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A CENTRAL-COMMUTATOR REGGE SEED
BCC SUBLATTICES, FACE-MATCHED HINGES, AND QUATERNIONIC SPECTRAL GAPS

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Abstract. (i) At the arithmetic parameter $\delta = 1/3$, the three short face diagonals of the negative-cycle ternary Gram form become mutually orthogonal and generate a simple-cubic sublattice of index two; the full lattice is body-centered cubic.

(ii) The positive- and negative-cycle Gram tetrahedra admit matching hinge faces. A four-tetrahedron Regge star consisting of one negative-cycle tetrahedron and three positive-cycle tetrahedra has an exactly computable positive deficit angle.

(iii) A figure-eight graph with quaternionic cycle holonomies i and j has a nonzero spectral gap; its characteristic polynomial is computed exactly and its smallest eigenvalue is isolated by rational bounds.

(iv) The commutator $[i, j] = -1$ identifies the graph's central $\mathbb{Z}_2$ obstruction with the negative signed-cycle class of the Gram construction.

The statements are finite-dimensional results about lattices, piecewise-Euclidean complexes, and unitary graph connections. They do not construct a closed shell, a Lorentzian spacetime, or a solution of the Einstein–matter equations; they do not prove the dominant energy condition or dynamical stability. The term "negative-cycle" refers only to a sign invariant and has no implication of negative physical energy.

1. The arithmetic point $\delta = 1/3$

For $\eta \in \{+1, -1\}$, set

$$G_\eta(\delta) = \begin{pmatrix} 1 & \delta & \delta \\ \delta & 1 & \eta\delta \\ \delta & \eta\delta & 1 \end{pmatrix}.$$

We use $G_- = G_{-1}(1/3)$ and $G_+ = G_{+1}(1/3)$.

Proposition 1.1 (Short-diagonal orthogonalization)**.** *Let a_1, a_2, a_3 have Gram matrix $\ell^2 G_\eta(\delta)$, and write $\sigma_{ij} = \operatorname{sgn}(a_i \cdot a_j)$ and $\eta = \sigma_{12}\sigma_{13}\sigma_{23}$. Define the short face diagonals*

$$d_{ij} = a_i - \sigma_{ij} a_j.$$

Then

$$\|d_{ij}\|^2 = 2\ell^2(1 - \delta), \qquad d_{ij} \cdot d_{ik} = \ell^2(1 - 2\delta + \eta\delta)$$

for distinct i, j, k. In particular, if $\eta = -1$ and $\delta = 1/3$, the three short face diagonals are mutually orthogonal, all have squared length $4\ell^2/3$, and generate a sublattice of index two. If $\eta = +1$ at $\delta = 1/3$, their pairwise inner products have absolute value $2\ell^2/3$, so they are not orthogonal.

Proof. The norm formula follows from

$$\|a_i - \sigma_{ij} a_j\|^2 = \ell^2 + \ell^2 - 2\sigma_{ij}(\sigma_{ij}\delta\ell^2) = 2\ell^2(1 - \delta).$$

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For distinct indices,

$$\begin{aligned} d_{ij} \cdot d_{ik} &= \ell^2 - \sigma_{ik}(a_i \cdot a_k) - \sigma_{ij}(a_j \cdot a_i) + \sigma_{ij}\sigma_{ik}(a_j \cdot a_k) \\ &= \ell^2(1 - 2\delta + \sigma_{ij}\sigma_{ik}\sigma_{jk}\delta) = \ell^2(1 - 2\delta + \eta\delta). \end{aligned}$$

At $\eta = -1$ and $\delta = 1/3$ this vanishes, while the common squared norm is $4\ell^2/3$. After switching to the canonical sign pattern $(\sigma_{12}, \sigma_{13}, \sigma_{23}) = (+1, +1, -1)$, the coefficient columns of (d_{12}, d_{13}, d_{23}) in the original basis are

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix},$$

whose determinant is 2. Thus they generate an index-two sublattice. For $\eta = +1$, the absolute values of the pairwise inner products at $\delta = 1/3$ equal $2\ell^2/3$. □

