Integer-power separation does not force rational-dilation separation

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Abstract

We construct an algebraically independent set with divergent reciprocal-logarithmic weight, arbitrary prescribed proximity to selected integers and growing integer-power separation on tails. Nevertheless, every tail contains arbitrarily close rational-dilation pairs for each member of a dense family. The necessary integer-power finite-state lemmas and their proofs are included. This supplies a contrasting construction to the bounded-denominator rational-power example in the companion paper.

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Attribution and status. Apicella's rational counterexample and private-prime construction are prior work. The contribution developed here is the additional theorem stated below, not a new resolution of the original convergence question. This consolidated research draft preserves the supplied mathematical arguments; no independent expert acceptance or historical priority is claimed.

1 Main theorem and its meaning

Put $\mathrm{w}(x) = 1/(x \log x)$ and $\Lambda = \{2^u/3^v : u, v \in \mathbb{N}\}$.

Theorem 1. *For every $\varepsilon : \mathbb{N} \to (0, \infty)$ there is a countably infinite $A = \{a_1, a_2, \ldots\} \subset (2, \infty)$, algebraically independent over $\mathbb{Q}$, with $\sum_{a \in A} \mathrm{w}(a) = \infty$, global $|ka - b| > 1$ for ordered distinct $a, b \in A$ and all $k \in \mathbb{N}$, and distinct integer anchors satisfying $|a_i - z_i| < \varepsilon(z_i)$. For every $d \in \mathbb{N}$ and $H > 0$, some finite removal gives*

$$|ka^r - b^s| > H \qquad (a \neq b,\ k \in \mathbb{N},\ 1 \leq r, s \leq d).$$

Nevertheless, for every $\lambda \in \Lambda$, every $\eta > 0$ and every finite $F \subset A$, there are distinct $a, b \in A \setminus F$ with $|\lambda a - b| < \eta$.

A dense family of obstructions. The set Λ is dense in $(0, \infty)$. Indeed $\log 2/\log 3$ is irrational by unique prime factorization. Density of an irrational rotation modulo one, including every tail of its orbit, gives for every real x and $\epsilon > 0$ arbitrarily large $u \in \mathbb{N}$ and an integer v with $|u \log 2 - v \log 3 - x| < \epsilon$. Taking u sufficiently large ensures $v > 0$. Exponentiating gives the claim. In contrast, the companion paper's rational-power theorem gives growing tail gaps at every fixed positive rational dilation [2]. The two examples therefore separate integer-power control from rational-dilation control.

2 Finite-state tools for integer powers

All inputs used for the mass-generating steps are proved in this section. They use only integer-power geometry, not the stronger bounded-denominator geometry of the companion paper. Every private prime may be taken greater than 200: all new private primes below exceed a grid containing a prime factor greater than 200, and preparation keeps the private primes fixed.

2.1 Squarefree level selection.

Let $\Omega(n)$ count prime factors with multiplicity.

Lemma 1. *If $E \subset \mathbb{N}$ is periodic with density $\delta > 3/4$, then beyond every threshold $N \geq 2$ there is a finite nonempty $P \subset E \cap [N, \infty)$ with $W(P) > 1/1000$ such that all its elements are squarefree and have the same value of Ω.*

Proof. Write $L = \log\log X$ and $S(X) = \sum_{2\leq n\leq X} \mathrm{w}(n) = L + O(1)$. For $\sigma = 1 + 1/\log X$, the Euler product and integral test give

$$\sum_{p\leq X} \frac{1}{p} \leq e \sum_{p\leq X} p^{-\sigma} \leq e \log \zeta(\sigma) \leq e \log(1 + \log X) \leq 3L + O(1).$$

Also $\sum_p \sum_{j\geq 2} p^{-j} \leq \sum_{m\geq 2} 1/(m(m-1)) = 1$. Consequently $T(X) = \sum_{p^j\leq X} p^{-j} \leq 3L + O(1)$, and

$$\sum_{2\leq n\leq X} \Omega(n)\mathrm{w}(n) = \sum_{p^j\leq X} p^{-j} \sum_{m\leq X/p^j} \frac{1}{m \log(p^j m)} \leq S(X)(T(X) + 1) \leq 3L^2 + O(L).$$

For this bound the terms $m \geq 2$ are dominated by $S(X)T(X)$; the $m = 1$ terms are at most $S(X)$ since prime powers are distinct.

