Realizer Bounds and Repetition Constraints at the First Coefficient Crossing in the $3x + 1$ Problem

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Abstract

Terras's coefficient stopping time conjecture asks whether the shortcut Collatz orbit of every integer $n \geq 2$ lies below n when its coefficient $\rho_k = 3^d/2^k$ first falls below one. Here k counts shortcut steps and d counts their odd inputs. We prove that every eligible realizer descends when the first-crossing word has at most 81 maximal runs of odd inputs. Any counterexample requires at least 82 such runs, 10 439 860 591 steps and 6 586 818 670 odd inputs. The proof uses a lower bound on the seed from the length of its odd runs, rational approximations to $\log_2 3$, the verified range of Rozier and Terracol, and Rhin's estimate for a linear form in logarithms. The run bound is applied not once but at several interior starts, selected by the size of their contributions to the correction term. We also prove descent for a family with arbitrarily many odd runs. Two further restrictions narrow the remaining words. An initial square UU in a failing word satisfies $|U| < 35 \log_2 k$. For a counterexample of shortest crossing length, the reciprocal-sum estimate of Nikzad, Frinkle and Bagherzadeh gives a lower bound on ρ_k whose value begins $0.9441733669\ldots$. The open inequality compares the least eligible starting value in each remaining parity class with an explicit threshold.

1 What is being asked

Write $T(n) = (3n + 1)/2$ at odd inputs and $T(n) = n/2$ at even inputs. Fix $n \geq 2$. For the parity word w of its first k input states $n, T(n), \ldots, T^{k-1}(n)$, let d count the odd inputs. Induction gives

$$2^k T^k(n) = 3^d n + C_w, \tag{1}$$

where the nonnegative integer C_w depends only on w. Write $\rho_k = 3^d/2^k$ with $\rho_0 = 1$. An odd step multiplies the coefficient by $3/2$ and an even step halves it. Let $\tau(n)$ be the first k with $\rho_k < 1$, and let $\chi(n)$ be the first k with $T^k(n) < n$. An empty defining set gives stopping time ∞. Equation (1) gives $\tau(n) \leq \chi(n)$, so any failure of Terras's conjecture $\tau(n) = \chi(n)$ has finite $k = \tau(n)$ and $T^k(n) \geq n$ [16, 5]. An even starting value descends at its first step, so a failure has odd n and $d \geq 1$.

A first-crossing word has $\rho_j \geq 1$ for every $j < k$ and $\rho_k < 1$. Its last step is even, so $\rho_k \in [1/2, 1)$. With $d \geq 1$, this also gives $2^{k-1} \leq 3^d < 2^k$ and $k = \lceil d \log_2 3 \rceil \leq 2d$. An *odd run* is a maximal block of consecutive odd input states.

2 The signed comparator

Dividing (1) by 2^k gives the form we use throughout.

Proposition 1. *With $M_w = C_w/3^d$,*

$$T^k(n) = \rho_k\,(n + M_w), \qquad \textit{so} \qquad T^k(n) < n \iff n\big(\rho_k^{-1} - 1\big) > M_w.$$

Moreover, if the odd runs have lengths $\ell_1, \dots, \ell_s$ and run i starts at time t_i,

$$M_w = \sum_{i=1}^{s} \frac{1 - (2/3)^{\ell_i}}{\rho_{t_i}}. \tag{2}$$

Proof. The first display is (1). For (2), each odd step at time j contributes $2^j 3^{-d_{j+1}} = 1/(3\rho_j)$ to M_w, where d_{j+1} counts odd steps through time j. Summing a run of ℓ consecutive odd steps from t gives $\frac{1}{3\rho_t}\sum_{r<\ell}(2/3)^r = (1 - (2/3)^\ell)/\rho_t$. □

If a first-crossing word contains $s \geq 1$ odd runs, every proper prefix has $\rho_t \geq 1$, so each run contributes less than one and each odd step contributes at most $1/3$. Thus

$$M_w < s, \qquad M_w \leq d/3. \tag{3}$$

The two bounds depend respectively on the number of runs and on the total number of odd inputs.

Remark 2 (Testing all eligible realizers of a word). A length-k parity word specifies one residue class modulo 2^k. Indeed, the two lifts of a given length-j class modulo 2^{j+1} have starting values that differ by 2^j. Their images after j steps differ by 3^{d_j}, an odd integer, so exactly one lift has the prescribed next parity. For a first-crossing word, let $r(w)$ be the least integer ≥ 2 in its class. Every eligible realizer descends if and only if $r(w)(\rho_k^{-1} - 1) > M_w$, since the left side increases with the seed. If a seed n fails, its endpoint satisfies $0 \leq T^k(n) - n < s$ by Proposition 1 and (3).

Remark 3 (What is classical here). Almost all of the above is in Lagarias's survey [10, §2.2–§2.3], and the reader should not take any of it as new.