Theorem 1.2 (Index-two cubic sublattice)**.** *Let*

$$D = \operatorname{diag}(1, -1, -1), \qquad U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $\det U = -2$ *and*

$$U^{\mathsf{T}} D G_- D U = \frac{4}{3} I_3.$$

Consequently, the lattice represented by G_- *contains an orthogonal equal-length sublattice of index two. Up to a Euclidean isometry,*

$$\Lambda_- = \frac{1}{\sqrt{3}} \left(2\mathbb{Z}^3 \ \cup \ \left((1,1,1) + 2\mathbb{Z}^3 \right) \right),$$

so Λ_- *is a scaled body-centered cubic lattice and its simple-cubic sublattice is* $(2/\sqrt{3})\mathbb{Z}^3$.

Proof. A direct multiplication gives

$$D G_- D = \frac{1}{3} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} = \frac{1}{3}(4I_3 - J),$$

where J is the all-ones matrix. The columns of U are $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$. Their pairwise inner products for the Gram matrix $4I_3 - J$ are zero and their squared lengths are four. Thus $U^{\mathsf{T}} D G_- D U = (4/3) I_3$. Since $|\det U| = 2$, the sublattice has index two.

For the explicit coordinate description, take

$$a_1 = \frac{(1,1,-1)}{\sqrt{3}}, \quad a_2 = \frac{(-1,1,1)}{\sqrt{3}}, \quad a_3 = \frac{(1,-1,1)}{\sqrt{3}}.$$

Their Gram matrix is $D G_- D$. Moreover,

$$a_1 + a_2 = \frac{(0,2,0)}{\sqrt{3}}, \quad a_1 + a_3 = \frac{(2,0,0)}{\sqrt{3}}, \quad a_2 + a_3 = \frac{(0,0,2)}{\sqrt{3}},$$

and $a_1 + a_2 + a_3 = (1,1,1)/\sqrt{3}$. These vectors generate the displayed union of the even-coordinate cubic lattice and its body-centred coset. The index-two assertion follows again. □

Remark 1.3. The determinant and covolume at this parameter are

$$\det G_+ = \frac{20}{27}, \qquad \det G_- = \frac{16}{27}, \qquad \frac{\operatorname{covol}(\Lambda_+)}{\operatorname{covol}(\Lambda_-)} = \frac{\sqrt{5}}{2}.$$

The negative-cycle lattice is therefore not merely an arbitrary rational Gram example: it has an exact body-centred-cubic realization and an index-two cubic sublattice.

2. Face geometry and a Regge hinge star

Let $\ell > 0$ and let $\Delta_\eta(\ell)$ be the Euclidean tetrahedron with one vertex O and three edge vectors a, b, c whose Gram matrix is $\ell^2 G_\eta(1/3)$. Positivity of $G_\eta(1/3)$ guarantees that these tetrahedra are nondegenerate.

Lemma 2.1 (Common hinge-face metric)**.** *In both $\Delta_+(\ell)$ and $\Delta_-(\ell)$, the two faces incident to the edge OA can be chosen to have side lengths*

$$\left(\ell, \ell, \frac{2\ell}{\sqrt{3}}\right).$$

Hence the two tetrahedra can be glued face-to-face along either of these faces by an isometry fixing the hinge edge endpoints.

Proof. Write $A = a$, $B = b$, $C = c$. In both Gram matrices, $|a| = |b| = |c| = \ell$ and $a \cdot b = a \cdot c = \ell^2/3$. Therefore

$$|a - b|^2 = |a - c|^2 = |a|^2 + |b|^2 - 2a \cdot b = \frac{4}{3}\ell^2.$$

The faces OAB and OAC consequently have the stated side lengths. A Euclidean triangle is determined up to isometry by its three side lengths. □

Lemma 2.2 (Hinge dihedral angle)**.** *Let θ_η be the interior dihedral angle of $\Delta_\eta(\ell)$ at the edge OA. Then*

$$\cos\theta_\eta = \frac{\eta\delta - \delta^2}{1 - \delta^2}, \qquad \delta = \frac{1}{3}.$$