The constant $C_2 = \sum_p p^{-2}$ is less than $1/2$. An elementary bound, counting all odd integers at least seven instead of primes, is

$$C_2 \leq \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{14} < \frac{1}{2}.$$

Indeed $\sum_{j\geq 0}(7 + 2j)^{-2} \leq 1/49 + \int_0^\infty (7 + 2t)^{-2}\, dt$. The union bound over $p^2 \mid n$ gives

$$\sum_{\substack{2\leq n\leq X \\ n \text{ not squarefree}}} \mathrm{w}(n) \leq C_2 S(X) + O(1).$$

Here $m \geq 2$ in $n = p^2 m$ contributes at most $p^{-2}S(X)$, and the $m = 1$ contributions have convergent total. Periodicity gives $\sum_{n\in E\cap[N,X]} \mathrm{w}(n) = \delta L + O_{E,N}(1)$. Thus the squarefree terms in this set have weight at least $(\delta - C_2)L + O_{E,N}(1)$, with $\delta - C_2 > 1/4$. Those with $\Omega(n) > \lfloor 60L \rfloor$ have weight at most $L/20 + O(1)$ by the first-moment bound. The remaining at most $60L$ positive levels have total weight greater than $L/5 + O_{E,N}(1)$. If every level had weight at most $1/1000$, their total would be at most $3L/50$, a contradiction for large X. A level of sufficient weight supplies P. □

Lemma 2. *Let P be squarefree, contained in one level of Ω, and coprime to an integer D. If $n, n' \in P$ are distinct, then*

$$|k(Dn)^r - (Dn')^s| \geq D \qquad (k, r, s \in \mathbb{N}).$$

Proof. Equality of the two powered expressions would imply that every prime dividing n divides Dn', hence divides n'. Squarefreeness and equality of the numbers of prime factors imply $n = n'$, a contradiction. Their nonzero difference is an integer divisible by D. □

Ordinary primitivity is insufficient for this lemma: $\{12, 18\}$ is primitive, but $27 \cdot 12 = 18^2$. The squarefree equal-level condition supplies the stronger prime-support antichain used here.

2.2 States and dense preparation.

A finite state has labeled rational points $x_i > 2$, positive integer tags τ_i, positive integer degree caps c_i, distinct odd private primes $p_1, \dots, p_t$, an assignment $h(i)$, and representations

$$x_i = p_{h(i)} u_i / v_i, \qquad u_i, v_i \in \mathbb{N}.$$

It is admissible if

$$|kx_i^r - x_j^s| > \min(\tau_i, \tau_j) \quad (i \neq j,\ k \in \mathbb{N},\ 1 \le r, s \le \min(c_i, c_j)), \tag{G}$$

$$x_i > 4\tau_i p_{h(i)}, \tag{S}$$

$$p_j \nmid u_i \quad \text{for all } i, j, \qquad p_{h(i)} \nmid v_i, \qquad \sum_j 1/p_j < 1/100. \tag{P}$$

It is prepared if, in addition, every u_i is a prime larger than every private prime and $\sum_i 1/u_i < 1/200$. The empty state is prepared. Empty maxima of point sets are zero.