The residue-class correspondence is Terras [16, Thm. 1.2, Cor. 1.3] and is Theorem B of [10]; Garner [3, §4] gives it in the coefficient-stopping-time form, modulo $2^{1+p(m)}$. Lagarias's equations (2.4)–(2.6) are Proposition 1: his $T^{(k)}(n) = \lambda_k(n)n + \rho_k(n)$ has $\lambda_k = \rho_k$ in our notation and $\rho_k(v) = C_w/2^k$, so that $M_w = \rho_k(v)/\lambda_k$. His (2.13)–(2.14) then state that the elements of a class with

$$n \;>\; \varepsilon_k^{-1}\rho_k(v), \qquad \varepsilon_k = 1 - \lambda_k(v),$$

have $\omega(n) = \sigma(n)$, and rearranging that inequality gives $n(\rho_k^{-1} - 1) > M_w$ *exactly.* **The descent criterion above is therefore Lagarias's, not ours.**

Sharper still, Theorem E of [10] states that there is an effectively computable k_0 such that for admissible $\mathbf{v}$ of length $k \geq k_0$, every element of $S(\mathbf{v})$ except possibly the smallest has stopping time k. Its proof is the one we use: a linear forms in logarithms bound (there Baker–Feldman, $|k\log 2 - l\log 3| \geq k^{-c_0}$; here Rhin's $k^{-13.3}$), fed into (2.14). **"Only the least element of the class can fail" is thus Theorem E, asymptotically in k.**

What the present section adds is only this. The criterion is stated with no "sufficiently large" and no threshold k_0: it is a single monotone comparison valid at every k, so one evaluation at $r(w)$ decides the class outright. And a failure is pinned to the s endpoints $n, \dots, n + s - 1$. The quantitative work is in §3, where the run decomposition (2) replaces Lagarias's cruder admissibility estimate $\rho_k(\mathbf{v}) \leq k2^{(1-\theta)k}$. We do *not* make Lagarias's k_0 explicit: his k_0 is a threshold on the number of *steps* k, whereas our hypothesis bounds the number of *runs*, and his conclusion exempts the smallest element of the class while ours covers it. The two statements are therefore distinct. What we do make explicit, in Corollary 9, is the exemption itself.

Example 4. The exact odd-to-odd valuation word $(1, 2, 1, 3, 1)$ from [8, App. A.6(b)] has eight shortcut input bits 11011001 and an odd terminal state. The eight input bits give the class 187 mod 256. Terminal oddness adds a ninth bit and selects 187 mod 512. Indeed, $T^8(187) = 179$, while the other lift, 443, has $T^8(443) = 179 + 3^5 = 422$. The first coefficient crossing already occurs at step 7. Its prefix 1101100 gives the class 59 mod 128, and

$$59 - T^7(59) = \frac{(2^7 - 3^4) \cdot 59 - 85}{2^7} = 21.$$

Thus $T^7(59) = 38$ and every eligible seed in $59 + 128\mathbb{Z}$ descends at step 7. Within a period of 512 there are four representatives of this crossing class, of which one realizes the full valuation word.

3 Words with at most 81 odd runs

Lemma 5 (run-length realizer bound)**.** *If an odd run starts at x and lasts ℓ steps, the endpoint plus one equals $(3/2)^\ell(x+1)$, so $2^\ell \mid x+1$. At each odd step $x+1$ is multiplied by $3/2$, and at each even step it decreases. Writing D_i for the number of odd steps before run i and $L = \log_2(n+1)$, we get $2^{\ell_i} \le (n+1)(3/2)^{D_i}$, hence $D_{i+1} \le L + \alpha D_i$ with $\alpha = \log_2 3$, and therefore*

$$d \le G_s L, \qquad G_s = \sum_{j=0}^{s-1} \alpha^j.$$

Neither ingredient is new. The divisibility $2^\ell \mid x + 1$ across a run of ℓ odd steps appears explicitly as Lemma 8 of Hercher [4], written there as $1 \le k_i \le x_i$ with $x_i = \log_2(n_i + 1)$; that is the antecedent we are aware of. Garner's contribution is separate and is recorded below: the coefficient stopping time itself [3, §4] and the Diophantine study of powers of 2 against powers of 3 [3, §5–§6]. The geometric sum $G_s = (\alpha^s - 1)/(\alpha - 1)$ is likewise the sum $(\delta^{m_2} - 1)/(\delta - 1)$ appearing in Hercher's Theorem 21, with the *same* constant $\delta = \log_2 3 = \alpha$; his recurrence $x_{i+1} < \delta x_i$ is the purely geometric form that cyclicity permits, while the affine form $D_{i+1} \le L + \alpha D_i$ above re-injects the seed size L at every run because a finite path has a fixed starting point.

The method of the next theorem — a Diophantine bound on powers of 2 against powers of 3, combined with a finite verification range — is Garner's [3, §6–§7]; see Remark 11.

Lemma 5 charges the seed size L once per run, so applying it from the seed alone costs a factor G_s. The next two lemmas apply it instead at several interior starts. Which starts are available is decided by the correction term: a run whose contribution m_i to M_w is large has small ρ_{t_i}, hence small x_i.

Lemma 6 (splitting)**.** *Let a first-crossing word have $s \le 81$ odd runs, let its realizer fail to descend, and suppose $0 < \lambda < \Lambda$. Then*

$$n + 1 < 1 + 81/\lambda. \tag{4}$$

If moreover $\lambda > r/B$ for an integer $r \ge 1$, then $s > r$, and among the $s - 1$ noninitial runs the r with the largest contributions m_i all begin at integers satisfying

$$x_i + 1 < 1 + \frac{(81 - r)\,(1 + 81/B)}{\lambda - r/B}. \tag{5}$$

The two bounds are proved together but neither contains the other: putting $r = 0$ in (5) would leave a spurious factor $1 + 81/B$, so (4) is the sharper statement for the initial run.

