \psi_{c'}(C_n(o))$.

Lemma 9.11. *(a) $C_n(c')$ is a cycle, its sites lie in $S_n(c') \cup S_{n+1}(c') \cup S_{n+2}(c')$, it contains $S_{n+1}(c')$, and the subgraph induced on its sites is this cycle. (b) A site x lies on $C_n(c')$ if and only if $|x|_{c'} = n+1$, or $|x|_{c'} \in \{n, n+2\}$ and x has at least two neighbours in $S_{n+1}(c')$. (c) A set Q of sites of $C_n(c')$ is a block if and only if $Q = \emptyset$ or the subgraph induced on Q is connected.*

Proof. Take $c' = o$. (a) Each c_i lies in $S_n \cup S_{n+2}$. If $c_i = c_j$ with $i \neq j$, it would have three neighbours in S_{n+1}; but sites of S_n have an inner neighbour and sites of S_{n+2} an outer one. The same argument excludes chords $c_i \sim s_j$, $j \notin \{i, i+1\}$, and sites whose levels differ by 0 or 2 are not adjacent. (b) If $|x| \in \{n, n+2\}$ and $s \neq s'$ are neighbours of x in S_{n+1}, then $d(s, s') = 2$, so s, s' are consecutive by Lemma 9.6, and x is their unique common neighbour. (c) follows from (a). □

Definition 9.12 (apex rays). For $\ell \geq 0$ let v_ℓ^j be vertex j of $S_\ell(o)$: $A(a_j)$ for $\ell = 2k \geq 2$, $B(b_j)$ for $\ell = 2k+1$, and $v_0^j = o$. The j-th *apex ray* of c' is $r_j(c') = (\psi_{c'}(v_0^j), \psi_{c'}(v_1^j), \dots)$, $j = 0, \dots, 5$.

Lemma 9.13. *(a) $r_j(c')$ is a path, and its ℓ-th site has $|\cdot|_{c'} = \ell$. (b) If u, v are sites of two distinct apex rays of c', then $d(u, v) \geq \max(|u|_{c'}, |v|_{c'}) - 3$.*

Proof. Take $c' = o$. (a) $a_j(k) - b_j(k)$ and $a_j(k+1) - b_j(k)$ lie in E for all j and k. (b) Write $a_j(k) = kw_j$ with $w_j \in U_6$. For distinct $w, w' \in U_6$ there is $\lambda \in \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3\}$ with $\lambda(w) = 1$ and $\lambda(w') \leq 0$, so $h(kw - k'w') \geq \max(k, k')$. Sites of the rays at levels ℓ, ℓ' are within distance 1 of $A(a_j(\lfloor \ell/2 \rfloor))$, $A(a_{j'}(\lfloor \ell'/2 \rfloor))$, so $d(u, v) \geq 2\max(k, k') - 2 \geq \max(\ell, \ell') - 3$. □

The following three local lemmas are proved in Section 9.8 by a computer search at finitely many radii ([S, Part Z, Section 2]), extended to all radii by Lemma 9.17.

Lemma 9.14 (port escapes). *Let $\rho \geq 200$ and let $a \neq g$ be sites of $S_\rho(o)$ with $\{a, g\} \notin X_\rho^{\mathrm{hex}}(o)$. Then $\{a, g\} = \{p_{\mathsf{R}}, p_{\mathsf{G}}\}$ for a labelling such that there are paths $W_{\mathsf{R}} = (p_{\mathsf{R}} = x_0, \dots, x_7)$ and $W_{\mathsf{G}} = (p_{\mathsf{G}} = y_0, \dots, y_{13})$ with $|x_i| = \rho - i$ $(0 \leq i \leq 7)$ and $|y_i| = \rho - i$ $(0 \leq i \leq 13)$, such that (P1) W_{R} and W_{G} do not touch, and (P2) $N[W_{\mathsf{G}}] \cap C_{\rho-8}(o)$ is a block of $C_{\rho-8}(o)$.*

Lemma 9.15 (terminal escapes). *Let c' be a site, $m \geq 1$ and (t_+, t_-) an escapable pair of $S_m(c')$. Then $\{t_+, t_-\} = \{t_{\mathsf{R}}, t_{\mathsf{G}}\}$ for a labelling such that there are paths $V_{\mathsf{R}} = (t_{\mathsf{R}} = x_0, \dots, x_{11})$ and $V_{\mathsf{G}} = (t_{\mathsf{G}} = y_0, \dots, y_5)$ with $|x_i|_{c'} = |y_i|_{c'} = m + i$, such that (T1) V_{R} and V_{G} do not touch, (T2) $N[V_{\mathsf{R}}] \cap C_{m+4}(c')$ is a block of $C_{m+4}(c')$, and (T3) for some j, no site of $r_j(c')$ with $|\cdot|_{c'} \geq m + 5$ lies in $N[V_{\mathsf{R}}]$.*

Lemma 9.16 (rays crossing tracks). *Let $d(o, c') \leq 1$, $0 \leq j \leq 5$ and $n \geq 4$. Then $r_j(c')$ contains a site of $C_n(o)$, and $N[r_j(c')] \cap C_n(o)$ is a block of $C_n(o)$.*

Let $W_0 = \{(z, i) : \lambda_1(z) \geq 0,\ \lambda_3(z) \geq 0\}$.

Lemma 9.17 (transfer). *(a) On W_0 the level is $|(z, i)| = 2\max(\lambda_1(z), \lambda_3(z)) + i$. Hence the translation $\tau_{(s,0)}$ changes the level by exactly $2s$ at every site v with $v, \tau_{(s,0)}v \in W_0$.*

(b) On Σ_0 the level is $2\lambda_3(z) + i$, and on Σ_5 it is $2\lambda_1(z) + i$. Since $\lambda_3(u_0) = 0$ and $\lambda_1(u_5) = 0$, the translation τ_{tu_0} preserves the level of every site (z, i) with $z, z + tu_0 \in \Sigma_0$, and τ_{tu_5} that of every site with $z, z + tu_5 \in \Sigma_5$.

(c) ϱ and σ_0 fix o, preserve levels, map each sphere $S_q(o)$ onto itself, vertices onto vertices and sides onto sides, map each track $C_n(o)$ onto itself, satisfy $\varrho(r_j(o)) = r_{j+2}(o)$ and $\sigma_0(r_j(o)) = r_{1-j}(o)$, and preserve X^{hex}.

(d) Let Θ be an automorphism, c_1, c_2 sites, Y a set of sites and $s \in \mathbb{Z}$ with $|\Theta v|_{c_2} = |v|_{c_1} + s$ for every $v \in N[Y]$. Then for $v \in Y$, every q and every $n \geq 1$ with $n + s \geq 1$: (i) $v \in S_q(c_1)$ iff $\Theta v \in S_{q+s}(c_2)$,

and Θ maps inner (outer) neighbours of v with respect to c_1 onto those of Θv with respect to c_2; (ii) $v \in C_n(c_1)$ iff $\Theta v \in C_{n+s}(c_2)$; (iii) a set $Q \subseteq Y \cap C_n(c_1)$ is a block of $C_n(c_1)$ iff $\Theta(Q)$ is a block of $C_{n+s}(c_2)$; (iv) if $N[v] \subseteq Y$, then v is a corner of $S_q(c_1)$ iff Θv is a corner of $S_{q+s}(c_2)$; (v) the same holds for $(\Theta^{-1}, \Theta(Y), c_2, c_1, -s)$.

Proof. (a) For $\lambda_1, \lambda_3 \geq 0$ we have $|\lambda_2| \leq \max(\lambda_1, \lambda_3)$, so $h = g = \max(\lambda_1, \lambda_3)$ by (9.1); the translation adds s to λ_1 and λ_3. (b) follows from Remark 9.5. (c) follows from Lemmas 9.2 and 9.4(d); tracks are built from consecutive polygon sites and their common neighbours, and X^{hex} is defined metrically. (d)(i) Θ preserves adjacency and shifts the levels of v and its neighbours by s. (ii) By Lemma 9.11(b), membership in a track depends only on the levels of v and its neighbours. (iii) By Lemma 9.11(c), being a block is a property of the induced subgraph. (iv) Being a corner depends only on levels of sites of $N[N[v]] \subseteq N[Y]$. (v) $N[\Theta(Y)] = \Theta(N[Y])$. □

So if Θ and Y are as in (d), every condition in Lemmas 9.14–9.16 that involves only sites of Y (membership in spheres and tracks, levels along a path, touching, blocks, corners, and membership in X^{hex} when $N[a] \cup N[g] \subseteq Y$) holds for a configuration in Y if and only if it holds for its image. The proofs in Section 9.8 construct, for each large radius, an automorphism Θ composed of an element of $\langle \varrho, \sigma_0 \rangle$, a translation along a side (Lemma 9.17(b)) and a translation $\tau_{(s,0)}$ (Lemma 9.17(a)), which shifts levels by a constant on a window of radius 35 (ports) or 29 (terminals) around the configuration and carries it to one of the radii checked by computer; for Lemma 9.16 the translation along the ray by two sites plays this role.

Theorem 9.18 ($\mathrm{TP}_{X^{\mathrm{hex}}}$). *HEX satisfies $TP_{X^{\mathrm{hex}}}$ with $\rho_{\mathrm{TP}} = 200$ and $C_0 = 30$.*

Proof. Let $\rho \geq 200$, $m \geq 1$ with $\rho \geq m + 30$, $d(o, c') \leq 1$, (t_+, t_-) an escapable pair of $S_m(c')$, and $a \neq g$ non-adjacent sites of $S_\rho(o)$ with $\{a, g\} \notin X^{\mathrm{hex}}_\rho(o)$. Put

$$T_1 = C_{\rho-8}(o), \qquad T_2 = C_{\rho-14}(o), \qquad T_3 = C_{m+10}(c'), \qquad T_4 = C_{m+4}(c').$$

Their sites have o-levels in $[\rho-8, \rho-6]$, $[\rho-14, \rho-12]$, and c'-levels in $[m+10, m+12]$, $[m+4, m+6]$.

Local pieces. Lemma 9.14 gives $p_{\mathsf{R}}, p_{\mathsf{G}}, W_{\mathsf{R}}, W_{\mathsf{G}}$, and Lemma 9.15 gives $t_{\mathsf{R}}, t_{\mathsf{G}}, V_{\mathsf{R}}, V_{\mathsf{G}}$. Let $x_{\mathsf{R}} \in S_{\rho-7}(o) \subseteq T_1$ and $x_{\mathsf{G}} \in T_2$ be the last sites of W_{R} and W_{G}, and $y_{\mathsf{R}} \in S_{m+11}(c') \subseteq T_3$ and $y_{\mathsf{G}} \in T_4$ the last sites of V_{R} and V_{G}.

Choice of rays. A ray r of c' is *blocked at the bottom* if $N[V_{\mathsf{R}}]$ contains a site of r of c'-level at least $m + 5$, and *blocked at the top* if $N[W_{\mathsf{G}}]$ contains a site of r. By (T3) some ray r_{G} is not blocked at the bottom. The sites of $N[W_{\mathsf{G}}]$ have c'-level at least $\rho - 15 \geq 185$ and mutual distances at most 15, so by Lemma 9.13(b) at most one ray is blocked at the top. Choose a ray $r_{\mathsf{R}} \neq r_{\mathsf{G}}$ not blocked at the top. On r_{R} let J_3 be the site of c'-level $m + 11$ (a vertex of $S_{m+11}(c')$, so $J_3 \in T_3$) and J_1 the first site on T_1 (Lemma 9.16); on r_{G} let J_4 be the site of c'-level $m + 5$ ($\in T_4$) and J_2 the first site on T_2. The *spokes* are Σ_{R}, the part of r_{R} from J_3 to J_1, and Σ_{G}, the part of r_{G} from J_4 to J_2. Note that $|J_1|_{c'} \geq \rho - 9$ and $|J_2|_{c'} \leq \rho - 11$.

Blocks. $Q_1 = N[W_{\mathsf{G}}] \cap T_1$ is a block by (P2). $Q_2 = N[\Sigma_{\mathsf{R}}] \cap T_2$ equals $N[r_{\mathsf{R}}] \cap T_2$, because sites of r_{R} below J_3 have o-level at most $m + 12 < \rho - 15$ and sites above J_1 have o-level at least $\rho - 9$; so Q_2 is a block by Lemma 9.16. Likewise $Q_3 = N[\Sigma_{\mathsf{G}}] \cap T_3 = N[r_{\mathsf{G}}] \cap T_3$ is a block by Lemma 9.16 for $c' = o$ after applying $\psi_{c'}^{-1}$ (c'-levels below J_4 are less than $m + 5$ and above J_2 at least $\rho - 14 \geq m + 16$). $Q_4 = N[V_{\mathsf{R}}] \cap T_4$ is a block by (T2).

Arcs. $x_{\mathsf{R}} \notin Q_1$ by (P1); $J_1 \notin Q_1$ as r_{R} is not blocked at the top; $x_{\mathsf{G}} \notin Q_2$ for the same reason; $J_2 \notin Q_2$ since sites of distinct rays at c'-level at least $m+5 \geq 6$ are at distance at least 3 (Lemma 9.13(b)); $J_3 \notin Q_3$ by Lemma 9.13(b); $y_{\mathsf{R}} \notin Q_3$ and $J_4 \notin Q_4$ as r_{G} is not blocked at the bottom; and $y_{\mathsf{G}} \notin Q_4$ by (T1). So there are arcs $\alpha_1 \subseteq T_1 \setminus Q_1$ from x_{R} to J_1, $\alpha_2 \subseteq T_2 \setminus Q_2$ from x_{G} to J_2, $\alpha_3 \subseteq T_3 \setminus Q_3$ from J_3 to y_{R} and $\alpha_4 \subseteq T_4 \setminus Q_4$ from J_4 to y_{G}.

The design. Let $\mathsf{R} = W_{\mathsf{R}} \cup \alpha_1 \cup \Sigma_{\mathsf{R}} \cup \alpha_3 \cup V_{\mathsf{R}}$ and $\mathsf{G} = W_{\mathsf{G}} \cup \alpha_2 \cup \Sigma_{\mathsf{G}} \cup \alpha_4 \cup V_{\mathsf{G}}$. For (D1), consecutive pieces share end sites. For (D2), apart from the ports, $W_{\mathsf{R}}, W_{\mathsf{G}}$ have o-level at most $\rho - 1$, and all other pieces at most $\rho - 4$. For (D3), V_{R} and V_{G} meet $B_m(c')$ only in t_{R} and t_{G}; $\alpha_3, \Sigma_{\mathsf{R}}$ have c'-level at least $m+10$, $\alpha_4, \Sigma_{\mathsf{G}}$ at least $m+4$, and $W_{\mathsf{R}}, W_{\mathsf{G}}, \alpha_1, \alpha_2$ have o-level at least $\rho - 14 \geq m+16$. For (D4), sites can be equal or adjacent only if their o-levels (and their c'-levels) differ by at most one. With $\rho \geq m+30$:

- W_{R} (levels $\geq \rho - 7$) is separated by levels from α_2, Σ_{G}, α_4, V_{G}, and does not touch W_{G} by (P1);
- α_1 avoids $Q_1 = N[W_{\mathsf{G}}] \cap T_1$, and is separated by levels from $\alpha_2, \Sigma_{\mathsf{G}}, \alpha_4, V_{\mathsf{G}}$;
- Σ_{R} does not touch W_{G} (r_{R} is not blocked at the top), α_2 avoids Q_2, Σ_{G} is at distance at least 3 (Lemma 9.13(b)), and α_4, V_{G} have c'-levels at most $m+6$ against at least $m+11$;
- α_3 is separated by levels from $W_{\mathsf{G}}, \alpha_2, \alpha_4, V_{\mathsf{G}}$, and avoids Q_3;
- V_{R} is separated by levels from W_{G}, α_2, does not touch Σ_{G} (r_{G} is not blocked at the bottom), α_4 avoids Q_4, and (T1) separates it from V_{G}.

So (R, G) is a design for $(p_{\mathsf{R}}, p_{\mathsf{G}}; t_{\mathsf{R}}, t_{\mathsf{G}})$, which after renaming is a design for $(a, g; t_+, t_-)$. □

Corollary 9.19. *For $\rho \geq \max(200, m+30)$, every $c' \in B_1(c)$ and every escapable pair of $S_m(c')$, a pair of distinct non-adjacent non-pocket sites of $S_\rho(c)$ has no design if and only if it belongs to $X_\rho^{\mathrm{hex}}(c)$, which has exactly 12 elements.*

Proof. This follows from Lemma 9.4(e), Proposition 9.9 and Theorem 9.18. □

9.6. The corner data

We now verify (X1)–(X4) for X^{hex}. Let $\Pi = \mathbb{Z}/6$, $\delta_0 = (1,0)$, $\delta_1 = (0,1)$, $\delta_2 = (-1,1)$, $\delta_3 = (-1,0)$, $\delta_4 = (0,-1)$, $\delta_5 = (1,-1)$, and let $(h(\pi), l(\pi), m(\pi))$ be $(3,1,2)$, $(3,2,1)$, $(1,2,3)$, $(1,3,2)$, $(2,3,1)$, $(2,1,3)$ for $\pi = 0, \ldots, 5$. With $\nu_1 = e_1$, $\nu_2 = e_2$, $\nu_3 = 0$ one has $\delta_\pi = \nu_{l(\pi)} - \nu_{h(\pi)}$ and $T(\delta_\pi) = f_{h(\pi)} - f_{l(\pi)}$. For $c = (z, i)$, $\rho \geq 2$ and $k = \lfloor \rho/2 \rfloor$ put

$$A_\rho(c, \pi) = \begin{cases} (z + k\delta_\pi,\ i) & \text{if } \rho = 2k, \\ (z + k\delta_\pi - \nu_{h(\pi)},\ 1) & \text{if } \rho = 2k+1,\ i = 0, \\ (z + k\delta_\pi + \nu_{l(\pi)},\ 0) & \text{if } \rho = 2k+1,\ i = 1. \end{cases}$$

Let $(H, L) = (k, -k)$ if $\rho = 2k$, $(k+1, -k)$ if $\rho = 2k+1$ and $i = 0$, and $(k, -k-1)$ if $\rho = 2k+1$ and $i = 1$. Then $H - L = \rho$, $H \geq 1$, $L \leq -1$, and $H, -L \geq \lfloor \rho/2 \rfloor$.

Theorem 9.20 (the corner data). *For every c and $\rho \geq 2$: (1) the corners of $S_\rho(c)$ are exactly the six distinct sites $A_\rho(c, \pi)$, and $q(A_\rho(c, \pi)) = q(c) + H f_{h(\pi)} + L f_{l(\pi)}$; (2) $N_\rho(c, \pi) = S_\rho(c) \cap S_2(A) = \{\tau_{\delta_{\pi+2}} A, \tau_{\delta_{\pi-2}} A\}$ for $A = A_\rho(c, \pi)$; (3) X^{hex} satisfies (X1), (X2) and (X3) with this Π, these A_ρ and $p_0 = 2$; (4) (X4) holds with $\rho_X = 2$, $p_0 = 2$ and the data $u = \delta_{\pi+2}$, $t_0 = 2$, $\lambda = \frac{1}{2}$ for $(\pi, \pi+1)$;*

$u = \delta_{\pi+2} - \delta_\pi$, $t_0 = 1$, $\lambda = \frac{1}{2}$ *for* $(\pi, \pi+2)$*;* $u = \delta_{\pi+3}$, $t_0 = 2$, $\lambda = 1$ *for* $(\pi, \pi+3)$*; and* $-u$ *with the same* t_0, λ *for the reversed pairs.*

The label π is the direction of the corner: with $F_\pi(v) = \langle T(\delta_\pi), q(v)\rangle$, the apex $A_\rho(c,\pi)$ is the unique site of $B_\rho(c)$ maximizing $F_\pi(v) - F_\pi(c)$, with value ρ. For $\rho \geq 4$ no partner is a corner, so $|X_\rho^{\mathrm{hex}}(c)| = 12$; one partner is the corner partner and the other the depth-two partner.

Proof. (1) By Lemma 9.1, $d(c, A_\rho(c,\pi)) = \|Hf_h + Lf_l\|_1 = H - L = \rho$, and the six vectors $Hf_a + Lf_b$, $a \neq b$, are distinct. Let $C \in S_\rho(c)$, $w = q(C) - q(c)$ and $\sigma = \sigma(C)$. By Lemma 9.1(b),(c), the neighbour of C with image $q(C) + \sigma f_j$ is inner if $\sigma w_j \leq -1$ and outer if $\sigma w_j \geq 0$. Hence C is a corner iff there are distinct indices j, j', j'' with $\sigma w_j \leq -1$, $w_{j'} = 0$ and $\sigma w_{j''} \geq 0$: exactly one inner neighbour $C' = C^{(j)}$, and the neighbour of C' with index j' lies in $S_\rho(c)$ iff $w_{j'} = 0$. Writing $\alpha = -\sigma w_j \geq 1$ and $\beta = \sigma w_{j''} \geq 0$, we get $\rho = \alpha + \beta$ and $s(w) = \sigma(\beta - \alpha) = i(C) - i(c)$, which forces $(\alpha, \beta) = (k, k)$ if $\rho = 2k$ and $(\alpha, \beta) = (k+1, k)$ if $\rho = 2k+1$, and in each case $w = Hf_a + Lf_b$ for an ordered pair (a, b); so $C = A_\rho(c,\pi)$ with $(h(\pi), l(\pi)) = (a, b)$. Conversely every $A_\rho(c,\pi)$ satisfies the criterion with $j' = m(\pi)$.

(2) The sites at distance 2 from A are those with images $q(A) + f_a - f_b$, $a \neq b$ (Lemma 9.1). With $w = Hf_h + Lf_l$, the norms $\|w + f_a - f_b\|_1$ are $\rho+2, \rho-2, \rho+2, \rho+2, \rho, \rho$ for $f_a - f_b = f_h - f_l, f_l - f_h, f_h - f_m, f_m - f_l, f_l - f_m, f_m - f_h$. So the partners have images $q(A) + f_l - f_m$ and $q(A) + f_m - f_h$. For $\pi = 0, \dots, 5$ the pair $(f_l - f_m, f_m - f_h)$ is $(T(\delta_2), T(\delta_4))$, $(T(\delta_5), T(\delta_3))$, $(T(\delta_4), T(\delta_0))$, $(T(\delta_1), T(\delta_5))$, $(T(\delta_0), T(\delta_2))$, $(T(\delta_3), T(\delta_1))$, which is $\{T(\delta_{\pi+2}), T(\delta_{\pi-2})\}$ in every case; by Lemma 9.1(a) the partners are $\tau_{\delta_{\pi\pm2}} A$.

(3) For (X1), translations are automorphisms. For (X2), $A_\rho(\tau_y c, \pi) = \tau_y A_\rho(c,\pi)$ by the formula; apexes lie on $S_\rho(c)$; every pair of $X_\rho^{\mathrm{hex}}(c)$ contains a corner, which is an apex by (1); and if $A_\rho(c_1,\pi) = A_\rho(c_2,\pi)$, then $i(c_1) = i(c_2)$ (the type of the apex determines that of the centre) and $q(c_1) = q(c_2)$. (X3) follows from (2).

(4) is proved in Section 9.9: by Lemma 9.1(d), membership of a site in $B_\rho(c_j)$ is a system of linear inequalities in cube coordinates, and (X4) reduces to checking that six explicit vectors lie in the box $\{x : s(x) = H + L,\ L \leq x_j \leq H\}$ for $t_0 \leq t \leq \lambda\rho$. □

9.7. The outer boundary property

We use the notation of Sections 9.1–9.6. For a site w let $\mathrm{ex}_w(y) = |w| + d(y,w) - |y|$. Then $\mathrm{ex}_w \geq 0$, ex_w is even, and along a geodesic ending at w it is non-increasing (each step changes it by 0 or -2). A path $(z_0, \dots, z_q)$ is *outward* if $|z_j| = |z_0| + j$.

Lemma 9.21. *(a) Let P be a geodesic between two sites of the same type, with sites of that type $s_i = (z_i, \iota)$, $0 \leq i \leq j$. Then all differences $z_{i+1} - z_i$ come from a pair $\{\delta, \delta'\}$ of vectors consecutive in the cyclic order $e_1, e_2, -e_1 + e_2, -e_1, -e_2, e_1 - e_2$ of U_6, and conversely every such sequence gives a geodesic; moreover $|s_i| \leq \max(|s_0|, |s_j|)$. (b) Let $Z = (w = z_0, \dots, z_q)$ be outward. Then $d(z_j, w) = j$, and if y equals or is adjacent to some z_j with $j \geq 1$, then $\mathrm{ex}_w(y) \in \{0, 2\}$; if $\mathrm{ex}_w(y) = 2$ then $y \sim z_j$ only for $j = d(y,w) - 1$, and if $\mathrm{ex}_w(y) = 0$ then $y = z_j$ only for $j = d(y,w)$ and $y \sim z_j$ only for $j = d(y,w) \pm 1$.*

Proof. (a) Consecutive s_i differ by vectors of U_6 and $j = h(z_j - z_0)$; for $\varepsilon\lambda_l(z_j - z_0) = j$ every step has $\varepsilon\lambda_l = 1$, and the vectors with this property form a consecutive pair. On a consecutive pair each λ_l has one weak sign, so the six affine functions $\pm\lambda_l$, $-\lambda_l - 1$ are monotone along the z_i, and the

level, given by (9.1), is bounded by its end values. (b) $j \geq d(z_j, w) \geq |z_j| - |w| = j$; the rest follows from Corollary 9.3(a). □

Proposition 9.22. *(B3) holds for HEX with $c_3 = 43$, $c_4 = 50$ and $K_3 = 1$.*

Proof. Let $K \geq 1$, $L \geq 4K + 200$ and $L - K - 10 \leq |v| \leq L$.

Step 1: choice of w. Let $n = L - K - 30 \geq 3K + 170$; the sides of the polygon S_n have at least 84 steps. Following inner neighbours from v we reach $w_0 \in S_n$ with $d(v, w_0) \leq K + 30$. Call a site of S_n *deep* if its polygon distance to each vertex is at least 6. If w_0 is deep let $w = w_0$; otherwise let w be the site of the same side at polygon distance 6 from the nearest vertex. Then $d(w_0, w) \leq 12$, $d(v, w) \leq K + 42$ and $K + 50 < |w| = L - K - 30 < L - K - 10$. By ϱ, σ_0 we may assume $w = (z_w, \iota)$ with $z_w = (p, q)$ and either ($\Sigma 0$) $p, q \geq 6$, or ($\Sigma 1$) $p \leq -6$ and $p + q \geq 5$. For z with $h(z - z_w) \leq 5$ we then have $z \in \Sigma_0$, resp. $z \in \Sigma_1$ with $z_2 \geq 0$, so the levels of $A(z), B(z)$ are $2\mu(z)$, $2\mu(z) + 1$ with $\mu = \lambda_3$, resp. $\mu = \lambda_2$ (Remark 9.5); and for $h(z - z_w) \leq 4$, z lies in the interior of the sector, so inner and outer neighbours are as in Remark 9.5.

Step 2: parity. If $d(u, w) = K + 50$ and $|u| < L$, then $|u| \equiv L \pmod 2$, so $|u| \leq L - 2$ and $\mathrm{ex}_w(u) \geq n + K + 50 - (L - 2) = 22$.

Step 3. Every geodesic F from u to w has all its sites at level at most $L - 1$: its sites of the parity of L have level at most $\max(|u|, |w| + 1) \leq L - 2$ by Lemma 9.21(a), and the others are adjacent to them.

Step 4: choice of F and of the start of the outward path. Let ι be the type of w, and let $u_1 = u$ if u has type ι, and otherwise a neighbour of u closer to w; then $\mathrm{ex}_w(u_1) \geq 20$. The type-$\iota$ sites of geodesics from u_1 to w come from multisets $\{\delta^a, \delta'^b\}$ with $a\delta + b\delta' = z_w - z_{u_1}$ (Lemma 9.21(a)). For real $\alpha, \beta \geq 0$ let $\mathcal{E}(\alpha, \beta) = f(z_w) + \alpha + \beta - f(z_w - \alpha\delta - \beta\delta')$, with $f = h$ for $\iota = 0$ and $f = g$ for $\iota = 1$. Then $2\mathcal{E}(\alpha, \beta)$ is ex_w of the type-ι site $(z_w - \alpha\delta - \beta\delta', \iota)$ for integers $0 \leq \alpha \leq a$, $0 \leq \beta \leq b$; $\mathcal{E}$ is concave, $\mathcal{E}(0, 0) = 0$ and $\mathcal{E}(a, b) \geq 10$. For $\alpha + \beta \leq 5$, Step 1 gives $\mathcal{E}(\alpha, \beta) = c\alpha + c'\beta$ with $c = 1 + \mu(\delta)$, $c' = 1 + \mu(\delta') \in \{0, 1, 2\}$, and concavity gives $ca + c'b \geq 10$. Call a direction *up*, *flat* or *down* if its coefficient is 2, 1 or 0.

(U) If a direction of positive multiplicity is up, put one step of it last; the type-ι site s_* one step before w has $\mathrm{ex}_w(s_*) = 4$.

(FF) Otherwise every direction of positive multiplicity is flat or down. The two flat directions are opposite ($\pm(e_1 - e_2)$ in case ($\Sigma 0$), $\pm e_1$ in case ($\Sigma 1$)) and hence not consecutive, so at most one of δ, δ' is flat; as down directions contribute 0, $ca + c'b \geq 10$ forces a flat direction of multiplicity at least 10. Put two of its steps last; the type-ι site s_* two steps before w has $\mathrm{ex}_w(s_*) = 4$.

Let F be the resulting geodesic from u (through u_1) to w, and T the set of its sites strictly between s_* and w (one site in case (U), three in case (FF)). All sites of F up to s_* have $\mathrm{ex}_w \geq 4$. In case (U) the site of T has level $|w| - 1$, distance 1 and $\mathrm{ex}_w = 2$, so by Lemma 9.21(b) it touches no site of $Z \setminus \{w\}$ for any outward Z from w. In case (FF) the following table lists the sites of T (from w backwards, with distance to w, level minus $|w|$, and ex_w) and the first two sites z_1, z_2 of the outward path Z that we choose:

case	w	**flat step into** w	T	z_1, z_2
$\Sigma 0$, A	$A(p,q)$	$-e_1+e_2$	$B(p,q-1)$ (1,−1,2), $A(p+1,q-1)$ (2,0,2), $B(p+1,q-2)$ (3,−1,4)	$B(p,q)$, any
$\Sigma 0$, A	$A(p,q)$	e_1-e_2	$B(p-1,q)$ (1,−1,2), $A(p-1,q+1)$ (2,0,2), $B(p-2,q+1)$ (3,−1,4)	$B(p,q)$, any
$\Sigma 0$, B	$B(p,q)$	$-e_1+e_2$	$A(p+1,q)$ (1,1,0), $B(p+1,q-1)$ (2,0,2), $A(p+2,q-1)$ (3,1,2)	$A(p,q+1)$, $B(p,q+1)$
$\Sigma 0$, B	$B(p,q)$	e_1-e_2	$A(p,q+1)$ (1,1,0), $B(p-1,q+1)$ (2,0,2), $A(p-1,q+2)$ (3,1,2)	$A(p+1,q)$, $B(p+1,q)$
$\Sigma 1$, A	$A(p,q)$	e_1	$B(p-1,q)$ (1,1,0), $A(p-1,q)$ (2,0,2), $B(p-2,q)$ (3,1,2)	$B(p,q)$, $A(p,q+1)$
$\Sigma 1$, A	$A(p,q)$	$-e_1$	$B(p,q)$ (1,1,0), $A(p+1,q)$ (2,0,2), $B(p+1,q)$ (3,1,2)	$B(p-1,q)$, $A(p-1,q+1)$
$\Sigma 1$, B	$B(p,q)$	e_1	$A(p,q)$ (1,−1,2), $B(p-1,q)$ (2,0,2), $A(p-1,q)$ (3,−1,4)	$A(p,q+1)$, any
$\Sigma 1$, B	$B(p,q)$	$-e_1$	$A(p+1,q)$ (1,−1,2), $B(p+1,q)$ (2,0,2), $A(p+2,q)$ (3,−1,4)	$A(p,q+1)$, any

In each row z_1 is an outer neighbour of w not in T and z_2 an outer neighbour of z_1 ("any": z_1 is the only outer neighbour of w and z_2 is arbitrary). By Lemma 9.21(b) a site $y \in T$ can touch $Z \setminus \{w\}$ only if $\mathrm{ex}_w(y) \in \{0,2\}$, and then only at z_1 or z_2. These finitely many non-adjacencies are read off from the neighbour lists $N(A(x,y)) = \{B(x,y), B(x-1,y), B(x,y-1)\}$ and $N(B(x,y)) = \{A(x,y), A(x+1,y), A(x,y+1)\}$. For instance, in row 3: $A(p+1,q) \neq z_1$, $A(p+1,q) \not\sim B(p,q+1)$, $B(p+1,q-1) \not\sim A(p,q+1)$ and $A(p+2,q-1) \not\sim B(p,q+1)$; in rows 1, 2, 7, 8 the only condition is that the site with $\mathrm{ex}_w = 2$ at distance 2 is not adjacent to z_1.

Step 5: the path. Continue Z by outer neighbours to S_L; then $Z = (z_0, \dots, z_{q'})$ with $q' = L - |w| = K + 30$. Let P' be F followed by $z_1, \dots, z_{q'}$. Every site of $F \setminus \{w\}$ with $\mathrm{ex}_w \leq 2$ lies in T, so by Lemma 9.21(b) and Step 4 no site of $F \setminus \{w\}$ touches $Z \setminus \{w\}$. Both F and Z are geodesics, hence induced, so P' is an induced path. Its sites on F have level less than L (Step 3), $|z_j| < L$ for $j < q'$, $z_{q'} \in S_L$, and w is an interior site. The sites of F other than u are within distance $K + 49$ of w, and $d(z_j, w) = j \leq K + 30$; so $P' \setminus \{u\} \subseteq B_{K+49}(w)$. □

9.8. Proofs of the three local lemmas

For sites v, w with coordinates z_v, z_w, (9.1) gives

$$d(v,w) \leq 2r+1 \implies |\lambda_l(z_v) - \lambda_l(z_w)| \leq r+1 \quad (l = 1,2,3), \tag{$*$}$$

and even $\leq r$ if v, w have the same type.

Proof of Lemma 9.14 (port escapes)

Sites of $S_\rho(o)$ have one type, so $d(a,g)$ is even; the pair is *close* if $d(a,g) \leq 20$ and *far* otherwise.

Computation. For $\rho^* \in \{200, 201\}$ the search of [S, Part Z, Section 2] provides, for every close non-exceptional pair, a labelling and paths $W_{\mathsf{R}}, W_{\mathsf{G}}$ with (P1) and (P2), and for every single site g a path W_{G} with (P2).

Windows. For a close pair put $Y = W_{\mathsf{R}} \cup N[W_{\mathsf{G}}] \cup N[a] \cup N[g]$. As $W_{\mathsf{R}} \subseteq B_7(p_{\mathsf{R}})$, $N[W_{\mathsf{G}}] \subseteq B_{14}(p_{\mathsf{G}})$ and $d(a, g) \le 20$, we have $N[Y] \subseteq B_{35}(a)$, whichever port is called a. For a single site, $N[N[W_{\mathsf{G}}] \cup N[g]] \subseteq B_{15}(g)$.

