A computer-assisted bound $\Lambda \leq 0.107964910079$ for the de Bruijn–Newman constant

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Abstract

In the Polymath normalisation, a computer-assisted comparison gives

$$\Lambda \leq \frac{427693072077942473661578098491}{396140812571321687967719751680 0} < 0.107964910079.$$

The initial squared zero height is bounded by 9/16, using OpenAI's ordinary-zeta 7/8 nonvanishing theorem formalised in Lean at the identified source revision. Finite signed translated-probe functionals, matched physical source estimates, and continuous-domain kernel inequalities supply a piecewise comparison for that height. A newly implemented exact rational consumer checks every one of the 221 010 arcs, their outward rounding and joins, and the complete final de Bruijn contraction. The fifteen-stage finite replay includes a direct reconstruction of all 67 source chords from their primitive payments, with explicit binding of the main terms and the complete derivative functionals. Separate consumers verify their continuous kernel certificates and the actual physical and arithmetic charges. The analytic argument includes the corrected 11/12 Gaussian remainder, finite-head localisation, attainment of a positive-time highest contact, and repeated complex contacts. The accompanying packet contains the mathematical sources, certificate data, checker programs, exact source identities, and replay instructions. The initial theorem is used at cited-theorem trust with authenticated external Lean-build evidence; a fresh local Lean build and an end-to-end formalisation of the present comparison are not claimed.

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1 Result and proof structure

Let

$$H_t(z) = \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,\mathrm{d}u, \tag{1}$$
$$\Phi(u) = \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})e^{-\pi n^2 e^{4u}}.$$

The decay of Φ makes H_t entire for every real t. These functions are even and real on the real axis, and satisfy

$$H_0(z) = \tfrac{1}{8}\xi((1+iz)/2), \qquad \partial_t H_t = -\partial_z^2 H_t, \qquad \xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{2}$$

The de Bruijn–Newman constant Λ is the real threshold for which all zeros of H_t are real exactly when $t \geq \Lambda$ [1, 2, 3].

Theorem 1.1. *For the normalisation* (1)–(2)*,*

$$0 \leq \Lambda \leq B_* := \frac{427693072077942473661578098491}{396140812571321687967719751680 0} < \frac{107964910079}{10^{12}}. \tag{3}$$

The upper bound is computer assisted and uses the ordinary-zeta 7/8 *theorem [24] and the finite-height zero theorem of Platt and Trudgian [4]. The exact finite certificates and analytic estimates used to obtain the upper bound are specified below and supplied in the accompanying packet.*

The lower inequality is the theorem of Rodgers and Tao [5]. The task here is to establish the upper inequality. The comparison does not use an assumption of the full Riemann hypothesis. The ordinary 7/8 input, including its source and external verification boundary, is stated separately so that a referee can check the imported theorem independently of the new comparison.

1.1 Initial data and terminal data

By [24], a nontrivial zeta zero ρ satisfies $\operatorname{Re}\rho \leq 7/8$. The functional equation also gives $\operatorname{Re}\rho \geq 1/8$. Under (2) its corresponding H_0 zero therefore has $|\operatorname{Im} z| \leq 3/4$. The strict squared-height barrier starts at

$$Q_0 = \frac{9}{16} + 10^{-12} = \frac{562500000001}{10^{12}}. \tag{4}$$

The strict cushion is retained in every rational computation. At the terminal comparison time the certified data are

$$T = \frac{22101}{204800} = 0.1079150390625, \qquad Q_T = \frac{7902377963830536747158643}{7922816251426433759354395 0336}. \tag{5}$$

Here $Q_T > 0$. Consequently a further time $Q_T/2$ is required for de Bruijn's strip contraction. Exact addition gives $T + Q_T/2 = B_*$; the displayed decimal in (3) is rounded upward. Stopping at T alone would omit part of the proof.

1.2 From a finite functional to a global height barrier

At a simple nonreal highest zero $x+iy$, set $q=y^2$. A finite signed combination of logarithmic-derivative probes supplies a physical lower bound J; its entire distinguished conjugate-pair contribution is the explicitly defined function $O_{\mathrm{pair}}(q)$. A kernel inequality then gives

$$q' \le -2 - 2q(J - O_{\mathrm{pair}}(q)). \tag{6}$$

The term -2 is the attraction to the distinguished conjugate zero. The subtraction $O_{\mathrm{pair}}(q)$ belongs to the whole signed functional and is paid exactly once. In particular, optimising a source floor alone does not optimise the contraction rate.

The proof has three distinct analytic obligations before one can integrate (6). First, the sources must apply to the actual function H_t at every declared probe and real coordinate, including arbitrarily large exterior coordinates. Second, the signed kernel inequalities must cover all real spacings and the full permitted heights of every other zero, including nonreal zeros. Third, the first-contact argument must handle compactness, time zero and repeated complex zeros. These obligations are proved in the following sections; the finite numerical histories check the subsequent scalar comparison.

Proof component	Content and verification boundary
Initial strip	Ordinary-zeta 7/8 theorem; exact strip map and strict seed (4).
Finite head	Published finite RH height, full continuous wall, and analytic zero continuation.
Exterior source	Corrected height-six approximation and signed finite, matched-source inequalities.
Kernel	Rational continuous-domain certificates, whole own pair, and harmonic extension to other-zero heights.
Global comparison	Simple and repeated contacts, remote compactness, all closed time cells and upward numerical resets.
Exact flow	221 010 rational arcs, independent checker, and full terminal $Q_T/2$ cleanup.

1.3 Edition and reproducibility

This edition supersedes referee edition 3 of 28 October 2026; the numerical plan, source table and comparison history are unchanged; Appendix B adds the direct source reconstruction.

1.4 Relation to earlier versions

The pinned public repository [9] states the ceiling 0.116397726408291. The present displayed ceiling is smaller by 0.008432816329291. This compares the stated numerical bounds; independent recognition of a world record is not asserted. An earlier research comparison gave 0.107936602217 under the narrower initial assertion $\operatorname{Re}\rho \le 0.8748867$. Xu and Sang [21, Theorem 1.1] obtain that boundary under the analytic inputs stated there. Liu's related refinements give 0.874957019420098946… [22] and 0.874886647039964… [23, Theorem 17.1], with their stated analytic inputs. Since $0.8748867 < 7/8$, the narrower assertion does not follow from the ordinary 7/8 theorem alone. The present argument therefore checks a new history from the literal seed (4); its input remains the ordinary 7/8 theorem.

The local native and confluent calculations retained in the research archive are not used in Theorem 1.1. Nor is an unproved uniform-current premise used. In particular, neither $\Lambda < 0.1$ nor the Riemann or Lindelof hypothesis is a conclusion of this manuscript.

1.5 Related work and the external strip input

The heat-flow and zero-motion framework follows de Bruijn, Newman and Polymath [1, 2, 3]. Ki, Kim and Lee [17] proved that, at every positive heat parameter, all but finitely many

zeros are real and simple, and obtained $\Lambda < 1/2$. The finite real-zero input used here is the verification of Platt and Trudgian [4]; the nonnegative lower bound is due to Rodgers and Tao [5].

Raymond's earlier time-zero construction and translated geometry are set out in the analytic companion [20]. The subsequent phase-correlated, translated-probe and retuned releases are [11, 12, 13]. The physical estimates and direct source reconstruction used here are stated in the present manuscript.

Planat [18, Section 8] reports a complete numerical reproduction of Raymond's frozen correlated-probe release, recovering 0.149395073827688, and a head-pair extension to 0.149344833997286. His later classical-strip comparison [19, Sections 4–5] reports a stored-source replay of the retuning archive, including all eight supplied comparison histories, and independently checks the 57-kernel catalogue and 19 023 steps of its refined comparison. These replications concern the earlier releases and retain the analytic premises stated in the cited theorems.

The 2026 entries in Tao's maintained catalogue [31] are the Mosaic Intelligence packages for 0.1965, 0.1875, 0.16 and 0.1595 [25, 26, 27, 28], and Kornaus's 0.172422 package [16]. The catalogue marks these as self-archived certificate packages awaiting external review. Further computer-assisted preprints include Gomila's 0.1787854 release [29] and Gordon's bound below 0.158 [30]. These references provide comparison context.

Recent comparisons also require attention to the initial theorem. Hakimoglu-Brown [14] states the 0.13963 ceiling under the seven-eighths hypothesis. The height-dependent region of Bellotti, Trudgian and Yang [15] is a different input from a fixed global seven-eighths half-plane. The OpenAI preprint [24] supplies the ordinary 7/8 input used here; its pinned Lean declaration is identified in Section 5.1.

2 Analytic foundations and the passage from source bounds to a global strip

This section fixes the normalisation and proves the analytic implications used to consume the finite certificates. The published inputs are the strip-contraction theorem of de Bruijn, the local zero-dynamics and positive-time tail theorems in [3], and the finite Riemann-hypothesis verification in [4]. The explicit wall at the numerical value of X below is a computation in this work. In particular, the published finite-height theorem is not being asserted at every later heat time.

2.1 Normalisation, symmetry, and the initial strip

Put

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{7}$$

$$\Phi(u) = \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})e^{-\pi n^2 e^{4u}}, \tag{8}$$

$$H_t(z) = \int_0^{\infty} e^{tu^2}\Phi(u)\cos(zu)\,\mathrm{d}u. \tag{9}$$

The apparent singularities in (7) are removed. Thus ξ is entire, and the standard Fourier identity gives

$$H_0(z) = \frac{1}{8}\xi\left(\frac{1+iz}{2}\right), \qquad \partial_t H_t = -\partial_z^2 H_t. \tag{10}$$

The super-exponential decay of Φ justifies differentiation under the integral on compact subsets of the (t, z) domain. In particular H_t is jointly analytic locally in t and z, and, for real t, is an entire function of order one. It is real on the real axis and even:

$$H_t(\overline{z}) = \overline{H_t(z)}, \qquad H_t(-z) = H_t(z). \tag{11}$$

All zero counts below include multiplicity. Every summand of (8) is positive when $u \geq 0$, since $2\pi n^2 e^{4u} - 3 > 0$. Hence

$$H_t(iy) = \int_0^\infty e^{tu^2} \Phi(u) \cosh(yu)\, \mathrm{d}u > 0 \quad (t, y \in \mathbb{R}). \tag{12}$$

In particular H_t is never the zero function and has no zeros on the imaginary axis.

The transformation between zeta zeros and H_0 zeros is worth keeping explicit. If $z = x + iy$, then

$$\frac{1 + iz}{2} = \frac{1 - y}{2} + i\frac{x}{2}. \tag{13}$$

Thus the classical critical strip gives $|y| \leq 1$ at $t = 0$. The ordinary seven-eighths theorem used here is the nonvanishing assertion

$$\zeta(s) \neq 0 \qquad \left(\operatorname{Re} s > \frac{7}{8},\ s \neq 1\right) \tag{14}$$

for the ordinary Riemann zeta function; the finite extension assigned to the pole by the formal library is irrelevant to this nontrivial-zero statement. Its precise source version and verification provenance are specified with the external theorem input [24]. By the functional equation, a nontrivial zero with real part below $1/8$ would give one with real part above $7/8$. Consequently all nontrivial zeta zeros have real parts in $[1/8, 7/8]$. The factors defining ξ add no zeros at $s = 0$ or $s = 1$, and its zeros are precisely the nontrivial zeta zeros. Equation (13) therefore gives the global initial strip

$$H_0(x + iy) = 0 \quad \Longrightarrow \quad |y| \leq \frac{3}{4}, \qquad y^2 \leq \frac{9}{16}. \tag{15}$$

The weaker seven-eighths input is used literally; a strip with boundary 0.8748867 is not inferred from it.

The following classical contraction theorem is used twice in the proof. It is [3, Theorem 3.2], quoting [1, Theorem 13].

Theorem 2.1 (de Bruijn strip contraction)**.** *Suppose $t_0 \in \mathbb{R}$ and $a > 0$, and every zero of H_{t_0} has imaginary part in $[-a, a]$. Then for $t \geq t_0$ every zero of H_t has imaginary part of modulus at most*

$$\sqrt{\max\{a^2 - 2(t - t_0), 0\}}.$$

In particular all zeros are real at every time $t \geq t_0 + a^2/2$.

The initial use of this theorem and (15) yields

$$H_t(x + iy) = 0 \quad \Longrightarrow \quad y^2 \leq \max\left\{\frac{9}{16} - 2t, 0\right\}, \qquad t \geq 0. \tag{16}$$

Only a very short initial segment of this elementary comparison is needed for strict startup. The quantitative improvement comes from the source and geometry inequalities, not from extrapolating the contraction theorem.

2.2 The stationary wall and finite-head theorem

The finite wall is at

$$X = 6000000185827, \qquad N = 690988, \qquad T_{\mathrm{w}} = \frac{41}{250}. \tag{17}$$

It is stationary in the z coordinate. Define

$$L = \log \frac{X}{4\pi}, \qquad v_n = \log n,$$

and use the leading gamma factor

$$M_0(s) = \frac{s(s-1)}{16}\pi^{-s/2}\sqrt{2\pi}(s/2)^{s/2-1/2}e^{-s/2}. \tag{18}$$

The powers use the logarithm analytic on the relevant upper or lower half-plane. Set

$$\theta_0 = \arg \frac{M_0((1+iX)/2)}{M_0((1-iX)/2)}.$$

The certified branch satisfies $-11/10 < \theta_0 < -1$. The wall computation also certifies

$$\sqrt{\frac{X}{4\pi}} \geq N, \qquad \sqrt{\frac{X}{4\pi} + \frac{T_{\mathrm{w}}}{16}} < N+1, \qquad \log N < \frac{L}{2}. \tag{19}$$

In particular the literal Riemann–Siegel cutoff is the same N throughout the wall interval.

Lemma 2.2 (Continuous prefix-to-wall transfer)**.** *For $0 \leq j \leq 3$, let*

$$t_j = \frac{41(2j+1)}{2000}, \qquad r = \frac{41}{2000}.$$

Suppose that for every $1 \leq J \leq N$ the following directed inequalities hold:

$$\sum_{n\leq J} \frac{1}{\sqrt{n}} \cos\left(\frac{Xv_n}{2} - \frac{\theta_0}{2} - \frac{\pi t_j(L-2v_n)}{16}\right) - r\sum_{n\leq J}\frac{\pi(L-2v_n)}{16\sqrt{n}} > \frac{79}{100}. \tag{20}$$

Then the real part of the unit-dephased ideal paired model at every $(t,y) \in [0,T_{\mathrm{w}}] \times [0,1]$ is greater than $79/100$.

Proof. The four closed cells $[t_j - r, t_j + r]$ cover $[0, T_{\mathrm{w}}]$. By $|\cos a - \cos b| \leq |a-b|$, (20) implies that every partial sum of

$$c_n(t) = \frac{1}{\sqrt{n}}\cos\psi_n(t), \qquad \psi_n(t) = \frac{Xv_n}{2} - \frac{\theta_0}{2} - \frac{\pi t(L-2v_n)}{16}$$

is greater than $79/100$ throughout the corresponding closed cell. The ideal dephased paired expression is

$$F_{\mathrm{id}}(t,y) = \sum_{n=1}^{N} \frac{e^{-tv_n(L-v_n)/4}}{\sqrt{n}}\left(e^{-yv_n/2}e^{i\psi_n(t)} + e^{-y(L-v_n)/2}e^{-i\psi_n(t)}\right). \tag{21}$$

Its real part is

$$\operatorname{Re} F_{\mathrm{id}}(t,y) = 2e^{-yL/4}\sum_{n=1}^{N} c_n(t)\, w_{t,y}(v_n), \quad w_{t,y}(v) = e^{-tv(L-v)/4}\cosh\left(\frac{y(L-2v)}{4}\right). \tag{22}$$

Both factors of $w_{t,y}$ are positive and nonincreasing on $[0,\log N]$. For the first this follows from $L-2v>0$; for the second it follows from $y\ge 0$ and $L-2v\ge 0$. Write $C_J=\sum_{n\le J} c_n(t)$ and $w_n = w_{t,y}(v_n)$. Finite Abel summation gives

$$\sum_{n=1}^{N} c_n(t)w_n = C_N w_N + \sum_{n=1}^{N-1} C_n(w_n - w_{n+1}) > \frac{79}{100}w_1.$$

Since $v_1=0$, multiplication by $2e^{-yL/4}$ gives

$$\operatorname{Re} F_{\mathrm{id}}(t,y) > \frac{79}{100}\,2e^{-yL/4}\cosh(yL/4) = \frac{79}{100}(1+e^{-yL/2}) \ge \frac{79}{100}.$$

No sampling in t or y appears in this deduction. □

The inequalities (20) comprise $4N = 2763952$ prefix tests. Their direct 512-bit replay checks every prefix and both cutoff guards, with a total of 3454987 gates including the ancillary wall checks. A ball-arithmetic lower endpoint must exceed the displayed rational threshold; overlap or a floating approximation to the threshold is not an acceptance criterion. The physical transfer uses the same nonzero normaliser B_t as the Riemann–Siegel model in Section 3. More explicitly, unit dephasing by $\exp[-i(\theta_0/2+\pi tL/16)]$ takes its leading pair to (21); the difference from the physical normalised function is charged as follows.

Write $v_N=\log N$, and bound the source-coordinate and gamma changes by

$$d_\alpha = 5/X, \qquad A_\alpha = L/2+1, \tag{23}$$

$$\epsilon_s = T_{\mathrm{w}} d_\alpha v_N/2, \qquad \epsilon_\gamma = T_{\mathrm{w}} d_\alpha(A_\alpha + d_\alpha/2) + (5+T_{\mathrm{w}}A_\alpha)/X. \tag{24}$$

The full displacement of the ideal paired sum is bounded by

$$E_{\mathrm{id}} = 2\sqrt{N}\left[(e^{\epsilon_s}-1) + (e^{\epsilon_s+\epsilon_\gamma}-1)\right] < \frac{1}{100}. \tag{25}$$

The inequality uses the complete sum of both branches, not a termwise comparison of selected low frequencies. The termwise heat/gamma error is separately bounded by $E_{\mathrm{AB}} < 10^{-6}$. It remains important to use a remainder valid on the entire wall time interval. The corrected broad-height transfer in the source supplement gives

$$E_{\mathrm{C}} \le 4\exp\left(-\frac{L}{4} - \frac{tL^2}{16} + \frac{1}{10}\right) \le 4e^{1/10}\left(\frac{X}{4\pi}\right)^{-1/4} < \frac{2}{369}. \tag{26}$$

For reference, the last strict rational bounds are

$$\frac{X}{4\pi} > \frac{7X}{88} > 820^4, \qquad e^{1/10} < \frac{10}{9}.$$

Both complete Riemann–Siegel branches and the unrefined remainder are included in this fee.

The wider-time normalisation in (26) is verified at $x_* = X-4$ and $t=T_{\mathrm{w}}$ with the bound

$$\frac{T_{\mathrm{w}}\pi^2}{32} + \frac{40}{x_*} + \frac{T_{\mathrm{w}}}{4}\left[\frac{20(26.90/2+1)}{x_*} + \frac{100}{x_*^2}\right] < \frac{4961}{98000} + \frac{40}{x_*} + \frac{T_{\mathrm{w}}}{4}\left[\frac{20(26.90/2+1)}{x_*} + \frac{100}{x_*^2}\right] < \frac{1}{10}. \tag{27}$$

Here $\pi < 22/7$ gives $T_{\mathrm{w}}(22/7)^2/32 = 4961/98000$ exactly. The broad remainder is established on $\sigma \in [-1, 2]$, $t \le 1/5$; it therefore covers the closed interval in (17). The Gaussian in that derivation has constant $11/12$, as proved in the source section. Specifically its full coefficient bracket is bounded by

$$\frac{1}{2}(1 + 10^{-10}) + \frac{(229/1000)110}{N - 1} + 10^{-100} < 1.$$

This displays the margin by which the corrected remainder, including its far Gaussian tail, retains coefficient one. Neither an old narrower remainder bound nor its old final wall-floor field is used.

Combining (25)–(26) gives

$$E_{\mathrm{AB}} + E_{\mathrm{C}} < 10^{-6} + \frac{2}{369} < \frac{1}{180}.$$

Consequently Lemma 2.2 implies the following explicit physical wall:

Proposition 2.3 (Full stationary wall)**.** *For every* $0 \le t \le 41/250$ *and* $0 \le y \le 1$,

$$\left|\frac{H_t(X + iy)}{B_t(X + iy)}\right| > \frac{79}{100} - \frac{1}{100} - \frac{1}{180} = \frac{697}{900} > \frac{3}{4}. \tag{28}$$

In particular the entire stated vertical wall is zero-free, including all its time and height endpoints.

Proposition 2.4 (Reality and simplicity in the finite head)**.** *Every zero of* H_t *with* $|\operatorname{Re} z| \le X$ *is real for* $0 \le t \le 41/250$*. Such zeros are simple for* $0 < t \le 41/250$*.*

Proof. Theorem 1 of [4] verifies the Riemann hypothesis through zeta ordinate 3000175332800. The conservative normalisation clearance is

$$3000175332800 - \frac{X + 1}{2} = 175239886 > 0. \tag{29}$$

Equations (13) and (29) therefore show that all time-zero zeros with $|\operatorname{Re} z| \le X$ are real. This uses the exact theorem endpoint, not only the rounded number $3 \cdot 10^{12}$ in the paper's title.

Take the compact rectangle $|\operatorname{Re} z| \le X$, $|\operatorname{Im} z| \le 2$. Classical strip contraction keeps all zeros in $|\operatorname{Im} z| \le 1$. Thus the horizontal edges and the vertical pieces with $1 < |\operatorname{Im} z| \le 2$ are zero-free. Proposition 2.3 and symmetry exclude zeros on the remaining vertical pieces. The boundary is zero-free for the whole closed time interval.

It remains to justify preservation of reality inside this rectangle. At any fixed time there are finitely many zeros inside. Surround them by disjoint small disks contained strictly inside the rectangle. Rouché's theorem preserves their total local multiplicities at nearby times, and a compact zero-free complement excludes new zeros elsewhere. A simple real zero persists as a real zero, by uniqueness and conjugation. At a real zero of multiplicity $m \ge 2$, Proposition 3.1(ii) of [3] gives the forward expansions

$$z_k(t_0 + \epsilon) = z_0 + \sqrt{2\epsilon}\,\lambda_k + O(\epsilon), \qquad \lambda_1 < \cdots < \lambda_m \in \mathbb{R},$$

where the λ_k are the distinct roots of the probabilists' Hermite polynomial. Each individual branch is localized in a disjoint disk centered on the real axis. Conjugation and the one-zero count in that disk force the zero to be real. Thus every all-real slice has an all-real right neighborhood; the compact zero-free boundary and local zero continuity also make the all-real property closed. Starting at zero and taking the supremum of times through which reality holds proves reality throughout the interval.