Proof. A failure gives $(e^{\Lambda}-1)n \le M_w$ by Proposition 1, and $e^t - 1 > t$, so $n\Lambda < M_w$. Each $m_i < 1$ by (2), so $M_w < s \le 81$, and $n\Lambda < 81$ already gives (4). The initial run needs this separate treatment: it has $\rho_{t_1} = 1$ and $x_1 = n$, so no height reduction is available for it.

Now let $r \ge 1$. With $n > B$ and $\lambda > r/B$,

$$s > M_w > n\Lambda > B\lambda > r,$$

so $s \ge r+1$ and at least r noninitial runs exist. Order the noninitial contributions $m_{(1)} \ge \cdots \ge m_{(s-1)}$ and put $m^* = m_{(r)}$, the smallest selected one. Bound by 1 exactly r contributions, namely the initial m_1 together with the $r-1$ larger selected ones $m_{(1)}, \dots, m_{(r-1)}$. Every one of the remaining $s-r$ contributions, which are $m_{(r)} = m^*$ itself and $m_{(r+1)}, \dots, m_{(s-1)}$, is at most m^*. Hence

$$n\Lambda < M_w < r + (s-r)\, m^*, \qquad \text{so} \qquad m^* > \frac{n\Lambda - r}{s-r}.$$

It matters that $m_{(r)}$ is counted in the second group: moving it to the first gives $r+1$ discards and replaces $\lambda - r/B$ by $\lambda - (r+1)/B$, which is negative for the three rows of Theorem 8 with $r = 2, 3, 4$ and positive only for the row with $r = 7$. The remaining row has $r = 0$: it selects no noninitial run, so there is no $m_{(r)}$ to move and the discard count does not arise for it. The hypothesis actually used, $\lambda > r/B$, holds in every row.

For a selected index i we have $m_i \ge m^*$, and $m_i = (1 - (2/3)^{\ell_i})/\rho_{t_i}$ gives $\rho_{t_i} < 1/m_i$. Since $x_i = \rho_{t_i}(n + M_i)$ with $M_i \le M_w < 81$,

$$x_i < \frac{n+81}{m^*} < \frac{(n+81)(s-r)}{n\Lambda - r} = \frac{(1+81/n)(s-r)}{\Lambda - r/n}.$$

The seed enters both parts, and $n > B$ improves both: $81/n < 81/B$ shrinks the numerator while $-r/n > -r/B$ enlarges the denominator, as does $\Lambda > \lambda$. With $s \le 81$ this is (5). □

The conclusion of Lemma 6 is an *absolute* height bound, not a statement about the relative drift ρ_{t_i}. Only one implication is used: a *selected* run has $m_i \ge m^*$, hence $\rho_{t_i} < 1/m^*$. The converse fails, because m_i depends on the run length as well as on ρ_{t_i}: a run with small ρ_{t_i} but small ℓ_i also has small m_i and may well be discarded.

Lemma 7 (restart)**.** *Order the r selected starts of Lemma 6 chronologically and prepend the initial start. If each of these $r+1$ starts satisfies $x + 1 < 2^H$, then*

$$d < H\big(G_{81-r} + r\big).$$

Proof. The $r+1$ starts cut the run sequence into $r+1$ consecutive segments, segment j beginning at one of them and containing $t_j \ge 1$ runs, with $\sum_j t_j = s$. Lemma 5 is local: the divisibility $2^{\ell} \mid x+1$ involves only the run's own starting value, and the recurrence $D_{i+1} \le L + \alpha D_i$ compares consecutive runs. Neither uses the first-crossing hypothesis or the identity of the seed, so the lemma may be restarted at an interior start with its own $L = H$: a segment of t runs beginning at x with $\log_2(x+1) < H$ contains fewer than $H\, G_t$ odd steps. Summing, $d < H \sum_j G_{t_j}$. Now $G_j = (\alpha^j - 1)/(\alpha - 1)$ is convex and increasing with $G_1 = 1$, so among compositions of s into $r+1$ positive parts the sum $\sum_j G_{t_j}$ is largest when one part takes everything available, giving $G_{s-r} + r\, G_1 = G_{s-r} + r$. Two adjacent selected starts, or a final segment holding a single run, contribute $G_1 = 1$ each. The crossing itself lies in the even block after the last run and adds to k but not to d. With $s \le 81$ the stated bound follows, and $r = 0$ recovers $d < H\, G_{81}$. □

Theorem 8. *Every eligible realizer of a first-crossing word with at most* 81 *odd runs descends at that crossing. Any counterexample to Terras's conjecture has at least* 82 *odd runs, crossing time $k \ge 10\,439\,860\,591$ and odd-input count $d \ge 6\,586\,818\,670$.*

Proof. Rozier and Terracol [14, Cor. 5.4] verify $\chi(n) = \tau(n)$ for $2 \le n \le B$ with $B = 28 \cdot 10^{18}$, so a failure has $n > B$. Put $\Lambda = k \ln 2 - d \ln 3 > 0$. By Proposition 1 a failure gives $(e^{\Lambda} - 1)n \le M_w$, hence with (3)

$$0 < \Lambda < s/n < s/B, \qquad 0 < \Lambda/d < 1/(3n) < 1/(3B). \tag{6}$$

The two large integers. Take the convergents of $\alpha = \log_2 3$ of indices 20 and 19, respectively:

$$(a, b) = (9\,809\,721\,694,\ 6\,189\,245\,291), \qquad (u, v) = (630\,138\,897,\ 397\,573\,379).$$