Transfer maps. Let $\rho \ge 202$, let $\rho^* \in \{200, 201\}$ have the parity of ρ, $k = \lfloor \rho/2 \rfloor \ge 101$, $k^* = 100$, and $a \in S_\rho(o)$. We construct an automorphism Θ_a with

$$|\Theta_a v| = |v| - (\rho - \rho^*) \qquad \text{for all } v \in B_{35}(a). \tag{$**$}$$

Step 1. If a is at polygon distance at most 40 from a vertex, let $\theta \in \langle \varrho, \sigma_0 \rangle$ map this vertex to vertex 0. Otherwise a is an interior point of a side j at polygon distance more than 40 from both ends; let $\theta \in \langle \varrho \rangle$ map side j onto side 0 or side 5. Then $b = \theta(a)$ satisfies (i) $z_b = (k - t, t)$ with $0 \le t \le 40$ or $z_b = (k, t)$ with $-40 \le t < 0$; or (ii) $z_b = (k - t, t)$ with $40 < t < k - 40$, or $z_b = (k, t)$ with $t < -40$ and $k + t \ge 40$. *Step 2* (case (ii) only). Let $\tau_2 = \tau_{(40-t)u_0}$, resp. $\tau_{(-40-t)u_5}$, so that $z_{\tau_2 b} = (k - 40, 40)$, resp. $(k, -40)$. By $(*)$ with $r = 17$, every site of $B_{35}(b)$ and of its image lies in Σ_0, resp. Σ_5 (with margins at least 22), so τ_2 preserves levels there by Lemma 9.17(b). In case (i) let $\tau_2 = \mathrm{id}$. *Step 3.* Let $\tau_3 = \tau_{(-(k-k^*),0)}$. Every site of $B_{35}(\tau_2 b)$ has $\lambda_1, \lambda_3 \ge k - 58 \ge 43$, and every site of its image has $\lambda_1, \lambda_3 \ge 42$; so both lie in W_0, and τ_3 lowers levels by $2(k - k^*) = \rho - \rho^*$ there (Lemma 9.17(a)). Then $\Theta_a = \tau_3 \tau_2 \theta$ satisfies $(**)$.

Close pairs. Let (a, g) be a close pair of $S_\rho(o)$, $\rho \ge 202$, with $\{a, g\} \notin X_\rho^{\mathrm{hex}}(o)$, and $\Theta = \Theta_a$. By Lemma 9.17(d) with $Y_0 = N[a] \cup N[g]$, the sites $\Theta a, \Theta g$ lie in $S_{\rho^*}(o)$, at distance $d(a, g) \le 20$, and $\{\Theta a, \Theta g\} \notin X_{\rho^*}^{\mathrm{hex}}(o)$ (by part (v)). The computation gives paths $W_{\mathsf{R}}^*, W_{\mathsf{G}}^*$ at ρ^* whose window satisfies $N[Y^*] \subseteq B_{35}(\Theta a) = \Theta B_{35}(a)$. Lemma 9.17(d) for Θ^{-1} shows that $W_{\mathsf{R}} = \Theta^{-1} W_{\mathsf{R}}^*$ and $W_{\mathsf{G}} = \Theta^{-1} W_{\mathsf{G}}^*$ start at a and g (in some order), that their i-th sites have level $\rho - i$, that they do not touch, and that $N[W_{\mathsf{G}}] \cap C_{\rho-8}(o) = \Theta^{-1}(N[W_{\mathsf{G}}^*] \cap C_{\rho^*-8}(o))$ is a block.

Far pairs. For every site g of $S_\rho(o)$, $\rho \ge 200$, there is a path W_{G} with (P2), by the computation and the same transfer. Let W_{R} be any path from a along inner neighbours. As $W_{\mathsf{R}} \subseteq B_7(a)$ and $W_{\mathsf{G}} \subseteq B_{13}(g)$, a touching would give $d(a, g) \le 21$. So (P1) holds. □

Proof of Lemma 9.15 (terminal escapes)

Applying $\psi_{c'}^{-1}$ we may take $c' = o$. The pair is *close* if $d(t_+, t_-) \le 16$ and *far* otherwise.

(T3) is automatic for $m \ge 12$. Sites of $N[V_{\mathsf{R}}]$ are at mutual distance at most 13. Two of them on distinct apex rays at levels at least $m + 5$ would be at distance at least $m + 2$ by Lemma 9.13(b). So for $m \ge 12$ at most one ray is touched at levels $\ge m + 5$.

Computation. The search of [S, Part Z, Section 2] covers every escapable pair for $1 \le m \le 40$, and every close escapable pair and every single site (with (T2), (T3)) for $41 \le m \le 160$.

Far pairs. Let V_{R} start at one terminal and satisfy (T2), (T3), and let V_{G} be any outward path from the other. As $V_{\mathsf{R}} \subseteq B_{11}(t_{\mathsf{R}})$ and $V_{\mathsf{G}} \subseteq B_5(t_{\mathsf{G}})$, a touching would give $d(t_+, t_-) \le 17$.

Windows and transfer for $m \ge 161$. For a close pair put $Y = N[V_{\mathsf{R}}] \cup V_{\mathsf{G}} \cup N[t_+] \cup N[t_-]$; then $N[Y] \subseteq B_{29}(t_+)$ for either labelling. Let $m^* \in \{159, 160\}$ have the parity of m. Define Θ_t for

$t \in S_m(o)$ as in the transfer maps of the proof of Lemma 9.14, with m, m^* in place of ρ, ρ^*, the threshold 30 in place of 40 and the window $B_{29}(t)$; by $(*)$ with $r = 14$ the windows lie in Σ_0, resp. Σ_5, with margins at least 15, and in W_0 with margins at least 34, so $|\Theta_t v| = |v| - (m - m^*)$ on $B_{29}(t)$. For a close escapable pair, $(\Theta t_+, \Theta t_-)$ is an escapable pair of $S_{m^*}(o)$ at the same distance (the escape witnesses are carried over by Lemma 9.17(d)(i)), and the paths found at m^* are carried back by Θ^{-1} as in the proof of Lemma 9.14; (T3) holds as $m \geq 12$. Single sites are treated in the same way with the window $B_{13}(t)$. □

Proof of Lemma 9.16 (rays crossing tracks)

The computation of [S, Part Z, Section 2] covers $4 \leq n \leq 240$ for all four $c' \in B_1(o)$ and all j. Let $n \geq 241$, assume the assertion for $n-2$, fix c' and j, and let R_ℓ be the site of $r_j(c')$ with $|R_\ell|_{c'} = \ell$. Let $w_j \in U_6$ with $a_j(k) = kw_j$ ($w_0 = e_1$, $w_1 = e_2$, $w_2 = -e_1 + e_2$, $w_{j+3} = -w_j$), and note that $b_j(k) = kw_j - f'_j$ with $f'_j = 0, 0, e_1, e_1, e_2, e_2$. Put $J = j$, $\delta_j = w_j$ if $c' = o$, and $J = j + 3$, $\delta_j = -w_j = w_J$ if c' is one of the three B-sites adjacent to o.

(a) $\tau_{\delta_j} R_\ell = R_{\ell+2}$. For $c' = o$, $R_{2k} = A(kw_j)$ and $R_{2k+1} = B(kw_j - f'_j)$. For $c' = B(w)$, $R_{2k} = B(w - kw_j)$ and $R_{2k+1} = A(w - kw_j + f'_j)$.

(b) The ray runs at offset 0 *or* 1 *from vertex* J. There are $\varepsilon \in \{-1, 0, 1\}$ and $\omega \in \{0, 1\}$, depending only on (c', j), such that for every $\ell \geq 1$, $|R_\ell| = \ell + \varepsilon$ and R_ℓ is the site of the polygon $S_{|R_\ell|}(o)$ at cyclic distance ω from its vertex J. For $c' = o$, $\varepsilon = \omega = 0$. For the B-neighbours the values are read off by writing R_{2k} and R_{2k+1} in terms of the vertices $a_J(\cdot), b_J(\cdot)$ of Lemma 9.4:

c'	j	J	R_{2k}	R_{2k+1}	ε	ω
$B(0,0)$	0	3	$B(b_3(k-1)) = B(-k, 0)$	$A(a_3(k)) = A(-k, 0)$	-1	0
$B(0,0)$	1	4	$B(b_4(k-1)) = B(0, -k)$	$A(a_4(k)) = A(0, -k)$	-1	0
$B(0,0)$	2	5	$B(b_5(k) + u_5) = B(k, -k)$	$A(a_5(k+1) + u_5) = A(k+1, -k)$	$+1$	1
$B(0,0)$	3	0	$B(b_0(k)) = B(k, 0)$	$A(a_0(k+1)) = A(k+1, 0)$	$+1$	0
$B(0,0)$	4	1	$B(b_1(k)) = B(0, k)$	$A(a_1(k+1)) = A(0, k+1)$	$+1$	0
$B(0,0)$	5	2	$B(b_2(k) - u_1) = B(-k, k)$	$A(a_2(k+1) - u_1) = A(-k, k+1)$	$+1$	1
$B(-1,0)$	0	3	$B(b_3(k)) = B(-k-1, 0)$	$A(a_3(k+1)) = A(-k-1, 0)$	$+1$	0
$B(-1,0)$	1	4	$B(b_4(k) - u_3) = B(-1, -k)$	$A(a_4(k+1) - u_3) = A(-1, -k)$	$+1$	1
$B(-1,0)$	2	5	$B(b_5(k-1)) = B(k-1, -k)$	$A(a_5(k)) = A(k, -k)$	-1	0
$B(-1,0)$	3	0	$B(b_0(k-1)) = B(k-1, 0)$	$A(a_0(k)) = A(k, 0)$	-1	0
$B(-1,0)$	4	1	$B(b_1(k) + u_1) = B(-1, k)$	$A(a_1(k+1) + u_1) = A(-1, k+1)$	$+1$	1
$B(-1,0)$	5	2	$B(b_2(k)) = B(-k-1, k)$	$A(a_2(k+1)) = A(-k-1, k+1)$	$+1$	0
$B(0,-1)$	0	3	$B(b_3(k) + u_3) = B(-k, -1)$	$A(a_3(k+1) + u_3) = A(-k, -1)$	$+1$	1
$B(0,-1)$	1	4	$B(b_4(k)) = B(0, -k-1)$	$A(a_4(k+1)) = A(0, -k-1)$	$+1$	0
$B(0,-1)$	2	5	$B(b_5(k)) = B(k, -k-1)$	$A(a_5(k+1)) = A(k+1, -k-1)$	$+1$	0
$B(0,-1)$	3	0	$B(b_0(k) - u_5) = B(k, -1)$	$A(a_0(k+1) - u_5) = A(k+1, -1)$	$+1$	1
$B(0,-1)$	4	1	$B(b_1(k-1)) = B(0, k-1)$	$A(a_1(k)) = A(0, k)$	-1	0
$B(0,-1)$	5	2	$B(b_2(k-1)) = B(-k, k-1)$	$A(a_2(k)) = A(-k, k)$	-1	0

Each entry is verified by substituting coordinates; a site $b_J(k) \pm u_i$ or $a_J(k) \pm u_i$ in the table is the polygon site next to the vertex along an adjacent side. Rows with $\varepsilon = -1$ hold for $k \geq 1$ (R_{2k}) and

$k \geq 0$ (R_{2k+1}), the others for all $k \geq 0$.

(c) On the window, τ_{w_J} raises levels by exactly 2. Let $\Theta_J \in \langle \varrho, \sigma_0 \rangle$ map vertex 0 to vertex J of every sphere around o; it acts on A-sites by a linear map L_J with $L_J e_1 = w_J$, and $\Theta_J \tau_u \Theta_J^{-1} = \tau_{L_J u}$. For $v \in W^J = \Theta_J(W_0)$, Lemma 9.17(a) gives $|\tau_{w_J} v| = |\tau_{e_1} \Theta_J^{-1} v| = |\Theta_J^{-1} v| + 2 = |v| + 2$. Moreover $B_4(v_q^0) \subseteq W_0$ for $q \geq 4$, hence $B_4(v_q^J) \subseteq W^J$, and by (b), $B_2(R_\ell) \subseteq B_4(v^J_{|R_\ell|})$. So τ_{δ_j} raises o-levels by exactly 2 on $B_2(R_\ell)$ whenever $|R_\ell| \geq 4$.

Induction step. A site of $C_n(o)$ has o-level in $[n, n+2]$, so a ray site equal or adjacent to it is R_ℓ with $n-1-\varepsilon \leq \ell \leq n+3-\varepsilon$. Put $Y = N[\{R_\ell : n-3-\varepsilon \leq \ell \leq n+1-\varepsilon\}]$. Then $N[Y]$ lies in the union of the balls $B_2(R_\ell)$ over these ℓ, which have o-levels at least $n-3 \geq 238$, so $\Theta = \tau_{\delta_j}$ raises levels by 2 on $N[Y]$, and $\Theta(Y) = N[\{R_\ell : n-1-\varepsilon \leq \ell \leq n+3-\varepsilon\}]$ by (a). Hence, by Lemma 9.17(d),

$$N[r_j(c')] \cap C_n(o) = \Theta\big(N[r_j(c')] \cap C_{n-2}(o)\big), \qquad r_j(c') \cap C_n(o) = \Theta\big(r_j(c') \cap C_{n-2}(o)\big),$$

and the assertion for n follows from that for $n-2$. □

9.9. Proof of (X4) for X^{hex}

Let $R_\pi = \{T(\delta_{\pi+2}), T(\delta_{\pi-2})\}$, so that the partners of $A_\rho(c,\pi)$ have images $q(A_\rho(c,\pi)) + r$, $r \in R_\pi$ (Theorem 9.20(2)); in the notation of its proof, $R_\pi = \{f_l - f_m, f_m - f_h\}$.

Lemma 9.23. *Let $A_1 = A_\rho(c_1,\pi_1)$ and $A_2 = A_\rho(c_2,\pi_2) = \tau_{tu} A_1$. Then $i(c_1) = i(c_2) =: i$. With (H,L) for this i, $a_\pi = H f_{h(\pi)} + L f_{l(\pi)}$, $U = T(u)$ and $\mathfrak{B} = \{x \in \mathbb{Z}^3 : s(x) = H + L,\ L \leq x_j \leq H\}$: $d(A_1,A_2) = |t|\,\|U\|_1$; $A_2 \in B_\rho(c_1)$ iff $a_{\pi_1} + tU \in \mathfrak{B}$, and $N_\rho(c_2,\pi_2) \subseteq B_\rho(c_1)$ iff $a_{\pi_1} + tU + r \in \mathfrak{B}$ for both $r \in R_{\pi_2}$; symmetrically $A_1 \in B_\rho(c_2)$ iff $a_{\pi_2} - tU \in \mathfrak{B}$, and $N_\rho(c_1,\pi_1) \subseteq B_\rho(c_2)$ iff $a_{\pi_2} - tU + r \in \mathfrak{B}$ for both $r \in R_{\pi_1}$.*

Proof. Translations preserve types, and the type of an apex determines that of its centre, so $i(c_1) = i(c_2)$. By Theorem 9.20(1), $q(A_j) = q(c_j) + a_{\pi_j}$, and $q(A_2) = q(A_1) + tU$. Hence $q(A_2) - q(c_1) = a_{\pi_1} + tU$ and $q(A_1) - q(c_2) = a_{\pi_2} - tU$. All the vectors have coordinate sum $H + L$, because $s(U) = s(r) = 0$, and by Lemma 9.1(d) a site with relative vector x, $s(x) = H + L$, lies in the ball of radius ρ iff $L \leq x_j \leq H$ for all j. □

For integers $1 \leq t \leq \lfloor \rho/2 \rfloor$ the following bounds hold, since $H - L = \rho$, $H \geq 1$, $L \leq -1$ and $H, -L \geq \lfloor \rho/2 \rfloor$: (E1) $L \leq -t \leq 1-t \leq 0 \leq t-1 \leq t \leq H$; (E2) $L \leq L+t-1 < L+t < L+t+1 \leq H$ and $L \leq H-t-1 < H-t < H-t+1 \leq H$; (E3) $L \leq L+2t-1 < L+2t \leq H$ and $L \leq H-2t < H-2t+1 \leq H$. Always (E4) $L \leq -1 < 0 < 1 \leq H$, $L \leq H-1$, $L+1 \leq H$; and for $1 \leq t \leq \rho$, (E5) $L \leq L+t-1 < L+t \leq H$ and $L \leq H-t < H-t+1 \leq H$.

Proof of Theorem 9.20(4). The conclusions of (X4) are symmetric in the indices 1 and 2, so data for (π_1,π_2) give data $(-u, t_0, \lambda)$ for (π_2,π_1); it suffices to treat $(\pi,\pi+1)$, $(\pi,\pi+2)$ ($\pi \in \Pi$) and $(\pi,\pi+3)$ ($\pi = 0,1,2$). Write (h,l,m) for the indices of π_1 and vectors of $\mathbb{Z}^3$ as triples (x_h, x_l, x_m). By Lemma 9.23 we must show that $t\|U\|_1 > 2$ and that six explicit vectors lie in $\mathfrak{B}$, i.e. that each coordinate lies in $[L,H]$. The indices of π_2 and the vectors U, R_{π_2} are read off from the table of labels.

Case A: $(\pi,\pi+1)$, π even; $u = \delta_{\pi+2}$, $t_0 = 2$, $\lambda = \frac{1}{2}$. Here $(h(\pi_2), l(\pi_2), m(\pi_2)) = (h,m,l)$, $U = f_l - f_m$, $R_{\pi_2} = \{f_m - f_l, f_l - f_h\}$, and the six vectors are $(H, L+t, -t)$, $(H, L+t-1, 1-t)$, $(H-1, L+t+1, -t)$, $(H, -t, L+t)$, $(H, 1-t, L+t-1)$, $(H-1, -t, L+t+1)$, whose coordinates lie in $[L,H]$ by (E1), (E2), (E4) for $2 \leq t \leq \rho/2$; and $d(A_1,A_2) = 2t \geq 4$.

Case B: $(\pi, \pi+1)$*,* π *odd; same data.* Here $(h(\pi_2), l(\pi_2), m(\pi_2)) = (m, l, h)$, $U = f_m - f_h$, $R_{\pi_2} = \{f_l - f_h, f_h - f_m\}$, and the vectors are $(H-t, L, t)$, $(H-t-1, L+1, t)$, $(H-t+1, L, t-1)$, $(t, L, H-t)$, $(t, L+1, H-t-1)$, $(t-1, L, H-t+1)$; $d(A_1, A_2) = 2t \geq 4$.

Case C: $(\pi, \pi+2)$*,* π *even;* $u = \delta_{\pi+2} - \delta_\pi$*,* $t_0 = 1$*,* $\lambda = \frac{1}{2}$. Here $(h(\pi_2), l(\pi_2), m(\pi_2)) = (l, m, h)$, $U = -f_h + 2f_l - f_m$, $R_{\pi_2} = \{f_m - f_h, f_h - f_l\}$, and the vectors are $(H-t, L+2t, -t)$, $(H-t-1, L+2t, 1-t)$, $(H-t+1, L+2t-1, -t)$, $(t, H-2t, L+t)$, $(t, H-2t+1, L+t-1)$, $(t-1, H-2t, L+t+1)$, in $\mathfrak{B}$ by (E1)–(E3) for $1 \leq t \leq \rho/2$; $d(A_1, A_2) = 4t \geq 4$.

Case D: $(\pi, \pi+2)$*,* π *odd; same data.* Here $(h(\pi_2), l(\pi_2), m(\pi_2)) = (m, h, l)$, $U = -2f_h + f_l + f_m$, $R_{\pi_2} = \{f_h - f_l, f_l - f_m\}$, and the vectors are $(H-2t, L+t, t)$, $(H-2t+1, L+t-1, t)$, $(H-2t, L+t+1, t-1)$, $(L+2t, -t, H-t)$, $(L+2t, 1-t, H-t-1)$, $(L+2t-1, -t, H-t+1)$; $d(A_1, A_2) = 4t \geq 4$.

Case E: $(\pi, \pi+3)$*;* $u = \delta_{\pi+3}$*,* $t_0 = 2$*,* $\lambda = 1$. Here $(h(\pi_2), l(\pi_2), m(\pi_2)) = (l, h, m)$, $U = f_l - f_h$, $R_{\pi_2} = \{f_h - f_m, f_m - f_l\}$, and the vectors are $(H-t, L+t, 0)$, $(H-t+1, L+t, -1)$, $(H-t, L+t-1, 1)$, $(L+t, H-t, 0)$, $(L+t, H-t+1, -1)$, $(L+t-1, H-t, 1)$, in $\mathfrak{B}$ by (E4), (E5) for $2 \leq t \leq \rho$; $d(A_1, A_2) = 2t \geq 4$.

In all cases $d(A_1, A_2) > 2 = p_0$ and the containments hold, for every $\rho \geq 2$. □

Remark 9.24. The tables give the exact sets of $t \geq 1$ for which the conclusions of (X4) hold with the chosen u: $[2, -L]$ in Case A, $[2, H]$ in Case B, $[1, \lfloor \rho/2 \rfloor]$ in Cases C and D, and $[2, \rho]$ in Case E. So $\lambda = \frac{1}{2}$ cannot be enlarged in Cases A and B, and $t_0 = 2$ cannot be lowered in Cases A, B and E (for $t = 1$, $d(A_1, A_2) = 2 = p_0$). In Cases A and B the apex A_2 stays on the sphere $S_\rho(c_1)$, so the containment of the partners is a genuinely discrete fact.

9.10. Proof of Theorem B-HEX, and the corner constant

Proof of Theorem B-HEX. (B1) and (B2) follow from Propositions 9.7 and 9.8, and (B3) is proved in Section 9.7. $\mathrm{TP}_{X^{\mathrm{hex}}}$ follows from Theorem 9.18, and (X1)–(X4) follow from Theorem 9.20; the statement on the exact set of failing pairs follows from Corollary 9.19. □

The corner constant. The constant of Theorem C for the honeycomb lattice is computed from $n = 2$, $\Delta = \delta = 3$, $|\Pi| = 6$, $T = 2$, $U = 2$ and $\lambda_* = \frac{1}{2}$ (Theorem 9.20(4)), and $\ell = 1$, $\gamma = 5$: by Lemma 9.1(c), sites in cells z, z' with $\|z - z'\|_\infty \leq 1$ are at distance at most $1 + 1 + 3 = 5$, with equality for $A(0)$ and $B(1, 1)$, and adjacent sites lie in cells at sup-distance at most 1. Then (8.1) gives $a = 31$, $r = 1\,245\,457$, $h = 1116\,r = 1\,389\,930\,012$ and $K_{\mathrm{CL}}(\rho_0) = 50h + r + 11 = 69\,497\,746\,068$ for every $\rho_0 \leq 1\,245\,457$.

10. The Kagomé lattice and bond percolation on the honeycomb lattice

Theorem B-KAG. *The Kagomé lattice satisfies (B1) with* $M(\mathrm{KAG}) = \{m \geq 2 : m \text{ even}\}$*, (B2) with* $M'(\mathrm{KAG}) = \{n \geq 2 : n \text{ even}\}$*,* $n_0 = 2$*,* $t_0 = 3$*,* $c_2 = 2$*, (B3) with* $c_3 = 22$*,* $c_4 = 30$*,* $K_3 = 1$*, and* $TP_\emptyset$ *with* $\rho_{\mathrm{TP}} = 8$ *and* $C_0 = 7$*. Moreover, for every site* x *and every odd* $m = 2k + 1 \geq 3$ *the sphere* $S_m(x)$ *has exactly two pockets, while* $S_1(x)$ *and the spheres of even radius have none.*

The Kagomé lattice lies in the regime that [BBR] call much easier. Site percolation on KAG is bond percolation on the honeycomb lattice HEX: the bijection ε of Lemma 10.1 identifies a set of open sites of KAG with a set of open edges of HEX. The combinatorial core of our proof of $\mathrm{TP}_\emptyset$

(Theorem 10.10) is the argument outlined in [BBR, §6] for bond percolation on $\mathbb{Z}^d$, transported to KAG through the identification KAG = L(HEX). Sets of sites of KAG that do not touch are edge sets of HEX with disjoint vertex sets (Lemma 10.2(a)), and the two required connections are obtained as vertex-disjoint paths from Menger's theorem with a 2-connected core. What is specific to KAG is the verification that this argument applies under the hypotheses of $\mathrm{TP}_\emptyset$. The ports are arbitrary distinct, non-adjacent, non-pocket sites of $S_\rho(c)$, which are radial edges (Lemma 10.8). The gadget sphere is centred at an arbitrary site c' with $d(c,c') \le 1$, so that two level functions have to be controlled at the same time. The terminals form an arbitrary escapable pair. The attachments of the ports and of the terminals to the core rest on an explicit level function (Lemmas 10.3 and 10.4) and on a lemma that excludes funnels (Lemma 10.7). Finally, the boundary properties (B2) and (B3) are proved in Proposition 10.11(b) and Section 10.4. Section 10.5 deduces Corollary 1.3.

Notation. For $t = (x,y) \in \mathbb{Z}^2$ let $N(t) = \max(|x|,|y|,|x+y|)$, $s(t) = x+y$ and $\bar t = (y,x)$, and let $U_6 = \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$; points of $\mathbb{Z}^2$ differing by an element of U_6 are *T-adjacent*. Put $\Delta(w) = \{w, w+e_1, w+e_2\}$. In the honeycomb lattice HEX, $A(z) \sim B(w)$ iff $z \in \Delta(w)$; we write d_H for its distance. $[\Phi]$ is the indicator of a statement Φ.

10.1. The line graph

Lemma 10.1. *The map $\varepsilon(a(z)) = \{A(z), B(z)\}$, $\varepsilon(b(z)) = \{A(z+e_1), B(z)\}$, $\varepsilon(c(z)) = \{A(z+e_2), B(z)\}$ is a bijection from the sites of KAG onto the edges of HEX that commutes with translations, and two distinct sites of KAG are adjacent if and only if their images share an endpoint. Thus KAG is the line graph of HEX.*

Proof. Every edge of HEX is $\{A(w+\eta), B(w)\}$ for a unique w and $\eta \in \{0, e_1, e_2\}$. The three edges at $B(w)$ are the images of $a(w), b(w), c(w)$, and the three edges at $A(z)$ are the images of $a(z), b(z-e_1), c(z-e_2)$. So the pairs of sites whose images share an endpoint are the pairs inside up triangles and down triangles, which are the edges of KAG. □

We identify a site f with the edge $\varepsilon(f)$, and write $V(f)$ for its two endpoints and $V(F) = \bigcup_{f\in F} V(f)$.

Lemma 10.2. *(a) Two sets F, F' of sites touch if and only if $V(F) \cap V(F') \neq \emptyset$. (b) A nonempty set F of sites is connected if and only if the subgraph $(V(F), F)$ of HEX is connected. (c) For distinct sites, $d(f,f') = 1 + \min\{d_H(p,p') : p \in V(f), p' \in V(f')\}$. (d) A sequence $(f_0, \dots, f_q)$, $q \ge 1$, is an induced path of KAG if and only if there are pairwise distinct vertices $p_0, \dots, p_{q+1}$ of HEX with $f_j = \{p_j, p_{j+1}\}$.*

Proof. (a) and (b) follow from Lemma 10.1. (c) Common endpoints of consecutive sites of a path $f = f_0, \dots, f_k = f'$ form a walk in HEX of length at most $k-1$; conversely a geodesic $r_0, \dots, r_\ell$ of HEX from $V(f)$ to $V(f')$ gives the walk $f, \{r_0, r_1\}, \dots, \{r_{\ell-1}, r_\ell\}, f'$. (d) If the p_j are distinct, sites at index distance at least 2 have disjoint vertex sets. Conversely, in an induced path let p_j be the common endpoint of f_{j-1} and f_j; $p_j = p_{j+1}$ would make f_{j-1} and f_{j+1} adjacent, and $p_i = p_j$ with $i < j$ would make f_i and f_{j-1} equal or adjacent. □

Automorphisms. With $R(x,y) = (-x-y, x)$, the maps τ_u, $\iota : A(z) \mapsto B(-z), B(w) \mapsto A(-w)$, $\sigma : A(z) \mapsto A(\bar z), B(w) \mapsto B(\bar w)$ and $\varphi : A(z) \mapsto A(Rz+e_1), B(w) \mapsto B(Rw)$ are automorphisms of HEX (for φ use that $t \mapsto Rt + e_1$ permutes $\{0, e_1, e_2\}$), hence of KAG. The maps ι and σ fix $o = a(0)$, and $\varphi(o) = b(0)$, $\varphi^2(o) = c(0)$. So $(z,i) = \tau_z \varphi^i(o)$, and KAG is vertex-transitive.

10.2. The level function

For a site c and a vertex p of HEX let $D_c(p) = \min_{r\in V(c)} d_H(p,r)$ and $D = D_o$. By Lemma 10.2(c),

$$d(c,f) = 1 + \min_{p\in V(f)} D_c(p) \qquad \text{for every site } f \neq c. \tag{10.1}$$

D_c changes by at most one along edges, and $D_{\psi(c)} \circ \psi = D_c$ for automorphisms ψ. A neighbour p' of p is *inner*, *tangential* or *outer* (with respect to c) if $D_c(p') - D_c(p)$ is -1, 0 or 1; $\mathrm{IN}_c(p)$ and $\mathrm{OUT}_c(p)$ are the sets of inner and outer neighbours. A site f is *radial* if its endpoints have different levels, with f^{in}, f^{out} the endpoints of smaller and larger level, and *tangential* otherwise.

Lemma 10.3. *$N(t)$ is the least number of elements of U_6 with sum t, $d_H(A(0),A(z)) = d_H(B(0),B(z)) = 2N(z)$ and $d_H(A(0),B(w)) = 1 + 2\min_{t\in\Delta(w)} N(t)$. Moreover $D(A(z)) = 2N(z) - [s(z) > 0]$ and $D(B(z)) = 2N(z) - [s(z) < 0]$.*

Proof. Each element of U_6 changes x, y and $x+y$ by at most one, and $N(t)$ steps suffice (for $x,y \geq 0$ use e_1, e_2; for $x \leq 0 \leq y$ use $(-1,1)$ together with e_2 or $-e_1$; the rest by symmetry). Two A-vertices at distance two along a path, $A(t) \sim B(w) \sim A(t')$, satisfy $t' - t \in U_6 \cup \{0\}$, and every T-adjacent pair lies in a common triangle $\Delta(w)$; this gives the A–A formula, and ι gives the B–B formula. The neighbours of $B(w)$ are the $A(t)$, $t \in \Delta(w)$.

For the levels, $D(A(z)) = \min\big(2N(z), 1 + 2N(z-e_1), 1 + 2N(z-e_2)\big)$, so it suffices to show that $\min(N(z-e_1), N(z-e_2)) \leq N(z) - 1$ if and only if $s(z) > 0$. Let $n = N(z)$, $z = (x,y)$, $s = x + y$. If $s \geq 1$ and $x \geq 1$, then $|y| \leq n-1$, $0 \leq x-1 \leq n-1$ and $0 \leq s-1 \leq n-1$, so $N(z-e_1) \leq n-1$; if $s \geq 1$ and $x \leq 0$, then $y \geq 1$ and symmetrically $N(z-e_2) \leq n-1$. If $s \leq 0$, we show $N(z-e_1), N(z-e_2) \geq n$, using that one of $|x|, |y|, |s|$ equals n. If $|s| = n$, then $|s-1| = n+1$. If $x = -n$, then $|x-1| = n+1$ and $N(z-e_2) \geq |x| = n$. If $x = n > 0$, then $y \leq -x$ forces $y = -n$, so $N(z-e_1) \geq |y| = n$ and $|y-1| = n+1$. The cases $y = \pm n$ are symmetric. The formula for $B(z)$ follows from $D \circ \iota = D$ and $\iota(A(-z)) = B(z)$. □

Lemma 10.4 (sign rule). *With respect to o, and for $z = (x,y)$: $B(z-e_1)$ is an inner neighbour of $A(z)$ if $x > 0$ and an outer one if $x \leq 0$; $B(z-e_2)$ is inner if $y > 0$ and outer if $y \leq 0$; $B(z)$ is inner, tangential or outer according as $s(z) < 0, = 0, > 0$. Equivalently, for $w = (x,y)$: $A(w+e_1)$ is inner for $B(w)$ iff $x < 0$; $A(w+e_2)$ is inner iff $y < 0$; $A(w)$ is inner, tangential or outer according as $s(w) > 0, = 0, < 0$.*

Proof. The statement about $B(z)$ is Lemma 10.3. For $B(z-e_1)$, $\Delta := D(B(z-e_1)) - D(A(z)) = 2(N(z-e_1) - N(z)) + 1 - 2[s \leq 0]$ is odd, so $\Delta = \pm 1$. Let $n = N(z)$. If $s \geq 1$: for $x \geq 1$, $N(z-e_1) = n-1$ by the proof of Lemma 10.3, so $\Delta = -1$; for $x \leq 0$, $n = \max(|x|, y) \leq N(z-e_1)$, so $\Delta = 1$. If $s \leq 0$: $N(z-e_1) - n \in \{0,1\}$, so $\Delta = -1$ iff $N(z-e_1) \neq n+1$, and since the term $|y|$ does not change, $N(z-e_1) = n+1$ iff $x = -n$ or $s = -n$. For $x \geq 1$ neither holds ($s = -n$ would give $y < -n$), so $\Delta = -1$. For $x \leq 0$ one has $n = -x$ if $y \geq 0$ (as $|y|, |s| \leq -x$) and $n = -s$ if $y < 0$ (as $|x|, |y| \leq |s|$), so $\Delta = 1$. The statement for $B(z-e_2)$ follows by σ, and the second form is the first applied to the three edges at $B(w)$. □

Corollary 10.5 (distance formula). *For $z = (x,y)$: $|a(z)| = 2N(z) + [x + y = 0 \neq x]$, $|b(z)| = \max(|2x+1|, 2|y|, |2x+2y+1|)$ and $|c(z)| = \max(2|x|, |2y+1|, |2x+2y+1|)$.*

Proof. By (10.1), $|f| = 1 + \min_{p\in V(f)} D(p)$ for $f \neq o$. For $a(z)$ use Lemma 10.3. For $b(z) = \{A(z+e_1), B(z)\}$, the inner endpoint is $A(z+e_1)$ iff $x < 0$ (Lemma 10.4). With $Q = \max(|2x+1|, 2|y|, |2s+$

1|): if $x \geq 0$, $s \geq 0$, then $|b(z)| = 1+2N(z)$, which equals Q because $|y| \leq \max(x,s)$; if $x \geq 0 > s$, then $N(z) = |y| \geq \max(x+1, |s|)$ and $|b(z)| = 2|y| = Q$; if $x < 0 \leq s$, then $|b(z)| = 2N(z+e_1) = 2y = Q$; if $x < 0$, $s < 0$, then $|b(z)| = 1+2N(z+e_1) = \max(-2x-1, 2|y|+1, -2s-1) = Q$, because $|y| \geq -x$ and $|y| \geq -s$ cannot both hold. Finally σ maps $b(z)$ to $c(\bar{z})$ and fixes o. □

Corollary 10.6. *(a) Every vertex p has an outer neighbour, and if $D_c(p) \geq 1$ it has one or two inner neighbours. (b) For $c = o$ the tangential sites are exactly the sites $a(k,-k)$, $k \in \mathbb{Z}$, both of whose endpoints have level $2|k|$.*

Proof. Take $c = o$. By Lemma 10.4, $A(z)$ has no outer neighbour iff $x > 0$, $y > 0$ and $s(z) \leq 0$, which is impossible, and similarly for $B(w)$. A vertex at level at least 1 has an inner neighbour on a geodesic, and at most two since it has an outer one. (b) is the case $s(z) = 0$ of Lemma 10.4. □

Lemma 10.7 (no funnels). *(a) No vertex p with $D_c(p) \geq 2$ has two distinct outer neighbours r, r' with $\mathrm{IN}_c(r) = \mathrm{IN}_c(r') = \{p\}$. (b) No vertex p has two distinct inner neighbours r, r' with $\mathrm{OUT}_c(r) = \mathrm{OUT}_c(r') = \{p\}$.*

Proof. Take $c = o$ and, applying ι, $p = A(z)$, $z = (x,y)$, $s = x+y$. (a) By Lemma 10.4: $B(z-e_1)$ is outer iff $x \leq 0$, and then $\mathrm{IN}(B(z-e_1)) = \{A(z)\}$ iff $y \geq 0$ and $s \leq 1$; $B(z-e_2)$ is outer iff $y \leq 0$, and then $\mathrm{IN} = \{A(z)\}$ iff $x \geq 0$ and $s \leq 1$; $B(z)$ is outer iff $s > 0$, and then $\mathrm{IN} = \{A(z)\}$ iff $x, y \geq 0$. Two of these three conditions hold together only for $z = 0$, $(0,1)$ or $(1,0)$, at levels 0 and 1. (b) $B(z-e_1)$ is inner iff $x > 0$, and then $\mathrm{OUT} = \{A(z)\}$ iff $y < 0$ and $s \geq 1$; $B(z-e_2)$ is inner iff $y > 0$, and then $\mathrm{OUT} = \{A(z)\}$ iff $x < 0$ and $s \geq 1$; $B(z)$ is inner iff $s < 0$, and then $\mathrm{OUT} = \{A(z)\}$ iff $x, y < 0$. No two of these hold together. □

Lemma 10.8 (pockets). *Let $\rho \geq 1$ and $f \in S_\rho(c)$. Then $\min_{p \in V(f)} D_c(p) = \rho - 1$, and f is a pocket of $S_\rho(c)$ if and only if it is tangential. So every non-pocket $f \in S_\rho(c)$ is radial with $D_c(f^{\mathrm{in}}) = \rho - 1$ and $D_c(f^{\mathrm{out}}) = \rho$. The sphere $S_\rho(c)$ has no pocket if ρ is even or $\rho = 1$, and exactly two if $\rho = 2k+1 \geq 3$: $a(z \pm (k,-k))$ for $c = a(z)$, $b(z \pm (0,k))$ for $c = b(z)$, and $c(z \pm (k,0))$ for $c = c(z)$.*

Proof. The first claim is (10.1). If f is tangential, every site sharing an endpoint with f has an endpoint at level $\leq \rho - 1$, hence lies in $B_\rho(c)$. If f is radial, f^{out} has an outer neighbour r (Corollary 10.6), and $\{f^{\mathrm{out}}, r\} \in S_{\rho+1}(c)$ is adjacent to f. For $c = o$, Corollary 10.6(b) gives the pockets; for $c = (z,i)$ apply $\tau_z \varphi^i$, using $\varphi(a(t)) = b(Rt)$, $\varphi(b(t)) = c(Rt)$, $R(\pm(k,-k)) = \pm(0,k)$ and $R(\pm(0,k)) = \pm(-k,0)$. □

10.3. Rings and the two-port property

For $n \geq 1$ let $H_n = \{t : N(t) = n\}$, listed cyclically as $t_0 = (n,0)$, $t_{j+1} = t_j + d_j$ (indices mod $6n$), where d_j equals $(-1,1)$, $(-1,0)$, $(0,-1)$, $(1,-1)$, $(1,0)$, $(0,1)$ for j in the six consecutive blocks of length n.