Finally, if a multiple real zero occurred at a positive time t_0, the backward version of the same expansion would have a nonreal branch $z_0 + i\sqrt{2\epsilon}\,\lambda_k + O(\epsilon)$ at $t_0 - \epsilon > 0$. That would contradict the reality just proved. No simplicity assumption about the verified time-zero zeros was used. □

2.3 Paired kernels and the complete signed probe functional

For $h > b \geq 0$ and $d \in \mathbb{R}$, put

$$K_h(d,b) = \frac{h-b}{d^2+(h-b)^2} + \frac{h+b}{d^2+(h+b)^2}. \tag{30}$$

This is the contribution of the conjugate pair $a \pm ib$, with $d = x - a$, to $-\operatorname{Im}(H_t'/H_t)(x+ih)$. For a real zero the same formula has to be given half weight: $K_h(d,0)/2 = h/(d^2+h^2)$.

Fix one of the source rows. Its signed linear functional $\mathcal{L}$ is a finite real linear combination of symmetric translations in the real variable and derivatives with respect to the probe height. For example, a node (h,a,w) contributes

$$\frac{w}{2}\left[S_t(x+a,h) + S_t(x-a,h)\right], \qquad S_t(x,h) = -\operatorname{Im}\frac{H_t'}{H_t}(x+ih).$$

Finite joint-jet combinations are included by applying the same linear operator to the h variable. The actual coefficients, factors of two, heights and shifts are defined in Section 3. Write

$$J(t,x) = \mathcal{L}[S_t](x), \qquad R(d,b) = \mathcal{L}[K_{(\cdot)}(d+(\cdot),b)]. \tag{31}$$

The notation for R means that exactly the same height, translation, and height-derivative operator is applied to the pair kernel. The operator is fixed within a source row; it is not differentiated along a moving zero. Its full own-pair function at a contact square $q = y^2$ is

$$O_{\mathrm{pair}}(q) = R(0,\sqrt{q}). \tag{32}$$

Here O_{pair} denotes the exact conjugate-pair contribution of the fixed finite functional, not a Landau order term. The row-indexed function is written $O_{\mathrm{pair},i}$ below. For an ordinary symmetric node it is explicitly

$$O_{\mathrm{pair};h,a,w}(q) = \frac{2wh(h^2+a^2-q)}{(h^2+a^2-q)^2+4a^2q}. \tag{33}$$

This is the contribution of the *entire* conjugate pair at $x \pm iy$. Subtracting only the lower member from the probe would use incorrect units. The lower member alone is removed separately in the zero-motion identity below.

At a first highest contact with height y, all other zero heights have modulus at most y. Every probe pole lies strictly outside this strip; indeed the complete contact caps satisfy $h^2 > q_{\max}$ for all original heights and all joint-jet centers. Therefore all quantities in (31) are well-defined at such a contact.

Lemma 2.5 (Convergence of paired imaginary logarithmic derivatives)**.** *Suppose the zeros of the real entire order-one function H_t have imaginary parts bounded in modulus. At every point away from its zeros, the imaginary part of its logarithmic derivative is the sum of the conjugately paired imaginary contributions, with real zeros assigned half-pair weight. The resulting sum is absolutely convergent. Finite signed combinations and finite height derivatives at probe points outside the zero strip may be interchanged with this sum.*

Proof. The order-one growth of H_t gives, by Jensen's formula, $N_t(r) = O(r^{1+\eta})$ for every fixed $\eta > 0$. In a strip of bounded height, this implies convergence of the zero sum of $(1+|\operatorname{Re}\rho|)^{-2}$, by a dyadic decomposition with, say, $\eta = 1/2$. At a fixed point $z = x+ih$, a conjugate pair at $a \pm ib$ contributes

$$-\operatorname{Im}\left(\frac{1}{z-a-ib} + \frac{1}{z-a+ib}\right) = K_h(x-a,b).$$

For large $|a|$, with b bounded, this is $O((1+|a|)^{-2})$. The same estimate, with a changed constant, holds after any fixed finite number of height derivatives on compact probe sets strictly outside the zero strip. The genus-one canonical product has a linear exponential factor whose logarithmic derivative is real: this follows from reality on the real axis, or directly from the symmetric product representation. The conjugate pairing also makes the imaginary parts of the canonical $1/\rho$ correction terms cancel. Thus there is no additional imaginary constant. Absolute convergence then gives the asserted exchanges. □

The boundary geometry can be extended to all other-zero heights by an explicit positive Poisson argument. This is needed because other zeros are not assumed real or to lie on the highest-contact boundary.

Lemma 2.6 (Boundary kernel inequality implies a full strip inequality)**.** *Fix $y > 0$. Suppose $R(d,b)$ is the signed finite probe kernel just defined, with every probe height strictly greater than y. If*

$$\frac{1}{2}R(d,y) \le \frac{2}{d^2+4y^2} \quad \text{for every } d \in \mathbb{R}, \tag{34}$$

then

$$K_y(d,b) \ge \frac{y}{2}R(d,b), \qquad 0 \le b \le y, \tag{35}$$

apart from the distinguished singular location $d = 0$, $b = y$.

Proof. As a function of (d,b), each $K_h(d+a,b)$ and each of its height derivatives is harmonic on $|b| < y$, is continuous and bounded on the closed strip, is even in b, and tends to zero as $|d| \to \infty$. So does $g = R/2$. It is the unique bounded harmonic extension of its equal boundary values $g(d,y) = g(d,-y)$.

Let $\mathcal{P}_y$ denote the positive Dirichlet extension operator for these equal boundary data. With Fourier convention $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-id\xi} f(d)\,\mathrm{d}d$, its multiplier at height b is

$$\frac{\cosh(b\xi)}{\cosh(y\xi)}.$$

In the sense of boundary measures the positive harmonic function $K_y(d,b)/y$ has, at *each* edge of the strip, the boundary data

$$\frac{2}{d^2+4y^2} + \frac{\pi}{y}\delta_0. \tag{36}$$

Indeed $\epsilon/(d^2+\epsilon^2) \to \pi\delta_0$ as $\epsilon \downarrow 0$. Equivalently the Fourier transform of the density in (36) is $(\pi/y)(e^{-2y|\xi|}+1)$, whose Poisson extension is

$$\frac{\pi}{y}(1+e^{-2y|\xi|})\frac{\cosh(b\xi)}{\cosh(y\xi)} = \frac{2\pi}{y}e^{-y|\xi|}\cosh(b\xi),$$

which is exactly the Fourier transform of $K_y(d,b)/y$.

By (34) and positivity of the Poisson operator,

$$g = \mathcal{P}_y[g(\cdot,y)] \le \mathcal{P}_y\left[\frac{2}{d^2+4y^2}\right] \le \mathcal{P}_y\left[\frac{2}{d^2+4y^2} + \frac{\pi}{y}\delta_0\right] = \frac{K_y(d,b)}{y}$$

in the interior. Continuity gives the nonsingular boundary points. The atoms may be discarded for the inequality but must be retained in any asserted identity for the target Poisson extension. □

The polynomial certificates in Theorem 3.4 establish (34) for every real spacing and for the entire closed q interval of each row, with the same signed coefficients as the source. Their rational compactification and Bernstein subdivisions are continuum sign proofs. Lemma 2.6 explains the additional analytic step from these two-variable boundary certificates to every possible other-zero height. Negative individual node coefficients cause no problem, because the boundary inequality is proved for their full signed sum before the positive extension is taken.

Proposition 2.7 (Current inequality at a simple highest zero)**.** *Let $z(t) = x(t) + iy(t)$ be a simple zero of H_t, with $y(t) > 0$, at a time when no zero has imaginary part exceeding $y(t)$. Let $q = y^2$. If a row functional satisfies* (35)*, then*

$$q' \le -2 - 2q\big[J(t,x) - O_{\mathrm{pair}}(q)\big]. \tag{37}$$

Proof. Proposition 3.1(i) of [3], valid when the other zeros may be complex or repeated, gives the local branch equation

$$z'(t) = 2 \sum_{\rho \neq z(t)} \frac{1}{z(t) - \rho},$$

with the symmetric summation prescribed there. Taking imaginary parts and pairing the other zeros with their conjugates is legitimate by Lemma 2.5. The own conjugate $x - iy$ contributes $2/(2iy) = -i/y$. All other conjugate pairs contribute $-2K_y(d,b)$, with real zeros assigned half weight. Hence

$$y' = -\frac{1}{y} - 2 \sum_{\text{other pairs}} K_y(d,b), \qquad q' = -2 - 4y \sum_{\text{other pairs}} K_y(d,b). \tag{38}$$

By Lemma 2.5, applying the signed probe functional gives

$$J(t,x) = O_{\mathrm{pair}}(q) + \sum_{\text{other pairs}} R(d,b).$$

The same other pairs, including their multiplicities and the half weight for real zeros, appear in both expressions. Apply (35) to each pair and sum absolutely to obtain

$$q' \le -2 - 2q \sum_{\text{other pairs}} R(d,b),$$

which is (37). The -2 is the own conjugate's exact motion contribution and is not counted again in the sum. □

2.4 Compact attainment and repeated nonreal contacts

A derivative formula for a single zero does not by itself control the supremum over an unbounded zero set. The compact-attainment and multiple-contact arguments are therefore supplied separately.

Lemma 2.8 (Uniform compactness after positive startup)**.** *For $0 < \tau < T \le 1/2$, all nonreal zeros of all H_t with $t \in [\tau, T]$ lie in a common compact rectangle.*

Proof. Theorem 1.5(i) of [3] has one sufficiently large absolute constant C, independent of $0 < t \le 1/2$, such that all zeros with $\operatorname{Re} z \ge \exp(C/t)$ are real. Symmetry handles the negative real tail. Thus possible nonreal zeros for $t \in [\tau, T]$ have $|\operatorname{Re} z| \le \exp(C/\tau)$. Classical strip contraction puts them in $|\operatorname{Im} z| \le 1$. The value of C need not be effective at the practical wall size: only existence of a common compact rectangle is used here. The theorem is not used to replace the exterior source proof by a numerical tail cutoff at X. □

The repeated-zero expansion is particularly transparent directly from the heat equation. If $H_{t_0}(z_0+w) = aw^m + O(w^{m+1})$, $a \neq 0$, then at the backward time $t_0 - \epsilon$ the heat scaling yields, uniformly for w in compact sets,

$$\frac{H_{t_0-\epsilon}(z_0+\sqrt{\epsilon}\,w)}{a\epsilon^{m/2}} \longrightarrow p_m(w) = m! \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{w^{m-2k}}{k!(m-2k)!}. \tag{39}$$

This is also a special case of [3, Proposition 3.1(ii)]. Writing He_m for the probabilists' Hermite polynomial,

$$p_m(w) = (i\sqrt{2})^m \,\mathrm{He}_m\left(\frac{w}{i\sqrt{2}}\right). \tag{40}$$

For $m \geq 2$, it has a simple root $i\sqrt{2}\,\lambda$ with $\lambda > 0$.

Lemma 2.9 (No repeated nonreal first contact)**.** *Let $Y(t) > 0$ be locally Lipschitz from the left at $t_0 > 0$. Suppose all zeros at earlier nearby times have imaginary part less than $Y(t)$, and $H_{t_0}(x + iY(t_0)) = 0$. Then this zero is simple.*

Proof. If its multiplicity were $m \geq 2$, Rouché's theorem applied to (39) near the positive imaginary root in (40) would give a zero of $H_{t_0-\epsilon}$ of the form

$$x + iY(t_0) + i\sqrt{2\epsilon}\,\lambda + O(\epsilon).$$

Its imaginary part is greater than $Y(t_0) + c\sqrt{\epsilon}$ for some $c > 0$ and all sufficiently small $\epsilon > 0$. But $Y(t_0 - \epsilon) = Y(t_0) + O(\epsilon)$. This contradicts the assumed earlier strict upper barrier. The argument is local at a complex zero; it does not require that this repeated zero be real. □

Theorem 2.10 (Global comparison from the row certificates)**.** *Let $0 < \tau < T < \min\{41/250, 1/2\}$. Suppose a positive upper barrier Q on $[\tau, T]$ consists of finitely many smooth closed incoming arcs, with only upward resets between successive arcs. On an arc beginning at a with initial ceiling q_a, suppose*

$$Q' = -2 - rQ, \qquad r > 0, \qquad 0 < Q \leq q_a \leq q_{\max}. \tag{41}$$

Suppose every actual highest contact in that arc has $|x| > X$, that the row source and kernel are valid on its complete closed time and q domains, and that

$$J(t,x) \geq B_a, \tag{42}$$

$$O_{\mathrm{pair}}(q) \leq O_{\mathrm{pair},a} \qquad (0 \leq q \leq q_a), \tag{43}$$

$$r < 2(B_a - O_{\mathrm{pair},a}). \tag{44}$$

Assume also a strict initial separation at τ, and exclusion of a violation during the preceding startup interval. Then every zero of H_t satisfies $(\operatorname{Im} z)^2 < Q(t)$ on the incoming arcs for $\tau \leq t \leq T$, and satisfies the corresponding outward ceiling at every reset.

Proof. Upward resets preserve the upper-barrier property, so it suffices to exclude a first contact on a closed incoming arc. The barrier is positive on every such arc. Hence its square root is Lipschitz there. By Lemma 2.8, all zeros capable of reaching this positive height lie in a common compact region. The zero sets vary locally continuously with time by Rouché's theorem. More explicitly, for any sequence of violation times decreasing to an infimum, their zeros have a convergent subsequence; continuity supplies a zero at the limiting time, and local persistence rules out a strict overshoot there. Strict startup separation and upward resets therefore force an attained first contact on the upper boundary. By symmetry it may be taken

at positive real part. Proposition 2.4 puts it at $x > X$. At this first contact every other zero is below the same upper height.

Lemma 2.9 excludes a repeated contact. At a simple contact Proposition 2.7 and (42)–(44) give

$$q' \le -2 - 2q(B_a - O_{\mathrm{pair},a}) < -2 - rq = Q'.$$

For a differentiable zero branch reaching $q - Q = 0$ for the first time from below, the incoming derivative of $q - Q$ is nonnegative. This contradicts the strict inequality. If the contact is at an arc endpoint or a row knot, its closed incoming arc supplies the same contradiction from the left. No differentiation of an unattained global supremum and no global simplicity assumption has been made. □

The exact numerical construction checks the hypotheses of Theorem 2.10 at every one of its 221010 arcs. The actual common-source theorem holds at every center $x \ge X$; its admitted translations obey $|a| \le 39/20 < 2$. Thus all translated probe coordinates at a contact $x > X$ are greater than $X - 2$ and stay within the theorem's physical domain. All probe circles and their complex padding are separately covered by the source bounds; compactness does not authorize any enlargement of them. On ordinary nodes and the three conic rows, the full own-pair rational function is proved nondecreasing on its whole contact interval, so the upper bound $O_{\mathrm{pair},a}$ may be evaluated at the outward q_a. Source lower chords are nondecreasing in time, so their value at the beginning of each small arc is a valid lower bound throughout that arc. These facts are used with the actual signed full functional, never with a sum of independently optimized flow rates.

2.5 The terminal global strip and the complete cleanup

Lemma 2.11 (Complete terminal cleanup)**.** *Suppose at a time T all zeros of H_T have imaginary part of modulus at most $\sqrt{q_e}$, where $q_e > 0$. Then*

$$\Lambda \le T + \frac{q_e}{2}. \tag{45}$$

No finite-head wall or exterior source estimate beyond T is required.

Proof. Apply Theorem 2.1 to the global strip at time T. At $T + q_e/2$ every zero is real. The defining characterization of the de Bruijn–Newman constant [1, 2] gives (45). The conclusion uses forward strip contraction only and makes no backward inference from a later all-real slice. □

For the certificate consumed here the endpoint data are the exact rationals

$$T = \frac{22101}{204800}, \qquad q_e = \frac{7902377963830536747158643}{792281625142643375935439503 36}. \tag{46}$$

Both the source horizon $19/125$ and the wall horizon $41/250$ are greater than T. The complete cost, including the positive terminal strip, is

$$T + \frac{q_e}{2} = \frac{427693072077942473661578098491}{396140812571321687967719751 6800} \tag{47}$$

$$< \frac{107964910079}{10^{12}} = 0.107964910079. \tag{48}$$

The exact-flow section proves the finite arithmetic leading to (46); this section establishes why those data control all zeros. Together with $\Lambda \ge 0$ from [5], the resulting interval is $0 \le \Lambda \le 0.107964910079$.

The analytic arguments above are ordinary mathematical proofs assembled with explicit interval and exact certificates. They are not claimed to be an end-to-end Lean formalisation of the Lambda bound. In particular, using a pinned external formal theorem for (14) does not turn the custom wall, source estimates, or contact argument into machine-checked Lean declarations. Their proofs and their finite certificate consumers are supplied separately for scrutiny.

3 The physical source and its exact finite certificates

The comparison uses a finite signed linear functional of logarithmic derivatives. Negative coefficients in that functional are essential: they must be combined before estimating the arithmetic head, whereas errors without a proved common representation must be charged with absolute coefficients. This section gives the analytic reductions behind the source file and the continuous kernel certificates. The finite rational data in the accompanying supplement are part of the proof. Theorem B.1 directly reconstructs the operational source for every row. Appendices C and D give the debit and credit derivations used in that reconstruction.

3.1 The functional, the certificate interface, and the data

Write

$$S_t(x,h) = -\operatorname{Im}\frac{H_t'}{H_t}(x+ih) = \partial_h \log|H_t(x+ih)|.$$

Here the prime on H_t denotes differentiation in its complex argument. For a node (h,a,w) with $a \geq 0$, the associated functional is

$$w\mathcal{A}_{h,a}S_t(x) := \frac{w}{2}\{S_t(x+a,h) + S_t(x-a,h)\}.$$

Thus a zero shift is one evaluation, not two evaluations. An ordinary row has

$$J_i(t,x) = \mathcal{L}_i S_t(x) := \sum_j w_{ij}\mathcal{A}_{h_{ij},a_{ij}}S_t(x). \tag{49}$$

The exact rational node lists, their closed time intervals, and their permitted squared contact-height intervals are the 67 records of `M8_LATE_PLAN.json`. These lists are not floating point proposals. The plan has SHA-256

`ec346b7c1136eda38021eb6323033b3b48ed21e6f7458eeeac7cc9579db2497a.`

Rows $0, 16, 44$ additionally contain a joint derivative functional, defined in Section 3.7; every other row is of the form (49).

The source file `MAIN_CLOCK_SOURCE.json` provides four operational rational numbers for each row:

$$M_i = \texttt{main_lo}, \quad N_i = \texttt{native_debit}, \quad A_i = \texttt{first_hi}, \quad B_i = \texttt{last_hi}.$$

If its time interval is $[t_i, t_{i+1}]$, put

$$\underline{J}_i(t) = M_i - N_i - (1-\theta)A_i - \theta B_i, \qquad \theta = \frac{t-t_i}{t_{i+1}-t_i}. \tag{50}$$

Historical fields in the source file are not alternative decompositions of this expression. In particular an earlier lower endpoint for an arithmetic head is not paired with a subsequently reduced upper debit.

Theorem 3.1 (Uniform physical source)**.** *For each declared row, throughout its entire closed time interval and for every actual central coordinate*

$$x \geq X := 6000000185827,$$

the physical functional specified by that row obeys

$$J_i(t,x) \geq \underline{J}_i(t).$$

All values and derivative evaluations in this assertion are of the same function H_t, with its actual physical phase, native cutoff, and normaliser. The assertion uses no hypothesis about an unbounded set of zeta zeros and no additional initial-strip assumption.

The proof is distributed below into the physical approximation, coefficient identities, complete analytic and physical charges, and finite phase/resource certificates. The final source file is bound by SHA-256

`b59905d7b994d3424ba14f9347be75a2429ae4d4b0de2c2404971ffda1d4d536.`

Its relation to the pinned plan is checked as an identity of the entire JSON objects, including each unchanged row, rather than inferred from matching displayed decimal values.

3.2 One finite analytic model at all the translations

Fix $x \geq X$ and $t \geq 0$. The notation in this subsection is local to this fixed physical center. Put

$$L = \log\frac{x}{4\pi}, \qquad \tau = \frac{x}{2} + \frac{\pi t}{8}, \qquad N = \left\lfloor \sqrt{e^L + t/16} \right\rfloor, \qquad b_t(n) = \exp\left(\frac{t}{4}\log^2 n\right).$$

For complex z near $x + i[0,6]$, set

$$a(z) = \frac{1 - iz}{2}, \qquad b(z) = 1 - a(z),$$

and, on the appropriate half-plane logarithm branches, define

$$\begin{aligned} M_0(s) &= \frac{s(s-1)}{16}\pi^{-s/2}\sqrt{2\pi}(s/2)^{s/2-1/2}e^{-s/2}, \\ \alpha(s) &= \frac{1}{2s} + \frac{1}{s-1} + \frac{1}{2}\operatorname{Log}\frac{s}{2\pi}, \\ M_t(s) &= M_0(s)\exp(t\alpha(s)^2/4), \qquad\qquad B_t(z) = M_t(a(z)). \end{aligned}$$

These functions are analytic and nonzero on every source disk used here. The logarithmic derivative of M_t is

$$\beta_t(s) = \alpha(s) + \frac{t}{2}\alpha(s)\alpha'(s).$$

Let

$$\Gamma = \frac{M_t((1+ix)/2)}{M_t((1-ix)/2)}.$$

In particular $|\Gamma| = 1$; it is determined by x, t and is not an independent phase. For $\zeta = h - iv$ the common finite model is

$$\begin{aligned} f_x(\zeta) = &\sum_{n\leq N} b_t(n) n^{-1/2-tL/4+i\tau} e^{-\zeta\log n/2} \\ &+ \Gamma \sum_{n\leq N} b_t(n) n^{-1/2-tL/4-i\tau} e^{-\zeta(L-\log n)/2}. \end{aligned} \tag{51}$$

All of N, L, τ, Γ in this formula are frozen at the central coordinate. The physical normaliser is still evaluated at the actual point $x + i\zeta = x + v + ih$. This distinction makes it possible to apply one holomorphic error bound or one spectral measure to multiple translations.

Write

$$\ell_* = \frac{2689}{100}, \qquad c_* = \frac{2689}{400}, \qquad X_* = X - 4.$$

The elementary anchor inequalities are

$$26.89 < \log \frac{X_*}{4\pi} < 26.90, \qquad \log \frac{X_*}{4\pi} > 2 \log 690988.$$

The latter inequality permits all subsequent comparisons with $L_0 = 2 \log N_0$, $N_0 = 690988$.

Proposition 3.2 (Complete common-model physical errors)**.** *Suppose a closed thermal row* $[t_-, t_+] \subset [0, 19/125]$ *and a native height* h *satisfy*

$$|v| \le \frac{39}{20}, \quad h - \frac{1}{100} \ge 0, \quad h + \frac{1}{100} \le \frac{601}{100}, \quad \frac{1 + h - 1/100}{2} + \frac{t_- \ell_*}{8} > \frac{5}{4}. \tag{52}$$

For every time in the row and all physical centers $x \ge X$*,*

$$\left| \frac{H_t(x + v + ih)}{B_t(x + v + ih)} - f_x(h - iv) \right| \le E_0(h, t_-), \tag{53}$$

$$\left| \partial_h \left(\frac{H_t}{B_t} \right)(x + v + ih) - f_x'(h - iv) \right| \le E_1(h, t_-), \tag{54}$$

where

$$\begin{aligned} \mathcal{S} &= (1 + h - 1/100)c_* + t_- c_*^2, \\ E_0(h, t_-) &= 8e^{-\mathcal{S}} + 5 \cdot 10^{-10}, \\ E_1(h, t_-) &= 800e^{-\mathcal{S}} + 7 \cdot 10^{-9}. \end{aligned} \tag{55}$$

The baseline bounds $e_0 = 5 \cdot 10^{-7}$ *and* $e_1 = 5 \cdot 10^{-5}$ *also hold on the same row domains. Every moving-cutoff term and both complete Riemann–Siegel remainders are included.*

Proof. The following estimates are especially material to the larger-height extension; the complete finite scalar inequalities and their exact endpoints are in the physical-source supplement. The individual heated term estimate is [3, Proposition 6.1]. Its use is valid for all real parts, before the narrower-height simplification in the printed numerical theorem. The two real parts needed here lie in $[-3, 4]$.