One certifies $a/b < \alpha$ and $(u/v) \ln 2 - \ln 3 > 1/(3B)$. With the second half of (6) this forces $a/b < \alpha < k/d < u/v$. The determinant $bu - av = 1$ makes $bk - ad$ and $ud - vk$ positive integers, and expanding gives $(k, d) = (ud - vk)(a, b) + (bk - ad)(u, v)$. Hence $k \ge a + u = 10\,439\,860\,591$ and $d \ge b + v = 6\,586\,818\,670$; this pair is the convergent of index 21, the mediant of the two used, and it is the pair that occurs in Eliahou's work on nontrivial cycle lengths [2].

Excluding at most 81 runs. Suppose now that $s \le 81$, so that $\Lambda < 81/B$. For $p/q < \alpha$ put $e = q \ln 3 - p \ln 2 > 0$. Since $k/d > \alpha$, $q\Lambda + de = (qk - pd) \ln 2 \ge \ln 2$. Four distinct convergents are used below: one for the lower bound on d and three in the table, namely

I	(9 881 527 843 552 324,	6 234 549 927 241 963),	$e_{\mathrm{I}} \in (5.22, 5.24) \cdot 10^{-18}$,
II	(79 641 170 620 168 673 833,	50 247 984 153 525 417 450),	$e_{\mathrm{II}} \in (6.71, 6.72) \cdot 10^{-21}$,
III	(6 724 555 128 221 608 268,	4 242 721 909 926 539 673),	$e_{\mathrm{III}} \in (1.24, 1.25) \cdot 10^{-19}$,
IV	(423 372 672 964 960 618,	267 118 416 222 671 843),	$e_{\mathrm{IV}} \in (1.64, 1.65) \cdot 10^{-18}$.

Convergent I gives $d > (0.69 - q \cdot 81/B)/(5.24 \cdot 10^{-18}) > 1.28 \cdot 10^{17}$. This is the lower bound that the rest of the proof must contradict.

The head. Read the same identity in the other direction: for a convergent $p/q < \alpha$ and any upper bound $d \le d^{\dagger}$ one gets $\Lambda \ge (\ln 2 - d^{\dagger} e)/q =: \lambda$. Each such λ feeds Lemma 6, whose bound and the initial-start bound give an H with $x + 1 < 2^H$ at all $r + 1$ segment starts, and then Lemma 7 returns a new upper bound $d < H(G_{81-r} + r)$. Only that upper bound is carried to the next row; the selected starts and the convergent may change between rows. Starting from $d < 2^{65}$:

r	H	$\lambda \ge$	new bound on d	q of the convergent used
0	73	$8.8606 \cdot 10^{-21}$	$1.9886 \cdot 10^{18}$	50 247 984 153 525 417 450
2	71	$1.0491 \cdot 10^{-19}$	$7.6987 \cdot 10^{17}$	4 242 721 909 926 539 673
3	71	$1.4074 \cdot 10^{-19}$	$4.8572 \cdot 10^{17}$	4 242 721 909 926 539 673
4	74	$1.4909 \cdot 10^{-19}$	$3.1940 \cdot 10^{17}$	4 242 721 909 926 539 673
7	68	$6.3202 \cdot 10^{-19}$	$7.3709 \cdot 10^{16}$	267 118 416 222 671 843

The last row gives $d < 7.3709 \cdot 10^{16}$, contradicting $d > 1.28 \cdot 10^{17}$. Each λ exceeds r/B, as Lemma 6 requires, and each is computed from the *printed* ceiling of the row above, so the table may be checked line by line.

The tail. For $d \ge 2^{65}$, Rhin's effective estimate [13] (in the form used by [14, Prop. 6.3]) gives $\Lambda \ge k^{-13.3}$, so $n < 81(2d)^{13.3}$ and $\log_2(n + 1) < 7 + 13.3 \log_2(2d)$. With $2^j \le d < 2^{j+1}$, Lemma 5 applied from the seed alone forces $2^j \le d < G_{81}\,[\,7 + 13.3(j + 2)\,]$ with $G_{81} < 2.7241 \cdot 10^{16}$. At $j = 65$ the bracket is 898.1 and the upper bound is $2.7241 \cdot 10^{16} \times 898.1 < 2.4466 \cdot 10^{19} < 2^{65}$, so the two printed constants give the inequality directly. It stays below 2^j by induction, since the bracket increases by 13.3 while 2^j doubles. At $j = 64$ the inequality fails, so 2^{65} is the first dyadic split that closes this tail estimate. This contradiction excludes the remaining values of d. □

The next corollary concerns *all* words, with no hypothesis on the run count. It is the effective form of the exemption in Lagarias's Theorem E: that theorem leaves the smallest element of a class unresolved for $k \ge k_0$, with k_0 effectively computable but not evaluated there, and the statement below removes the length restriction altogether.