Lemma 10.9 (rings). *Let $n \geq 1$. (a) Two points of H_n are T-adjacent iff they are cyclically consecutive. (b) For each j there is a unique w_j with $\{t_j, t_{j+1}\} \subseteq \Delta(w_j)$, and $\Gamma_n = (A(t_0), B(w_0), A(t_1), \ldots, A(t_{6n-1}), B(w_{6n-1}))$ is a cycle of HEX of length $12n$. (c) The vertex sets of the Γ_n are pairwise disjoint and cover all vertices except $A(0)$. (d) With $\mathrm{ind}(A(0)) = 0$ and $\mathrm{ind}(p) = n$ for $p \in \Gamma_n$, adjacent vertices have indices differing by at most one. (e) Every $p \in \Gamma_n$ satisfies $2n - 2 \leq D(p) \leq 2n + 1$.*

Proof. (a) The linear maps $t \mapsto -t$, $t \mapsto \bar{t}$ and $(x,y) \mapsto (-y, x+y)$ preserve N and U_6 and permute the corners and sides of the hexagon, so it suffices to check the corner $(0,n)$ and a point (x,n) with

$-n < x < 0$. The six T-neighbours of $(0,n)$ are $(1,n),(-1,n),(0,n+1),(0,n-1),(1,n-1),(-1,n+1)$, with $N = n+1, n, n+1, n-1, n, n+1$, and the two in H_n are its cyclic neighbours. For (x,n), the points $(x \pm 1, n)$ are its cyclic neighbours, $(x, n+1)$ and $(x-1, n+1)$ have $N \geq n+1$, and $(x, n-1)$ and $(x+1, n-1)$ have $N = n-1$. (b) A T-adjacent pair lies in exactly one triangle $\Delta(w)$. If $w_j = w_k$ with $j \neq k$, then $\Delta(w_j)$ would contain three pairwise T-adjacent points of H_n, impossible by (a). (c) The three points of $\Delta(w)$ are pairwise T-adjacent, so their values of N lie in $\{k, k+1\}$ for some $k \geq 0$. They are not all equal, by (a) (a cycle of length $6n \geq 6$ has no triangle) and since $H_0 = \{0\}$. So exactly two of them have a common value $n \geq 1$; by (a) they are consecutive in H_n, say t_j, t_{j+1}, and $B(w) = B(w_j)$ by (b). A triangle cannot contain two points of H_n and two of $H_{n'}$ with $n \neq n'$, so $B(w)$ lies on exactly one ring. (d) If $A(t) \sim B(w) \in \Gamma_n$, then $N(t) \in \{n-1, n, n+1\}$. (e) $D(A(t)) \in \{2n-1, 2n\}$ for $t \in H_n$ by Lemma 10.3, and for $B(w) \in \Gamma_n$, $D(B(w)) = \min(1 + 2\min_{\Delta(w)} N, 2N(w)) \in [2n-2, 2n+1]$. □

We use Menger's theorem [Me] (see [D, Theorem 3.3.1]): for vertex sets $\mathcal{X}, \mathcal{Y}$ of a finite graph, the maximum number of pairwise vertex-disjoint $\mathcal{X}$–$\mathcal{Y}$ paths equals the minimum size of a set of vertices separating $\mathcal{X}$ from $\mathcal{Y}$.

Theorem 10.10 ($\mathrm{TP}_\emptyset$). *Let $d(c,c') \leq 1$, $m \geq 1$ and $\rho \geq m+7$, let (t_+, t_-) be an escapable pair of $S_m(c')$, and let a, g be distinct, non-adjacent, non-pocket sites of $S_\rho(c)$. Then there is a design for $(a, g; t_+, t_-)$ in $B_\rho(c)$. Hence $TP_\emptyset$ holds with $\rho_{\mathrm{TP}} = 8$ and $C_0 = 7$.*

Proof. Take $c = o$, $D = D_o$ and $D' = D_{c'}$. If $c' \neq o$, then c' and o share an endpoint, so $|D' - D| \leq 1$; also $B_m(c') \subseteq B_{\rho-1}(o)$.

Step 1. By Lemma 10.8, $a = \{a^{\mathrm{in}}, a^{\mathrm{out}}\}$ with $D(a^{\mathrm{in}}) = \rho - 1$, $D(a^{\mathrm{out}}) = \rho$, and likewise for g; $V(a) \cap V(g) = \emptyset$ as $a \not\sim g$. An escapable site of $S_m(c')$ is not a pocket, so $t_\pm = \{t_\pm^{\mathrm{in}}, t_\pm^{\mathrm{out}}\}$ with D'-levels $m-1, m$, and $V(t_+) \cap V(t_-) = \emptyset$.

Step 2. Let $\mathcal{A} = \{p : D(p) \leq \rho - 2,\ D'(p) \geq m\}$, $\mathcal{X} = \{a^{\mathrm{in}}, g^{\mathrm{in}}\}$ and $\mathcal{Y} = \{t_+^{\mathrm{out}}, t_-^{\mathrm{out}}\}$, and let H be the subgraph of HEX induced by $\mathcal{A} \cup \mathcal{X}$. Then $\mathcal{Y} \subseteq \mathcal{A}$ and $\mathcal{X} \cap \mathcal{A} = \emptyset$.

Step 3: two disjoint $\mathcal{X}$–$\mathcal{Y}$ paths give a design. Let P_1 from a^{in} to t_σ^{out} and P_2 from g^{in} to $t_{\sigma'}^{\mathrm{out}}$ be vertex-disjoint paths in H, $\{\sigma, \sigma'\} = \{+, -\}$. Let R consist of a, the edges of P_1 and t_σ, and G of g, the edges of P_2 and $t_{\sigma'}$. (D1) follows from Lemma 10.2(b). For (D2), every edge of P_1 has an endpoint at level $\leq \rho - 2$, so it lies in $B_{\rho-1}(o)$ by (10.1), and $t_\sigma \in B_{m+1}(o) \subseteq B_{\rho-1}(o)$. For (D3), all vertices of P_1 have $D' \geq m$ (for a^{in}, $D' \geq \rho - 2 \geq m$), so its edges and a lie outside $B_m(c')$ by (10.1). For (D4), by Lemma 10.2(a) we need $V(\mathsf{R}) \cap V(\mathsf{G}) = \emptyset$, where $V(\mathsf{R}) = V(P_1) \cup \{a^{\mathrm{out}}, t_\sigma^{\mathrm{in}}\}$. The vertex a^{out} has $D = \rho$, so it is not in $\mathcal{A}$, it is not in $V(g)$, and it is not $t_{\sigma'}^{\mathrm{in}}$ (which has $D \leq m$); the vertex t_σ^{in} has $D' = m - 1$, so it is not in $\mathcal{A}$, not in $V(g)$ (where $D' \geq \rho - 2$) and not in $V(t_{\sigma'})$. The rest is symmetric.

Step 4: a core. Choose n with $m + 3 \leq 2n \leq \rho - 3$; then $n \geq 2$. By Lemma 10.9(e), every vertex of Γ_n has $D \leq 2n+1 \leq \rho - 2$ and $D' \geq 2n - 3 \geq m$, so $\Gamma_n \subseteq \mathcal{A}$; vertices of index $\geq n+1$ have $D \geq 2n$, and vertices of index $\leq n-1$ have $D \leq 2n-1$.

Step 5: attaching the ports. Starting from $x_0 = a^{\mathrm{in}}$ and $y_0 = g^{\mathrm{in}}$, choose inductively $x_{k+1} \in \mathrm{IN}(x_k)$ and $y_{k+1} \in \mathrm{IN}(y_k)$ with $x_{k+1} \neq y_{k+1}$, down to level $2n-2$. This is possible by Corollary 10.6(a) and Lemma 10.7(a), because the common level $j-1$ of the next step is at least $2n - 2 \geq 2$. The two sequences are disjoint paths, they start at index $\geq n+1$ and end at index $\leq n$, so by Lemma 10.9(d) they meet Γ_n; let P_a and P_g be their initial segments up to the first vertex on Γ_n. Their vertices

other than the first lie in $\mathcal{A}$.

Step 6: attaching the terminals. In the same way, using outer neighbours with respect to c' and Lemma 10.7(b), we obtain disjoint paths $Q_\pm$ from $t_\pm^{\mathrm{out}}$ up to D'-level $2n+3$; they start at index $\le n$ and end at index $\ge n+1$, and their initial segments up to Γ_n lie in $\mathcal{A}$.

Step 7: no vertex separates. For a vertex z of H choose $P \in \{P_a, P_g\}$ and $Q \in \{Q_+, Q_-\}$ avoiding z. Their ends on Γ_n are joined in the cycle Γ_n minus z. So no set of size at most 1 separates $\mathcal{X}$ from $\mathcal{Y}$ in H, and Menger's theorem gives two vertex-disjoint $\mathcal{X}$–$\mathcal{Y}$ paths. Step 3 gives the design. Finally $\rho \ge m+7 \ge 8$ and $C_0 = 7 \ge \kappa + 1$. □

In the designs constructed, R and G are induced paths of KAG (Lemma 10.2(d)). The proof uses only that the terminals are distinct, non-adjacent and not pockets. As explained at the start of this section, the ring Γ_n here plays the role of the 2-connected set in [BBR, §6], and Lemma 10.7 provides the disjoint attachments of arbitrary ports and terminals.

Remark on the constant C_0. The constant $C_0 = 7$ is probably not optimal: a computer search over all instances of a range of radii finds designs whenever $\rho \ge m+4$ ([S, Part Z, Section 5]). For $\rho = m+3$ and $c' \ne c$ designs can fail. Here is such an instance: $c = o$, $c' = b(0)$, $m = 4$, $\rho = 7$, $(t_+, t_-) = (b(-2,0), b(-2,1))$, $a = b(-4,0)$ and $g = b(2,1)$. Distances from o are given by Corollary 10.5. Distances from $b(0) = \varphi(o)$ are $d(b(0), v) = |\varphi^{-1}(v)|$, where $\varphi^{-1}(a(u)) = c(R^{-1}u)$, $\varphi^{-1}(b(u)) = a(R^{-1}u)$, $\varphi^{-1}(c(u)) = b(R^{-1}u)$ and $R^{-1}(x,y) = (y, -x-y)$; indeed $\varphi(a(t)) = b(Rt)$, $\varphi(b(t)) = c(Rt)$ and $\varphi(c(t)) = a(Rt)$. Thus $d(b(0), t_+) = |a(0,2)| = 4$ and $d(b(0), t_-) = |a(1,1)| = 4$. Two sites of the orbit b are never adjacent, so (t_+, t_-) is escapable by Proposition 10.11(a). Next, $|a| = |g| = 7$, a and g are not adjacent, and they are not pockets of $S_7(o)$, whose pockets are $a(\pm(3,-3))$ (Lemma 10.8). The neighbours of g are $a(2,1)$, $c(2,1)$, $a(3,1)$ and $c(3,0)$, with $|\cdot| = 6, 7, 8, 7$. The neighbours of $a(2,1)$ other than g are $c(2,1)$, with $|c(2,1)| = 7$, and $b(1,1)$, $c(2,0)$, with $d(b(0), b(1,1)) = |a(1,-2)| = 4$ and $d(b(0), c(2,0)) = |b(0,-2)| = 4$. Suppose that (R, G) is a design for $(a, g; t_+, t_-)$. By (D3), $\mathsf{G} \cap B_4(b(0)) = \{t\}$ with $t \in \{t_+, t_-\}$; by (D2), $\mathsf{G} \setminus \{g\} \subseteq B_6(o)$; and by (D1), G contains a path from g to t. Its second site is a neighbour of g in $B_6(o)$, that is $a(2,1)$, which is not t. Its third site is a neighbour of $a(2,1)$ in $B_6(o)$ other than g, that is $b(1,1)$ or $c(2,0)$. Both lie in $B_4(b(0))$, so the third site is t, which is impossible since $t \in \{b(-2,0), b(-2,1)\}$. Hence there is no design for $(a, g; t_+, t_-)$ in $B_7(o)$.

Proposition 10.11 ((B1) and (B2)). *(a) For every site x and every even $m \ge 2$, $S_m(x)$ has no pockets and every pair of distinct non-adjacent sites of $S_m(x)$ is escapable. (b) (B2) holds with $M' = \{n \ge 2 : n \text{ even}\}$, $n_0 = 2$, $t_0 = 3$, $c_2 = 2$.*

Proof. (a) The sphere $S_m(x)$ has no pockets by Lemma 10.8, and all sites are radial with levels $m-1, m$. For distinct non-adjacent $t_\pm$, $V(t_+) \cap V(t_-) = \emptyset$. By Corollary 10.6(a) and Lemma 10.7(b) there are distinct outer neighbours $r_\pm$ of $t_\pm^{\mathrm{out}}$; the sites $u_\pm = \{t_\pm^{\mathrm{out}}, r_\pm\}$ lie in $S_{m+1}(x)$, and $\{t_+, u_+\}$, $\{t_-, u_-\}$ have disjoint vertex sets, since $r_\pm$ (level $m+1$) differs from the vertices of $t_\mp$ (levels $m-1, m$).

(b) The sphere S_n has no pockets by Lemma 10.8. Let $\mathcal{N} = S_{n+1} \cup S_{n+2} \cup S_{n+3}$, the set of sites f with $\min_{V(f)} D \in \{n, n+1, n+2\}$, and $k = n/2+1 \ge 2$. The edges of the ring Γ_k lie in $\mathcal{N}$ (Lemma 10.9(e)) and form a connected set. Let $f \in \mathcal{N}$ and $p \in V(f)$ of level $j = \min_{V(f)} D$. If $\mathrm{ind}(p) \ge k$, follow a geodesic from p towards $V(o)$ until its first vertex on Γ_k; if $\mathrm{ind}(p) < k$, follow outer neighbours (Corollary 10.6(a)) until the first vertex on Γ_k. In both cases the edges passed have minimal level in $[n, n+1]$ (by Lemma 10.9(d),(e)), so they lie in $\mathcal{N}$ and join f to Γ_k; hence $\mathcal{N}$ is connected. For

the distance clause: from a radial site of S_i the site $\{f^{\mathrm{out}}, r\}$, with r an outer neighbour of f^{out}, is radial, adjacent and in S_{i+1}; a pocket $v = \{p, p'\}$ is adjacent to the radial site $\{p, r\} \in S_{|v|}$; and o is adjacent to the radial site $b(0) \in S_1$. So $d(v, S_n) \le n - |v| + 1$. □

Proof of Theorem B-KAG. $\mathrm{TP}_\emptyset$ follows from Theorem 10.10, (B1) and (B2) follow from Proposition 10.11, the statement on the pockets follows from Lemma 10.8, and (B3) is proved in Section 10.4. □

For odd $m \ge 3$ the two pockets of $S_m(x)$ prevent (B1) at these radii, and this is why $M(\mathrm{KAG})$ and $M'(\mathrm{KAG})$ consist of even radii. The restriction costs nothing: (B1) is used only through Lemma G, at a single radius (Remark 7.16(1)), and the proof of Proposition 10.11(a) shows that for every $m \ge 1$ every pair of distinct, non-adjacent, non-pocket sites of $S_m(x)$ is escapable.

10.4. The outer boundary property

For a vertex p of HEX let $\pi(p) \in \mathbb{Z}^2$ be its coordinate; by Lemma 10.3, $2N(\pi(p)) - 1 \le D(p) \le 2N(\pi(p))$.

Lemma 10.12 (T-geodesics). *Let $(t_0, \dots, t_k)$ be a sequence in $\mathbb{Z}^2$ with $t_{i+1} - t_i \in U_6$ and $k = N(t_k - t_0)$. Then $N(t_i) \le \max(N(t_0), N(t_k))$ for all i, and the sequence lifts to a geodesic $A(t_0), B(\cdot), A(t_1), \dots, A(t_k)$ of HEX.*

Proof. For a form $\ell \in \{\pm x, \pm y, \pm(x+y)\}$ with $\ell(t_k - t_0) = k$, every step satisfies $\ell(d) = 1$, so the steps come from a set $\{d \in U_6 : \ell(d) = 1\}$, which is $\{e_1, e_2\}$, $\{e_1, e_1 - e_2\}$, $\{e_2, e_2 - e_1\}$ or its negative; on each such set every form ℓ' takes values of one weak sign, so ℓ' is monotone along the sequence. Consecutive points lie in a common triangle, which gives a walk of length $2k = d_H(A(t_0), A(t_k))$. □

Lemma 10.13 (sign rules at $a(q)$ and $b(q)$). *Let $q \in \mathbb{Z}^2$. (a) With respect to $a(q)$ and $(X, Y) = z - q$: $B(z - e_1)$ is inner for $A(z)$ iff $X > 0$; $B(z - e_2)$ is inner iff $Y > 0$; $B(z)$ is inner, tangential, outer according as $X + Y < 0, = 0, > 0$; and with $(X', Y') = w - q$, $A(w + e_1)$ is inner for $B(w)$ iff $X' < 0$, $A(w + e_2)$ iff $Y' < 0$, and $A(w)$ is inner, tangential, outer according as $X' + Y' > 0, = 0, < 0$. (b) With respect to $b(q)$ and $(X, Y) = z - q$: $B(z - e_2)$ is inner for $A(z)$ iff $Y > 0$; $B(z)$ is inner iff $X + Y \le 0$ (outer otherwise); $B(z - e_1)$ is inner iff $X \ge 2$, tangential iff $X = 1$, outer iff $X \le 0$; and with $(X', Y') = w - q$, $A(w + e_1)$ is inner for $B(w)$ iff $X' \le -1$, tangential iff $X' = 0$, outer iff $X' \ge 1$; $A(w + e_2)$ is inner iff $Y' \le -1$; $A(w)$ is inner iff $X' + Y' \ge 1$ (outer otherwise).*

Proof. (a) follows from $D_{a(q)} = D \circ \tau_{-q}$ and Lemma 10.4. (b) $\tau_q \varphi$ maps o to $b(q)$, and $(\tau_q \varphi)^{-1}(A(z)) = A(\tilde{z})$ with $\tilde{z} = (Y, 1 - X - Y)$, while $B(z)$, $B(z - e_1)$, $B(z - e_2)$ go to $B(\tilde{z} - e_2)$, $B(\tilde{z})$, $B(\tilde{z} - e_1)$. Apply Lemma 10.4 at $A(\tilde{z})$; the second part is the first applied to the three edges at $B(w)$. □

For $q \in \mathbb{Z}^2$ define the site $w(q)$ by $w(q) = b(q)$ if $q_x, q_y \ge 0$; $w(q) = a(q)$ if $q_x < 0 \le q_y$; and $w(q) = \iota(w(-q))$ if $q_y < 0$ (so that $w(q) = b(q - e_1)$ if $-q_x, -q_y \ge 0$ and $w(q) = a(q)$ otherwise).

Lemma 10.14. *Let $w = w(q)$. (a) $d(a(q), w) \le 1$, both endpoints r of w satisfy $N(\pi(r)) \ge N(q) - 1$, and $N(q) \le (|w| + 2)/2$. (b) For each endpoint r of w there is an infinite path $r = \zeta_0, \zeta_1, \dots$ of HEX avoiding the other endpoint, with $D(\zeta_k) = D(r) + k$ and $D_w(\zeta_k) = k$. (c) Every vertex p with $N(\pi(p)) \ge N(q) + 2$ has a neighbour p' with $D(p') = D(p) - 1$ and $D_w(p') = D_w(p) - 1$.*

Proof. (a) $a(q)$ and $b(q)$ share $B(q)$, and $a(q)$, $b(q - e_1)$ share $A(q)$; the endpoints of w have coordinates within N-distance 1 of q, so $D(r) \ge 2N(q) - 3$ and $|w| = 1 + \min_r D(r)$ gives the bound. For (b) and (c) it suffices, by ι (which fixes o), to treat $q_y \ge 0$. (b) If $q_x, q_y \ge 0$ and $w = b(q)$: from $B(q)$ take $\zeta_{2k} = B(q + ke_2)$, $\zeta_{2k+1} = A(q + (k+1)e_2)$, and from $A(q + e_1)$ take $\zeta_{2k} = A(q + (k+1)e_1)$,

$\zeta_{2k+1} = B(q + (k+1)e_1)$; by Lemmas 10.4 and 10.13(b) each step is outer for D and for D_w. If $q_x < 0 \le q_y$ and $w = a(q)$: from $A(q)$ take $\zeta_{2k} = A(q + (-k, k))$, $\zeta_{2k+1} = B(q + (-k-1, k))$, and from $B(q)$ take $\zeta_{2k} = B(q + (-k, k))$, $\zeta_{2k+1} = A(q + (-k, k+1))$; the steps $A(p) \to B(p - e_1)$ with $p_x \le q_x \le 0$ and $B(r) \to A(r + e_2)$ with $r_y \ge q_y \ge 0$ are outer for both by Lemmas 10.4 and 10.13(a). (c) If $w = a(q)$ with $q_x < 0 \le q_y$ and $p = A(t)$, a common inner neighbour exists if $t_x > 0$, $t_y > q_y$ or $s(t) < \min(0, s(q))$; if none holds, then $N(t) \le N(q)$. For $p = B(t)$ a common inner neighbour exists if $t_x < q_x$, $t_y < 0$ or $s(t) > \max(0, s(q))$, and otherwise $N(t) \le N(q)$. If $w = b(q)$ with $q_x, q_y \ge 0$: for $p = A(t)$ a common inner neighbour exists if $t_y > q_y$, $s(t) < 0$ or $t_x \ge q_x + 2$, and otherwise $N(t) \le N(q) + 1$; for $p = B(t)$ if $t_x < 0$, $t_y < 0$ or $s(t) > s(q)$, and otherwise $N(t) \le N(q)$. □

Proposition 10.15. *(B3) holds for KAG with $c_3 = 22$, $c_4 = 30$ and $K_3 = 1$.*

Proof. Let $K \ge 1$, $L \ge 4K + 120$ and $L - K - 10 \le |v| \le L$. Put $h = L - K - 20$. A geodesic from o to v contains $w_0 = (q, i)$ with $|w_0| = h$ and $d(w_0, v) \le K + 20$. Let $w = w(q)$; as $a(q), b(q), c(q)$ are pairwise adjacent, Lemma 10.14(a) gives $d(w_0, w) \le 2$, so $d(v, w) \le K + 22$, $L - K - 22 \le |w| \le L - K - 18$ (in particular $K + 30 < |w| < L - K - 10$) and $N(q) \le (L - K - 16)/2$.

Let $d(u, w) = K + 30$ and $|u| < L$. By (10.1), $\min_{V(u)} D_w = K + 29$ and $D \le L - 1$ on $V(u)$. Let $x_1 \in V(u)$ with $D_w(x_1) = K + 29$ and x_0 the other endpoint.

Part 1. We choose $x_1, \dots, x_\ell$ ($1 \le \ell \le 4$), each x_{i+1} an inner neighbour of x_i with respect to w, ending at an A-vertex $x_\ell = A(z_2)$ with $D(A(z_2)) \le L - 3$ and with $D(x_{i+1}) \le L - 2$ for $i < \ell$: if x_1 is an A-vertex with $D(x_1) \le L - 3$, take $\ell = 1$; if x_1 is a B-vertex with $D(x_1) \le L - 4$, take any inner neighbour with respect to w; otherwise $D(x_1) \ge L - 3$, and we take common inner neighbours (with respect to o and w) until an A-vertex with $D \le L - 3$ is reached, at most three steps, each at a vertex with $N(\pi(x_i)) \ge (L - 3)/2 \ge N(q) + 2$, so Lemma 10.14(c) applies.

Part 2. Let w_1 be the endpoint of w with $d_H(A(z_2), w_1) = D_w(A(z_2))$ (unique, as HEX is bipartite and the endpoints of w are adjacent) and w_2 the other. Take a T-geodesic from z_2 to the coordinate t^* of w_1 (if $w_1 = A(r)$, $t^* = r$; if $w_1 = B(r')$, $t^* \in \Delta(r')$ minimizing the distance, and then append $B(r')$), and lift it by Lemma 10.12. Its A-vertices $A(t)$ have $N(t) \le \max(N(z_2), N(t^*))$, with $2N(z_2) \le L - 2$ and $2N(t^*) \le 2N(q) + 4 \le L - 2$, so $D(A(t)) \le L - 2$. Let seg_1 be the concatenation of Parts 1 and 2. It is a geodesic from x_1 to w_1 along which D_w decreases strictly, and $d_H(p, w_1) = D_w(p)$ for its vertices.

Part 3. Let $\mathrm{seg}_2 = (\zeta_0 = w_2, \dots, \zeta_{k^*})$ be the initial part of the path of Lemma 10.14(b) from w_2, with $k^* = L - D(w_2) \in [K + 18, K + 23]$.

The path. Let P' be the sequence of sites $\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{\cdot, w_1\}, \{w_1, w_2\} = w, \{\zeta_0, \zeta_1\}, \dots, \{\zeta_{k^*-1}, \zeta_{k^*}\}$. Its underlying vertices are pairwise distinct: D_w decreases strictly along seg_1 and D increases along seg_2; vertices of seg_1 are closer to w_1 than to w_2, and vertices of seg_2 closer to w_2; and $D_w(x_0) \ge K + 29$ exceeds D_w on all other vertices except x_1. By Lemma 10.2(d), P' is an induced path. Its last site has endpoints at levels $L - 1$ and L, so it lies in S_L by (10.1); $|u| < L$ by assumption, and every other site before the last has an endpoint at level at most $L - 2$, so it has norm less than L; w is an interior site of P'; and every site after u has an endpoint with $D_w \le K + 28$, so $P' \setminus \{u\} \subseteq B_{K+29}(w)$. □

10.5. Bond percolation on the honeycomb lattice

Let $\mathcal{E}$ be the edge set of HEX. Translations act on edges, and every edge is a translate of exactly one of $\varepsilon_0 = \{A(0), B(0)\}$, $\varepsilon_1 = \{A(e_1), B(0)\}$, $\varepsilon_2 = \{A(e_2), B(0)\}$, which are the images of $a(0), b(0), c(0)$ under the map ε of Lemma 10.1. Let d_L be the distance of the line graph of HEX: for distinct edges e, f, $d_L(e,f) = 1 + \min\{d_H(x,y) : x \in e, y \in f\}$. Put $\mathfrak{B}_r(e) = \{f : d_L(e,f) \le r\}$.

A *bond configuration* is a set $\eta \subseteq \mathcal{E}$ of open edges. An *enhancement of bond percolation on HEX activated at edges* is a family $F = (F_j)_{j=0,1,2}$ of maps $\mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$ with a common range r: $F_j(\eta)$ depends only on $\eta \cap \mathfrak{B}_r(\varepsilon_j)$ and $F_j(\eta) \subseteq \mathfrak{B}_r(\varepsilon_j)$. For $e = \tau_z \varepsilon_j$ put $F_e(\eta) = \tau_z F_j(\tau_{-z}\eta)$, and for $\beta \subseteq \mathcal{E}$ put $F(\eta, \beta) = \eta \cup \bigcup_{e \in \beta} F_e(\eta)$. A *doubly infinite open path* in η is an injective $x : \mathbb{Z} \to V(\mathrm{HEX})$ with $\{x_t, x_{t+1}\} \in \eta$ for all t, and F is *essential* if for some j there is η without a doubly infinite open path such that $\eta \cup F_j(\eta)$ contains one. With η, β independent Bernoulli(p), Bernoulli(s) sets of edges, $\theta_F^{\mathrm{bond}}(p,s)$ is the probability that $A(0)$ lies in an infinite open cluster of $F(\eta,\beta)$, $\theta^{\mathrm{bond}}(p)$ is the same for η, and $p_c^{\mathrm{bond}}(\mathrm{HEX}) = \sup\{p : \theta^{\mathrm{bond}}(p) = 0\}$. These are the definitions of [BBR, §1] transcribed to bonds ([BBR] omit the details). Other natural choices of the edge metric change ranges by at most an additive constant, so the class of enhancements of finite range does not depend on this choice. We write $\Phi(\omega) = \varepsilon(\omega)$ for sets of sites of KAG; by Lemma 10.1, Φ is a bijection onto $\mathcal{P}(\mathcal{E})$ that commutes with unions and translations, and $\Phi(B_r(x)) = \mathfrak{B}_r(\varepsilon(x))$.

Lemma 10.16. *A bond configuration η contains a doubly infinite open path if and only if $\Phi^{-1}(\eta)$ contains a doubly infinite path of KAG.*

Proof. The edges of a doubly infinite open path are distinct and consecutive ones share a vertex, so they form a doubly infinite path of KAG. Conversely, let (f_t) be a doubly infinite path in $\Phi^{-1}(\eta)$, $e_t = \varepsilon(f_t)$, and let v_t be the common endpoint of e_t and e_{t+1}. Since HEX is cubic, $v_{t-1} = v_t$ implies $v_t \neq v_{t+1}$ (otherwise four distinct edges would contain v_t), and if $v_{t-1} \neq v_t$ then $e_t = \{v_{t-1}, v_t\}$. Let $T = \{t : v_{t-1} \neq v_t\} = \{t_j : j \in \mathbb{Z}\}$ (increasing), which is unbounded in both directions, and put $w_j = v_{t_j}$. Then $e_{t_{j+1}} = \{w_j, w_{j+1}\}$ is open, and $w_j = w_l$ with $l \ge j+2$ would give four distinct edges $e_{t_j}, e_{t_{j+1}}, e_{t_l}, e_{t_{l+1}}$ at one vertex. So (w_j) is a doubly infinite open path. □

The converse direction fails on graphs with vertices of degree at least 4: in the triangular lattice, the edge set of a one-way infinite chain of triangles in which consecutive triangles share exactly one vertex and non-consecutive triangles are disjoint contains no doubly infinite path, while it contains a doubly infinite path of the line graph.

Proposition 10.17. *Let F be an enhancement of bond percolation on HEX activated at edges, with range r, and let $E_a = \Phi^{-1} F_0 \Phi$, $E_b = \Phi^{-1} F_1 \Phi$, $E_c = \Phi^{-1} F_2 \Phi$. Then:*

1. *E is an enhancement of site percolation on KAG with range r, and every such enhancement arises from exactly one F;*
2. *$E(\omega, \alpha) = \Phi^{-1} F(\Phi\omega, \Phi\alpha)$;*
3. *E is essential if and only if F is;*
4. *if ω, α are independent Bernoulli(p), Bernoulli(s) sets of sites of KAG, then $\Phi\omega, \Phi\alpha$ are independent Bernoulli(p), Bernoulli(s) sets of edges;*
5. *$A(0)$ lies in an infinite open cluster of η if and only if one of the sites $a(0)$, $b(-e_1)$, $c(-e_2)$ lies in an infinite component of $\Phi^{-1}(\eta)$; hence $\theta_F^{\mathrm{bond}}(p,s) \ge \theta_E(p,s)$ and $\theta^{\mathrm{bond}}(p) \ge \theta^{\mathrm{KAG}}(p)$, where θ_E and θ^{KAG} refer to the root $o = a(0)$;*

6. $\theta^{\mathrm{bond}}(p) \le (1+2/p)\,\theta^{\mathrm{KAG}}(p)$ *for* $p \in (0,1]$, *so* $p_c^{\mathrm{bond}}(\mathrm{HEX}) = p_c(\mathrm{KAG})$.

Proof. (1) $\Phi^{-1}(\mathfrak{B}_r(\varepsilon_j))$ is the ball of radius r around $a(0)$, $b(0)$ or $c(0)$. (2) For $v = \tau_z x_j$, $E_v(\omega) = \Phi^{-1}F_{\varepsilon(v)}(\Phi\omega)$, since Φ commutes with translations. (3) follows from Lemma 10.16 and (2). (4) Φ is a bijection. (5) The three edges at $A(0)$ are $\varepsilon(a(0)), \varepsilon(b(-e_1)), \varepsilon(c(-e_2))$. An open path from $A(0)$ to x gives a path of KAG in $\Phi^{-1}(\eta)$ starting at one of these three sites; as the cluster is infinite and each edge has two endpoints, one of the three components is infinite. Conversely, the edges of an infinite component are joined to $A(0)$ by open edges and have infinitely many endpoints. Apply this to $\eta = F(\Phi\omega, \Phi\alpha) = \Phi E(\omega, \alpha)$ and use (4). (6) The sites $b(-e_1)$, $c(-e_2)$ are adjacent to $a(0)$, and for adjacent $x \sim y$ the event that x lies in an infinite component contains $\{x \in \omega\} \cap \{y$ lies in an infinite component$\}$, whose probability is at least p times that of the second event by Harris's inequality [G, §2.2]. With (5) and a union bound, $\theta^{\mathrm{bond}}(p) = 0$ iff $\theta^{\mathrm{KAG}}(p) = 0$. □

Proof of Corollary 1.3, edge activation. Let F be essential and $E = \Phi^{-1}F\Phi$. By Proposition 10.17(1),(3), E is an essential enhancement of site percolation on KAG. Theorem 1.2 for KAG gives $\pi(s) < p_c(\mathrm{KAG}) = p_c^{\mathrm{bond}}(\mathrm{HEX})$ with $\theta_E(p,s) > 0$ for $p \in (\pi(s), p_c(\mathrm{KAG}))$, and $\theta_F^{\mathrm{bond}} \ge \theta_E$ by Proposition 10.17(5). □

Vertex activation. Let $F^{\mathrm{A}}, F^{\mathrm{B}}$ be rules attached to $A(0)$ and $B(0)$, of range r in the sense that $F^{\mathrm{X}}(\eta)$ depends only on, and lies in, the set of edges with an endpoint within distance r of the vertex. Such edges lie in $\mathfrak{B}_{r+1}(\varepsilon_0)$, resp. $\mathfrak{B}_{r+1}(\varepsilon_1)$. So $E_a = \Phi^{-1}F^{\mathrm{A}}\Phi$, $E_b = \Phi^{-1}F^{\mathrm{B}}\Phi$, $E_c \equiv \emptyset$ is an enhancement of site percolation on KAG with range $r+1$ (Definition 6.2 allows different rules on different orbits, including the empty rule). Assigning $A(z)$ to $a(z)$ and $B(z)$ to $b(z)$, a Bernoulli(s) set of sites of KAG induces a Bernoulli(s) set of vertices of HEX, independent of $\Phi\omega$, and the enhanced configurations correspond. By Lemma 10.16 essentialness corresponds, and Proposition 10.17(5),(6) apply unchanged. This proves Corollary 1.3 for vertex activation. □

The crossing events of the two models do not correspond exactly, but they are not needed: Claim 5 and Theorem 6.5 are applied on KAG, and only θ and p_c are transferred. The bond case outlined in [BBR, §6] uses the 2-connectedness of the boundaries of ℓ^∞ boxes of $\mathbb{Z}^d$; the spheres of HEX are independent sets, so that outline does not apply to HEX as it stands.