The full Gaussian calculation must use the constant $11/12$. Indeed, with $D = T - 3$ and real u,

$$\frac{u^2 + 6|u| + 2/3}{4D} \le \frac{u^2 + 11/12}{D},$$

because, after multiplication by $4D$, the difference is $3(|u| - 1)^2$. Completing the square therefore gives the exact majorant

$$G_\sigma(c) = \frac{\exp\left\{ \frac{11}{12D} + \frac{\sigma^2/D + \sigma c + tc^2/4}{1 - t/D} \right\}}{\sqrt{1 - t/D}}. \tag{56}$$

Convexity in σ and endpoint comparison on $[-3, 4]$ give

$$G_4(\log 9) < 8104, \qquad G_{-3}(-\log 2) < 8.5$$

throughout the stated time and coordinate domain. In particular, with the deliberately smaller native lower bound 690000, the complete remainder bracket is bounded by

$$\frac{1}{2}G_4(0)+\frac{0.2}{690000-.865}G_4(\log 9)+\frac{.029}{690000-.353}G_{-3}(-\log 2)+10^{-100}<1.$$

The omitted negative Gaussian range in the intermediate expansion is not discarded: for its variable $r\geq 100$ its complete integrand is bounded by

$$e^{-4r^2+40r+20+(r^2+1)/D}\leq e^{-3r^2},\qquad \int_{100}^{\infty}e^{-3r^2}\,\mathrm{d}r\leq e^{-30000}/600.$$

The extra 1/12 relative to the erroneous 5/6 value in an earlier source note is paid in (56); that earlier value is not used. Consequently each complete remainder is bounded by $e^{t\pi^2/64}|M_0(iT')|$ at the corresponding shifted ordinate.

For $|\operatorname{Re}s|\leq 4$ and $|\operatorname{Im}s|=x'/2$, direct estimates in the displayed formula for α give

$$|\alpha(s)-(L'/2\pm i\pi/4)|<10/x',\quad |\alpha'(s)|<2/x',\quad |\alpha(s)|<L'/2+1.$$

After division by the actual B_t, both remainders together are thus bounded by

$$e_C<4\exp\left(-\frac{(1+y)L'}{4}-\frac{t(L')^2}{16}+\frac{1}{10}\right). \tag{57}$$

The 1/10 includes the positive correction $t\pi^2/32$ and all coordinate and α corrections on this source domain.

The full first and reflected native sums have, per branch, zeroth, first-logarithmic, and second-logarithmic moments at most $6, 22, 180$; the deliberately enlarged total value mass bound is 18. For example the uninflated majorants are supplied by $\sum n^{-5/4}$ and its first two logarithmic moments, followed by the row guard (52). The complete individual term errors from Proposition 6.1 of [3] total less than $e_{AB}=5.5\cdot 10^{-12}$.

Across a translated disk of radius 1/100, every real displacement from the central coordinate is less than 2. Consecutive activation coordinates are $4\pi(m^2-t/16)$ and have spacing $4\pi(2m+1)>4$. At most one native pair can change. For a changed index the exact inequality

$$\left|\log m-\frac{L'}{2}\right|<10^{-12}$$

holds on the whole disk. Expanding the terminal thermal exponent before bounding its linear term gives the complete crossing-pair cost $3e^{10^{-5}}e^{-\mathcal{S}}$. This covers both directions of crossing and both native branches. The normaliser and fixed-index comparison are holomorphic on the disk, so Cauchy's formula gives a factor 100 for its height derivative. In view of

$$4e^{.1}+3e^{10^{-5}}<8,$$

this accounts for the first terms in (55).

For completeness, the passage from exact local α exponents to the local leading model has value error at most

$$\frac{440t+12yC_\gamma}{x'},\qquad C_\gamma=10+t(L'/2+1),$$

and derivative error at most

$$\frac{yC_\gamma(6L'+22)+5t(22L'+360)+12C_\gamma+44t}{x'}.$$

These are respectively less than $2 \cdot 10^{-10}$ and $2.5 \cdot 10^{-9}$ at the worst anchor, and decrease at larger x'. Finally freezing the common gamma, clock, and frequency gives, with

$$B_{\mathrm{fr}} = \frac{30 + 2t(L+3)}{x-2},$$

the separate charges

$$36B_{\mathrm{fr}} < 2.35 \cdot 10^{-10}, \qquad 18B_{\mathrm{fr}}(L+1) + \frac{36}{x-2} < 3.3 \cdot 10^{-9}.$$

They include the change of reflected frequency as well as the common phase. Adding these costs and e_{AB} proves (53)–(55). Applying the same complete calculation without the height/time refinement gives value error less than $4.167459 \cdot 10^{-7}$ and derivative error less than $4.637616 \cdot 10^{-5}$, so the stated baseline allowances also hold. All endpoint inequalities used here are directly outward-verified by the supplied corrected physical-source checkers. □

3.3 Finite thermal coefficients and exact matching

Let D denote the formal Dirichlet derivation $(Da)(n) = (\log n)a(n)$. Formal multiplication, logarithms, and inverses are coefficientwise: the coefficient at any fixed integer uses finitely many divisors. Define

$$d_t = D \log_* b_t, \qquad E_t(s) = \prod_{p \in \{2,3,5\}} (1 - b_t(p)p^{-s}), \qquad c_t = m_t * b_t,$$

where m_t is the eight-term coefficient sequence of E_t, and put $\widetilde{d}_t = D \log_* c_t$. No positive-time infinite numerical Dirichlet series with coefficients $b_t(n)$ is used.

Lemma 3.3 (Coefficient positivity and recurrences)**.** *For $n \geq 2$ and $t \geq 0$, the coefficients $d_t(n), \widetilde{d}_t(n), c_t(n)$ are nonnegative and nondecreasing in t. Their exact triangular recurrences are*

$$d_t(n) = (\log n)b_t(n) - \sum_{\substack{r \mid n \\ 2 \leq r < n}} d_t(r)b_t(n/r), \tag{58}$$

$$(\log n)c_t(n) = \sum_{\substack{r \mid n \\ r \geq 2}} \widetilde{d}_t(r)c_t(n/r), \tag{59}$$

$$\widetilde{d}_t(n) = d_t(n) - \sum_{\substack{p \in \{2,3,5\},\ j \geq 1 \\ n = p^j}} (\log p)e^{jt \log^2 p/4}. \tag{60}$$

Proof. Formal heat differentiation and the derivation rule give

$$\partial_t d_t(n) = \frac{\log^2 n}{4} d_t(n) + \frac{\log n}{4} \sum_{ab=n,\ a,b \geq 2} d_t(a)d_t(b).$$

At time zero these are von Mangoldt coefficients. Induction on n and the integrating-factor formula prove nonnegativity and monotonicity. In (60), at $n = p^j$ the subtracted term has derivative $j(\log^2 p)/4$ times itself, whereas the first linear coefficient in the equation for $d_t(p^j)$ is $j^2(\log^2 p)/4$. The extra forcing is therefore nonnegative, proportional to $j(j-1)$. The same induction applies to $\widetilde{d}_t$, whose initial coefficients are nonnegative. Equation (59) then proves positivity and monotonicity of c_t. The triangular identities follow from $Da = (D \log_* a) * a$. □

At the fixed physical center introduce the finite Euler multiplier

$$E_x(\zeta) = \prod_{p\in\{2,3,5\}} \left(1 - b_t(p)p^{-1/2-tL/4+i\tau}e^{-\zeta\log p/2}\right), \qquad Q = E_x f_x = 1 + U, \quad Q' = -V/2.$$

The first branch is collected only through N_0. Every product label above N_0, reflected label, and newly activated index is retained separately, including its sign. Their absolute amplitudes and their frequency-weighted amplitudes form finite positive majorants. All these frequencies are nonnegative. Reflected labels have frequency $L - \log n + \log d$ for $d \mid 30$.

The matched logarithmic-derivative polynomial is

$$T_K(\zeta) = \sum_{2\le k\le K} \widetilde{d}_t(k)k^{-1/2-tL/4+i\tau}e^{-\zeta\log k/2}, \qquad K = 4096.$$

For the collected first head, the coefficient remaining after matching QT_K against V is exactly

$$(\log n)c_t(n) - \sum_{\substack{k\mid n\\ 2\le k\le K}} \widetilde{d}_t(k)c_t(n/k) = \sum_{\substack{k\mid n\\ k>K}} \widetilde{d}_t(k)c_t(n/k) \ge 0. \tag{61}$$

The physical integer phases multiply correctly in this identity; no independently selectable phase is assigned to a divisor product.

At a native height, let A, B denote the complete nonconstant amplitude and frequency moment of the fixed head and its retained exterior labels. Let D_K be the positive norm of T_K and let Z_K be its overlap with the nonconstant collected head:

$$Z_K = \sum_{2\le k\le K} \widetilde{d}_t(k)k^{-\sigma} \sum_{2\le m\le\lfloor N_0/k\rfloor} c_t(m)m^{-\sigma}, \qquad \sigma = \frac{1+h}{2} + \frac{tL}{4}.$$

The full positive residual decomposition gives the bound

$$|V - QT_K| \le R_* = B - (1 - A)D_K - 2Z_K \tag{62}$$

before activation is added. This difference is nonnegative because it equals a sum of the residual coefficients (61), overflow products, and absolute exterior and reflected labels. It is not the difference of unrelated monotone estimates. The complete prefixes through $\lfloor N_0/k\rfloor$ are used in Z_K; they are not truncated to K.

For a block, coefficient monotonicity permits the artificial common comparison point with thermal parameter t_+ and abscissa $(1 + h)/2 + t_-L_0/4$. All matching masses in (62) are evaluated at that same point. Upper bounds for A, B and lower bounds for D_K, Z_K then give an upper bound for R_*, provided $A < 1$. Beyond L_0, the actual positive residual decreases term by term. For example a reflected weighted label obeys

$$\frac{\mathrm{d}}{\mathrm{d}L}(\lambda w) = w\left[1 - \lambda\left(\frac{h}{2} + \frac{t}{4}\log(dn)\right)\right] \le 0,$$

since $\lambda \ge \log N_0$ and $h\log N_0/2 > 1$ on every row. This proves frequency uniformity after exact matching, without applying an anchor lower mass at an arbitrary larger frequency.

3.4 All future cutoffs and the retained main slope

Put $v_0 = \log N_0$ and write $L = 2(v_0 + s)$, $\log n = v_0 + w$ for a newly active index $N_0 < n \le N$. Then

$$0 < w \le s + \delta, \qquad \delta = 10^{-12}.$$

The small extension is required by $N = \lfloor\sqrt{e^L + t/16}\rfloor$. After inserting counting measure, the two positive densities are bounded by

$$C_0 e^{-a_0 s} e^{-dw}, \qquad C_0 e^{-(a_0+h)s} e^{pw},$$

where

$$a_0 = t_- v_0/2, \quad d = (h-1)/2 > 0, \quad p = (h+1)/2, \quad C_0 = e^{-t_- v_0^2/4 - dv_0}.$$

The dropped thermal factor is at most $e^{t\delta^2/4}$. The full adjacent-integer and reflected-growth inflation is less than 10^{-4} in logarithm; the saved prefactors pay it before any integral comparison. In particular the frequency moment includes the derivative of the Euler factor, not just the derivative of a native term.

On a finite near interval it is convenient to use positive rational endpoints $v_- < v_0 < v_+$ and an upper constant $C \geq e^{-t_- v_-^2/4 - dv_- + 1/5000}$. For $u = s + \delta$, integration over $0 \leq w \leq u$ gives the simple complete bounds

$$A_{\mathrm{act}}(s) \leq 2Cu, \qquad D_{\mathrm{act}}(s) \leq C(v_+ + 2u)u. \tag{63}$$

If P, P_1 are the positive Euler norm and log moment, set

$$a_a(s) = 2PC(s+\delta), \qquad b_a(s) = 2PC(v_+ + 2(s+\delta))(s+\delta) + 2P_1 C(s+\delta).$$

These are polynomials with nonnegative coefficients. The full analytic and physical charges at that height are bounded by

$$E_{\mathrm{an}}(s) = \frac{1}{2}\left(\frac{R_* + b_a(s) + a_a(s)D^+}{g(s)} + T\right), \tag{64}$$

$$E_{\mathrm{ph}}(s) = \frac{r_1}{g(s) - r_0} + \frac{(B + b_a(s))r_0}{2g(s)(g(s) - r_0)} + 2 \cdot 10^{-12}, \tag{65}$$

$$g(s) = 1 - A - a_a(s), \qquad r_0 = e_0 P, \quad r_1 = e_1 P + (e_0/2)P_1.$$

Here T is the complete omitted geometric Euler-power tail. The strict condition $g > r_0$ precedes division. Formula (65) is precisely the difference-of-quotients bound, with $|Q| \geq g$ and $|Q'| \leq (B + b_a)/2$.

Both charges are increasing and convex as long as that gap remains positive: expand their reciprocal denominators in geometric power series in s, whose coefficients are nonnegative. For a row choose one fixed analytic spectral recombination, add absolute-weighted physical charges and the fixed coordinate allowance, and call the result $N(s)$. The same fixed choice is used throughout the near interval. It is not asserted that a pointwise minimum of convex caps is convex.

For the complementary half-line one needs global activation constants that remain valid when the near denominator is no longer positive. Appendix A proves the constants actually consumed by the source certificates. With $d = (h-1)/2$, $p = (h+1)/2$, $a = t_- V_-/2$, $k = a + d$, $c = a + h$, $U = 1/d - 1/p$, and the complete integer/end-point factor C defined there, the combined branch integrals are

$$A(s) = \frac{e^{-as}}{d} - Ue^{-ks} - \frac{e^{-cs}}{p}, \tag{66}$$

$$D(s) = A_a e^{-as} - (A_k + B_k s)e^{-ks} - (A_c + B_c s)e^{-cs}. \tag{67}$$

The rational coefficients are specified in the appendix. Factored derivative monotonicity proves a unique maximum on $s \geq 0$ when $a > 0$ and gives the exact limiting supremum when $a = 0$. Outward evaluation on rational critical-point enclosures proves, at every actual row and height,

$$C \sup_{s \geq 0} A(s) \leq \overline{A}_{\mathrm{act}}, \qquad C \sup_{s \geq 0} D(s) \leq \overline{D}_{\mathrm{act}},$$

where the right sides are the actual stored rational bounds. In particular the two native branches are combined before maximizing. At $t_- = 0, h = 2$ this gives $\sup A = 2$, rather than the weaker sum of two separate suprema. These inequalities cover every future finite activation set on the entire unbounded clock domain.

After multiplying by the Euler factor put

$$\bar{a} = P\overline{A}_{\mathrm{act}}, \qquad \bar{b} = 2P\overline{D}_{\mathrm{act}} + P_1\overline{A}_{\mathrm{act}}, \qquad \bar{g} = 1 - A - \bar{a}.$$

The complete global analytic and physical caps at that height are

$$\overline{E}_{\mathrm{an}} = \frac{1}{2}\left(\frac{R_* + \bar{b} + \bar{a}D^+}{\bar{g}} + T\right), \tag{68}$$

$$\overline{E}_{\mathrm{ph}} = \frac{r_1}{\bar{g} - r_0} + \frac{(B + \bar{b})r_0}{2\bar{g}(\bar{g} - r_0)} + 2 \cdot 10^{-12}, \qquad \bar{g} > r_0. \tag{69}$$

The gap is explicitly checked; it is not inferred from a favorable near-s gap. In particular the Euler derivative term $P_1\overline{A}_{\mathrm{act}}$ is present. At the scalar-parent row level, one valid global debit is the absolute-weight sum of these analytic and physical caps plus the fixed coordinate debit. A valid shared spectral envelope from Section 3.5 can instead replace that analytic sum by its certified nonnegative combination of the global analytic moments, while retaining the same absolute physical sum and coordinate charge. Either fixed branch proves a constant debit for all $s \geq 0$; taking the smaller of two proved constants is also safe. The directed consumers apply these formulas to the sharp activation constants just proved, including the full Euler derivative moment. Conic rows inherit the complete scalar parent bound with their stated positive multiplier and pay their jet component separately, as in Section 3.7.

Let $M > 0$ be the signed node mass. The finite source certificates give a rational $0 < \beta < 1$ and $R > 0$ such that

$$N(R) \leq N(0) + \frac{\beta MR}{2}, \qquad G \leq N(0) + \frac{\beta MR}{2}.$$

Convexity for $s \leq R$ and the global cap for $s \geq R$ prove

$$N_{\mathrm{actual}}(L) \leq N(0) + \frac{\beta M}{4}(L - L_0), \qquad L \geq L_0. \tag{70}$$

Subtracting from $ML/4$ leaves the unspent main slope $(1 - \beta)M(L - L_0)/4$. This is how infinitely many possible future activation states are covered: each is still a finite native sum, and its entire added mass is dominated by the positive integral estimates. No infinite heat-weighted Dirichlet series has been introduced.

The refined errors of Proposition 3.2 may safely refund part of the physical cap. If $E_0 \leq \kappa e_0$ and $E_1 \leq \kappa e_1$ with $0 \leq \kappa \leq 1$, then

$$\frac{\kappa r_1}{g - \kappa r_0} + \frac{W\kappa r_0}{2g(g - \kappa r_0)} \leq \kappa\left\{\frac{r_1}{g - r_0} + \frac{Wr_0}{2g(g - r_0)}\right\}.$$

The fixed $2 \cdot 10^{-12}$ and coordinate charges are retained. For an affine row this refund is taken from the anchor cap and is valid on both its near and far branch. It changes neither β nor its paid slope. The supplied consumer rebuilds all 469 native-height caps and all 67 row refunds from their rational primitives.

3.5 One common positive-spectrum analytic remainder

The preceding scalar analytic caps are moments of one measure, which allows the signed translations to be combined more efficiently. At the fixed actual x, t, N, let ν assign absolute U-label coefficients to their frequencies. Let ρ assign the nonnegative matched head residual coefficients of (61), and absolute unmatched exterior, reflected, and activated residual coefficients, to their frequencies. Both are positive atomic measures attached to the same finite analytic model.

At the lowest licensed native height h_0, $\int e^{-h_0\lambda/2}\,\mathrm{d}\nu < 1$. The reciprocal of $Q = 1 + U$ therefore has an absolutely convergent geometric expansion in the weighted total-variation norm. Including the positive measure ϵ of omitted Euler prime powers gives the complete majorant

$$\mu_{x,t} = \frac{1}{2}\rho * \sum_{m\geq 0} \nu^{*m} + \frac{1}{2}\epsilon. \tag{71}$$

The analytic logarithmic-derivative remainder is the Laplace transform of a complex measure dominated by $\mu_{x,t}$. Every residual frequency is at least $\log 4097$, so the same is true after convolution. All seven native moment bounds for the row thus concern this one measure:

$$\int e^{-h_l\lambda/2}\,\mathrm{d}\mu_{x,t}(\lambda) \leq r_l = \frac{1}{2}\left(\frac{R_l}{g_l} + T_l\right).$$

Only the reciprocal of a finite function with a proved gap and convergent finite-prime geometric tails are summed infinitely here.

For the signed scalar functional define

$$F_i(\lambda) = \sum_j w_{ij} e^{-h_{ij}\lambda/2}\cos(a_{ij}\lambda/2).$$

Any nonnegative rational numbers c_l satisfying

$$|F_i(\lambda)| \leq \sum_l c_l e^{-h_l\lambda/2} \quad \text{for every } \lambda \geq \log 4096 \tag{72}$$

give combined analytic charge at most $\sum_l c_l r_l$. The slightly larger half-line in (72) is safe. Physical approximation errors are not put into this measure; their quotient charges continue to have absolute node weights.

For clarity, the continuous proof of the finite envelope is as follows. Factor out its lowest exponential and form, for both signs,

$$D_\pm(\lambda) = \sum_l c_l e^{-\beta_l\lambda} \pm \sum_j w_{ij} e^{-\alpha_j\lambda}\cos(\delta_j\lambda).$$

On a rational interval of midpoint m and radius r, compute outward enclosures for $D_\pm(m), D'_\pm(m)$ and an upper bound B_2 for $|D''_\pm|$ throughout it. The sufficient check is

$$D_\pm(m)_{\mathrm{lo}} - r|D'_\pm(m)|_{\mathrm{hi}} - r^2 B_2/2 \geq 0.$$

Exact endpoint chaining proves a complete interval cover. Beyond its last endpoint, retain the nondecaying positive envelope coefficient and bound every decaying numerator exponential at that endpoint. This proves the entire tail. Thus the envelope certificates are continuous inequalities in the spectral variable, not samples of a physical orbit.

The finite thermal head is handled separately. Its exact signed filter is

$$F_i(n) = \sum_j w_{ij} n^{-h_{ij}/2}\cos(a_{ij}\log n/2),$$

and its absolute charge is

$$\frac{1}{2}\sum_{2\le n\le 4096} d_t(n)n^{-1/2-tL/4}|F_i(n)|. \tag{73}$$

The signed node sum is formed before its modulus at each integer. For the basic chord one first replaces $d_t(n)$ by $d_{t_+}(n)$ and $L/4$ by $c_0 = L_0/4$. Convexity of the positive exponential gives, for $\theta = (t-t_-)/(t_+-t_-)$,

$$d_t(n)n^{-tc_0} \le d_{t_+}(n)\{(1-\theta)n^{-t_-c_0} + \theta n^{-t_+c_0}\}.$$

After multiplication by $n^{-1/2}|F_i(n)|/2$ and summation, the two complete endpoint costs therefore bound the whole thermal cell. The endpoint certificates round the first of these costs upwards to its first head field and the second upwards to its last head field. Any further retained time majorant is checked on its complete declared cell. Thus (50) is a proved chord, not interpolation of sampled lower bounds. Later credits described next reduce this same complete charge, with an explicit disjoint resource ledger.

3.6 Arithmetic resource allocation and one common physical clock

This subsection describes the 39 retained-source improvements entering the final source file. It is also useful for distinguishing a genuine all-center certificate from a numerical calculation at one center. Let

$$c = L/4, \quad c_0 = \tfrac{1}{2}\log N_0, \quad \Delta = c - c_0 \ge 0.$$

After the native slope in (70) has been paid, put $A = (1-\beta)M > 0$. Exact finite coefficient allocation gives simultaneously, in its declared near resource domain,

$$J_i \ge B_i^{\mathrm{comp}}(t) + A\Delta + \sum_p \alpha_p(1-\cos\phi_p), \qquad \alpha_p \ge 0, \tag{74}$$

and globally $J_i \ge B_i^{\mathrm{comp}}(t) + A\Delta$. Here the composite credit is already in B_i^{comp}. For $n = \prod p^{e_p}$ an allocated coefficient r_n expends $r_n\Omega(n)e_p$ of prime p's deficit capacity. Subtracting every such expenditure from the original prime capacity defines the remaining α_p. This finite accounting proves that the composite credit and the displayed prime deficits coexist; the capacities are never reset after a credit is taken. Their lower bounds use the lower thermal endpoint and attenuation at $L = 27$, so they remain valid throughout the licensed near domain. The composite allocation spends none of the retained main slope A.