Corollary 9. *Every counterexample to Terras's conjecture is the least eligible seed of its parity class.*

Proof. Let n fail at crossing length k. By Proposition 1 and (3), $n\Lambda < n(e^{\Lambda}-1) \le M_w \le d/3$, and Rhin's estimate gives $\Lambda \ge k^{-13.3}$, so $n < (d/3)k^{13.3} \le k^{14.3}/3$. A length-$k$ parity word specifies a single class modulo 2^k, so every eligible seed of that class except the least exceeds 2^k. For $k \ge 128$ one has $14.3\log_2 k < k$, hence $k^{14.3} < 2^k$ and so $n < 2^k$; and $k \ge 10\,439\,860\,591$ by Theorem 8. Therefore n is the least element of its class. □

Remark 10 (Dependencies of Theorem 8). The Rozier–Terracol verification supplies the seed bound $n > B$. Combined with (6), the Farey interval then gives the two integer bounds. Rhin's estimate enters only in the case $d \ge 2^{65}$; its hypothesis is $H = \max(|u_1|, |u_2|) \ge 2$ [14, Prop. 6.3], and with $u_1 = k$, $u_2 = -d$ and $k > d$ one has $H = k$, so the form used is $\Lambda \ge k^{-13.3}$. Beyond these, the proof of Theorem 8 uses only Proposition 1 with (2) and (3), Lemma 5, and certified enclosures of $\ln 2$ and $\ln 3$. It uses no formalised or externally computed bound on individual runs; in particular it is independent of the two-run bounds discussed in Remark 11. Whether 81 is optimal for this method is open: the thresholds depend on the available convergents and on the choice of r in each row, and we have not searched that space exhaustively.

Remark 11 (Historical context, and what is claimed here). Every method used above is older than this note; what is new is the combination and the resulting explicit constants for *finite first crossings*.

Garner [3] is the direct precedent. He defined the coefficient stopping time, also formulated the conjectured equality with the stopping time — he cites Terras [16] as his reference [3] — and verified it for $n < 1\,150\,000$, identified the realizing residue class modulo $2^{1+p(m)}$ with $p(m) = \lfloor m\log_2 3 \rfloor$, and proved (his §6) that $\sigma(n) = \kappa(n)$ whenever the number of odd *terms* m is below a bound assembled from a study of the powers of 2 and 3 — his tabulated breakpoints are denominators of convergents of $\log_2 3$ — together with a verified range. His §7 applies this to cycles, obtaining at least 35 400 terms from $n > 6 \cdot 10^7$. Theorem 8 is that argument with two changes: the hypothesis bounds the number of odd *runs* rather than odd terms, which is strictly weaker because a single run may contain arbitrarily many odd steps, and the modern verification range and an explicit Farey sandwich replace his tabulated constants.

Simons and de Weger [15] developed the run-growth approach for nontrivial cycles, bounding the number of local minima; *Hercher [4]* strengthened that line to $m \ge 92$. Their counts and ours are the same kind of quantity but not the same quantity: in a cycle the odd runs close up, so the number of local minima equals the number of runs, whereas in a finite first-crossing word the first run may begin at the seed, which is not a local minimum, and the path stops instead of closing. The last odd run is complete: were step k odd we would have $2^{k-1} \le 3^{d-1}$ and $2^k > 3^d$, whence $3^d < 2^k \le 2 \cdot 3^{d-1} < 3^d$. The crossing therefore occurs in the even block that follows the last run, as the proof of Lemma 7 uses. Their bounds also use cyclicity, through the closing of the product around the cycle, which a finite path does not provide. With a modern verification range their cycle bound becomes $m \ge 83$ [4].

A different route to a run threshold proceeds from a bound on the first two runs of every small seed. Justin Alick has formalised such local two-run bounds in Lean 4 [1], among them the declaration `allPairs70`, which establishes $\ell_1 + \ell_2 \le 76$ for every odd $n < 2^{70}$ and is sharp at $n = 806\,134\,805\,793\,934\,253\,823$ with runs of lengths 8 and 68. Combined with the present estimates that bound yields a threshold of 76 runs. The proof of Theorem 8 does not use it, and the two routes are independent. **Nothing in this note raises the first-crossing threshold to 92, and Hercher's cycle conclusion must not be read as applying to finite first crossings.**

Lagarias [10] is the closest predecessor of all, and Remark 3 gives the locators. His (2.14) is our descent criterion verbatim after rearrangement, and his Theorem E already proves, by exactly our route — linear forms in logarithms fed into that criterion — that for admissible

words of length $k \geq k_0$ only the smallest element of the class can fail. Two things are left to us, and they are different from each other: Corollary 9 removes the length restriction $k \geq k_0$ from that exemption, while Theorem 8 proves outright descent, for the smallest element included, under a hypothesis on the run count rather than on k. Lagarias also records [10, §2.3] that Terras [16] and "later Garner" [3] conjectured the equality, which settles the question of independence raised above.

Rozier and Terracol [14] supply the verification range and, in their Remark 5.5, credit Simons with proposing classification by local maxima. Three classes of object must be kept apart. A *nontrivial cycle* is the subject of [3, 15, 4]. A *paradoxical sequence* [14, Def. 1.1] is a finite trajectory with $T^j(n) \geq n$ and coefficient below one at step j, for *any* such j. A *first-crossing failure*, the object here, is a paradoxical sequence with $j = \tau(n)$, hence a proper subclass. Their Theorem 5.3 excludes paradoxical sequences with $93 \leq j \leq 301\,993$ outright, and all with $4615 \leq n \leq 2.8 \cdot 10^{19}$; the 301 993 is the extent of a finite computation, whereas the bound $k \geq 10\,439\,860\,591$ of Theorem 8 is a lower bound on a counterexample derived from the Farey sandwich. A counterexample must satisfy both, and ours is the stronger constraint; there is no conflict.