Lemma 3. *Fix the tags, caps, private primes and assignments, with the prime sum less than* $1/100$*. The set of prepared rational states is dense in every nonempty open geometric region satisfying* (G)–(S)*. It can avoid the zero sets of any finite collection of nonzero real polynomials.*

Proof. Let $M = \prod_j p_j$, and prescribe any sufficiently large lower bound B for the prime quotients u_i. For an assigned prime p, numbers pu/v with $u > B$ prime and $v \equiv 1 \pmod M$ are dense in $(0, \infty)$. To approximate $z > 0$, let $u \to \infty$ through primes and choose a positive $v \equiv 1 \pmod M$ nearest pu/z. Then $|pu/v - z| \le zM/(2v) \to 0$. Take $B > \max(200m, \max_j p_j)$, where m is the number of coordinates. The resulting representations satisfy (P) and the preparation bound. The coordinatewise product of these dense scalar sets is dense. The complement of finitely many nonzero polynomial zero sets is open and dense, which proves the final assertion. □

Lemma 4. *For fixed finite tags, caps and assignments, the geometric region* (G)–(S) *is open.*

Proof. Restrict coordinates locally to $[a, b]$ with $1 < a < b$. Put $C = \max_i c_i$ and $T = \max_i \tau_i$. For $k > (b^C + T)/a$ all relevant inequalities are automatic, since $x_i^r \ge a$ and $x_j^s \le b^C$. Only finitely many strict continuous inequalities remain. □

2.3 A residue estimate for two powers.

Lemma 5. *Let $D \ge 2$, $R, C \ge 1$ be integers and p, q be distinct primes larger than both D and C. Put $b = Dpq/(pq-1)$. There is a periodic set $\mathcal{B}_{p,q} \subset \mathbb{N}$ of density at most $3RC^4/D$ with the following property: if ℓ is coprime to pq and outside $\mathcal{B}_{p,q}$, then*

$$|b^r h - (D\ell)^s| > R \qquad (h \in \mathbb{Z},\ 1 \le r, s \le C).$$

Proof. Fix r, s and put $m = \min(r, s)$, $Q = (pq)^r$. Failure of the inequality gives an integer

$$z = D^{r-m}(pq)^r h - D^{s-m}(pq-1)^r \ell^s, \qquad |z| \le \frac{R(pq-1)^r}{D^m}.$$

Modulo Q the coefficient of ℓ^s is a unit. For each z, the resulting congruence has at most s^2 unit solutions modulo Q. Indeed modulo p a degree-s polynomial has at most s roots; each unit root lifts uniquely from p^j to p^{j+1} because $p > s$ and the derivative $s\ell^{s-1}$ is a unit. The same argument applies to q, and the Chinese remainder theorem gives s^2. This defines a periodic exceptional set, restricted to units, of density at most

$$s^2 \left(\frac{2R}{D^m} + \frac{1}{(pq)^r} \right) \le \frac{3Rs^2}{D}.$$

Take the union over $1 \le r, s \le C$ and use $C \sum_{s=1}^{C} s^2 \le C^4$. □

2.4 The finite extension.

Lemma 6. *Fix $\varepsilon : \mathbb{N} \to (0, \infty)$ and a prepared finite state. Let integers $R, C \geq 1$ be at least, respectively, every old tag and every old cap. There is a finite admissible extension, preserving all old values and arithmetic data, whose new points all have tag R and cap C, and whose total new weight exceeds*

$$\frac{\gamma}{RC^4}, \qquad \gamma = \frac{1}{128000}.$$

Each new point y has an integer anchor $z \geq 2$ with $0 < y - z < \min(\varepsilon(z), 1/4)/4$. The resulting state need not be prepared.

Proof. Choose an odd prime $l > 200$ different from all private primes and all numerator primes of the prepared state. For a sufficiently large $K \in \mathbb{N}$ set

$$D = l^K \prod_i u_i^{c_i} \geq 64RC^4.$$

Every private prime avoids D, and $\sum_{\pi \mid D,\ \pi \text{ prime}} 1/\pi < 1/100$. For an initial point $x_i = pu_i/v_i$, an exponent $1 \leq r \leq c_i$, and a label ℓ not divisible by p, write $w = D/u_i^r \in \mathbb{N}$. For every $s \geq 1$,

$$|kx_i^r - (D\ell)^s| = (x_i/p)^r |kp^r - D^{s-1} w v_i^r \ell^s| > 4\tau_i \quad (k \in \mathbb{Z}). \tag{1}$$

The integer factor is nonzero modulo p and $(x_i/p)^r > (4\tau_i)^r \geq 4\tau_i$.