11. The triangular lattice

Theorem B-TRI. *The triangular lattice satisfies (B1) with* $M(\mathrm{TRI}) = \{m \ge 3\}$, *(B2) with* $M'(\mathrm{TRI}) = \{n \ge 1\}$, $n_0 = 1$, $t_0 = 1$, $c_2 = 1$, *(B3) with* $c_3 = 20$, $c_4 = 30$, $K_3 = 1$, *and* $\mathrm{TP}_\emptyset$ *with* $\rho_{\mathrm{TP}} = 16$ *and* $C_0 = 15$.

Theorem 1.2 for the triangular lattice is stated in [BBR, Theorem 3], and its proof is outlined in [BBR, §6]. We claim nothing new here. We include a complete proof because it is the simplest instance of the method: there are no exceptional pairs, and $\mathrm{TP}_\emptyset$ follows the pattern of Section 3 (a local escape at the ports, the first step of [BBR, §6]; a step that moves arbitrary terminals outward and separates them; and a routing between two rings). Together with Theorem A and Proposition 6.6, Theorem B-TRI gives Theorem 1.2 for the triangular lattice (Section 12.11).

Since $\kappa(\mathrm{TRI}) = 0$, the gadget centre is $c' = c$. We identify V with $\mathbb{Z}^2$, so that $z \sim w$ iff $z - w \in U = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$. Two sites whose norms differ by at least 2 are neither equal nor adjacent; we call this the *layer rule*. For $x \in S_k$ let $\mathrm{in}(x)$ and $\mathrm{out}(x)$ be its sets of neighbours in S_{k-1}

and S_{k+1}.

11.1. Cones, distances and spheres

Generators and cones. Let $\xi_0 = (1,0)$, $\xi_1 = (0,1)$, $\xi_2 = (-1,1)$, $\xi_3 = (-1,0)$, $\xi_4 = (0,-1)$, $\xi_5 = (1,-1)$, indices modulo 6. Then $U = \{\xi_0, \dots, \xi_5\}$,

$$\xi_{i+3} = -\xi_i, \qquad \xi_{i+1} = \xi_i + \xi_{i+2}, \tag{11.1}$$

and each pair (ξ_i, ξ_{i+1}) is a basis of $\mathbb{Z}^2$. The linear map $\phi(z_1, z_2) = (-z_2, z_1 + z_2)$ satisfies $\phi(\xi_i) = \xi_{i+1}$, so it is an automorphism fixing o; so are the reflection $(z_1, z_2) \mapsto (z_2, z_1)$ and the translations. Let $f_0(z) = z_1 + z_2$, $f_1(z) = z_2$, $f_2(z) = -z_1$ and $f_{i+3} = -f_i$. Then $f_i = f_0 \circ \phi^{-i}$ and

$$f_i(\xi_i) = f_i(\xi_{i+1}) = 1, \qquad f_i(\xi_{i+2}) = f_i(\xi_{i+5}) = 0, \qquad f_i(\xi_{i+3}) = f_i(\xi_{i+4}) = -1. \tag{11.2}$$

The *cone* Σ_i is the set of sites $\alpha\xi_i + \beta\xi_{i+1}$ with integers $\alpha, \beta \geq 0$.

Lemma 11.1 (distance formula). *For all z, w and i: (1) $|f_i(z) - f_i(w)| \leq d(z,w)$; (2) every site lies in some cone, and if $z = \alpha\xi_i + \beta\xi_{i+1}$ with $\alpha, \beta \geq 0$ then $|z| = \alpha + \beta = f_i(z)$; (3) $|z| = \max_j f_j(z) = \max(|z_1|, |z_2|, |z_1 + z_2|)$.*

Proof. (1) By (11.2), f_i changes by at most one along an edge. (2) If $z_1, z_2 \geq 0$ then $z = z_1\xi_0 + z_2\xi_1$; if $z_1 \leq 0 \leq z_1 + z_2$ then $z = (z_1 + z_2)\xi_1 + (-z_1)\xi_2$; if $z_1 + z_2 \leq 0 \leq z_2$ then $z = z_2\xi_2 + (-z_1 - z_2)\xi_3$; the other three sign patterns follow by $z \mapsto -z$. A walk of α steps ξ_i and β steps ξ_{i+1} gives $|z| \leq \alpha + \beta$, and (1) gives $|z| \geq f_i(z) = \alpha + \beta$. (3) follows from (1) and (2). □

Consequently, for fixed z, h the function $t \mapsto |z + th| = \max_j(f_j(z) + tf_j(h))$ is convex, and on an integer interval it attains its maximum at an end.

Lemma 11.2 (spheres are hexagonal cycles). *Let $k \geq 1$. For $s \in \mathbb{Z}$ write $s = ik + j$ with $0 \leq j < k$ and put $P_k(s) = (k - j)\xi_i + j\xi_{i+1}$. Then $P_k(s + 6k) = P_k(s)$, the formula also holds for $j = k$, $s \mapsto P_k(s)$ is a bijection from $\mathbb{Z}/6k$ onto S_k, and $P_k(s) \sim P_k(s + 1)$. So S_k is a cycle of length $6k$.*

Proof. $|P_k(s)| = k$ by Lemma 11.1(2). For $s = ik + j$, (11.2) gives $f_{i-1}(P_k(s)) = k - j$, $f_i(P_k(s)) = k$, $f_{i+1}(P_k(s)) = j < k$, while $f_{i+2}, f_{i+3}, f_{i+4}$ take the values $-(k - j), -k, -j$. So $i \pmod 6$ is the index with $f_i(P_k(s)) = k$ and $f_{i+1}(P_k(s)) \neq k$, and then $j = f_{i+1}(P_k(s))$; hence $s \pmod{6k}$ is determined by $P_k(s)$. Every $x \in S_k$ is $\alpha\xi_i + \beta\xi_{i+1}$ with $\alpha + \beta = k$, hence of the form $P_k(s)$. Finally $P_k(s + 1) - P_k(s) = \xi_{i+1} - \xi_i = \xi_{i+2}$. □

The sites $P_k(ik) = k\xi_i$ are the *corners* of S_k. For $x = P_k(s)$ and $y = P_k(t)$ the *ring distance* $r_k(x, y)$ is the distance of s and t in $\mathbb{Z}/6k$. For $s = ik + j$ $(0 \leq j < k)$ put $\varphi_k(s) = i(k - 1) + j$ and $\psi_k(s) = i(k + 1) + j$, and let $c_k(I)$ be the number of multiples of k in an integer interval I.

Lemma 11.3 (neighbours). *Let $k \geq 1$ and $x = P_k(s)$ ($P_0 \equiv o$). The neighbours of x in S_k are $P_k(s \pm 1)$. If $k \mid s$, then $\mathrm{in}(x) = \{P_{k-1}(\varphi_k(s))\}$ and $\mathrm{out}(x) = \{P_{k+1}(\psi_k(s) + e) : e = -1, 0, 1\}$. If $k \nmid s$, then $\mathrm{in}(x) = \{P_{k-1}(\varphi_k(s) - 1), P_{k-1}(\varphi_k(s))\}$ and $\mathrm{out}(x) = \{P_{k+1}(\psi_k(s)), P_{k+1}(\psi_k(s) + 1)\}$. In particular no sphere has pockets.*

Proof. Write $x = \alpha\xi_i + \beta\xi_{i+1}$ with $\alpha = k - j \geq 1$, $\beta = j$. The six neighbours are $x \pm \xi_i$, $x \pm \xi_{i+1}$, $x \pm \xi_{i+2}$. If $j \geq 1$, then $x + \xi_i = P_{k+1}(i(k+1) + j)$, $x + \xi_{i+1} = P_{k+1}(i(k+1) + j + 1)$, $x - \xi_i = P_{k-1}(i(k-1) + j)$, $x - \xi_{i+1} = P_{k-1}(i(k-1) + j - 1)$ and $x \pm \xi_{i+2} = P_k(s \pm 1)$, all with non-negative coefficients. If $j = 0$,

then $x+\xi_i$, $x+\xi_{i+1}$ and, by (11.1), $x-\xi_{i+2} = \xi_{i-1}+k\xi_i$ are the three outer neighbours $P_{k+1}(i(k+1)+e)$, $e = 0, 1, -1$; $x - \xi_i = (k-1)\xi_i$ is the inner neighbour; and $x + \xi_{i+2}$, $x - \xi_{i+1} = \xi_{i-1} + (k-1)\xi_i$ are $P_k(s \pm 1)$. □

Lemma 11.4. *(a) $\varphi_k(s+t) - \varphi_k(s) = t - c_k(s, s+t]$ and $\psi_k(s+t) - \psi_k(s) = t + c_k(s, s+t]$ for $t \geq 0$. (b) For $k \geq 1$ and $x, y \in S_k$: $x \sim y$ iff $r_k(x,y) = 1$, and $d(x,y) \geq \min(r_k(x,y), k)$. (c) Two consecutive sites of S_k have exactly one common outer neighbour. (d) If $k \geq 2$ and $x, y \in S_k$ are distinct and non-adjacent, then* $\mathrm{in}(y) \not\subseteq \mathrm{in}(x)$.

Proof. (a) Both functions increase by one at each step, except that φ_k stays and ψ_k increases by two when the step reaches a multiple of k. (b) By the rotation ϕ^{-i} we may take $x = P_k(j)$, $0 \leq j < k$, and $y = P_k(t)$, $-k \leq t < 5k$. Let $-k \leq t \leq 2k$; then $r_k(x,y) = |t-j|$, as $|t-j| \leq 2k$. If $0 \leq t \leq k$, then $y - x = (t-j)\xi_2$; if $t = k + t'$ with $0 \leq t' \leq k$, then $y - x = (k-j)\xi_2 + t'\xi_3 \in \Sigma_2$; and if $t = -k + t'$ with $0 \leq t' \leq k$, then $y - x = j\xi_5 + (k-t')\xi_4 \in \Sigma_4$ (all by (11.1)). In each case Lemma 11.1(2) gives $d(x,y) = |t-j|$. For $2k < t < 5k$, $y = (k-t')\xi_{i'} + t'\xi_{i'+1}$ with $i' \in \{2,3,4\}$, so $f_{i'}(y) = k$, while $f_{i'}(x) \leq 0$; Lemma 11.1(1) gives $d(x,y) \geq k$. Adjacency on S_k follows (for $k = 1$ check $S_1 = U$ directly). (c) Let $\psi = \psi_k(p)$; positions on S_{k+1} are taken modulo $6(k+1) \geq 12$. By Lemma 11.3, $\mathrm{out}(P_k(p))$ contains $\psi, \psi+1$ and is contained in $\{\psi-1, \psi, \psi+1\}$. If $k \nmid p+1$, then $\psi_k(p+1) = \psi+1$ by (a) and $\mathrm{out}(P_k(p+1)) = \{\psi+1, \psi+2\}$. If $k \mid p+1$, then $\psi_k(p+1) = \psi+2$ and $\mathrm{out}(P_k(p+1)) = \{\psi+1, \psi+2, \psi+3\}$, while $\mathrm{out}(P_k(p)) = \{\psi, \psi+1\}$ for $k \geq 2$ (as $P_k(p)$ is not a corner) and $\{\psi-1, \psi, \psi+1\}$ for $k = 1$. In all cases the common outer neighbour is $P_{k+1}(\psi+1)$ only. (d) If y is a corner, $\mathrm{in}(y) = \{(k-1)\xi_i\}$, whose outer neighbours are y and its two neighbours on S_k; so $\mathrm{in}(y) \subseteq \mathrm{in}(x)$ would force $r_k(x,y) \leq 1$. Otherwise $\mathrm{in}(y)$ consists of two consecutive sites of S_{k-1}, and $\mathrm{in}(y) \subseteq \mathrm{in}(x)$ forces $\mathrm{in}(x) = \mathrm{in}(y)$; then x and y are common outer neighbours of two consecutive sites, and $x = y$ by (c). □

11.2. Outward steps, (B1) and (B2)

Lemma 11.5 (outward step). *Let $k \geq 1$ and $x, y \in S_k$ with $r := r_k(x,y) \geq 2$. There are $x' \in \mathrm{out}(x)$ and $y' \in \mathrm{out}(y)$ with $r_{k+1}(x', y') \geq r+1$ such that $\{x, x'\}$ and $\{y, y'\}$ do not touch.*

Proof. We may write $x = P_k(s)$, $y = P_k(s+r)$ with $2 \leq r \leq 3k$. Let $\varepsilon = 1$ if $k \mid s$ and $\varepsilon = 0$ otherwise, and take the first outer neighbour of x and the last of y in the cyclic order: $x' = P_{k+1}(\psi_k(s) - \varepsilon)$, $y' = P_{k+1}(\psi_k(s+r)+1)$. With $c = \varepsilon + c_k(s, s+r]$, Lemma 11.4(a) gives $D := (\psi_k(s+r)+1) - (\psi_k(s) - \varepsilon) = r + 1 + c$. The integers strictly between $s+r$ and $s+6k$ contain at least two multiples of k, so $c \leq 4$, $3 \leq D \leq 3k+5$, and $6(k+1) - D \geq (6k-r) + 1 \geq r+1$. So both arcs between x' and y' have length at least $r+1$. Since $\mathrm{out}(y) \subseteq \{P_{k+1}(\psi_k(s+r)+e) : |e| \leq 1\}$ and $x' = P_{k+1}(\psi_k(s+r)+1-D)$ with $1 - D - e \in [-(3k+5), -1]$, $x' \not\sim y$; symmetrically $y' \not\sim x$. Finally $x \not\sim y$ as $r \geq 2$. □

Proposition 11.6 ((B1) and (B2)). *For every $m \geq 1$, S_m has no pockets and every pair of distinct non-adjacent sites of S_m is escapable. (B2) holds with $M' = \{n \geq 1\}$, $n_0 = 1$, $t_0 = 1$ and $c_2 = 1$.*

Proof. By Lemma 11.3, S_m has no pockets. A pair of distinct non-adjacent sites has ring distance at least 2 (Lemma 11.4(b)), and Lemma 11.5 gives the escape. For (B2), S_{n+1} is a cycle, hence connected, and following outer neighbours from v reaches S_n in $n - |v|$ steps. □

11.3. The two-port property

Lemma 11.7 (escape at the ports). *Let $k \geq 3$ and let $a, g \in S_k$ be distinct and non-adjacent. There are $a_1 \in \mathrm{in}(a)$, $g_1 \in \mathrm{in}(g)$ and $g_2 \in \mathrm{in}(g_1)$ such that $\{a, a_1\}$ and $\{g, g_1, g_2\}$ do not touch.*

Proof. After exchanging the names if necessary, $a = P_k(s)$ and $g = P_k(s+d)$ with $2 \leq d \leq 3k$ (the exchange is used only to choose a_1 and g_1, and it is undone before g_2 is chosen). Let $\varepsilon = 0$ if $k \mid s$ and $\varepsilon = 1$ otherwise, and take the first inner neighbour of a and the last of g: $a_1 = P_{k-1}(q)$ with $q = \varphi_k(s) - \varepsilon$, and $g_1 = P_{k-1}(q + D)$ with $D = d - c_k(s, s+d] + \varepsilon$ (Lemma 11.4(a)). Then $2 \leq D \leq 6(k-1) - 2$: the upper bound holds as $D \leq 3k + 1$ and $k \geq 3$; for the lower bound, if $d < k$ then either $c_k(s, s+d] = 0$ or it is 1 with $\varepsilon = 1$; if $d = k$ then $D = k - 1 + \varepsilon \geq 2$; and if $d > k$ then $D \geq d - d/k - 1 \geq (k+1)(k-1)/k - 1 > 1$. Hence a_1 and g_1 are distinct and non-adjacent. By Lemma 11.3, $\mathrm{in}(g) \subseteq \{P_{k-1}(q + D - 1), P_{k-1}(q + D)\}$ and $\mathrm{in}(a) \subseteq \{P_{k-1}(q + \varepsilon - 1), P_{k-1}(q + \varepsilon)\}$, and $D - 1, D, D - \varepsilon, D - \varepsilon + 1$ lie in $[1, 6(k-1) - 1]$; so $a_1 \not\sim g$ and $g_1 \not\sim a$. Since $a_1, g_1 \in S_{k-1}$ are distinct and non-adjacent and $k - 1 \geq 2$, Lemma 11.4(d) gives $g_2 \in \mathrm{in}(g_1) \setminus \mathrm{in}(a_1)$; it is not adjacent to a_1 (a site of S_{k-2} adjacent to $z \in S_{k-1}$ lies in $\mathrm{in}(z)$) nor to a (layer rule). □

In contrast with $\mathbb{Z}^d$ there is no corner obstruction: the only sites of S_k adjacent to the inner neighbour $(k-1)\xi_i$ of a corner $k\xi_i$ are the corner and its two neighbours on the cycle S_k, which are adjacent to the corner.

Lemma 11.8 (moving the terminals outward). *Let $\rho \geq 3$ and $1 \leq k \leq \rho - 2$, let $a \neq g$ be sites of S_ρ, let $x, y \in S_k$ be distinct, and let $x' \in \mathrm{out}(x)$, $y' \in \mathrm{out}(y)$ be such that $\{x, x'\}$ and $\{y, y'\}$ do not touch. If $(\mathsf{R}', \mathsf{G}')$ is a design for $(a, g; x', y')$ in B_ρ with gadget ball B_{k+1}, then adding to each colour the site of $\{x, y\}$ adjacent to its terminal gives a design for $(a, g; x, y)$ with gadget ball B_k.*

Proof. Say $\mathsf{R}' \cap B_{k+1} = \{x'\}$ and $\mathsf{G}' \cap B_{k+1} = \{y'\}$, and put $\mathsf{R} = \mathsf{R}' \cup \{x\}$, $\mathsf{G} = \mathsf{G}' \cup \{y\}$. (D1)–(D3) are clear, using $|x'| = k + 1$. For (D4): x and all its neighbours lie in B_{k+1}, where G consists of y' and y, and $\{x, x'\}$, $\{y, y'\}$ do not touch; similarly for y. □

Lemma 11.9 (routing between two rings). *Let $n \geq 1$, $\rho = n + 4$, let $T_1, T_2 \in S_n$ with $d(T_1, T_2) \geq 12$, and let $a, g \in S_\rho$ be distinct and non-adjacent. Then there is a design for $(a, g; T_1, T_2)$ in B_ρ with gadget ball B_n.*

Proof. Ports. Lemma 11.7 gives a_1, g_1, g_2; choose $g_3 \in \mathrm{in}(g_2)$, and let $A = (a, a_1)$ and $B = (g, g_1, g_2, g_3)$. They do not touch ($g_3 \in S_{n+1}$ by the layer rule). *Terminals.* For $\nu = 1, 2$ choose an outward path $\Lambda_\nu = (T_\nu, \lambda_\nu^1, \lambda_\nu^2, \lambda_\nu^3)$, $\lambda_\nu^l \in S_{n+l}$. If both touched B, then $d(T_1, T_2) \leq 3+1+3+1+3 = 11$; so some $\Lambda = \Lambda_\mu$ does not touch B. Let μ' be the other index and $M = (T_{\mu'}, \lambda_{\mu'}^1)$; Λ and M do not touch, since otherwise $d(T_1, T_2) \leq 5$. *Upper ring.* $I = N[B] \cap S_{n+3}$ consists of at most five consecutive sites of the cycle S_{n+3} around g_1 (Lemma 11.3), and $|S_{n+3}| \geq 24$, so $S_{n+3} \setminus I$ is a path. It contains a_1 and λ_μ^3; let Γ be the subpath between them. *Lower ring.* Likewise $I' = N[\Lambda] \cap S_{n+1}$ consists of at most five consecutive sites, and $S_{n+1} \setminus I'$ contains g_3 and $\lambda_{\mu'}^1$; let Γ' be the subpath between them. *Design.* $\mathsf{R} = A \cup \Gamma \cup \Lambda$ and $\mathsf{G} = B \cup \Gamma' \cup M$ satisfy (D1)–(D3), with $\mathsf{R} \cap B_n = \{T_\mu\}$ and $\mathsf{G} \cap B_n = \{T_{\mu'}\}$. For (D4): A–B, Λ–B, Λ–M, Γ–B and Γ'–Λ were arranged above, and the remaining pairs lie in $S_{n+3} \cup S_{n+4}$ and $S_n \cup S_{n+1}$ respectively. □

Theorem 11.10 ($\mathrm{TP}_\emptyset$). *Let $m \geq 1$ and $\rho \geq m + 15$, let (t_+, t_-) be an escapable pair of $S_m(c)$, and let a, g be distinct non-adjacent sites of $S_\rho(c)$. Then there is a design for $(a, g; t_+, t_-)$ in $B_\rho(c)$. Hence $\mathrm{TP}_\emptyset$ holds with $\rho_{\mathrm{TP}} = 16$ and $C_0 = 15$.*

Proof. Take $c = o$ and $n = \rho - 4 \geq m + 11$. Let $u_\pm$ be escape witnesses, $x_{m+1} = u_+$, $y_{m+1} = u_-$; they are distinct and non-adjacent. Applying Lemma 11.5 for $k = m+1, \dots, n-1$ gives $x_k, y_k \in S_k$ with $r_k(x_k, y_k) \geq k - m + 1$, such that consecutive steps do not touch. Then $r_n(x_n, y_n) \geq 12$ and $n \geq 12$, so $d(x_n, y_n) \geq 12$ by Lemma 11.4(b). Lemma 11.9 gives a design for $(a, g; x_n, y_n)$ with gadget ball B_n, and Lemma 11.8, applied for $k = n-1, \dots, m+1$ and finally to (t_+, t_-, u_+, u_-), gives a design for $(a, g; t_+, t_-)$. Spheres of TRI have no pockets, $X = \emptyset$, and $C_0 = 15 \geq \kappa + 1$. □

The constant $C_0 = 15$ is not optimal, but $C_0 \geq 3$ is necessary: for $\rho = m+2$, the ports $a = P_\rho(i\rho - 1)$, $g = P_\rho(i\rho + 1)$ and a terminal pair containing $m\xi_i$ admit no design, because both coloured sets must avoid the common inner neighbour $(\rho - 1)\xi_i$ of a and g, so they contain $P_{\rho-1}(i(\rho-1) \pm 1)$, both of which are adjacent to $m\xi_i$. Compared with the outline of [BBR, §6], the terminals here are an arbitrary escapable pair, as produced by Lemma G, rather than two fixed antipodal sites; Lemmas 11.5 and 11.8 move them outward and separate them.

Proof of Theorem B-TRI. (B1) and (B2) follow from Proposition 11.6 and $\mathrm{TP}_\emptyset$ from Theorem 11.10, and (B3) is proved in Section 11.4 (a geodesic from u to w followed by a straight outward segment, separated by one of the functionals f_i). □

11.4. The outer boundary property

Proposition 11.11. *(B3) holds for TRI with $c_3 = 20$, $c_4 = 30$ and $K_3 = 1$.*

Proof. Let $K \geq 1$, $L \geq 4K + 120$ and $L - K - 10 \leq |v| \leq L$. Put $W = L - K - 20$ and $K' = K + 30$. Following inner neighbours from v we reach $w \in S_W$ with $d(v, w) \leq K + 20$, and $K + 30 < W < L - K - 10$. Let $d(u, w) = K'$ and $|u| < L$. By a rotation ϕ^{-i}, which fixes o, we may assume $w \in \Sigma_0$, i.e. $w = (w_1, w_2)$ with $w_1, w_2 \geq 0$ and $w_1 + w_2 = W$. By Lemma 11.1(2): (i) if $z \in \Sigma_0$ then $|w + z| = |w| + |z|$; (ii) for $e \in \{\xi_0, \xi_1\}$ and $t \geq 0$, $|w + te| = W + t$ and $d(w, w + te) = t$.

Write $u - w = a\xi_j + b\xi_{j+1}$ with $a, b \geq 0$ and $a + b = K'$. If $j = 0$, then $|u| = W + K' = L + 10$ by (i), which is excluded; so $j \in \{1, \dots, 5\}$. The table fixes, for each j, the order of the steps of the first segment Q from u to w (its steps are a steps $-\xi_j = \xi_{j+3}$ and b steps $-\xi_{j+1} = \xi_{j+4}$), a functional F, an outward direction e with $F(e) = 1$, and the turning site u^* after the first group of steps:

j	**steps of Q, in order**	F	**increments of F**	e	u^*
1	$a \times \xi_4$, then $b \times \xi_5$	$f_5 = z_1$	0, then 1	ξ_0	$w + b\xi_2$
2	$a \times \xi_5$, then $b \times \xi_0$	$f_5 = z_1$	1, 1	ξ_0	$w + b\xi_3$
3	$a \times \xi_0$, then $b \times \xi_1$	$f_0 = z_1 + z_2$	1, 1	ξ_0	$w + b\xi_4$
4	$a \times \xi_1$, then $b \times \xi_2$	$f_1 = z_2$	1, 1	ξ_1	$w + b\xi_5$
5	$b \times \xi_3$, then $a \times \xi_2$	$f_1 = z_2$	0, then 1	ξ_1	$w + a\xi_5$

The increments are read off from (11.2). In rows 1 and 5 the group with increment 1 is nonempty, because $b = 0$ in row 1 or $a = 0$ in row 5 would put $u - w$ in Σ_0. Let $Q = (u = q_0, \dots, q_{K'} = w)$, $T = L - W = K + 20$, $\sigma_t = w + te$ for $0 \leq t \leq T$, and $P' = (q_0, \dots, q_{K'} = \sigma_0, \sigma_1, \dots, \sigma_T)$.

Q is an induced geodesic, since it has $K' = d(u, w)$ steps. *The segment σ* satisfies $|\sigma_t| = W + t$ and $d(\sigma_t, \sigma_{t'}) = |t - t'|$, so it is an induced path with $\sigma_T \in S_L$ and $|\sigma_t| < L$ for $t < T$. *Separation:* F is non-decreasing along Q and increases at its last step, so $F(q_l) \leq F(w) - 1$ for $l < K'$, while $F(\sigma_t) = F(w) + t \geq F(w) + 1$ for $t \geq 1$; by Lemma 11.1(1), no q_l with $l < K'$ equals or is adjacent

to any σ_t with $t \geq 1$. So P' is an induced path through w. *Norms along Q:* Q consists of two straight segments, so by convexity its norms are at most $\max(|u|, |u^*|, |w|)$, and Lemma 11.1(3) gives $|u^*| \leq \max(W, K', |u|) < L$ in each row (for instance in row 1, $u^* = (w_1 - b, w_2 + b)$ and $u = (w_1 - b, w_2 + a + b)$). *Location:* $d(q_l, w) = K' - l \leq K' - 1$ for $l \geq 1$ and $d(\sigma_t, w) = t \leq K + 20$, so $P' \setminus \{u\} \subseteq B_{K+29}(w)$. □

12. Further lattices

This section treats the six further lattices of Theorem 1.2 (Cases 4–9), abbreviated ELTRI (elongated triangular), SNUBSQ (snub square), SNUBHEX (snub hexagonal), RTH (rhombitrihexagonal), KING (square lattice with diagonals) and FCC (face-centred cubic). Section 12.1 introduces the generic corner family and the corner pairs, Section 12.2 states the results, and Section 12.3 describes the scheme common to the six proofs; Tables 5 and 6 in Appendix C list the finite computations used in them and give for each lattice the statements of [S] in which the steps of the scheme are carried out. Sections 12.4–12.9 treat the lattices one at a time: coordinates, distance formula, exceptional family, and the facts that are specific to the lattice. The supplementary file [S] contains the complete proofs. For each lattice there are two parts of [S]: one on the distance formula, (B1)–(B3), the failing port pairs and TP_X ([S, Parts K2, E2, Q2, H2, R2, F2]), and one on the corner data (X1)–(X4) ([S, Parts KC, EC, QC, HC, RC, FC]). Part QH of [S] treats the exceptional families of the two snub lattices, and Part Z describes the computations. The parts of [S] use the definitions of Sections 6–8 and are self-contained otherwise. A reference such as [S, Part K2, Lemma 7.1] is to Lemma 7.1 of Part K2 of [S]. Sections 12.10 and 12.11 contain remarks and the proof of Theorem 1.2. Section 12.12 treats the body-centred cubic lattice (BCC) and proves Theorem 1.4; [S, Parts B2 and BC] give the complete proofs. Sections 12.13, 12.14, 12.15 and 12.16 treat the truncated square lattice 4.8^2, the truncated hexagonal lattice 3.12^2, the truncated trihexagonal lattice 4.6.12 and the diamond lattice and prove Theorem 1.5; the complete proofs are in [S, Parts S2, X2, T2 and D2].

All six lattices are periodic lattices in the sense of Definition 6.1. On the square lattice with diagonals and on the face-centred cubic lattice the translations act transitively; on the other four, the translations together with the one further automorphism ϕ given in Sections 12.4–12.7 act transitively. So Remark 6.4 applies.

12.1. Corner pairs and the generic corner family

The natural candidate for the exceptional family on a lattice without a special structure is defined metrically. For a site c and $\rho \geq 2$, let $\mathrm{Apex}_\rho(c)$ be the set of sites $C \in S_\rho(c)$ that have exactly one neighbour C' in $S_{\rho-1}(c)$, this C' having a neighbour in $S_\rho(c)$ other than C, and for an integer $p_0 \geq 1$ let

$$X^{\mathrm{gen}}_\rho(c; p_0) = \big\{\{C, N\} :\ C \in \mathrm{Apex}_\rho(c),\ N \in S_\rho(c) \setminus \{C\},\ d(C, N) \leq p_0\big\},$$

the *generic corner family*. For the honeycomb lattice, $X^{\mathrm{gen}}(2)$ coincides with X^{hex} up to pairs of adjacent sites, which are never port pairs (Proposition 7.8(1)). If $C \in \mathrm{Apex}_\rho(c)$ and $N \in S_\rho(c)$ is a neighbour of C' other than C, then $d(C, N) \leq 2$; if moreover $N \not\sim C$, the pair $\{C, N\}$ admits no design, for every gadget, by the argument of [BBR, §2]: by (D1)–(D3) the colour that contains C contains a path from C into $B_m(c')$, whose second site lies in $B_{\rho-1}(c)$ and is therefore C', which is adjacent to N, contrary to (D4) (compare Proposition 9.9 and, in [S], Part K2, Proposition 6.1; Part

E2, Proposition 6.1; Part Q2, Proposition 7.1; Part R2, Proposition 9.1; and Part FC, Remark 3.3). Such pairs are *corner pairs*. So every family X for which TP_X holds contains all corner pairs of large spheres whose sites are not pockets (on the six lattices of this section no sphere has a pocket), and $p_0 \geq 2$ is needed for $X^{\mathrm{gen}}(p_0)$.

On KING, ELTRI, RTH and FCC the apexes are exactly the corners of the ball, which is a polygon (a polytope for FCC), and $X^{\mathrm{gen}}(2)$ satisfies (X1)–(X4) (Proposition 12.1). On the two snub lattices this fails. These lattices are not bipartite, and a site C in the interior of a side of the ball can have a unique inner neighbour C' whose other neighbour on the sphere is adjacent to C. Such a site is an apex in the above sense, although the corner obstruction does not occur at it. On both lattices the number of apexes of $S_\rho(c)$ grows linearly in ρ ([S, Part Q2, Lemma 2.3]; [S, Part HC, Theorem 3.3]). Each such apex lies in a pair of $X^{\mathrm{gen}}_\rho(c; p_0)$, for instance with an adjacent site of the sphere, so the number of pairs of $X^{\mathrm{gen}}_\rho(c; p_0)$ is unbounded in ρ ([S, Part Q2, Corollary 2.4]; [S, Part HC, Theorem 3.3]). On the other hand (X2) and (X3) bound $|X_\rho(c)|$ by $|\Pi| \cdot \max_x |B_{p_0}(x)|$. So $X^{\mathrm{gen}}(p_0)$ *violates (X2)–(X3) on SNUBSQ and on SNUBHEX, for every* $p_0 \geq 1$. We use explicit smaller families instead: the dead-first-step pairs X^{snub} on SNUBSQ (Section 12.5) and the sharp corner pairs X^{sc} on SNUBHEX (Section 12.6).

12.2. The results

Proposition 12.1 (corner data). *Let G and X^G be as in Table 2. Then X^G satisfies (X1)–(X4) with the label set Π and the constants ρ_X, p_0 of Table 2, the apex maps of Sections 12.4–12.9, 12.13 and 12.16 (summarized in Table 2), and the (X4) data of Table 3.*

G	X^G	Π	**apexes**	**partners of an apex**	ρ_X	p_0
KING	$X^{\mathrm{gen}}(2)$	$\mathbb{Z}/4$	the 4 corners $c + \rho\delta_\pi$, $\delta_\pi \in \{\pm 1\}^2$	4 sites	2	2
ELTRI	$X^{\mathrm{gen}}(2)$	$\mathbb{Z}/6$	the 6 corners of a hexagon (period 2 in ρ)	2 or 3 sites	10	2
RTH	$X^{\mathrm{gen}}(2)$	$\mathbb{Z}/6$	the 6 corners of a hexagon (period 3 in ρ)	2 sites	4	2
FCC	$X^{\mathrm{gen}}(2)$	the 12 steps π	the 12 vertices $c + \rho\pi$ of a cuboctahedron	12 sites	3	2
SNUBSQ	X^{snub}	$\{E, N, W, S\}$	four poles; for $\rho \geq 3$ two of them lie in pairs	1 or 0 sites	2	2
SNUBHEX	X^{sc}	$\mathbb{Z}/6$ (long corners)	at the 2 or 3 sharp long corners	1 site	2	2
4.8^2	X^{cor}	$\{\mathrm{NE}, \mathrm{NW}, \mathrm{SW}, \mathrm{SE}\}$	the 4 diagonal corners of the octagon (period 4 in ρ)	1 site	8	4
DIA	X^{vtx}	the 24 triples (i, j, ε)	the 12 vertex sites of the sphere	4 sites	6	2

Table 2. The partner counts are for ρ larger than a small constant. For ELTRI, RTH, SNUBSQ, SNUBHEX and 4.8^2 the apex maps are given by explicit tables depending on the orbit of c and on ρ

modulo 2, 3, 2, 5 and 4 ([S, Parts EC, RC, QH, HC, S2]). For DIA the apexes are the twelve vertex sites of the sphere, and the apex of a label (i, j, ε) is defined for the centres of parity ε ([S, Part D2, Definition 4.1]).

G	**pairs of labels**	u	t_0	λ
KING	all (π_1, π_2)	$\frac{1}{2}(\delta_{\pi_2} - \delta_{\pi_1})$	3	2
ELTRI	all (π_1, π_2)	[S, Part EC, Table 0.3], $\|u\|_\infty \le 3$	3	$\frac{1}{4}$
RTH	$(\pi, \pi+1)$; $(\pi, \pi+2)$; $(\pi, \pi+3)$	$\delta_{\pi+2}$; $\delta_{\pi+2} - \delta_\pi$; $\delta_{\pi+3}$	1	$\frac{1}{4}$; $\frac{1}{4}$; $\frac{2}{3}$
FCC	$\pi_2 - \pi_1$ a step; $\pi_2 = -\pi_1$; others	$\pi_2 - \pi_1$; π_2; $\pi_2 - \pi_1$	3; 3; 2	1; 2; 1
SNUBSQ	$(E, W), (N, S)$; $(E, N), (N, W), (W, S), (S, E)$	[S, Part QH, Corollary 2.3]	2	1; $\frac{1}{2}$
SNUBHEX	$(k, k+1)$; $(k, k+2)$; $(k, k+3)$	$R^k(-2, 1)$; $R^k(-1, 0)$; $R^k(-1, -1)$	2; 1; 1	$\frac{1}{10}$; $\frac{1}{5}$; $\frac{1}{5}$
4.8^2	all (π_1, π_2)	$\frac{1}{2}(s(\pi_2) - s(\pi_1))$	2	$\frac{1}{8}$
DIA	same parity; different parities	[S, Part D2, Proposition 4.2], $\|u\|_\infty \le 2$; $(1, 0, 0)$	1 or 2; 1	$\frac{1}{4}$

Table 3. (X4) data; for the reversed pair (π_2, π_1) use $-u$ and the same t_0, λ. For RTH, $\delta_0, \dots, \delta_5$ are the six vectors $(1, 0), (0, 1), (-1, 1), (-1, 0), (0, -1), (1, -1)$; for SNUBHEX, $R(x, y) = (-y, x + y)$. For 4.8^2, $s(\pi) \in \{\pm 1\}^2$ is the direction of the corner π: $s(\mathrm{NE}) = (1, 1)$, $s(\mathrm{NW}) = (-1, 1)$, $s(\mathrm{SW}) = (-1, -1)$, $s(\mathrm{SE}) = (1, -1)$.