Hikawa [5] reviews the problem and singles out exactly the triple studied here — the correction term q_v, the coefficient difference $2^k - 3^d$, and the least positive representative r_v of the residue class. **Priority for the specific first-crossing statements of this note is therefore not established**; we have not located a match, which is not the same as there being none.

4 A family with arbitrarily many runs

Proposition 12. *Repeat the block* 110 *exactly $h \geq 1$ times and append zeros up to the first coefficient crossing. Every eligible realizer descends at that crossing.*

Proof. One block acts by $H(x) + 5 = 9(x + 5)/8$, so -5 is its fixed point. The block's full coefficient is $9/8$ and its two nonempty proper-prefix coefficients are $3/2$ and $9/4$. Consequently the first crossing occurs among the appended zeros at $k = \lceil h \log_2 9 \rceil \leq 4h$. Iteration gives $H^h(n) + 5 = 9^h(n + 5)/8^h$. Integrality implies $8^h \mid n + 5$, hence $n \geq 8^h - 5$. The normalized memory is $5(1 - (8/9)^h) < 5$ and the appended zeros leave it unchanged.

For $h \leq 63$, $k \leq 252$ is below the crossing-length bound in Theorem 8. For $h \geq 64$, Rhin's estimate gives $n\Lambda \geq (8^h - 5)/(4h)^{13.3} > 5$. Indeed, $8^{64} - 5 > 5 \cdot 256^{14}$, and $((h + 1)/h)^{14} \leq (65/64)^{14} < 8$, so $8^h - 5 > 5(4h)^{14}$ persists by induction. A failure requires $n\Lambda < n(e^{\Lambda} - 1) \leq M_w < 5$, a contradiction. □

The prefix $(110)^h$ and the completed word generally have different least seeds. Write $m = k - 3h$ and $n = 8^h t - 5$. All these n form the binary prefix class, since $2^{3h} = 8^h$ is its exact modulus. Its least positive representative is $8^h - 5$. To append m zeros we also need $2^m \mid 9^h t - 5$, the block endpoint. At $h = 12$ this gives $k = 39$, $m = 3$ and $t \equiv 5 \pmod 8$. The prefix and completed-word least seeds are therefore 68 719 476 731 and 343 597 383 675, respectively.

5 A finite-word repetition bound

Proposition 13. *Let a first-crossing word from n begin with UU, where $|U| = \ell \geq 1$. Then $(4/3)^{\ell} < n + 1$. If the crossing fails to give descent, then*

$$\ell < \frac{\log_2(n+1)}{\log_2(4/3)} < \frac{14.3}{\log_2(4/3)} \log_2 k < 35 \log_2 k.$$

Proof. Set $x = T^\ell(n)$. Both n and x start with U, so the parity-class congruence gives $x \equiv n \pmod{2^\ell}$. Since $\ell < k$, $\rho_\ell \geq 1$. Equality would give $3^{d_\ell} = 2^\ell$, impossible for $\ell \geq 1$. Thus $x \geq \rho_\ell n > n$, so $x - n$ is a positive multiple of 2^ℓ. Equivalently $2^\ell(x-n)$ is a positive multiple of $2^{2\ell}$. At each step $T(y) + 1 \leq (3/2)(y+1)$, whence

$$2^\ell \leq x - n < x + 1 \leq (3/2)^\ell (n+1).$$

Dividing gives the first assertion, independently of whether the seed descends.

For a failure set $\Lambda = k \ln 2 - d \ln 3 > 0$. Then $n\Lambda < n(e^\Lambda - 1) \leq M_w \leq d/3$ by Proposition 1 and (3). Since $k \geq 2$ and $d \leq k$, Rhin's bound [14, Prop. 6.3] gives $\Lambda \geq k^{-13.3}$. It follows that $n < (d/3)k^{13.3} \leq k^{14.3}/3$ and hence $n + 1 < k^{14.3}$. Taking logarithms completes the proof, since $14.3/\log_2(4/3) < 35$. □

The restriction concerns repetition inside a single finite first-crossing prefix. The periodic-approximation results in [6] concern infinite parity words.

6 A restriction from reciprocal sums

Write $x_j = T^j(n)$ and form the product over the odd input values of the first k steps,

$$Q = \prod_{\substack{0 \leq j < k \\ x_j \text{ odd}}} \left(1 + \frac{1}{3x_j}\right).$$

Multiplying the single-step factors gives $x_k/n = \rho_k Q$. Hence Proposition 1 gives $Q = 1 + M_w/n$.

Let $D = 28\,019\,077\,177\,231\,758\,495$. For any finite shortcut path with distinct inputs all exceeding D, let R be the sum of the reciprocals of its odd inputs. Nikzad, Frinkle and Bagherzadeh prove $R < B_0$ in [12, Theorem 1A.1], where

$$B_0 = \frac{45}{256} + 10^{-6} + 10^{-15}.$$

Their proof gives a second rational upper bound by retaining the coefficient in (153) before the rounding in (154). Set

$$K = \frac{6\,719\,267}{13\,693\,680}, \qquad c = \frac{93\,657\,291\,203\,213}{10^{15}}, \qquad B_1 = \frac{15}{4}\,Kc + 10^{-6} + 10^{-15}.$$

Indeed, for each short gap bracket of total length m, (153) bounds the sum over its admissible starting values by $K2^{-m}$. The one-step balance bound (142) contributes the factor $15/4$. Equation (164) bounds the weighted bracket count by the exact dyadic $\sum_{m=61}^{1000} \widehat{C}_m 2^{-m} = 0.093657291203212905 56\ldots < c$. Adding the long-shell allowance from (139) and the boundary allowance from (149), as in (151)–(154), gives $R < B_1 < B_0$. These equation numbers refer to [12]. The unpublished verification [9] independently reconstructs the high-path estimate and reports an exact reproduction of the finite count in (164).