For each integer exponent relation $R_j = \prod_p p^{k_{jp}}$, let

$$a_j = 2N_0^2\log R_j, \qquad v = e^{4\Delta} - 1, \qquad q_{\mathrm{clk}} = \frac{t}{16N_0^2}.$$

The gauged phases and an authenticated integer h_j satisfy the exact identity

$$\frac{k_j\cdot\phi}{\pi} = 2m_j + 1 + \delta_j(v, q_{\mathrm{clk}}), \qquad \delta_j = a_j - h_j + a_j(v + q_{\mathrm{clk}}). \tag{75}$$

The odd parity includes the actual gauges of all primes in the relation. It follows from the physical clock $\tau = \pi(2e^L + t/8)$, rather than being imposed on a relaxed independent phase vector. If $|\delta_j| < 1$, the true sign $\sigma_j = \mathrm{sgn}(k_j\cdot\phi)$ obeys

$$\sigma_j k_j\cdot\phi \ge \pi(1+\sigma_j\delta_j).$$

For fixed nonnegative rational multipliers λ_j, set $b_p = \sum_j \lambda_j \sigma_j k_{jp}$ and use the exact principal-angle conjugate

$$C^*(\alpha, b) = \max_{-\pi \le u \le \pi} \{bu - \alpha(1 - \cos u)\}.$$

Then the sum of prime deficits is bounded below by

$$\pi \sum_j \lambda_j (1 + \sigma_j \delta_j) - \sum_p C^*(\alpha_p, b_p).$$

The maximum defining each conjugate includes both principal endpoints and every interior stationary maximum. In particular it is not replaced by a local quadratic near a preferred angle.

Choose a total desired angular/main credit $d > 0$ and put

$$\omega = 4d/A < 1, \qquad V = \frac{\omega}{1 - \omega}.$$

For every one of the full sign branches certify $|\delta_j| < 1$ on the rectangle

$$0 \le v \le V, \qquad \frac{t_-}{16N_0^2} \le q_{\mathrm{clk}} \le \frac{t_+}{16N_0^2},$$

and verify $L_0 + \log(1 + V) < 27$. The whole lower payment is

$$\mathcal{F}(v, q_{\mathrm{clk}}) = \pi \sum_j \lambda_j (1 + \sigma_j \delta_j(v, q_{\mathrm{clk}})) - \sum_p C^*(\alpha_p, b_p) + \frac{A}{4} \log(1 + v). \tag{76}$$

For fixed multipliers it is jointly concave on this rectangle: its phase part is affine and its only nonlinear term is the displayed positive multiple of $\log(1 + v)$. Consequently its minimum is at least the minimum of its four corner values. The same physical v, q_{clk} is used in every relation before evaluation; separate worst clocks for individual relations are not substituted.

If all four values are at least d in every sign branch, then $J_i \ge B_i^{\mathrm{comp}} + d$ in the near interval $0 \le \Delta \le d/A$, since $e^{4\Delta} - 1 \le \omega/(1 - \omega)$. For the complementary far interval $\Delta \ge d/A$, the global phase-free main bound gives $A\Delta \ge d$. The two closed domains cover every physical center and spend main growth once.

For the seven-relation rows all 128 signs are checked. Row 8 uses an eighth relation and all 256 signs. Its new exact relation is

$$\frac{3^4 \cdot 5 \cdot 11 \cdot 19^2 \cdot 23 \cdot 47 \cdot 71}{2^4 \cdot 7^5 \cdot 13 \cdot 17 \cdot 31 \cdot 67} = \frac{123435179505}{123435179504}, \qquad h_8 = 7.$$

Its total new credit is $11043047769/10^{12}$, replacing the original $8143047769/10^{12}$; the source improvement is therefore $29/10000$. The exact remaining 38 operational decrements are in the supplied delta table. They are applied relative to their pinned common parent, not added to intermediate improvements for the same prime resources. In particular row 55's earlier $5 \cdot 10^{-6}$ native-zero credit is removed when its pure-source credit replaces it. Mixed rows $0, 16$ multiply their old-component gain by their positive conic coefficient; their separately charged derivative component is untouched.

3.7 Joint derivative rows with one physical logarithm

For a real center $p > 0$ and rational coefficients $b_0, \dots, b_m$, define

$$T_b S_t(x, p) = \frac{1}{p} \sum_{r=0}^{m} \frac{b_r p^r}{r!} \partial_p^r S_t(x, p), \qquad P_b(z) = \sum_{r=0}^{m} \frac{b_r z^r}{r!}. \tag{77}$$

The three derivative rows have the form

$$J_i = \alpha_i \mathcal{L}_i^{\mathrm{old}} S_t + 2T_{b_i} S_t(x, p_i), \qquad \alpha_i \geq 0, \quad b_{i,0} \geq 0.$$

The exact coefficient vectors are included in the plan. Their orders are $4, 4, 8$ for rows $0, 16, 44$ respectively; $p_0 = p_{16} = 61/20$ and $p_{44} = 4$. A single summed kernel, not an addition of separately proved rates, allocates the available zero-motion force.

Let $G_x(\zeta) = H_t(x + i\zeta)/B_t(x + i\zeta)$, and choose a licensed circle $|\zeta - p| \leq r$ above a native height h_0. The same finite Euler gap supplies $|f_x| \geq g/P$ throughout it. Hence

$$W = (G_x - f_x)/f_x, \qquad |W| \leq \delta < 1$$

defines one holomorphic logarithm $\log(1 + W)$ on the full disk, bounded by $\eta = \delta/(1 - \delta)$. It is this logarithm that supplies all physical derivative errors. Cauchy's formula gives the safe joint charge

$$|T_b(\log(1 + W))'(p)| \leq \frac{\eta}{pr} \sum_{r'=0}^{m} (r' + 1)|b_{r'}|(p/r)^{r'}. \tag{78}$$

The earlier Parseval version with the square root of the corresponding sum of squares is also valid where its certificate uses it. Neither formula assigns independent errors to the derivatives.

The baseline quotient ledger implies $e_0 P/g \leq E_{\mathrm{ph,old}}(h_0)/100$. Proposition 3.2 therefore permits

$$\delta \leq \min\{1, E_0(h_0, t_-)/e_0\} \cdot E_{\mathrm{ph,old}}(h_0)/100. \tag{79}$$

Indeed the direct new bound is $|(G_x - f_x)/f_x| \leq E_0 P/g$; no extra factor of P is inserted. No E_1 estimate is needed for this value-log route. The normaliser is still present: on all these circles

$$|\partial_\zeta \log B_t(x + i\zeta) - (L/4 - i\pi/8)| < 10^{-10}.$$

Its separately paid charge is $(10^{-10}/p) \sum_{r'} |b_{r'}|(p/r)^{r'}$.

For the analytic remainder, use the same measure (71). If $h_0 < p$ and $a = (p - h_0)/2$, its complete charge is bounded by

$$\frac{E_{\mathrm{an}}(h_0)}{p} \sup_{\lambda \geq \log 4097} |P_b(-p\lambda/2)|e^{-a\lambda}. \tag{80}$$

A monomial-by-monomial bound follows from the maximum of $\lambda^{r'} e^{-a\lambda}$ at $\max(\log 4097, r'/a)$. The selected row-44 bound instead keeps the whole polynomial before absolute value. It evaluates its endpoint and every zero on the half-line of

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} P_b(-p\lambda/2) - aP_b(-p\lambda/2).$$

An exact rational Sturm sequence proves that the disjoint root isolators exhaust the half-line; directed interval evaluation bounds each critical value. The endpoint is rounded down below $\log 4097$, which enlarges the domain, and the value tends to zero at infinity. Zeros of the polynomial itself contribute zero. This proves the full semi-infinite envelope, with no sampled-maximum assumption.

The finite derivative head is charged as

$$\frac{1}{2p} \sum_{2 \leq n \leq 4096} d_{t_+}(n) n^{-1/2 - t_- c_0 - p/2} |P_b(-p \log n/2)|, \tag{81}$$

with the appropriate end-time attenuation for each endpoint chord. All 4095 coefficient enclosures are reconstructed by the proper-divisor recurrence (58). The main term is $b_0 c_0/p$, frozen downwards only because $b_0 \geq 0$.

Combining the main term with (78), (80), (81), and the separate normaliser charge gives the new jet source. Every old source charge is multiplied by α_i, and every jet source field by 2. Thus the operational source fields are exactly α_i times the complete old fields plus twice the new fields. The full old record is retained in each certificate; an old diagnostic split is not used as a decomposition of the conic source.

The selected row-16 circle uses native floor 51/20 and radius 1/2; row 44 uses 51/25 and radius 39/20. Each circle independently passes $p - r \geq h_0$, $p + r \leq 6$, $r \leq 39/20$, $p + r < 7$, and the thermal guard. The independent source consumers reconstruct 4128 gates for mixed row 0, 8363 for the later conic extensions, and 4215 for the final order-eight row. These counts describe finite proof obligations; they do not replace the preceding analytic argument.

3.8 Continuous signed kernel and the entire own pair

For an upper/lower pair at horizontal spacing d and height b, define

$$K_h(d,b) = \frac{h-b}{d^2+(h-b)^2} + \frac{h+b}{d^2+(h+b)^2}.$$

Apply the same row functional to this kernel, including all signed nodes and all derivatives for a joint row, and denote the result by $R_i(d,b)$. Its own-pair contribution at squared height $q = y^2$ is

$$O_{\mathrm{pair},i}(q) = R_i(0,y). \tag{82}$$

For an ordinary row this is the explicit rational function

$$O_{\mathrm{pair},i}(q) = \sum_j \frac{2w_{ij}h_{ij}(h_{ij}^2 + a_{ij}^2 - q)}{(h_{ij}^2 + a_{ij}^2 - q)^2 + 4a_{ij}^2 q}. \tag{83}$$

For a conic row add

$$2\sum_{r=0}^{m} \frac{b_r p^{r-1}}{r!} \partial_p^r \frac{2p}{p^2-q}$$

to α_i times the old function. The signs stay inside the entire sum; neither individual own terms nor their derivatives are presumed positive.

Theorem 3.4 (Complete continuous kernel certificates)**.** *For every row of the pinned plan and every q in its closed declared contact interval, $h_{ij}^2 > q$ at every node and*

$$R_i(d,\sqrt{q}) \leq \frac{4}{d^2+4q} \qquad (d \in \mathbb{R},\ (d,q) \neq (0,0)). \tag{84}$$

The corresponding inequality at every other-zero height $0 \leq b \leq y = \sqrt{q}$ is

$$K_y(d,b) \geq \frac{y}{2} R_i(d,b), \tag{85}$$

away from the distinguished singular pair. Each complete $O_{\mathrm{pair},i}$ is nondecreasing on its declared contact interval.

Proof. The finite part of the proof is an exact rational polynomial certificate, regenerated directly from the displayed Poisson fractions and the plan. Symmetric shifts make the rational expression even in d. Put $D = d^2$ and, away from $(D,q) = (0,0)$, clear all positive denominators in $4/(D+4q) - R_i(d,\sqrt{q})$. Their positivity follows from $h_{ij}^2 > q$; for a derivative row the corresponding powers have the same sign. Compactify $D = z/(1-z)$ and multiply by the

required positive power of $1-z$. The endpoint $z=1$ represents the leading far-field coefficient and is included. The result is a rational polynomial on the compact rectangle in (z,q).

On every rational subdivision cell the checker converts this polynomial to its tensor-product Bernstein basis. If its local degree is (m,n), the basis functions are

$$\binom{m}{r}u^r(1-u)^{m-r}\binom{n}{s}v^s(1-v)^{n-s}\geq 0 \qquad (0\leq u,v\leq 1).$$

The stored rational coefficients are all checked to be nonnegative, and the checker verifies the complete subdivision tree and its boundary ownership. Thus their sum proves the inequality at every point of every cell. It independently regenerates the cleared polynomial, so an irrelevant nonnegative coefficient table would not pass. At $(d,q)=(0,0)$ the target fraction is not defined; the denominator-cleared polynomial is still checked on that closed corner. Actual nonreal contacts have $q>0$. This proves (84), including q and spacing continua and the far-field endpoint.

The same procedure applied to the rational derivative $O'_{\mathrm{pair},i}(q)$ proves aggregate monotonicity. For example the final row-44 boundary polynomial has degree 118 in each of its two variables and 29 certified cells; its own-derivative polynomial has degree 116 and one cell. Degree size does not add an analytic assumption: all its arithmetic is rational.

Finally, as a function of (d,b) each probe kernel is harmonic throughout $|b|\leq y$, since its poles are outside the strip. The same holds after real translations and differentiation in a probe height. The positive strip-Poisson formula extends the nonnegative boundary defect to the whole interior. The target kernel carries a nonnegative boundary atom at the distinguished point, which only increases the interior defect. This proves (85). The explicit extension, convergence of its zero sums, and treatment of the distinguished pair are given in the analytic foundations section. □

3.9 Closing the finite source assertion

Here is the exact dependency order in Theorem 3.1. First, the corrected physical calculation proves the common finite model and the absolute errors on every actual source disk. Finite thermal recurrences and complete native labels give a denominator gap, the matched residual, and full future-activation charges. The common analytic measure permits only the explicitly certified spectral recombinations; physical quotient charges remain separate. The finite head is summed with signs before moduli. The affine native bound retains an authenticated portion of the main growth. Disjoint composite capacities and the complete physical clock branches yield the licensed finite head decrements. Selected derivative rows are charged by one analytic physical logarithm and a complete spectral envelope. Each conic row uses one combined source and kernel.

The final source-assembly consumer reconstructs the M8 parent from its physical and analytic primitives and then applies the 39 globally proved pure-source gains to `first_hi` and `last_hi`. Every other operational field is preserved. In row 55, an earlier local credit is replaced, rather than retained in addition to the new global credit. The direct M8-to-final reconstruction therefore has no local native exclusion as a premise. Rows $0, 16$ carry the exact positive conic scaling of their scalar parent. Whole-object comparisons establish that the resulting rational fields are precisely those in (50). Together with the estimates above, this proves Theorem 3.1.

The finite supplement contains the raw rational primitives, branch multipliers, phase factorizations, interval envelopes, conic coefficients, kernel subdivision trees, source assembly and independent consumers. Checks of file digests identify those objects but do not prove the analytic inequalities. Conversely, a report saying that a numerical optimizer passed is not one

of the source premises: the consumers reconstruct the declared finite inequalities using exact rational arithmetic or outward Arb enclosures. The comparison section uses only Theorems 3.1 and 3.4 for these pinned versions.

4 An exact positive comparison and its complete terminal cleanup

This section specifies the finite calculation that turns the analytic source inequalities into an upper bound for Λ. The inputs are the rational plan, the rational source ledger, and the list of rational arcs. The list is a certificate: the optimization and integration procedures that found it are unnecessary for its verification. In particular, rounding toward a larger comparison height is permitted only after the whole preceding ideal arc has been shown to remain in its licensed domain. The endpoint below includes the additional time needed to remove the *entire* terminal strip.

4.1 The interface with the analytic estimates

Write $q = y^2$ for the square of the height of a highest nonreal zero. There are 67 rows, indexed by $i = 0, \dots, 66$. Row i supplies a closed time cell $[a_i, b_i]$, a contact cap $c_i > 0$, a signed probe functional, and rational constants

$$M_i, \qquad N_i, \qquad A_i \geq B_i > 0.$$

The cells meet at their endpoints. They cover $[0, 87/800]$; the comparison will stop before the right endpoint of the last cell. Put

$$\theta_i(t) = \frac{t - a_i}{b_i - a_i}, \qquad S_i(t) = M_i - N_i - (1 - \theta_i(t))A_i - \theta_i(t)B_i. \tag{86}$$

The meanings of the four stored fields are respectively `main_lo`, `native_debit`, `first_hi`, and `last_hi`. Thus S_i is a lower bound, rather than a numerical estimate of the actual source. Since $A_i \geq B_i$, it is nondecreasing on its entire cell.

For reference, the exact own-pair function for the retained translated-probe part of a row is the following rational function. If the row has nodes (h_j, v_j, w_j), with signed w_j, define

$$O^{\mathrm{old}}_{\mathrm{pair},i}(q) = \sum_j \frac{2w_j h_j (h_j^2 + v_j^2 - q)}{(h_j^2 + v_j^2 - q)^2 + 4v_j^2 q}. \tag{87}$$

The signs in this sum are retained. Replacing them separately by absolute values would change the functional and invalidate the numerical rate. All node poles lie outside the complete contact cap: $h_j^2 > c_i$.

Rows 0, 16, and 44 also have joint-jet terms. A row of this form has rational $\alpha \geq 0$, centres p, and coefficient vectors $(b_0, \dots, b_m)$. In the units of the stored source its whole own pair is

$$O_{\mathrm{pair},i}(q) = \alpha O^{\mathrm{old}}_{\mathrm{pair},i}(q) + 2 \sum_{\text{centres } p} \sum_{m=0}^{d_p} b_m p^{m-1} \frac{1}{m!} \frac{\partial^m}{\partial p^m} \left(\frac{2p}{p^2 - q} \right). \tag{88}$$

For the other rows $O_{\mathrm{pair},i} = O^{\mathrm{old}}_{\mathrm{pair},i}$. The poles satisfy $p^2 > c_i$. The geometry certificate proves that the *whole signed* $O_{\mathrm{pair},i}$ is nondecreasing on $0 \leq q \leq c_i$. This is not inferred from the monotonicity of individual signed summands. The same coefficient vector and the same scaling occur in the source and kernel certificates.

The source theorem (Theorem 3.1) and kernel theorem (Theorem 3.4), together with the all-complex first-contact argument in the analytic foundations, give the following interface at a simple highest contact in the exterior region:

$$q'(t) \le -2 - 2q(t)(S_i(t) - O_{\mathrm{pair},i}(q(t))), \quad t \in [a_i, b_i], \quad 0 < q(t) \le c_i. \tag{89}$$

The finite head is covered by its separate wall theorem. Compactness and the treatment of multiple contacts are also supplied by the analytic foundations. Equation (89) is the only analytic zero-motion input to the computation here; the finite consumer cannot establish it by examining an arc file.

4.2 A comparison arc with an upward reset

An arc record consists of

$$(i, t_0, \delta, q_0, q_1, s, o, r, \ell),$$

all rational except the row index i. Here q_0 is the stored initial square, s a source lower bound, o an own-pair upper bound, and $r > 0$ the comparison rate. The ideal arc is

$$Q(u) = \left(q_0 + \frac{2}{r}\right) e^{-ru} - \frac{2}{r}, \qquad 0 \le u \le \delta. \tag{90}$$

It solves $Q' = -2 - rQ$ and $Q(0) = q_0$. Its outgoing ideal value need not be rational. The record instead stores an outward rational endpoint $q_1 \ge Q(\delta)$, which becomes the next arc's initial value. The following finite conditions are enough:

$$a_i \le t_0 < t_0 + \delta \le b_i, \qquad 0 < q_0 \le c_i, \tag{91}$$

$$s \le S_i(t_0), \qquad o \ge O_{\mathrm{pair},i}(q_0), \qquad 0 < r \le 2(s - o) - \eta, \tag{92}$$

$$\ell = q_0 - (2 + rq_0)\delta > 0, \qquad Q(\delta) \le q_1 < q_0, \tag{93}$$

where the fixed strictness allowance is

$$\eta = 10^{-12}. \tag{94}$$

Each q_1 has denominator a power of two at most 2^{96}. The allowance η has the units of the rate in (92); it is not a lower bound for the source itself.

Lemma 4.1 (A positive ideal arc)**.** *If $q_0 > 0$, $r > 0$, and ℓ in* (93) *is positive, then on $0 \le u \le \delta$*

$$0 < \ell \le Q(u) \le q_0, \qquad Q'(u) < 0.$$

Proof. The elementary inequality $e^{-ru} \ge 1 - ru$ gives

$$Q(u) \ge q_0 - (2 + rq_0)u \ge \ell > 0.$$

Consequently $Q' = -2 - rQ < 0$, and $Q(u) \le Q(0) = q_0$. This proof controls the entire closed ideal arc. Positivity of the stored outgoing value alone would not prove the assertion. □

Lemma 4.2 (Strict exclusion of a simple contact on an arc)**.** *Assume* (89) *and the whole-row own-pair monotonicity. A zero branch starting below Q cannot first meet Q at a simple highest contact on the arc satisfying* (91)–(93)*.*

Proof. At such a contact, $q = Q(u) > 0$ and $q \le q_0 \le c_i$. Since S_i is nondecreasing and $O_{\mathrm{pair},i}$ is nondecreasing on the cap,

$$S_i(t_0 + u) - O_{\mathrm{pair},i}(q) \ge s - o \ge \frac{r+\eta}{2}.$$

Thus (89) yields

$$q' \le -2 - (r+\eta)q < -2 - rq = Q'(u),$$

contrary to a first meeting from below. The strict inequality is valid at every positive contact height, however small it is. Lemma 4.1 guarantees positivity through the incoming end. □

At $t_0 + \delta$ the replacement of $Q(\delta)$ by q_1 is an *upward* jump of the comparison, so it preserves a previously established strict separation from all zeros. A terminal contact at the incoming ideal value is excluded by the same strict differential inequality. The next cell may have a different signed functional; that change is legal because the next record checks its own initial cap and its original absolute time cell. No continuous interpolation of the functionals is required.

4.3 Exact exponential enclosures and rational own-pair evaluation

For an exact rational $x \ge 0$ put

$$P(x) = 1 + \frac{x}{2} + \frac{5x^2}{44} + \frac{x^3}{66} + \frac{x^4}{792} + \frac{x^5}{15840} + \frac{x^6}{665280}. \tag{95}$$

All coefficients are positive. Repeated integration by parts gives

$$e^{-x}P(x) - P(-x) = -\frac{x^{13}}{12!}\int_0^1 e^{-tx}t^6(1-t)^6\,\mathrm{d}t \le 0.$$

Thus the even diagonal [6/6] Padé expression is an exact upper bound,

$$e^{-x} \le \frac{P(-x)}{P(x)}. \tag{96}$$

The consumer evaluates the two polynomials with exact rational Horner arithmetic and checks

$$\left(q_0 + \frac{2}{r}\right)\frac{P(-r\delta)}{P(r\delta)} - \frac{2}{r} \le q_1. \tag{97}$$

Together with (96), this establishes the outgoing inequality in (93). No floating approximation to an exponential and no trust in the endpoint producer is needed.