Proof. Proposition 12.1 is proved for KING in [S, Part KC, Theorem K and Corollary K], for ELTRI in [S, Part EC, Theorem E], for RTH in [S, Part RC, Theorem R and Corollary R′] and for FCC in [S, Part FC, Theorem F and Corollary F2]. For SNUBSQ it follows from [S, Part QC, Theorem III] and [S, Part QH, Corollary 2.3]. Indeed, the family X^{snub} is a translation covariant subfamily of the family $X^{\mathrm{pole}}(2)$ of [S, Part QC], and (X1)–(X4) pass to such subfamilies ([S, Part QH, Theorem 1]). For SNUBHEX it follows from [S, Part HC, Theorems 5.4 and 6.1] and [S, Part QH, Theorem 3.10]. Indeed, the family $X^{\mathrm{sc}} = X^{\mathrm{sh}} \cap X^{\mathrm{SH}}$ is a translation covariant subfamily of the corner family X^{SH} of [S, Part HC]. For 4.8^2 it is [S, Part S2, Proposition 3.1]. For DIA it is [S, Part D2, Proposition 4.2].

Each of the six lattices of Theorem 12.2 has minimum degree at least 4, so Theorem C gives CL_{X^G}. On 4.8^2 and on DIA the families X^{cor} and X^{vtx}, respectively, are used instead, with the corner lemma for deep-occupied balls of [S, Part BC, Theorem 3.2] (Sections 12.13 and 12.16). For KING, (8.1) gives $K_{\mathrm{CL}}(\rho_0) = 6\,021\,188$ for every $\rho_0 \le 21\,633$ ([S, Part KC, Remark 7.2]).

Theorem 12.2. *Let $G \in \{\mathrm{KING}, \mathrm{ELTRI}, \mathrm{SNUBSQ}, \mathrm{SNUBHEX}, \mathrm{RTH}, \mathrm{FCC}\}$ and let X^G be as in Table 2. Then G satisfies (B1) with $M(G) = \{m \ge 1\}$, (B2) with $M'(G) = \{n \ge 1\}$, $n_0 = 1$, $c_2 = 1$ and the value of t_0 in Table 4, (B3) with the values of c_3, c_4 in Table 4 and $K_3 = 1$, and TP_{X^G} with the constants ρ_{TP}, C_0 of Table 4. Moreover, for $\rho \ge \max(\rho_{\mathrm{TP}}, m + C_0)$, every site c' with $d(c, c') \le \kappa(G)$ and every escapable pair of $S_m(c')$, the pairs of distinct non-adjacent sites of $S_\rho(c)$ that admit no design are exactly those listed in the last column of Table 4. None of these spheres has a pocket.*

G	κ	**(B2)** t_0	**(B3)** c_3, c_4	ρ_{TP}, C_0	**failing pairs for large** ρ
KING	0	1	11, 12	9, 8	the 8 non-adjacent pairs of $X_\rho^{\mathrm{gen}}(c;2)$
ELTRI	1	2	13, 14	164, 20	the 4 corner pairs $\{C, N\}$, $N \sim C'$
SNUBSQ	1	2	98, 110	195, 20	the 2 pairs of $X_\rho^{\mathrm{snub}}(c)$
SNUBHEX	1	3	99, 100	227, 23	the 2 or 3 pairs of $X_\rho^{\mathrm{sc}}(c)$
RTH	2	3	46, 50	195, 31	the 6 corner pairs $\{C, N_C\}$, $N_C \sim C'$
FCC	0	1	42, 50	13, 8	the 24 corner pairs $\{C, N_\pm\}$, $N_\pm \sim C'$

Table 4. On ELTRI the other four non-adjacent pairs of $X_\rho^{\mathrm{gen}}(c;2)$, on RTH the other two, and on FCC the other 72 (of 96), have designs. Theorem 12.2 does not apply to 4.8^2, 3.12^2 and 4.6.12, whose proofs do not use (B1); their constants are those of Theorems 12.6, 12.9 and 12.10.

Proof. Theorem 12.2 is proved for KING in [S, Part K2, Propositions 3.1, 4.1, 5.1, Theorem 7.4 and Corollary 7.5], for ELTRI in [S, Part E2, Propositions 3.1, 4.1, 5.4, Theorem 7.9 and Corollary 7.10], for RTH in [S, Part R2, Propositions 5.1, 6.1, 7.3, Theorem 9.2 and Corollary 9.3] and for FCC in [S, Part F2, Propositions 3.1, 4.1, 5.4, Theorem 7.8 and Corollary 7.9]. For SNUBSQ it follows from the Main Theorem of [S, Part Q2] (Propositions 5.1, 6.1, Theorems 8.3, 9.4 and Corollary 8.4 there) and [S, Part QH, Theorem 2]. The latter identifies the family of [S, Part Q2] with X^{snub}. For SNUBHEX it follows from [S, Part H2, Propositions 3.1, 4.3, 5.1, Theorem 7.9 and Corollary 7.10] and [S, Part QH, Theorem 3]: TP holds for X^{sh}, and since $X_\rho^{\mathrm{sc}}(c) = X_\rho^{\mathrm{sh}}(c)$ for $\rho \geq 4$ and $\rho_{\mathrm{TP}} = 227$, $\mathrm{TP}_{X^{\mathrm{sc}}}$ holds with the same constants. The statements for the two snub lattices are assembled in [S, Part QH, Theorems B-SNUBSQ and B-SNUBHEX].

12.3. The common scheme of the proofs

The proofs for the six lattices follow the scheme of Section 9. The exceptional families are those of Table 2, and the constants are those of Proposition 12.1, Theorem 12.2 and Tables 2–4. Appendix C contains Table 5, which lists for each lattice of this section the statements whose proofs use a finite computation and the arguments that extend them to all radii (Section 13.2 discusses the computations, and [S, Part Z] describes each of them), and Table 6, which lists the statements of [S] that describe the spheres and prove (B1)–(B3), the two-port property and the corner data, and those that prove the bounds on p_c. In Sections 12.4–12.9 and in Tables 5 and 6, a statement number that follows a citation of a part of [S] in the same sentence or table cell refers to a statement of that part.

(a) Distance formula. In each case the distance is a maximum of finitely many affine forms in the coordinates (Sections 12.4–12.9). Such a formula is proved by a Lipschitz property of the forms along the finitely many edge types, together with a descent property: every site other than the centre has a neighbour that lowers the maximum. For SNUBSQ, SNUBHEX and RTH the descent property is reduced to finitely many sites and checked there by computer. The balls are lattice polygons: squares (KING), hexagons (ELTRI, RTH), octagons (SNUBSQ), dodecagons with six long and six short corners (SNUBHEX), and cuboctahedra (FCC).

(b) Spheres, pockets, (B1) and (B2). The spheres have no pockets, every pair of distinct non-adjacent sites of every sphere is escapable, and a fixed number t_0 of consecutive spheres beyond S_n is connected (one sphere on KING and FCC, two on ELTRI and SNUBSQ, three on SNUBHEX and RTH).

(c) (B3). The site w is chosen a fixed number of levels inside S_L, and the path P' consists of a path from u to w (a geodesic, except for a short detour near w on SNUBSQ) followed by an outward path from w, chosen so that the two parts do not touch; the separation is read off from the distance formula, as for the triangular lattice (Section 11.4) and the honeycomb lattice (Section 9.7).

(d) The failing pairs. The corner pairs admit no design, by the argument recalled in Section 12.1. On SNUBSQ the corner pairs are exactly the dead-first-step pairs. On SNUBHEX the failing pairs are the pairs whose first inner steps are all blocked ([S, Part H2, Definition 2.5 and Proposition 6.1]), which for $\rho \geq 4$ are exactly the sharp corner pairs.

(e) The two-port property. On ELTRI, SNUBSQ, SNUBHEX and RTH the design is assembled as in Theorem 9.18: four *tracks*, cycles near $S_\rho(c)$ and near $S_m(c')$; two *spokes*, pieces of geodesic rays of c' through corners; and *arcs* along the tracks, each of which avoids one block created by the other colour. Three local lemmas separate the two colours near the ports (port escapes), near the terminals (terminal escapes), and where a ray crosses a track. On KING the tracks are the square spheres themselves and the spokes are diagonals, and all three local lemmas are proved by hand. On FCC the tracks are whole spheres and the spokes are vertex rays of the cuboctahedron; each arc avoids a small set created by the other colour, and that a sphere minus such a set is connected follows from a local criterion ([S, Part F2, Lemma 2.5 and Corollary 2.6]).

(f) Finite checks and their extension to all radii. On ELTRI, SNUBSQ, SNUBHEX, RTH and FCC the local lemmas, and some facts about spheres and tracks, are checked by computer at finitely many radii. They are extended to all radii by a transfer principle of the kind of Lemma 9.17: a translation between two sites with the same *local type* (the set of nearly maximal forms of the distance formula, with their gaps) shifts all levels by the same constant on a ball around the site, so every condition that is expressed through adjacency and levels in that ball is preserved; and every local type that occurs at a large level already occurs at two consecutive bounded levels. This principle is proved in [S, Part E2, Lemmas 1.4 and 1.5] (every type of radius R at a level $\geq 8R+4$ occurs at the levels n^*, n^*+1 for every $n^* \geq 8R+4$), [S, Part R2, Section 3], [S, Part Q2, Proposition 3.3] (the window reduction, which transfers every statement about a window of bounded radius around a site of large level to bounded levels), [S, Part H2, Lemma 2.3], and [S, Part F2, Lemmas 7.2 and 7.3] (every R-type at a level $\geq 3R+3$ occurs at the level $3R+3$).

(g) Corner data. The apexes are identified from the distance formula, and (X4) reduces to a finite list of linear inequalities in the coordinates of the apexes and partners, valid for all t with $t_0 \leq t \leq \lambda\rho$, as in Section 9.9. On KING, ELTRI, RTH and FCC this is done by hand for all radii. On SNUBSQ the description of the apex sets is checked by exhaustive computation for $2 \leq \rho \leq 7$ and proved by hand for $\rho \geq 8$, and the base case $D \leq 7$ of the descent in its distance formula is checked by computation ([S, Part QC, Theorem 1.6]). On SNUBHEX, (X4) is checked by computer for $2 \leq \rho \leq 52$ ([S, Part HC, Proposition 6.5]) and proved by hand for $\rho \geq 53$ ([S, Part HC, Proposition 6.4]).

12.4. The elongated triangular lattice

Coordinates. For ELTRI ($3^3.4^2$) we have $n = 2$ and $k = 2$. The neighbours of $((x,b),0)$ are $((x \pm 1, b), 0)$, $((x,b),1)$, $((x,b-1),1)$, $((x+1,b),1)$, and those of $((x,b),1)$ are $((x \pm 1, b), 1)$, $((x,b),0)$, $((x,b+1),0)$, $((x-1,b),0)$. In *row coordinates* $[x,y]$ with $y = 2b + r$ for the site $((x,b),r)$, the rows are paths, and consecutive rows bound alternately a strip of triangles and a strip of squares. Every

site has degree 5, and $\kappa = 1$. The map $\phi((x,b),r) = ((-x,-b),1-r)$ is an automorphism with $\phi \circ \tau_y = \tau_{-y} \circ \phi$ that exchanges the two orbits ([S, Part E2, Lemma 1.3]).

Distance formula. $d(u,v) = \max_l |\vartheta_l(v) - \vartheta_l(u)|$, where $\vartheta(((a,b),i)) = (a-2b-i,\ a+b,\ 2b+i)$ ([S, Part E2, Lemma 1.2]; [S, Part EC, Lemma 1.1]). The balls are hexagons $\{v : |\vartheta_j(v) - \vartheta_j(c)| \le \rho,\ j = 0,1,2\}$.

Exceptional family and corner data. The apexes of $S_\rho(c)$ are exactly the six corners of the hexagon, given by an explicit table that depends on the orbit of c and on $\rho \bmod 2$ ([S, Part EC, Definition 0.1 and Theorem 3.3]); each has two or three partners for $\rho \ge 6$ ([S, Part EC, Proposition 4.2]). Of the eight non-adjacent pairs of $X^{\mathrm{gen}}_\rho(c;2)$ ($\rho \ge 6$), the four corner pairs $\{C,N\}$ with $N \sim C'$ fail ([S, Part E2, Proposition 6.1]), and for $\rho \ge \max(164, m+20)$ the other four have designs ([S, Part E2, Corollary 7.10]).

12.5. The snub square lattice

Coordinates. For SNUBSQ ($3^2.4.3.4$) we have $n = 2$ and $k = 4$, and the edges are the translates of $(z,0)(z,1)$, $(z,0)(z,3)$, $(z,1)(z,2)$, $(z,2)(z,3)$, $(z,0)(z+e_1,1)$, $(z,0)(z+e_1,2)$, $(z,0)(z-e_2,3)$, $(z,1)(z-e_2,2)$, $(z,1)(z-e_2,3)$ and $(z,2)(z-e_1,3)$. Every site has degree 5, and $\kappa = 1$. The map $\phi(z,i) = (Mz, i-1 \bmod 4)$ with $M(x,y) = (-y,x)$ is an automorphism with $\phi \circ \tau_y = \tau_{My} \circ \phi$ ([S, Part Q2, Lemma 1.1]).

Distance formula. $d((w,j),(z,k)) = \max_{m \in \mathbb{Z}/8}\big(\lambda_m(z-w) + c_{m+2j,\,k-j}\big)$, with the forms $(\lambda_0,\dots,\lambda_7)(x,y) = (2x+y,\ x+2y,\ -x+2y,\ -2x+y,\ -2x-y,\ -x-2y,\ x-2y,\ 2x-y)$ and the offsets $(c_{0,j},\dots,c_{7,j})$ equal to $(0,0,0,0,0,0,0,0)$, $(-1,-1,0,1,1,1,0,-1)$, $(-1,0,1,2,1,0,-1,-2)$, $(0,1,1,1,0,-1,-1,-1)$ for $j = 0,1,2,3$ (in $c_{m+2j,\,k-j}$ the first index is taken modulo 8 and the second modulo 4; [S, Part Q2, Lemma 1.2 and Theorem 1.3]). Equivalently, $d(v,w) = \|H(v) - H(w)\|_\infty$ for four height functions $H = (h_1,\dots,h_4)$ ([S, Part QC, Theorem 1.6]). The balls are octagons.

Exceptional family. $X^{\mathrm{snub}}_\rho(c)$ is the set of pairs $\{C,N\}$ of sites of $S_\rho(c)$ such that C has exactly one neighbour C' in $S_{\rho-1}(c)$, $N \sim C'$, and N is neither equal nor adjacent to C (the *dead-first-step pairs*, [S, Part Q2, Definition 2.5]). For $\rho \ge 3$ it consists of exactly two pairs, at two of the eight corners of the octagonal ball, and their two sites are at distance 2 ([S, Part Q2, Lemma 2.6]; [S, Part QH, Proposition 2.2]). By [S, Part QH, Lemma 2.1] it equals the family X^{frc} of forced port pairs of [S, Part QC]. It is contained in $X^{\mathrm{gen}}(2)$, and its pairs fail ([S, Part Q2, Proposition 7.1]).

Corner data. The labels are the four poles E, N, W, S of [S, Part QC, Definition 0.1]; for $\rho \ge 3$ exactly two of the four poles of $B_\rho(c)$ lie in pairs of $X^{\mathrm{snub}}_\rho(c)$, each with one partner at distance 2, and the other two have no partner. Properties (X1)–(X4) are inherited from the corner family $X^{\mathrm{pole}}(2)$ of [S, Part QC, Definition 0.2 and Theorem III] by [S, Part QH, Theorem 1 and Lemma 1.1]. In [S, Part QC] the distance formula is proved in height coordinates (descent by hand for $D \ge 8$, Lemma 1.7), the poles are located (Lemmas 2.1, 2.2), all apexes are described (Theorem II and Theorems 3.3, 3.4), and (X4) is proved by a box criterion (Lemma 5.1 and Proposition 5.3).

12.6. The snub hexagonal lattice

Coordinates. For SNUBHEX ($3^4.6$) we have $n = 2$ and $k = 6$, and the neighbours (s,j) of $(0,i)$ are

i	**neighbours** (s,j)
0	$((0,0),1)$, $((0,0),5)$, $((0,1),4)$, $((1,0),2)$, $((1,0),3)$
1	$((-1,1),5)$, $((0,0),0)$, $((0,0),2)$, $((0,1),3)$, $((0,1),4)$
2	$((-1,0),0)$, $((-1,1),4)$, $((-1,1),5)$, $((0,0),1)$, $((0,0),3)$
3	$((-1,0),0)$, $((-1,0),5)$, $((0,-1),1)$, $((0,0),2)$, $((0,0),4)$
4	$((0,-1),0)$, $((0,-1),1)$, $((0,0),3)$, $((0,0),5)$, $((1,-1),2)$
5	$((0,0),0)$, $((0,0),4)$, $((1,-1),1)$, $((1,-1),2)$, $((1,0),3)$

The six sites of a cell form a hexagonal face; all other faces are triangles. Every site has degree 5, and $\kappa = 1$. The map $\phi(z,i) = (Rz, i+1 \bmod 6)$ with $R(x,y) = (-y, x+y)$ is an automorphism with $\phi \circ \tau_y = \tau_{Ry} \circ \phi$ ([S, Part H2, Lemma 1.1]).

Distance formula. $d((w,i),(z,j)) = f(R^{-i}(z-w), j-i)$ with $f(z,j) = \max_{q \in \mathbb{Z}/12}(\ell_q(z) + \beta_{q,j})$, where $(\ell_0, \dots, \ell_{11}) = ((3,2),(2,3),(1,3),(-1,2),(-2,1),(-3,-1),(-3,-2),(-2,-3),(-1,-3),(1,-2),(2,-1),(3,1))$ act by the scalar product and the offsets $(\beta_{q,0}, \dots, \beta_{q,5})$ are, for $q = 0, \dots, 11$: $(0,-1,-2,-3,-2,-1)$, $(0,0,-1,-2,-2,-1)$, $(0,1,0,-1,-2,-1)$, $(0,1,1,0,-1,-1)$, $(0,1,2,1,0,-1)$, $(0,1,2,2,1,0)$, $(0,1,2,3,2,1)$, $(0,0,1,2,2,1)$, $(0,-1,0,1,2,1)$, $(0,-1,-1,0,1,1)$, $(0,-1,-2,-1,0,1)$, $(0,-1,-2,-2,-1,0)$; the orbit index is taken modulo 6 ([S, Part H2, Lemma 1.2]). [S, Part HC] proves the formula again in a triangular model (Lemma 1.3 and Theorem 2.1). The balls are lattice dodecagons with six long and six short corners.

Exceptional family. Depending on the orbit of the centre and on $\rho \bmod 5$, a long corner is *sharp* or not. A sharp long corner carries one *sharp corner pair*, two sites of the sphere at distance 2 for which no design exists, and every ball has two or three sharp corners (three if $\rho \equiv 0, 3 \pmod 5$). In [S, Part HC] the long corners and the sharp corner pairs are defined from the triangular model (Definitions 4.1 and 4.4), and the sharp corner pairs are shown to fail (Proposition 4.5 for $\rho \geq 32$; for $\rho \geq 4$ by [S, Part H2, Proposition 6.1] and [S, Part QH, Theorem 3]). Let $X^{\mathrm{sc}}_\rho(c)$ be the set of sharp corner pairs of $S_\rho(c)$ ([S, Part QH, Definition 3.2]). For $\rho \geq 4$ it coincides with the set $X^{\mathrm{sh}}_\rho(c)$ of pairs $\{a,g\}$ of distinct non-adjacent sites of $S_\rho(c)$ such that $\{a,a'\}$ and $\{g,g'\}$ touch for all neighbours a' of a and g' of g in $S_{\rho-1}(c)$ ([S, Part H2, Definition 2.5]; [S, Part QH, Theorem 3]). Explicitly ([S, Part QH, Section 4]): for $c = (z,i)$, $k \in \mathbb{Z}/6$ and $\rho = 5q + e \geq 2$ with $0 \leq e \leq 4$, put $(z', i') = \phi^{-k}(c)$. The long corner k of $B_\rho(c)$ is sharp if and only if (e,i') is one of the twelve pairs of the following table, and then it carries the pair $\{A_\rho(c,k), \bar{A}_\rho(c,k)\}$ with $A_\rho(c,k) = \phi^k((z' + (q,q) + \alpha, j))$ and $\bar{A}_\rho(c,k) = \phi^k((z' + (q,q) + \beta, j'))$:

(e,i')	$(0,0)$	$(0,3)$	$(0,4)$	$(1,1)$	$(1,5)$	$(2,2)$	$(2,4)$	$(3,0)$	$(3,1)$	$(3,3)$	$(4,2)$	$(4,5)$
(α, j)	$(1,-1),1$	$(-1,0),0$	$(0,0),4$	$(0,1),4$	$(1,-1),1$	$(0,1),4$	$(1,-1),1$	$(1,1),4$	$(1,0),1$	$(-1,1),0$	$(1,0),1$	$(1,1),4$
(β, j')	$(0,0),0$	$(0,-1),1$	$(-1,0),0$	$(-1,1),0$	$(0,0),0$	$(-1,1),0$	$(0,0),0$	$(0,1),0$	$(0,1),0$	$(0,1),4$	$(0,1),0$	$(0,1),0$

So $X^{\mathrm{sc}}_\rho(c)$ has three pairs if $\rho \equiv 0, 3 \pmod 5$ and two otherwise; $A_\rho(c,k)$ is the apex of label k in Proposition 12.1, and $\bar{A}_\rho(c,k)$ its partner. The number of apexes in the sense of Section 12.1 is $(8\rho + \varepsilon_{\rho \bmod 5})/5$, with a correction $\varepsilon_{\rho \bmod 5}$ depending on $\rho \bmod 5$, for $2 \leq \rho \leq 60$ (computed, [S, Part HC, Remark 3.5]), a count that no proof uses.

Corner data. The labelling needs care. Two adjacent centres can share the same sharp corner pair

([S, Part HC, Proposition 4.6]). Then a labelling that uses the corner site as the apex for both centres violates the injectivity required in (X2), and two labels at one corner would violate (X4) ([S, Part HC, Corollary 4.7]). For one of the two centres the other site of the shared pair is used as the apex; this is the corner family X^{SH} of [S, Part HC, Definition 5.1], for which (X1)–(X3) and (X4) are proved in [S, Part HC, Theorems 5.4 and 6.1], respectively. These properties pass to $X^{\mathrm{sc}} = X^{\mathrm{sh}} \cap X^{\mathrm{SH}}$ ([S, Part QH, Theorem 3.10]).

12.7. The rhombitrihexagonal lattice

Coordinates. For RTH (3.4.6.4) we have $n = 2$ and $k = 6$, and the edges are the translates of the six hexagon edges $(z, j)(z, j+1)$ (j modulo 6) and of $(z, 0)(z+e_2, 4)$, $(z, 0)(z+e_1, 2)$, $(z, 1)(z-e_1+e_2, 5)$, $(z, 1)(z + e_2, 3)$, $(z, 2)(z - e_1 + e_2, 4)$ and $(z, 3)(z - e_1, 5)$. Every site has degree 4, and $\kappa = 2$. The map $\phi(z, j) = (Rz, j+1 \bmod 6)$ with $R(x, y) = (-y, x+y)$ is an automorphism with $\phi \circ \tau_y = \tau_{Ry} \circ \phi$ ([S, Part R2, Lemma 1.1]).

Distance formula. $d(u, v) = \|P(u)-P(v)\|_\infty$ with $P(z, j) = (3z_1+\beta_1(j),\ 3z_2+\beta_2(j),\ 3(z_1+z_2)+\beta_3(j))$ and $(\beta_1, \beta_2, \beta_3)(j) = (0, 0, 0)$, $(-1, 1, -1)$, $(-2, 0, -2)$, $(-1, -1, -3)$, $(0, -2, -2)$, $(1, -1, -1)$ for $j = 0, \dots, 5$ ([S, Part R2, Theorem 1.3]). [S, Part RC] uses the cube coordinates $x(v) = (P_3(v), -P_2(v), -P_1(v))$, which give the same formula (Theorem 1.7). The balls are hexagons.

Exceptional family and corner data. The apexes of $S_\rho(c)$ are exactly the six corner sites of the hexagon, given by an explicit table that depends on the orbit of c and on $\rho \bmod 3$ ([S, Part RC, Definition 0.1 and Corollary 2.5]; [S, Part R2, Lemma 8.5]); each has two partners. Of the eight non-adjacent pairs of $X^{\mathrm{gen}}_\rho(c; 2)$, the six corner pairs $\{C, N_C\}$, where N_C is the other neighbour on $S_\rho(c)$ of the unique inner neighbour of C, fail ([S, Part R2, Proposition 9.1]), and for $\rho \geq \max(195, m+31)$ the other two have designs ([S, Part R2, Corollary 9.3]). $\mathrm{TP}_{X^{\mathrm{gen}(2)}}$ holds with the constants of Table 4 for all gadget centres $c' \in B_2(c)$.

12.8. The square lattice with diagonals

Coordinates. For KING we have $n = 2$ and $k = 1$, and $z \sim z + s$ for $s \in \{-1, 0, 1\}^2 \setminus \{0\}$. Every site has degree 8, and $\kappa = 0$.

Distance formula. $d(u, v) = \|z_v - z_u\|_\infty$ ([S, Part K2, Lemma 1.2]); the balls are squares and the spheres are their boundaries ([S, Part K2, Corollary 1.3]). For every $m \geq 1$, the escapable pairs of $S_m(c)$ are exactly the pairs of distinct non-adjacent sites.

Exceptional family and corner data. The apexes of $S_\rho(c)$ are the four corners $c + \rho\delta_\pi$, $\delta_\pi \in \{\pm 1\}^2$ ([S, Part KC, Proposition 2.1]); each has four partners, two of them adjacent to it. The eight non-adjacent pairs of $X^{\mathrm{gen}}_\rho(c; 2)$, formed by a corner C and a site $N \in S_\rho(c)$ with $d(C, N) = 2$, fail ([S, Part K2, Proposition 6.1]), and for $\rho \geq m + 8$ they are the only failing pairs ([S, Part K2, Corollary 7.5]). By (8.1), $K_{\mathrm{CL}}(\rho_0) = 6\,021\,188$ for every $\rho_0 \leq 21\,633$ ([S, Part KC, Remark 7.2]).

12.9. The face-centred cubic lattice

Coordinates. For FCC we have $n = 3$ and $k = 1$, and $z \sim z + s$ for $s \in \{\pm e_1, \pm e_2, \pm e_3, \pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3)\}$. Every site has degree 12, and $\kappa = 0$. [S, Part F2] works in the isomorphic A_3 model (Lemma 1.1), and [S, Part FC] in tetrahedral coordinates (Lemma 1.1).

Distance formula. $d(x,y) = \frac{1}{2}\big(|\Delta_1| + |\Delta_2| + |\Delta_3| + |\Delta_1 + \Delta_2 + \Delta_3|\big)$, $\Delta = y - x$ ([S, Part F2, Lemma 1.3]; [S, Part FC, Lemma 1.2]). The balls are cuboctahedra.

Exceptional family and corner data. The apexes of $S_\rho(c)$ are the twelve vertices $c + \rho\pi$ of the cuboctahedron, π a step ([S, Part FC, Proposition 2.1]); each has twelve partners ([S, Part FC, Proposition 3.1]), and $X^{\mathrm{gen}}_\rho(c;2)$ has 144 pairs, of which 96 are non-adjacent. Each apex C has exactly two sites $N_\pm \in S_\rho(c)$ that are adjacent to its inner neighbour C' but not to C; the resulting 24 corner pairs fail ([S, Part F2, Definition 2.3 and Proposition 6.1]; [S, Part FC, Remark 3.3]). For $\rho \geq \max(13, m+8)$ the other 72 non-adjacent pairs have designs ([S, Part F2, Corollary 7.9]). $\mathrm{TP}_{X^{\mathrm{gen}}(2)}$ holds with the constants of Table 4 even for the subfamily of the 24 corner pairs ([S, Part F2, Theorem 7.8]).

12.10. Remarks on the further lattices

1. *The size of C_0.* Some gap between the gadget ball and the host sphere is necessary: on KING no family of the form $X^{\mathrm{gen}}(p_0)$ admits TP with $C_0 = 2$ ([S, Part K2, Proposition 7.7]), as on the triangular lattice (Section 11.3). The values of ρ_{TP} in Table 4 come from the windows of the local lemmas and are not optimized.
2. *Larger families.* TP_X is monotone in X, so TP also holds for every family that contains X^G. For instance on SNUBSQ and SNUBHEX it holds for the corner families $X^{\mathrm{pole}}(2)$ and X^{SH} of [S, Parts QC and HC] (for X^{SH} because $X^{\mathrm{sh}}_\rho(c) \subseteq X^{\mathrm{SH}}_\rho(c)$ for $\rho \geq 4$), and these satisfy (X1)–(X4) as well ([S, Part QH, Section 4]). The families X^{snub} and X^{sc} are the smallest possible: for large ρ they consist exactly of the failing pairs (Theorem 12.2). On KING, ELTRI, RTH and FCC the family $X^{\mathrm{gen}}(2)$ also contains adjacent pairs, which are never port pairs, and on ELTRI, RTH and FCC also non-adjacent pairs that have designs. ([S, Part F2] proves TP on FCC even for the subfamily of the 24 corner pairs.)
3. *Small radii on SNUBHEX.* For $\rho \in \{2,3\}$ the family $X^{\mathrm{sh}}_\rho(c)$ of [S, Part H2] is not contained in $X^{\mathrm{SH}}_\rho(c)$; for instance $\{((0,0),4),((1,0),4)\} \in X^{\mathrm{sh}}_2(o) \setminus X^{\mathrm{SH}}_2(o)$ ([S, Part QH, Proposition 3.9]). This is why the family $X^{\mathrm{sc}} = X^{\mathrm{sh}} \cap X^{\mathrm{SH}}$ is used; it agrees with X^{sh} at all radii that matter for TP.
4. *Thin pairs on 4.8^2.* On the truncated square lattice the port pairs without a design found by exact computer searches include the thin pairs, whose number grows linearly with the radius ([S, Part S2, Remark 2.5]), so no exceptional family that contains them satisfies (X2)–(X3) ([S, Part S2, Remark 4.3]). They are not treated by a corner lemma: thin pairs are never the ports of a deep-occupied ball, and the gadget terminals are never thin (Section 12.13).

12.11. Proof of Theorem 1.2

Let G be one of the lattices of Theorem 1.2 and E an essential enhancement of site percolation on G. By Theorems B-HEX, B-KAG, B-TRI and 12.2, G satisfies (B1), (B2), (B3) and TP_X, with $X = X^{\mathrm{hex}}$ for HEX, $X = \emptyset$ for KAG and TRI, and $X = X^G$ of Table 2 for the lattices of this section. $\mathrm{CL}_\emptyset$ holds trivially (Section 7.2). For the other families, G has minimum degree at least 3 and X satisfies (X1)–(X4) (Theorem 9.20 for HEX and Proposition 12.1), so CL_X holds by Theorem C. By Theorem A, Claim 5 holds for E. By Proposition 6.6, $p_c(G) < 1$, and Theorem 6.5 gives Conjecture 1 for E. □

All constants in this proof are explicit: those of Theorem A are given by (7.1) and (7.2) in terms of

the constants of Theorems B-HEX, B-KAG, B-TRI and 12.2, of the corner constant K_{CL} of Theorem C and of the gadget radius m, which depends on the enhancement.

Theorem 1.5 is not obtained in this way. It is proved by the variant of Theorem A in Theorem 12.5 on 4.8^2 (Section 12.13), by Theorems 12.7 and 12.8 on 3.12^2 (Section 12.14), by the variant of Theorem A in Theorem 12.10 on 4.6.12 (Section 12.15) and, as on the body-centred cubic lattice, by the variant of Theorem A with history hosts in Theorem 12.11 on DIA (Section 12.16). The obstacles differ from lattice to lattice. On 4.8^2 the property (B1) fails. On 3.12^2 it fails as well, (B3) is proved only for odd L, and no exceptional family satisfying TP_X satisfies (X2)–(X3) (Section 12.14). The proof for 4.6.12 does not use (B1) (non-escapable pairs exist for $3 \le m \le 250$, by computation), and on DIA (B1) holds, but no corner lemma is used for the edge pairs.

12.12. The body-centred cubic lattice

Coordinates. For BCC we have $n = 3$ and $k = 1$, and $z \sim z+s$ for $s \in \{\pm e_1, \pm e_2, \pm e_3, \pm(e_1+e_2+e_3)\}$. Every site has degree 8, and $\kappa = 0$; the translations act transitively, so Remark 6.4 applies. The map $\Phi(z) = (-z_1 + z_2 + z_3,\ z_1 - z_2 + z_3,\ z_1 + z_2 - z_3)$ is an isomorphism onto the graph on $\Lambda = \{x \in \mathbb{Z}^3 : x_1 \equiv x_2 \equiv x_3 \pmod 2\}$ in which $x \sim y$ iff $y - x \in \{\pm 1\}^3$ ([S, Part B2, Definition 1.1]).

Distance formula. $d(u, v) = \|\Phi(v - u)\|_\infty$ ([S, Part B2, Lemma 1.2]). The balls are cubes, the lattice is bipartite, and no two sites of a sphere are adjacent.

Dead pairs. A pair of distinct sites of $S_\rho(c)$ is *dead* if every neighbour of one of them in $S_{\rho-1}(c)$ is adjacent to the other; let $X^{\mathrm{bcc}}_\rho(c)$ be the set of dead pairs. For $\rho \ge 2$ they are the 48 *corner pairs*, formed by a corner of the cube and one of six sites at distance 2 from it, and the $24(\rho-1)$ *edge pairs*, formed by a site in the interior of an edge of the cube and one of the two sites obtained from it by moving one of its two extreme coordinates two steps inwards; no dead pair has a design ([S, Part B2, Proposition 2.1]). A site in the interior of an edge is the apex of the same edge pair for $\rho - 1$ centres, so X^{bcc} violates (X2) ([S, Part B2, Proposition 2.3]), and Theorems A and C do not apply to it. The family X^{cor} of the 48 corner pairs satisfies (X1)–(X4) with $\rho_X = 2$ and $p_0 = 2$ ([S, Part BC, Proposition 3.1]).

History hosts. For an edge pair $\{A, N\}$ of $S_\rho(c)$, with A in the interior of an edge of the cube, the *history hosts* are the sets $D = B_\rho(c) \setminus Z$ with $Z = \{A\}$ and, if A is at distance 2 from a corner C of the cube, also $Z \in \{\{A, C\}, \{A, M\}, \{A, C, M\}\}$, where $M = N + (C - A)$. Let $\rho \ge 8$, and let $B = B_\rho(c)$ be an admissible ball whose ports form such an edge pair and which contains a site of the path at distance at most $\rho - 2$ from c. Then the path is forced near A and N ([S, Part B2, Lemma 4.2]), and one of the history hosts, D, is admissible, with ports N and a site $a(D) \in S_\rho(c)$, such that neither port has all its neighbours in $\operatorname{int} D$ adjacent to the other; the removed sites other than A are sites of the path ([S, Part B2, Lemmas 4.3 and 4.4]). Theorem 7.10 holds for a history host D in place of a ball, with designs in D defined by (D1), (D3), (D4) and the condition $\mathsf{R} \setminus \{a\}, \mathsf{G} \setminus \{g\} \subseteq \operatorname{int} D$ in place of (D2), because its proof uses the ball only through this inclusion ([S, Part B2, Lemma 4.1]). The use of the sites visited by the path near a port is in the spirit of the path-history argument of [BBR, Section 5].