In particular,

$$\theta_+ = \arccos\frac{1}{4}, \qquad \theta_- = \frac{2\pi}{3}.$$

Proof. Let $n_1 = a \times b$ and $n_2 = a \times c$. The cross-product identity gives

$$n_1 \cdot n_2 = |a|^2(b \cdot c) - (a \cdot b)(a \cdot c) = \ell^4(\eta\delta - \delta^2),$$

while

$$|n_1|^2 = |n_2|^2 = \ell^4(1 - \delta^2).$$

With the outward orientations of the two faces, the interior dihedral angle has the cosine displayed above. Substitution of $\delta = 1/3$ yields $1/4$ for $\eta = +1$ and $-1/2$ for $\eta = -1$. □

Theorem 2.3 (A face-matched positive-deficit Regge star)**.** *There is a piecewise-Euclidean 3-dimensional simplicial complex consisting of one copy of $\Delta_-(\ell)$ and three copies of $\Delta_+(\ell)$ arranged cyclically around a common edge h of length ℓ, such that all shared triangular faces are isometric and h is the unique interior hinge. Its Regge deficit is*

$$\boxed{\epsilon_h = \frac{4\pi}{3} - 3\arccos\frac{1}{4} > 0.}$$

Numerically, $\epsilon_h \approx 0.2344419898279373$ radians, or approximately 13.4325365578 degrees.

Proof. Take four tetrahedra indexed cyclically by $r \in \mathbb{Z}/4\mathbb{Z}$, each with common hinge endpoints O, A and two further vertices B_r, B_{r+1}. Give one tetrahedron the Gram type G_- and the other three type G_+. For each tetrahedron, choose the two faces incident to OA to be the faces in Lemma 2.1; identify consecutive tetrahedra along the resulting isometric triangles OAB_r. This produces a cyclic link of four sectors around $h = OA$. All other faces are boundary faces, so h is the only interior hinge.

The sum of the four interior dihedral angles is $\theta_- + 3\theta_+$. By the Regge definition of deficit angle,

$$\epsilon_h = 2\pi - (\theta_- + 3\theta_+) = \frac{4\pi}{3} - 3\arccos\frac{1}{4}.$$

To see that this is positive, note that

$$\cos\frac{4\pi}{9} = \sin\frac{\pi}{18} < \frac{\pi}{18} < \frac{1}{4}.$$

Since cosine is strictly decreasing on $[0, \pi]$, this implies $\arccos(1/4) < 4\pi/9$, and hence $\epsilon_h > 0$. □

Corollary 2.4 (Flat and curved local hinge configurations)**.** *Three copies of $\Delta_-(\ell)$ can be arranged around a common edge using the same hinge-face isometries, and their dihedral angles sum to 2π; the resulting hinge deficit is zero. By contrast, the one-negative/three-positive configuration in Theorem 2.3 has positive deficit.*

Proof. The negative-cycle dihedral angle is $2\pi/3$, so three sectors sum to 2π. The second assertion is Theorem 2.3. □

Proposition 2.5 (An irrational dihedral angle)**.** *The ratio θ_+/π is irrational. In particular, no positive integer number of identical Δ_+ sectors can close a hinge with zero deficit.*

Proof. If θ_+/π were rational, then $\zeta = e^{i\theta_+}$ would be a root of unity. Hence $\zeta + \zeta^{-1} = 2\cos\theta_+ = 1/2$ would be an algebraic integer. A rational algebraic integer is an integer, which $1/2$ is not. Thus θ_+/π is irrational. If $m\theta_+ = 2\pi$ for some positive integer m, then $\theta_+/\pi = 2/m$ would be rational, a contradiction. □

Remark 2.6 (What is and is not matched). Each two-dimensional parallelogram face spanned by a pair of Gram vectors is a rhombus with side length ℓ, area $2\sqrt{2}\ell^2/3$, and diagonals $2\ell/\sqrt{3}$ and $2\sqrt{2/3}\ell$. Thus the individual face metrics are congruent as unlabelled rhombi. The choice of which diagonal is shorter depends on the edge sign. The product of the three edge signs prevents these facewise identifications from being made by one global vertex-sign switching. The Regge star above uses only the two hinge faces for which the side-length matching is explicit; it does not claim that the entire boundaries of Δ_+ and Δ_- are isometric.