Keep this D fixed while adding $J = \lfloor D/(16RC^4) \rfloor$ blocks. Each new block has

$$y_n = \frac{Dnpq}{pq - 1} \quad (n \in P), \tag{2}$$

and tag R, cap C, private prime p, numerator quotient $u_n = qDn$, and denominator $v_n = pq - 1$. Maintain auxiliary prime sum $\sum 1/q < 1/100$ for the blocks inserted in this round; no such sum is required for earlier rounds.

After $j < J$ blocks, choose a new prime p larger than D, C, all current private primes, all current numerator quotients, and $\left(1/100 - \sum_i 1/p_i\right)^{-1}$. The enlarged private-prime sum stays below $1/100$. Let E be the periodic set of positive labels coprime to D, divisible by no current private prime (including this p) and no auxiliary prime q of this round, and avoiding the j sets $\mathcal{B}_{p_i, q_i}$ from Lemma 5. The density satisfies

$$\mathrm{d}(E) > 1 - \frac{1}{100} - \frac{1}{100} - \frac{1}{100} - \frac{3jRC^4}{D} \geq \frac{97}{100} - \frac{3}{16} > \frac{3}{4}. \tag{3}$$

Equation (1) controls every initial point and Lemma 5 controls every point in a block already inserted at this scale: in the latter case take $h = kn^r$. Consequently, for every current point x of tag τ_x and cap c_x,

$$|kx^r - (D\ell)^s| > \tau_x \quad (k \in \mathbb{Z},\ \ell \in E,\ 1 \leq r, s \leq c_x). \tag{4}$$

Use Lemma 1 to choose $P \subset E$ in one squarefree Ω-level with $W(P) > 1/1000$, beyond a threshold ensuring

$$\min P \geq 2D, \qquad D \min P > \max\left(4Rp,\ \max_{x \text{ current}} x^{c_x} + R + 2\right). \tag{5}$$

At the integer vector $(Dn)_{n \in P}$, Lemma 2 gives all required within-block distances at least $D > R$. By Lemma 4, the within-block geometric inequalities remain valid when (2) is sufficiently close to this vector.

Write each current rational point in lowest terms as A_x/B_x. Choose the auxiliary prime q arbitrarily large, greater than $D, C, p, \max P$, every current private prime, and $(1/100 - \sum_{\text{current round}} 1/q_i)^{-1}$. Primes are unbounded, so additionally impose, for every $n \in P$,

$$0 < y_n - Dn < \tfrac{1}{4}\min(\varepsilon(Dn), 1/4), \tag{6}$$

$$|y_n^s - (Dn)^s| < \min\left(1, \min_{\substack{x \text{ current} \\ 1 \le r \le c_x}} B_x^{-r}\right) \qquad (1 \le s \le C), \tag{7}$$

omitting the second minimum for an empty state, and impose all the within-block inequalities just obtained. These are finitely many conditions valid for all sufficiently large q, since $y_n \to Dn$.

For a current point x, integer tags and the lattice $kx^r - (Dn)^s \in B_x^{-r}\mathbb{Z}$ upgrade (4) to $|kx^r - (Dn)^s| \ge \tau_x + B_x^{-r}$. Condition (7) therefore gives $|kx^r - y_n^s| > \tau_x$ for all $k \in \mathbb{Z}$ and $1 \le r, s \le c_x$. In the reverse direction (5) implies $ky_n^r - x^s > R + 2 \ge \tau_x$ for all positive k and all required r, s, since $y_n^r \ge y_n > x^{c_x} + R + 2$. Together with the within-block inequalities these prove (G). Condition (5) also gives $y_n/p > 4R$.