Theorem 12.3 (Theorem A with history hosts). *Let m_1, ρ_{TP} and $C_0 \ge 2$ be integers such that for every gadget radius $m \ge m_1$, every $\rho \ge \max(\rho_{\mathrm{TP}}, m + C_0)$ and every escapable pair of $S_m(c)$ as gadget terminals the following hold on BCC. ($\mathrm{TP}^{\mathrm{bcc}}$) Every pair of distinct sites of $S_\rho(c)$ that is not*

dead has a design in $B_\rho(c)$. (HTP$^{\mathrm{bcc}}$) For every edge pair $\{A, N\}$ of $S_\rho(c)$, with A in the interior of an edge, and every history host D of $\{A, N\}$, every site $a \neq N$ of $S_\rho(c) \cap D$ such that neither of a, N has all its neighbours in $\operatorname{int} D$ *adjacent to the other has a design for (a, N) in D. Then Claim 5 holds for every essential enhancement of site percolation on BCC.*

The proof ([S, Part B2, Theorem 5.1]) follows that of Theorem A in Sections 7.3–7.6, except for two changes in Proposition 7.13. First, CL_X is replaced by a version of Theorem C for X^{cor} that, in addition, finds a ball containing a site of the path at distance at most $\rho - 2$ from its centre ([S, Part BC, Theorem 3.2 and Corollary 3.4]); its proof is that of Theorem C, with a larger constant. Second, if the ports of this ball form an edge pair, the design is placed in the history host instead of the ball. The gadget radius may be chosen with $m \geq m_1$, since Lemma G gives a gadget for every $m \in M(G)$ with $m \geq m_E$.

Theorem 12.4. *BCC satisfies (B1) with $M(\mathrm{BCC}) = \{m \geq 1\}$, (B2) with $M'(\mathrm{BCC}) = \{n \geq 1\}$, $n_0 = 1$, $t_0 = 2$, $c_2 = 1$, and (B3) with $c_3 = 20$, $c_4 = 21$, $K_3 = 1$. For every gadget radius $m \geq 20$, every $\rho \geq m + 12$ and every pair of distinct sites of $S_m(c)$ as gadget terminals, TP$^{\mathrm{bcc}}$ and HTP$^{\mathrm{bcc}}$ hold. Moreover $1/7 \leq p_c(\mathrm{BCC}) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/2} < 1$.*

Proof. Theorem 12.4 is proved in [S, Part B2, Lemmas 5.3, 5.4, 5.5, Theorem 9.9 and Remark 5.2]. Table 6 shows which statement proves which claim. The two-port statements are proved by the scheme of [S, Part F2, Section 7]. Since no two sites of a sphere are adjacent, the tracks are unions of two consecutive spheres, and the spokes are the eight rays through the corners of the cubes. Four local statements (port escapes, terminal escapes, port escapes in history hosts, and the connectivity of a track minus a spoke) rest on finite computations, which Table 5 lists with their extension to all radii.

Proof of Theorem 1.4. By Theorem 12.4 the hypotheses of Theorem 12.3 hold with $m_1 = 20$, $\rho_{\mathrm{TP}} = 32$ and $C_0 = 12$, so Claim 5 holds for every essential enhancement E of site percolation on BCC. Since $p_c(\mathrm{BCC}) < 1$, Theorem 6.5 gives Conjecture 1 for E ([S, Part B2, Corollary 9.10]). □

For $m \geq 20$ and $\rho \geq m + 12$ the port pairs of $S_\rho(c)$ without a design are exactly the $24\rho + 24$ dead pairs. The constant of the corner lemma of [S, Part BC, Corollary 3.4] is $K'(\rho_0) = 960\,005\,453\,344$ for $\rho_0 \leq 5\,333\,334$. In the proof of Theorem 1.4 the constants of Claim 5 are given by (7.1) and (7.2), with $K'(\rho_0)$ in place of $K_{\mathrm{CL}}(\rho_0)$ and $\rho_0 = m + 12$, where $m \geq 20$ is the gadget radius.

12.13. The truncated square lattice

Coordinates. For the truncated square lattice 4.8^2 we have $n = 2$ and $k = 4$: the set of sites is $V = \mathbb{Z}^2 \times \mathbb{Z}/4$, and a site (z, k) is adjacent to $(z, k+1)$, $(z, k-1)$ (*square edges*) and to $b(z, k) := (z + \epsilon_k, k + 2)$ (its *bridge neighbour*), where $\epsilon_0 = (1, 0)$, $\epsilon_1 = (0, 1)$, $\epsilon_2 = (-1, 0)$, $\epsilon_3 = (0, -1)$. The four sites of a cell form a square $Q(z)$, and the other faces are octagons. Every site has degree 3, and $\kappa = 1$; the lattice is bipartite, and no two sites of a sphere are adjacent. The map $R(z, k) = (\varrho z, k+1)$ with $\varrho(z_1, z_2) = (-z_2, z_1)$ is an automorphism with $R\tau_w = \tau_{\varrho w} R$, and the translations together with R act transitively (see Remark 6.4; [S, Part S2, Definition 1.1, Lemma 1.1 and Corollary 1.3]).

Distance formula. For a site $x = (z, k)$ put $p(x) = 3z + \epsilon_k$. For sites $x = (z, k)$ and $y = (z', k')$, with $P = p(y) - p(x)$ and $\Delta = z' - z$,

$$d(x, y) = \max\bigl(|P_1| + |\Delta_2|,\ |P_2| + |\Delta_1|\bigr)$$

([S, Part S2, Lemma 1.2]). The balls are octagons, with four *axis corners* and four *diagonal corners*.

Thin pairs. A pair $\{x, y\}$ of sites is *thin* if x and y are opposite vertices of a square, i.e. $\{x, y\} = \{(z, k), (z, k+2)\}$. The number of thin pairs in S_n grows linearly in n. Most thin pairs of a sphere are not escapable, and (B1) fails for every large radius ([S, Part S2, Remark 1.6 and Section 9.3]). Two facts replace (B1), and a corner lemma, for these pairs. First, the terminals of the gadget of Lemma G are never a thin pair, and every pair of distinct sites of a sphere that is not thin is escapable ([S, Part S2, Lemma 4.1 and Proposition 7.2]). Second, thin pairs are never the ports of a *deep-occupied* ball, a ball $B_\rho(c)$ in which the path has a site x with $d(x, c) \le \rho - 2$ ([S, Part S2, Lemma 2.2]; [S, Part BC, Definition 3.2]).

Corner data. For $\rho \ge 8$ the family $X^{\mathrm{cor}}_\rho(c)$ consists of four pairs (apex, partner), one at each diagonal corner of the octagon $B_\rho(c)$, given by an explicit table that depends on the orbit of c and on ρ mod 4. The two sites of each pair are distinct sites of $S_\rho(c)$ at distance 2 or 4, and the pairs are not thin. X^{cor} satisfies (X1)–(X4) with the label set $\Pi = \{\mathrm{NE}, \mathrm{NW}, \mathrm{SW}, \mathrm{SE}\}$, $\rho_X = 8$, $p_0 = 4$, and for $\pi_1 \ne \pi_2$ the data $u(\pi_1, \pi_2) = \frac{1}{2}\big(s(\pi_2) - s(\pi_1)\big)$, $t_0 = 2$, $\lambda = \frac{1}{8}$, where $s(\pi) \in \{\pm 1\}^2$ is the direction of the corner π ([S, Part S2, Definition 2.3, Lemma 2.4 and Proposition 3.1]; Tables 2 and 3). (X4) is verified by an exact symbolic computation, valid for all radii. The corner lemma for deep-occupied balls ([S, Part BC, Theorem 3.2]) applies to X^{cor}: for every $\rho_0 \ge 8$, every induced path through a site v with both end sites outside $B_{K'}(v)$ meets a deep-occupied ball $B_\rho(c)$ with $\rho \ge \rho_0$ and $B_{\rho+1}(c) \subseteq B_{K'}(v)$ whose ports are not in $X^{\mathrm{cor}}_\rho(c)$, and for $\rho_0 \le 1\,881\,601$, $K' = 113\,801\,110\,100$ ([S, Part S2, Corollary 3.2]).

Exact computer searches at the radii $8 \le \rho \le 17$ and $20 \le \rho \le 23$, which the proof does not use, found that the port pairs without a design were exactly the thin pairs and the four pairs of X^{cor}, two of them dead and two blocked after two steps ([S, Part S2, Remark 2.5]).

The reduction follows the route of the body-centred cubic lattice (Section 12.12), with the thin pairs in place of the edge pairs.

Theorem 12.5 (Theorem A for 4.8^2). *Suppose that* 4.8^2 *satisfies*

- *(B1′) for every* $m \ge 1$ *and every site* x, $S_m(x)$ *has no pockets, and every pair of distinct sites of* $S_m(x)$ *that is not thin is escapable;*
- *(B2) and (B3) of Section 7.2;*
- *(TP*) there are integers* m_1, ρ_{TP} *and* $C_0 \ge 2$ *such that for every* $m \ge m_1$, *every* $\rho \ge \max(\rho_{\mathrm{TP}}, m + C_0)$, *all sites* c, c' *with* $d(c, c') \le 1$, *every escapable pair* (t_+, t_-) *of* $S_m(c')$ *that is not thin, and every pair* $\{a, g\}$ *of distinct sites of* $S_\rho(c)$ *that is neither thin nor in* $X^{\mathrm{cor}}_\rho(c)$, *there is a design for* $(a, g; t_+, t_-)$ *in* $B_\rho(c)$.

Then Claim 5 holds for every essential enhancement E *of site percolation on* 4.8^2, *and Conjecture 1 holds for* E.

The proof ([S, Part S2, Theorem 4.2]) differs from that of Theorem A in Sections 7.3–7.6 in three points. First, the terminals given by Lemma G are not thin, hence escapable by (B1′), so property 1 of Lemma G holds. Second, K is defined by (7.1) with $K_{\mathrm{CL}}(\rho_0)$ replaced by the constant $K'(\rho_0)$ of the corner lemma for deep-occupied balls, applied to X^{cor}. Third, in Proposition 7.13 this corner lemma gives a deep-occupied admissible ball whose ports are not in X^{cor}; they are not thin, as the ball is deep-occupied, and (TP*) gives a design. The family $X^{\mathrm{cor}} \cup X^{\mathrm{thin}}$ itself does not satisfy (X2)–(X3), and, unlike on the body-centred cubic lattice, no history hosts are needed ([S, Part S2, Remarks 4.3

and 4.4]).

Theorem 12.6. 4.8^2 *satisfies (B1′), (B2) with* $M' = \{n \geq 1\}$, $n_0 = 1$, $t_0 = 4$ *and* $c_2 = 1$, *(B3) with* $c_3 = 180$, $c_4 = 185$, $K_3 = 1$, *and (TP*) with* $m_1 = 411$, $\rho_{\mathrm{TP}} = 387$ *and* $C_0 = 30$. *Moreover* $\frac{1}{2} \leq p_c(4.8^2) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/4} < 1$.

Proof. Theorem 12.6 is proved in [S, Part S2, Propositions 7.2, 7.3, 7.10, Theorem 8.6 and Proposition 1.5] (see Table 6). The tracks are the boundary cycles $Z_c(n)$ of the unions of the faces contained in $B_n(c)$; for every centre c and every $n \geq 4$, $Z_c(n)$ is a single cycle. The spokes are pieces of the four axis rays, and, as in Theorem 9.18, the design is put together from port escapes, terminal escapes and crossings of rays with tracks ([S, Part S2, Sections 6 and 8]).

Finite checks and their extension to all radii. Besides the corner data, the following steps are checked by computer at bounded levels: the ascent and descent of faces used for the tracks, a lemma on unique outer neighbours used for (B1′), two finite inputs of (B3), the crossings of rays with tracks, and the port and terminal escapes (Table 5 lists the statements of [S, Part S2] and the ranges checked). As in Section 12.3(f), a transfer lemma carries these checks to all radii: a translation between two sites of the same R-type shifts the levels on a ball around the site by a constant, and every R-type that occurs at a level $\geq N_R - 12$ occurs at a level in $[N_R - 12, N_R - 1]$, where $N_R = 8R + 111$ ([S, Part S2, Section 5]). The levels along the axis rays ([S, Part S2, Lemma 8.1(a)]) are checked for the first 262 sites of every ray and extended to all sites by hand. All decisive checks have a second, independent encoding (Section 13.2); for the corner data the claim for all radii rests on the symbolic encoding, and the second encoding is a finite cross-check ([S, Part Z, Section Z.23]).

Proof of Theorem 1.5 for 4.8^2. By Theorem 12.6 the hypotheses of Theorem 12.5 hold, so Claim 5 and Conjecture 1 hold for every essential enhancement E of site percolation on 4.8^2 ([S, Part S2, Theorem 9.1]). □

For a given E, the gadget radius is any $m \geq \max(m_E, 411)$; then $\rho_0 = m + 30$, $K' = K'(\rho_0)$ of [S, Part S2, Corollary 3.2] ($K' = 113\,801\,110\,100$ when $\rho_0 \leq 1\,881\,601$), $K = \max(K', \rho_0 + 1, 1, 1) = K'$, $C = 20 + 2c_2 + t_0 + 2c_4^* = 396$, $R = 3K + 396$ and $L_0 = 100R$ ([S, Part S2, Section 9.1]).

12.14. The truncated hexagonal lattice

Coordinates. For the truncated hexagonal lattice 3.12^2, abbreviated TH, we have $n = 2$ and $k = 6$: TH is the graph on $V = \mathbb{Z}^2 \times \{0, 1, \ldots, 5\}$ in which $(z, i) \sim (z + d, j)$ for the offsets (d, j) in the following lists N_i ([S, Part X2, Definition 1.1]):

$$N_0 = \{(0,1),(0,2),(0,3)\},\quad N_1 = \{(0,0),(0,2),(-e_1,4)\},\quad N_2 = \{(0,0),(0,1),(-e_2,5)\},$$
$$N_3 = \{(0,4),(0,5),(0,0)\},\quad N_4 = \{(0,3),(0,5),(e_1,1)\},\quad N_5 = \{(0,3),(0,4),(e_2,2)\}.$$

The sites are the *darts* of the honeycomb lattice HEX: TH is the truncation of HEX, each vertex of HEX replaced by a triangle, and its faces are the triangles and the dodecagons coming from the hexagons of HEX. Every site v lies in exactly one triangle and has exactly one neighbour outside its triangle, its *partner* v^*. Every site has degree 3, and $\kappa(\mathrm{TH}) = 2$. The translations together with two automorphisms ϱ, ι act transitively (see Remark 6.4; [S, Part X2, Lemma 1.2 and Theorem 9.1]).

Distance formula. In the cube coordinates of Section 9.1 a site is a pair (x, j) of a vertex $x \in \mathbb{Z}^3$ of HEX and a direction j; its neighbours are its triangle mates (x, j'), $j' \neq j$, and its partner

$(x+\sigma(x)f_j, j)$, where $s(x)=x_1+x_2+x_3$, $\sigma(x)=1-2s(x)$ ([S, Part X2, Lemma 1.3]). For darts at distinct vertices $x\neq y$, with $D=\|x-y\|_1$, [S, Part X2, Lemma 1.4] gives

$$d((x,j),(y,k)) = 2D-1+\min_{(l,l')\in\Gamma(x,y)}\big([l\neq j]+[l'\neq k]\big).$$

Here $\Gamma(x,y)$ is the set of pairs (l,l') such that some honeycomb geodesic from x to y has first direction l and last direction l'. So $2D-1\le d((x,j),(y,k))\le 2D+1$, and the levels are given explicitly ([S, Part X2, Corollary 1.5]). No sphere $S_r(o)$ with $r\not\equiv 2 \pmod 4$ or $r=2$ has a pocket, and $S_{4K+2}(o)$ has exactly two pockets for $K\ge 1$ ([S, Part X2, Corollary 1.7]).

The triangle-blocked pairs. For a site c and $\rho\ge 2$ let

$$X^{\mathrm{T}}_\rho(c)=\big\{\{a,g\}:\ a\neq g\in S_\rho(c),\ a^*,g^*\in B_{\rho-1}(c),\ a^*\neq g^*,\ a^*\sim g^*\big\}$$

([S, Part X2, Definition 3.3]). The partners a^*, g^* are two corners of one triangle inside the ball. If a port pair lies in $X^{\mathrm{T}}_\rho(c)$, then a^* is the only neighbour of a in $S_{\rho-1}(c)$ and g^* the only neighbour of g in $S_{\rho-1}(c)$, so it has no design, for any gadget ([S, Part X2, Proposition 3.5]). For every site c and every $\rho\ge 4$, at least $\lfloor(\rho+1)/4\rfloor$ port pairs of $S_\rho(c)$ have no design, and at the radius $\rho=4K+2$ one such pair is exceptional for $K+1$ distinct centres ([S, Part X2, Theorem 3.4 and Corollary 3.6]), as the edge pairs of the body-centred cubic lattice (Section 12.12). So no exceptional family on TH satisfies TP_X together with (X2) and (X3) ([S, Part X2, Theorem 3.7]), and Theorems A and C cannot be combined. By [S, Part X2, Corollary 8.4], for $\rho\ge\max(111, m+28)$ the port pairs without a design are exactly the port pairs in $X^{\mathrm{T}}_\rho(c)$.

However, by [S, Part X2, Lemma 4.1], an induced path can never enter a ball through such a pair and reach two levels below the sphere: if the first and last sites of the path in $B_\rho(c)$ form a pair of $X^{\mathrm{T}}_\rho(c)$, the path has no site in $B_{\rho-2}(c)$. Hence, by [S, Part X2, Corollary 4.2], $\mathrm{CL}_{X^{\mathrm{T}}}$ holds on TH with $K_{\mathrm{CL}}(\rho_0)=\rho_0+1$ for every $\rho_0\ge 2$.

Gadget radii. (B1) fails: for every $m\ge 5$ the sphere $S_m(o)$ contains a pair of distinct non-adjacent sites that is not escapable ([S, Part X2, Theorem 4.3]). A pair $\{t,t'\}$ of distinct sites of $S_m(c')$ is *tail-excluded* ([S, Part X2, Definition 4.4]) if for one of its orderings (s,s') we have $d(c',s^*)\ge m$ and either $s^*\sim s'$, or $d(c',s'^*)\ge m$ and $s^*\sim s'^*$. The terminals of the gadget constructed as in Lemma G with induced tails ([S, Part X2, Lemma 4.5]) are never tail-excluded. Let M_{TH} be the set of $m\ge 1$ such that every pair of distinct non-adjacent sites of $S_m(o)$ is escapable or tail-excluded. Then $M_{\mathrm{TH}}\supseteq\{m\ge 1: m\not\equiv 2 \pmod 4\}\cup\{2\}$, and Lemma G holds on TH with $M(G)$ replaced by M_{TH} ([S, Part X2, Theorem 4.6]).

The outer boundary. The argument for (B3) used on the other lattices needs geodesics with levels bounded by the levels of their ends, which do not exist on TH. TH satisfies $(\mathrm{B3}_{\mathrm{odd}})$: the statement (B3) of Section 7.2 for all $K\ge K_3$, all odd $L\ge 4K+4c_4$ and every site v with $L-K-10\le|v|\le L$. The proof, given in [S, Part X2, Section 6], is a reroute through a deepest possible site, made of the lift of a honeycomb path, which is automatically induced. By the following variant of Theorem 6.5 ([S, Part X2, Theorem 4.8]), Claim 5 along the odd values of L suffices.

Theorem 12.7 (the probabilistic step along a subsequence). *Let G be a periodic lattice with $p_c(G)<1$, E an enhancement of site percolation on G, and $\mathcal{M}\subseteq\mathbb{N}$ an infinite set. If there are L_0, R such that the statement of Claim 5 holds for every $L\in\mathcal{M}$ with $L\ge L_0$, then Conjecture 1 holds for E.*

Theorem 12.8 (Theorem A on 3.12^2). *Suppose that TH satisfies (B2) of Section 7.2, ($B3_{\mathrm{odd}}$) and $TP_{X^{\mathrm{T}}}$ with constants ρ_{TP}, $C_0 \geq 3$. Let E be an essential enhancement of site percolation on TH, m_E as in [S, Part X2, Lemma 4.5], and $m \in M_{\mathrm{TH}}$ with $m \geq m_E$. Then the statement of Claim 5 holds for E for every odd $L \geq L_0$, with the constants (7.1), (7.2) computed with this m, with $K_{\mathrm{CL}}(\rho_0) = \rho_0 + 1$ and with the constants of ($B3_{\mathrm{odd}}$) in place of those of (B3): $\rho_0 = \max(\rho_{\mathrm{TP}}, m + C_0)$, $K = \max(\rho_0 + 1, K_3, n_0^*)$, $R = 3K + C$, $L_0 = 100R$.*

Its proof, given in [S, Part X2, Theorem 4.7], is that of Theorem A, with Lemma G on TH and $\mathrm{CL}_{X^{\mathrm{T}}}$ in place of CL_X.

Theorem 12.9. *TH satisfies (B2) with $M'(\mathrm{TH}) = \{n \geq 1 : n \text{ odd}\}$, $n_0 = 1$, $t_0 = 5$ and $c_2 = 3$, ($B3_{\mathrm{odd}}$) with $c_3 = 39$, $c_4 = 40$ and $K_3 = 1$, and $TP_{X^{\mathrm{T}}}$ with $\rho_{\mathrm{TP}} = 111$ and $C_0 = 28$: for every site c, every $\rho \geq 111$ and $m \geq 1$ with $\rho \geq m + 28$, every c' with $d(c, c') \leq 2$, every escapable pair (t_+, t_-) of $S_m(c')$ and every port pair $\{a, g\}$ of $S_\rho(c)$ with $\{a, g\} \notin X_\rho^{\mathrm{T}}(c)$ there is a design for $(a, g; t_+, t_-)$ in $B_\rho(c)$. Moreover $\frac{1}{2} \leq p_c(\mathrm{TH}) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/6} < 1$.*

Proof. Theorem 12.9 is proved in [S, Part X2, Theorems 5.1, 6.2, 8.3 and Lemma 1.8]. The tracks are the lifts $T_n(X)$ of the honeycomb tracks $C_n(X)$ of Definition 9.10 ([S, Part X2, Definition 7.2]). By [S, Part X2, Lemma 7.1] the lift of a honeycomb path is an induced path of TH, and the lift of a closed honeycomb path is an induced cycle. The spokes are the rays $\varrho_j(X)$, the lifts of the apex rays of Definition 9.12 ([S, Part X2, Definition 7.4]). The design is assembled from the same three local ingredients as in Section 12.13 and Theorem 9.18 ([S, Part X2, Lemmas 7.7, 8.1, 8.2 and Theorem 8.3]).

Finite checks and their extension to all radii. Some statements are checked by computer at finitely many radii, in two encodings (Table 5). The gadget radii of [S, Part X2, Theorem 4.6] are checked for $m \leq 64$, and the port and terminal escapes of [S, Part X2, Lemmas 8.1 and 8.2] for $111 \leq \rho \leq 115$ and $1 \leq m \leq 115$; for both, a transfer principle covers all radii. As in Section 12.3(f), a translation between two vertices of the same *type* shifts all levels by the same constant on a window around them, and every vertex x with $\|x\|_1 \geq 4R+4$ has the same R-type as a vertex x° with $\|x^\circ\|_1 \in \{4R+4, 4R+5\}$ ([S, Part X2, Lemmas 2.2 and 2.3]). The crossings of rays with tracks of [S, Part X2, Lemma 7.7] are checked for $2 \leq n \leq 40$ and extended by a translation along the ray. The second bound in the separation of rays of [S, Part X2, Lemma 7.6(c)] is checked for levels ≤ 32; it concerns only levels at most 30, and larger levels are covered by the first bound, which is proved by hand.

Proof of Theorem 1.5 for 3.12^2. By Theorem 12.9, by [S, Part X2, Corollary 4.2] and by Theorem 12.8, the statement of Claim 5 holds for every essential enhancement E of site percolation on TH for every odd $L \geq L_0$. Since $p_c(\mathrm{TH}) < 1$, Theorem 12.7 with $\mathcal{M}$ the set of odd integers gives Conjecture 1 for E ([S, Part X2, Theorem 9.1]). □

Here $\rho_0 = \max(111, m + 28)$, $K = \max(K_{\mathrm{CL}}(\rho_0), \rho_0 + 1, K_3, n_0^*) = \rho_0 + 1$, $C = 20 + 2c_2 + t_0 + 2\max(c_4, 10) = 20 + 6 + 5 + 80 = 111$, $R = 3K + 111$ and $L_0 = 100R$, where $m \in M_{\mathrm{TH}}$, $m \geq m_E$, is the gadget radius ([S, Part X2, Section 9]).

12.15. The truncated trihexagonal lattice

Coordinates. For the truncated trihexagonal lattice 4.6.12, abbreviated TT, we have $n = 2$ and $k = 12$. The sites are the *flags* (x, j, k) of the honeycomb lattice HEX, with $s(x) \in \{0, 1\}$ and $j \neq k \in \{1, 2, 3\}$: the vertex x, its edge in direction j, and the face of HEX at x between the

edges j and k. With $l = 6 - j - k$ the flag (x, j, k) is adjacent to exactly (x, j, l), (x, k, j) and $(x + \sigma(x) f_j, j, l)$. Every site has degree 3, $\kappa(\mathrm{TT}) = 3$, TT is vertex-transitive, and its automorphisms g_i satisfy $g_i \circ \tau_y = \tau_{M_i y} \circ g_i$ (see Remark 6.4). The distance is the maximum of twelve linear forms with offsets that depend on the orbit of the site, and the balls are dodecagons ([S, Part T2, Definition 1.1, Lemma 1.2 and Theorem 2.1]).

Locally blocked pairs. Let $a \neq g$ be sites of $S_r(c)$ and $L \geq 1$. The pair $\{a, g\}$ is *blocked at depth* L if there are no induced paths $(a, x_1, \ldots, x_L)$ and $(g, y_1, \ldots, y_L)$ with all $x_i, y_i \in B_{r-1}(c)$ that are disjoint and do not touch, and *locally blocked* if it is blocked at some depth $L \leq 3$. For $m \geq 577$, $\rho \geq m + 32$, $d(c, c') \leq 3$ and gadget terminals that satisfy the tail condition TF_2 of the next paragraph, the port pairs without a design are exactly the locally blocked pairs ([S, Part T2, Corollary 10.2]). They fall into four classes: *thin* pairs (two sites with two common neighbours), *dead* pairs, *hexagon* pairs and the remaining pairs ([S, Part T2, Definitions 3.1 and 3.3 and Proposition 3.4]). Dead and hexagon pairs grow linearly with the radius ([S, Part T2, Theorem A.5.2]), so they cannot be put into a corner family; for thin pairs the classification for $8 \leq r \leq 356$ gives evidence of the same growth ([S, Part Z, Section Z.25]). None of this is used in the proof. Thin and dead pairs are never the ports of a ball that the path enters two levels deep. At a hexagon pair the path is forced as soon as it enters the ball three levels deep. For most hexagon pairs (the *generic* ones) the host obtained from the ball by removing two sites of the path has two ports with a design ([S, Part T2, Lemmas 4.4 and 4.6 and Theorem 10.4]). The remaining pairs form the *corner family* X^{cor}: two or three clusters of two pairs each, at corners of the dodecagonal balls. X^{cor} satisfies (X1)–(X4) on the radii $\rho \in \mathcal{R}$, where $\mathcal{R} = \{\rho \geq 209 : \ \rho \bmod 10 \in \{2, 3, 4, 7, 8, 9\}\}$, with 48 labels, $p_0 = 4$, $\rho_X = 209$, $t_0 = 5$ and $\lambda = 1/32$; on the other residues (X4) fails. So the corner lemma is proved along $\mathcal{R}$, for balls that the path enters three levels deep ([S, Part T2, Definition 5.1, Proposition 5.2, Remark 5.3, Theorem 6.2 and Corollary 6.3]).

Gadget radii. (B1) fails for the radii computed: for $3 \leq m \leq 250$ the sphere $S_m(o)$ contains non-escapable pairs (computed; not used in the proof). The gadget is built from induced tails, and its terminals satisfy the tail condition TF_2 ([S, Part T2, Lemma 6.1]); the two-port statements are proved for all terminal pairs with TF_2.

Theorem 12.10 (Theorem A for TT). *Let E be an essential enhancement of site percolation on TT. Then Claim 5 holds for E, with the constants $R = 3K + C$ and $L_0 = 100R$ of (7.1), (7.2) computed with: a gadget radius $m \geq \max(m'_E, 577)$; $\rho_0 = m + 32$; $K_{\mathrm{CL}}(\rho_0)$ replaced by the constant $K'(\rho_0)$ of [S, Part T2, Corollary 6.3]; $n_0 = 1$, $t_0 = 6$, $c_2 = 1$; $c_3 = 344$, $c_4 = 345$, $K_3 = 1$. Consequently Conjecture 1 holds for E.*

Proof. Theorem 12.10 is [S, Part T2, Theorem 12.1]. Its proof follows that of Theorem A in Sections 7.3–7.6, with the gadget described above, the corner lemma along $\mathcal{R}$, (B2) with $t_0 = 6$, (B3) with $c_3 = 344$, $c_4 = 345$ and the two-port properties TP and HTP for $m \geq 577$, $\rho \geq m + 32$ and $d(c, c') \leq 3$ ([S, Part T2, Propositions 7.4 and 11.4 and Theorems 10.1 and 10.4]). In the main case a generic hexagon pair is replaced by the ports of its host. The tracks are the boundary cycles of unions of faces ([S, Part T2, Lemma 7.2]). The statements that rest on computations, and the arguments that extend them to all radii, are listed in Section 13.2 and Table 5. Moreover $\frac{1}{2} \leq p_c(\mathrm{TT}) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/12} < 1$ ([S, Part T2, Lemma 1.3]).

Proof of Theorem 1.5 for TT. Theorem 12.10 gives Conjecture 1 for every essential enhancement E of site percolation on TT ([S, Part T2, Main Theorem]). □

12.16. The diamond lattice

Coordinates. For the diamond lattice, abbreviated DIA, we have $n = 3$ and $k = 2$: DIA is the graph on $V = \mathbb{Z}^3 \times \{0, 1\}$ in which $(n, 0) \sim (n - e_i, 1)$ for $i = 0, 1, 2, 3$, where $e_0 = 0$ and e_1, e_2, e_3 are the unit vectors of $\mathbb{Z}^3$. Let $f_0, \dots, f_3$ be the unit vectors of $\mathbb{Z}^4$, $\Sigma(p) = p_0 + p_1 + p_2 + p_3$, and

$$W = \{p \in \mathbb{Z}^4 : \Sigma(p) \in \{0, -1\}\}, \qquad \varphi(n, 0) = (-(n_1 + n_2 + n_3), n_1, n_2, n_3), \qquad \varphi(n, 1) = \varphi(n, 0) - f_0.$$

Then φ is a bijection from V onto W with $\varphi(o) = 0$, and $u \sim v$ iff $\|\varphi(u) - \varphi(v)\|_1 = 1$. DIA is 4-regular and bipartite with the two orbits as classes, and $\kappa(\mathrm{DIA}) = 1$. Every permutation of the four coordinates is an automorphism fixing o, and the translations together with $\iota(p) = -p - f_0$ act transitively (see Remark 6.4; [S, Part D2, Definition 1.1 and Lemmas 1.2 and 1.4]).

Distance formula. $d(u, v) = \|\varphi(u) - \varphi(v)\|_1$. So $S_r(o) = \{p \in W : \|p\|_1 = r\}$; no two sites of a sphere are adjacent, and no sphere has a pocket ([S, Part D2, Lemma 1.2 and Corollary 1.3]). For $r \geq 2$ the twelve sites of $S_r(o)$ with two zero coordinates are its *vertex sites*.

Dead pairs and vertex pairs. A port pair $\{a, g\}$ of $S_\rho(c)$ is *dead* if every inner neighbour of a is adjacent to g (then a is an *apex* of the pair), or every inner neighbour of g is adjacent to a; $D_\rho(c)$ is the set of dead pairs. A dead port pair has no design. For $r \geq 3$ each dead pair of $S_r(o)$ has exactly one apex. Exactly 24 of them have a vertex site as apex, and the others are the *edge pairs*, $12(\lfloor r/2 \rfloor - 1)$ in number, so that $|D_r(o)| = 6r + 12$ (r even), $6r + 6$ (r odd) ([S, Part D2, Lemma 2.1 and Theorem 2.3]). Moreover the set $V_\rho(c)$ of 24 *vertex pairs*, each formed by a vertex site and a site at ℓ^1-distance 2 from it, consists of pairs that are not dead but have no design either, whenever $\rho \geq m + 4$ ([S, Part D2, Proposition 2.4]). For $m \geq 98$ and $\rho \geq m + 17$ every port pair outside $D_\rho(c) \cup V_\rho(c)$ has a design ([S, Part D2, Theorem 7.9]), so the port pairs without a design are exactly the pairs of $D_\rho(c) \cup V_\rho(c)$. As on the body-centred cubic lattice (Section 12.12), the edge pairs are handled without a corner lemma.

Corner data. The family $X^{\mathrm{vtx}}_\rho(c)$ of the 48 pairs of $D_\rho(c) \cup V_\rho(c)$ that contain a vertex site of $S_\rho(c)$ satisfies (X1)–(X4) with 24 labels, $\rho_X = 6$, $p_0 = 2$ and $\lambda = \frac{1}{4}$ ([S, Part D2, Definition 2.5 and Proposition 4.2]; Tables 2 and 3). As on 4.8^2 (Section 12.13), the corner lemma for deep-occupied balls ([S, Part BC, Theorem 3.2]) applies to X^{vtx}: every induced path through a site v with both end sites outside $B_{K'}(v)$ meets a deep-occupied ball $B_\rho(c)$ with $\rho \geq \rho_0$ and $B_{\rho+1}(c) \subseteq B_{K'}(v)$ whose ports are not in $X^{\mathrm{vtx}}_\rho(c)$. For $\rho_0 \leq 3\,811\,055\,617$ the constant is $K' = 45\,203\,452\,787\,856\,551$ ([S, Part D2, Corollary 4.3]).

History hosts. If the ports of a deep-occupied admissible ball form an edge pair, sites of the path on the sphere are removed from the ball one at a time, as long as the ports of the resulting host form a dead pair of the host. The possible removals are recorded in *history trees*, which define the set $\mathcal{H}_\rho(c)$ of *history hosts*. A finite computation shows that the removal ends in a recorded host whose ports are not dead in it ([S, Part D2, Definitions 5.4 and 5.8, Theorem 5.9 and Corollary 5.10]). Designs in a host are defined as in Section 12.12. Let $m \geq 1$ be a gadget radius, $\rho \geq m + 3$, c' a site with $d(c, c') \leq 1$ and (t_+, t_-) an escapable pair of $S_m(c')$. ($\mathrm{TP}^{\mathrm{dia}}$) Every port pair of $S_\rho(c)$ that is not in $D_\rho(c) \cup V_\rho(c)$ has a design in $B_\rho(c)$ for the terminals (t_+, t_-) at c'. ($\mathrm{HTP}^{\mathrm{dia}}$) For every $(Z, y) \in \mathcal{H}_\rho(c)$, with $D = B_\rho(c) \setminus Z$, and every $z \in S_\rho(c) \setminus (Z \cup \{y\})$ such that $\{z, y\}$ is not D-dead, there is a design for (z, y) in the host D for the terminals (t_+, t_-) at c' ([S, Part D2, Definitions 5.1 and 6.1]).

Theorem 12.11 (Theorem A with history hosts for DIA). *Suppose that for some integers $m_1 \geq 1$ and $C_0 \geq 3$, $\mathrm{TP}^{\mathrm{dia}}$ and $\mathrm{HTP}^{\mathrm{dia}}$ hold for every gadget radius $m \geq m_1$, every $\rho \geq m + C_0$, every c,*

every c' with $d(c,c') \le 1$ and every escapable pair of $S_m(c')$. Then Claim 5 holds for every essential enhancement E of site percolation on DIA, and Conjecture 1 holds for E.

The proof of Theorem 12.3 applies, with the corner lemma for X^{vtx} and the history hosts of DIA in place of those of BCC ([S, Part D2, Corollary 4.3 and Theorem 6.2]).