3. The quaternionic central commutator

Let

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}, \qquad i^2 = j^2 = k^2 = ijk = -1.$$

Let $\rho : Q_8 \to U(2)$ be the faithful representation

$$R_i = \rho(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad R_j = \rho(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad R_k = R_i R_j.$$

Lemma 3.1 (Central commutator)**.** *The center of Q_8 is $Z(Q_8) = \{\pm 1\}$, the quotient is $Q_8/Z(Q_8) \cong \mathbb{Z}_2^2$, and*

$$[i, j] := iji^{-1}j^{-1} = -1.$$

Thus the commutator of the two independent nontrivial quotient classes is the nontrivial central element.

Proof. The defining relation $ij = -ji$ gives $iji^{-1}j^{-1} = -1$. The only elements commuting with both i and j are ± 1, so the center is $\{\pm 1\}$. Modulo this center, the images of i and j commute and each has order two; they generate a group of order four, isomorphic to $\mathbb{Z}_2^2$. □

Definition 3.2 (Central-commutator matching rule)**.** Let $H_1, H_2 \in Q_8$ be noncommuting order-four elements. Let $\chi : Z(Q_8) \to \{\pm 1\}$ be the nontrivial character, with $\chi(-1) = -1$. We set

$$\eta := \chi([H_1, H_2]).$$

For $H_1 = i$ and $H_2 = j$, this gives $\eta = -1$, and we associate the negative-cycle Gram block $G_-(1/3)$ to the pair of graph holonomies.

Remark 3.3. This rule identifies the same central $\mathbb{Z}_2$ obstruction in two finite structures: the switching class of a signed Gram triangle and the commutator of two quaternionic loop holonomies. It is an explicit compatibility rule. It is not yet derived from a common variational principle or a continuum gauge-gravity field equation.

4. A non-Abelian figure-eight graph and its exact spectrum

Consider the graph with vertices $0, 1, 2, 3, 4$ and two edge-disjoint triangles sharing vertex 0. Orient the first triangle as $0 \to 1 \to 2 \to 0$ and assign labels $1, 1, i$; orient the second as $0 \to 3 \to 4 \to 0$ and assign labels $1, 1, j$. The cycle holonomies are $H_1 = i$ and $H_2 = j$, so they do not commute.

For an oriented edge $u \to v$ with unitary label U_{uv}, define the connection energy

$$\mathcal{E}(f) = \sum_{\{u,v\}} \|f_u - U_{uv} f_v\|^2, \qquad f_v \in \mathbb{C}^2,$$

with the reverse orientation labelled by U_{uv}^*. Let L be its self-adjoint connection Laplacian.

Theorem 4.1 (Exact characteristic polynomial of the non-Abelian figure-eight)**.** *For unit edge weights, the characteristic polynomial of the* 10×10 *complex Hermitian connection Laplacian is*

$$\boxed{\det(\lambda I - L) = (\lambda - 1)^2(\lambda - 3)^2(\lambda^3 - 8\lambda^2 + 15\lambda - 4)^2.}$$

Consequently its smallest eigenvalue is α*, where* α *is the unique root of*

$$p(\lambda) = \lambda^3 - 8\lambda^2 + 15\lambda - 4$$

in the interval

$$\frac{3}{10} < \alpha < \frac{8}{25}.$$

Numerically, $\alpha \approx 0.318669356395023$*. In particular, the connection Laplacian has a strict spectral gap.*

Proof. Use the vertex order $0, 1, 2, 3, 4$ and put $d = \lambda - 2$. The matrix $\lambda I - L$ has diagonal blocks $(\lambda - 4)I_2$ at vertex 0 and dI_2 elsewhere. The two branches have identical internal block

$$B = \begin{pmatrix} dI_2 & I_2 \\ I_2 & dI_2 \end{pmatrix}, \qquad \det B = (d^2 - 1)^2.$$

