Algebraically independent counterexamples with rational-power separation

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Abstract

Starting from Apicella's private-prime construction, we give a proof draft of divergent reciprocal-logarithmic-weight examples that are algebraically independent, admit any prescribed positive integer-anchor error and have growing mixed-power gaps for rational dilations with bounded denominators. The integer numerator remains unrestricted. We record transcendence and field consequences and retain an elementary reconstruction of Apicella's rational example in an appendix, with its prior-work attribution explicit.

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Attribution and status. Apicella's rational counterexample and private-prime construction are prior work. The contribution developed here is the additional theorem stated below, not a new resolution of the original convergence question. This consolidated research draft preserves the supplied mathematical arguments; no independent expert acceptance or historical priority is claimed.

1 Introduction and main theorem

The question motivating this work is whether integer-dilation separation forces convergence of $\sum 1/(a \log a)$. Apicella's rational counterexample already answers that question negatively [1]. Our main statement adds algebraic independence and a quantified rational-power separation property. The rational case and its elementary presentation are retained in Appendix A rather than presented as another new solution. Put $\mathbb{N} = \{1, 2, \ldots\}$, $\mathrm{w}(x) = 1/(x \log x)$ for $x > 1$, and $W(B) = \sum_{b \in B} \mathrm{w}(b)$ for a countable $B \subset (1, \infty)$. All logarithms are natural.

Theorem 1. *For every function $\varepsilon : \mathbb{N} \to (0, \infty)$ there is a countably infinite set $A = \{a_1, a_2, \ldots\} \subset (2, \infty)$, algebraically independent over $\mathbb{Q}$, with*

$$|ka_i - a_j| > 1 \quad (i \neq j,\ k \in \mathbb{N}), \qquad W(A) = \infty,$$

and distinct integers $z_i \geq 2$ satisfying $|a_i - z_i| < \varepsilon(z_i)$. Furthermore, for every $d \in \mathbb{N}$ and $H > 0$ there is a finite $F_{d,H} \subset A$ such that

$$\left| \frac{m}{n} a^r - b^s \right| > H \quad (a, b \in A \setminus F_{d,H},\ a \neq b, \quad m \in \mathbb{N}, \quad 1 \leq n, r, s \leq d)\,.$$

The quantifiers concern one set A and every fixed finite exponent and denominator bound d; no assertion is made about a single tail working for unbounded exponents or denominators simultaneously. The denominator qualification is necessary: even for one fixed pair $a, b > 0$, rational numbers m/n approximate b^s/a^r arbitrarily closely. Algebraic independence means that every nonzero rational-coefficient polynomial in finitely many distinct members of A has nonzero value.

2 Squarefree level selection

Let $\Omega(n)$ count prime factors with multiplicity.

Lemma 1. *Let $E \subset \mathbb{N}$ have a period $M \in \mathbb{N}$, density $\delta > 0$, and $\gcd(n, M) = 1$ for every $n \in E$. Beyond every threshold $N \geq 2$ there is a finite nonempty $P \subset E \cap [N, \infty)$ with $W(P) > \delta^2/1000$ such that all its elements are squarefree and have the same value of Ω.*

Proof. Write $L = \log\log X$ and $S(X) = \sum_{2 \leq n \leq X} \mathrm{w}(n) = L + O(1)$. For $\sigma = 1 + 1/\log X$, the Euler product and integral test give

$$\sum_{p \leq X} \frac{1}{p} \leq e \sum_{p \leq X} p^{-\sigma} \leq e \log \zeta(\sigma) \leq e \log(1 + \log X) \leq 3L + O(1).$$

Also $\sum_p \sum_{j \geq 2} p^{-j} \leq \sum_{m \geq 2} 1/(m(m-1)) = 1$. Consequently $T(X) = \sum_{p^j \leq X} p^{-j} \leq 3L + O(1)$, and

$$\sum_{2 \leq n \leq X} \Omega(n)\mathrm{w}(n) = \sum_{p^j \leq X} p^{-j} \sum_{m \leq X/p^j} \frac{1}{m \log(p^j m)} \leq S(X)(T(X) + 1) \leq 3L^2 + O(L).$$

For this bound the terms $m \geq 2$ are dominated by $S(X)T(X)$; the $m = 1$ terms are at most $S(X)$ since prime powers are distinct.