Proposition 14. *If a counterexample exists, let n have the least crossing length k among all counterexamples. For $i = 0, 1$ it satisfies*

$$0 < \Lambda = -\ln \rho_k < B_i/3, \qquad \rho_k > \exp(-B_i/3).$$

Proof. We use the external height record $H(y) \leq 2456$ through D, reported in [12, eq. (34)]. Here H counts ordinary Collatz operations to first arrival at 1. For $2 \leq y \leq D$, this gives $\tau(y) \leq \chi(y) \leq H(y) \leq 2456$, since shortcut steps combine ordinary operations. The lower bound on k in Theorem 8 therefore gives $n > D$. For $0 \leq j < k$, the prefix coefficient is at least one and its affine correction is nonnegative, so $x_j \geq n > D$.

If two of $x_0, \dots, x_{k-1}$ coincided, the intervening segment would contain a cycle of length $L < k$. Let m be its minimum. One full turn from m has positive affine correction, since any positive cycle has an odd input. The return to m therefore forces its coefficient below one. Thus $\tau(m) \le L$, while every value on the cycle is at least m, giving a failure of shorter crossing length. This proves that the k inputs are distinct. The high-path estimate applies even when the endpoint repeats an input. Finally $x_k \ge n$ gives $1 \le \rho_k Q$, and

$$\Lambda \le \ln Q < \frac{1}{3} \sum_{\substack{0 \le j < k \\ x_j \text{ odd}}} \frac{1}{x_j} < B_i/3.$$

□

In the coordinate $-\log_2 \rho_k$, the nominal first-crossing interval has length one. For each bound the surviving interval has length $B_i/(3 \ln 2)$. The values below are rounded for display.

	$\exp(-B_i/3)$	surviving width	excluded width
B_0	0.9430895072	8.45334%	91.54666%
B_1	0.9441733669	8.28763%	91.71237%

These percentages measure intervals of log coefficients. The restriction concerns a counterexample of minimal crossing length.

7 The open question

A shortest counterexample would have at least 82 odd runs, obey the coefficient bound in Proposition 14, and have no initial square whose period exceeds the bound in Proposition 13. For each remaining first-crossing word w, its coefficient and its additive correction are known exactly. The open estimate for its least eligible realizer $r(w)$ is

$$r(w) > \frac{C_w}{2^k - 3^d} = \frac{M_w}{\rho_k^{-1} - 1}.$$

Equality already gives a counterexample. A lower bound for $r(w)$ or an upper bound for the right side can certify a word if it preserves this strict inequality. The rational-shadow estimates of [7] supply bounds for individual blocks. Applying one to this question requires transporting it back to the starting seed and comparing it with the threshold of the actual first-crossing prefix.

Status of the results

It is worth separating three things explicitly.

Established before this note. The residue-class correspondence for parity words (Terras [16]); the affine form, the descent criterion $n > \varepsilon_k^{-1}\rho_k(v)$, and — in Theorem E — the statement that for admissible words of length $k \geq k_0$ only the smallest element of a class can fail, proved from linear forms in logarithms (Lagarias [10, §2.2–§2.3, Thm. E]); the coefficient stopping time, the conjecture $\kappa = \sigma$, and the method of combining a Diophantine bound on 2^a against 3^b with a finite verification range, including a descent theorem under a bound on the number of odd *terms* (Garner [3]); run-growth bounds for the number of local minima in a nontrivial cycle (Simons–de Weger [15], Hercher [4]); verification of $t = \tau$ to $2.8 \cdot 10^{19}$ and the exclusion of paradoxical sequences with $93 \leq j \leq 301\,993$ (Rozier–Terracol [14]); the reciprocal-sum estimate (Nikzad–Frinkle–Bagherzadeh [12]).

Offered here. The transfer of Garner's method to finite first crossings with the hypothesis weakened from odd *terms* to odd *runs*, which is what Lemma 5 buys; the resulting explicit constants $k \geq 10\,439\,860\,591$, $d \geq 6\,586\,818\,670$ and a threshold of 82 runs; the splitting and restart lemmas, which apply the run bound at several interior starts rather than once from the seed; descent for the whole repeated-110 family, which has arbitrarily many runs; the finite-word square bound $|U| < 35 \log_2 k$; Corollary 9, which removes the length restriction from the conclusion that only the least eligible seed can fail; and the reciprocal-sum restriction $\rho_k > 0.9441733669$ on a shortest counterexample. Subject to the priority caveat of Remark 11.