All private primes avoid $u_n = qDn$: they avoid D, avoid $n \in E$, and avoid q by its size. The new private prime avoids its own denominator $pq - 1$ and all old numerator quotients by its choice. Thus (P) is preserved. The auxiliary prime sum remains below $1/100$, and all primes p, q in this round exceed D, C, as needed for the residue lemma. This completes one block insertion.

Finally $y_n < 2Dn \le n^2$ gives $\mathrm{w}(y_n) > \mathrm{w}(n)/(4D)$. Each block adds weight $> 1/(4000D)$. Since $J \ge D/(32RC^4)$, the whole round adds weight $> 1/(128000RC^4)$. The anchors $z = Dn$ satisfy (6). □

3 Insertion of a prescribed ratio pair

Use the states of the preceding section, with their private-prime assignments. Lemma 6 gives a birth-weight gain greater than $\gamma/(RC^4)$, where $\gamma = 1/128000$. Lemmas 3 and 4 provide the dense preparation and finite geometric openness used below. The ratio-pair insertion need only return an admissible state; preparation is performed afterwards in an open approximation band and is not required to preserve an exact rational ratio.

Lemma 7 (Adding a prescribed ratio pair)**.** *Let a prepared state as above be given, let R, C dominate its tags and caps, and fix $\lambda = 2^u/3^v \in \Lambda$. It has a finite admissible extension by two points α, β of tag R and cap C, preserving the old points, such that $\beta = \lambda\alpha$. The points have integer anchors z_α, z_β with $0 < \alpha - z_\alpha < e(z_\alpha)/4$ and $0 < \beta - z_\beta < e(z_\beta)/4$, where $e(z) = \min(\varepsilon(z), 1/4)$.*

Proof. Choose, as in Lemma 6, a grid $D = l^K \prod_i U_i^{c_i} > R$, with $l > 200$ prime different from all numerator and private primes. Thus D is coprime to 6 and to every private prime. For an initial point $x_i = p_i U_i/V_i$, any $r \le c_i$ and $s \ge 1$, and any label ℓ not divisible by p_i, the grid identity from Lemma 6 is

$$|kx_i^r - (D\ell)^s| = (x_i/p_i)^r \left|kp_i^r - D^{s-1}(D/U_i^r)V_i^r\ell^s\right| > 4\tau_i.$$

It holds for all integer k by reduction of the integer factor modulo p_i.

Choose a fresh private prime $p > D, C$, larger than every old numerator quotient and large enough to preserve the private reciprocal sum. Choose a prime N larger than D, p and every old private prime, and large enough that both $D3^vN$ and $D2^uN$ exceed $4Rp$ and $\max_i x_i^{c_i} + R + 2$. The two labels 3^vN and 2^uN avoid all private primes: the latter exceed 200, and N is a larger distinct prime. The grid identity therefore controls both labels against every old source power, and the ordering controls the reverse direction.

At the integer pair $(D3^vN, D2^uN)$ there is no exact integer power-multiple relation in either order. A source involving 3^vN has prime factor 3, absent from $D2^uN$; in the reverse order the distinguishing prime is 2. Every nonzero powered difference is an integer multiple of D, hence has absolute value at least $D > R$.

Now take an auxiliary prime q arbitrarily large and set

$$\alpha = \frac{D3^vNpq}{pq-1}, \qquad \beta = \frac{D2^uNpq}{pq-1}.$$

Then $\beta = \lambda\alpha$ exactly. As $q \to \infty$, both coordinates approach their integer anchors. Finite geometric openness protects all their within-pair conditions. For an old point with reduced denominator B_i, the grid distance exceeds the integer tag by at least B_i^{-r}. Choose q large enough that the target powers, for all exponents at most C, move by less than the minimum of these finitely many lattice margins. This protects all old-source multipliers at once; the reverse ordering persists. Also impose the stated anchor bounds and q larger than all private primes. The new quotients $qD3^vN, qD2^uN$ avoid every private prime, and the own denominator $pq-1$ avoids p. The new private prime avoids old quotients by its size. All state invariants hold. □

4 Fusion and proof of Theorem 1

Enumerate the members of Λ in a sequence λ_j in which each appears infinitely often. At macro-stage j set $R = C = t_j$, where $t_j^5 \le j < (t_j+1)^5$. Perform two finite steps, with one preparation-and-box operation after each.