The constant $C_2 = \sum_p p^{-2}$ is less than $1/2$. An elementary bound, counting all odd integers at least seven instead of primes, is

$$C_2 \leq \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{14} < \frac{1}{2}.$$

Indeed $\sum_{j \geq 0} (7+2j)^{-2} \leq 1/49 + \int_0^\infty (7+2t)^{-2}\, dt$. Choose a fixed Y so large that $\sum_{p > Y} p^{-2} < \delta/8$. For $p \leq Y$ dividing M, no point of E is divisible by p^2. For $p \leq Y$ not dividing M, the Chinese remainder theorem shows that $E \cap p^2\mathbb{N}$ is periodic of density δ/p^2. Periodicity and summation by parts therefore give, for this finite list of primes, total weight at most $\delta C_2 L + O_{E,Y,N}(1)$. For the primes $p > Y$ the unconditional bound

$$\sum_{\substack{2 \leq n \leq X \\ p^2 \mid n}} \mathrm{w}(n) \leq p^{-2} S(X) + \mathrm{w}(p^2)$$

shows that their total contribution is at most $(\delta/8)L + O(1)$, since $\sum_p \mathrm{w}(p^2) < \infty$. The union bound and $\sum_{n \in E \cap [N,X]} \mathrm{w}(n) = \delta L + O_{E,N}(1)$ now give squarefree weight at least $(3\delta/8)L + O_{E,Y,N}(1)$. Those with $\Omega(n) > \lfloor 48L/\delta \rfloor$ have weight at most $(\delta/16)L + O_\delta(1)$ by the first-moment bound. The remaining at most $48L/\delta$ positive levels have total weight at least $(5\delta/16)L + O_{E,Y,N,\delta}(1)$. If every level had weight at most $\delta^2/1000$, their total would be at most $(48\delta/1000)L$, a contradiction for large X. A level of sufficient weight supplies P. □

Lemma 2. *Let P be squarefree, contained in one level of Ω, and coprime to an integer D. Suppose also that every prime factor of every member of P exceeds $C \in \mathbb{N}$. If $n, n' \in P$ are distinct, then*

$$|k(Dn)^r - h(Dn')^s| \geq D \qquad (k, r, s \in \mathbb{N},\ 1 \leq h \leq C).$$

Proof. Equality of the two powered expressions would imply that every prime dividing n divides hDn', hence divides n' since it divides neither D nor h. Squarefreeness and equality of the numbers of prime factors imply $n = n'$, a contradiction. Their nonzero difference is an integer divisible by D. □

Ordinary primitivity is insufficient for this lemma: $\{12, 18\}$ is primitive, but $27 \cdot 12 = 18^2$. The squarefree equal-level condition supplies the stronger prime-support antichain used here. The additional roughness condition handles target coefficients: when $\gcd(D, 6) = 1$, the pair $2D, 3D$ obeys all the integer two-power exclusions of the earlier lemma without roughness, yet $3(2D) = 2(3D)$.

3 Finite states, preparation and openness

A finite state has labeled rational points $x_i > 2$, positive integer tags τ_i, positive integer degree caps c_i, distinct odd private primes $p_1, \dots, p_t$, an assignment $g(i)$, and representations

$$x_i = p_{g(i)} u_i / v_i, \qquad u_i, v_i \in \mathbb{N}.$$

It is admissible if

$$|kx_i^r - hx_j^s| > \min(\tau_i, \tau_j) \quad (i \neq j,\ k \in \mathbb{N},\ 1 \le r, s, h \le \min(c_i, c_j)), \tag{G}$$

$$x_i > 4\tau_i p_{g(i)}, \qquad p_{g(i)} > c_i, \tag{S}$$

$$p_j \nmid u_i \quad \text{for all } i, j, \qquad p_{g(i)} \nmid v_i, \qquad \sum_j 1/p_j < 1/100. \tag{P}$$

It is prepared if, in addition, every u_i is a prime larger than every private prime and $\sum_i 1/u_i < 1/200$. The empty state is prepared. Empty maxima of point sets are zero.

Lemma 3. *Fix the tags, caps, private primes and assignments, with the prime sum less than* $1/100$*. The set of prepared rational states is dense in every nonempty open geometric region satisfying* (G)–(S)*. It can avoid the zero sets of any finite collection of nonzero real polynomials.*

Proof. Let $M = \prod_j p_j$, and prescribe any sufficiently large lower bound B for the prime quotients u_i. For an assigned prime p, numbers pu/v with $u > B$ prime and $v \equiv 1 \pmod M$ are dense in $(0, \infty)$. To approximate $z > 0$, let $u \to \infty$ through primes and choose a positive $v \equiv 1 \pmod M$ nearest pu/z. Then $|pu/v - z| \le zM/(2v) \to 0$. Take $B > \max(200m, \max_j p_j)$, where m is the number of coordinates. The resulting representations satisfy (P) and the preparation bound. The coordinatewise product of these dense scalar sets is dense. The complement of finitely many nonzero polynomial zero sets is open and dense, which proves the final assertion. □