The joint derivative terms also admit an independent rational evaluation. Formally write $q = y^2$ and set

$$u = \frac{1}{p-y}, \quad v = \frac{1}{p+y}, \quad A = u + v = \frac{2p}{p^2-q}, \quad B = uv = \frac{1}{p^2-q}.$$

The symmetric power sums $S_j = u^j + v^j$ are rational in p, q and obey

$$S_0 = 2, \qquad S_1 = A, \qquad S_{j+1} = AS_j - BS_{j-1}.$$

Consequently

$$\frac{1}{m!}\frac{\partial^m}{\partial p^m}\frac{2p}{p^2-q} = (-1)^m S_{m+1}. \tag{98}$$

The fresh consumer uses this Newton recurrence, independently of the power-series derivative calculation used to construct the candidate record. The identity is exact for all rational inputs in the licensed domain $p^2 > q$. Every signed term of the whole own pair is added before testing its proposed upper bound.

4.4 Initialization at the ordinary seven-eighths strip

The ordinary zeta theorem used in the analytic foundations excludes nontrivial zeros with real part greater than 7/8. The functional equation gives the reflected bound. Under the coordinate conversion between zeta and H_0, the initial imaginary halfwidth is therefore $2(7/8) - 1 = 3/4$, and its square is 9/16. The comparison is initialized strictly outside that strip:

$$q_{\mathrm{init}} = \frac{9}{16} + 10^{-12} = \frac{562500000001}{10^{12}}. \tag{99}$$

The cushion is part of the exact computation. It is not permission to use a narrower unproved initial zeta plane.

To enter the positive-time compact first-contact setting, take

$$\tau = 10^{-14}. \tag{100}$$

Classical de Bruijn strip contraction, applied from $t = 0$, bounds every actual squared height at $t \le \tau$ by $9/16 - 2t$. On the first ideal arc, Lemma 4.1 gives

$$Q(t) \ge q_{\mathrm{init}} - (2 + r_0 q_{\mathrm{init}})t.$$

The separation from the classical upper strip is consequently at least

$$Q(t) - \left(\frac{9}{16} - 2t\right) \ge 10^{-12} - r_0 q_{\mathrm{init}} t \ge 10^{-12} - r_0 q_{\mathrm{init}} \tau. \tag{101}$$

Here the first rate in the exact certificate is

$$r_0 = \frac{1226153390478175752291909362361}{9903520314283042199192993792}.$$

The last expression in (101) is exactly

$$\frac{92138090321378421847855254529664559063763 9}{99035203142830421991929937920000000000000000000000000000} > \frac{1}{2 \cdot 10^{12}}. \tag{102}$$

It is approximately $9.3035695790 \cdot 10^{-13}$. Also $0 < \tau < 1/2048000$, so it lies strictly inside the first arc. This proves a strict starting separation at an actual positive time; it does not assume positive-time simplicity of every zero at $t = 0$. Theorem 2.10 justifies passing from this separation to the first-contact argument on $[\tau, T]$.

4.5 The finite certificate and its coverage

The exact computation consists of 221010 arcs. Every arc has

$$\delta = \frac{1}{2048000}. \tag{103}$$

The first 20 time rows have length 1/400 and therefore contain 5120 arcs each. Rows 20 through 65 have length 1/800 and contain 2560 arcs each. The calculation uses 850 arcs of row 66. In particular,

$$20 \cdot 5120 + 46 \cdot 2560 + 850 = 221010.$$

The finer decomposition is data, rather than an inference from a plot or a numerical time mesh: exact stored times and states are checked at every join.

The following table gives several checkpoints. The height columns are decimal summaries for orientation; the certificate stores their exact rational values, and none of the displayed rounded entries is used as an inequality in the proof.

Row	Time cell	Arcs	Entry square	Exit square	Number of nodes
0	$[0, 1/400]$	5120	.562500000001	.540389791362	9
1	$[1/400, 1/200]$	5120	.540389791362	.518757332785	9
16	$[1/25, 17/400]$	5120	.263984448758	.249128464055	9
44	$[2/25, 13/160]$	2560	.075504237976	.071222671787	25
55	$[3/32, 19/200]$	2560	.033261081999	.029931516439	10
65	$[17/160, 43/400]$	2560	.003496950134	.000934680690	10
66	$[43/400, 87/800]$	850	.000934680690	.000099742033	9

For row 66, the exit is at the stopping time, before the right endpoint of the displayed licensed cell. In all rows the argument stays below the cap supplied by the plan; it is not necessary that the cap equal the comparison's entry value.

The minimum exact entry-cap clearance, over the complete list, is

$$\min_{\text{arcs}}(c_i - q_0) = \frac{21059999}{10^{12}} > 0. \tag{104}$$

The smallest comparison rate occurs at the first arc and is r_0 above, approximately 12.3809852615. The rates range up to approximately 22.6946264088. These are the rates r in $Q' = -2 - rQ$, rather than values of the undoubled source $S_i - O_{\text{pair},i}$.

The checker additionally verifies that all 2560 closed ideal arcs of row 55, including their outward jumps, remain inside

$$\frac{289}{10000} \le Q \le \frac{361}{10000}.$$

This retained tube check is conservative. The old native improvement associated with that tube has already been replaced in the final source ledger and is not credited a second time. Likewise, every source change is compared against its specified parent in its original functional units; a flow-only verifier must not silently rescale a source or widen a cap.

4.6 A fully specified first record

The first arc makes the normalisation and directions of all roundings concrete. It belongs to row 0, whose licensed cap is

$$c_0 = \frac{28126053}{50000000} = 0.56252106.$$

Its exact inputs are $t_0 = 0$, $\delta = 1/2048000$, and $q_0 = 562500000001/10^{12}$. The stored source and own bounds are

$$s_0 = \frac{189158635366668237198319364 83}{247588007857076054979824844 8},$$
$$o_0 = \frac{114846276982028443785727385843}{792281625142643375935439503 36}.$$

They are approximately 7.64005643908 and 1.44956380834, respectively. The difference between twice their difference and the stored rate is exactly

$$2(s_0 - o_0) - r_0 = \frac{39614081257132169}{396140812571321687967719751 68} > 10^{-12}.$$

Thus the strictness allowance is paid before the comparison is used. The record's positive linear lower bound is

$$\ell_0 = \frac{1140876662390550393698814138598645989687665 3639}{2028240960365167042394725128601600000000000000 00} > 0,$$

and its stored outgoing value is

$$q_1 = \frac{5570686828209889305941576421}{9903520314283042199192993792}.$$

These are approximately 0.5624956228993352 and 0.5624956229125658. The two inequalities are used for different purposes: $\ell_0 > 0$ bounds the entire ideal arc from below, while (97) bounds its final value from above by q_1. The initial cap clearance is $c_0 - q_0 = 21059999/10^{12}$, which also equals the smallest clearance in the entire certificate. Every later record is verified by precisely the same rational conditions; it need not have the same source or functional as this example.

4.7 What the complete consumer proves

A successful numerical verification establishes the following finite statement, independently of the method used to construct the records.

Proposition 4.3 (Finite comparison certificate)**.** *For the pinned plan and source ledger, the* 221010 *records start at* (99)*, satisfy* (91)–(93) *with* $\eta = 10^{-12}$*, meet exactly in stored time and outward state, and end at the rational pair* (T, q_T) *in* (105) *below. The strict startup inequality* (102) *holds. Every ideal arc lies within a licensed source time cell and contact cap.*

Here is a complete logical outline of the finite consumer. The pseudocode uses exact integer fractions throughout.

1. Check the input hashes, schemas, and the 67 row identities. For every row check its absolute time interval, common source/kernel cap, source-functional scaling, and the separate source and geometry proof bindings. In particular, verify $A_i \geq B_i > 0$ and poles outside the full cap. Check the strict far-field condition needed by the harmonic extension in the analytic kernel proof.
2. Set $(t, q) = (0, q_{\text{init}})$. For each arc check that its incoming (t_0, q_0) equals the current exact pair, its row is the current or next row, and (91) holds.
3. Evaluate (86) at t_0 and (87)–(88) at q_0 by (98). Verify the downward source and upward own bounds and the strict rate inequality (92).
4. Check the exact positive lower bound ℓ and the outgoing exponential enclosure (97). Check $q_1 < q_0$ and the 96-bit dyadic denominator. Where recorded, check the complete closed tube as well. Update $(t, q) = (t_0 + \delta, q_1)$.
5. Verify that the resulting (t, q) equals the stated terminal pair, that $0 < q \leq 10^{-4}$, and that all stated time horizons hold. Check (102), compute $t + q/2$ exactly, and round that sum outward to the claimed decimal.

The consumer `verification/arcs/verify_arcs_fresh.py` evaluates all of these inequalities directly with arbitrary-precision rational arithmetic. Its execution performs 5 797 314 finite checks on all 221 010 arcs. The count records the scope of an execution; the proof consists of the inequalities and their analytic implications.

For each arc, the supplied compact transcript gives six exact 128-bit dyadic lower bounds: source margin, own-pair margin, strict rate margin, outgoing-state margin, positive linear floor, and cap clearance. The first four are nonnegative and the final two strictly positive. These actual witnesses are in `verification/arcs/COMPACT_ALL_ARC_MARGIN_LOWER_BOUNDS.jsonl.gz`. They permit direct inspection of the inequalities and are not a substitute for recomputing the arc from its rational data.

The exact numerical inputs have the following SHA-256 identities. Paths are relative to the evidence workspace in the proof packet.

```
lambda_round14/material/m8_late/M8_LATE_PLAN.json
ec346b7c1136eda38021eb6323033b3b48ed21e6f7458eeeac7cc9579db2497a
lambda_round15/flow_main_v2/MAIN_CLOCK_SOURCE.json
b59905d7b994d3424ba14f9347be75a2429ae4d4b0de2c2404971ffda1d4d536
lambda_round16/ordinary_7_8_flow/full_flow/FLOW.json.gz
a664e829a3681d96e474bcf875d80cba7b16fe8f65e050c69f01b5ea07de4737
```

The compressed flow identity is checked before decompression. The producer identity is also bound, but correctness does not depend on trusting the producer: the output inequalities are re-evaluated.

4.8 Terminal values and the upper bound

The exact terminal pair is

$$T = \frac{22101}{204800} = 0.1079150390625, \qquad q_T = \frac{790237796383053674715864 3}{792281625142643375935439503 36}. \tag{105}$$

Here q_T is approximately 0.00009974203254313493. Both source and wall time restrictions are respected: $T < 19/125$, and even the eventual cleanup endpoint is below $41/250$. Only the source/wall bounds through T are actually needed; after T, the classical contraction theorem is applied to the global strip already obtained.

Proposition 4.4 (Assembly of the global terminal strip)**.** *Assume the ordinary seven-eighths theorem and the analytic source, whole-kernel, finite-head, compactness, and multiple-contact results proved in the preceding sections. Then every zero of H_T has imaginary part at most $\sqrt{q_T}$ in absolute value.*

Proof. Classical contraction and (101) give strict separation at τ. Suppose that a zero first reaches the piecewise ideal comparison before its terminal time. Positive-time remote-tail compactness produces an attained highest contact; the finite-head wall puts a nonreal contact in the exterior region covered by the source estimates. A multiple contact is excluded by the backward complex Hermite splitting argument in the analytic foundations. A simple one contradicts Lemma 4.2. All ideal arcs are positive and within their licensed caps by Proposition 4.3. Upward resets preserve the separation, including time-row joins. Hence there is no first contact, and at time T all zeros lie within the outward terminal strip. □

Corollary 4.5 (Exact comparison-plus-cleanup endpoint)**.** *The de Bruijn–Newman constant satisfies*

$$\Lambda \le T + \frac{q_T}{2} = \frac{427693072077942473661578098491}{396140812571321687967719751 6800} < \frac{107964910079}{10^{12}}. \tag{106}$$

Proof. De Bruijn contraction applied at time T to the complete global strip $|\operatorname{Im} z| \le \sqrt{q_T}$ makes every zero real by time $T + q_T/2$. The rational addition in (106) is exact. Finally,

$$T + q_T/2 \approx 0.10796491007877157,$$

and the displayed fraction with denominator 10^{12} is its upward decimal ceiling. In particular the full cleanup has been paid, rather than omitting $q_T/2$ or treating a small positive terminal height as a real-zero conclusion. □

The computation is initialized at exactly $7/8$. An earlier, numerically smaller endpoint used a narrower initial zeta strip and cannot be relabelled as this theorem. Similarly, the local fixed-head confluent certificates and hypothetical uniform-current comparisons play no role in Proposition 4.4 or Corollary 4.5. The bound here uses the proved row source ledger and its complete finite comparison.

5 Certificate interfaces and reproducibility

The proof has three distinct verification layers. First, it uses ordinary mathematical theorems, including a particular externally verified formal nonvanishing theorem. Second, it uses the analytic estimates proved in this manuscript, with their domains and normalisations stated explicitly. Third, it uses finite rational or outward-rounded interval certificates. A successful numerical producer is not an additional mathematical premise: its output is used only through the inequalities checked by the corresponding consumer. This section specifies these interfaces and the evidence retained for them.

5.1 The imported ordinary nonvanishing theorem

The only new initial-strip input is the ordinary statement

$$\forall s \in \mathbb{C} \setminus \{1\}, \qquad \operatorname{Re} s > \frac{7}{8} \quad \Longrightarrow \quad \zeta(s) \neq 0. \tag{107}$$

The formal source [6] accompanying the preprint [24] exports the Lean declaration

```
OAI.riemannZeta_ne_zero_of_seven_eighths_lt_re.
```

Its type has the variable s : Complex and the single assumption $(7/8 : \text{Real}) < s$.re; the conclusion concerns Mathlib's `riemannZeta`. In particular, it does not accept a source certificate, an additional zero-free strip, a moment estimate, or a detector inequality as an argument. The certificate used by its final assembly is constructed in the imported development. The usual completed-zeta functional equation, conjugation and zero correspondence then give precisely $|\operatorname{Im} z| \leq 3/4$ for zeros of H_0; no strictly smaller halfwidth is imported. The convention assigning a finite value to the meromorphic zeta function at its pole does not affect any zero of the completed function.

The retained source and build identities are as follows. Long hexadecimal identifiers below are Git commit or tree identifiers, rather than numerical error bounds.

Object	Pinned identity
Official source commit	`fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`
External rebuild source commit	`adc7f1241b42e322a6451854ab7e4b4c146bf78a`
Shared proof subtree	`1e340710a6e6220842ad878bb89a1de4b1989af4`
Mathlib commit	`d13f23b723b8a846827a245b89c10fc7d3f11612`
External verification report commit	`b83a73c6cb626a391e551baafdb7d0cceafaa55a`

The proof subtree, toolchain, dependency manifest and relevant compatibility patches agree at the two official-source commits. The downloaded ordinary proof closure consists of 2924 imported OAI modules. There are 2926 downloaded source files in the saved subtree, two of which are outside this ordinary-zeta dependency path. Their saved SHA-256 values and Git blob identities are recorded separately. A static inspection of the closure finds neither unproved declarations nor a replacement of the target zeta function, real numbers or complex numbers. Static inspection is useful for identifying the claimed statement; it does not replace kernel checking.

The separately pinned report [7] records a complete Lean 4.34.1 build of the 2924 OAI modules and the independently retyped ordinary-zeta challenge. Its retained `#print axioms` output lists only

```
propext,    Classical.choice,    Quot.sound.
```

The report used the pinned official Mathlib compiled cache; it did not re-elaborate every Mathlib declaration. Its additional `leanchecker` replay completed for the final two modules. The optional additional replay of all 2924 modules was stopped and is not reported as complete.

This manuscript imports (107) as that identified external theorem. The work reported here did not perform a fresh full local Lean build. Neither the custom source estimates nor the zero-dynamics assembly nor the final Λ inequality has been formalised end to end in Lean. The distinction is a verification boundary, not an undisclosed RH hypothesis in (107). A reader wishing to replace the external build evidence by a new kernel run should build the pinned target `OAI.NumberTheory.DirichletL.Nonvanishing`, then run the retained root-qualified `RootZetaChallenge.lean` and save both its elaborated type and axiom output.

5.2 Finite mathematical obligations

Every row of the source plan binds one functional, one full contact-height interval, one thermal interval and one own-pair expression. The certificate interface requires all of the following.

1. The signed boundary-kernel defect is nonnegative for every real spacing throughout the declared closed contact interval. Its denominators are positive, its far-field coefficient has the required sign, and the whole signed own-pair function has the stated monotonicity. Rational polynomial identities and Bernstein/conic certificates certify a continuum, rather than a set of sampled spacings.

2. The physical source lower bound applies to that same functional, at every admitted exterior physical center and every time of its closed thermal cell. Its complete native labels, analytic tails, common normalisation, translated cutoffs and physical-error charges are retained. A source refund is applied only to the reserved variable charge it actually bounds.

3. The wall certificate applies to the complete fixed wall in time and height. Its finite prefix computation is combined with Abel summation and the physical-error transfer proved in the text. In particular, the usable wall floor is the corrected 697/900, rather than the superseded narrower historical error field.

4. The flow consumer checks each arc's source debit, full signed own-pair upper bound, strict comparison rate, complete domain caps, positive ideal arc, outward endpoint and adjoining state and time. Startup and final global strip contraction are checked separately.

The words "complete" and "same" in these obligations are essential. For example, a proof for a single physical phase, an individual unsigned probe, or an initial state outside a certified kernel cap would not satisfy this interface. The local confluent-current experiments are not used in any of the four obligations or in the theorem.

The three operational input identities are

Object	SHA-256
Source ledger	`b59905d7b994d3424ba14f9347be75a2429ae4d4b0de2c2404971ffda1d4d536`
Kernel plan	`ec346b7c1136eda38021eb6323033b3b48ed21e6f7458eeeac7cc9579db2497a`
Complete flow	`a664e829a3681d96e474bcf875d80cba7b16fe8f65e050c69f01b5ea07de4737`

The complete bundle inventory supplies byte counts and SHA-256 digests for the actual input files, checker programs and mathematical explanations. Hashes identify the inputs; the consumers re-evaluate their inequalities.

5.3 Independent finite consumers

The finite gates were reimplemented from the actual coefficients and primitive data. Historical success fields and proposed critical roots are not acceptance premises. All consumers use the same trusted FLINT/Arb arithmetic implementation for their transcendental enclosures; the distinct implementations concern the mathematical reductions and finite algorithms. The principal gates are as follows.

Gate	Continuous or finite claim checked
Native coefficients	All 469 heights; full support $1 \le n \le 690988$; 27 484 independently generated bins and 1658 exact cutoffs.
Global activation	Both native branches on the complete half-line; unique critical-point enclosures or exact limiting suprema.
Spectral envelopes	All 67 rows on $\lambda \ge \log 4096$, including 14 727 closed interval pieces and the unbounded tails.
Arithmetic head	All 4095 finite labels; 28 shared-resource rows and 39 retained rows; all allocation, phase and Gram inequalities.
Physical source	All 469 quotient caps and all 67 row refunds; actual common model; three complete derivative circles; all 67 operational rows.
Kernel	All 67 rational boundary functions and whole own-pair derivatives; complete subdivision coverage and 2 716 496 positive terminal integer coefficients.
Wall	Direct 512-bit cosine sums for all 2 763 952 prefixes, followed by the corrected broad physical transfer.
Exact flow	Every source, signed-own, rate, positivity, outgoing and cap inequality on all 221 010 arcs; strict startup and full cleanup.
Interfaces	Same actual functional, time/contact domains and source fields at every source-to-kernel-to-flow join.
Direct reconstruction	All 67 actual main bounds; the complete scalar and jet functional identities; every primitive debit and arithmetic payment; all 268 final source-field identities.

The actual source table is `verification/source_overlays/ALL67_OPERATIONAL_SOURCE_TABLE.json`. It includes every signed rational scalar node, conic derivative component, absolute thermal/contact domain and the four operational source fields. The polynomial data are the 67 files `verification/kernel/ROW_000_ACTUAL_CERTIFICATE.json.gz` through `verification/kernel/ROW_066_ACTUAL_CERTIFICATE.json.gz`. Each contains the full initial integer tensor and its certified closed leaf boxes, rather than only a reported number of passed tests. The exact arc data and the all-arc margin transcript accompany them.

Rational checks use arbitrary-precision integer fractions. Transcendental comparisons accept only when the entire outward-rounded ball lies on the required side of the rational threshold. Midpoint agreement or an inconclusive overlapping interval is insufficient. Consumers that use Python assertions reject optimized execution; they must be run without `-O` or `-OO`.

5.4 Replay and mathematical review

The bundle supplies editable and flattened manuscript sources, actual primitive and operational inputs, continuous-domain certificates, new consumers, source explanations and the imported ordinary-theorem provenance. The custom certificate data and their consumers are included in the accompanying archive, together with the retained ordinary-theorem source and attributed verification evidence. The reference-location ledger supplies exact URLs and hashes for contextual source articles and external report files that are not redistributed. Its top-level replay guide specifies installation and the commands for an extracted bundle. The raw input layout is preserved under `inputs/workspace/`; portability wrappers are separate from the

immutable data. No consumer may silently read an original research workspace that is absent from the bundle.

The final stage is `direct_source/verify_direct_source.py`. It evaluates Theorem B.1 from the primitive data, verifies all main-term and derivative-operator bindings, and checks complete endpoint/critical-value coverage for the order-eight source. The fifteen stages run in four dependency groups; their exact names and execution results appear in the matching verification guide and receipt.

The authoritative full replay command, run from the extracted root, is

```
python3.12 replay/run.py -all -jobs 1 -work ../lambda-review-replay
```

The custom finite proof can be replayed without network access on the supported platform using the bundled runtime. The included python-flint 0.8.0/FLINT 3.3.1 runtime supports Linux x86-64 with CPython 3.12; the system interpreter and its platform libraries remain software dependencies. Other platforms can install the pinned package as specified in `replay/README.txt`. Writable output is directed to the chosen external work directory; the bundled scientific checker files and raw inputs remain unchanged.

A referee should proceed in dependency order:

1. Verify the inventory and the exact terminal arithmetic. Establish that the declared source and kernel rows, their thermal/contact domains, and the actual flow bytes agree with the inputs just identified.
2. Inspect the precise imported ordinary theorem in §5.1. A fresh formal build is distinct from accepting the retained external theorem and its authenticated build evidence.
3. Replay the primitive, activation, arithmetic, physical and spectral source gates; then the 67 continuous kernel certificates and full wall. All source debits and credits must be derived in the same functional units, including the near/far joins and every unused capacity reserve.
4. Traverse every exact comparison arc and require strict startup, complete ideal-arc coverage, outward resets and the entire terminal $q_T/2$ cleanup. A metadata-only or endpoint-only run does not establish this step.
5. Review the analytic implications linking the finite inequalities to all actual complex zeros: finite-head continuation, harmonic kernel extension, compact positive-time contact and repeated-zero splitting. The finite arithmetic does not replace these arguments.

The ordinary formal theorem, analytic implications, arithmetic software and finite input checks are separate trust boundaries. The packet is a self-contained record of the present analytic and computational proof, with its software requirements disclosed; it is not an offline Lean kernel build of all imported libraries.

5.5 Status of review

The proof packet is organised for external mathematical review. The separate producer and consumer inspections described above were conducted within the same research session; they are not journal peer review and do not constitute independent public recognition of a record. The theorem claims the stated numerical bound at ordinary cited-theorem and computer-assisted-proof trust. It claims neither $\Lambda < 1/10$, the Riemann hypothesis, the Lindelof hypothesis, nor a complete formalisation of the assembled argument.