First apply Lemma 6, keeping all old centers. It adds birth weight $> \gamma/t_j^5 \ge \gamma/j$. Prepare a new center and choose nested closed boxes exactly as in the nested-box construction below, retaining all birth windows and avoiding the first $2j-1$ polynomial zero sets whose variables are present.

Second apply Lemma 7 with $\lambda = \lambda_j$. For its new labels require the additional open condition

$$|\lambda_j x_{\alpha_j} - x_{\beta_j}| < 2^{-j}.$$

This open region contains the just-extended rational center, because its ratio is exact. Intersect it with the old interval interiors, new birth windows, and the strict geometric region. Dense preparation gives a center there avoiding the first $2j$ currently applicable polynomial zero sets. Choose nested rational-endpoint closed boxes inside that open region, with diameters at most 2^{-2j}. Use diameters at most $2^{-(2j-1)}$ in the first step. Preserve all previous boxes and conditions.

Here is the limit argument explicitly. Enumerate all nonzero integer-coefficient polynomials in finitely many variables among the coordinate labels. After each finite extension, intersect the old interval interiors, the new birth windows and the open geometric region. This intersection contains the extended rational center. Lemma 3 supplies a prepared center off all currently required polynomial zero sets. Choose positive-length rational-endpoint closed boxes inside that open intersection, inside the polynomial complements and, for a ratio insertion, inside its prescribed approximation band. Each old coordinate interval is nested and its diameter tends to zero, so it has a unique limiting coordinate. For any fixed finite constraint, all its final coordinates lie in a closed box already contained in its strict solution set. Every polynomial is eventually treated; clearing denominators proves algebraic independence. Tags and caps t_j tend to infinity, so discarding the finitely many early labels gives every prescribed integer-power gap and exponent bound. Each point stays within $e(z)/4$ of its rational birth value and the birth value within $e(z)/4$ of its anchor. Thus it satisfies the prescribed error. Equal anchors would contradict the global distance greater than one. It yields distinct algebraically independent limits, the prescribed integer closeness, global integer separation and the growing integer-power tail gaps. Each limit weight is greater than one quarter of its birth weight, so the mass steps alone give $\sum_{j\le J} \gamma/(4j)$ and divergence.

The limiting pair born in the second step obeys $|\lambda_j\alpha_j - \beta_j| < 2^{-j}$, since its entire ratio-stage box lies in that strict open band. These pairs have disjoint labels. For fixed $\lambda \in \Lambda$, infinitely many indices have $\lambda_j = \lambda$. A finite F meets only finitely many of these pairs. Choose a later such index with $2^{-j} < \eta$. This proves the final assertion of Theorem 1.

In particular the example fails the bounded-denominator conclusion already at $d = 2$, $H = 1$: use $\lambda = 2/3$ and swap the arbitrarily late pairs to obtain distances for $3/2$ tending to zero: the swapped error is $3/2$ times the original error. The stronger construction and this contrast example therefore give two different rational-dilation behaviours under the same divergent weight, algebraic independence, integer closeness and integer-power tail conditions.

Attribution, provenance and limits

Apicella's rational counterexample and private-prime method are prior work. The companion paper treats the rational-power-separated construction; the present article proves a different example with contrasting rational-dilation behaviour. The algebraic and field consequences of algebraic independence apply here as well. This manuscript was consolidated with ChatGPT/Codex assistance from the supplied research drafts and their finite-state companion. No independent expert acceptance, Lean verification or historical-priority certification is claimed for this version.

References

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