Lemma 4. *For fixed finite tags, caps and assignments, the geometric region* (G)–(S) *is open.*

Proof. Restrict coordinates locally to $[a, b]$ with $1 < a < b$. Put $C = \max_i c_i$ and $T = \max_i \tau_i$. For $k > (Cb^C + T)/a$ all relevant inequalities are automatic, since $x_i^r \ge a$ and $hx_j^s \le Cb^C$. Only finitely many strict continuous inequalities remain. □

4 A residue estimate for two powers

Lemma 5. *Let $D \ge 2$, $R, C \ge 1$ be integers and p, q be distinct primes larger than both D and C. Put $b = Dpq/(pq-1)$. There is a periodic set $\mathcal{B}_{p,q} \subset \mathbb{N}$ of density at most $3RC^5/D$ with the following property: if ℓ is coprime to pq and outside $\mathcal{B}_{p,q}$, then*

$$|b^r k - h(D\ell)^s| > R \qquad (k \in \mathbb{Z},\ 1 \le r, s, h \le C).$$

Proof. Fix r, s, h and put $m = \min(r, s)$, $Q = (pq)^r$. Failure of the inequality gives an integer

$$z = D^{r-m}(pq)^r k - hD^{s-m}(pq-1)^r \ell^s, \qquad |z| \le \frac{R(pq-1)^r}{D^m}.$$

Modulo Q the coefficient of ℓ^s is a unit, since $h \le C < p, q$. For each z, the resulting congruence has at most s^2 unit solutions modulo Q. Indeed modulo p a degree-s polynomial has at most s roots; each unit root lifts uniquely from p^j to p^{j+1} because $p > s$ and the derivative $s\ell^{s-1}$ is a unit. The same argument applies to q, and the Chinese remainder theorem gives s^2. This defines a periodic exceptional set, restricted to units, of density at most

$$s^2 \left(\frac{2R}{D^m} + \frac{1}{(pq)^r} \right) \le \frac{3Rs^2}{D}.$$

Take the union over $1 \le r, s, h \le C$ and use $C^2 \sum_{s=1}^{C} s^2 \le C^5$. □

5 The finite extension

Lemma 6. *Fix $\varepsilon : \mathbb{N} \to (0,\infty)$ and a prepared finite state. Let integers $R, C \geq 1$ be at least, respectively, every old tag and every old cap. There is a finite admissible extension, preserving all old values and arithmetic data, whose new points all have tag R and cap C, and whose total new weight exceeds*

$$\frac{\gamma}{RC^8}, \qquad \gamma = \frac{1}{256000}.$$

Each new point y has an integer anchor $z \geq 2$ with $0 < y - z < \min(\varepsilon(z), 1/4)/4$. The resulting state need not be prepared.

Proof. Put $Q_C = \prod_{\pi \leq C,\ \pi \text{ prime}} \pi$ and $\delta_C = \prod_{\pi \leq C,\ \pi \text{ prime}} (1 - 1/\pi)$. These empty products are one when $C = 1$. Since $\delta_C \geq \prod_{j=2}^{C} (1 - 1/j) = 1/C$, it suffices to obtain gain $\delta_C^3/(256000RC^5)$. Choose an odd prime $l > 200$ different from all private primes and all numerator primes of the prepared state. For a sufficiently large $K \in \mathbb{N}$ set

$$D = l^K \prod_i u_i^{c_i} \geq 64RC^5/\delta_C.$$

Every private prime avoids D, and $\sum_{\pi \mid D,\ \pi \text{ prime}} 1/\pi < 1/100$. For an initial point $x_i = pu_i/v_i$, an exponent $1 \leq r \leq c_i$, and a label ℓ not divisible by p, write $w = D/u_i^r \in \mathbb{N}$. For every $s \geq 1$ and $1 \leq h \leq c_i$,

$$|kx_i^r - h(D\ell)^s| = (x_i/p)^r |kp^r - hD^{s-1} w v_i^r \ell^s| > 4\tau_i \quad (k \in \mathbb{Z}). \tag{1}$$

The integer factor is nonzero modulo p, since $h \leq c_i < p$, and $(x_i/p)^r > (4\tau_i)^r \geq 4\tau_i$.