6 The assembled proof and its scope

The ordinary 7/8 theorem and the functional equation initialize the strict barrier (4). The finite wall and finite-height zero theorem localize every possible nonreal contact to the exterior source region. The signed continuous-domain kernels and the corrected physical sources imply the comparison inequality on each closed cell. The positive-time compactness and repeated-zero arguments make the first-contact comparison global. The rational consumer verifies every arc and every reset, giving the global strip $|\operatorname{Im} z| \le \sqrt{Q_T}$ at the time T in (5). De Bruijn contraction then makes all zeros real by $T + Q_T/2 = B_*$. This proves the upper bound of Theorem 1.1; Rodgers–Tao supplies its lower bound.

The proof is a conventional analytic proof with finite exact and interval-arithmetic certificates, using an externally verified formal theorem as an input. It is not an end-to-end Lean proof. The machine checks certify the finite inequalities and joins described here; they do not replace the analytic implications linking those inequalities to all complex zeros. Source hashes identify the reviewed versions; hash agreement alone is not mathematical verification.

Provenance and assistance. This manuscript assembles the author's retained research, public source materials, and the exact-version proof records. Its preparation, code inspection and internal cross-checking used AI assistants. Such internal checking is distinguished from an independent external referee's judgment. The full source packet is supplied so that the analytic derivations and computational boundaries can be assessed directly.

A Sharp all-frequency activation

The global source cap requires a joint bound for both activated native branches, retaining their common activation length until after integration. This appendix proves the 469 actual activation bounds used by all 67 thermal rows. It is the quantitative justification of the constants in `ACTIVATION.json`.

Parameters and the original densities. Let $N_0 = 690988$, $v_0 = \log N_0$, and let $[t_-, t_+]$ be one of the actual closed time rows. Let $h > 1$ be one of its seven native heights. Write $L = 2(v_0 + s)$, $s \geq 0$, and $\log n = v_0 + w$ for an activated integer. The literal native cutoff gives $0 < w \leq s + \delta$, where $\delta = 10^{-12}$. Put

$$V_- = \frac{2689}{200}, \qquad V_+ = \frac{6723}{500}, \qquad a = \frac{t_- V_-}{2},$$
$$d = \frac{h-1}{2}, \quad p = \frac{h+1}{2}, \quad k = a + d, \quad c = a + h.$$

Thus $p - d = 1$, $p + d = h$, $V_- < v_0 < V_+$, and $d, p, k, c > 0$.

After the counting measure $\mathrm{d}n = e^{v_0+w}\mathrm{d}w$ is inserted, the first-branch amplitude is exactly the exponential of

$$-\frac{tv_0^2}{4} + \frac{1-h}{2}v_0 - \frac{tv_0 s}{2} + \frac{1-h}{2}w + \frac{t}{4}(w^2 - 2sw).$$

The reflected amplitude has the same first two terms and last term, but its s, w part is $-(tv_0/2 + h)s + pw$. On the activated support, $w^2 - 2sw \leq \delta^2$. Their height-derivative frequencies are $(v_0 + w)/2$ and $(v_0 + 2s - w)/2$.

The adjacent-integer and endpoint correction is bounded by

$$\rho = \frac{t_+\delta^2}{4} + \left(\frac{t_+ v_0}{2} + h\right)\delta + \frac{h-1}{2N_0} + \frac{t_+\delta}{2N_0}. \tag{108}$$

These are the complete positive-density bounds for the finite activated sum. They contain both branches, including the reflected derivative frequency; no positive-time infinite Dirichlet series is used.

Let $u = s + \delta$. Enlarging the derivative frequency to $(V_+ + 2u - w)/2$ is safe. Replacing s by u in the exponent costs at most $\exp[(t_+ v_0/2 + h)\delta]$. The directed consumer verifies

$$\rho + \left(\frac{t_+ v_0}{2} + h\right)\delta < \frac{1}{5000} \tag{109}$$

for every actual row and height. Since the prefactor decreases with v_0 and t, the single constant

$$C = \exp\left(-\frac{t_- V_-^2}{4} - dV_- + \frac{1}{5000}\right) \tag{110}$$

therefore pays all the corrections. In what follows the variable u is renamed s and is allowed to range over the larger half-line $[0, \infty)$.

Define $I_j(z, s) = \int_0^s w^j e^{zw}\,\mathrm{d}w$. The complete activation amplitude and its absolute height derivative are at most $CA(s)$ and $CD(s)$, where

$$A(s) = e^{-as} I_0(-d, s) + e^{-cs} I_0(p, s), \tag{111}$$
$$D(s) = \frac{1}{2}e^{-as}\{V_+ I_0(-d, s) + I_1(-d, s)\} + \frac{1}{2}e^{-cs}\{(V_+ + 2s)I_0(p, s) - I_1(p, s)\}. \tag{112}$$

Each integrand in these expressions is nonnegative on its declared support. In particular the reflected derivative factor is retained.

Lemma A.1 (Three-exponential reduction)**.** *Put $U = 1/d - 1/p = 1/(dp)$ and*

$$A_a = \frac{V_+}{2d} + \frac{1}{2d^2}, \qquad A_k = \frac{V_+U}{2} + \frac{1}{2}\left(\frac{1}{d^2} - \frac{1}{p^2}\right), \qquad B_k = \frac{U}{2},$$

$$A_c = \frac{V_+}{2p} + \frac{1}{2p^2}, \qquad B_c = \frac{1}{p}.$$

Then, exactly for $s \geq 0$,

$$A(s) = \frac{1}{d}e^{-as} - Ue^{-ks} - \frac{1}{p}e^{-cs}, \tag{113}$$

$$D(s) = A_a e^{-as} - (A_k + B_k s)e^{-ks} - (A_c + B_c s)e^{-cs}. \tag{114}$$

Proof. Use $I_0(z,s) = (e^{zs} - 1)/z$ and $I_1(z,s) = [e^{zs}(zs-1)+1]/z^2$, then collect the three exponentials. Here $p - d = 1$ and $p + d = h$, so all displayed coefficients follow by rational algebra. No numerical approximation enters this identity. □

Proposition A.2 (The global amplitude maximum)**.** *If $a = 0$, then A is strictly increasing and $\sup_{s\geq 0} A(s) = 1/d$. If $a > 0$, A has exactly one positive critical point and its global maximum occurs there.*

Proof. By (113),

$$e^{as}A'(s) = -\frac{a}{d} + kUe^{-ds} + \frac{c}{p}e^{-hs}.$$

The last two terms are positive and strictly decreasing. At $s = 0$ their sum with the first term is 2, either by direct algebra or by differentiating (111). For $a > 0$ the limit is $-a/d < 0$. Thus there is exactly one sign change, from positive to negative. For $a = 0$ the derivative is positive everywhere, and the limit in (113) is $1/d$. □

Proposition A.3 (The global derivative maximum)**.** *Let*

$$H_k = kA_k - B_k, \quad J_k = kB_k, \qquad H_c = cA_c - B_c, \quad J_c = cB_c.$$

For each actual native height the following rational inequalities hold:

$$H_k, J_k, H_c, J_c > 0, \qquad J_k - dH_k < 0, \qquad J_c - hH_c < 0. \tag{115}$$

Under these inequalities, if $a = 0$ then D is strictly increasing with $\sup_{s\geq 0} D(s) = A_a$. If $a > 0$, it has exactly one positive critical point, which is its global maximum.

Proof. Differentiating (114) gives

$$e^{as}D'(s) = -aA_a + (H_k + J_k s)e^{-ds} + (H_c + J_c s)e^{-hs}.$$

The two nonconstant terms are positive. Their derivatives have signs of $J_k - dH_k - dJ_k s$ and $J_c - hH_c - hJ_c s$, so (115) makes both strictly decreasing throughout $s \geq 0$. At zero the expression equals $V_+ > 0$, directly from (112). Its limit is $-aA_a$. For $a > 0$ this proves a unique maximum. For $a = 0$ it proves strict increase, with limiting value A_a by (114). □

The actual directed inequalities. The rational parameters in (115) are checked exactly for all 469 heights. For each positive a, the new checker independently starts with $[0,1]$, doubles its right endpoint until the factored derivative is negative, and performs 110 rational bisections. Both endpoint signs are checked by outward Arb balls. The unique critical point is therefore contained in the resulting rational interval. Evaluation of (113) or (114) on that entire interval bounds the global maximum. The $a=0$ case uses the exact limiting constants proved above. This is a proof on the complete unbounded activation domain, not a sampled search.

The final directed comparisons are, separately at every row and height,

$$C \le C_{\mathrm{stored}}, \qquad C \sup_{s\ge 0} A(s) \le A_{\mathrm{act,stored}}, \qquad C \sup_{s\ge 0} D(s) \le D_{\mathrm{act,stored}}.$$

The exact lower and upper rational endpoints, stored constants, critical-root brackets and source-time ownership are in `verification/source_primitives/ACTIVATION_PRIMITIVES_FRESH_384.json`. The consumer is `verification/source_primitives/check_activation_primitives.py`. All 67 rows and 469 heights pass at 384 Arb bits. The saved historical critical-root proposals and success fields are not used to establish any inequality.

At the diagnostic corner $t_- = 0, h = 2$, $d = 1/2$ and $p = 3/2$, so

$$A(s) = 2 - \frac{4}{3}e^{-s/2} - \frac{2}{3}e^{-2s} < 2, \qquad \sup_{s\ge 0} D(s) = V_+ + 2.$$

This explicitly proves the actual $2C$ bound and explains why summing separate pointwise suprema into $4C$ would lose the required constant.

How these bounds enter the global source. Let P, P_1 be the independently verified Euler absolute norm and its logarithmic moment. The complete added value and frequency moment are

$$\bar{a} = PA_{\mathrm{act,stored}}, \qquad \bar{b} = 2PD_{\mathrm{act,stored}} + P_1 A_{\mathrm{act,stored}}.$$

The P_1 term is the Euler-factor derivative and must be included. These are exactly the activation fields consumed by the affine near/far source and physical-refund constructions. A separate independent source composition proves their quotient gaps, rational cap and near/far joins. The remaining source composition uses these actual constants; it is checked separately from this activation gate.

Verification boundary. This is a mathematical derivation and directed computation of the stated activation inequalities. It uses python-flint/Arb for outward transcendental arithmetic, not a second implementation of that arithmetic library. It does not by itself prove the heat approximation, the finite source capacities, kernel inequality, ordinary 7/8 theorem, or final comparison; those have separate proof and verification gates.

B Direct reconstruction of the 67 physical source chords

This appendix assembles the operational source directly from the primitive native moments, finite coefficient capacities, arithmetic certificates, physical error refunds and derivative functionals. A previously assembled source table is not an additional source theorem. In particular, the validity of `WEIGHTED_SOURCE.json` is not assumed in this argument. The four rational fields in the final table are the output of the formulas below, whose ingredients have the analytic meanings established in Sections 3.2–3.7 and Appendix D.

For row i, write its exact closed thermal interval as $I_i = [t_i^-, t_i^+]$, and put

$$\theta_i(t) = \frac{t - t_i^-}{t_i^+ - t_i^-}, \qquad c_0 = \frac{1}{2}\log N_0, \quad N_0 = 690988, \qquad c = \frac{1}{4}\log\frac{x}{4\pi}.$$

Let $\mathcal{L}_i^{\mathrm{sc}}$ be the scalar functional specified by the complete signed list (h_{ij}, a_{ij}, w_{ij}), with symmetric translations interpreted as in (49). Define

$$m_i^{\mathrm{mass}} = \sum_j w_{ij} > 0, \qquad F_i(n) = \sum_j w_{ij} n^{-h_{ij}/2}\cos(a_{ij}\log n/2).$$

The two sets of scalar rows are

$$\mathcal{R} = \{0,\dots,26\}\cup\{52,\dots,61\}\cup\{65,66\}, \qquad \mathcal{S} = \{27,\dots,51\}\cup\{62,63,64\}.$$

Thus $|\mathcal{R}| = 39$ and $|\mathcal{S}| = 28$; they are disjoint and exhaust the row set. The derivative rows are $\mathcal{J} = \{0, 16, 44\}$.

The following quantities are defined from their primitive certificates. Their definitions are mathematical inequalities and rational formulas, rather than a reading of an upstream success flag.

(i) m_i is the actual rational scalar main field, checked to satisfy

$$m_i \le m_i^{\mathrm{mass}} c_0. \tag{116}$$

This comparison concerns the value used in the final reconstruction, not a main field in a different source record.

(ii) $H_{i,-}$ and $H_{i,+}$ are complete raw finite-head upper bounds:

$$H_{i,-} \ge \frac{1}{2}\sum_{n=2}^{4096} d_{t_i^+}(n) n^{-1/2-t_i^- c_0}|F_i(n)|,$$
$$H_{i,+} \ge \frac{1}{2}\sum_{n=2}^{4096} d_{t_i^+}(n) n^{-1/2-t_i^+ c_0}|F_i(n)|. \tag{117}$$

For $i \in \mathcal{S}$, they are the sums of the 4095 respective dyadic upper capacities, denoted by $U_{i,n,-}$ and $U_{i,n,+}$. For $i \in \mathcal{R}$, they are the directed upper sums checked from the literal scaled probes and the finite Dirichlet-log recurrence. They contain no arithmetic payment.

(iii) D_i is the scalar analytic, physical and coordinate debit before the matched physical refund. It is reconstructed from the native moment formulas (124) below. For $i \in \mathcal{R}$, it is the intercept of the verified native affine bound; for $i \in \mathcal{S}$, it is the verified global debit.

(iv) ρ_i is the simultaneous physical refund in (126). It refunds the variable quotient error only. The fixed allowance in every native physical cap and the coordinate allowance remain charged.

(v) The complete scalar arithmetic payment is

$$C_i = \begin{cases} C_i^{\mathrm{comp}} + d_i, & i \in \mathcal{R}, \\ \Delta_i^{\mathrm{fix}} + \mu_i \kappa_0 + C_i^{\mathrm{orbit}}, & i \in \mathcal{S}, \end{cases} \qquad \kappa_0 = \frac{1}{5000000}. \tag{118}$$

Here d_i is the *total replacement* common-clock payment. If the earlier angular payment is $d_i^{(0)}$ and the certified additional payment is γ_i, then $d_i = d_i^{(0)} + \gamma_i$. The three payments in the second line use the fixed lower-capacity allocation, the coordinate reserve, and the remaining-prime weighted Gram certificate, respectively. Their simultaneous validity is proved below.

(vi) For $i \in \mathcal{J}$, let the actual derivative functional have the coefficient vector and centre (b_i, p_i) and nonnegative scalar multiplier α_i recorded in the final plan. Write its independently charged source fields as

$$(m_i^{\mathrm{j}}, D_i^{\mathrm{j}}, H_{i,-}^{\mathrm{j}}, H_{i,+}^{\mathrm{j}}).$$

They bound $T_{b_i} S_t(x, p_i)$ in the units of (77), before multiplication by 2. The orders are $4, 4, 8$ for rows $0, 16, 44$; the order-eight row-44 certificate is used once. For $i \notin \mathcal{J}$, set $\alpha_i = 1$ and set all four jet fields equal to zero.

Theorem B.1 (Direct reconstruction of all physical source chords)**.** *For every row i, define four rational numbers by*

$$M_i = \alpha_i m_i + 2m_i^{\mathrm{j}}, \tag{119}$$

$$N_i = \alpha_i (D_i - \rho_i) + 2D_i^{\mathrm{j}}, \tag{120}$$

$$A_i = \alpha_i (H_{i,-} - C_i) + 2H_{i,-}^{\mathrm{j}}, \tag{121}$$

$$B_i = \alpha_i (H_{i,+} - C_i) + 2H_{i,+}^{\mathrm{j}}. \tag{122}$$

With the actual signed functional

$$J_i(t, x) = \alpha_i \mathcal{L}_i^{\mathrm{sc}} S_t(x) + 2T_{b_i} S_t(x, p_i),$$

where the second term is absent outside $\mathcal{J}$, one has

$$J_i(t, x) \geq M_i - N_i - (1 - \theta_i(t)) A_i - \theta_i(t) B_i \quad (t \in I_i,\ x \geq 6000000185827). \tag{123}$$

The numbers (119)–(122) *are exactly the* `main_lo`, `native_debit`, `first_hi` *and* `last_hi` *fields, respectively, in the supplied final* `MAIN_CLOCK_SOURCE.json`. *The complete signed nodes, derivative operators, thermal intervals and contact intervals used for this identification agree with the final kernel and arc plan.*

Proof. Step 1: identify the finite source and reconstruct its debit. Fix an actual x, t in the stated domain and freeze the common finite model (51) at that centre. The exact coefficient identity (61), the finite Euler multiplier and its restored prime-power terms identify the finite head with

$$P_{i,t,c}(\chi) = \frac{1}{2} \sum_{n=2}^{4096} d_t(n) n^{-1/2 - tc} F_i(n) \chi_n, \qquad |\chi_p| = 1, \quad \chi_{mn} = \chi_m \chi_n.$$

The physical tuple is used only after the inequalities for this finite polynomial have been established. The leading normaliser contributes $m_i^{\mathrm{mass}} c$; its discrepancy is charged in the coordinate allowance. The finite polynomial and the error measure come from the same common model, so an independently phased or differently truncated model is not substituted.

To give the debit explicitly, at a native height write the verified primitive bounds as $A_h, B_h, P_h, P_{1,h}, D_h^+, D_h^-, Z_h^-, T_h$. For the licensed activation amplitude and half-frequency moment $A_h^{\mathrm{act}}, D_h^{\mathrm{act}}$, put

$$a_h = P_h A_h^{\mathrm{act}}, \quad b_h = 2P_h D_h^{\mathrm{act}} + P_{1,h} A_h^{\mathrm{act}}, \quad g_h = 1 - A_h - a_h,$$

$$R_h = B_h - (1 - A_h) D_h^- - 2Z_h^-, \qquad r_{0,h} = \frac{P_h}{2000000}, \quad r_{1,h} = \frac{P_h}{20000} + \frac{P_{1,h}}{4000000}.$$

The primitive consumers check $R_h \geq 0$ and $g_h > r_{0,h} > 0$ before any division. The exact charged caps are

$$E_{\mathrm{an},h} = \frac{R_h + b_h + a_h D_h^+}{2g_h} + \frac{T_h}{2},$$
$$E_{\mathrm{ph},h} = \frac{r_{1,h}}{g_h - r_{0,h}} + \frac{(B_h + b_h) r_{0,h}}{2g_h(g_h - r_{0,h})} + 2 \cdot 10^{-12}. \tag{124}$$

The shared spectral envelope is applicable to the analytic caps by (71)–(72). Physical caps are combined with absolute node weights. Thus the shared-row global debit is exactly

$$D_i = \sum_l c_{i,l} E_{\mathrm{an},h_l}^{\mathrm{global}} + \sum_j |w_{ij}| E_{\mathrm{ph},h_{ij}}^{\mathrm{global}} + \frac{\sum_j |w_{ij}|}{125000000000} \quad (i \in \mathcal{S}).$$

The $c_{i,l} \geq 0$ are the coefficients of the verified full-half-line envelope of the actual scalar filter.

For a retained row, the prescribed fixed near branch combines either the absolute analytic caps or its licensed common spectral envelope, and adds the absolute physical caps and the coordinate allowance 10^{-9}. Denote this combined expression by $\mathcal{N}_i(s)$, where $s = (L - L_0)/2 = 2(c - c_0)$, and set $D_i = \mathcal{N}_i(0)$. The verified near/far inequalities and convexity in (70) give

$$\mathcal{N}_{i,\mathrm{actual}}(x,t) \leq D_i + \beta_i m_i^{\mathrm{mass}}(c - c_0), \qquad 0 < \beta_i < 1.$$

The remaining main slope is therefore $A_i^{\mathrm{rem}} = (1 - \beta_i) m_i^{\mathrm{mass}} > 0$. This reconstructs the actual intercept and slope from native moments; it does not infer them from an assembled source record.

The separate coordinate fees have an explicit uniform interpretation. Write $\zeta = h - iv$, $x' = x + v$, $L' = \log(x'/(4\pi))$ and $s_* = (1 + h)/2 - ix'/2$. The full padded native disks satisfy $0 \leq h \leq 6.01$ and $|v| < 2$; the jet disks satisfy the smaller $0 \leq h \leq 6$ and $|v| \leq 39/20$. Hence $|\Re s_*| \leq 4$ on all of these complex disks. For $s_* = u - ix'/2$, the chosen logarithm has the exact decomposition

$$\operatorname{Log} \frac{s_*}{2\pi} = L' - \frac{i\pi}{2} + \operatorname{Log}\left(1 + \frac{2iu}{x'}\right).$$

The inequality $|\operatorname{Log}(1 + w)| \leq |w|/(1 - |w|)$ for $|w| < 1$, and the displayed formula for α, give

$$|\alpha(s_*) - (L'/2 - i\pi/4)| \leq \frac{3}{x'} + \frac{4}{x' - 8} < \frac{10}{x'}, \qquad |\alpha'(s_*)| \leq \frac{1}{x'} + \frac{6}{(x')^2} < \frac{2}{x'},$$

and $|\alpha(s_*)| < L'/2 + 1$. Since $\partial_\zeta \log B_t = \alpha(s_*)/2 + t\alpha(s_*)\alpha'(s_*)/4$, it follows that

$$\begin{aligned} |\partial_\zeta \log B_t(x + i\zeta) - (L/4 - i\pi/8)| &< \frac{5 + t(L'/2 + 1)/2}{x'} + \frac{|L' - L|}{4} \\ &\leq \frac{32991}{5000(X - 2)} < 2 \cdot 10^{-12}. \end{aligned} \tag{125}$$

For the second line, the first term decreases for $x' \geq X - 2$, $t \leq 19/125$, and $\log((X-2)/(4\pi)) < 26.90$; also $|L' - L| \leq 2/(x - 2)$. This proves the bound on the entire complex disks, not just their real-height diameters.

Assign the scalar normaliser discrepancy entirely to the separate coordinate allowance. Its bound is $2\cdot10^{-12}\sum_j|w_{ij}|$. For a shared row this is at most the already charged $8\cdot10^{-12}\sum_j|w_{ij}|$. For every actual retained row, exact summation gives $\sum_j|w_{ij}|<100$, so it is less than the charged 10^{-9}. The direct consumer checks the required domination for each actual functional. The additional fixed $2\cdot10^{-12}$ in every native physical cap remains intact; it is not spent again in this argument or included in a refund. For a jet, the separately charged 10^{-10} bound is valid on its whole disk by (125); its height derivatives are paid by the independent jet Cauchy allowance.

Step 2: prove both complete raw head chords. Lemma 3.3 gives $0\le d_t(n)\le d_{t_i^+}(n)$ on I_i. Convexity of the exponential then gives, term by term,

$$d_t(n)n^{-tc_0}\le d_{t_i^+}(n)\big((1-\theta_i)n^{-t_i^-c_0}+\theta_i n^{-t_i^+c_0}\big).$$

After forming the signed filter and taking its modulus at each integer, (117) bounds the full triangle head by $H_i(t):=(1-\theta_i)H_{i,-}+\theta_iH_{i,+}$. No interpolation of sampled source values is involved.