For the branch with holonomy matrix R_i, the coupling from the root to that branch is $C_i = (I_2, R_i^*)$. Its Schur contribution is

$$C_i B^{-1} C_i^* = \frac{2dI_2 - (R_i + R_i^*)}{d^2 - 1} = \frac{2d}{d^2 - 1} I_2,$$

because $R_i^* = -R_i$. The same formula holds for R_j. Thus the root Schur complement is

$$S(\lambda) = \left((\lambda - 4) - \frac{4d}{d^2 - 1} \right) I_2.$$

Taking determinants and simplifying gives

$$\det(\lambda I - L) = (d^2 - 1)^2 \big((\lambda - 4)(d^2 - 1) - 4d \big)^2,$$

which is the displayed factorization because

$$d^2 - 1 = (\lambda - 1)(\lambda - 3), \quad (\lambda - 4)(d^2 - 1) - 4d = p(\lambda).$$

The identity first holds when $d^2 \neq 1$ and then holds identically as a polynomial.

Finally,

$$p(3/10) = -193/1000 < 0, \qquad p(8/25) = 212/15625 > 0.$$

Moreover $p'(\lambda) = 3\lambda^2 - 16\lambda + 15 > 0$ on $[0, 8/25]$. Hence there is a unique root in that interval and no smaller nonnegative root. Since L is positive semidefinite and has no zero mode (either nontrivial cycle holonomy has no fixed vector in this representation), this root is the smallest eigenvalue. □

Theorem 4.2 (Two-cycle non-Abelian spectral-gap bound)**.** *Let Γ be a connected finite graph with N vertices. Suppose it contains two edge-disjoint cycles C_1, C_2 based at a common root r, of lengths m_1, m_2, whose Q_8 holonomies are noncommuting order-four elements. Let T be a spanning tree rooted at r with maximal depth D, and suppose every edge in $C_1 \cup C_2 \cup T$ has weight at least $c_0 > 0$. In the faithful two-dimensional representation above,*

$$\boxed{\lambda_{\min}(L) \geq \frac{c_0}{N(\frac{1}{2}\max(m_1, m_2) + 2D)}.}$$

Proof. For an oriented edge $e : u \to v$, write $d_e = f_u - U_e f_v$. Telescoping around a cycle of length m_a gives

$$(I - R_{H_a})f(r) = \sum_{e \in C_a} \widetilde{d}_e,$$

where each $\widetilde{d}_e$ is a unitary transport of d_e. Hence, by Cauchy–Schwarz,

$$\|(I - R_{H_a})f(r)\|^2 \leq m_a \sum_{e \in C_a} \|d_e\|^2.$$

For every order-four element of Q_8, its image is skew-Hermitian and squares to $-I_2$, so

$$(I - R_{H_a})^*(I - R_{H_a}) = 2I_2.$$

Adding the two cycle estimates and using edge-disjointness yields

$$4\|f(r)\|^2 \leq \max(m_1, m_2) \sum_{e \in C_1 \cup C_2} \|d_e\|^2 \leq \frac{m_*}{c_0}\mathcal{E}(f),$$

where $m_* := \max(m_1, m_2)$. Thus $\|f(r)\|^2 \leq m_*\mathcal{E}(f)/(4c_0)$.

For any vertex v, follow the unique tree path from r to v, of length at most D. Unitary transport and Cauchy–Schwarz give

$$\|f(v)\|^2 \leq 2\|f(r)\|^2 + 2D \sum_{e \in T} \|d_e\|^2 \leq \frac{1}{c_0}\left(\frac{m_*}{2} + 2D\right)\mathcal{E}(f).$$

Summing over all N vertices gives

$$\|f\|^2 \leq \frac{N}{c_0}\left(\frac{m_*}{2} + 2D\right)\mathcal{E}(f),$$

which is the claimed Rayleigh-quotient bound. □

Remark 4.3 (What drives the two-cycle gap). The lower bound uses the fact that each individual order-four holonomy has no fixed vector in the chosen representation. Noncommutativity is needed to certify that the two-loop connection is genuinely non-Abelian and to produce the central commutator class; it does not, by itself, improve the numerical constant in the estimate. A sharper commutator-sensitive bound is a separate open problem.