Keep this D fixed while adding $J = \lfloor \delta_C D/(16RC^5) \rfloor$ blocks. Each new block has

$$y_n = \frac{Dnpq}{pq - 1} \quad (n \in P), \tag{2}$$

and tag R, cap C, private prime p, numerator quotient $u_n = qDn$, and denominator $v_n = pq - 1$. Maintain auxiliary prime sum $\sum 1/q < 1/100$ for the blocks inserted in this round; no such sum is required for earlier rounds.

After $j < J$ blocks, choose a new prime p larger than D, C, all current private primes, all current numerator quotients, and $\left(1/100 - \sum_i 1/p_i\right)^{-1}$. The enlarged private-prime sum stays below $1/100$. Let E be the periodic set of positive labels coprime to $Q_C D$, divisible by no current private prime (including this p) and no auxiliary prime q of this round, and avoiding the j sets $\mathcal{B}_{p_i,q_i}$ from Lemma 5. The set of labels obeying just the prime exclusions has density at least δ_C times the product of $(1 - 1/\pi)$ over excluded primes $\pi > C$. The latter product exceeds $97/100$, by the three reciprocal-prime bounds. The residue union bound consequently gives

$$\delta := \mathrm{d}(E) > \frac{97}{100}\delta_C - \frac{3jRC^5}{D} \geq \left(\frac{97}{100} - \frac{3}{16}\right)\delta_C > \frac{3}{4}\delta_C. \tag{3}$$

The set E is coprime to a period of itself: its period uses only primes dividing $Q_C D$, the private primes, and the auxiliary primes of this round, all of which E avoids. Equation (1) controls every initial point and Lemma 5 controls every point in a block already inserted at this scale: in the latter case take the free source multiplier to be kn^r. Consequently, for every current point x of tag τ_x and cap c_x,

$$|kx^r - h(D\ell)^s| > \tau_x \quad (k \in \mathbb{Z},\ \ell \in E,\ 1 \leq r, s, h \leq c_x). \tag{4}$$

Use Lemma 1 to choose $P \subset E$ in one squarefree Ω-level with $W(P) > \delta^2/1000 > \delta_C^2/2000$, beyond a threshold ensuring

$$\min P \geq 2D, \qquad D \min P > \max\left(4Rp,\ \max_{x \text{ current}} c_x x^{c_x} + R + 2\right). \tag{5}$$

Every prime factor of a member of P exceeds C, because $P \subset E$ is coprime to Q_C. At the integer vector $(Dn)_{n\in P}$, Lemma 2 gives all required within-block distances at least $D > R$. By Lemma 4, the within-block geometric inequalities remain valid when (2) is sufficiently close to this vector.

Write each current rational point in lowest terms as A_x/B_x. Choose the auxiliary prime q arbitrarily large, greater than $D, C, p, \max P$, every current private prime, and $(1/100 - \sum_{\text{current round}} 1/q_i)^{-1}$. Primes are unbounded, so additionally impose, for every $n \in P$,

$$0 < y_n - Dn < \tfrac{1}{4}\min(\varepsilon(Dn), 1/4), \tag{6}$$

$$|y_n^s - (Dn)^s| < \frac{1}{C}\min\left(1, \min_{\substack{x \text{ current} \\ 1\le r\le c_x}} B_x^{-r}\right) \qquad (1 \le s \le C), \tag{7}$$

omitting the second minimum for an empty state, and impose all the within-block inequalities just obtained. These are finitely many conditions valid for all sufficiently large q, since $y_n \to Dn$.

For a current point x, integer tags and the lattice $kx^r - h(Dn)^s \in B_x^{-r}\mathbb{Z}$ upgrade (4) to $|kx^r - h(Dn)^s| \ge \tau_x + B_x^{-r}$. Since $h \le c_x \le C$, condition (7) therefore gives $|kx^r - hy_n^s| > \tau_x$ for all $k \in \mathbb{Z}$ and $1 \le r, s, h \le c_x$. In the reverse direction (5) implies $ky_n^r - hx^s > R + 2 \ge \tau_x$ for all positive k and all required r, s, h, since $y_n^r \ge y_n > c_x x^{c_x} + R + 2$. Together with the within-block inequalities these prove (G). Condition (5) also gives $y_n/p > 4R$.

All private primes avoid $u_n = qDn$: they avoid D, avoid $n \in E$, and avoid q by its size. The new private prime avoids its own denominator $pq - 1$ and all old numerator quotients by its choice. Thus (P) is preserved. The auxiliary prime sum remains below $1/100$, and all primes p, q in this round exceed D, C, as needed for the residue lemma. This completes one block insertion.