Theorem 12.12. *DIA satisfies (B1) with $M(\mathrm{DIA}) = \{m \ge 1\}$, (B2) with $M'(\mathrm{DIA}) = \{n \ge 1\}$, $n_0 = 1$, $t_0 = 3$ and $c_2 = 1$, and (B3) with $c_3 = 55$, $c_4 = 60$, $K_3 = 1$. For every gadget radius $m \ge 98$, every $\rho \ge m + 17$, every site c, every c' with $d(c,c') \le 1$ and every pair of distinct sites t^+, t^- of $S_m(c')$ as gadget terminals, $\mathrm{TP}^{\mathrm{dia}}$ and $\mathrm{HTP}^{\mathrm{dia}}$ hold. Moreover $\frac{1}{3} \le p_c(\mathrm{DIA}) \le p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/2} < 1$.*

Proof. Theorem 12.12 is proved in [S, Part D2, Propositions 3.3, 3.4, 3.9, Theorem 7.9 and Lemma 1.5] (see Table 6). Since the union of two consecutive spheres is disconnected, the tracks are unions of three consecutive spheres, whose sites far from a small box lie in one connected component of the complement of the box ([S, Part D2, Lemma 3.1, Corollary 3.2 and Remark 3.5]). The design is assembled from these tracks and spokes whenever the local escapes exist ([S, Part D2, Proposition 7.6]). The local escapes are constructed by hand for far pairs ([S, Part D2, Lemma 7.7]). For close port pairs, close terminal pairs and history hosts they are checked by computer ([S, Part D2, Proposition 7.8]), and so is the bound on the history chains ([S, Part D2, Theorem 5.9]); these checks are made at finitely many class representatives and radii, in two encodings each, and transfer lemmas carry them to all radii ([S, Part D2, Lemmas 1.8 and 7.2]; Table 5).

Proof of Theorem 1.5 for DIA. By Theorem 12.12 the hypotheses of Theorem 12.11 hold with $m_1 = 98$ and $C_0 = 17$, so Claim 5 and Conjecture 1 hold for every essential enhancement E of site percolation on DIA ([S, Part D2, Theorem D]). □

For a given E, let $m \ge \max(m_E, 98)$ be the gadget radius; then $\rho_0 = m + 17$, the constants K and C are given by (7.1) and (7.2) with $K'(\rho_0)$ in place of $K_{\mathrm{CL}}(\rho_0)$, $K = K'(m+17)$ and $C = 20 + 2c_2 + t_0 + 2\max(c_4, 10) = 145$, and Claim 5 holds with $R = 3K + C$ and $L_0 = 100R$; for $m \le 3\,811\,055\,600$: $K = 45\,203\,452\,787\,856\,551$, $R = 135\,610\,358\,363\,569\,798$ ([S, Part D2, Theorem D]).

13. Remarks on the periodic lattices

13.1. Constants

All constants are explicit. TP holds with $(\rho_{\mathrm{TP}}, C_0) = (200, 30)$ on the honeycomb lattice, $(8, 7)$ on the Kagomé lattice, $(16, 15)$ on the triangular lattice, and with the values of Table 4 on the lattices of Sections 12.4–12.9. For the five lattices of Theorems 1.4 and 1.5 the two-port statements hold with $\rho \ge m + 12$ for gadget radii $m \ge 20$ ($\mathrm{TP}^{\mathrm{bcc}}$ and $\mathrm{HTP}^{\mathrm{bcc}}$, body-centred cubic lattice), with $m_1 = 411$, $\rho_{\mathrm{TP}} = 387$ and $C_0 = 30$ ((TP*), truncated square lattice), with $(\rho_{\mathrm{TP}}, C_0) = (111, 28)$ ($\mathrm{TP}_{X^{\mathrm{T}}}$, truncated hexagonal lattice), with $\rho \ge m + 32$ for gadget radii $m \ge 577$ (TP and HTP, truncated trihexagonal lattice) and with $\rho \ge m + 17$ for gadget radii $m \ge 98$ ($\mathrm{TP}^{\mathrm{dia}}$ and $\mathrm{HTP}^{\mathrm{dia}}$, diamond lattice). Their corner lemmas hold with $K'(\rho_0) = 960\,005\,453\,344$ for $\rho_0 \le 5\,333\,334$ (Section 12.12), with $K'(\rho_0) = 113\,801\,110\,100$ for $\rho_0 \le 1\,881\,601$ (for deep-occupied balls, Section 12.13), with $K_{\mathrm{CL}}(\rho_0) = \rho_0 + 1$ ($\mathrm{CL}_{X^{\mathrm{T}}}$, Section 12.14), with the constant $K'(\rho_0)$ given explicitly in [S, Part T2, Corollary 6.3] (Section 12.15) and with $K'(\rho_0) = 45\,203\,452\,787\,856\,551$ for $\rho_0 \le 3\,811\,055\,617$ (for deep-occupied balls, Section 12.16). $\mathrm{CL}_{X^{\mathrm{hex}}}$ holds with $K_{\mathrm{CL}}(\rho_0) = 69\,497\,746\,068$ for $\rho_0 \le 1\,245\,457$

(Section 9.10), and on KING $\mathrm{CL}_{X^{\mathrm{gen}}(2)}$ holds with $K_{\mathrm{CL}}(\rho_0) = 6\,021\,188$ for $\rho_0 \le 21\,633$ (Section 12.8); for the other families, (8.1) gives K_{CL} from the data of Tables 2 and 3 and the two lattice constants ℓ, γ of Section 8.2. Theorem A then gives Claim 5 with $R = 3K + C$ and $L_0 = 100R$, where K and C are given by (7.1) and (7.2) in terms of these constants, of the constants of (B2) and (B3), and of the gadget radius m, which depends on the enhancement. We made no attempt to make any of the constants of this section smaller. The cross-checks described in [S, Part Z] indicate how far they are from the truth in the two-port property: in the tested ranges, every admissible port pair has a design when $\rho \ge m + 3$ on the triangular lattice (tested for $m \le 4$; and $C_0 \ge 3$ is necessary, Section 11.3), when $\rho \ge m + 4$ on the Kagomé lattice (tested for $m \le 4$ and $\rho \le 12$; and designs can fail for $\rho = m + 3$, Section 10.3), and, for pairs outside X^{hex}, when $\rho \ge m + 6$ on the honeycomb lattice (tested for $m = 2$ with $\rho \le 14$ and for $m \in \{3, 4\}$ with $\rho \le 11$).

13.2. The role of computation

No step of the proof for $\mathbb{Z}^d$ (Sections 2–4), of the reduction (Section 7), of the abstract corner lemma (Theorem C, Section 8), of the proofs for the Kagomé and triangular lattices (Sections 10 and 11) or of the probabilistic step (Appendix B) uses a computer. Every computation used in a proof has a second, independent encoding ([S, Part Z, Section 0]), except that for the corner data of the truncated square lattice the second encoding covers only $8 \le \rho \le 160$ and the claim for all radii rests on the symbolic encoding (Section 12.13), and that the second encoding of the distance-formula check of the truncated trihexagonal lattice imports the coordinate map ψ of [S, Part T2, Lemma A.1.2(e)] from a program of the first. Every statement used in a proof that is not named in this section or in Table 5 as resting on a computation is proved by hand. For the honeycomb lattice only Lemmas 9.14, 9.15 and 9.16 use a computer, at the finitely many radii given in Section 9.8, and the transfer lemma (Lemma 9.17) extends these searches to all radii. For the lattices of Section 12, Table 5 lists the statements that rest on finite computations and the arguments that extend them to all radii. On KING, and in the corner data for KING, ELTRI, RTH and FCC ([S, Parts KC, EC, RC, FC]), everything is proved by hand.

On the body-centred cubic lattice only the four local statements of [S, Part B2, Propositions 9.5, 9.7 and 9.8] use a computer; Theorem 12.3, the history hosts and the corner data ([S, Part B2, Sections 4 and 5, and Part BC]) are proved by hand.

The corner data of the truncated square lattice ([S, Part S2, Proposition 3.1]) are verified by an exact symbolic computation valid for all radii, and [S, Part S2, Lemmas 6.3, 7.1, 7.8, 7.9, 8.1(a), 8.2, 8.4 and 8.5] rest on finite computer checks, extended to all radii as described in Section 12.13. Sections 1, 2, 4 and 5 of [S, Part S2], which contain Theorem 12.5 and the treatment of the thin pairs, are proved by hand, and Sections 6–8 there combine proofs by hand with these checks.

On the truncated hexagonal lattice only [S, Part X2, Theorem 4.6, Lemma 7.6(c) (second bound), Lemma 7.7 and Lemmas 8.1 and 8.2] use a computer, at finitely many radii; Section 12.14 says how [S, Part X2, Theorem 4.6, Lemma 7.7 and Lemmas 8.1 and 8.2] are extended to all radii, and the second bound of [S, Part X2, Lemma 7.6(c)] concerns only levels at most 30.

In the proof for the truncated trihexagonal lattice the statements that rest on computations are the lattice structure ([S, Part T2, Lemma 1.2(a)–(d)]), the distance formula ([S, Part T2, Theorem 2.1 and Corollary 2.2]), the transfer bounds ([S, Part T2, Lemma 2.4(b)]), the classification of the locally blocked pairs ([S, Part T2, Proposition 3.4]), the corner data ([S, Part T2, Proposition 5.2]), a

descent property of the tracks ([S, Part T2, Lemma 7.3(a)]), the rays ([S, Part T2, Lemma 8.2]) and their crossings with the tracks ([S, Part T2, Lemma 8.4]). They also include the local escapes ([S, Part T2, Lemmas 9.1, 9.2 and 10.3]), the local steps of (B3) ([S, Part T2, Lemmas 11.1–11.3]) and, in the appendix of [S, Part T2], the absence of pockets, the thresholds $M^*(c)$ ([S, Part T2, Lemma A.5.4]) and the dead-pair test of [S, Part T2, Proposition A.4.6(a)], on which [S, Part T2, Theorem 4.3] rests. The distance formula is extended to all radii by induction ([S, Part T2, Theorem A.5.3]) and by an exact identity of affine functions; the transfer bounds rest on a complete enumeration; the rays are extended by the cone lemma and periodicity, and the partner rays of (B3) through them; the absence of pockets and the dead-pair test are extended by [S, Part T2, Lemma A.5.5]; the lattice structure and the thresholds concern finitely many objects; the corner data are verified exactly for all radii at which they are used; and the remaining checks are extended by the transfer lemmas of [S, Part T2, Section 2]. The statements proved by hand, given the computed statements they cite, are [S, Part T2, Lemmas 1.2(e), 1.3, 2.4(a), 2.5, 2.6, 3.2, 4.2, 4.4, 4.6, 4.8 and 6.1, Theorem 6.2, Corollary 6.3, Lemmas 7.1, 7.2 and 7.3(b), Proposition 7.4, Theorems 10.1 and 10.4, Lemma 11.1, Proposition 11.4 and Theorem 12.1], [S, Part T2, Main Theorem] and the cone lemma ([S, Part T2, Lemma A.5.1]) ([S, Part T2, Section 13]).

Only the bound on the history chains ([S, Part D2, Theorem 5.9]) and the local escapes of [S, Part D2, Proposition 7.8] use a computer on the diamond lattice, at finitely many radii and class representatives, with the extension to all radii described in Section 12.16; the corner data and the reduction are proved by hand.

The computations described in [S, Part Z] as cross-checks (SAT classifications of the two-port property, executions of the constructions of the two-port theorems with direct verification of (D1)–(D4), and checks of the distance formulas against breadth-first search) are not used in any proof.

Section 14 describes the formalization.

13.3. Comparison with $\mathbb{Z}^d$, and open problems

On $\mathbb{Z}^d$ the reduction of Section 2 and Theorem A differ in the treatment of the gadget terminals and of the boundary cases (the table in Section 7.2 and Remark 7.16(2)), and Corollary 8.15 recovers the Corner Lemma CL(d) of Section 4 from Theorem C with an explicit constant. Theorem A is not applied to $\mathbb{Z}^d$: the two-port theorem of Section 3 (Corollary 3.3) treats the terminals $c \pm me_1$ of the normalized gadget of Lemma 2.2, whereas TP_X requires every escapable terminal pair. On the triangular and Kagomé lattices no corner lemma is needed at all. On the honeycomb lattice and the lattices of Sections 12.4–12.9 the corner lemma is needed, but it comes from Theorem C as soon as the exceptional pairs are identified and shown to satisfy (X1)–(X4), which, given an explicit distance formula, are linear inequalities. The snub lattices show that the identification needs care: the metric definition of an apex that works on the other lattices produces unboundedly many apexes there, and the exceptional family has to be read off from the actual failing pairs.

On the body-centred cubic lattice, where the bounded searches of Section 1.5 found port pairs without a design along the edges of the cubic balls, host sets obtained from graph balls by removing sites visited by the path deal with these pairs (Section 12.12); on the diamond lattice such history hosts handle the dead pairs along the edges of the balls in the same way, and the corner lemma comes from the version of Theorem C for deep-occupied balls, applied to the 48 pairs of each sphere that contain a vertex site, which satisfy (X1)–(X4) (Section 12.16). On the three truncated lattices

the pairs found in numbers growing with the radius are the thin pairs (4.8^2), the triangle-blocked pairs (3.12^2) and the locally blocked pairs (4.6.12). On 4.8^2, where the ports of a deep-occupied ball never form a thin pair, the same version of Theorem C ([S, Part BC, Theorem 3.2]) gives the corner lemma, this time applied to the four corner pairs of each sphere, which satisfy (X1)–(X4) (Section 12.13). As shown in Section 12.14, the port pairs among the triangle-blocked pairs are exactly the port pairs without a design for $\rho \geq \max(111, m + 28)$, and no exceptional family containing them satisfies (X2)–(X3); the corner lemma $\mathrm{CL}_{X^{\mathrm{T}}}$ still holds trivially on 3.12^2, because an induced path that enters a ball through such a pair cannot reach two levels below its sphere. The locally blocked pairs are treated by the depth to which the path enters the ball, by host sets and by a version of Theorem C along a set of radii with bounded gaps, for balls that the path enters three levels deep (Section 12.15). Bond percolation on lattices other than the honeycomb lattice is not treated here (for $\mathbb{Z}^d$ see [BBR, Theorem 3], outlined in [BBR, §6]). On lattices with vertices of degree at least 4 the line-graph argument of Section 10.5 does not apply, because the analogue of Lemma 10.16 fails, and a direct treatment with a bond version of the two-port property would be needed.

14. Formal verification

The proof of Theorem 1.1 has been verified in the Lean 4 proof assistant [dMU], using the Mathlib library [Mat], and so have Theorem 1.2 for all nine of its lattices and Corollary 1.3. The formalization consists of the libraries `BBR` and `BBRAG` for $\mathbb{Z}^d$ and the library `BBRP` for periodic lattices. The archive cited under Data availability contains all three libraries, with instructions for checking them. This section describes what the formalization proves; Appendix D gives the Lean names of the main formal statements mentioned here.

Site percolation on $\mathbb{Z}^d$. Theorem 1.1 is proved with no hypothesis beyond Lean's axioms. The formalization proves Claim 5 for every essential enhancement of site percolation on $\mathbb{Z}^d$ and every $d \geq 2$, including the gadget (Lemma 2.2), Lemma 2.3, the deletion step (Lemmas 2.4 and 2.5), Theorem 2.10, Proposition 2.12 and Theorem 2.13, the two-port theorem (Lemma 3.1, Theorem 3.2 and Corollary 3.3) and the Corner Lemma (Lemma 4.5 and Theorem 4.8). It also proves Theorem 2.1. The comparison along curves in that proof needs $p_c^{\mathrm{site}}(\mathbb{Z}^d) < 1$, and the formalization proves the classical bounds $0 < p_c^{\mathrm{site}}(\mathbb{Z}^d) < 1$, by counting paths and by a Peierls argument; one formal theorem states Theorem 1.1 together with these bounds. Three of the formal proofs take a different route from the text. The formal proof of Lemma 2.2 works directly in the ℓ^1 form: the two sites given by the first step of the proof of [BBR, Lemma 6] are routed along explicit paths made of monotone pieces to $-me_1$ and me_1, with $m = d(r_E + 6)$, instead of deriving the ℓ^1 form from the ℓ^∞ form. The formal proof of the Corner Lemma does not follow Sections 4.2–4.6: instead of a stationary limit and mass transport it counts in a large cube, using Lemma 4.5 at a single radius and the density bound from the proof of Proposition 4.1. The formal proof of Case 3 of Theorem 2.13 uses the crossing directly instead of a deletion step, as Proposition 7.15 does.

Periodic lattices: the general results. The library `BBRP` proves the following statements of Sections 6–8: Theorem 6.5, and the bounds $0 < p_c(G) < 1$ of Proposition 6.6 for all nine lattices of Theorem 1.2; Lemma G, whose formal proof uses only the escapability clause of (B1), so that Lemma G follows from this clause alone, for an arbitrary set of radii and without the condition on pockets; Lemmas 7.3 and 7.4 and Theorem 7.10; Theorem A without its explicit constants: under the hypotheses of Theorem A, Claim 5 holds with some constants R and L_0, and Conjecture 1 holds if moreover $p_c(G) < 1$; and

Theorem C, which assumes (X1)–(X4) as in the text and gives CL_X for every $\rho_0 \geq 2$. The hypothesis (X1) is not used in the proof of Theorem C, and a general form of it assumes connectivity, minimum degree at least 3 and (X2)–(X4), but not (X1), in every dimension $n \geq 1$, and gives CL_X for every $\rho_0 \geq \rho_X$. The library also proves $\mathrm{CL}_\emptyset$, and two statements that compare the formal definitions with the text: every graph as in Definition 6.1 is an instance of the formal notion of a periodic lattice, and the formal versions of (B2) and (B3) are equivalent to their literal forms.

In the formal statements on periodic lattices the constants of (B2), (B3), TP_X, CL_X and Claim 5, the sets of radii in (B2), and the label sets, apex maps, constants and data of (X2)–(X4) are existentially quantified. The explicit values that this paper gives for them, for instance in (7.1), (7.2), (8.1), Theorems B-HEX, B-KAG and B-TRI and Table 4, are therefore not part of the formal results. The only exceptions are the two-port properties of the snub square and snub hexagonal lattices, which are also proved with the constants of Table 4: $\rho_{\mathrm{TP}} = 195$ and $C_0 = 20$, and $\rho_{\mathrm{TP}} = 227$ and $C_0 = 23$.

Periodic lattices: the instances. For the triangular, Kagomé and honeycomb lattices `BBRP` proves the properties (B1), (B2), (B3) and TP_X required by Theorem A, with $X = \emptyset$ on TRI and KAG and $X = X^{\mathrm{hex}}$ on HEX. In these statements (B1) holds with $M = \{m \geq 3\}$ on TRI and HEX and with $M = \{m \geq 2 : m \text{ even}\}$ on KAG. For HEX the library also proves (X1)–(X4) for X^{hex} and the two-port property $\mathrm{TP}_{X^{\mathrm{hex}}}$ of Theorem 9.18. In the proof of the latter the three local lemmas of Section 9.5 are proved by a direct argument, without the finite searches of Section 9.8: Lemmas 9.14 and 9.15 for all radii in their statements, and Lemma 9.16 for all $n \geq 10$. The Lean files that contain these proofs use neither `decide` nor `native_decide`. From these results the formalization obtains Theorem 1.2 for the three lattices, together with $0 < p_c < 1$, and Corollary 1.3, for enhancements activated at edges and at vertices. The identity $p_c^{\mathrm{bond}}(\mathrm{HEX}) = p_c(\mathrm{KAG})$ and the inequality $\theta^{\mathrm{bond}} \geq \theta^{\mathrm{KAG}}$ of Proposition 10.17 are proved as well.

For the elongated triangular, snub square and rhombitrihexagonal lattices, the square lattice with diagonals and the face-centred cubic lattice, `BBRP` proves (B1), (B2), (B3) and TP_X, and (X1)–(X4), for an exceptional family X: a generic corner family $X = X^{\mathrm{gen}}(p_0)$ on the four lattices other than the snub square lattice, and the family $X = X^{\mathrm{snub}}$ of Section 12.5 on the snub square lattice. These statements are Proposition 12.1 and the part of Theorem 12.2 that concerns (B1), (B2), (B3) and TP_X, for these five lattices, except that, besides the quantities listed above (among them the data u, t_0, λ of (X2)–(X4)), the sets of radii in (B1) and the parameter p_0 of $X^{\mathrm{gen}}(p_0)$ are existentially quantified; the proofs use $p_0 = 2$, and on the snub square lattice the proof of (B1) uses the radii $m \geq 12$. Theorem 12.2 also identifies the pairs that admit no design (the last column of Table 4); this identification is not part of the formal statements. The two-port property of the snub square lattice is also proved in the form of [S, Part Q2, Theorem 8.3], with $\rho_{\mathrm{TP}} = 195$ and $C_0 = 20$ as noted above; in its proof the local lemmas [S, Part Q2, Lemmas 8.1 and 8.2] are checked by Lean's kernel on bounded windows around a base site, and the condition (T3) of [S, Part Q2, Lemma 8.2] is proved for every $m \geq 1$. The last sentence of Theorem 12.2, on pockets, is proved for every sphere of the snub square lattice. Combined with CL_X from Theorem C, these results give formal proofs of Theorem 1.2 and of $0 < p_c < 1$ for the five lattices. For the face-centred cubic lattice the port check in the proof of the two-port property is reduced by the 48 point symmetries of the lattice to 103 representative sites of the sphere of radius 21.

For the snub hexagonal lattice `BBRP` proves (B1), (B2), (B3) and TP_X, and (X1)–(X4), for some exceptional family X; the proof uses the family X^{sc} of Section 12.6, and (B1) holds with $M = \{m \geq$

1}. As for the five lattices above, the sets of radii, the label set, the apex maps and the constants and data of (B2), (B3), TP_X and (X2)–(X4) are existentially quantified, and so is the family X. The two-port property for X^{sc} is also proved in the form of [S, Part H2, Theorem 7.9], with $\rho_{\mathrm{TP}} = 227$ and $C_0 = 23$. In its proof the close pairs of [S, Part H2, Lemmas 7.5 and 7.6] are reduced by translations to finitely many representative positions, which Lean's kernel checks, and the connectivity of the tracks [S, Part H2, Lemma 4.2] is proved by a descent along the track, without the polygonal Jordan curve theorem. Theorem C again supplies CL_X, and with it the formalization proves Theorem 1.2 and $0 < p_c < 1$ for the snub hexagonal lattice.

For these six lattices the formal proofs use general arguments or finite checks evaluated by Lean's kernel in place of the computations listed for them in Section 13.2 and Table 5, and of the extension of those computations to all radii.

Parts of statements that are not formalized. None of the following is used by the formal proofs of Theorem 1.1, of Theorem 1.2 for the nine lattices and of Corollary 1.3: conclusion 4 of Lemma 2.5 (the formal version of Lemma 2.5 is stated for a ball that contains v and not the origin); the values of R and L_0 that (2.2) gives in Theorem 2.13; the explicit bounds on $p_c(G)$ in Proposition 6.6 (only $0 < p_c(G) < 1$ is stated); the last sentence of Lemma G; in Lemma 7.4, the inclusion $\omega' \subseteq \omega \cup \{v\}$ and conclusion 4; in the last sentence of Theorem C, the claim that the covariance clause of (X2) is not used; the last sentence of Theorem B-HEX and parts 1 and 2 of Theorem 9.20; Lemma 9.16 for $4 \le n \le 9$; the last sentence of Theorem B-KAG except its clause on spheres of even radius; parts of Proposition 10.17 (the formalization proves what the proof of Corollary 1.3 uses from it: the range and essentialness of the transferred enhancement, the transfer of the Bernoulli laws and the inequality $\theta_F^{\mathrm{bond}} \ge \theta_E$, and it proves the identity of the critical points from the equivalence of $\theta^{\mathrm{bond}}(p) = 0$ and $\theta^{\mathrm{KAG}}(p) = 0$); and the second-to-last sentence of Theorem 12.2, on the pairs without a design (its last sentence, on pockets, is proved separately, for KING and FCC for the spheres around the root, and for SNUBHEX for the spheres around the six sites $(0, i)$).

Not formally verified. For the body-centred cubic lattice and the truncated square, truncated hexagonal, truncated trihexagonal and diamond lattices no result is formally verified: Theorems 1.4, 12.3 and 12.4, Theorem 1.5 and the statements of Sections 12.13, 12.14, 12.15 and 12.16, including the bounds $0 < p_c < 1$, rest on the proofs of Section 12.12, of these sections and of [S, Parts B2, BC, S2, X2, T2 and D2], and the proof of Theorem 1.4 and the proofs for these lattices are computer-assisted.

Trusted base. Every theorem listed in Appendix D depends only on Lean's standard axioms `propext`, `Classical.choice` and `Quot.sound`. In particular none of them depends on `sorry` or on `native_decide`. The proofs are checked by Lean's kernel. Some finite verifications are carried out by the kernel itself, with `decide`, among them the port check for the face-centred cubic lattice and the checks of the local lemmas for the snub square and snub hexagonal lattices, all described above. The trusted base is the kernel, including its built-in arithmetic on natural numbers, together with the definitions used in the formal statements. The definitions used in the statements of the main theorems are those of Mathlib and those in the files `BBR/Defs.lean` and `BBR/Main.lean` for $\mathbb{Z}^d$ and in `BBRP/Graph.lean`, `BBRP/Lattice.lean`, `BBRP/Defs.lean`, `BBRP/Lattices.lean`, `BBRP/LatticesMore.lean` and `BBRP/BondDefs.lean` for the periodic lattices. The statements of the formal versions of the intermediate results mentioned in this section also use definitions from `BBR/Setting.lean`, `BBRAG/Defs.lean`, `BBRP/Setting.lean`, `BBRP/FaithfulB.lean`, `BBRP/HexTracks.lean`, `BBRP/SnubhexGeom.lean` and `BBRP/SnubhexX.lean`.

Appendix A. Computational cross-checks for $\mathbb{Z}^d$

The computations below concern Sections 2–5. They are cross-checks, and no step of any proof depends on them. The searches in A.1.1 and A.2.4 do not use the proofs. The programs of A.1.2–A.1.5 follow the constructions of the proofs and check their output against the statements, and those of A.2.1–A.2.3 test the statements directly.

A.1. The two-port theorem.

1. *Exhaustive escape search.* For all non-separated pairs on S_r, up to the symmetries of $\mathbb{Z}^d$, escape walks with $k \le 4$ and length at most 7 were searched for, without using the case analysis of Theorem 3.2. The pairs were: $d = 2$, $r = 20$ (20 pairs); $d = 3$, $r = 20$ (222 pairs); $d = 4$, $r = 12, 16, 20, 25$ (328, 682, 1244, 2309 pairs); $d = 5$, $r = 13$ (953 pairs). Some of these radii are below the threshold of Theorem 3.2. In every run the only pairs without an escape were the corner pairs, in both orders.
2. *Implementation of the construction.* The case analysis of Theorem 3.2 was applied to every non-separated, non-corner pair up to symmetry. In Case II-A the program took the first step of Y not adjacent to X, and in Case II-B the first pair of steps of g that passed the check of (E1)–(E3), instead of following the rule of the proof (A.1.3 covers every choice the proof allows). The resulting walks were checked against (E1)–(E3): $d = 2, \dots, 9$ at $r = 20$; $d = 10$ at $r = 21 = 2d + 1$; $d = 4$ at $r = 25, 31, 40$; $d = 5$ at $r = 31$; $d = 3$ at $r = 41$. No pair failed. For example, there are 1,071,277 such pairs for $d = 9$ and 3,705,055 for $d = 10$.
3. *Complete designs.* Every escape that the proof of Theorem 3.2 allows (every admissible choice of coordinates and of the port X) was completed by an implementation of Lemma 3.1 at radius $r - k$, for every separating coordinate and both extreme admissible values of s. The resulting sets were checked against (D1)–(D4), with $m = r - 13$, so that Lemma 3.1 runs at radius exactly $m + 10$ when $k = 3$. This was done exhaustively for $d = 2$ ($r = 20$), $d = 3$ (each r from 20 to 40), $d = 4$ (each r from 20 to 30, and $r = 31$), $d = 5$ ($r = 20, 26$) and $d = 6, 7$ ($r = 20$), and on samples of pairs and escapes for $d = 8, 9$ ($r = 20$) and $d = 10, 11, 12$ ($r = 2d + 1$). No design failed.
4. *Lemma 3.1 alone.* Exhaustively up to symmetry for $d = 2$ ($r = 20$) and $d = 3$ ($r = 20, \dots, 23$), and on random samples for $d = 4, 5, 6$, with no failures.
5. *Theorem 2.10 on random paths.* Random induced walks in a box played the role of P. Random balls $B_\rho(c)$ with $\rho \in [\rho_0, \rho_0 + 4]$ and random centres (almost all of them off the walk), meeting the walk in at least three sites and with non-corner ports, had their intersection with the walk replaced by a design. The gadget was modelled by its two terminals, with an axis segment playing the role of X. For 8734, 4500 and 3000 balls in $d = 3, 4, 5$, the two ends of the walk were disconnected without X and connected with X, in every case.

A.2. The Corner Lemma.

1. Lemma 4.3 was checked on all geometric cases for $d = 2, \dots, 5$ and $R \le 9$, and on random pairs $(V, \prec)$ in boxes (6564 corner-paired balls), with no failures for $r \ge 2$. For $r = 1$ the map Ψ has collisions, as expected.
2. Lemma 4.5 was checked for all ordered pairs of signed poles and all partners, $d = 2, \dots, 5$, $r_1, r_2 \in [2, 9]$ and all t in the stated ranges: 1,760,576 cases, none of which failed.

3. Example 5.1 was checked for $d = 2, 3, 4$: every tested ball is corner-paired with pole e_1, the conclusions of Proposition 4.4 and Lemma 4.6 hold, and the averaged measure is invariant on random translations.
4. K-windows (Section 5.2) were searched for by a SAT solver for $d = 2$, with a separate checker for every satisfying assignment. The largest K admitting a K-window and the smallest K admitting none were $(4, 5)$ for $\rho_0 = 2$, $(7, 8)$ for $\rho_0 = 3$ and $(9, 10)$ for $\rho_0 = 4$. For $\rho_0 = 2$ an independent backtracking search confirmed that a 4-window exists. The non-existence of 5-, 8- and 10-windows rests on the solver's unsatisfiability answers, which were not checked independently. Subject to this, CL(2) holds with $K = 5, 8, 10$ for $\rho_0 = 2, 3, 4$.

Appendix B. The probabilistic step on periodic lattices

In this appendix G is a periodic lattice and E an enhancement with range r_E. We prove Theorem 6.5, following [AG, §3], [BBR, pp. 4–6] and Section 2.2. Essentialness of E is not used.

B.1. Locality

Lemma B.1. *(1) Each τ_z is an automorphism, so $\tau_z B_r(c) = B_r(\tau_z c)$. (2) $\deg(z, i) = \deg(x_i)$, so $\Delta = \max_i \deg(x_i) < \infty$, and $\Delta \geq 2$. (3) $B_{r+1}(c) = N[B_r(c)]$ and $|B_r(c)| \leq (\Delta + 1)^r$, so $M_r := \max_c |B_r(c)| = \max_i |B_r(x_i)|$ is finite.*

Proof. (1) and (2) are clear ($\Delta \geq 2$ because G is infinite and connected). (3) is (F1) together with $|N[A]| \leq (\Delta + 1)|A|$ and (1). □

Lemma B.2. *(1) $E_v(\omega) \subseteq B_{r_E}(v)$, and $E_v(\omega) = E_v(\omega')$ if ω, ω' agree on $B_{r_E}(v)$. (2) $\omega \subseteq E(\omega, \alpha) \subseteq E(\omega, \alpha')$ for $\alpha \subseteq \alpha'$. (3) If ω, ω' agree on $B_{2r_E}(x)$ and α, α' agree on $B_{r_E}(x)$, then $x \in E(\omega, \alpha)$ iff $x \in E(\omega', \alpha')$.*

Proof. (1) is (F8), and (2) holds by definition. (3) If $x \in \omega$, then $x \in \omega'$. If $x \in E_u(\omega)$ with $u \in \alpha$, then $d(u, x) \leq r_E$, so $u \in \alpha'$, and $B_{r_E}(u) \subseteq B_{2r_E}(x)$ gives $E_u(\omega') = E_u(\omega)$. □

Lemma B.3. *Let $A, A' \subseteq V$. (1) If A contains a path from o to a site y with $|y| \geq L$, then $A \cap B_L$ contains a path from o to S_L. (2) $R_L(A)$ iff $R_L(A \cap B_L)$. (3) R_L is increasing, and $R_{L+1}(A)$ implies $R_L(A)$. (4) o lies in an infinite component of A iff $R_L(A)$ holds for every L.*

Proof. (1) is (F6). (2) and (3) follow. (4) If the component $C \ni o$ is infinite, then $R_0(A)$ holds as $o \in A$, and for $L \geq 1$ the component has sites outside the finite ball B_{L-1}, so (1) applies. Conversely, sites $y_L \in S_L$ joined to o in A are distinct and lie in C. □

Let $F_L(\omega, \alpha) = 1$ if $R_L(E(\omega, \alpha))$ and 0 otherwise.

Lemma B.4. *(1) If ω, ω' agree on B_{L+2r_E} and α, α' agree on B_{L+r_E}, then $F_L(\omega, \alpha) = F_L(\omega', \alpha')$. (2) $F_L(\omega, \alpha) \leq F_L(\omega, \alpha')$ for $\alpha \subseteq \alpha'$. (3) A p-pivotal site v satisfies $|v| \leq L + 2r_E$, and an s-pivotal site w satisfies $|w| \leq L + r_E$.*

Proof. (1) By Lemma B.2(3), $E(\omega, \alpha) \cap B_L = E(\omega', \alpha') \cap B_L$; apply Lemma B.3(2). (2) follows from Lemma B.2(2). (3) Otherwise $\omega \cup \{v\}$ and $\omega \setminus \{v\}$ agree on B_{L+2r_E}, and (1) applies; similarly for w. □

B.2. The crossing polynomial

Let $\mathbb{P}_{p,s} = \mu_p \otimes \mu_s$ on $\mathcal{P}(V) \times \mathcal{P}(V)$, with μ_q the Bernoulli product measure. For a finite $\Lambda \subseteq V$ and $\eta \subseteq \Lambda$ put $w_q^\Lambda(\eta) = q^{|\eta|}(1-q)^{|\Lambda\setminus\eta|}$, and for $f : \mathcal{P}(\Lambda)^2 \to \mathbb{R}$ put $\mathbb{E}_{p,s}^\Lambda[f] = \sum_{\eta,\zeta\subseteq\Lambda} f(\eta,\zeta) w_p^\Lambda(\eta) w_s^\Lambda(\zeta)$, a polynomial in (p,s) that equals $\mathbb{E}_{p,s}[f(\omega\cap\Lambda, \alpha\cap\Lambda)]$ for $p,s \in [0,1]$.

Lemma B.5. *Let $\Lambda \supseteq B_{L+2r_E}$ be finite. (1) $\tau_L(p,s) := \mathbb{P}_{p,s}(R_L(E(\omega,\alpha))) = \mathbb{E}_{p,s}^\Lambda[F_L]$, so τ_L is (the restriction of) a polynomial. (2) $\theta_E(p,s) = \lim_L \tau_L(p,s) = \inf_L \tau_L(p,s)$. (3) $\theta(p) = \theta_E(p,0) \le \tau_L(p,0)$.*

Proof. (1) follows from Lemma B.4(1), (2) from Lemma B.3(3),(4) and continuity from above, and (3) from $E(\omega,\emptyset) = \omega$. □

B.3. Claim 5 implies Claim 4

Proposition B.6 ([BBR, pp. 5–6]). *If Claim 5 holds with constants L_0, R, then Claim 4 holds with the same constants.*

Proof. Let v be p-pivotal in (ω,α) and $\alpha^- = \alpha \setminus B_R(v)$. If $F_L(\omega\cup\{v\},\alpha^-) = 1$, then v is p-pivotal in (ω,α^-) by Lemma B.4(2), and Claim 5 applies to (ω,α^-); the resulting state agrees with (ω,α) outside $B_R(v)$. Otherwise list the finite set $\alpha\cap B_R(v)$ as $a_1,\dots,a_m$, put $\alpha_j = \alpha\setminus\{a_1,\dots,a_j\}$, and let $j \ge 1$ be least with $F_L(\omega\cup\{v\},\alpha_j) = 0$. Then $w = a_j$ is s-pivotal in $(\omega\cup\{v\},\alpha_j)$, a state that agrees with (ω,α) outside $B_R(v)$. □

B.4. The Margulis–Russo formula

Lemma B.7. *For finite Λ, any $f : \mathcal{P}(\Lambda)^2 \to \mathbb{R}$ and $\tau = \mathbb{E}_{p,s}^\Lambda[f]$,*

$$\partial_p \tau = \sum_{v\in\Lambda} \mathbb{E}_{p,s}^\Lambda\big[f(\eta\cup\{v\},\zeta) - f(\eta\setminus\{v\},\zeta)\big], \qquad \partial_s \tau = \sum_{w\in\Lambda} \mathbb{E}_{p,s}^\Lambda\big[f(\eta,\zeta\cup\{w\}) - f(\eta,\zeta\setminus\{w\})\big].$$

Proof. By the product rule, $\frac{d}{dp} w_p^\Lambda(\eta) = \sum_x \varepsilon_x(\eta) w_p^{\Lambda\setminus\{x\}}(\eta\setminus\{x\})$ with $\varepsilon_x(\eta) = \pm 1$ according as $x \in \eta$ or not. Pairing η' and $\eta'\cup\{x\}$ for $\eta' \subseteq \Lambda\setminus\{x\}$, and using $w_p^\Lambda(\eta'\cup\{x\}) + w_p^\Lambda(\eta') = w_p^{\Lambda\setminus\{x\}}(\eta')$, both sides of the one-parameter identity equal $\sum_x \sum_{\eta'} [h(\eta'\cup\{x\}) - h(\eta')] w_p^{\Lambda\setminus\{x\}}(\eta')$. Apply this to $h_s(\eta) = \sum_\zeta f(\eta,\zeta) w_s^\Lambda(\zeta)$, and symmetrically in s. No monotonicity of f is needed. □

For $\Lambda \supseteq B_{L+2r_E}$ let $N_p(\eta,\zeta)$ and $N_s(\eta,\zeta)$ be the numbers of p-pivotal and s-pivotal sites in Λ. By Lemma B.4 these are all p- and s-pivotal sites, and pivotality in (ω,α) depends only on $(\omega\cap\Lambda, \alpha\cap\Lambda)$.