Step 3: pay the scalar arithmetic improvement globally. For $i\in\mathcal{S}$, every fixed triad allocation spends at most the verified lower coefficient capacity, with repeated labels charged twice. Summing (131) therefore gives the simultaneous payment Δ_i^{fix}. The unused prime capacities, rather than their pre-allocation values, define the weighted Gram matrix. Its exact Loewner/LDL certificate and (132) give C_i^{near} on the complete physical near-coordinate band. At a fixed phase tuple, finite differentiation gives the remaining positive coordinate margin

$$\mu_i=m_i^{\mathrm{mass}}-t_i^+\sum_{n=2}^{4096}(\log n)^+U_{i,n,-}>0.$$

The first coordinate payment is $\mu_i\kappa_0$. Beyond the near band the remaining coordinate reserve gives C_i^{far} after subtracting that first payment. The verified inequality $C_i^{\mathrm{orbit}}\le\min(C_i^{\mathrm{near}},C_i^{\mathrm{far}})$ covers the two alternative coordinate domains. It follows that

$$m_i^{\mathrm{mass}}c-\Re P_{i,t,c}(\chi_{x,t})\ge m_i-H_i(t)+\Delta_i^{\mathrm{fix}}+\mu_i\kappa_0+C_i^{\mathrm{orbit}}.$$

The derivative argument is first uniform in the phase tuple and only then evaluated at $\chi_{x,t}$; it does not differentiate the oscillating physical tuple with respect to x.

For $i\in\mathcal{R}$, the composite allocations charge prime p the full amount $r_n\Omega(n)e_p$ for every allocated composite n. They give C_i^{comp} and leave the verified residual capacities $\alpha_{i,p}$. The same finite prime polynomial extends the composite bound beyond $L=27$ by the polydisc maximum principle, as proved in Appendix D. This spends none of A_i^{rem}.

For completeness, the total replacement d_i is paid as follows. Put $\Delta=c-c_0$, $\omega=4d_i/A_i^{\mathrm{rem}}<1$, $V=\omega/(1-\omega)$, $v=e^{4\Delta}-1$, and $q_{\mathrm{clk}}=t/(16N_0^2)$. For each actual relation and every complete sign branch, use its nonnegative multipliers to form

$$\mathcal{F}_i(v,q_{\mathrm{clk}})=\pi\sum_j\lambda_j(1+\sigma_j\delta_j(v,q_{\mathrm{clk}}))-\sum_p C^*(\alpha_{i,p},b_p)+\frac{A_i^{\mathrm{rem}}}{4}\log(1+v).$$

The checked odd aliases and the principal-angle conjugate include both endpoints and all stationary maxima. Joint concavity makes the four checked corner values a lower bound on the entire common-clock rectangle. All of them are at least d_i. If $\Delta\le d_i/A_i^{\mathrm{rem}}$, then $v\le e^{\omega}-1\le V$ and the near bound applies. If $\Delta\ge d_i/A_i^{\mathrm{rem}}$, then the global unused main term $A_i^{\mathrm{rem}}\Delta$ itself pays d_i. These are alternative domains, so the same main reserve is spent once. After the native affine debit has been subtracted, the result is

$$\mathcal{L}_i^{\mathrm{sc}}S_t(x)\ge m_i-D_i-H_i(t)+C_i^{\mathrm{comp}}+d_i\quad(i\in\mathcal{R}),$$

before the refund. This proves the first line of (118) from the total payment. Merely adding a new payment to the old source inequality would not justify this step.

Step 4: make the physical refund without changing the paid slope. At every scalar probe height the matched common-model theorem supplies one value/derivative error pair bounded by $\kappa_{i,h}(e_0, e_1)$, with $0 \le \kappa_{i,h} \le 1$. The quotient identity (129) shows that its variable physical cap scales by at most $\kappa_{i,h}$. Accordingly

$$\rho_i = \sum_j |w_{ij}|(1 - \kappa_{i,h_{ij}})(E^{\mathrm{paid}}_{\mathrm{ph},h_{ij}} - 2 \cdot 10^{-12}). \tag{126}$$

For a shared row the paid cap is global. For a retained row it is the anchor cap. The near physical cap increases with s, and the global physical cap dominates the anchor, so this anchor refund is valid on both branches of the already proved affine inequality. In particular, neither β_i nor its remaining main slope changes. Combining Steps 1–4 gives the direct scalar bound

$$\mathcal{L}^{\mathrm{sc}}_i S_t(x) \ge m_i - (D_i - \rho_i) - H_i(t) + C_i \quad (i = 0, \dots, 66). \tag{127}$$

Step 5: add the actual derivative functional. For each of the three derivative rows, the complete physical disk has one analytic common model, one Euler gap and one holomorphic logarithm of the physical/model quotient. Equations (78) and (79) charge all derivatives of that one logarithm. The positive-spectrum envelope (80), its full endpoint and critical-point proof, the complete finite head (81), and the separately paid normaliser give

$$T_{b_i} S_t(x, p_i) \ge m^{\mathrm{j}}_i - D^{\mathrm{j}}_i - (1 - \theta_i) H^{\mathrm{j}}_{i,-} - \theta_i H^{\mathrm{j}}_{i,+}.$$

The downward main bound uses $b_{i,0} \ge 0$. Exact equality of the final plan's α_i, p_i, b_i, old scalar probes, time interval and contact interval with the corresponding checked proposal is required here. Equality of a source table and a kernel table alone would not establish that the source proof concerned the same derivative operator.

Multiply (127) by $\alpha_i \ge 0$ and the jet bound by 2, then add. The coefficients are exactly (119)–(122), proving (123). For row 44, the order-eight jet is the jet in this formula; its order-four predecessor is not added.

Step 6: identify the operational data. The direct rational consumer `verification/direct_source/verify_direct_source.py` evaluates (119)–(122) from the primitive inputs and checks every one of the $4 \cdot 67$ identities against the actual final source fields. It also verifies (116) for the actual scalar mains and checks the functional and domain identities described in Step 5. The primitive native, activation, spectral, head, phase and jet consumers establish the inequalities used above; the direct consumer establishes their assembly and data identity. In particular, the row-44 critical-value list is checked to contain its endpoint and every isolated critical interval, so a shorter list cannot silently omit an extremum during paired iteration.

As a useful cross-check, on a retained row the old scalar head was $H_{i,\pm} - C^{\mathrm{comp}}_i - d^{(0)}_i$. The direct formula therefore changes that complete head by exactly $-\alpha_i(d_i - d^{(0)}_i)$, and leaves the main and native fields unchanged after their own prescribed refund/conic assembly. At row 8 the scalar difference is 29/10000. The former local native credit in row 55 occurs nowhere in (119)–(122). These identities confirm that no successive historical credit is being counted twice. □

Verification boundary. The reconstruction has three different kinds of input: the written analytic identification of the physical model and its continuous-domain extensions; exact or directed primitive inequalities checked by the native, activation, spectral, affine, head, common-clock and jet consumers; and the exact final assembly and functional identities. A

digest identifies the data, and a successful assembly check establishes the stated identities. Neither substitutes for the first two inputs. The present theorem concerns the source chords. The boundary kernels, global contact argument, initial ordinary-zeta theorem and exact flow remain the separate ingredients of the main theorem.

C The complete physical debit ledger

This appendix specifies how the physical approximation is consumed. A check of the rational ledger proves the advertised debit only after the common-model error theorem has been established. In particular, neither a saved success field nor an enclosure of a final debit proves that analytic theorem. Throughout, all functions, cutoffs, carrier phases, and errors refer to the same actual central pair (x, t).

C.1 From the common model to a logarithmic derivative

Let $G(\zeta) = H_t(x + i\zeta)/B_t(x + i\zeta)$ and let f_x be the finite common model in (51). At a native height let E denote the finite Euler multiplier, $Q = Ef_x$, and let $\Delta = E(G - f_x)$. Use the positive Euler norm P and its logarithmic moment P_1. Since the Euler frequencies are $\log d/2$,

$$|E| \le P, \qquad |E'| \le P_1/2.$$

Consequently a single physical error satisfying $|G - f_x| \le e_0$ and $|G' - f_x'| \le e_1$ gives

$$|\Delta| \le r_0 := Pe_0, \qquad |\Delta'| \le r_1 := Pe_1 + P_1e_0/2. \tag{128}$$

The term $P_1e_0/2$ is compulsory. It differentiates the same Euler multiplier used in the value comparison.

Let A, B be the nonconstant amplitude and frequency moment of the fixed native labels, and let a, b be their additional activation amplitude and frequency moment after multiplication by E. Thus

$$|Q| \ge g := 1 - A - a, \qquad |Q'| \le (B + b)/2.$$

Every denominator below is used only after proving $g > r_0$. Direct subtraction, before any estimate, gives

$$\frac{Q' + \Delta'}{Q + \Delta} - \frac{Q'}{Q} = \frac{\Delta'Q - Q'\Delta}{Q(Q + \Delta)}.$$

It follows that the logarithmic-derivative discrepancy is bounded by

$$\mathcal{P}(g, B + b; r_0, r_1) := \frac{r_1}{g - r_0} + \frac{(B + b)r_0}{2g(g - r_0)}. \tag{129}$$

The ledger stores $E_{\mathrm{ph}} = \mathcal{P} + 2 \cdot 10^{-12}$; the fixed last allowance is not part of the refundable error. The normaliser identity

$$\partial_h \log H_t = \partial_h \log B_t + G'/G$$

accounts separately for the leading main term and its coordinate allowance. It is not permissible to apply (129) to an unnormalised H_t and keep the normalised main term.

C.2 The value and derivative errors have one origin

For the licensed row $[t_-, t_+]$, let

$$S = (1 + h - 1/100)\frac{2689}{400} + t_- \left(\frac{2689}{400}\right)^2.$$

The uniform physical theorem supplies

$$E_0 = 8e^{-S} + 5 \cdot 10^{-10}, \qquad E_1 = 800e^{-S} + 7 \cdot 10^{-9}.$$

For clarity, its four contributions to the value and derivative estimates are as follows. These are deterministic bounds, not independent random errors.

Contribution	Value	Height derivative
Both complete heat remainders and the possible crossing pair	$< 8e^{-S}$	$< 800e^{-S}$
All individual heated-term errors	$< 5.5 \cdot 10^{-12}$	$< 5.5 \cdot 10^{-10}$
Exact local exponents to the local leading model	$< 2 \cdot 10^{-10}$	$< 2.5 \cdot 10^{-9}$
Local leading model to the central common model	$< 2.35 \cdot 10^{-10}$	$< 3.3 \cdot 10^{-9}$

The factor 100 in the first two derivative entries follows from Cauchy's formula on the radius-1/100 disk, applied to the discrepancy with the *fixed central cutoff*. No discontinuous moving truncation is differentiated. The remaining derivative entries are obtained by the product rule on their explicit finite models.

The complete heat remainder uses the corrected square-completion constant 11/12. In particular,

$$4\left(u^2 + \frac{11}{12}\right) - (u^2 + 6|u| + 2/3) = 3(|u| - 1)^2 \geq 0.$$

The Gaussian is integrated on the full real line. Its separate negative tail is bounded by $e^{-30000}/600$; neither Riemann–Siegel branch is removed. The estimate is the all-real-abscissa calculation of the cited Polymath expansion on $[-3, 4]$, with the explicit constants and domain guards in Proposition 3.2. The narrower printed numerical simplification alone would not license the present height-six probes.

To make the two leading-model transfers auditable, put $L' = \log(x'/(4\pi))$ and $C_\gamma = 10 + t(L'/2 + 1)$. The exact finite exponents and the local leading exponents have discrepancies at most

$$\delta_s = 5t/x', \quad \delta_s' = t/(2x'), \quad \delta_\gamma = yC_\gamma/x', \quad \delta_\gamma' = C_\gamma/x'.$$

The logarithm in δ_γ is integrated from height zero on one branch. Using the per-branch moments $6, 22, 180$, the product rule gives

$$\begin{aligned} \text{value} &\leq 4\delta_s\, 22 + 2\delta_\gamma\, 6, \\ \text{derivative} &\leq \delta_\gamma(6L' + 22) + \delta_s(22L' + 360) + 12\delta_\gamma' + 88\delta_s'. \end{aligned}$$

For example, the first-branch derivative error is at most $\delta_s M_2 + 2\delta_s' M_1$; the second branch supplies the other terms. This explains the second moment and prevents a value estimate from being differentiated without justification.

Finally, with

$$B_{\text{fr}} = \frac{30 + 2t(L + 3)}{x - 2},$$

the common-clock transfer pays $36B_{\mathrm{fr}}$ in value and $18B_{\mathrm{fr}}(L+1)+36/(x-2)$ in derivative. The last term pays the change in the reflected frequency. All these quantities are bounded at the anchor and then decrease: they are positive logarithmic polynomials of degree at most two divided by x' or $x-2$, and $L'>26.89$. This proves all-center coverage, rather than extrapolating from a numerical evaluation at $x=X$.

C.3 Exact primitive assembly and all future cutoffs

Let D^+, D^- and Z^- be the matched finite-head upper norm, lower norm and overlap lower bound, respectively. The positive residual identity gives

$$R_*=B-(1-A)D^- -2Z^- \geq 0.$$

For any of the licensed near or global activation bounds,

$$E_{\mathrm{an}}=\frac{R_*+b+aD^+}{2g}+\frac{T}{2},\quad E_{\mathrm{ph}}=\mathcal{P}(g,B+b;r_0,r_1)+2\cdot 10^{-12}. \tag{130}$$

Here T is the complete omitted Euler-power tail. The frequency moment must use

$$a=PA_{\mathrm{act}},\qquad b=2PD_{\mathrm{act}}+P_1A_{\mathrm{act}}.$$

Thus the derivative of the Euler factor is present in the activation ledger as well as in (128).

Near the anchor, A_{act} and D_{act} are positive polynomials in the activation length. Their composition with positive denominators gives increasing convex caps. Beyond the near split, the sharp combined-branch activation maxima in the activation appendix supply a bound for every $s\geq 0$. The old termwise $4C$ estimate does not prove the actual $2C$ table entry at $t_-=0,h=2$; the sharp combined-branch proof is required. With that correction, the same stored 469 activation entries are valid on the whole half-line.

Only analytic residual errors share the positive spectral measure. A licensed spectral envelope replaces their absolute node sum by a fixed nonnegative combination of the seven moments of that measure. Physical errors in (129) always use absolute node weights. In particular, cancellation in a signed spectral filter does not refund a physical error with no corresponding common-spectrum representation.

For a retained row, a fixed near branch $N(s)$ and a global constant G_* satisfy

$$N(R),G_*\leq N(0)+\beta MR/2.$$

Convexity for $0\leq s\leq R$ and the global cap for $s\geq R$ yield

$$N_{\mathrm{actual}}(L)\leq N(0)+\beta M(L-L_0)/4.$$

The remaining main slope is precisely $(1-\beta)M$; only that slope is available to the later arithmetic credit.

C.4 Refunds preserve both affine branches

The baseline errors are $e_0=5\cdot 10^{-7}$ and $e_1=5\cdot 10^{-5}$. For each actual native height choose a rational

$$\kappa\geq\max(E_0/e_0,E_1/e_1),\qquad 0\leq\kappa\leq 1.$$

Scaling the *same* value and derivative errors in (129) gives

$$\mathcal{P}(g,B+b;\kappa r_0,\kappa r_1)\leq\kappa\mathcal{P}(g,B+b;r_0,r_1).$$

The refund per unit absolute node weight is therefore

$$(1-\kappa)(E_{\mathrm{ph,old}} - 2\cdot 10^{-12}).$$

For retained rows this uses the anchor physical cap. The near physical cap increases in s and the global activation cap dominates the anchor, so this one refund is valid on both branches of the existing affine proof. Thus β and its paid slope are unchanged. For a shared row the refund uses its global physical cap. Fixed coordinate charges and the $2\cdot 10^{-12}$ allowance remain in the ledger.

C.5 Joint derivatives and the final 67-row identity

For a joint row, let $|G - f_x|/|f_x| \le \delta < 1$ on its whole licensed circle. There is one holomorphic logarithm $\ell = \log(G/f_x)$, with $|\ell| \le \eta := \delta/(1-\delta)$. Cauchy's formula gives

$$\left|\frac{1}{p}\sum_{m=0}^{M}\frac{b_m p^m}{m!}\ell^{(m+1)}(p)\right| \le \frac{\eta}{pr}\sum_{m=0}^{M}(m+1)|b_m|(p/r)^m.$$

All derivatives are of this one logarithm. The direct bound is $\delta \le E_0 P/g$. Since $e_1/e_0 = 100$, the baseline quotient ledger implies $e_0 P/g \le E_{\mathrm{ph,old}}/100$; this is the source of the circle estimate. There is no additional multiplication by P. The normaliser discrepancy is analytic on the same disk and is charged separately. Row 44 uses radius $39/20$, so its horizontal translations remain inside the physical-error domain; radius $49/25$ would require an additional domain proof.

Let D_i^{sc} be the scalar native debit after its proved refund. For rows other than $0, 16, 44$ the final native debit is exactly D_i^{sc}. For those three rows it is

$$D_i^{\mathrm{final}} = \alpha_i D_i^{\mathrm{sc}} + 2D_i^{\mathrm{jet}}.$$

The order-eight row-44 jet replaces its order-four predecessor; those jets are not added. The later main-clock gains change only the first and last finite-head costs. They do not change this native debit. The final source is consequently

$$\underline{J}_i(t) = M_i - D_i^{\mathrm{final}} - (1-\theta)A_i - \theta B_i.$$

The accompanying machine-readable debit map gives every rational quantity in these identities for all 67 rows, including the scalar baseline, refund, conic multiplier, jet debit, and final operational native debit. Their equality is checked with exact rational arithmetic. This ledger does not infer a physical theorem from a hash or a reported check count.

D Derivation and accounting of the arithmetic source payments

This section supplies the mathematical implications behind the arithmetic payments in all 67 source rows. It distinguishes the 28 redesigned scalar functionals from the 39 retained scalar functionals. The later conic derivative additions do not create a new arithmetic payment. All lower capacities, integer allocations, relation vectors and rational multipliers are data whose defining inequalities must be verified; an optimizer's status or an earlier acceptance flag is not a premise.

D.1 One finite polynomial and its coefficients

Put $K = 4096$, $N_0 = 690988$, $c_0 = \log N_0/2$, and $L_0 = 4c_0$. For the actual signed probe list (h_j, v_j, w_j) define

$$M = \sum_j w_j > 0, \qquad F(n) = \sum_j w_j n^{-h_j/2} \cos(v_j \log n/2).$$

The signed sum defining $F(n)$ is formed before taking its modulus. Let $b_t(n) = \exp(t \log^2 n/4)$. Formal Dirichlet convolution defines $d_t = (b_t \log) * b_t^{-1}$ coefficient by coefficient. Thus

$$d_t(n) = (\log n) b_t(n) - \sum_{\substack{r|n \\ 2 \le r < n}} d_t(r) b_t(n/r).$$

No infinite positive-time Dirichlet series is used. Differentiation of this finite coefficient identity gives

$$\partial_t d_t(n) = \frac{\log^2 n}{4} d_t(n) + \frac{\log n}{4} \sum_{\substack{ab=n \\ a,b \ge 2}} d_t(a) d_t(b).$$

Since $d_0(n) = \Lambda(n) \ge 0$, induction and an integrating factor prove nonnegativity and monotonicity on every closed thermal interval.

Write $s_n = \operatorname{sgn} F(n)$ and

$$a_n(t,c) = \tfrac{1}{2} d_t(n) n^{-1/2-tc} |F(n)|, \qquad P_{t,c}(\chi) = \sum_{n=2}^{K} s_n a_n(t,c) \chi_n.$$

Here $\chi_{mn} = \chi_m \chi_n$ is the actual completely multiplicative unit phase. The common finite-model and remainder calculation in Section 3.2 and Step 1 of Theorem B.1 reduces the source to $Mc - \Re P_{t,c}$ minus its fully paid analytic and physical debit. This section proves the arithmetic improvements of that finite polynomial bound. The source identification follows from the common model and remainder formulas, without assuming the completed uniform source theorem.

If $t \in [t_-, t_+]$ and $c = c_0$, monotonicity gives

$$a_n(t, c_0) \le (1-\theta) U_{n,-} + \theta U_{n,+}, \quad \theta = \frac{t - t_-}{t_+ - t_-},$$

where the stored upward capacities dominate

$$\tfrac{1}{2} d_{t_+}(n) n^{-1/2-t_- c_0} |F(n)|, \qquad \tfrac{1}{2} d_{t_+}(n) n^{-1/2-t_+ c_0} |F(n)|,$$

respectively. This follows from convexity of $e^{-tc_0 \log n}$ after freezing the same nonnegative $d_{t_+}(n)$. Both endpoint sums include every $2 \le n \le 4096$.

D.2 The redesigned 28 rows: fixed lower capacities

For these rows the stored lower capacities obey

$$0 \le \ell_n \le \tfrac{1}{2} d_{t_-}(n) n^{-1/2-t_+c_0} |F(n)| \le a_n(t, c_0).$$

Every one of the 28 actual rows selects the fixed-lower-capacity branch. The earlier endpoint amplitude-box branch is therefore not needed as a premise for their final support bounds.

For a frustrated edge (m, n, mn), $s_m s_n s_{mn} = -1$. Put $u = s_m \chi_m$ and $v = s_n \chi_n$, so the third gauged phase is $-uv$. A template $(A, B, C; p)$ proves

$$A(1 - \Re u) + B(1 - \Re v) + C(1 + \Re uv) \ge p. \tag{131}$$

The actual templates are validated by one of the following exact identities. For $A, B, C > 0$,

$$A\Re u + B\Re v - C\Re uv \le \tfrac{1}{2}(AB/C + AC/B + BC/A),$$

because the difference between the right and left sides is

$$\frac{C}{2} \left| \sqrt{A/B}\, u + \sqrt{B/A}\, \overline{v} - \frac{\sqrt{AB}}{C} \right|^2 .$$

Alternatively, $C(1/A + 1/B) \le 1$ and the weighted Cauchy inequality give the stronger deficit $2C$. For repeated labels $m = n$, the same phase occurs twice; the exact maximum of $(A + B)\cos u - C\cos(2u)$ is

$$\begin{cases} C + (A + B)^2/(8C), & A + B \le 4C, \\ A + B - C, & A + B \ge 4C. \end{cases}$$

The checker verifies that each stored p is no larger than the corresponding proved deficit. A dyadic allocation g spends Ag, Bg, Cg at its three labels and supplies pg.

Summing allocations is legal only when their complete cost at each label is at most ℓ_n. Both occurrences of a repeated label are charged. The remaining actual coefficients are nonnegative; their individual triangle deficits are nonnegative. Consequently the simultaneous fixed payment is

$$\Delta_{\mathrm{fix}} = \sum_e p_e g_e, \qquad \Re P_{t,c_0} \le \sum_n a_n(t, c_0) - \Delta_{\mathrm{fix}}.$$

The global coordinate reserve. For a fixed phase tuple, finite differentiation gives

$$\partial_c \{Mc - \Re P_{t,c}(\chi)\} \ge M - t_+ \sum_{n=2}^{K} (\log n)^+ U_{n,-} =: \mu > 0 \quad (c \ge c_0).$$

This does not differentiate the physical phase as x varies. At each physical point its actual tuple is inserted into a statement already proved for every tuple. Since $c = \log(x/(4\pi))/4$,

$$c - c_0 \ge \tfrac{1}{4}(1 - 4\pi N_0^2/x).$$

For $x \ge X_0 = 6 \cdot 10^{12}$ the right side exceeds $\kappa_0 = 1/5000000$. The first coordinate payment is precisely $\mu\kappa_0$; it is subsequently subtracted from every far reserve.