Finally $y_n < 2Dn \le n^2$ gives $\mathrm{w}(y_n) > \mathrm{w}(n)/(4D)$. Each block adds weight $> \delta_C^2/(8000D)$. Since $J \ge \delta_C D/(32RC^5)$, the whole round adds weight $> \delta_C^3/(256000RC^5) \ge 1/(256000RC^8)$. The anchors $z = Dn$ satisfy (6). □

6 Fusion and proof of Theorem 1

Fix ε and let $e(z) = \min(\varepsilon(z), 1/4)$. Enumerate all nonzero integer-coefficient polynomials in finitely many variables from $X_1, X_2, \ldots$ as $f_1, f_2, \ldots$. For $v \ge 1$ let t_v be the positive integer with $t_v^9 \le v < (t_v + 1)^9$. Thus t_v is nondecreasing, tends to infinity, and $t_v^9 \le v$.

Inductively construct a prepared rational state at stage v, with labels $1, \ldots, m_v$ and positive-length rational-endpoint closed intervals $I_{i,v}$ around its rational coordinates. Require:

- the full product of the intervals lies inside the geometric region (G)–(S);
- each interval has diameter at most 2^{-v}, and each old interval is contained in the interior of its preceding interval;
- the product avoids the zeros of f_j for $j \le v$ whenever all the variables of f_j have labels at most m_v.

Every point born at stage v has fixed tag and cap $\tau_i = c_i = t_v$, a recorded rational birth value b_i, and an integer anchor z_i. Its first interval must lie inside $(b_i - e(z_i)/4, b_i + e(z_i)/4)$.

Start from the empty prepared state. At stage v, apply Lemma 6 to the preceding prepared state with $R = C = t_v$. The old tags and caps are at most t_v. The extension preserves every old center and adds points of total birth weight

$$\sum_{\text{born at } v} \mathrm{w}(b_i) > \frac{\gamma}{t_v^9} \ge \frac{\gamma}{v}. \tag{8}$$

Assign consecutive new labels and record their anchors and birth values. The product of the previous interval interiors for the old labels and the prescribed birth windows for the new labels contains the extended state's vector. Its intersection with the geometric region is therefore nonempty and open by Lemma 4. Apply Lemma 3 in that open region, also avoiding the finitely many currently required polynomial zero sets. Keep all tags, caps, private primes and assignments, and replace the rational values and their numerator/denominator representations by those of this prepared state. Around its coordinates choose rational-endpoint closed intervals of positive length, diameter at most 2^{-v}, whose product lies in the indicated open region and avoids those zero sets. The centers lie in the interiors of their intervals. This completes the induction. Every stage adds at least one point, so $m_v \to \infty$.

For each label i, the nested compact intervals $I_{i,v}$ have diameters tending to zero. Let a_i be their unique common point. For every fixed pair $i \neq j$, the closed product box at the stage when both labels first exist contains their final coordinates and lies in the strict geometric region. Hence

$$|ka_i^r - ha_j^s| > \min(\tau_i, \tau_j) \quad (k \in \mathbb{N},\ 1 \le r, s, h \le \min(c_i, c_j))\,. \tag{9}$$

This preserves strict inequalities, rather than merely non-strict limits: the entire closed box lies in their strict solution set. Also $a_i > 4\tau_i p_{g(i)} > 2$. In particular $r = s = h = 1$ in (9) gives separation greater than one. Thus the limit points are distinct, and A is countably infinite and locally finite.

The birth window gives $|a_i - b_i| < e(z_i)/4 \le 1/16$. Together with the extension's $|b_i - z_i| < e(z_i)/4$, this yields $|a_i - z_i| < e(z_i)/2 < \varepsilon(z_i)$. If two anchors were equal, their two limit points would differ by less than $1/4$, contrary to separation; the anchors are distinct.

Each polynomial f_j is handled once its index and all its variable labels are present, and its avoidance persists inside that stage's closed box. Therefore every such polynomial is nonzero at the final coordinates. Clearing coefficient denominators proves algebraic independence over $\mathbb{Q}$.

Since $b_i > 2$ and $|a_i - b_i| < 1/16$, we have $a_i < 2b_i$ and $\log a_i < 2 \log b_i$, so $\mathrm{w}(a_i) > \mathrm{w}(b_i)/4$. Consequently the points born in the first V stages have total limit weight greater than

$$\frac{\gamma}{4} \sum_{v=1}^{V} \frac{1}{v},$$

by (8). This tends to infinity and proves $W(A) = \infty$.