Lemma B.8. *For $(p,s) \in [0,1]^2$: $\partial_s \tau_L = \mathbb{E}_{p,s}^\Lambda[N_s] \ge 0$ and $\partial_p \tau_L \le \mathbb{E}_{p,s}^\Lambda[N_p]$. In particular $\tau_L(p,\cdot)$ and $\theta_E(p,\cdot)$ are non-decreasing.*

Proof. In Lemma B.7 with $f = F_L$, the s-differences are $\mathbf{1}[w \text{ s-pivotal}]$ by Lemma B.4(2), and the p-differences lie in $\{-1,0,1\}$ and equal 1 exactly when v is p-pivotal. □

Since E need not be monotone, $\partial_p \tau_L$ may be negative; only the upper bound is used.

B.5. The modification map

Assume Claim 4 with constants L_0, R. Put $M = M_R$ and $c(p,s) = \min(p, 1-p)\min(s, 1-s)$.

Proposition B.9 ([BBR, (2)]). *Let $L \geq L_0$ and $\Lambda = B_{L+2r_E+R}$. For $(p,s) \in (0,1)^2$,*

$$\mathbb{E}_{p,s} N_s(L) \geq g(p,s)\, \mathbb{E}_{p,s} N_p(L), \qquad g(p,s) = \frac{c(p,s)^M}{M\, 4^M}.$$

Proof. Let $\mathcal{A}$ be the set of triples (v,η,ζ) with $\eta,\zeta \subseteq \Lambda$ and $v \in \Lambda$ p-pivotal in (η,ζ), and $\mathcal{B}$ the analogous set for s-pivotal sites; weight a triple by $W(u,\eta,\zeta) = w_p^\Lambda(\eta) w_s^\Lambda(\zeta)$. For $a = (v,\eta,\zeta) \in \mathcal{A}$ we have $|v| \leq L + 2r_E$, so $B_R(v) \subseteq \Lambda$, and Claim 4 gives $(w,\eta',\zeta') \in \mathcal{B}$ with $d(v,w) \leq R$ and (η',ζ') agreeing with (η,ζ) outside $B_R(v)$; choose one and call it $T(a)$. Given $T(a) = b$, there are at most M choices for $v \in B_R(w)$ and at most 4^M for $(\eta \cap B_R(v), \zeta \cap B_R(v))$, so $|T^{-1}(b)| \leq M4^M$. The weights differ only on $B_R(v)$, so $W(T(a)) \geq c(p,s)^M W(a)$. Hence $c^M \sum_{\mathcal{A}} W \leq \sum_{\mathcal{A}} W(T(a)) \leq M4^M \sum_{\mathcal{B}} W$. □

The function g is continuous and positive on $(0,1)^2$ and does not depend on L, because M depends only on G and R. With Lemma B.8 this gives $\partial_s \tau_L \geq g\, \partial_p \tau_L$ on $(0,1)^2$ for $L \geq L_0$ [BBR, (3)].

B.6. Proof of Theorem 6.5

Let $p_c = p_c(G) \in (0,1)$ ($p_c \geq 1/(\Delta-1)$ by Lemma 6.7) and let Claim 5, hence Claim 4, hold with L_0, R. Fix $s \in (0,1]$ and put

$$s_0 = \min(s, \tfrac{1}{2}), \quad p_1 = \tfrac{p_c}{2}, \quad p_2 = \tfrac{1+p_c}{2}, \quad c_* = \tfrac{\min(p_c, 1-p_c)}{2} \cdot \tfrac{s_0}{2}, \quad g_0 = \frac{c_*^M}{M4^M}, \quad T = \min\Big(\frac{s_0}{2}, \frac{1-p_c}{2g_0}\Big),$$

and $\pi(s) = \max(p_c/2,\ p_c - g_0 T) < p_c$. On $\mathcal{W} = [p_1,p_2] \times [s_0/2, s_0]$ we have $c \geq c_*$ and $g \geq g_0$. Let $p \in (\pi(s), p_c)$, $L \geq L_0$, and $\varphi(t) = \tau_L(p + g_0 t, s_0 - t)$ for $t \in [0,T]$; the curve stays in $\mathcal{W}$. By Lemma B.8 and Proposition B.9,

$$\varphi'(t) = g_0 \partial_p \tau_L - \partial_s \tau_L \leq g_0\, \mathbb{E}^\Lambda[N_p] - g\, \mathbb{E}^\Lambda[N_p] \leq 0,$$

so $\tau_L(p', s_0 - T) \leq \tau_L(p, s_0)$ with $p' = p + g_0 T > p_c$. Since $p' > p_c$, $\theta(p') > 0$. By Lemma B.5(3) and Lemma B.8,

$$\theta(p') \leq \tau_L(p',0) \leq \tau_L(p', s_0 - T) \leq \tau_L(p, s_0) \leq \tau_L(p,s) \qquad (L \geq L_0),$$

and Lemma B.5(2) gives $\theta_E(p,s) \geq \theta(p') > 0$. □

The uniformity of R is used through $M = M_R$, and that of L_0 in the last display. The hypothesis $p_c < 1$ is used to find $p' \in (p_c, p_2]$ inside a compact window on which $g \geq g_0 > 0$.

B.7. Roots

Finite energy (Remark 6.3). If π is the set of sites of a path from y to x, then $\theta_y(p) \geq p^{|\pi|} \theta_x(p)$. Indeed, on $\{\pi \subseteq \omega\}$ the event that x lies in an infinite component of ω coincides with the event I that x lies in an infinite component of $\omega \cup \pi$, and on both it implies that y lies in the same component. The event I depends only on $\omega \setminus \pi$, so it is independent of $\{\pi \subseteq \omega\}$, and it contains the event that x lies in an infinite component of ω. Hence $\theta_y(p) \geq \mathbb{P}_p(I)\, p^{|\pi|} \geq p^{|\pi|} \theta_x(p)$. So positivity of θ does not depend on the root, and neither does $p_c(G)$. The analogous statement for θ_E can fail on general periodic lattices when the enhancement is not monotone (Remark 6.3): on $\mathbb{Z}^2$ with a pendant site

attached to every site, the non-essential enhancement that opens a site of $\mathbb{Z}^2$ and its four neighbours in $\mathbb{Z}^2$ whenever its ball of radius 3 contains no open site has $\theta_E > 0$ at the sites of $\mathbb{Z}^2$ for small $p > 0$ and s close to 1, while $\theta_E = 0$ at the pendant sites for $p < \frac{1}{4}$.

Remark 6.4. Let ϕ be an automorphism with $\phi\tau_y = \tau_{Ay}\phi$. Then $E_{\tau_y u}(\tau_y\omega) = \tau_y E_u(\omega)$ gives $E^{\phi}_{\tau_z v}(\tau_z\omega) = \tau_z E^{\phi}_v(\omega)$, so E^{ϕ} is the enhancement with rules $E^{\phi}_j = E^{\phi}_{(0,j)}$, of range r_E. If $\phi^{-1}(0,j) = (y,i)$, then $\omega \cup E^{\phi}_j(\omega) = \phi\tau_y(\tilde{\omega} \cup E_i(\tilde{\omega}))$ with $\tilde{\omega} = \tau_{-y}\phi^{-1}\omega$, so E^{ϕ} is essential iff E is. Finally $E^{\phi}(\phi\omega, \phi\alpha) = \phi E(\omega, \alpha)$, and $(\phi\omega, \phi\alpha)$ has law $\mathbb{P}_{p,s}$. For HEX, $\phi(z,i) = (-z, 1-i)$ works with $A = -I$. For KAG, the map $a(z) \mapsto b(Mz)$, $b(z) \mapsto c(Mz)$, $c(z) \mapsto a(Mz)$ with $Me_1 = -e_1 + e_2$, $Me_2 = -e_1$ maps the edges $a(z)b(z)$, $b(z)c(z)$, $c(z)a(z)$, $a(z)b(z-e_1)$, $a(z)c(z-e_2)$, $b(z)c(z+e_1-e_2)$ to edges of KAG; it is bijective and maps the four edges at each site injectively, hence onto the four edges at the image, so it is an automorphism, with $A = M$. Its powers permute the three orbits cyclically.

Appendix C. Tables for the periodic lattices

This appendix contains three tables referred to in the text: Table 1, on the exceptional families of the two-port property (Section 1.4), and Tables 5 and 6, on the computations and on the sources in [S] of the proofs for the lattices of Section 12 (Section 12.3, whose convention for statement numbers of [S] applies to them).

Table 1 lists the exceptional families of the two-port property (Section 7.2) used on the lattices of Theorem 1.2 and on the four lattices of Theorem 1.5, and the number of port pairs on a large sphere that admit no design (Definition 7.2). For the honeycomb, snub square and snub hexagonal lattices the family consists exactly of these pairs. The families $X^{\mathrm{gen}}(2)$ contain them and further pairs. On $\mathbb{Z}^d$ the exceptional pairs are the corner pairs of Definition 2.11, $4d(d-1)$ on every sphere of radius at least 2.

lattice	k	**degree**	**exceptional family**	**failing non-adjacent pairs per large sphere**	**section**
honeycomb	2	3	X^{hex} (corner pairs)	12	9
Kagomé	3	4	none	0	10
triangular	1	6	none	0	11
elongated triangular $3^3.4^2$	2	5	$X^{\mathrm{gen}}(2)$	4	12.4
snub square $3^2.4.3.4$	4	5	X^{snub}	2	12.5
snub hexagonal $3^4.6$	6	5	X^{sc}	2 or 3	12.6
rhombitrihexagonal 3.4.6.4	6	4	$X^{\mathrm{gen}}(2)$	6	12.7
square with diagonals	1	8	$X^{\mathrm{gen}}(2)$	8	12.8
face-centred cubic	1	12	$X^{\mathrm{gen}}(2)$	24	12.9
truncated square 4.8^2	4	3	X^{cor} and the thin pairs (TP*)	the thin pairs and 4 (computed)	12.13
truncated hexagonal 3.12^2	6	3	X^{T} (triangle-blocked pairs)	at least $\lfloor(\rho+1)/4\rfloor$	12.14

lattice	k	degree	exceptional family	failing non-adjacent pairs per large sphere	section
truncated trihexagonal 4.6.12	12	3	X^{cor} on $\mathcal{R}$ and the locally blocked pairs	the locally blocked pairs; the dead and hexagon pairs among them grow linearly	12.15
diamond	2	4	$D_\rho(c) \cup V_\rho(c)$; corner family X^{vtx}	the dead pairs, growing linearly, and the 24 vertex pairs	12.16

Table 1. Here k is the number of orbits of sites under the translations of the coordinates used (Definition 6.1). The fifth column gives the number of pairs of distinct non-adjacent non-pocket sites of a large sphere that admit no design (Theorems B-HEX, B-KAG and B-TRI of Sections 9–11, and Theorem 12.2). For the families $X^{\mathrm{gen}}(2)$ these are some of the non-adjacent pairs of the family (all of them on the square lattice with diagonals). On the elongated triangular, rhombitrihexagonal and face-centred cubic lattices the other non-adjacent pairs of the family have designs (Section 12.2). For the truncated square lattice the family consists of the four corner pairs of X^{cor} and the thin pairs (two opposite vertices of a square), whose number grows linearly with the radius; the fifth column gives the result of exact computations at the radii $8 \le \rho \le 17$ and $20 \le \rho \le 23$ ([S, Part S2, Remark 2.5]), which is not proved for other radii and is not used (Section 12.13). For the truncated hexagonal lattice the port pairs in the family X^{T} are exactly the port pairs without a design when $\rho \ge \max(111, m+28)$ ([S, Part X2, Corollary 8.4]); their number grows linearly with the radius, and the corner lemma $\mathrm{CL}_{X^{\mathrm{T}}}$ (Section 1.4) holds for another reason (Section 12.14). For the truncated trihexagonal lattice the port pairs without a design are, for gadget radii $m \ge 577$, $\rho \ge m + 32$, gadget centres c' with $d(c, c') \le 3$ and terminals that satisfy the tail condition TF_2, exactly the locally blocked pairs ([S, Part T2, Corollary 10.2]); the corner lemma is applied to the subfamily X^{cor} along a set $\mathcal{R}$ of radii, and the other classes are treated as described in Section 12.15. For the diamond lattice the port pairs without a design are, for $m \ge 98$ and $\rho \ge m + 17$, exactly the dead pairs and the 24 vertex pairs; the corner lemma is applied to the subfamily X^{vtx} of the 48 pairs that contain a vertex site of the sphere, and the edge pairs are treated with history hosts (Section 12.16).

lattice	statements whose proofs use a finite computation	extension to all radii
KING	none	—
ELTRI	apexes ([S, Part E2, Lemma 2.4]), (B1) ([S, Part E2, Proposition 3.1(b)]), tracks ([S, Part E2, Lemma 7.2(b)]), the three local lemmas ([S, Part E2, Lemmas 7.5–7.7]: ports at $\rho \in \{164, 165\}$, terminals for $m \le 125$, rays for $n \le 60$)	local types, [S, Part E2, Lemmas 1.4, 1.5 and Corollary 1.6]

lattice	statements whose proofs use a finite computation	extension to all radii
SNUBSQ	descent in the distance formula, spheres and apexes for $3 \le \rho \le 69$, tracks, rays, (B1), the local lemmas of [S, Part Q2, Theorem 8.3], parts of (B3) (certificates C0–C4, C6–C8 of [S, Part Q2]); apexes for $2 \le \rho \le 7$ and the base case $D \le 7$ of the distance formula ([S, Part QC, Theorem 1.6])	window reduction, [S, Part Q2, Proposition 3.3]
SNUBHEX	descent in the distance formula ([S, Part H2, Lemmas 1.0, 1.2]), [S, Part H2, Lemma 2.1], the family X^{sh} ([S, Part H2, Proposition 2.6 and Remark 2.7]), (B1) ([S, Part H2, Proposition 3.1]), tracks ([S, Part H2, Lemma 4.2(c)]), parts of (B3) ([S, Part H2, Lemmas 5.2–5.4]), the local lemmas ([S, Part H2, Lemmas 7.4–7.7]); the descent of [S, Part HC, Lemma 2.6], [S, Part HC, Lemma 5.3] for $2 \le \rho \le 16$, and (X4) for $2 \le \rho \le 52$ ([S, Part HC, Proposition 6.5])	transfer lemma, [S, Part H2, Lemma 2.3]; by hand in [S, Part HC]
RTH	the 24 sites of level 1 to 3 ([S, Part R2, Theorem 1.3]), (B1) for $m \le 40$, tracks, apexes for $3 \le n \le 60$, the local lemmas ([S, Part R2, Lemmas 8.6–8.8]: ports at $\rho \in \{195, 196, 197\}$), a step of (B3)	transfer principle, [S, Part R2, Section 3] (Lemmas 3.1, 3.2, 3.4, 3.5 and Corollary 3.3)
FCC	(B1) for $m \le 11$ ([S, Part F2, Proposition 3.1]), the local lemmas ([S, Part F2, Lemmas 7.5–7.7]); none in [S, Part FC]	R-types, [S, Part F2, Definition 7.1 and Lemmas 7.2, 7.3]
BCC	the connectivity of a track minus a spoke for $3 \le k \le 30$ and a local form of it at $k = 7$ ([S, Part B2, Proposition 9.5]), the local port escapes for $13 \le \rho \le 53$ and terminal escapes for $20 \le m \le 53$ ([S, Part B2, Proposition 9.7]), the local port escapes in history hosts for $30 \le \rho \le 59$ ([S, Part B2, Proposition 9.8]); none in [S, Part BC]	connectivity of tracks minus a ball and transfer lemma, [S, Part B2, Lemmas 9.4, 9.6]

lattice	**statements whose proofs use a finite computation**	**extension to all radii**
4.8^2	(X2)–(X4) for X^{cor} ([S, Part S2, Proposition 3.1], exact for all radii), ascent and descent of faces with top ≤ 250 (Lemma 6.3), unique outer neighbours for levels 1–140 (Lemma 7.1), the finite inputs of (B3) at levels 491–502 and for 8 representatives (Lemmas 7.8, 7.9), rays crossing tracks for $10 \leq n \leq 250$ (Lemmas 8.1(a), 8.2), the local escapes (Lemmas 8.4, 8.5: ports at $387 \leq \rho \leq 398$, terminals for $411 \leq m \leq 422$)	exact for the corner data; R-types, [S, Part S2, Lemmas 5.1, 5.3 and Corollary 5.2], and translation along the rays (proof of Lemma 8.2)
3.12^2	the gadget radii ([S, Part X2, Theorem 4.6]: $m \leq 64$, $m \not\equiv 2 \pmod 4$), the second bound of Lemma 7.6(c) (levels ≤ 32), rays crossing tracks (Lemma 7.7, $2 \leq n \leq 40$), the local escapes (Lemmas 8.1, 8.2: ports at $111 \leq \rho \leq 115$, terminals for $1 \leq m \leq 115$)	transfer principle, [S, Part X2, Lemmas 2.2, 2.3]; translation along the rays (Lemma 7.7); the first bound of Lemma 7.6(c), by hand, for levels ≥ 10
4.6.12	the lattice structure ([S, Part T2, Lemma 1.2(a)–(d)]), the distance formula ([S, Part T2, Theorem 2.1 and Corollary 2.2]), the transfer bounds (Lemma 2.4(b)), the classification of the locally blocked pairs (Proposition 3.4: $8 \leq r \leq 356$), the corner data (Proposition 5.2), the descent property of the tracks (Lemma 7.3(a): $n \leq 250$), rays (Lemmas 8.2, 8.4: $n \leq 280$), the local escapes (Lemmas 9.1, 9.2, 10.3: ports at $513 \leq \rho \leq 522$, terminals for $577 \leq m \leq 586$), the local steps of (B3) (Lemmas 11.1–11.3); in the appendix of [S, Part T2]: the absence of pockets ($r \leq 300$), the thresholds $M^*(c)$ ([S, Part T2, Lemma A.5.4], and $M^*(32) = 266$) and the dead-pair test of [S, Part T2, Proposition A.4.6(a)] ($\rho \leq 300$), used in [S, Part T2, Theorem 4.3]	the lattice structure and $M^*(c)$: finite objects, no extension needed; the distance formula: induction ([S, Part T2, Theorem A.5.3]) and an exact identity of affine functions; the transfer bounds: complete enumeration; the classification, the descent property, the crossings of rays with tracks, the local escapes and the local steps of (B3): transfer lemmas, [S, Part T2, Lemmas 2.5 and 2.6]; the rays: cone lemma and periodicity, and the partner rays of (B3) also through them; the absence of pockets and the dead-pair test: [S, Part T2, Lemma A.5.5]; the corner data exactly for all $\rho \in \mathcal{R}$
DIA	the bound on the history chains ([S, Part D2, Theorem 5.9]) and the local escapes for close port pairs, close terminal pairs and history hosts ([S, Part D2, Proposition 7.8]), at finitely many radii and class representatives	transfer lemmas, [S, Part D2, Lemmas 1.8 and 7.2]

Table 5. The certificates (programs, commands, outputs and the second encodings) are described in

[S, Part Z] and archived as stated under Data availability.

lattice	**spheres**	**(B1); (B2); (B3)**	**two-port property**	**corner data**	p_c
KING	[S, Part K2, Lemma 2.2] (neighbours in adjacent spheres), Lemmas 2.4, 2.5 (outward rays, diagonals)	[S, Part K2, Propositions 3.1, 4.1, 5.1]	[S, Part K2]: port escapes, Lemmas 7.1, 7.2; terminal escapes, Lemma 7.3; crossing of blocks by paths of increasing level, Lemma 2.6; design, Theorem 7.4	[S, Part KC, Theorem K and Corollary K]; Propositions 2.1, 3.1, 4.1, 5.2	$1/7 \leq p_c(\mathrm{KING}) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2) < 1$, [S, Part K2, Proposition 8.1]
ELTRI	[S, Part E2, Lemmas 2.1, 2.2] (spheres), Lemma 2.4 (apexes)	[S, Part E2, Propositions 3.1, 4.1, 5.4]; (B3) also uses Lemmas 5.1–5.3	[S, Part E2]: tracks, Definition 7.1, Lemma 7.2; apex rays, Definition 7.3, Lemma 7.4; port escapes, terminal escapes, rays crossing tracks, Lemmas 7.5, 7.6, 7.7; design, Theorem 7.9	[S, Part EC, Theorem E]; via the functionals F_π, Lemmas 2.2, 2.3, Proposition 3.2, Theorem 3.3, Propositions 4.1, 5.1, 6.3	$\frac{1}{4} \leq p_c(\mathrm{ELTRI}) \leq p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/2} < 1$, [S, Part E2, Proposition 8.1]

lattice	**spheres**	**(B1); (B2); (B3)**	**two-port property**	**corner data**	p_c
SNUBSQ	[S, Part Q2, Lemma 2.1] (no pockets), Lemma 2.3 and Corollary 2.4 (number of apexes)	[S, Part Q2, Propositions 5.1, 6.1, Theorem 9.4]; (B3) also uses Lemmas 9.1–9.3	[S, Part Q2]: tracks are induced cycles, Theorem 4.2; corner rays and their crossings with tracks, Lemmas 4.4, 4.5; port and terminal escapes, Lemmas 8.1, 8.2; design, Theorem 8.3	[S, Part QC, Theorem III]; [S, Part QH, Theorem 1, Lemma 1.1, Corollary 2.3]	$\frac{1}{4} \leq p_c(\text{SNUBSQ}) \leq p_c^{\text{site}}(\mathbb{Z}^2)^{1/4} < 1$, [S, Part Q2, Proposition 10.1]
SNUBHEX	[S, Part H2, Lemma 2.1] (inner and outer neighbours), Proposition 2.6 (dead pairs)	[S, Part H2, Propositions 3.1, 4.3, 5.1]; (B2) uses tracks, Definition 4.1, Lemma 4.2, and (B3) also uses Lemmas 5.2–5.4	[S, Part H2]: apex rays, Definition 7.3, Lemma 7.4; local lemmas, Lemmas 7.5–7.7; design, Theorem 7.9	[S, Part HC, Theorems 5.4, 6.1]; [S, Part QH, Theorem 3.10]	$\frac{1}{4} \leq p_c(\text{SNUBHEX}) \leq p_c^{\text{site}}(\mathbb{Z}^2)^{1/6} < 1$, [S, Part H2, Proposition 8.1]
RTH	[S, Part R2, Definition 2.1, Lemma 2.2] (hexagonal spheres with a boundary coordinate)	[S, Part R2, Propositions 5.1, 6.1, 7.3]; (B3) also uses Lemmas 7.0–7.2	[S, Part R2]: tracks, Definition 8.1, Theorem 8.2; corner rays, Definition 8.3, Lemma 8.4; local lemmas, Lemmas 8.6–8.8; design, Theorem 9.2	[S, Part RC, Theorem R, Proposition 5.3]; in the cube coordinates, Lemma 2.2, Propositions 2.3, 2.4, 3.1, Lemma 5.1	$\frac{1}{3} \leq p_c(\text{RTH}) \leq p_c^{\text{site}}(\mathbb{Z}^2)^{1/6} < 1$, [S, Part R2, Proposition 4.1]

lattice	**spheres**	**(B1); (B2); (B3)**	**two-port property**	**corner data**	p_c
FCC	[S, Part F2, Lemma 2.2] (spheres), Lemma 2.5 and Corollary 2.6 (a sphere minus a small box around a site is connected)	[S, Part F2, Propositions 3.1, 4.1, 5.4]; (B3) also uses Lemmas 5.1–5.3	[S, Part F2]: vertex rays, Lemma 7.4; port and terminal escapes, Lemmas 7.5, 7.6; connectivity of a sphere minus a vertex ray, Lemma 7.7; design, Theorem 7.8	[S, Part FC, Theorem F, Corollary F2, Proposition 5.4]; Lemma 3.2, Lemmas 5.1–5.3	$1/11 \le p_c(\mathrm{FCC}) \le p_c^{\mathrm{site}}(\mathbb{Z}^2) < 1$, [S, Part F2, Proposition 8.1]
BCC	[S, Part B2, Lemmas 1.2 and 1.3]	[S, Part B2, Lemmas 5.3, 5.4, 5.5]	[S, Part B2]: design, Proposition 9.2; $\mathrm{TP}^{\mathrm{bcc}}$ and $\mathrm{HTP}^{\mathrm{bcc}}$, Theorem 9.9	[S, Part BC, Proposition 3.1] (for X^{cor})	$1/7 \le p_c(\mathrm{BCC}) \le p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/2} < 1$, [S, Part B2, Remark 5.2]
4.8^2	[S, Part S2, Lemma 1.2] (distance formula), Corollary 1.3 (inner and outer neighbours, no pockets), Remark 1.6 (spheres along a side)	[S, Part S2, Propositions 7.2 ((B1′)), 7.3, 7.10]; (B2) uses tracks, Proposition 6.4, and (B3) also uses Lemmas 7.4–7.9	[S, Part S2]: tracks, Definition 6.1, Lemma 6.3, Proposition 6.4; axis rays, Lemmas 8.1, 8.2; port and terminal escapes, Lemmas 8.4, 8.5; design, Theorem 8.6; thin pairs, Lemma 2.2; the reduction, Theorem 4.2	[S, Part S2, Proposition 3.1] (for X^{cor}), Corollary 3.2 (deep-occupied corner lemma)	$\frac{1}{2} \le p_c(4.8^2) \le p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/4} < 1$, [S, Part S2, Proposition 1.5]

lattice	**spheres**	**(B1); (B2); (B3)**	**two-port property**	**corner data**	p_c
3.12^2	[S, Part X2, Lemmas 1.3, 1.4] (darts, distance formula), Corollaries 1.5, 1.7 (levels, pockets)	[S, Part X2, Theorem 4.6 and Lemma 4.5] (gadget radii in place of (B1)); Theorem 5.1; Theorem 6.2 ($(\mathrm{B3}_{\mathrm{odd}})$), with Theorem 4.8 (Claim 5 along odd L)	[S, Part X2]: tracks, Definition 7.2, Lemmas 7.1, 7.3; rays, Definition 7.4, Lemmas 7.6, 7.7; local escapes, Lemmas 8.1, 8.2; design, Theorem 8.3; failing pairs, Proposition 3.5 and Corollary 8.4	none: $\mathrm{CL}_{X^{\mathrm{T}}}$ by [S, Part X2, Lemma 4.1 and Corollary 4.2]; no family with (X2)–(X3), Theorem 3.7	$\frac{1}{2} \le p_c(\mathrm{TH}) \le p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/6} < 1$, [S, Part X2, Lemma 1.8]
4.6.12	[S, Part T2, Theorem 2.1 and Corollary 2.2] (distance formula)	(B1) not used (non-escapable pairs exist for $3 \le m \le 250$, by computation); the terminals satisfy TF_2 ([S, Part T2, Lemma 6.1]); [S, Part T2, Propositions 7.4, 11.4]	[S, Part T2]: locally blocked pairs, Definition 3.1, Proposition 3.4; hexagon pairs and their hosts, Lemmas 4.4, 4.6; tracks, Lemma 7.2; TP and HTP, Theorems 10.1, 10.4; the reduction, Theorem 12.1	[S, Part T2, Proposition 5.2] (for X^{cor} on $\mathcal{R}$), Theorem 6.2 and Corollary 6.3 (corner lemma along $\mathcal{R}$)	$\frac{1}{2} \le p_c(\mathrm{TT}) \le p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/12} < 1$, [S, Part T2, Lemma 1.3]

lattice	spheres	(B1); (B2); (B3)	two-port property	corner data	p_c
DIA	[S, Part D2, Lemma 1.2] (distance formula), Corollary 1.3 (spheres, no pockets)	[S, Part D2, Propositions 3.3, 3.4, 3.9]	[S, Part D2]: dead pairs and vertex pairs, Theorem 2.3, Proposition 2.4; three-layer tracks, Lemma 3.1, Corollary 3.2; history hosts, Definition 5.8, Theorem 5.9, Corollary 5.10; assembly, Proposition 7.6; far pairs, Lemma 7.7; $\mathrm{TP}^{\mathrm{dia}}$ and $\mathrm{HTP}^{\mathrm{dia}}$, Theorem 7.9; the reduction, Theorem 6.2	[S, Part D2, Proposition 4.2] (for X^{vtx}), Corollary 4.3 (deep-occupied corner lemma)	$\frac{1}{3} \le p_c(\mathrm{DIA}) \le p_c^{\mathrm{site}}(\mathbb{Z}^2)^{1/2} < 1$, [S, Part D2, Lemma 1.5]

Table 6. The failing pairs are treated in the subsections on the individual lattices, and the finite computations and their extension to all radii are listed in Table 5. For the lattices of Theorem 1.2 the bounds on p_c are those of Proposition 6.6.

Appendix D. Names of the formal statements

The table gives, for the main statements of the paper whose formal versions Section 14 mentions, the names of the corresponding theorems of the Lean formalization, in code font and exactly as in the formalization. Section 14 says how far each formal statement goes.

statement	Lean names
Theorem 1.1	`BBR.theorem_1_1_unconditional`; with the bounds $0 < p_c^{\mathrm{site}}(\mathbb{Z}^d) < 1$: `BBR.theorem_1_1_full`
Claim 5 on $\mathbb{Z}^d$	`BBR.claim5`
Lemma 2.2 (the gadget)	`BBR.gadget`, `BBR.essential_pair`
Lemma 2.3	`BBR.two_layers`, `BBR.two_layers_connected`

statement	**Lean names**
Lemmas 2.4 and 2.5 (the deletion step)	`BBR.locality`, `BBR.deletion`
Theorem 2.10	`BBR.design_spivotal`
Proposition 2.12 and Theorem 2.13	`BBR.main_case`, `BBR.case3`, `BBR.claim5_of_CL`
Lemma 3.1, Theorem 3.2 and Corollary 3.3	`BBR.lemma_3_1`, `BBR.escape`, `BBR.two_port`
Lemma 4.5 and Theorem 4.8	`BBR.two_ball`, `BBR.corner_lemma`
Theorem 2.1	`BBR.agStep`
$0 < p_c^{\mathrm{site}}(\mathbb{Z}^d)$ and $p_c^{\mathrm{site}}(\mathbb{Z}^d) < 1$	`BBR.pcSite_pos`, `BBR.pcSite_lt_one`
Theorem 6.5	`BBRP.agStep`
$0 < p_c(G) < 1$ (Proposition 6.6) for TRI, HEX, KAG, KING, ELTRI, RTH, SNUBSQ, SNUBHEX, FCC	`BBRP.propP_tri`, `BBRP.propP_hex`, `BBRP.propP_kag`, `BBRP.propP_king`, `BBRP.propP_eltri`, `BBRP.propP_rth`, `BBRP.propP_snubsq`, `BBRP.propP_snubhex`, `BBRP.propP_fcc`
Lemma G; Lemma G from the escapability clause of (B1) alone	`BBRP.lemmaG`; `BBRP.lemmaG_esc`
Lemmas 7.3 and 7.4 and Theorem 7.10	`BBRP.locality`, `BBRP.deletion`, `BBRP.design_spivotal`
Theorem A without its explicit constants: Claim 5; Conjecture 1 if $p_c(G) < 1$	`BBRP.theoremA_claim5`; `BBRP.theoremA`
Theorem C; its general form	`BBRP.theoremC`; `BBRP.CornerAbs.cornerLemmaX_of_apexLabelling`
$\mathrm{CL}_\emptyset$	`BBRP.cornerLemma_empty`
Definition 6.1 and the formal notion of a periodic lattice	`BBRP.exists_plattice_eq`
(B2) and (B3) and their literal forms	`BBRP.PLattice.innerBoundary_iff_lit`, `BBRP.PLattice.outerBoundary_iff_lit`
(B1), (B2), (B3) and TP_X on TRI, KAG and HEX	`BBRP.B_TRI`, `BBRP.B_KAG`, `BBRP.B_HEX`
(X1)–(X4) for X^{hex}	`BBRP.hex_X1`, `BBRP.hex_X234`
$\mathrm{TP}_{X^{\mathrm{hex}}}$ (Theorem 9.18)	`BBRP.hex_TP`
Lemmas 9.14 and 9.15; Lemma 9.16 for $n \geq 10$	`BBRP.Hex.port_escapes`, `BBRP.Hex.terminal_escapes`; `BBRP.Hex.ray_crosses_track`

statement	Lean names
Theorem 1.2 for TRI, KAG and HEX, with $0 < p_c < 1$	`BBRP.theorem_1_1'`
Corollary 1.3, for activation at edges and at vertices	`BBRP.corollary_1_2'`, `BBRP.corollary_1_2'_vertex`
$p_c^{\mathrm{bond}}(\mathrm{HEX}) = p_c(\mathrm{KAG})$ and $\theta^{\mathrm{bond}} \geq \theta^{\mathrm{KAG}}$ (Proposition 10.17)	`BBRP.pcBond_eq_pc_kagome`, `BBRP.thetaBond_ge_theta_kagome`
(B1), (B2), (B3), TP_X and (X1)–(X4) on ELTRI, RTH, KING, FCC and SNUBSQ	`BBRP.B_ELTRI`, `BBRP.B_RTH`, `BBRP.B_KING`, `BBRP.B_FCC`, `BBRP.B_SNUBSQ`
$\mathrm{TP}_{X^{\mathrm{snub}}}$ with $\rho_{\mathrm{TP}} = 195$ and $C_0 = 20$	`BBRP.SnubSq.twoPortWith`
no pockets on the spheres of SNUBSQ	`BBRP.SnubSq.not_pocket`
Theorem 1.2 for ELTRI, SNUBSQ, RTH, KING and FCC, with $0 < p_c < 1$	`BBRP.theorem_1_1''_eltri`, `BBRP.theorem_1_1''_snubsq`, `BBRP.theorem_1_1''_rth`, `BBRP.theorem_1_1''_king`, `BBRP.theorem_1_1''_fcc`
(B1), (B2), (B3), TP_X and (X1)–(X4) on SNUBHEX	`BBRP.B_SNUBHEX`
$\mathrm{TP}_{X^{\mathrm{sc}}}$ with $\rho_{\mathrm{TP}} = 227$ and $C_0 = 23$	`BBRP.snubhexXsc_twoPortWith`
Theorem 1.2 for SNUBHEX, with $0 < p_c < 1$	`BBRP.theorem_1_1''_snubhex`

Data availability

The Lean formalization is archived, with instructions for checking it, at https://doi.org/10.5281/zenodo.23692474. The supplementary proofs [S] and the programs and logs of the computational certificates (Part Z of [S]) are those of version 4 (27 October 2026), unchanged in versions 5 and 6; they are included in the record of this paper, https://doi.org/10.5281/zenodo.23716333, as the files `essential_enhancements_v4_supplementary_proofs.pdf` and `essential_enhancements_v4_certificates.tar.gz`.

Use of AI tools

Generative AI tools were used to find and check the proofs, to write and run the computational checks, to write the Lean formalization, and to draft the text. The author is responsible for all statements.

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