Unresolved. Combining Theorem 8 with Corollary 9, what remains is exactly this: **the least eligible seed of a first-crossing word with** 82 **or more odd runs**. A lower bound on that seed for a word with a shallow crossing and no long initial repetition — the inequality of §7 — is open. Whether 81 is optimal for the present method is also open; the thresholds depend on the available convergents and on the choice of r in each row, and that space has not been searched exhaustively. Whether Hercher's cycle techniques transfer to finite first-crossing trajectories is a separate question, not addressed here. Terras's conjecture itself is open, as is the Collatz conjecture.

Reproducibility

The companion verifier checks the numerical inequalities used in the proofs, recomputes the finite bracket count, and verifies the worked examples using integer and rational arithmetic. It lives in the companion repository `github.com/innerlightr-wq/terras-first-crossing`, with the verification code pinned at the immutable commit `a93b1e8`. Run the following commands from the project root.

```
python3 scripts/verify_crossing.py
python3 scripts/verify_81_runs.py
```

The first script is the pinned release verifier and is unchanged. The second checks the inputs of Theorem 8: the certified enclosures of $\ln 2$ and $\ln 3$, the sign of $e = q \ln 3 - p \ln 2$ for each convergent used, the four printed e-enclosures, the lower bound $d > 1.28 \cdot 10^{17}$, the whole tail $d \geq 2^{65}$ together with the failure at $j = 64$, the hypothesis of Corollary 9, that the crossing step is even, and that the displayed rounded form of the lower bound on d also closes. It reports 31 checks.

Its treatment of the five-row table should be described precisely, since an earlier wording overstated it. For each row the script searches all admissible convergents and keeps the one giving the smallest new ceiling — which turns out to be the convergent printed in the table — and it then compares its own computed H against a target that is looser than the printed height, the targets being $75, 72, 72, 74, 68$ against the printed $73, 71, 71, 74, 68$. It chains its own

computed ceilings from row to row, and it verifies the *printed*-ceiling chain explicitly only in the last row, where the printed ceiling $d < 3.1940 \cdot 10^{17}$ is shown to give $\lambda > 6.3202 \cdot 10^{-19}$ and again $H = 68$. For the initial start it uses the more conservative $(1 + 81/B) \cdot 81/\lambda$ in place of (4). Its 31 passing checks therefore certify that the table's chain closes, but they do not by themselves confirm the printed heights or the printed intermediate ceilings row by row.

That row-by-row confirmation is supplied by the independent verification of Loeb [11], which recomputes every row from the printed ceiling of the row above as well as from its own exact ceilings, obtains the printed heights $73, 71, 71, 74, 68$ in every row, and finds each printed λ and each printed ceiling on the safe side of its rigorous value; see the acknowledgments.

Two divisions of labour are worth stating. The numerical content of Theorem 8 — the convergents and their e-enclosures, the lower bound on d, the five rows, the final contradiction and the tail — is finite and has been recomputed in exact arithmetic, here and in [11]. The content of Lemmas 6 and 7 is analytic and is established by the proofs in the text; no computation replaces them, and the finite tests reported in [11] are evidence on data, not proof. Separately, the finite seed verification of Rozier–Terracol, Rhin's estimate, the high-path accounting of Nikzad–Frinkle–Bagherzadeh and the height record cited above are accepted as external inputs and are not reproved here or in [11]. The script checks the numerical inequalities with integers and rationals, encloses $\ln 2$ and $\ln 3$ by a 96-term series with a geometric tail, recomputes the finite bracket sum in (164) of [12], and encloses each exponential cutoff by alternating Taylor bounds. The SHA-256 digest of `verify_crossing.py` begins `f852c88311071a0d`; that of `verify_81_runs.py` begins `ca338694ac39ff93`. Both scripts are included in the deposit bundle. The orbit and residue-class examples are finite checks. The proofs for arbitrary word lengths are in the text.

The general case of Terras's conjecture remains open.

AI assistance and author responsibility

The author directed this research and used Claude (Anthropic) and ChatGPT (OpenAI) for proof exploration and revision, cross-checking the reciprocal-sum estimates against the source paper, developing and testing the exact-arithmetic verifiers, and drafting the manuscript. For Theorem 8 specifically, the splitting and restart lemmas were developed in author-directed work with ChatGPT, and the numerical certificate was then reconstructed independently by Claude from the lemma statements rather than from the author's script: the discard count in Lemma 6 was confirmed to be load-bearing, the convexity step in Lemma 7 was checked against every composition of $s \leq 13$, and the five rows were recomputed by optimising over all certified convergents, which improved the first three height bounds and reproduced the last two. The author takes responsibility for the mathematical claims, their interpretation, and any errors.

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I thank Franklin Loeb Jr. (Fellow, Ronin Institute for Independent Scholarship 2.0) for his independent mathematical review and computational verification of the 81-run argument [11].

Using his own exact-arithmetic implementation, he reconstructed all five numerical rows of Theorem 8, verified the printed bounds and rounding directions, and confirmed the dyadic tail cutoff at 2^{65}. He identified the narrowest numerical margin (0.0013 bits) and three matters corrected or clarified in this revision: the G_{81} tail calculation, the discard-count remark following Lemma 6, and the scope of the original verification script.

He examined Lemmas 6 and 7 line by line and tested the applicable proof steps that do not require the hypothetical failure assumption, as well as Lemma 7 in full, finding no violations.

His report [11] documents the verification methods and limitations. The external estimates

of Rozier–Terracol and Rhin were accepted as inputs, and §6 was not independently reviewed. Responsibility for the mathematical statements and any remaining errors rests with the author.

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