Finally, for fixed $d \in \mathbb{N}$ and $H > 0$, let $F_{d,H}$ consist of all points born in stages for which $t_v < d$ or $t_v \le dH$. There are finitely many such stages and finitely many points per stage. Every remaining point has cap at least d and tag greater than dH. For $m \in \mathbb{N}$ and $1 \le n, r, s \le d$, formula (9), with $k = m$ and $h = n$, gives

$$\left| \frac{m}{n} a^r - b^s \right| = \frac{|ma^r - nb^s|}{n} > \frac{dH}{n} \ge H.$$

This establishes all assertions of Theorem 1.

7 Relation to the rational construction

Apicella's private-prime grid and finite-block extension are the starting point. Arbitrary prescribed proximity to integers is a routine extra denominator bound, not an independent novelty claim. For rational power separation, a squarefree common Ω-level with all prime factors greater than C gives integer blocks with no exact $k(Dn)^r = h(Dn')^s$ relation for $h \le C$. The positive-density selection lemma retains a uniform weight depending only on the allowed density, even with these prime exclusions. Prime numerator quotients keep the reset grid's prime exclusions small. The power-map root count bounds each current block's exceptional labels by $3RC^5/D$.

The resulting gain is at least $1/(256000RC^8)$, using the elementary bound $\delta_C \geq 1/C$. Taking tags and caps to grow as the integer ninth root of the stage number leaves a divergent harmonic lower bound. Dense arithmetic re-encoding and nested closed boxes then add algebraic independence while preserving every previously protected inequality. No rational arithmetic representation is asserted for the limit points.

8 Consequences and comparison

Corollary 1. *Every element of the set in Theorem 1 is transcendental, as is a/b whenever a, b are distinct elements. The field $\mathbb{Q}(A)$ is a purely transcendental extension of $\mathbb{Q}$ of countably infinite transcendence degree.*

Proof. A polynomial relation for one element contradicts algebraic independence. If a/b were algebraic and $P(T) = \sum_{j=0}^{d} c_j T^j$ were a nonzero polynomial in $\mathbb{Q}[T]$ vanishing there, then $\sum_{j=0}^{d} c_j a^j b^{d-j} = 0$ would be a nonzero polynomial relation between the two distinct elements a, b. Here $b > 2$, so denominators may be cleared. Finally the elements of A form an algebraically independent countable generating set for $\mathbb{Q}(A)$, proving the field assertion. □

Corollary 2. *For every fixed $\lambda \in \mathbb{Q}_{>0}$, fixed $r, s \in \mathbb{N}$ and $H > 0$, there is a finite $F \subset A$ such that $|\lambda a^r - b^s| > H$ for all distinct $a, b \in A \setminus F$.*

Proof. Write $\lambda = m/n$ with $m, n \in \mathbb{N}$ and apply Theorem 1 with $d \geq \max(n, r, s)$. The numerator m is unrestricted in that theorem. □

The companion paper *Integer-power separation does not force rational-dilation separation* constructs a different algebraically independent set for which the integer-power conditions persist but this fixed-rational conclusion fails on a dense family of rational dilations. These are alternative examples, not two simultaneous descriptions of the same set. The corollaries above are elementary consequences, not separate priority claims.

A Apicella's rational counterexample, reconstructed

This appendix retains the rational result and the elementary modifications from the original exposition. The underlying counterexample, block method and disproof of the convergence assertion are due to Apicella [1]. The elementary selection bound and composite auxiliary parameter are recorded without a claim of priority or improved constants. The points in this appendix are rational; they are not the algebraically independent points of Theorem 1.

Theorem 2 (Apicella's counterexample, reconstructed)**.** *There is a countably infinite $A \subset \mathbb{Q} \cap (2, \infty)$ such that*

$$|ka - b| > 1 \quad (a, b \in A,\ a \neq b,\ k \in \mathbb{N}), \qquad W(A) = \infty.$$

In particular this refutes the full convergence assertion of the catalog target Erdős 143(ii). The set is locally finite.

A.1 An elementary primitive-selection lemma

The following estimate replaces the analytic selection argument in [1]. Its constants are deliberately nonoptimal.