Corollary 4.4 (Uniform bound for a refined two-loop graph)**.** *Let Γ_n be the bouquet of two n-cycles sharing one vertex, with holonomies i and j, and constant edge weight $c_n = n^2/L_0^2$. Then*

$$\lambda_{\min}(L_{\Gamma_n}) \geq \frac{n^2}{L_0^2(2n-1)(5n/2-2)},$$

so

$$\liminf_{n\to\infty} \lambda_{\min}(L_{\Gamma_n}) \geq \frac{1}{5L_0^2} > 0.$$

Proof. Here $N = 2n-1$, $m_1 = m_2 = n$, and a natural spanning tree has maximal depth $D = n-1$. Apply Theorem 4.2. □

5. The initial-data bridge and its limits

In three-dimensional Regge geometry, curvature is concentrated on interior edges (hinges), with deficit equal to 2π minus the sum of incident tetrahedral dihedral angles [1]. For the star in Theorem 2.3, the integrated scalar-curvature measure in a neighborhood containing only that hinge is

$$\int R^{(3)}\, dV = 2\ell\epsilon_h.$$

If one seeks time-symmetric initial data with $K_{ij} = 0$ and no cosmological constant, the Hamiltonian constraint in physical units is [3]

$$R^{(3)} = \frac{16\pi G}{c^4}\rho.$$

At the level of this scalar constraint, the hinge therefore corresponds to the positive integrated energy-density measure

$$\int \rho\, dV = \frac{c^4}{8\pi G}\ell\epsilon_h > 0.$$

Remark 5.1 (Necessary distinction). The last equation is a constraint-level observation, not a matter-model construction. It does not determine the spatial stresses, prove the dominant energy condition, establish a smooth regularization of the hinge source, or show that the configuration is dynamically stable. A closed shell would require a globally face-compatible triangulation whose interior and exterior boundary data match; the present star is only a local hinge generator.

Proved / Computed / Open

Status	Statement	Scope
Proved	At $\delta = 1/3$, the negative-cycle Gram lattice is BCC and contains a simple-cubic sublattice of index two.	Theorem 1.2.
Proved	The hinge faces of Δ_+ and Δ_- are congruent, and the one-negative/three-positive star has the exact positive deficit $4\pi/3 - 3\arccos(1/4)$.	Theorems 2.3 and Lemmas 2.1–2.2.
Proved	The Q_8 figure-eight Laplacian has the stated characteristic polynomial and a gap $\alpha > 3/10$.	Theorem 4.1.
Proved	Two edge-disjoint noncommuting order-four cycle holonomies give the stated graph-size-dependent spectral lower bound.	Theorem 4.2.
Computed	The numerical hinge deficit is approximately 0.2344419898279373 radians; the exact graph gap is approximately 0.318669356395023.	Numerical evaluation of exact algebraic expressions.
Open	A globally closed face-compatible shell assembled from the local hinge stars.	Requires a separate global gluing construction.
Open	A common variational principle coupling the Q_8 connection and Regge edge lengths.	No coupled action is established here.
Open	A smooth Lorentzian limit, a complete matter stress tensor satisfying DEC, and nonlinear dynamical stability.	Not proved by the present finite model.

Acknowledgements and Author's Statement

This module develops a finite mathematical construction motivated by earlier work on ternary lattices and finite graph spectra. The author is responsible for all statements, calculations, interpretations, and the final form of the text. No claim of a physical warp configuration or of a completed spacetime solution is made.

References

[1] T. Regge, *General relativity without coordinates*, Il Nuovo Cimento **19** (1961), 558–571.
[2] K. Conrad, *Generalized Quaternions*, expository notes, available at kconrad.math.uconn.edu/blurbs/grouptheory/quaternion.pdf.
[3] E. Gourgoulhon, *3+1 Formalism in General Relativity*, Lecture Notes in Physics, vol. 846, Springer, 2012.
[4] P. Shi, *Finite Truncated Lattice-Length Data on Three-Dimensional Flat Tori: Explicit Obstructions, Finite Determination, and an Exact Data-Fiber Classification*, research manuscript, 2026.