The remaining prime capacities and one Gram bound. After the complete fixed allocation, each used prime has a positive unspent lower capacity r_p. For the actual prime gauges let $\theta_p \in [-\pi, \pi]$ be their principal phases. A relation vector k_j is checked by multiplying its integer prime powers, checking its gauge parity, and enclosing its actual logarithmic ratio. On the whole band $X_0 \le x \le X_1 = 6.6 \cdot 10^{12}$, with the same thermal cell, the actual clock $\tau = x/2 + \pi t/8$ then gives

$$|k_j \cdot \theta| \ge \rho_j > 0.$$

The clock interval is checked continuously at its extreme x, t values, not on a phase grid. Let $R = \operatorname{diag}(r_p)$, let $s_j > 0$ be the stored rational row scales, and put

$$V_{jp} = s_j k_{jp}/\sqrt{r_p}, \quad G = VV^{\mathsf{T}}, \quad z_p = \sqrt{r_p}\,\theta_p.$$

All Gram entries are reconstructed exactly from the same unspent capacities. The certificate rounds G entrywise downward to a symmetric $\widetilde{G}$ on grid 2^{-64}. For m relations,

$$U = \widetilde{G} + m2^{-64} I \succeq G.$$

This uses the entrywise error bound and the row-sum bound on the operator norm. Its exact certificate is

$$U - UWU = \mathcal{L}\mathcal{D}\mathcal{L}^{\mathsf{T}}, \qquad W = \operatorname{diag}(w_j), \quad w_j \ge 0, \quad \mathcal{D} \succ 0.$$

Every multiplication identity and every diagonal sign is checked with fractions. Since $U = \mathcal{L}\mathcal{D}\mathcal{L}^{\mathsf{T}} + UWU$ is positive definite, this implies $W^{1/2}UW^{1/2} \preceq I$. Therefore $W^{1/2}GW^{1/2} \preceq I$ and $V^{\mathsf{T}}WV \preceq I$. This proof also covers singular G; no inverse of G is taken. It follows that

$$\sum_p r_p\theta_p^2 \ge E := \sum_j w_j(s_j\rho_j)^2.$$

The relations are one combined operator constraint, not separately funded expenditures from reset prime capacities.

From angle energy to an actual source payment. Define $r_* = \min_p r_p$ and $u = \min(E/r_*, 6)$. Since $u \le 6 < \pi^2$, the following bounds hold:

$$\sum_p r_p(1 - \cos\theta_p) \ge r_*(1 - \cos\sqrt{u}) \ge r_*\left(\frac{u}{2} - \frac{u^2}{24} + \frac{u^3}{720} - \frac{u^4}{40320}\right) =: C_{\mathrm{near}}. \tag{132}$$

Indeed, if some $|\theta_p| \ge \sqrt{u}$, that one term suffices. Otherwise $(1 - \cos v)/v^2$ decreases on $[0, \pi]$, and multiplication by the angle-energy bound proves the first inequality. The second is the alternating cosine bound on $0 \le u \le 6$.

For $x \ge X_1$, the global coordinate derivative instead supplies

$$C_{\mathrm{far}} = \mu\left\{\tfrac{1}{4}(1 - 4\pi N_0^2/X_1) - \kappa_0\right\} > 0.$$

Thus the exact additional orbit payment must obey $0 < C_{\mathrm{orbit}} \le \min(C_{\mathrm{near}}, C_{\mathrm{far}})$. The near and far arguments are alternative domain bounds, so their payments are not added. The redesigned scalar endpoints are exactly

$$H_\pm = \sum_n U_{n,\pm} - \Delta_{\mathrm{fix}} - \mu\kappa_0 - C_{\mathrm{orbit}}.$$

D.3 The retained 39 rows: composites and common clock

For these rows the native proof has already charged an affine growth cost $\beta M(c-c_0)$, with $0<\beta<1$. The main slope still available is $A=(1-\beta)M>0$. Lower capacities at thermal endpoint t_- and attenuation $t_+27/4$ are valid on the whole resource rectangle $t\in[t_-,t_+]$, $L_0\le L\le 27$.

For a composite $n=\prod p^{e_p}$ with $\Omega(n)=\sum e_p$ and a negative gauged coefficient, allocate r_n no larger than its actual available magnitude. Writing z_p for the gauged unit prime phases,

$$1-\Re\prod_p z_p^{e_p}\le\Omega(n)\sum_p e_p(1-\Re z_p)$$

by telescoping and Cauchy. Hence its full triangle deficit is at least

$$r_n\{1+\Re\prod_p z_p^{e_p}\}\ge 2r_n-r_n\Omega(n)\sum_p e_p(1-\Re z_p).$$

Prime p is therefore charged $r_n\Omega(n)e_p$. Summing every allocation proves the composite payment $C_{\mathrm{comp}}\le 2\sum_n r_n$ while leaving the exact residual prime capacities $\alpha_p>0$. These capacities are not restored before the angular duals are evaluated.

Why the composite payment is global. At fixed t, the complete finite head is the real part of a polynomial in its finitely many prime variables. The composite allocation gives a torus-uniform upper bound at $L=27$. For $L>27$ the same polynomial is evaluated at the radial contraction

$$z_p\longmapsto p^{-t(L-27)/4}z_p.$$

The maximum principle in each variable preserves that upper bound throughout the polydisc. The baseline endpoint chord at L_0 bounds the triangle mass at 27 as well. Thus the composite improvement continues globally without assuming positive constant lower capacities at arbitrarily large L, and without spending any of the main slope A.

The final total angular/main payment. Let d be the total angular payment that will replace the old angular payment. Put $\Delta=c-c_0$, $v=e^{4\Delta}-1$ and $q=t/(16N_0^2)$. A checked prime relation $R_j=\prod p^{k_{jp}}>1$ has $a_j=2N_0^2\log R_j$, a checked gauged odd target h_j, and

$$\frac{k_j\cdot\theta}{\pi}=2m_j+1+\delta_j,\qquad \delta_j=a_j(1+v+q)-h_j.$$

The exact source clock used here is $\tau=\pi(2e^L+t/8)$. On the verified alias rectangle $|\delta_j|<1$, the actual sign $\sigma_j=\operatorname{sgn}(k_j\cdot\theta)$ gives

$$\sigma_j k_j\cdot\theta\ge\pi(1+\sigma_j\delta_j).$$

For each complete sign branch use its fixed nonnegative rational multipliers λ_j, and form the combined coefficient $b_p=\sum_j\lambda_j\sigma_j k_{jp}$. The exact conjugate is

$$C^*(\alpha,b)=\max_{|u|\le\pi}\{bu-\alpha(1-\cos u)\}.$$

For $|b|<\alpha$ its maximum is the larger of

$$\pi|b|-2\alpha,\qquad \alpha\left[\frac{|b|}{\alpha}\arcsin\frac{|b|}{\alpha}+\sqrt{1-\frac{b^2}{\alpha^2}}-1\right];$$

for $|b| \geq \alpha$ only the endpoint maximum is needed. In particular, the global principal-angle endpoint is never dropped.

Choose $\omega = 4d/A < 1$ and $V = \omega/(1-\omega)$. On $0 \leq v \leq V$ and $q \in [t_-/(16N_0^2), t_+/(16N_0^2)]$, with $L_0 + \log(1+V) < 27$, define

$$\mathcal{F}(v,q) = \pi \sum_j \lambda_j(1+\sigma_j\delta_j) - \sum_p C^*(\alpha_p, b_p) + \frac{A}{4}\log(1+v).$$

This explicit lower payment is jointly concave, since its only nonlinear term has second derivative $-A/[4(1+v)^2]$. Its four corner values therefore bound it from below throughout the rectangle. All 128 branches of every seven-relation row, and all 256 branches of the eight-relation row 8, are verified. The same v, q is used in all relations before taking this bound.

If the four corner payments are at least d, the near interval $0 \leq \Delta \leq d/A$ is covered because $e^{4\Delta} - 1 \leq V$. On the complementary far interval, $A\Delta \geq d$. These alternatives cover every physical center and spend the retained main slope once. This argument asserts concavity of $\mathcal{F}$, not concavity of the native current itself.

D.4 Exact operational composition and review data

The retained scalar endpoints are their complete raw triangle endpoints minus $C_{\mathrm{comp}} + d$. The old angular payment d_0 is replaced by d, so the decrement relative to the pinned M8 parent is $d - d_0$, not another copy of d. For row 8, $d - d_0 = 29/10000$. The former row-55 local native credit is not used by the direct final construction.

At rows $0, 16, 44$ the source is a conic functional α_i times its complete scalar functional plus twice its separately charged derivative functional, with $\alpha_i > 0$. An arithmetic payment belonging to the scalar component is multiplied by α_i; it is not assigned to the derivative component. All main, native and head fields are combined in exactly these same units.

The companion files `review/arithmetic_credit/ALL67_ARITHMETIC_CREDIT_EXACT.csv` and `review/arithmetic_credit/ALL67_ARITHMETIC_CREDIT_AUDIT.json` give every scalar payment, conic factor, effective payment and final operational field as an exact rational. Their composition script performs 650 exact cross-table checks, including that all 28 redesigned rows actually use the fixed branch and that the final MAIN decrement changes neither main nor native debit. This composition audit does not replace the directed capacity checks or the exact allocation/dual feasibility checks. Those are provided by the fresh head, phase and source-overlay consumers in the bundle. Decimal displays in the associated table are explanatory only; the exact rational files are the review data.

Table 1: Actual arithmetic payments and operational values. R denotes a retained scalar row; S denotes a shared-resource scalar row. Decimal displays are for orientation; the exact CSV and JSON are authoritative. C_i is the total arithmetic payment in final conic units; g_i is the additional MAIN payment relative to the M8 parent, already included in C_i.

Row	Branch	C_i	g_i	Native debit	First head
0	R	0.007992754	0.000692071	0.007352276	0.830909465
1	R	0.008310455	0.000725020	0.004473392	0.841700097
2	R	0.008846060	0.000770300	0.005000931	0.860827810
3	R	0.009463726	0.000825200	0.005536952	0.886329286
4	R	0.009990737	0.000871920	0.006038487	0.910052483
5	R	0.010501562	0.000901010	0.006562747	0.931199801
6	R	0.011031045	0.000933730	0.007162690	0.954090179
7	R	0.011500752	0.001028680	0.007488877	0.981247488
8	R	0.013366970	0.002900000	0.007973102	0.999627216
9	R	0.012873033	0.001803780	0.010538224	1.028942841
10	R	0.013415637	0.001963190	0.010856963	1.059829839
11	R	0.014507521	0.002086750	0.017702977	1.098120775
12	R	0.015070784	0.002180770	0.018940933	1.126986096
13	R	0.015516128	0.002259060	0.019991563	1.152390263
14	R	0.016097677	0.002390950	0.021443059	1.186140203
15	R	0.015868172	0.002476250	0.014065630	1.203282544
16	R	0.016257247	0.002585420	0.014594437	1.231512530
17	R	0.016874786	0.002756000	0.015690624	1.269936367
18	R	0.017335611	0.002905560	0.016811373	1.305060028
19	R	0.017754214	0.003051390	0.017389834	1.339761469
20	R	0.018851274	0.003321610	0.018241455	1.366940774
21	R	0.018937267	0.003356330	0.018384089	1.378608746
22	R	0.019269513	0.003417300	0.019359209	1.395682437
23	R	0.019602610	0.003492240	0.020547219	1.414513496
24	R	0.019711483	0.003546330	0.021992492	1.429308430
25	R	0.020027084	0.003657460	0.023867433	1.452776191
26	R	0.020343147	0.003778830	0.026095236	1.477717729
27	S	0.028242099	0.000000000	0.044129721	1.467643936
28	S	0.026777735	0.000000000	0.045465678	1.484603418
29	S	0.027790520	0.000000000	0.045820570	1.498670787
30	S	0.028185726	0.000000000	0.046303642	1.511838457
31	S	0.026407072	0.000000000	0.047195919	1.528413131
32	S	0.028429880	0.000000000	0.051003043	1.559002153
33	S	0.028595615	0.000000000	0.051558876	1.572185886
34	S	0.028575250	0.000000000	0.052568393	1.586742435
35	S	0.028771523	0.000000000	0.052868914	1.600817782
36	S	0.029248344	0.000000000	0.057294700	1.635240981
37	S	0.029229774	0.000000000	0.058289519	1.650966508
38	S	0.028561564	0.000000000	0.058648779	1.665852839
39	S	0.028473963	0.000000000	0.059021661	1.679704793
40	S	0.029112812	0.000000000	0.063734400	1.717221416
41	S	0.027063314	0.000000000	0.064156008	1.734529499
42	S	0.027886168	0.000000000	0.065097015	1.747862654
43	S	0.027395467	0.000000000	0.066042023	1.761963696
44	S	0.026592799	0.000000000	0.069865179	1.805833736
45	S	0.026269794	0.000000000	0.069789725	1.818446555
46	S	0.025493330	0.000000000	0.071248660	1.833463484
47	S	0.024785208	0.000000000	0.070803068	1.847371066
48	S	0.025626963	0.000000000	0.074972823	1.891059020

Row	Branch	C_i	g_i	Native debit	First head
49	S	0.024393778	0.000000000	0.075222858	1.905153435
50	S	0.023581134	0.000000000	0.075283539	1.920317573
51	S	0.023000336	0.000000000	0.075384956	1.934066267
52	R	0.025088829	0.005539620	0.052801917	1.983306678
53	R	0.025257545	0.005551530	0.055196710	1.998797936
54	R	0.025171524	0.005460310	0.056277883	1.999146894
55	R	0.025408491	0.005538960	0.056820089	2.024425013
56	R	0.025653990	0.005623970	0.057063553	2.051073524
57	R	0.025835110	0.005664110	0.058453267	2.071897773
58	R	0.025775362	0.005604400	0.059338358	2.078165784
59	R	0.025996625	0.005645030	0.061728418	2.099323832
60	R	0.025858250	0.005533830	0.062200549	2.096631518
61	R	0.025859724	0.005563150	0.063602791	2.102464680
62	S	0.016206921	0.000000000	0.082599160	2.188309328
63	S	0.015637810	0.000000000	0.081251399	2.199655740
64	S	0.014936423	0.000000000	0.079801036	2.210346408
65	R	0.026341046	0.005533790	0.060396113	2.172074803
66	R	0.026560964	0.005592840	0.058847507	2.199420129

References

[1] N. G. de Bruijn, *The roots of trigonometric integrals*, Duke Math. J. **17** (1950), no. 3, 197–226. doi:10.1215/S0012-7094-50-01720-0.

[2] C. M. Newman, *Fourier transforms with only real zeros*, Proc. Amer. Math. Soc. **61** (1976), no. 2, 245–251. doi:10.1090/S0002-9939-1976-0434982-5.

[3] D. H. J. Polymath, *Effective approximation of heat flow evolution of the Riemann ξ function, and a new upper bound for the de Bruijn–Newman constant*, Research in the Mathematical Sciences **6** (2019), article 31. doi:10.1007/s40687-019-0193-1. Numbered references use arXiv:1904.12438v2.

[4] D. J. Platt and T. S. Trudgian, *The Riemann hypothesis is true up to* $3 \cdot 10^{12}$, Bull. Lond. Math. Soc. **53** (2021), 792–797. doi:10.1112/blms.12460; arXiv:2004.09765. Theorem 1 states the exact verification height 3000175332800.

[5] B. Rodgers and T. Tao, *The de Bruijn–Newman constant is non-negative*, Forum of Mathematics, Pi **8** (2020), e6, 62 pp. doi:10.1017/fmp.2020.6; arXiv:1801.05914.

[6] OpenAI, `lean/OAI/NumberTheory/DirichletL/Nonvanishing.lean`, Lean source for the ordinary-zeta declaration `OAI.riemannZeta_ne_zero_of_seven_eighths_lt_re`, in the official `openai/math` repository, revision `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`. `https://github.com/openai/math/blob/fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb/lean/OAI/NumberTheory/DirichletL/Nonvanishing.lean`. The independent rebuild in [7] uses revision `adc7f1241b42e322a6451854ab7e4b4c146bf78a`; the relevant proof subtree and build configuration match the retained source pin.

[7] tomoto0, *Quasi-Riemann hypothesis* 7/8 *verification*, source/build report and explicit ordinary-zeta challenge, `https://github.com/tomoto0/quasi-riemann-hypothesis-7-8-verification`, commit `b83a73c6cb626a391e551baafdb7d0cceafaa55a`. The external report is supplied as verification evidence; the present work did not repeat the full Lean build locally.

[8] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Transactions on Computers **66** (2017), 1281–1292. doi:10.1109/TC.2017.2690633.

[9] D. J. Raymond, *Retuned translated probes and the bound for the de Bruijn–Newman constant*, public manuscript and proof materials, `https://github.com/Dale-Dug/de-bruijn-newman-0116398`, commit `4694f9ab2e8bc88aeeae1949d2e690c2360d2eba`. The pinned repository states $\Lambda \leq 0.116397726408291$; its October 26 update adds the ordinary 7/8 formal-input evidence.

[10] D. J. Raymond, *Proof and replay materials for* $\Lambda \leq 0.107964910079$, exact-version research and publication packet, version 1.0, 29 October 2026. DOI: 10.5281/zenodo.23752103. The accompanying manifest specifies each source file, certificate, checker, size and SHA-256 digest. The proof does not depend on the historical names of the research rounds.

[11] D. J. Raymond, *Phase-correlated certificates and the bound* $\Lambda < 0.163297$ *for the de Bruijn–Newman constant*, Zenodo preprint, 8 September 2026. doi:10.5281/zenodo.22626161.

[12] D. J. Raymond, *Correlated translated probes and the bound* $\Lambda < 0.149396$ *for the de Bruijn–Newman constant*, Zenodo preprint, 21 September 2026. doi:10.5281/zenodo.22867547.

[13] D. J. Raymond, *Retuned translated probes and the conditional bound $\Lambda < 0.116398$ for the de Bruijn–Newman constant*, Zenodo preprint, 10 October 2026. doi:10.5281/zenodo.23272725.

[14] Cetin Hakimoglu-Brown. *A conditional bound $\Lambda \leq 0.13963$ for the de Bruijn–Newman constant*. 8 October 2026. DOI: 10.48550/arXiv.2610.12803. arXiv:2610.12803v1.

[15] Chiara Bellotti, Tim Trudgian, Andrew Yang. *Zero-free regions inspired by work of Heath-Brown*. 23 March 2026. DOI: 10.48550/arXiv.2603.21490. arXiv:2603.21490v1.

[16] Daniel Martin Kornaus. *Collision Exclusion in the de Bruijn–Newman Heat Flow*. 8 September 2026. DOI: 10.5281/zenodo.22658444. Zenodo preprint, record 22658444.

[17] H. Ki, Y.-O. Kim and J. Lee, *On the de Bruijn–Newman constant*, Advances in Mathematics **222** (2009), no. 1, 281–306. doi:10.1016/j.aim.2009.04.003.

[18] M. Planat, *Collective Contraction in the de Bruijn–Newman Heat Flow: Off-Zero Logarithmic Derivatives and Certified Barrier Refinements*, preprint, version 2, 7 October 2026, 54 pp.; first version 29 September 2026. arXiv:2609.37164v2; doi:10.48550/arXiv.2609.37164.

[19] M. Planat, *A certified classical-strip comparison for the de Bruijn–Newman constant*, preprint, 13 October 2026, 19 pp. arXiv:2610.18214v1; doi:10.48550/arXiv.2610.18214.

[20] D. J. Raymond, *Analytic foundations for the time-zero upper-bound construction*, version 1.0, 21 September 2026, 19 pp. Companion manuscript in the deposit [12], doi:10.5281/zenodo.22867547; supplied as `Prior_Analytic_Foundations.pdf`. The original public companion is also retained in `https://github.com/Dale-Dug/de-bruijn-newman-0149396`.

[21] Y. Xu and H. Sang, *Amplified moments and a refinement of the quasi-Riemann hypothesis*, preprint, version 2, 15 October 2026; first version 11 October 2026. arXiv:2610.15291v2; doi:10.48550/arXiv.2610.15291.

[22] B. Liu, *Improved zero-free half-planes for the quasi-Riemann hypothesis*, preprint, version 2, 12 October 2026; first version 8 October 2026. arXiv:2610.12234v2; doi:10.48550/arXiv.2610.12234.

[23] B. Liu, *Zero-free half-planes for Hecke L-functions over number fields*, preprint, submitted 19 October 2026. arXiv:2610.26835v1; doi:10.48550/arXiv.2610.26835.

[24] OpenAI. *The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane* $\mathrm{Re}(s) > 7/8$. 30 September 2026. `https://github.com/openai/math/blob/main/preprints/The-Quasi-Riemann-Hypothesis-September-30-2026/paper.pdf`.

[25] Mosaic Intelligence. *A certified unconditional upper bound $\Lambda \leq 0.1965$ for the de Bruijn–Newman constant*. 2026. DOI: 10.5281/zenodo.20724170.

[26] Mosaic Intelligence. *A certified unconditional upper bound $\Lambda \leq 0.1875$ for the de Bruijn–Newman constant*. 2026. DOI: 10.5281/zenodo.21175533.

[27] Mosaic Intelligence. *A certified unconditional upper bound $\Lambda < 0.16$ for the de Bruijn–Newman constant*. 2026. `https://github.com/463464q435q43/optimizationproblems/tree/6baae4c15dccec1e94ac011de399ec382fcea7b6/certificates/certified0160`. Repository revision `6baae4c15dccec1e94ac011de399ec382fcea7b6`.

[28] Mosaic Intelligence. *A certified unconditional upper bound $\Lambda < 0.1595$ for the de Bruijn–Newman constant*. 2026. https://github.com/463464q435q43/optimizationproblems/tree/cc8107854fedd5a96d7eb695178f405568712f35/certificates/certified01595. Repository revision cc8107854fedd5a96d7eb695178f405568712f35.

[29] Jude Gomila. *A computer-assisted proof of the bound $\Lambda \leq 0.1787854$ for the de Bruijn–Newman constant*. 20 August 2026. https://github.com/judegomila/dbn-lambda-01787854-candidate-audit. Author's primary release manuscript; not yet peer reviewed.

[30] Stefan Gordon. *A computer-assisted upper bound for the de Bruijn–Newman constant*. 12 September 2026. DOI: 10.2139/ssrn.7608638. Preprint; author-hosted PDF dated September 2026. SSRN posting 14 October 2026, revised 15 October 2026.

[31] T. Tao and contributors, *de Bruijn–Newman constant*, *Optimization Problems and Constants*, maintained catalogue, accessed 29 October 2026. https://teorth.github.io/optimizationproblems/constants/21a.html.