Lemma 7 (Elementary primitive selection)**.** *Let $E \subset \mathbb{N}$ be periodic with natural density $\delta > 2/3$. For every $N \geq 2$, there is a finite primitive $P \subset E \cap [N, \infty)$ with $W(P) > 1/48$.*

Proof. Use the moment estimate already established in the proof of Lemma 1: with $L = \log\log X$, $\sum_{2\le n\le X}\Omega(n)\mathrm{w}(n) \le 3L^2+O(L)$. Periodicity gives weight $\delta L+O_{E,N}(1)$ in $E\cap[N,X]$. The levels with $\Omega(n) > \lfloor 12L\rfloor$ have weight at most $L/4+O(1)$. Thus the first $\lfloor 12L\rfloor$ positive levels have weight at least $(\delta-1/4)L+O_{E,N}(1)$. If every level had weight at most $1/48$, their combined weight would be at most $L/4$, contradicting $\delta > 2/3$ for large X. A single Ω-level is primitive, and so supplies the desired finite set. □

A.2 The rational-block construction

We reconstruct the private-prime and rational-block mechanism of [1]. The structural construction and its use to obtain a counterexample are due to Apicella. The auxiliary parameter q_i below is allowed to be composite.

A.2.1 Finite configurations

A block has data (D_i, p_i, q_i, P_i), where $D_i \ge 48$ is an integer, $p_i > 2$ is prime, $q_i \ge 2$ is an integer, and $P_i \subset \mathbb{N}\setminus\{1\}$ is finite and primitive. Its points are

$$a_{i,n} = \frac{D_i n p_i q_i}{p_i q_i - 1} \quad (n \in P_i).$$

A finite ordered list of blocks is admissible if its union F satisfies:

1. $|ka-b| > 1$ for all distinct $a, b \in F$ and $k \in \mathbb{N}$, and every block lies more than one to the right of all earlier blocks;
2. the p_i are distinct and $p_j \nmid q_i D_i n$ for every i, j and every $n \in P_i$;
3. $a_{i,n}/p_i > 4$ for every point, and $\sum_i 1/p_i < 1/100$.

The empty list is admissible.

A.2.2 Extension with a uniform weight gain

Lemma 8 (Finite extension)**.** *Every admissible finite configuration F_0 extends, without changing any old data, to an admissible F' with $W(F'\setminus F_0) > 1/4608$.*

Proof. Write t for the number of old blocks. Choose $D \ge 48$ as a sufficiently large power of two times the product of all old integers $q_i D_i n$. Then every such integer divides D, and none of the old private primes p_i divides D.

For an old point $a = a_{i,n}$ put $v = D/(q_i D_i n) \in \mathbb{N}$. If $p_i \nmid \ell$, then for every $k \in \mathbb{Z}$

$$|ka - D\ell| = \frac{a}{p_i}|kp_i - v(p_i q_i - 1)\ell| > 4. \tag{A1}$$

The integer in absolute values is nonzero modulo p_i, since neither v nor ℓ is divisible by p_i.

We add $J = \lfloor D/12\rfloor$ new blocks, all with scale D. For parameters p, q of any such block define the bad residue classes

$$B_{p,q} = \{z \bmod pq : z \in \mathbb{Z},\ |z| \le \lfloor (pq-1)/D\rfloor\}.$$

If $\ell \bmod pq \notin B_{p,q}$, then

$$\left|\frac{Dpq}{pq-1}h - D\ell\right| > 1 \quad (h \in \mathbb{Z}). \tag{A2}$$

Otherwise $z = pqh-(pq-1)\ell$ would have $D|z| \le pq-1$ and residue ℓ modulo pq. If $pq \ge 2D$, the proportion of bad classes is

$$\frac{|B_{p,q}|}{pq} \le \frac{2}{D} + \frac{1}{pq} \le \frac{3}{D}. \tag{A3}$$

Suppose $s < J$ blocks have been added, preserving admissibility and $p_j \nmid D$ for every private prime now present. Choose a new prime p larger than D, every old private prime, every currently stored integer $q_j D_j n$, and

$$\left(1/100 - \sum_j 1/p_j\right)^{-1}.$$

This is possible because these are finitely many finite bounds. The private-prime sum, now including p, is still less than $1/100$. Let E consist of the labels $\ell \in \mathbb{N}$ which are divisible by none of these private primes and which avoid B_{p_j,q_j} for each of the s blocks already added during this extension. This is periodic. The union bound on residue classes, without any coprimality assumption on the periods, gives

$$\mathrm{d}(E) \geq 1 - \sum_j 1/p_j - 1/p - 3s/D > 99/100 - 1/4 = 37/50 > 2/3. \tag{A4}$$

Here the first sum is over the primes present before choosing p. By (1) for the initial blocks and (2) for the new ones,

$$|ka - D\ell| > 1 \quad (a \in F,\ \ell \in E,\ k \in \mathbb{Z}), \tag{A5}$$

where F is the current finite union.

Apply Lemma 7 with a sufficiently large threshold to obtain finite primitive $P \subset E$ with $W(P) > 1/48$ and

$$\min P \geq 2D, \qquad D \min P > \max\{4p, \max F + 1\}.$$

For $F = \varnothing$ take $\max F = 0$. Put $U = D \max P$. For every old point a write $a = A_a/B_a$ with integers A_a, B_a and $B_a > 0$. Choose an integer $q \geq 2$ coprime to all current private primes, including p, sufficiently large that

$$pq \geq 2D, \qquad \frac{U}{pq-1} < 1, \qquad \frac{U}{pq-1} < \frac{1}{B_a} \quad (a \in F). \tag{A6}$$

Such q exist: take $q = 1 + K \prod_j p_j$, with the product now also including p, and increase $K \in \mathbb{N}$. Add the block $a_n = Dnpq/(pq-1) = Dn + \eta_n$, $n \in P$. Then $0 < \eta_n \leq U/(pq-1)$.

For an old $a = A_a/B_a$ and any integer k, (5) and $ka - Dn \in B_a^{-1}\mathbb{Z}$ imply $|ka - Dn| \geq 1 + 1/B_a$. Using (6), the triangle inequality gives $|ka - a_n| > 1$ for every k. In the other direction $ka_n - a > 1$ for every positive k, since $a_n > Dn > \max F + 1$. Inside the new block, primitivity gives

$$|ka_n - a_{n'}| = \frac{Dpq}{pq-1}|kn - n'| > D > 1 \quad (n \neq n',\ k \in \mathbb{N}).$$

The new p divides none of the old stored integers by its size. Every private prime, old or new, avoids the new qDn: it avoids q by (6), D by the induction hypothesis and $p > D$, and n because $P \subset E$. Also $a_n/p > Dn/p > 4$. Thus all the admissibility conditions and the local condition $p_j \nmid D$ persist.

Finally $a_n < 2Dn \leq n^2$ and $\log a_n \leq 2 \log n$, so

$$W(\{a_n : n \in P\}) > \frac{W(P)}{4D} > \frac{1}{192D}.$$

The blocks are disjoint and above all earlier ones. This completes the induction through all J insertions. Since $D \geq 48$ gives $J \geq D/24$,

$$W(F' \setminus F_0) > \frac{J}{192D} \geq \frac{1}{4608}.$$

The final configuration is admissible, so the same argument can be applied again with a new common scale. □

A.3 The infinite set and convergence assertion

Proof of Theorem 2. Start from the empty configuration and repeatedly apply Lemma 8. The resulting finite unions satisfy

$$F_0 = \varnothing \subset F_1 \subset F_2 \subset \cdots, \qquad W(F_R) > R/4608 \quad (R \geq 1).$$

Let $A = \bigcup_{R \geq 1} F_R$. All points are rational and exceed 2. Each distinct pair belongs to one F_R, where its separation holds for every positive integer multiplier, proving the separation assertion with its full quantifiers. The union is countable, and its finite subsums are unbounded, so it is infinite and $W(A) = \infty$. Taking $k = 1$ shows distinct points are more than one apart, which also proves local finiteness. □

A.3.1 Effectivity

The existence proof can be made a recursive description of rational points. Search over X and Ω levels for the finite P in Lemma 7, certifying its strict weight bound by rational lower approximations to the logarithmic weights. Dovetail the precision of these approximations; the strict bound proved above guarantees termination. All remaining inequalities involve rationals and finite primality or divisibility tests. This supplies a description, not a practical size bound for the very large parameters.

Attribution, provenance and limits

The private-prime framework is credited to Mattia Apicella. This paper was prepared with extensive ChatGPT/Codex assistance, including consolidation of the supplied exposition and extension manuscripts and elementary deductions from their stated theorems. Algebraic independence, rational-power tail separation and the rational reconstruction are kept distinct throughout. No Lean verification, independent human refereeing or certified historical priority is claimed for this consolidated version. Finite arithmetic checks support local identities; they do not certify the analytic estimates or the infinite construction. The author retains responsibility for correctness, attribution and any decision to submit the work.

References

[1] M. Apicella, *A rational counterexample to the reciprocal-logarithmic assertion in Erdős problem 143*, revised 26 October 2026. `https://github.com/mapice/erdos-143-candidate-disproof/blob/3a9189c3e0c6a400e368948fc47ab2d449d1c1de/paper/erdos143-candidate.tex`.

[2] D. Koukoulopoulos, Y. Lamzouri, J. D. Lichtman, *Erdős's integer dilation approximation problem and GCD graphs*, 2025. `https://arxiv.org/abs/2502.09539`.